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The Image of the Lepowsky Homomorphism for the Group F_4^{-20}

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Let G_o be a semisimple Lie group, let K_o be a maximal compact subgroup of G_o and let $\mathfrak{k} \subset \mathfrak{g}$ denote the complexification of their Lie algebras. Let G be the adjoint group of $\mathfrak{g}$ and let K be the connected Lie subgroup of G with Lie algebra $\mathrm{ad}(\mathfrak{k})$. If $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$, then $U(\mathfrak{g})^K$ will denote the centralizer of K in $U(\mathfrak{g})$. Also let $P : U(\mathfrak{g}) \longrightarrow U(\mathfrak{k}) \otimes U(\mathfrak{a})$ be the projection map corresponding to the direct sum $U(\mathfrak{g}) = (U(\mathfrak{k}) \otimes U(\mathfrak{a})) \oplus U(\mathfrak{g})\mathfrak{n}$ associated to an Iwasawa decomposition of G_o adapted to K_o. In this paper, we give a characterization of the image of $U(\mathfrak{g})^K$ under the injective antihomorphism $P : U(\mathfrak{g})^K \longrightarrow U(\mathfrak{k})^M \otimes U(\mathfrak{a})$, considered by Lepowsky in [12], when G_o is isomorphic to the rank 1 real form F_4^{-20} of the exceptional Lie group F_4.

1 Introduction

Let G_o be a connected, noncompact, real semisimple Lie group with finite center, and let K_o denote a maximal compact subgroup of G_o. We denote with $\mathfrak{g}_o$ and $\mathfrak{k}_o$ the Lie algebras of G_o and K_o, and $\mathfrak{k} \subset \mathfrak{g}$ will denote the respective complexified Lie algebras. Let G be the adjoint group of $\mathfrak{g}$ and let K be the connected Lie subgroup of G with Lie algebra $\mathrm{ad}(\mathfrak{k})$. Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$ and let $U(\mathfrak{g})^K$ denote the centralizer of K in $U(\mathfrak{g})$.

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In order to contribute to the understanding of $U(\mathfrak{g})^K$ B. Kostant suggested to consider the projection map $P : U(\mathfrak{g}) \longrightarrow U(\mathfrak{k}) \otimes U(\mathfrak{a})$, corresponding to the direct sum $U(\mathfrak{g}) = (U(\mathfrak{k}) \otimes U(\mathfrak{a})) \oplus U(\mathfrak{g})\mathfrak{n}$ associated to an Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ adapted to $\mathfrak{k}$. In [12], Lepowsky studied the restriction of P to $U(\mathfrak{g})^K$ and proved, among other things, that one has the following exact sequence:

$$0 \longrightarrow U(\mathfrak{g})^K \xrightarrow{P} U(\mathfrak{k})^M \otimes U(\mathfrak{a}),$$

where $U(\mathfrak{k})^M$ denotes the centralizer of M in $U(\mathfrak{k})$, M being the centralizer of $\mathfrak{a}$ in K. Moreover, if $U(\mathfrak{k})^M \otimes U(\mathfrak{a})$ is given the tensor product algebra structure, then P becomes an antihomomorphism of algebras. Hence to go any further in this direction it is necessary to determine the image of P.

To determine the image $P(U(\mathfrak{g})^K)$, Tirao introduced in [15] a subalgebra B of $U(\mathfrak{k})^M \otimes U(\mathfrak{a})$ which is described in detail in §2. This subalgebra B is defined by a set of linear equations in $U(\mathfrak{k})$ derived from certain embeddings between Verma modules and he proved, among other things, that $P(U(\mathfrak{g})^K) \subset B$ for any G_o. We point out that B is defined in a uniform way for any noncompact real semisimple Lie group G_o with finite center (see Theorem 2.3). When G_o has real rank 1 the definition of B becomes very transparent and it is given below in Definition 2.5.

More recently, in [3, 4], we proved that $P(U(\mathfrak{g})^K) = B$ for $G_0 = \mathrm{Sp}(n, 1)$ and for $G_0 = \mathrm{SO}(n,1)$ or $\mathrm{SU}(n,1)$ (see also [11, 15]). Hence these results established that $P(U(\mathfrak{g})^K) = B$ for every classical real rank 1 semisimple Lie group with finite center. This paper is devoted to proving that this result also holds for the rank 1 real form F_4^{-20} of F_4. The main result of this paper is the following:

Theorem 1.1. If G_o is isomorphic to F_4^{-20}, then $P(U(\mathfrak{g})^K) = B$. □

This result confirms our old belief that the image of P can be described in a uniform way for all real rank 1 semisimple Lie groups, as it is stated in the following theorem.

Theorem 1.2. Let G_o be a real rank 1 semisimple Lie group. Then the image of the Lepowsky homomorphism P is the algebra B. □

The proof of Theorem 1.2 follows a general pattern in all cases, however, at certain points in the argument there are some differences. Certainly, the cases of $\mathrm{Sp}(n, 1)$ and F_4^{-20} are the most difficult to handle.

Originally, one of the main motivations to study the image of P was the hope that understanding it would lead to a classification of the irreducible $(\mathfrak{g}, K)$-modules. However, due to the difficulties encountered in the characterization of $P(U(\mathfrak{g})^K)$ and the enormous progress achieved in the classification of the irreducible $(\mathfrak{g}, K)$-modules this is no longer our main motivation. Nevertheless, taking into account that P is a remarkable injective antihomomorphism between two very important algebras associated to a semisimple Lie group, we believe that our description of the image of P might have other applications yet to be discovered. For instance, we think that a new proof of the remarkable result of Knop (see [9]), describing the center of $U(\mathfrak{g})^K$ as $Z(U(\mathfrak{g})^K) = Z(U(\mathfrak{g})) \otimes Z(U(\mathfrak{k}))$, could be obtained by using the map P and the algebra B, at least for G_o of real rank 1. In fact, in [16] this program was carried out by Tirao for the groups SO(n,1) and SU(n,1). Also, it is reasonable to expect $P(U(\mathfrak{g})^K)$ to be the subalgebra of invariants under the action of certain group acting on $U(\mathfrak{k})^M \otimes U(\mathfrak{a})$ and since B is the solution space of a system of linear equations in $U(\mathfrak{k})^M \otimes U(\mathfrak{a})$, this system might help us to discover this group action.

The proof of Theorem 1.1 follows the same ideas used to prove the analog theorem for the group $\mathrm{Sp}(n, 1)$, however, we had to overcome some difficulties to establish the transversality results needed and the a priori estimates of the Kostant degrees. In Section 6, we give a new and simplified version of the corresponding transversality results obtained in the symplectic case (see [3, Section 4]). This version is sufficient because of the introduction of a simplifying hypothesis called the *degree property*, which is done in Section 7. In this section, we use this property to obtain an a priori estimate of the Kostant degree of certain elements $b \in B$. This allows us to reduce the proof of Theorem 1.1 to proving Theorem 7.12 (see Proposition 7.14). The proof of this last theorem is given in Section 8 following the ideas developed in the symplectic case. In fact, most of the results proved in [3, Section 6] hold in this case with appropriate changes.

2 The Algebra B and the Image of $U(\mathfrak{g})^K$

Let G_o be a connected, noncompact, real semisimple Lie group with finite center, and let K_o be a maximal compact subgroup of G_o. Let $\mathfrak{g}_o$ and $\mathfrak{k}_o$ be the Lie algebras of G_o and K_o, and let $\mathfrak{k} \subset \mathfrak{g}$ be their respective complexifications. Also let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ corresponding to (G_o, K_o) and let θ denote the associated Cartan involution. Let $\mathfrak{t}_o$ be a Cartan subalgebra of the Lie algebra $\mathfrak{m}_o$ of M_o. Set $\mathfrak{h}_o = \mathfrak{t}_o \oplus \mathfrak{a}_o$ and let $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ be the corresponding complexification. Then $\mathfrak{h}_o$ and $\mathfrak{h}$ are Cartan subalgebras

of $\mathfrak{g}_o$ and $\mathfrak{g}$, respectively. Let Δ be the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Choose a Borel subalgebra $\mathfrak{t} \oplus \mathfrak{m}^+$ of the complexification $\mathfrak{m}$ of $\mathfrak{m}_o$ and take $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{m}^+ \oplus \mathfrak{n}$ as a Borel subalgebra of $\mathfrak{g}$. Let Δ^+ be the corresponding set of positive roots and set, $\mathfrak{g}^+ = \mathfrak{m}^+ \oplus \mathfrak{n}$ and $\mathfrak{g}^- = \sum_{\alpha \in \Delta^+} \mathfrak{g}_{-\alpha}$. Let $\langle\ ,\ \rangle$ denote the Killing form of $\mathfrak{g}$ and, for each $\alpha \in \Delta$, let $H_\alpha \in \mathfrak{h}$ be the unique element such that $\phi(H_\alpha) = 2\langle \phi, \alpha \rangle / \langle \alpha, \alpha \rangle$ for all $\phi \in \mathfrak{h}^*$, and let X_α denote a nonzero root vector associated to α.

