Baramidze et al. *Journal of Inequalities and Applications* (2023) 2023:66
https://doi.org/10.1186/s13660-023-02970-w

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Some weak type inequalities and almost everywhere convergence of Vilenkin–Nörlund means

Davit Baramidze[1,2], Nato Nadirashvili[1], Lars-Erik Persson[2,3*] and George Tephnadze[1]

*Correspondence:
larserik6pers@gmail.com
[2]UiT The Arctic University of Norway, P.O. Box 385, N-8505, Narvik, Norway
[3]Department of Mathematics and Computer Science, Karlstad University, 65188 Karlstad, Sweden
Full list of author information is available at the end of the article

Abstract

We prove and discuss some new weak type (1, 1) inequalities of maximal operators of Vilenkin–Nörlund means generated by monotone coefficients. Moreover, we use these results to prove a.e. convergence of such Vilenkin–Nörlund means. As applications, both some well-known and new inequalities are pointed out.

MSC: 26015; 42C10; 42B30

Keywords: Vilenkin system; Vilenkin group; Vilenkin–Fourier series; Nörlund means; Martingale Hardy space; *Weak* – L^p spaces; Maximal operator; Weak type inequality

1 Introduction

In 1947 Vilenkin [31] actually introduced a large class of compact groups (now called Vilenkin groups) and the corresponding characters. In particular, Vilenkin investigated the group G_m, which is a direct product of the additive groups $Z_{m_k} := \{0, 1, \dots, m_k - 1\}$ of integers modulo m_k, where $m := (m_0, m_1, \dots)$ are positive integers not less than 2, and introduced the Vilenkin systems $\{\psi_j\}_{j=0}^{\infty}$ as follows:

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x), \qquad r_k(x) := \exp(2\pi i x_k/m_k), \quad \left(i^2 = -1, x \in G_m, k \in \mathbb{N}\right),$$

where $\mathbb{N}_+$ denotes the set of positive integers and $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$. In this paper we discuss bounded Vilenkin groups only, that is, $\sup_{n\in\mathbb{N}} m_n < \infty$. The Vilenkin system is orthonormal and complete in $L^2(G_m)$ (see [31]). Specifically, we call this system the Walsh–Paley system when $m \equiv 2$.

It is well known (see e.g. the books [1] and [27]) that if $f \in L^1(G_m)$ and the Vilenkin series $T(x) = \sum_{j=0}^{\infty} c_j \psi_j(x)$ converges to f in L^1-norm, then

$$c_j = \int_{G_m} f \overline{\psi}_j \, d\mu := \widehat{f}(j), \quad j = 0, 1, 2, \dots,$$

where c_j is called the jth Vilenkin–Fourier coefficient and μ is the Haar measure on the locally compact abelian groups G_m, which coincide with the direct product of measures $\mu_k(\{j\}) := 1/m_k$ $(j \in Z_{m_k})$.

The classical theory of Hilbert spaces (for details, see e.g. the books [1, 27]) implies that if we consider the partial sums S_n, defined by

$$S_n f := \sum_{k=0}^{n-1} \widehat{f}(k)\psi_k,$$

with respect to any orthonormal systems and among them to Vilenkin systems, then the inequality $\|S_n f\|_2 \leq \|f\|_2$ holds. It follows that for every $f \in L^2$,

$$\|S_n f - f\|_2 \to 0 \quad \text{as } n \to \infty.$$

Since

$$S_n f(x) = \int_{G_m} f(t) D_n(x-t)\, d\mu(t)$$

and the Dirichlet kernels

$$D_n := \sum_{k=0}^{n-1} \psi_k \quad (n \in \mathbb{N}_+)$$

are not uniformly bounded in $L^1(G_m)$, the boundedness of partial sums does not hold from $L^1(G_m)$ to $L^1(G_m)$.

The analogue of Carleson's theorem for the Walsh system was proved by Billard [4] for $p = 2$ and by Sjölin [29] for $1 < p < \infty$, while for bounded Vilenkin systems it was proved by Gosselin [13]. In each proof, they show that the maximal operator of the partial sums is bounded on $L^p(G_m)$, i.e., there exists an absolute constant c_p such that

$$\left\| S^* f \right\|_p \leq C_p \|f\|_p, \quad \text{when } f \in L^p, 1 < p < \infty.$$

A recent proof of almost everywhere convergence of subsequences of Walsh–Fourier series was given by Demeter [7] in 2015. Hence, if $f \in L^p(G_m)$ for $p > 1$, then

$$S_n f \to f \quad \text{a.e. on } G_m.$$

Persson, Schipp, Tephnadze, and Weisz [22] (see also [25]) gave a new and shorter proof of almost everywhere convergence of Vilenkin–Fourier series of $f \in L^p(G_m)$, which was based on the theory of martingales.

The nth Nörlund mean L_n is defined by

$$L_n f := \frac{1}{l_n} \sum_{k=0}^{n-1} \frac{S_k f}{n-k}, \quad \text{where } l_n := \sum_{k=1}^{n} \frac{1}{k}.$$

In [9] Gát and Goginava proved some properties of the Nörlund logarithmic means of integrable functions in L^1 norm. Moreover, in [10] they proved that weak type $(1,1)$ inequality does not hold for the maximal operator of Nörlund logarithmic means L^*, defined by

$$L^*f := \sup_{n\in\mathbb{N}} |L_n f|,$$

but there exists an absolute constant c_p such that the inequality

$$\left\| L^*f \right\|_p \le c_p \|f\|_p \quad \text{when } f \in L^p, p > 1$$

holds.