If $\mu \in \mathfrak{h}^*$ consider the Verma module $M(\mu) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_{\mu-\rho}$, where $\mathbb{C}_{\mu-\rho}$ denotes the 1-dimensional $\mathfrak{b}$-module where $\mathfrak{h}$ acts by $\mu - \rho$ and $\mathfrak{g}^+$ acts trivially. Then $M(\mu)$ is a $U(\mathfrak{g})$-module by left multiplication in the first factor with canonical generator $1_\mu = 1 \otimes 1$.

For the sake of completeness and in the benefit of the reader we will now summarize the results obtained by Tirao in [15, Section 2]. These results give a family of equations that are satisfied by every element of $P(U(\mathfrak{g})^K)$, moreover, these equations have proved to be enough to obtain an a priori description of $P(U(\mathfrak{g})^K)$ as a certain subalgebra of $U(\mathfrak{k})^M \otimes U(\mathfrak{a})$. We are going to consider embeddings $M(\mu_1) \subset M(\mu_2)$ between Verma modules. The pairs (μ_1, μ_2) for which $\mathrm{Hom}_{U(\mathfrak{g})}(M(\mu_1), M(\mu_2)) = 1$ are described by the B-G-G Theorem (see [1]). In particular, if $\mu(H_\alpha) = 2\langle \mu, \alpha \rangle / \langle \alpha, \alpha \rangle = n \in \mathbb{N}$ for some $\alpha \in \Delta^+$, it is shown in [1] that $M(\mu - n\alpha) \subset M(\mu)$. Moreover, every embedding $M(\mu_1) \subset M(\mu_2)$ is a composition of embeddings of this kind. The following proposition, due to Shapovalov (see [13]), is a refinement of results contained in [1].

Proposition 2.1. For every $\alpha \in \Delta^+$ and $n \in \mathbb{N}$ there exists an element $\theta_{\alpha,n} \in U(\mathfrak{g}^- \oplus \mathfrak{h})$ of weight $-n\alpha$ with the following properties:

(i) $[X_\gamma, \theta_{\alpha,n}] \in U(\mathfrak{g})(H_\alpha + \rho(H_\alpha) - n) + U(\mathfrak{g})\mathfrak{g}^+$ for all $\gamma \in \Delta^+$.

(ii) If $\{\alpha_1, \ldots, \alpha_r\} \subset \Delta^+$ is the set of simple roots and $\alpha = \sum_i \ell_i \alpha_i$, then

$$\theta_{\alpha,n} = \prod_i X_{-\alpha_i}^{n\ell_i} + \sum_j a_j b_j,$$

where $a_j \in U(\mathfrak{g}^-)$ is of weight $-n\alpha$, $b_j \in U(\mathfrak{h})$ and the degree of a_j is less than $n\sum_i \ell_i$.

(iii) The element $\theta_{\alpha,n}$ is uniquely determined by properties (i) and (ii) modulo the left ideal of $U(\mathfrak{g}^- \oplus \mathfrak{h})$ generated by $H_\alpha + \rho(H_\alpha) - n$. □

Remarks.

(i) If α is a simple root, then we may choose $\theta_{\alpha,n} = X_{-\alpha}^n$.

(ii) If $1_\mu = 1 \otimes 1$ is the canonical generator of $M(\mu)$, then $\theta_{\alpha,n} \cdot 1_\mu$ can be identified with the canonical generator $1_{\mu - n\alpha}$ of $M(\mu - n\alpha) \subset M(\mu)$.

If $\mu \in \mathfrak{h}^*$, let $A_\mu = \{X \in U(\mathfrak{k}) : X \cdot 1_\mu = 0\}$ be the annihilator of 1_μ in $U(\mathfrak{k})$. The algebra $U(\mathfrak{a})$ is just the symmetric algebra $S(\mathfrak{a})$, which can be identified with $S(\mathfrak{a}^*)$, hence we may regard every element $b \in U(\mathfrak{k}) \otimes U(\mathfrak{a})$ as a polynomial function on $\mathfrak{a}$ with coefficients in $U(\mathfrak{k})$. Next, we recall [15, Proposition 2] and refer the reader to [15, Section 2] for a detailed proof of this result. □

Proposition 2.2. (i) If $\alpha \in \Delta^+$ and $\mu(H_\alpha) = n \in \mathbb{N}$. Then,

$$P(\theta_{\alpha,n})(\mu - \rho)P(u)(\mu - \rho) \equiv P(u)(\mu - n\alpha - \rho)P(\theta_{\alpha,n})(\mu - \rho) \tag{1}$$

for every $u \in U(\mathfrak{g})^K$. Here, the congruence is modulo A_μ.

(ii) The annihilator of 1_μ in $U(\mathfrak{k})$ is given as follows:

$$A_\mu = U(\mathfrak{k})\mathfrak{m}^+ + \sum_{H \in \mathfrak{t}} U(\mathfrak{k})(H - \mu(H) + \rho(H)). \qquad \square$$

For $\alpha \in \Delta^+$ write $H_\alpha = Y_\alpha + Z_\alpha$ where $Y_\alpha \in \mathfrak{t}$ and $Z_\alpha \in \mathfrak{a}$, and let $P_+ = \{\alpha \in \Delta^+ : Z_\alpha \neq 0\}$. If $\alpha \in P_+$ let $\mathfrak{a}_\alpha = \{H \in \mathfrak{a} : \alpha(H) = 0\}$. Then $\mathfrak{a} = \mathfrak{a}_\alpha \oplus \mathbb{C}Z_\alpha$ and we can consider the elements in $U(\mathfrak{k}) \otimes U(\mathfrak{a})$ as polynomials in Z_α with coefficients in $U(\mathfrak{k}) \otimes U(\mathfrak{a}_\alpha)$. In the next theorem, the congruence modulo A_μ in (1) is replaced by a congruence modulo $U(\mathfrak{k})\mathfrak{m}^+ \otimes U(\mathfrak{a}_\alpha)$, a detailed proof of this result can be found in [15, Theorem 4]. On the other hand, since $\alpha \in \Delta^+$ is a simple root, in view of Remark (i), we can replace $\theta_{\alpha,n}$ by $X^n_{-\alpha}$ for every $n \in \mathbb{N}$. For the proof of part (ii), we refer the reader to [15, Theorem 5].

Theorem 2.3. (i) Let $\alpha \in P_+$ be a simple root and $n \in \mathbb{N}$. Then for every $u \in U(\mathfrak{g})^K$ the element $b = P(u)$ satisfies

$$P(X^n_{-\alpha})(n - Y_\alpha - 1)b(n - Y_\alpha - 1) \equiv b(-n - Y_\alpha - 1)P(X^n_{-\alpha})(n - Y_\alpha - 1), \tag{2}$$

where the congruence is modulo $U(\mathfrak{k})\mathfrak{m}^+ \otimes U(\mathfrak{a}_\alpha)$.

(ii) Let B be the set of all $b \in U(\mathfrak{k})^M \otimes U(\mathfrak{a})$ that satisfy (2) for every simple root $\alpha \in P_+$ and every $n \in \mathbb{N}$. Then B is a subalgebra of $U(\mathfrak{k})^M \otimes U(\mathfrak{a})$. □

If $\alpha \in P_+$ is a simple root set $E_\alpha = X_{-\alpha} + \theta X_{-\alpha}$. One can show that if $\alpha \in P_+$ is simple and $Y_\alpha \neq 0$, then $P(X_{-\alpha}^n) = E_\alpha^n$. We refer the reader to [15, Corollary 6] for a proof of this result. Then from Theorem 2.3 we obtain the following theorem:

Theorem 2.4. Let $\alpha \in P_+$ be a simple root such that $Y_\alpha \neq 0$. Then for every $n \in \mathbb{N}$ and $u \in U(\mathfrak{g})^K$ the element $b = P(u)$ satisfies

$$E_\alpha^n b(n - Y_\alpha - 1) \equiv b(-n - Y_\alpha - 1)E_\alpha^n, \tag{3}$$

where the congruence is modulo $U(\mathfrak{k})\mathfrak{m}^+ \otimes U(\mathfrak{a}_\alpha)$. □

If G_o has real rank 1, we have $\mathfrak{a} = \mathbb{C}Z_\alpha$ for any $\alpha \in P_+$, hence the congruence in Equation (3) is modulo the left ideal $U(\mathfrak{k})\mathfrak{m}^+$ of $U(\mathfrak{k})$. Also, in this paper, we are interested in the case when G_o is isomorphic to the rank 1 real form F_4^{-20} of the Lie group F_4. In this case, there is only one simple root $\alpha \in P_+$ and $Y_\alpha \neq 0$, therefore we can restate the definition of the algebra B (see Theorem 2.3(ii)) as follows:

Definition 2.5. The algebra B is the set of all $b \in U(\mathfrak{k})^M \otimes U(\mathfrak{a})$ such that

$$E_\alpha^n b(n - Y_\alpha - 1) \equiv b(-n - Y_\alpha - 1)E_\alpha^n \bmod (U(\mathfrak{k})\mathfrak{m}^+), \tag{4}$$

for all simple roots $\alpha \in P_+$ and all $n \in \mathbb{N}$. □

In view of Theorem 2.4, we have $P(U(\mathfrak{g})^K) \subset B$, in this paper, we will show that equality holds for the rank 1 real form F_4^{-20} of F_4.