If we define the so-called generalized number system based on m in the following way:

$$M_0 := 1, \qquad M_{k+1} := m_k M_k \quad (k \in \mathbb{N}),$$

then every $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum_{j=0}^{\infty} n_j M_j$, where $n_j \in Z_{m_j}$ $(j \in \mathbb{N})$ and only a finite number of n_js differ from zero. Moreover, if we consider the following restricted maximal operator $\widetilde{L}^*_{\#}$, defined by

$$\widetilde{L}^*_{\#} f := \sup_{n\in\mathbb{N}} |L_{M_n} f|,$$

then

$$y\mu\left\{\widetilde{L}^*_{\#} f > y\right\} \le c\|f\|_1, \quad f \in L^1(G_m), y > 0.$$

Hence, if $f \in L^1(G_m)$, then $L_{M_n} f \to f$ a.e. on G_m.

If we consider the Fejér means σ_n and Fejér kernels K_n, defined by

$$\sigma_n f := \frac{1}{n}\sum_{k=1}^{n} S_k f \quad \text{and} \quad K_n := \frac{1}{n}\sum_{k=0}^{n-1} D_k,$$

it is obvious that

$$\sigma_n f(x) = (f * K_n)(x) = \int_{G_m} f(t) K_n(x-t)\, d\mu(t).$$

Since $\|K_n\|_1 \le c < \infty$, we obtain that the Fejér means are bounded from the space L^p to the space L^p for $1 \le p \le \infty$. The a.e. convergence of Fejér means is due to Schipp [26] for Walsh series and Pál, Simon [21] (see also Simon, Weisz [28] and Weisz [28, 32–34]) for bounded Vilenkin series proved that the maximal operator of Fejér means σ^*, defined by

$$\sigma^* f := \sup_{n\in\mathbb{N}} |\sigma_n f|,$$

is of weak type $(1,1)$, from which the a.e. convergence follows by standard argument (see [14]). Another well-known summability method is the so-called (C,α)-means (denoted by

σ_n^α), which are defined by

$$\sigma_n^\alpha f := \frac{1}{A_n^\alpha}\sum_{k=1}^{n} A_{n-k}^{\alpha-1} S_k f, \qquad A_0^\alpha := 0, \qquad A_n^\alpha := \frac{(\alpha+1)\cdots(\alpha+n)}{n!}, \quad \alpha \neq -1,-2,\ldots.$$

It is well known that for $\alpha = 1$ this summability method coincides with the Fejér summation and for $\alpha = 0$ we just have the partial sums of the Vilenkin–Fourier series. Moreover, if we consider the maximal operator of the Cesáro means $\sigma^{\alpha,*}$, defined by

$$\sigma^{\alpha,*} f := \sup_{n\in\mathbb{N}} \left|\sigma_n^\alpha f\right| \quad \text{for } 0 < \alpha \leq 1,$$

then the following weak type inequality holds (for details, see [23]):

$$y\mu\left\{\sigma^{\alpha,*} f > y\right\} \leq c\|f\|_1, \quad f \in L^1(G_m), y > 0.$$

The boundedness of the maximal operator of the Cesáro means does not hold from $L^1(G_m)$ to the space $L^1(G_m)$. However,

$$\left\|\sigma_n^\alpha f - f\right\|_p \to 0, \quad \text{when } n \to \infty, \left(f \in L^p(G_m), 1 \leq p < \infty\right).$$

The nth Nörlund mean t_n for the Fourier series of f is defined by

$$t_n f := \frac{1}{Q_n}\sum_{k=1}^{n} q_{n-k} S_k f,$$

where $\{q_k : k \in \mathbb{N}\}$ is a sequence of nonnegative numbers and $Q_n := \sum_{k=0}^{n-1} q_k$.

If we assume that $q_0 > 0$ and $\lim_{n\to\infty} Q_n = \infty$, then it is well known (see [15]) that the summability method generated by $\{q_k : k \geq 0\}$ is regular if and only if $\lim_{n\to\infty} \frac{q_{n-1}}{Q_n} = 0$. The representation

$$t_n f(x) = \int_{G_m} f(t) F_n(x-t)\, d\mu(t), \quad \text{where } F_n := \frac{1}{Q_n}\sum_{k=1}^{n} q_{n-k} D_k$$

plays a central role in the sequel. The Nörlund means are generalizations of the Fejér, Cesàro, and Nörlund logarithmic means.

Móricz and Siddiqi [16] investigated the approximation properties of some special Nörlund means of Walsh–Fourier series of L^p functions in norm. Similar problems for the two-dimensional case can be found in papers by Nagy [17–20] (see also [5]).

Let us define the maximal operator t^* of Nörlund means by

$$t^* f := \sup_{n\in\mathbb{N}} |t_n f|,$$

and if $\{q_k : k \in \mathbb{N}\}$ is nonincreasing and satisfying the condition

$$\frac{1}{Q_n} = O\left(\frac{1}{n}\right) \quad \text{as } n \to \infty, \tag{1}$$

then in [23] it was proved that the weak type inequality

$$y\mu\left\{t^*f > y\right\} \leq c\|f\|_1, \quad f \in L^1(G_m), y > 0 \tag{2}$$

holds. When the sequence $\{q_k : k \in \mathbb{N}\}$ is nonincreasing, then the weak type $(1,1)$ inequality (2) holds for every maximal operator of Nörlund means. The boundedness of the maximal operator of the Nörlund means does not hold from $L^1(G_m)$ to the space $L^1(G_m)$. However,

$$\|t_nf - f\|_p \to 0 \quad \text{as } n \to \infty \left(f \in L^p(G_m), 1 \leq p < \infty\right).$$

Moreover, if $\{q_k : k \in \mathbb{N}\}$ is nondecreasing and satisfying the condition

$$\frac{q_{n-1}}{Q_n} = O\left(\frac{1}{n}\right) \quad \text{as } n \to \infty, \tag{3}$$

or $\{q_k : k \in \mathbb{N}\}$ is nonincreasing, then for any $f \in L^1(G_m)$ we have that

$$\lim_{n\to\infty} t_nf(x) = f(x)$$

for all Vilenkin–Lebesgue points of f.