In order to prove Theorem 1.1 we will now introduce some notation and recall known results. Let Γ denote the set of all equivalence classes of irreducible holomorphic finite-dimensional K-modules V_γ such that $V_\gamma^M \neq 0$. Any $\gamma \in \Gamma$ can be realized as a submodule of all harmonic polynomial functions on $\mathfrak{p}$, homogeneous of degree d, for a uniquely determined $d = d(\gamma)$ (see [10]). We shall refer to the nonnegative integer $d(\gamma)$ as the *Kostant degree* of γ. If V is any K-module and $\gamma \in \hat{K}$, then V_γ will denote the isotypic component of V corresponding to γ. Let V be a locally finite K-module such that $V^M \neq 0$ and let $v \in V^M$, $v \neq 0$. Since V is locally finite, we can decompose v into K-isotypic M-invariants as follows:

$$v = \sum_{\gamma \in \Gamma} \mathbf{v}_\gamma,$$

where $\mathbf{v}_\gamma \in V_\gamma$ denotes the γ-isotypic component of v. Then we define the *Kostant degree* of v by,

$$d(v) = \max\{d(\gamma) : \mathbf{v}_\gamma \neq 0\}. \tag{5}$$

Since we are mainly concerned with representations $\gamma \in \Gamma$ that occur as subrepresentations of $U(\mathfrak{k})$ we set,

$$\Gamma_1 = \{\gamma \in \Gamma : \gamma \text{ is a subrepresentation of } U(\mathfrak{k})\}. \tag{6}$$

If $0 \neq b \in U(\mathfrak{k}) \otimes U(\mathfrak{a})$, we can write $b = b_m \otimes Z^m + \cdots + b_0$ in a unique way with $b_j \in U(\mathfrak{k})$ for $j = 0, \ldots, m$, $b_m \neq 0$ and $Z = Z_\alpha$ for any $\alpha \in P_+$ simple. We shall refer to b_m (resp. $\tilde{b} = b_m \otimes Z^m$) as the *leading coefficient* (resp. *leading term*) of b and to m as the *degree* of b. Also, let 0 be the leading coefficient and the leading term of $b = 0$.

Let M_o' be the normalizer of A_o in K_o and let $W = M_o'/M_o$ be the Weyl group of (G_o, K_o). Then $(U(\mathfrak{k})^M \otimes U(\mathfrak{a}))^W$ denotes the ring of W-invariants in $U(\mathfrak{k})^M \otimes U(\mathfrak{a})$, under the tensor product action of the natural actions of W on $U(\mathfrak{k})^M$ and $U(\mathfrak{a})$, respectively.

At this point, it is convenient to state the following result. Its proof is given in [3, Proposition 2.6], using the techniques and the notation of [15, Section 3].

Proposition 2.6. If $\tilde{b} = b_m \otimes Z^m \in (U(\mathfrak{k})^M \otimes U(\mathfrak{a}))^W$ and $d(b_m) \leq m$, then there exits $u \in U(\mathfrak{g})^K$ such that $\tilde{b}$ is the leading term of $b = P(u)$. □

Last proposition suggests using an inductive argument to prove Theorem 1.1. To do this, it is sufficient to establish the following theorem. In fact, in Proposition 2.8, we prove that Theorem 2.7 implies Theorem 1.1.

Theorem 2.7. If $b = b_m \otimes Z^m + \cdots + b_0 \in B$ and $b_m \neq 0$, then $d(b_m) \leq m$ and its leading term $b_m \otimes Z^m \in (U(\mathfrak{k})^M \otimes U(\mathfrak{a}))^W$. □

Remark. In F_4^{-20} the nontrivial element of W can be represented by an element in M_o' which acts on $\mathfrak{g}$ as the Cartan involution. Hence, to prove that the leading term $b_m \otimes Z^m$ is W-invariant it is enough to show that m is even. □

Proposition 2.8. Theorem 2.7 implies Theorem 1.1. □

Proof. Assume that Theorem 2.7 holds. From Theorem 2.4, we know that $P(U(\mathfrak{g})^K) \subset B$. Then let us prove by induction on the degree m of $b \in B$, that $B \subset P(U(\mathfrak{g})^K)$. If $m = 0$,

we have $b = b_0 \in U(\mathfrak{k})^M$ and Theorem 2.7 implies that $d(b_0) = 0$. If $\gamma \in \Gamma$ and $d(\gamma) = 0$, then γ can be realized by constant polynomial functions on $\mathfrak{p}$ and these functions are K-invariant. Thus, $b_0 \in U(\mathfrak{k})^K$ and therefore $b = b_0 = P(b_0) \in P(U(\mathfrak{g})^K)$.

If $b \in B$ and $m > 0$, from Theorem 2.7 and Proposition 2.6, we know that there exists $v \in U(\mathfrak{g})^K$ such that $\widetilde{P(v)} = \tilde{b}$. Then $b - P(v)$ lies in B and the degree of $b - P(v)$ is strictly less than m. Hence, by the induction hypothesis, there exists $u \in U(\mathfrak{g})^K$ such that $P(u) = b - P(v)$ and $b = P(u + v) \in P(U(\mathfrak{g})^K)$. This completes the induction argument and we obtain that $B \subset P(U(\mathfrak{g})^K)$, as we wanted to prove. ■

In view of this result the main objective of this paper is to prove Theorem 2.7 when G_0 is isomorphic to the rank 1 real form F_4^{-20} of F_4.

3 The Equations Defining B

From now on, we shall write $u \equiv v$ instead of $u \equiv v \bmod (U(\mathfrak{k})\mathfrak{m}^+)$, for any $u, v \in U(\mathfrak{k})$. Next result was proved in [15, Lemma 29] for G_o of arbitrary rank.

Lemma 3.1. Let $\alpha \in P_+$ be a simple root. Set $H_\alpha = Y_\alpha + Z_\alpha$ where $Y_\alpha \in \mathfrak{t}$, $Z_\alpha \in \mathfrak{a}$ and let $c = \alpha(Y_\alpha)$. If $\lambda = \alpha|_{\mathfrak{a}}$ and $m(\lambda)$ is the multiplicity of λ, then $c = 1$ when 2λ is not a restricted root and $m(\lambda)$ is even, or when $m(\lambda)$ is odd, and $c = \frac{3}{2}$ when 2λ is a restricted root and $m(\lambda)$ is even. □

In particular, if G_o is isomorphic to F_4^{-20} we have $c = \frac{3}{2}$. To simplify the notation set $E = E_\alpha$, $Y = Y_\alpha$ and $Z = Z_\alpha$ for any simple root $\alpha \in P_+$. Note that $[E, Y] = cE$, where c is as in Lemma 3.1. Also, since $E_\alpha = X_{-\alpha} + \theta X_{-\alpha}$ and α is a simple root in P_+ it follows that E is $\mathfrak{m}^+$- dominant.

We shall identify $U(\mathfrak{k}) \otimes U(\mathfrak{a})$ with the polynomial ring in one variable $U(\mathfrak{k})[x]$, replacing Z by the indeterminate x. To study Equation (4), we change $b(x) \in U(\mathfrak{k})[x]$ by $c(x) \in U(\mathfrak{k})[x]$ defined by

$$c(x) = b(x + H - 1), \tag{7}$$

where H is an appropriate vector in $\mathfrak{t}$ to be chosen later, depending on the simple root $\alpha \in P_+$ and such that $[H, E] = \frac{1}{2}E$ (see (19)). Now, if $\tilde{Y} = Y + H$, we have $[E, \tilde{Y}] = E$. Then $b(x) \in U(\mathfrak{k})[x]$ satisfies (4) if and only if $c(x) \in U(\mathfrak{k})[x]$ satisfies

$$E^n c(n - \tilde{Y}) \equiv c(-n - \tilde{Y}) E^n \tag{8}$$

for all $n \in \mathbb{N}$. Note that (8) is an equation in the noncommutative ring $U(\mathfrak{k})$.