In this paper we investigate a wider class of Nörlund means and prove that if $\{q_k : k \in \mathbb{N}\}$ is nondecreasing and satisfying the conditions

$$\frac{1}{Q_n} = O\left(\frac{1}{n^\alpha}\right) \quad \text{and} \quad q_n - q_{n+1} = O\left(\frac{1}{n^{2-\alpha}}\right) \quad \text{as } n \to \infty, \tag{4}$$

then the weak type inequality (2) holds. In particular, from this result follows almost everywhere convergence of such Nörlund means.

The paper is organized as follows: In Sect. 3 we present and prove the main results. Moreover, in order not to disturb our discussions in this section, some preliminaries are given in Sect. 2. Also some of these results are new and of independent interest.

2 Preliminaries

Lemma 1 (see [1, 12]) *Let $n \in \mathbb{N}$. Then*

$$D_{M_n}(x) = \begin{cases} M_n, & x \in I_n, \\ 0, & x \notin I_n. \end{cases}$$

Moreover, if $n \in \mathbb{N}$ and $1 \leq s_n \leq m_n - 1$, then

$$D_{s_nM_n} = D_{M_n}\sum_{k=0}^{s_n-1}\psi_{kM_n} = D_{M_n}\sum_{k=0}^{s_n-1}r_n^k$$

and

$$D_n = \psi_n\left(\sum_{j=0}^{\infty}D_{M_j}\sum_{k=m_j-n_j}^{m_j-1}r_j^k\right) \quad \text{for } n = \sum_{i=0}^{\infty}n_iM_i,$$

where $n = \sum_{i=0}^{\infty}n_iM_i$. We note that $\sum_{k=m_j-n_j}^{m_j-1}r_j^k \equiv 0$ for all $n_j = 0$.

Lemma 2 (see [8]) *Let $n > t$, $t, n \in \mathbb{N}$. Then*

$$K_{M_n}(x) = \begin{cases} \frac{M_t}{1-r_t(x)}, & x \in I_t \backslash I_{t+1}, x - x_t e_t \in I_n, \\ \frac{M_n+1}{2}, & x \in I_n, \\ 0, & \textit{otherwise}. \end{cases}$$

Lemma 3 (see [3, 6, 23, 24, 30]) *If $n \geq M_N$ and $\{q_k : k \in \mathbb{N}\}$ is a sequence of nondecreasing numbers, then there exists an absolute constant c such that*

$$\left| \frac{1}{Q_n} \sum_{j=M_N}^{n} q_{n-j} D_j \right| \leq \frac{c}{M_N} \left\{ \sum_{j=0}^{|n|} M_j |K_{M_j}| \right\}.$$

If the sequence $\{q_k : k \in \mathbb{N}\}$ is either nondecreasing and satisfying condition (3) or nonincreasing and satisfying condition (1), then the inequality

$$|F_n| \leq \frac{c}{n} \left\{ \sum_{j=0}^{|n|} M_j |K_{M_j}| \right\}$$

holds. On the other hand, if $\{q_k : k \in \mathbb{N}\}$ is a sequence of nonincreasing numbers satisfying (4) for $0 < \alpha < 1$, then there exists a constant c_α, depending only on α, such that the following inequality holds:

$$|F_n| \leq \frac{c_\alpha}{n^\alpha} \left\{ \sum_{j=0}^{|n|} M_j^\alpha |K_{M_j}| \right\}. \tag{5}$$

Lemma 4 (see [3, 6, 23, 24]) *Let $\{q_k : k \in \mathbb{N}\}$ be either a sequence of nondecreasing numbers or nonincreasing numbers satisfying condition (1) or nonincreasing numbers satisfying the conditions in (4). Then, for any $n, N \in \mathbb{N}_+$,*

$$\int_{G_m} F_n(x)\, d\mu(x) = 1,$$

$$\sup_{n\in\mathbb{N}} \int_{G_m} \left|F_n(x)\right| d\mu(x) \leq c < \infty,$$

$$\sup_{n\in\mathbb{N}} \int_{G_m \backslash I_N} \left|F_n(x)\right| d\mu(x) \to 0 \quad \textit{as } n \to \infty,$$

where

$$I_0(x) := G_m, \qquad I_n(x) := \{y \in G_m | y_0 = x_0, \dots, y_{n-1} = x_{n-1}\}$$

for any $x \in G_m$, $n \in \mathbb{N}$.

The next lemma is very important to study problems concerning almost everywhere convergence.

Lemma 5 (see [32]) *Suppose that the σ-sublinear operator V is bounded from L^{p_1} to L^{p_1} for some $1 < p_1 \leq \infty$ and*

$$\int_{\overline{I}} |Vf| \, d\mu \leq C\|f\|_1$$

for $f \in L^1$ and Vilenkin interval I, which satisfy

$$\operatorname{supp} f \subset I, \qquad \int_{G_m} f \, d\mu = 0. \tag{6}$$

Then the operator V is of weak type $(1,1)$, i.e., the following inequality holds:

$$\sup_{y>0} y\mu\big(\{Vf > y\}\big) \leq \|f\|_1.$$

Lemma 6 (see [14]) *Let*

$$T, T_n : L^p(G_m) \to L^p(G_m)$$

be sublinear operators for some $1 \leq p < \infty$ with T bounded and

$$T_n f \to Tf \quad \text{a.e. on } G_m \text{ as } n \to \infty,$$

for each $f \in X_0$, where X_0 is dense in $L^p(G_m)$. Set

$$T^* f := \sup_{n\in\mathbb{N}} |T_n f|, \quad f \in X.$$

If there is a constant $C > 0$, independent of f and n, such that the weak type inequalities

$$y^p \mu\big(\big\{|Tf| > y\big\}\big) \leq C\|f\|_X^p$$

and

$$y^p \mu\big(\big\{T^* f > y\big\}\big) \leq C\|f\|_X^p$$

hold for all $y > 0$ and $f \in L^p(G_m)$, then

$$Tf = \lim_{n\to\infty} T_n f \quad \text{a.e. on } G_m$$

for every $f \in L^p(G_m)$.

Next we prove a new lemma of independent interest, which is very important to prove almost everywhere convergence of Nörlund means generated by nondecreasing sequences $\{q_k : k \in \mathbb{N}\}$.

Lemma 7 *Let $n \in \mathbb{N}$ and $\{q_k : k \in \mathbb{N}\}$ be a sequence of nondecreasing numbers. Then*

$$\int_{\overline{I_N}} \sup_{n>M_N} \left| \frac{1}{Q_n} \sum_{j=M_N}^{n} q_{n-j} D_j(x) \right| d\mu(x) \leq c < \infty,$$

where c is an absolute constant.

Proof If we define

$$I_N^{k,l} := \begin{cases} I_N(0,\dots,0,x_k \neq 0,0,\dots,0,x_l \neq 0,x_{l+1},\dots,x_{N-1},\dots), \\ \quad \text{for } k<l<N, \\ I_N(0,\dots,0,x_k \neq 0,x_{k+1}=0,\dots,x_{N-1}=0,x_N,\dots), \\ \quad \text{for } l=N. \end{cases}$$

then we can decompose $\overline{I_N} := G_m \backslash I_N$ as

$$G_m \backslash I_N = \bigcup_{s=0}^{N-1} I_s \backslash I_{s+1} = \left(\bigcup_{k=0}^{N-2} \bigcup_{l=k+1}^{N-1} I_N^{k,l}\right) \cup \left(\bigcup_{k=0}^{N-1} I_N^{k,N}\right). \tag{7}$$

Let $n > M_N$ and

$$x \in I_N^{k,l}, \quad k=0,\dots,N-2, l=k+1,\dots,N-1.$$

By using Lemma 3, we get that

$$\begin{aligned} \left|\frac{1}{Q_n}\sum_{j=M_N}^{n} q_{n-j}D_j(x)\right| &\leq \frac{c}{M_N}\sum_{i=0}^{l} M_i \left|K_{M_i}(x)\right| \\ &\leq \frac{c}{M_N}\sum_{i=0}^{l} M_i M_k \\ &\leq \frac{cM_l M_k}{M_N} \end{aligned}$$

so that

$$\begin{aligned} \sup_{n>M_N}\left|\frac{1}{Q_n}\sum_{j=M_N}^{n} q_{n-j}D_j(x)\right| &\leq \frac{c}{M_N}\sum_{i=0}^{|n|} M_i \left|K_{M_i}(x)\right| \\ &\leq \frac{cM_l M_k}{M_N}. \end{aligned} \tag{8}$$

Let $n > M_N$ and $x \in I_N^{k,N}$. By using Lemma 1, we can conclude that

$$\left|D_n(x)\right| \leq cM_k$$

and

$$\begin{aligned} \left|\frac{1}{Q_n}\sum_{j=M_N}^{n} q_{n-j}D_j(x)\right| &\leq \frac{c}{Q_n}\sum_{j=M_N}^{n} q_{n-j}M_k \\ &\leq \frac{cQ_{n-M_N}}{Q_n} M_k \leq cM_k, \end{aligned}$$

so that

$$\sup_{n>M_N}\left|\frac{1}{Q_n}\sum_{j=M_N}^{n} q_{n-j}D_j(x)\right| \leq cM_k. \tag{9}$$

Hence, if we apply estimates (8) and (9), then we get that

$$\begin{aligned}
&\int_{\overline{I_N}} \sup_{n>M_N}\left|\frac{1}{Q_n}\sum_{j=M_N}^{n} q_{n-j}D_j(x)\right| d\mu \\
&= \sum_{k=0}^{N-2}\sum_{l=k+1}^{N-1}\sum_{x_j=0,j\in\{l+1,\dots,N-1\}}^{m_j-1} \int_{I_N^{k,l}} \sup_{n>M_N}\left|\frac{1}{Q_n}\sum_{j=M_N}^{n} q_{n-j}D_j(x)\right| d\mu \\
&\quad + \sum_{k=0}^{N-1}\int_{I_N^{k,N}} \sup_{n>M_N}\left|\frac{1}{Q_n}\sum_{j=M_N}^{n} q_{n-j}D_j(x)\right| d\mu \\
&\leq c\sum_{k=0}^{N-2}\sum_{l=k+1}^{N-1}\frac{m_{l+1}\cdots m_{N-1}}{M_N}\frac{M_l M_k}{M_N} + c\sum_{k=0}^{N-1}\frac{M_k}{M_N} \\
&\leq c\sum_{k=0}^{N-2}\frac{(N-k)M_k}{M_N} + c < C < \infty.
\end{aligned}$$

The proof is complete. □

We also need the following new lemmas.

Lemma 8 *Let $\{q_k : k \in \mathbb{N}\}$ be a sequence of nonincreasing numbers satisfying condition (1). Then there exists an absolute constant c such that*

$$\int_{\overline{I_N}} \sup_{n>M_N} |F_n|\, d\mu \leq c < \infty.$$

Proof The proof is analogous to that of Lemma 7. Hence, we leave out the details. □

Lemma 9 *Let $\{q_k : k \in \mathbb{N}\}$ be a sequence of nondecreasing numbers satisfying condition (3). Then there exists an absolute constant c such that*

$$\int_{\overline{I_N}} \sup_{n>M_N} |F_n|\, d\mu \leq c < \infty.$$

Proof Also in this case the proof is analogous to that of Lemma 7, so we leave out the details. □

Finally, we prove the following new estimate of independent interest.