Now, if p is a polynomial in one indeterminate x with coefficients in a ring let $p^{(n)}$ denote the nth discrete derivative of p. That is, $p^{(n)}(x) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} p(x + \frac{n}{2} - j)$. In particular, if $p = p_m x^m + \cdots + p_0$, we have

$$p^{(n)}(x) = \begin{cases} 0 & \text{if } n > m, \\ m!\, p_m & \text{if } n = m. \end{cases}$$

Also, if $X \in \mathfrak{k}$, we shall denote with $\dot{X}$ the derivation of $U(\mathfrak{k})$ induced by $\mathrm{ad}(X)$. Moreover, if D is a derivation of $U(\mathfrak{k})$, we shall denote with the same symbol the unique derivation of $U(\mathfrak{k})[x]$ which extends D and such that $Dx = 0$. Thus for $b \in U(\mathfrak{k})[x]$ and $b = b_m x^m + \cdots + b_0$, we have $Db = (Db_m)x^m + \cdots + (Db_0)$. Observe that these derivations commute with the operation of taking the discrete derivative in $U(\mathfrak{k})[x]$.

Next theorem gives a triangularized version of the system (8), and in turn, of the system (4) that defines the algebra B. A proof of it is given in [2], where the system (8) is studied in a more abstract setting and in particular the LU-decomposition of its coefficient matrix is given.

Theorem 3.2. Let $c \in U(\mathfrak{k})[x]$. Then the following systems of equations are equivalent:

(i) $E^n c(n - \tilde{Y}) \equiv c(-n - \tilde{Y}) E^n$, $(n \in \mathbb{N}_0)$;

(ii) $\dot{E}^{n+1}(c^{(n)})\big(\frac{n}{2} + 1 - \tilde{Y}\big) + \dot{E}^n(c^{(n+1)})\big(\frac{n}{2} - \frac{1}{2} - \tilde{Y}\big)E \equiv 0$, $(n \in \mathbb{N}_0)$.
Moreover, if $c \in U(\mathfrak{k})[x]$ is a solution of one of the above systems, then for all $\ell, n \in \mathbb{N}_0$, we have

(iii) $(-1)^n \dot{E}^\ell(c^{(n)})\big(-\frac{n}{2} + \ell - \tilde{Y}\big)E^n - (-1)^\ell \dot{E}^n(c^{(\ell)})\big(-\frac{\ell}{2} + n - \tilde{Y}\big)E^\ell \equiv 0$. □

Observe that if $c \in U(\mathfrak{k})[x]$ is of degree m and $c = c_m x^m + \cdots + c_0$, then all the equations of the system (ii) corresponding to $n > m$ are trivial, because $c^{(n)} = 0$. Moreover, the equation corresponding to $n = m$ reduces to $\dot{E}^{m+1}(c_m) \equiv 0$, and more generally the equation associated to $n = j$ only involves the coefficients $c_m, \ldots, c_j$. In this sense, the system (ii) is a triangular system of $m + 1$ linear equations in the $m + 1$ unknowns $c_m, \ldots, c_0$.

If $0 \neq b(x) \in U(\mathfrak{k})[x]$ and $c(x) \in U(\mathfrak{k})[x]$ is defined by $c(x) = b(x + H - 1)$, where H is as in (19), we find it convenient to write, in a unique way, $b = \sum_{j=0}^{m} b_j x^j$ with $b_j \in U(\mathfrak{k})$,

$b_m \neq 0$, and $c = \sum_{j=0}^{m} c_j \varphi_j$ where $c_j \in U(\mathfrak{k})$ and $\{\varphi_n\}_{n\geq 0}$ is the basis of $\mathbb{C}[x]$ defined by,

(i) $\varphi_0 = 1$,
(ii) $\varphi_n^{(1)} = \varphi_{n-1}$ if $n \geq 1$,
(iii) $\varphi_n(0) = 0$ if $n \geq 1$.

Moreover it is easy to prove that such a family is given by

$$\varphi_n(x) = \frac{1}{n!} x \left(x + \frac{n}{2} - 1\right)\left(x + \frac{n}{2} - 2\right) \cdots \left(x - \frac{n}{2} + 1\right), \quad n \geq 1. \tag{9}$$

Next lemma contains the results of [3, Lemmas 3.3 and 3.5]. Its proof is the same as that of the corresponding lemmas in [3].

Lemma 3.3. Let $b = \sum_{j=0}^{m} b_j x^j \in U(\mathfrak{k})[x]$ and set $c(x) = b(x + H - 1)$. Then, if $c = \sum_{j=0}^{m} c_j \varphi_j$ with $c_j \in U(\mathfrak{k})$, we have

$$c_i = \sum_{j=i}^{m} b_j t_{ij}, \quad 0 \leq i \leq m,$$

where

$$t_{ij} = \sum_{k=0}^{i} (-1)^k \binom{i}{k} \left(H + \frac{i}{2} - 1 - k\right)^j.$$

Thus, t_{ij} is a polynomial in H of degree $j - i$. Moreover,

$$\dot{E}^{j-i}(t_{ij}) = (-\tfrac{1}{2})^{j-i} j! E^{j-i}. \qquad \square$$

From these results and Theorem 3.2, we obtain the following theorem and its corollary in the same way as in [3].

Theorem 3.4. If $b = b_m \otimes Z^m + \cdots + b_0 \in B$, then $\dot{E}^{m+1}(c_j) \equiv 0$ for all $0 \leq j \leq m$. $\square$

Corollary 3.5. If $b = b_m \otimes Z^m + \cdots + b_0 \in B$, then $\dot{E}^{2m+1-j}(b_j) \equiv 0$ for all $0 \leq j \leq m$. $\square$

Next we rewrite equation (iii) of Theorem 3.2 for later reference. Given $b = \sum_{j=0}^{m} b_j x^j \in B$ and $c(x) = b(x + H - 1)$ as above, it follows from Theorem 3.4 that equation (iii) of Theorem 3.2 is satisfied if $\ell > m$ or $n > m$, and it is trivial when $\ell = n$. Also note that the equation corresponding to (n, ℓ) is equivalent to the one corresponding to (ℓ, n).

Theorem 3.6. Let $b=\sum_{j=0}^{m} b_j x^j \in U(\mathfrak{k})[x]$ and $c(x)=b(x+H-1)$. If $c=\sum_{j=0}^{m} c_j\varphi_j$ with $c_j \in U(\mathfrak{k})$ and $0 \le \ell, n$, we set

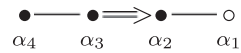
$$\epsilon(\ell,n)=(-1)^n \sum_{n\le i\le m} \dot{E}^{\ell}(c_i)\varphi_{i-n}\left(-\frac{n}{2}+\ell-\tilde{Y}\right)E^n-(-1)^{\ell}\sum_{\ell\le i\le m}\dot{E}^{n}(c_i)\varphi_{i-\ell}\left(-\frac{\ell}{2}+n-\tilde{Y}\right)E^{\ell}.$$

Then, if $b\in B$, we have $\epsilon(\ell,n)\equiv 0 \bmod (U(\mathfrak{k})\mathfrak{m}^+)$ for all $0\le \ell, n$. □

Proof. The assertion follows from equation (iii) of Theorem 3.2 and the fact that $c^{(k)}=\sum_{i=k}^{m} c_i\varphi_{i-k}$ for all $0\le k\le m$. ■

4 The Group $\mathbf{F}_4^{-20}$

Let G_o be isomorphic to the rank 1 real form F_4^{-20} of F_4. Then the Dynkin–Satake diagram of $\mathfrak{g}$ is



We can choose an orthonormal basis $\{\epsilon_i\}_{i=1}^4$ of $\mathfrak{h}_{\mathbb{R}}^*$ such that $\alpha_4=\epsilon_2-\epsilon_3$, $\alpha_3=\epsilon_3-\epsilon_4$, $\alpha_2=\epsilon_4$, $\alpha_1=\frac{1}{2}(\epsilon_1-\epsilon_2-\epsilon_3-\epsilon_4)$. Moreover, if σ denotes the conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_o$, then $\epsilon_1^{\sigma}=\epsilon_1$ and $\epsilon_i^{\sigma}=-\epsilon_i$ if $2\le i\le 4$. Also, we have $\epsilon_1^{\theta}=-\epsilon_1$ and $\epsilon_i^{\theta}=\epsilon_i$ for $2\le i\le 4$. From the diagram it follows that

$$\Delta^+(\mathfrak{g},\mathfrak{h})=\{\epsilon_i : 1\le i\le 4\}\cup\{\epsilon_i\pm\epsilon_j : 1\le i<j\le 4\}\cup\{\tfrac{1}{2}(\epsilon_1\pm\epsilon_2\pm\epsilon_3\pm\epsilon_4)\},$$

$$P_+=\{\epsilon_1,\epsilon_1\pm\epsilon_2,\epsilon_1\pm\epsilon_3,\epsilon_1\pm\epsilon_4\}\cup\{\tfrac{1}{2}(\epsilon_1\pm\epsilon_2\pm\epsilon_3\pm\epsilon_4)\},$$

$$P_-=\{\epsilon_2,\epsilon_3,\epsilon_4,\epsilon_2\pm\epsilon_3,\epsilon_2\pm\epsilon_4,\epsilon_3\pm\epsilon_4\},$$

where the signs may be chosen independently. Here, P_- denotes the set of roots in $\Delta^+(\mathfrak{g},\mathfrak{h})$ that vanish on $\mathfrak{a}$. Hence, $P_-=\Delta^+(\mathfrak{m},\mathfrak{t})$ and from this it follows that $\mathfrak{m}\simeq\mathfrak{so}(7,\mathbb{C})$.