Lemma 10 *Let $n \in \mathbb{N}$ and $\{q_k : k \in \mathbb{N}\}$ be a sequence of nonincreasing numbers satisfying the conditions in (4). Then there exists an absolute constant c such that*

$$\int_{\overline{I_N}} \sup_{n>M_N}\left|\frac{1}{Q_n}\sum_{j=M_N}^{n} q_{n-j}D_j(x)\right| d\mu(x) \leq c < \infty. \tag{10}$$

Proof Let $n > M_N$ and $x \in I_N^{k,l}$, $k = 0,\dots,N-2$, $l = k+1,\dots,N-1$. By combining Lemma 2 and (5) in Lemma 3, we get that

$$\left|\frac{1}{Q_n}\sum_{j=M_N}^{n} q_{n-j}D_j(x)\right| \leq \frac{cM_l^{\alpha}M_k}{M_N^{\alpha}},$$

so that

$$\sup_{n>M_N}\left|\frac{1}{Q_n}\sum_{j=M_N}^{n} q_{n-j}D_j(x)\right| \leq \frac{cM_l^{\alpha}M_k}{M_N^{\alpha}}. \tag{11}$$

Let $n > M_N$ and $x \in I_N^{k,N}$. By using Lemma 1, we can conclude that

$$\left|\frac{1}{Q_n}\sum_{j=M_N}^{n} q_{n-j}D_j(x)\right| \leq \frac{c}{Q_n}\sum_{j=M_N}^{n} q_{n-j}M_k \leq cM_k,$$

so that

$$\sup_{n>M_N}\left|\frac{1}{Q_n}\sum_{j=M_N}^{n} q_{n-j}D_j(x)\right| \leq cM_k. \tag{12}$$

By combining (7), (11), and (12), we can conclude that

$$\begin{aligned}
&\int_{\overline{I_N}}\sup_{n>M_N}\left|\frac{1}{Q_n}\sum_{j=M_N}^{n} q_{n-j}D_j\right| d\mu \\
&= \sum_{k=0}^{N-2}\sum_{l=k+1}^{N-1}\sum_{x_j=0,j\in\{l+1,\dots,N-1\}}^{m_j-1}\int_{I_N^{k,l}}\sup_{n>M_N}|F_n|\,d\mu \\
&\quad + \sum_{k=0}^{N-1}\int_{I_N^{k,N}}\sup_{n>M_N}|F_n|\,d\mu \\
&\leq c\sum_{k=0}^{N-2}\sum_{l=k+1}^{N-1}\frac{m_{l+1}\cdots m_{N-1}}{M_N}\frac{M_l^{\alpha}M_k}{M_N^{\alpha}} + c\sum_{k=0}^{N-1}\frac{M_k}{M_N} \\
&\leq c\sum_{k=0}^{N-2}\sum_{l=k+1}^{N-1}\frac{M_l^{\alpha-1}M_k}{M_N^{\alpha}} + c\sum_{k=0}^{N-1}\frac{M_k}{M_N} \\
&\leq c\sum_{l=k+1}^{N-1}\frac{M_k^{\alpha}}{M_N^{\alpha}} + c \\
&< C < \infty,
\end{aligned}$$

so (10) holds and the proof is complete. □

3 The main results

Our first main result reads as follows.

Theorem 1 *Let t_n be the Nörlund means and F_n be the corresponding Nörlund kernels such that*

$$\int_{\overline{I_N}} \sup_{n>M_N} \left| \frac{1}{Q_n} \sum_{k=M_N+1}^{n} q_{n-k} D_k(x) \right| d\mu(x) < c < \infty.$$

If the maximal operator t^* of Nörlund means is bounded from L^{p_1} to L^{p_1} for some $1 < p_1 \leq \infty$, then the operator t^* is of weak type $(1,1)$, i.e., for all $f \in L^1(G_m)$, the following weak type inequality holds:

$$\sup_{y>0} y\mu\{t^* f > y\} \leq \|f\|_1.$$

Proof In view of Lemma 5 we obtain that the proof is complete if we prove that

$$\int_{\overline{I}} |t^* f(x)| \, d\mu(x) \leq c\|f\|_1 \tag{13}$$

for every function f, which satisfies the conditions in (6), where I denotes the support of the function f.

Without loss of generality we may assume that f is a function with support I and $\mu(I) = M_N$. We may also assume that $I = I_N$. It is easy to see that

$$t_n f = 0 \quad \text{when } n \leq M_N.$$

Therefore, we can suppose that $n > M_N$. Moreover,

$$S_n f = 0 \quad \text{for } n \leq M_N,$$

so that

$$\frac{1}{Q_n} \left(\sum_{k=0}^{M_N} q_{n-k} S_k f(x) \right) = 0,$$

which implies that

$$\int_{I_N} \frac{1}{Q_n} \left(\sum_{k=0}^{M_n} q_{n-k} D_k(x-t) \right) f(t) \, d\mu(t) = 0.$$

Hence,

$$\begin{aligned} &|t^* f(x)| \\ &\quad \leq \sup_{n>M_N} \left| \int_{I_N} \frac{1}{Q_n} \left(\sum_{k=0}^{M_N} q_{n-k} D_k(x-t) \right) f(t) \, d\mu(t) \right| \\ &\qquad + \sup_{n>M_N} \left| \int_{I_N} \frac{1}{Q_n} \left(\sum_{k=M_N+1}^{n} q_{n-k} D_k(x-t) \right) f(t) \, d\mu(t) \right| \\ &\quad = \sup_{n>M_N} \left| \int_{I_N} \frac{1}{Q_n} \left(\sum_{k=M_N+1}^{n} q_{n-k} D_k(x-t) \right) f(t) \, d\mu(t) \right|. \end{aligned} \tag{14}$$

Let $t \in I_N$ and $x \in \overline{I_N}$. Then $x - t \in \overline{I_N}$ and (14) implies that

$$\begin{aligned}
&\int_{\overline{I_N}} \left| t^* f(x) \right| d\mu(x) \\
&\leq \int_{\overline{I_N}} \sup_{n > M_N} \int_{I_N} \left| \frac{1}{Q_n} \left(\sum_{k=M_N+1}^{n} q_{n-k} D_k(x-t) \right) f(t) \right| d\mu(t)\, d\mu(x) \\
&\leq \int_{\overline{I_N}} \int_{I_N} \sup_{n > M_N} \left| \frac{1}{Q_n} \left(\sum_{k=M_N+1}^{n} q_{n-k} D_k(x-t) \right) f(t) \right| d\mu(t)\, d\mu(x) \\
&\leq \int_{I_N} \int_{\overline{I_N}} \sup_{n > M_N} \left| \frac{1}{Q_n} \left(\sum_{k=M_N+1}^{n} q_{n-k} D_k(x-t) \right) f(t) \right| d\mu(x)\, d\mu(t) \\
&\leq \int_{I_N} \int_{\overline{I_N}} \sup_{n > M_N} \left| \frac{1}{Q_n} \left(\sum_{k=M_N+1}^{n} q_{n-k} D_k(x) \right) f(t) \right| d\mu(x)\, d\mu(t) \\
&\leq \int_{I_N} \left| f(t) \right| d\mu(t) \int_{\overline{I_N}} \sup_{n > M_N} \left| \frac{1}{Q_n} \left(\sum_{k=M_N+1}^{n} q_{n-k} D_k(x) \right) \right| d\mu(x) \\
&= \|f\|_1 \int_{\overline{I_N}} \sup_{n > M_N} \left| \frac{1}{Q_n} \left(\sum_{k=M_N+1}^{n} q_{n-k} D_k(x) \right) \right| d\mu(x) \\
&\leq c \|f\|_1.
\end{aligned}$$

Thus (13) holds, so the proof is complete. □

By using the same technique of proof, we obtain in a similar way the following result.

Theorem 2 *Let t_n be Nörlund means and F_n be the corresponding Nörlund kernels such that*

$$\int_{\overline{I_N}} \sup_{n > M_N} \left| F_n(t) \right| d\mu(t) < c < \infty.$$

If the maximal operator t^* of the Nörlund means is bounded from L^{p_1} to L^{p_1} for some $1 < p_1 \leq \infty$, then the operator t^* is of weak type $(1,1)$, i.e., the following weak type inequality

$$\sup_{y>0} y\mu\left\{ t^* f > y \right\} \leq \|f\|_1$$

holds for all $f \in L^1(G_m)$.

Next, we present a new related result concerning almost everywhere convergence of some summability methods. The study of almost everywhere convergence is one of the most difficult topics in Fourier analysis.

Theorem 3 *Let $f \in L^1(G_m)$ and t_n be the regular Nörlund means with nondecreasing sequences $\{q_k : k \in \mathbb{N}\}$. Then*

$$t_n f \to f \quad \text{a.e. as } n \to \infty.$$

Proof Since

$$S_n P = P \quad \text{for every } P \in \mathcal{P}$$

according to the regularity of Nörlund means with nondecreasing sequence $\{q_k : k \in \mathbb{N}\}$, we obtain that

$$t_n P \to P \quad \text{a.e. as } n \to \infty,$$

where $P \in \mathcal{P}$ is dense in the space L^1.

On the other hand, by combining Lemma 4, Lemma 7, and Theorem 1, we obtain that the maximal operator t^* of the Nörlund means with nondecreasing sequence $\{q_k : k \in \mathbb{N}\}$ is bounded from the space L^1 to the space *weak* – L^1, that is, the following weak type inequality holds:

$$\sup_{y>0} y\mu\left\{x \in G_m : \left|t^* f(x)\right| > y\right\} \le \|f\|_1.$$

Hence, according to Lemma 6, we obtain the claimed almost everywhere convergence of Nörlund means with nondecreasing sequence $\{q_k : k \in \mathbb{N}\}$:

$$t_n f \to f \quad \text{a.e. as } n \to \infty.$$

The proof is complete. □

Theorem 4 *Let $f \in L^1$ and t_n be the Nörlund means with nondecreasing sequence $\{q_k : k \ge 0\}$ satisfying the conditions in (3). Then*

$$t_n f \to f, \quad \text{a.e., as } n \to \infty.$$

Proof The proof is similar to the proof of Theorem 3 if we instead apply Lemma 4, Lemma 9, and Theorem 1, so we omit the details. □

Next we consider almost everywhere convergence of Nörlund means with nonincreasing sequence $\{q_k : k \in \mathbb{N}\}$.

Theorem 5 *Let $f \in L^1$ and t_n be the Nörlund means with nonincreasing sequence $\{q_k : k \in \mathbb{N}\}$ satisfying condition (1). Then*

$$t_n f \to f \quad \text{a.e. as } n \to \infty.$$

Proof The proof is quite analogous to that of Theorem 3 if we apply Lemma 4, Lemma 8, and Theorem 1, so we omit the details. □

Theorem 6 *Let $f \in L^1$ and t_n be Nörlund means with nonincreasing sequence $\{q_k : k \in \mathbb{N}\}$ satisfying the conditions in (4). Then*

$$t_n f \to f \quad \text{a.e. as } n \to \infty.$$

Proof The proof is similar to the proof of Theorem 3 if we instead apply Lemma 4, Lemma 10, and Theorem 1, so we omit the details. □