We have $\mathfrak{t}=\ker(\epsilon_1)$ and ϵ_1 is the only root in P_+ that vanishes on $\mathfrak{t}$. If we set $\mu=\epsilon_1$, then $H_\mu=Z_\mu\in\mathfrak{a}$. Choose the root vector X_μ so that $\langle X_\mu,\theta X_\mu\rangle=2$ and define $X_{-\mu}=\theta X_\mu$. Then the ordered set $\{H_\mu,X_\mu,X_{-\mu}\}$ is an s-triple. This choice characterizes X_μ up to a sign. On the other hand, it can be established that for any choice of nonzero root vectors X_{α_1} and $X_{-\alpha_1}$ we have $[X_\mu,\theta X_{\alpha_1}]=tX_{\alpha_1}$ and $[X_\mu,X_{-\alpha_1}]=-t\theta X_{-\alpha_1}$ with $t^2=1$. Then

normalize X_μ so that,

$$[X_\mu, \theta X_{\alpha_1}] = -X_{\alpha_1} \quad \text{and} \quad [X_\mu, X_{-\alpha_1}] = \theta X_{-\alpha_1}. \tag{10}$$

Now consider the Cayley transform χ of $\mathfrak{g}$ defined by

$$\chi = \mathrm{Ad}\left(\exp \frac{\pi}{4}(\theta X_\mu - X_\mu)\right).$$

It is easy to see that

$$\mathrm{Ad}(\exp t(\theta X_\mu - X_\mu))H_\mu = \cos(2t)H_\mu + \sin(2t)(X_\mu + \theta X_\mu).$$

Then $\chi(H_\mu) = X_\mu + \theta X_\mu$ and, since $\mu_{|_\mathfrak{t}} = 0$, χ fixes all elements of $\mathfrak{t}$. Therefore, $\mathfrak{h}_\mathfrak{k} = \chi(\mathfrak{t} \oplus \mathfrak{a}) = \mathfrak{t} \oplus \mathbb{C}(X_\mu + \theta X_\mu) \subset \mathfrak{k}$ is a Cartan subalgebra of both $\mathfrak{g}$ and $\mathfrak{k}$.

For any $\phi \in \mathfrak{h}^*$ define $\tilde{\phi} \in \mathfrak{h}_\mathfrak{k}^*$ by $\tilde{\phi} = \phi \cdot \chi^{-1}$. Then $\Delta(\mathfrak{g}, \mathfrak{h}_\mathfrak{k}) = \{\tilde{\alpha} : \alpha \in \Delta(\mathfrak{g}, \mathfrak{h})\}$ and $\mathfrak{g}_{\tilde{\alpha}} = \chi(\mathfrak{g}_\alpha)$. A root $\tilde{\alpha} \in \Delta(\mathfrak{g}, \mathfrak{h}_\mathfrak{k})$ is said to be compact (respectively noncompact) if $\mathfrak{g}_{\tilde{\alpha}} \subset \mathfrak{k}$ (respectively, $\mathfrak{g}_{\tilde{\alpha}} \subset \mathfrak{p}$). Let $\Delta(\mathfrak{k}, \mathfrak{h}_\mathfrak{k})$ and $\Delta(\mathfrak{p}, \mathfrak{h}_\mathfrak{k})$ denote, respectively, the sets of compact and noncompact roots.

Using [3, Lemma 3.1] it follows that $\tilde{\alpha}_3$ and $\tilde{\alpha}_4$ are compacts roots, and that $\tilde{\alpha}_2$ is a noncompact root. Also, since X_μ was chosen so that (10) holds, we obtain that $\tilde{\alpha}_1$ is a noncompact root. From this it follows that

$$\Delta(\mathfrak{k}, \mathfrak{h}_\mathfrak{k}) = \{\pm(\tilde{\epsilon}_i \pm \tilde{\epsilon}_j) : 1 \le i < j \le 4\} \cup \{\tfrac{1}{2}(\pm\tilde{\epsilon}_1 \pm \tilde{\epsilon}_2 \pm \tilde{\epsilon}_3 \pm \tilde{\epsilon}_4) : \text{ even number of minus signs}\},$$

$$\Delta(\mathfrak{p}, \mathfrak{h}_\mathfrak{k}) = \{\pm\tilde{\epsilon}_i : 1 \le i \le 4\} \cup \{\tfrac{1}{2}(\pm\tilde{\epsilon}_1 \pm \tilde{\epsilon}_2 \pm \tilde{\epsilon}_3 \pm \tilde{\epsilon}_4) : \text{ odd number of minus signs}\}.$$

Next we construct a particular Borel subalgebra $b_\mathfrak{k} = \mathfrak{h}_\mathfrak{k} \oplus \mathfrak{k}^+$ of $\mathfrak{k}$ that will be useful later on to describe the set Γ, as well as some of the properties of the elements of Γ (see Proposition 5.1). For more details on the construction of the subalgebra $b_\mathfrak{k}$ and its relation with Γ we refer the reader to [6].

Since $\alpha_1 = \frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4)$ is the only simple root in P_+ set, as in the previous section, $E = X_{-\alpha_1} + \theta X_{-\alpha_1}$. Let $H_+ \in \mathfrak{t}_\mathbb{R}$ be such that $\alpha(H_+) > 0$ for all $\alpha \in \Delta^+(\mathfrak{m}, \mathfrak{t})$. We say that H_+ is $\mathfrak{k}$-regular if in addition $\alpha(H_+) \neq 0$ for all α with $\tilde{\alpha} \in \Delta(\mathfrak{k}, \mathfrak{h}_\mathfrak{k})$. Since μ is the only root in $\Delta^+(\mathfrak{g}, \mathfrak{h})$ that vanishes on $\mathfrak{t}$ and since $\tilde{\mu}$ is a noncompact root, it follows that

$\mathfrak{k}$-regular vectors exist. Given a $\mathfrak{k}$-regular vector H_+ consider the positive system

$$\Delta^+(\mathfrak{k}, \mathfrak{h}_\mathfrak{k}) = \{\tilde{\alpha} \in \Delta(\mathfrak{k}, \mathfrak{h}_\mathfrak{k}) : \alpha(H_+) > 0\}.$$

If $\lambda_0 = \alpha_1|_\mathfrak{a}$ is the simple restricted root and H_+ is a $\mathfrak{k}$-regular vector, we consider the following set:

$$P_+(\lambda_0)^- = \{\alpha \in P_+ : \alpha|_\mathfrak{a} = \lambda_0 \quad \text{and } \alpha(H_+) < 0\}.$$

Definition 4.1. A positive system $\Delta^+(\mathfrak{k}, \mathfrak{h}_\mathfrak{k})$ defined by a $\mathfrak{k}$-regular vector H_+ (see (4)) is said to be compatible with E if $\alpha - \alpha_1$ is a root for every $\alpha \in P_+(\lambda_0)^-$ such that $\alpha \neq \alpha_1$. □

The $\mathfrak{k}$-regular vectors in $\mathfrak{t}_\mathbb{R}$, for $\mathfrak{g}_o \simeq \mathfrak{f}_4$, are all of the form $H_+ = (0, t_2, t_3, t_4)$ with $t_2 > t_3 > t_4 > 0$ and $t_2 \neq t_3 + t_4$. Different vectors H_+ define two different positive systems, they depend only on whether $\pm(t_2 - t_3 - t_4) > 0$, and they are both compatible with E. From now on, fix a $\mathfrak{k}$-regular vector $H_+ = (0, t_2, t_3, t_4)$ with $t_2 > t_3 > t_4 > 0$ and $t_2 > t_3 + t_4$. The corresponding positive system in $\Delta(\mathfrak{k}, \mathfrak{h}_\mathfrak{k})$ is,

$$\begin{aligned}\Delta^+(\mathfrak{k}, \mathfrak{h}_\mathfrak{k}) = {} & \{\tilde{\epsilon}_i \pm \tilde{\epsilon}_j : 2 \leq i < j \leq 4\} \cup \{\tilde{\epsilon}_i \pm \tilde{\epsilon}_1 : 2 \leq i \leq 4\} \\ & \cup \{\tfrac{1}{2}(\pm\tilde{\epsilon}_1 + \tilde{\epsilon}_2 \pm \tilde{\epsilon}_3 \pm \tilde{\epsilon}_4) : \text{ even number of minus signs}\},\end{aligned}$$

and $b_\mathfrak{k} = \mathfrak{h}_\mathfrak{k} \oplus \mathfrak{k}^+$ is the associated Borel subalgebra. A simple system in $\Delta^+(\mathfrak{k}, \mathfrak{h}_\mathfrak{k})$ is given by,

$$\Pi(\mathfrak{k}, \mathfrak{h}_\mathfrak{k}) = \{\tilde{\epsilon}_4 + \tilde{\epsilon}_1, \tilde{\epsilon}_3 - \tilde{\epsilon}_4, \tilde{\epsilon}_4 - \tilde{\epsilon}_1, \tfrac{1}{2}(\tilde{\epsilon}_1 + \tilde{\epsilon}_2 - \tilde{\epsilon}_3 - \tilde{\epsilon}_4)\}. \tag{11}$$

Hence $\mathfrak{k} \simeq \mathfrak{so}(9, \mathbb{C})$.