Theorem 7 *Let $f \in L^1$ and t_n be Nörlund means with nonincreasing sequence $\{q_k : k \in \mathbb{N}\}$. Then*

$$t_{M_n} f \to f \quad a.e.\ as\ n \to \infty.$$

Proof If we apply the fact that (see [8–10], and [25])

$$F_{M_n}(x) = D_{M_n}(x) - \psi_{M_n - 1}(x)\overline{F^{-1}}_{M_n}(x),$$

we can prove that if $\{q_k : k \in \mathbb{N}\}$ is a sequence of nonincreasing numbers, then, for any $N \in \mathbb{N}_+$,

$$\int_{G_m} F_{M_n}(x)\, d\mu(x) = 1,$$
$$\sup_{n \in \mathbb{N}} \int_{G_m} \left|F_{M_n}(x)\right| d\mu(x) \le c < \infty,$$
$$\sup_{n \in \mathbb{N}} \int_{G_m \setminus I_N} \left|F_{M_n}(x)\right| d\mu(x) \to 0 \quad \text{as } n \to \infty,$$

and

$$\int_{\overline{I_N}} \sup_{n > N} \left| \frac{1}{Q_n} \sum_{j=M_N}^{M_n} q_j D_{n-j}(x) \right| d\mu(x) \le c < \infty,$$

and also in this case the proof is absolutely analogous to that of Theorem 3, so we can omit the details. □

A number of special cases of our results are of particular interest and give both well-known and new information. We just give the following examples of such corollaries.

In particular, since σ_n and σ_n^α are regular Nörlund means with nondecreasing sequence $\{q_k : k \in \mathbb{N}\}$, we have the following consequences of our Theorems:

Corollary 1 (see [23] and [32]) *Let $f \in L^1$. Then*

$$\sigma_n f \to f, \quad a.e.,\ as\ n \to \infty$$

and

$$\sigma_n^\alpha f \to f, \quad a.e.,\ as\ n \to \infty,\ when\ 0 < \alpha < 1.$$

Corollary 2 (see [2] and [11]) *Let $f \in L^1$. Then*

$$L_{M_n} f \to f \quad a.e.\ as\ n \to \infty.$$

We also give the following examples of new consequences.

Corollary 3 *Let $f \in L^1$ and the summability method V_n^α be defined by*

$$V_n^\alpha f := \frac{1}{Q_n} \sum_{k=1}^{n} (n-k-1)^{\alpha-1} S_k f.$$

Then

$$V_n^\alpha f \to f \quad a.e.\ as\ n \to \infty, as\ 0 < \alpha < 1.$$

Proof Since V_n^α are Nörlund means with nonincreasing sequences $\{q_k : k \in \mathbb{N}\}$ satisfying the conditions in (4). Hence, the proof is complete by just using Theorem 6. □

Corollary 4 *Let $f \in L^1$ and the summability method β_n^α be defined by*

$$\beta_n^\alpha f := \frac{1}{Q_n} \sum_{k=1}^{n} \log^\alpha (n-k-1) S_k f.$$

Then

$$\beta_n^\alpha f \to f \quad a.e.\ as\ n \to \infty.$$

Proof We note that β_n^α are Nörlund means with nondecreasing sequences $\{q_k : k \in \mathbb{N}\}$. Hence, the proof is complete by just using Theorem 3. □

Corollary 5 *Let $f \in L^1$ and B_n be the Nörlund means with monotone and bounded sequence $\{q_k : k \in \mathbb{N}\}$. Then*

$$B_n f \to f \quad a.e.\ as\ n \to \infty.$$

Proof The proof follows from Theorems 4 and 5. □

Corollary 6 *Let $f \in L^1$ and the summability method U_n^α be defined by*

$$U_n^\alpha f := \frac{1}{Q_n} \sum_{k=1}^{n} \frac{S_k f}{(n-k-3) \ln^\alpha (n-k-3)}.$$

Then

$$U_{M_n}^\alpha f \to f \quad a.e., as\ n \to \infty.$$

Proof Obviously, U_n^α are regular Nörlund means with nonincreasing sequences $\{q_k : k \in \mathbb{N}\}$, the proof follows from Theorem 7. □

Acknowledgements
The work of Davit Baramidze was supported by Shota Rustaveli National Science Foundation grant PHDF-21-1702. The publication charges for this article have been funded by a grant from the publication fund of UiT The Arctic University of Norway. We thank both careful referees for some generous advice, which has improved the final version of this paper.

Funding
The publication charges for this manuscript are funded by UiT The Arctic University of Norway. Open Access funding provided by UiT The Arctic University of Norway (incl University Hospital of North Norway).

Availability of data and materials
Not applicable.

Declarations

Competing interests
The authors declare no competing interests.

Author contributions
DB and NN gave the idea and initiated the writing of this paper. LEP and GT followed up this with some complementary ideas. All authors read and approved the final manuscript.

Author details
[1]School of Science and Technology, The University of Georgia, 77a Merab Kostava St, Tbilisi 0128, Georgia. [2]UiT The Arctic University of Norway, P.O. Box 385, N-8505, Narvik, Norway. [3]Department of Mathematics and Computer Science, Karlstad University, 65188 Karlstad, Sweden.