Fix nonzero root vectors $X_{\epsilon_i + \epsilon_1}$ $(2 \leq i \leq 4)$, $X_{\epsilon_i \pm \epsilon_j}$ $(2 \leq i < j \leq 4)$ and define,

$$X_{\tilde{\epsilon}_i + \tilde{\epsilon}_1} = \chi(X_{\epsilon_i + \epsilon_1}), \quad X_{\tilde{\epsilon}_i - \tilde{\epsilon}_1} = \chi(\theta X_{\epsilon_i + \epsilon_1}), \quad X_{\tilde{\epsilon}_i \pm \tilde{\epsilon}_j} = \chi(X_{\epsilon_i \pm \epsilon_j}). \tag{12}$$

Then it follows from [6, Proposition 2.4] that,

$$\begin{aligned} X_{\tilde{\epsilon}_i \pm \tilde{\epsilon}_j} &= X_{\epsilon_i \pm \epsilon_j}, \\ X_{\tilde{\epsilon}_i + \tilde{\epsilon}_1} &= \tfrac{1}{2}(X_{\epsilon_i + \epsilon_1} + [X_\mu, \theta X_{\epsilon_i + \epsilon_1}] + \theta X_{\epsilon_i + \epsilon_1}), \end{aligned} \tag{13}$$

and

$$X_{\tilde{\epsilon}_i-\tilde{\epsilon}_1} = \tfrac{1}{2}(X_{\epsilon_i+\epsilon_1} - [X_\mu, \theta X_{\epsilon_i+\epsilon_1}] + \theta X_{\epsilon_i+\epsilon_1}).$$

Hence,

$$X_{\tilde{\epsilon}_i+\tilde{\epsilon}_1} - X_{\tilde{\epsilon}_i-\tilde{\epsilon}_1} = [X_\mu, \theta X_{\epsilon_i+\epsilon_1}] = X_{\epsilon_i} \in \mathfrak{m}^+. \tag{14}$$

Then from (13) and (14) it follows that:

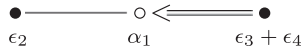
$$\mathfrak{m}^+ = \langle \{X_{\tilde{\epsilon}_i \pm \tilde{\epsilon}_j} : 2 \le i < j \le 4\} \cup \{X_{\tilde{\epsilon}_i+\tilde{\epsilon}_1} - X_{\tilde{\epsilon}_i-\tilde{\epsilon}_1} : 2 \le i \le 4\} \rangle, \tag{15}$$

where $\langle S \rangle$ denotes the linear space spanned by the set S.

Next we define, as in the case of Sp(n,1) (see [3, Section 3]), a Lie subalgebra $\tilde{\mathfrak{g}}$ of $\mathfrak{g}$ that it is both σ and θ stable and its real form $\tilde{\mathfrak{g}}_o = \mathfrak{g}_o \cap \tilde{\mathfrak{g}}$ is isomorphic to $\mathfrak{sp}(2,1)$. Recall that $\alpha_1 = \frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4)$ is the only simple root in P_+. Let $\tilde{\mathfrak{g}}$ be the complex Lie subalgebra of $\mathfrak{g}$ generated by the following nonzero root vectors:

$$\{X_{\pm\epsilon_2}, X_{\pm\alpha_1}, X_{\pm(\epsilon_3+\epsilon_4)}\}.$$

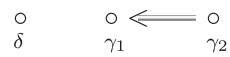
Then $\tilde{\mathfrak{g}}$ is a simple Lie algebra stable under σ and θ. Therefore, $\tilde{\mathfrak{g}}$ is the complexification of the real subalgebra $\tilde{\mathfrak{g}}_o = \mathfrak{g}_o \cap \tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{g}} = \tilde{\mathfrak{k}} \oplus \tilde{\mathfrak{p}}$ is a Cartan decomposition of $\tilde{\mathfrak{g}}$, where $\tilde{\mathfrak{k}} = \mathfrak{k} \cap \tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{p}} = \mathfrak{p} \cap \tilde{\mathfrak{g}}$. Moreover, $\tilde{\mathfrak{h}} = (\mathfrak{t} \cap \tilde{\mathfrak{g}}) \oplus \mathfrak{a}$ is a Cartan subalgebra of $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{m}} = \mathfrak{m} \cap \tilde{\mathfrak{k}}$ is the centralizer of $\mathfrak{a}$ in $\tilde{\mathfrak{k}}$. That $\tilde{\mathfrak{g}}_o \simeq \mathfrak{sp}(2,1)$ follows from the Dynkin–Satake diagram of $\tilde{\mathfrak{g}}_o$,



Since the root vectors X_μ and θX_μ are in $\tilde{\mathfrak{g}}$, it follows that $\tilde{\mathfrak{g}}$ is stable under the Cayley transform χ of the pair $(\mathfrak{g}, \mathfrak{h})$. Hence the restriction of χ to $\tilde{\mathfrak{g}}$ is the Cayley transform associated to $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}})$. Then $\mathfrak{h}_{\tilde{\mathfrak{k}}} = \chi(\tilde{\mathfrak{h}}) = \mathfrak{h}_{\mathfrak{k}} \cap \tilde{\mathfrak{k}}$ is a Cartan subalgebra of $\tilde{\mathfrak{k}}$ and $\tilde{\mathfrak{g}}$. The positive system $\Delta^+(\mathfrak{k}, \mathfrak{h}_{\mathfrak{k}})$ determines a positive system $\Delta^+(\tilde{\mathfrak{k}}, \mathfrak{h}_{\tilde{\mathfrak{k}}}) = \{\tilde{\alpha}_{|_{\mathfrak{h}_{\tilde{\mathfrak{k}}}}} \in \Delta(\tilde{\mathfrak{k}}, \mathfrak{h}_{\tilde{\mathfrak{k}}}) : \tilde{\alpha} \in \Delta^+(\mathfrak{k}, \mathfrak{h}_{\mathfrak{k}})\}$ in $\Delta(\tilde{\mathfrak{k}}, \mathfrak{h}_{\tilde{\mathfrak{k}}})$. Moreover,

$$\Pi(\tilde{\mathfrak{k}}, \mathfrak{h}_{\tilde{\mathfrak{k}}}) = \{\delta = \tilde{\epsilon}_2 - \tilde{\epsilon}_1, \gamma_1 = \tfrac{1}{2}(\tilde{\epsilon}_1 + \tilde{\epsilon}_2 - \tilde{\epsilon}_3 - \tilde{\epsilon}_4), \gamma_2 = \tilde{\epsilon}_3 + \tilde{\epsilon}_4\}$$

is a simple system in $\Delta^+(\tilde{\mathfrak{k}}, \mathfrak{h}_{\tilde{\mathfrak{k}}})$ and the corresponding Dynkin diagram is



Then $\Delta^+(\tilde{\mathfrak{k}}, \mathfrak{h}_{\tilde{\mathfrak{k}}}) = \{\delta, \gamma_1, \gamma_2, \gamma_3, \gamma_4\}$, where $\gamma_3 = \gamma_1 + \gamma_2 = \frac{1}{2}(\tilde{\epsilon}_1 + \tilde{\epsilon}_2 + \tilde{\epsilon}_3 + \tilde{\epsilon}_4)$ and $\gamma_4 = 2\gamma_1 + \gamma_2 = \tilde{\epsilon}_1 + \tilde{\epsilon}_2$. Hence, $\tilde{\mathfrak{k}} \simeq \mathfrak{sp}(1, \mathbb{C}) \times \mathfrak{sp}(2, \mathbb{C})$.

A simple calculation shows that $\chi(\theta X_{-\alpha_1}) = \frac{\sqrt{2}}{2} E$, thus E is a root vector in $\tilde{\mathfrak{k}}^+$ corresponding to γ_3. Then set $X_{\gamma_3} = E$. Now define $\varphi_1 = \tilde{\epsilon}_3 + \tilde{\epsilon}_1$, $\delta_1 = \tilde{\epsilon}_3 - \tilde{\epsilon}_1$, $\varphi_2 = \tilde{\epsilon_4} + \tilde{\epsilon}_1$ and $\delta_2 = \tilde{\epsilon}_4 - \tilde{\epsilon}_1$. Then in view of (12) we have,

$$X_{\gamma_4} = \chi(X_{\epsilon_2+\epsilon_1}), \quad X_\delta = \chi(\theta X_{\epsilon_2+\epsilon_1}), \quad X_{\varphi_1} = \chi(X_{\epsilon_3+\epsilon_1}) \tag{16}$$

and

$$X_{\delta_1} = \chi(\theta X_{\epsilon_3+\epsilon_1}), \quad X_{\varphi_2} = \chi(X_{\epsilon_4+\epsilon_1}), \quad X_{\delta_2} = \chi(\theta X_{\epsilon_4+\epsilon_1}). \tag{17}$$

It follows from (14) that $X_{\gamma_4} - X_\delta$ and $X_{\varphi_i} - X_{\delta_i}$ are in $\mathfrak{m}^+$ for $i = 1, 2$.

Normalize $X_{-\gamma_4}$, $X_{-\delta}$, $X_{-\varphi_i}$, and $X_{-\delta_i}$ so that $\langle X_{\gamma_4}, X_{-\gamma_4}\rangle = \langle X_\delta, X_{-\delta}\rangle = \langle X_{\varphi_i}, X_{-\varphi_i}\rangle = \langle X_{\delta_i}, X_{-\delta_i}\rangle = 1$, for $i = 1, 2$. Then it follows that:

$$\langle X_{\gamma_4} - X_\delta, X_{-\gamma_4} + X_{-\delta}\rangle = \langle X_{\varphi_i} - X_{\delta_i}, X_{-\varphi_i} + X_{-\delta_i}\rangle = 0. \tag{18}$$

Hence, $X_{-\gamma_4} + X_{-\delta}$ and $X_{-\varphi_i} + X_{-\delta_i}$ $(i = 1, 2)$ are in $(\mathfrak{m}^+)^\perp$, the orthogonal complement of $\mathfrak{m}^+$ in $\mathfrak{k}$ with respect to the Killing form of $\mathfrak{k}$.

To simplify the notation set, $X_{\pm 1} = X_{\pm\gamma_1}$, $X_{\pm 2} = X_{\pm\gamma_2}$, $X_{\pm 3} = X_{\pm\gamma_3}$, and $X_{\pm 4} = X_{\pm\gamma_4}$. Let $H_1 \in [\mathfrak{k}_{\gamma_1}, \mathfrak{k}_{-\gamma_1}]$ be such that $\gamma_1(H_1) = 2$, and normalize X_1 and X_{-1} so that $\{H_1, X_1, X_{-1}\}$ is an $\mathfrak{s}$-triple. Next normalize X_2 and X_4 (and accordingly X_δ), so that

$$[X_1, X_2] = E \quad \text{and} \quad [X_1, E] = X_4.$$

From this, and the fact that $\gamma_2(H_1) = -2$, it follows that

$$[X_{-1}, E] = 2X_2 \quad \text{and} \quad [X_{-1}, X_4] = 2E.$$

Now choose $H_2 \in [\mathfrak{k}_{\gamma_2}, \mathfrak{k}_{-\gamma_2}]$ such that $\gamma_2(H_2) = 2$ and normalize X_{-2} so that $\{H_2, X_2, X_{-2}\}$ is an $\mathfrak{s}$-triple. Since $[\mathfrak{k}_{\gamma_2}, \mathfrak{k}_{-\gamma_2}] \subset \mathfrak{t}$ and $\gamma_1(H_2) = -1$, if we define

$$H = \tfrac{1}{2} H_2, \tag{19}$$

we obtain a vector $H \in \mathfrak{t}$ such that $\dot{H}(E) = \frac{1}{2} E$. This vector H is the one used in (7). Also, since $\delta(H_2) = 0$, we have $[X_\delta, H] = 0$.

As in the previous sections, set $Z = Z_{\alpha_1}$, $Y = Y_{\alpha_1}$ and $\tilde{Y} = Y + H$. From Lemma 3.1, it follows that $\dot{E}(Y) = \frac{3}{2}E$, hence $\dot{E}(\tilde{Y}) = E$. Now, since $(\epsilon_1 + \epsilon_2)(H_{\alpha_1}) = 0$, we have $(\epsilon_1 + \epsilon_2)(Y) = -(\epsilon_1 + \epsilon_2)(Z) = -1$ because $(\epsilon_1 + \epsilon_2)|_{\mathfrak{a}} = 2\alpha_1|_{\mathfrak{a}}$ and $\alpha_1(Z) = \frac{1}{2}$ (see Lemma 3.1). Then $\dot{X}_\delta(Y) = X_\delta$, and therefore $\dot{X}_\delta(\tilde{Y}) = X_\delta$.

5 The M-Spherical K-Modules

In this section, we describe the main properties of the K-modules in the classes Γ and Γ_1 (see (6)). In the following proposition, we collect several results that will be very useful later on, and in Proposition 5.3 we will prove some important properties of the Kostant degree $d(u)$ for $u \in U(\mathfrak{k})^M$ that make use of these results.

Proposition 5.1. Let G_o be isomorphic to F_4^{-20} and let $\mathfrak{b}_\mathfrak{k} = \mathfrak{h}_\mathfrak{k} \oplus \mathfrak{k}^+$ be the Borel subalgebra of $\mathfrak{k}$ defined before. Then $\mathfrak{m}^+ \subset \mathfrak{k}^+$ and E is a root vector in $\mathfrak{k}^+$. Moreover:

(i) For any $\gamma \in \hat{K}$ let ξ_γ denote its highest weight. Then, $\gamma \in \Gamma$ if and only if $\xi_\gamma = \frac{k}{2}(\gamma_4 + \delta) + \ell\gamma_3$ with $k, \ell \in \mathbb{N}_o$. In this context, we write $\gamma = \gamma_{k,\ell}$, $\xi_\gamma = \xi_{k,\ell}$ and $V_{k,\ell}$ for the corresponding representation space. Also we shall refer to any $v \in V_{k,\ell}^M$ as an M-invariant element of type (k, ℓ).

(ii) For any $\gamma_{k,\ell} \in \Gamma$, we have $d(\gamma_{k,\ell}) = k + 2\ell$.

(iii) If $\gamma \in \Gamma$, we have $\gamma \in \Gamma_1$ if and only if $\xi_\gamma = \xi_{k,\ell}$ with k even.

(iv) For any $\gamma_{k,\ell} \in \Gamma$, we have $X_\delta^k E^\ell(V_{k,\ell}^M) = V_{k,\ell}^{\mathfrak{k}^+}$ and $X_\delta^p E^q(V_{k,\ell}^M) = \{0\}$ if and only if $p > k$ or $p + q > k + \ell$. □

For a proof of this proposition, we refer the reader to [6]. The construction of the Borel subalgebra $\mathfrak{b}_\mathfrak{k}$ is contained in [6, Section 3] and the statements in (i), (ii) and (iv) follow from [6, Proposition 4.4, Theorem 4.5 and Theorem 5.3], respectively. On the other hand (iii) is a well-known general fact. We point out that some of these results were first established in [8], others were proved in [5] and they were generalized in [6] to any real rank 1 semisimple Lie group.

The following proposition is the analog of part (ii) of [3, Proposition 3.11]. We omit its proof since, up to minor changes, is the same as that of Proposition 3.11.

Proposition 5.2. Let G_o be isomorphic to F_4^{-20}. Let $\gamma_{k,\ell} \in \Gamma$ and let $V_{k,\ell}$ be a K-module in the class $\gamma_{k,\ell}$. Then if $0 \neq v \in V_{k,\ell}^M$, the set

$$\{X_\delta^{k-j} E^{\ell+j}(v) : \ 0 \leq j \leq k\}$$

is a basis of the irreducible $\{H_1, X_1, X_{-1}\}$-module of dimension $k+1$ generated by any nontrivial highest weight vector of $V_{k,\ell}$. Moreover, $X_\delta^{k-j}E^{\ell+j}(v)$ is a weight vector of weight $\xi_{k,\ell} - j\gamma_1$ and the following identities hold:

$$X_1 X_\delta^{k-j} E^{\ell+j}(v) = \frac{(j+\ell)}{2} X_\delta^{k-j+1} E^{\ell+j-1}(v), \quad 0 \le j \le k, \tag{20}$$

$$X_{-1} X_\delta^{k-j} E^{\ell+j}(v) = \frac{2(j+1)(k-j)}{\ell+j+1} X_\delta^{k-j-1} E^{\ell+j+1}(v), \quad 0 \le j \le k, \tag{21}$$

$$X_{-1}^j(u_{k,\ell}) = 2^j j! \binom{k}{j} \binom{\ell+j}{\ell}^{-1} X_\delta^{k-j} E^{\ell+j}(v), \quad 0 \le j \le k, \tag{22}$$

where $u_{k,\ell}$ is the highest weight vector $X_\delta^k E^\ell(v)$. □

In the following proposition, we prove some important properties of the Kostant degree $d(u)$ for $u \in U(\mathfrak{k})^M$. Even though we give the proof for F_4^{-20}, since our argument relies heavily on Proposition 5.1, the same proof hold for the other real rank 1 groups, $\mathrm{SO}(n,1)$, $\mathrm{SU}(n,1)$ and $\mathrm{Sp}(n,1)$, with the appropriate changes. These result will be used in Section 8.