Received: 16 May 2022 Accepted: 12 April 2023 Published online: 04 May 2023

References

1. Agaev, G., Vilenkin, N., Dzahafarly, G., Rubinstein, A.: Multiplicative Systems of Functions and Harmonic Analysis on Zero-Dimensional Groups. Ehim, Baku (1981)
2. Baramidze, D., Persson, L.-E., Singh, H., Tephnadze, G.: Some new results and inequalities for subsequences of Nörlund logarithmic means of Walsh-Fourier series. J. Inequal. Appl. (2022). https://doi.org/10.1186/s13660-022-02765-5
3. Baramidze, L., Persson, L.E., Tephnadze, G., Wall, P.: Sharp $H_p - L_p$ type inequalities of weighted maximal operators of Vilenkin-Nörlund means and its applications. J. Inequal. Appl. (2016). https://doi.org/10.1186/s13660-016-1182-1
4. Billard, P.: Sur la convergence presque partout des séries de Fourier-Walsh des fonctions de l'espace $L^2(0,1)$. Stud. Math. **28**, 363–388 (1967)
5. Blahota, I., Nagy, K., Tephnadze, G.: Approximation by Marcinkiewicz Θ-means of double Walsh-Fourier series. Math. Inequal. Appl. **22**(3), 837–853 (2019)
6. Blahota, I., Persson, L.E., Tephnadze, G.: On the Nörlund means of Vilenkin-Fourier series. Czechoslov. Math. J. **65**(4), 983–1002 (2015)
7. Demeter, C.: A guide to Carleson's theorem. Rocky Mt. J. Math. **45**(1), 169–212 (2015)
8. Gát, G.: Investigations of certain operators with respect to the Vilenkin system. Acta Math. Hung. **61**, 131–149 (1993)
9. Gát, G., Goginava, U.: Uniform and L-convergence of logarithmic means of Walsh-Fourier series. Acta Math. Sin. Engl. Ser. **22**(2), 497–506 (2006)
10. Gát, G., Goginava, U.: On the divergence of Nörlund logarithmic means of Walsh-Fourier series. Acta Math. Sin. Engl. Ser. **25**(6), 903–916 (2009). (English summary)
11. Goginava, U.: Almost everywhere convergence of subsequences of logarithmic means of Walsh-Fourier series. Acta Math. Acad. Paedagog. Nyházi. **21**, 169–175 (2005)
12. Golubov, B.I., Efimov, A.V., Skvortsov, V.A.: Walsh Series and Transforms. Mathematics and Its Applications, vol. 64. Nauka, Moscow (1987). (Russian), English transl, Kluwer Academic, Dordrecht, 1991
13. Gosselin, J.: Almost everywhere convergence of Vilenkin-Fourier series. Trans. Am. Math. Soc. **185**, 345–370 (1973)
14. Marcinkiewicz, I., Zygmund, A.: On the summability of double Fourier series. Fundam. Math. **32**, 112–132 (1939)
15. Moore, C.N.: Summable Series and Convergence Factors, Summable Series and Convergence Factors. Dover, New York (1966)
16. Móricz, F., Siddiqi, A.: Approximation by Nörlund means of Walsh-Fourier series. J. Approx. Theory **70**(3), 375–389 (1992). (English summary)
17. Nagy, K.: Approximation by Nörlund means of quadratical partial sums of double Walsh-Fourier series. Anal. Math. **36**(4), 299–319 (2010)
18. Nagy, K.: Approximation by Cesáro means of negative order of Walsh-Kaczmarz-Fourier series. East J. Approx. **16**(3), 297–311 (2010)
19. Nagy, K.: Approximation by Nörlund means of Walsh-Kaczmarz-Fourier series. Georgian Math. J. **18**(1), 147–162 (2011)
20. Nagy, K.: Approximation by Nörlund means of double Walsh-Fourier series for Lipschitz functions. Math. Inequal. Appl. **15**(2), 301–322 (2012)
21. Pál, J., Simon, P.: On a generalization of the concept of derivative. Acta Math. Acad. Sci. Hung. **29**(1–2), 155–164 (1977)
22. Persson, L.-E., Schipp, F., Tephnadze, G., Weisz, F.: An analogy of the Carleson-Hunt theorem with respect to Vilenkin systems. J. Fourier Anal. Appl. (2022). https://doi.org/10.1007/s00041-022-09938-2
23. Persson, L.E., Tephnadze, G., Wall, P.: On the maximal operators of Vilenkin-Nörlund means. J. Fourier Anal. Appl. **21**(1), 76–94 (2015)
24. Persson, L.E., Tephnadze, G., Wall, P.: Some new (H_p, L_p) type inequalities of maximal operators of Vilenkin-Nörlund means with non-decreasing coefficients. J. Math. Inequal. **9**(4), 1055–1069 (2015)
25. Persson, L.E., Tephnadze, G., Weisz, F.: Martingale Hardy Spaces and Summability of One-Dimensional Vilenkin-Fourier Series. Springer, Basel (2022)
26. Schipp, F., Certain, F.: Rearrangements of series in the Walsh system. Mat. Zametki **18**(2), 193–201 (1975). (Russian)
27. Schipp, F., Wade, W.R., Simon, P., Pál, J.: Walsh Series. An Introduction to Dyadic Harmonic Analysis. Hilger, Bristol (1990)

28. Simon, P., Weisz, F.: Weak inequalities for Cesáro and Riesz summability of Walsh-Fourier series. J. Approx. Theory **151**(1), 1–19 (2008)
29. Sjölin, P.: An inequality of Paley and convergence a.e. of Walsh-Fourier series. Ark. Mat. **7**, 551–570 (1969)
30. Tephnadze, G.: Martingale Hardy Spaces and Summability of the One Dimensional Vilenkin-Fourier Series. PhD thesis, Department of Engineering Sciences and Mathematics, LuleåUniversity of Technology (2015). ISSN 1402-1544
31. Vilenkin, N.Y.: On a class of complete orthonormal systems. Am. Math. Soc. Transl. **28**(2), 1–35 (1963)
32. Weisz, F.: Martingale Hardy Spaces and Their Applications in Fourier Analysis. Lecture Notes in Mathematics, vol. 1568. Springer, Berlin (1994)
33. Weisz, F.: Cesáro summability of one- and two-dimensional Walsh-Fourier series. Anal. Math. **22**(3), 229–242 (1996)
34. Weisz, F.: (C,α) summability of Walsh-Fourier series, summability of Walsh-Fourier series. Anal. Math. **27**(2), 141–155 (2001)

Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.