Proposition 5.3. Let G_o be isomorphic to F_4^{-20}. If $u, v \in U(\mathfrak{k})^M$ are nonzero vectors, then

(1) $d(u+v) \le \max\{d(u), d(v)\}$,
(2) $d(uv) = d(u) + d(v)$,
(3) $d(u) = 0$ if and only if $u \in U(\mathfrak{k})^K$. □

Proof. The assertions (a) and (c) follow directly from the definition of the Kostant degree. We start the proof of (b) by showing that $d(uv) \le d(u) + d(v)$ for any $0 \ne u, v \in U(\mathfrak{k})^M$. Let us begin by considering $u \in V_{r,s} \subset U(\mathfrak{k})^M$ and $v \in V_{r',s'} \subset U(\mathfrak{k})^M$ where $V_{r,s}$ and $V_{r',s'}$ are, respectively, irreducible finite-dimensional K-modules in the classes $\gamma_{r,s}$ and $\gamma_{r',s'}$ of Γ_1. Then $u \otimes v \in (V_{r,s} \otimes V_{r',s'})^M$ and we decompose it as follows:

$$u \otimes v = \sum_{i,j} \mathbf{w}_{i,j}, \tag{23}$$

where $\mathbf{w}_{i,j} \neq 0$ is the $\gamma_{i,j}$-isotypic component of $u \otimes v$. We recall that if $\gamma_{i,j} \in \Gamma$, then its highest weight is $\xi_{i,j} = \frac{i}{2}(\gamma_4 + \delta) + j\gamma_3$ and $d(\gamma_{i,j}) = i + 2j$, see Proposition 5.1. We will show that $d(\mathbf{w}_{i,j}) \leq d(u) + d(v)$ for any $\mathbf{w}_{i,j}$ that occurs in (23).

In view of (11) a simple system of roots in $\Delta^+(\mathfrak{k}, \mathfrak{h}_\mathfrak{k})$ is given by

$$\Pi(\mathfrak{k}, \mathfrak{h}_\mathfrak{k}) = \{\tilde{\epsilon}_4 + \tilde{\epsilon}_1, \tilde{\epsilon}_3 - \tilde{\epsilon}_4, \tilde{\epsilon}_4 - \tilde{\epsilon}_1, \gamma_1 = \tfrac{1}{2}(\tilde{\epsilon}_1 + \tilde{\epsilon}_2 - \tilde{\epsilon}_3 - \tilde{\epsilon}_4)\}. \tag{24}$$

Then it follows that

$$\gamma_4 + \delta = (\tilde{\epsilon}_4 + \tilde{\epsilon}_1) + 2(\tilde{\epsilon}_3 - \tilde{\epsilon}_4) + 3(\tilde{\epsilon}_4 - \tilde{\epsilon}_1) + 4\gamma_1$$

and

$$\gamma_3 = (\tilde{\epsilon}_4 + \tilde{\epsilon}_1) + (\tilde{\epsilon}_3 - \tilde{\epsilon}_4) + (\tilde{\epsilon}_4 - \tilde{\epsilon}_1) + \gamma_1.$$

If $V_{i,j} \subset U(\mathfrak{k})^M$ occurs in the decomposition of $V_{r,s} \otimes V_{r',s'}$ it is known (see [7]) that its highest weight $\xi_{i,j} = \frac{i}{2}(\gamma_4 + \delta) + j\gamma_3$ is given by

$$\xi_{i,j} = \xi_{r+r',s+s'} - [c_1(\tilde{\epsilon}_4 + \tilde{\epsilon}_1) + c_2(\tilde{\epsilon}_3 - \tilde{\epsilon}_4) + c_3(\tilde{\epsilon}_4 - \tilde{\epsilon}_1) + c_4\gamma_1], \tag{25}$$

where $c_i \in \mathbb{N}_o$ for $1 \leq i \leq 4$. Hence comparing the coefficients of the simple root $\tilde{\epsilon}_4 + \tilde{\epsilon}_1$ on the left- and the right-hand sides of (25) it follows that

$$\frac{i}{2} + j = \frac{r + r'}{2} + s + s' - c_1.$$

Then, since $c_1 \geq 0$, we have

$$d(\mathbf{w}_{i,j}) = r + r' + 2(s + s') - 2c_1 = d(u) + d(v) - 2c_1 \leq d(u) + d(v).$$

Therefore, using the definition (5) and (23) it follows that:

$$d(u \otimes v) = \max\{d(\mathbf{w}_{i,j})\} \leq d(u) + d(v).$$

Now, using that the map $u \otimes v \in U(\mathfrak{k})^M \otimes U(\mathfrak{k})^M \to uv \in U(\mathfrak{k})^M$ is a K-homomorphism it follows that $d(uv) \leq d(u) + d(v)$.

Now let $u \in V_{r,s} \oplus \cdots \oplus V_{r,s}$ (m summands) and $v \in V_{r',s'} \oplus \cdots \oplus V_{r',s'}$ (n summands), where $V_{r,s}$ and $V_{r',s'}$ are irreducible finite-dimensional K-submodules of $U(\mathfrak{k})^M$ as above.

Write $u = u_1 + \cdots + u_m$ with $u_k \in V_{r,s}$ $(1 \le k \le m)$ and $v = v_1 + \cdots + v_n$ with $v_\ell \in V_{r',s'}$ $(1 \le \ell \le n)$. Then using the above calculation, we obtain:

$$d(uv) = d\left(\sum_{k,\ell} u_k v_\ell\right) \le \max\{d(u_k v_\ell) : 1 \le k \le m,\ 1 \le \ell \le n\}$$

$$\le \max\{d(u_k) + d(v_\ell)) : 1 \le k \le m,\ 1 \le \ell \le n\} = d(u) + d(v). \tag{26}$$

Consider now $u, v \in U(\mathfrak{k})^M$ such that $d(u) = p$ and $d(v) = q$. It follows from (5) that,

$$u = \sum_{d(\gamma) \le p} \mathbf{u}_\gamma \quad \text{and} \quad v = \sum_{d(\tau) \le q} \mathbf{v}_\tau, \tag{27}$$

where $\mathbf{u}_\gamma$ and $\mathbf{v}_\tau$ denote, respectively, the K-isotypic components of u and v corresponding to the classes γ and τ of Γ_1. Then using (26), we obtain,

$$d(uv) = d\left(\sum_{\gamma,\tau} \mathbf{u}_\gamma \mathbf{v}_\tau\right) \le \max\{d(\mathbf{u}_\gamma \mathbf{v}_\tau) : \mathbf{u}_\gamma \ne 0,\ \mathbf{v}_\tau \ne 0\}$$

$$\le \max\{d(\mathbf{u}_\gamma) + d(\mathbf{v}_\tau) : \mathbf{u}_\gamma \ne 0,\ \mathbf{v}_\tau \ne 0\}$$

$$= \max\{d(\gamma) + d(\tau) : \mathbf{u}_\gamma \ne 0,\ \mathbf{v}_\tau \ne 0\}$$

$$\le p + q = d(u) + d(v).$$

Our next goal is to show that $d(uv) = d(u) + d(v)$ for any $u, v \in U(\mathfrak{k})^M$. Assume that $d(u) = p$ and $d(v) = q$. Then, using (27) and the fact that $d(uv) \le d(u) + d(v)$ for any $u, v \in U(\mathfrak{k})^M$ it follows that:

$$uv = \sum_{d(\gamma)=p,\ d(\tau)=q} \mathbf{u}_\gamma \mathbf{v}_\tau + w,$$

where $w \in U(\mathfrak{k})^M$ is such that $d(w) < p + q$. Then, in view of (5), we may assume that

$$u = \sum_{i+2j=p} \mathbf{u}_{i,j} \quad \text{and} \quad v = \sum_{r+2s=q} \mathbf{v}_{r,s}, \tag{28}$$

where $\mathbf{u}_{i,j}$ and $\mathbf{v}_{r,s}$ denote, respectively, the K-isotypic components of u and v corresponding to the classes $\gamma_{i,j}$ and $\gamma_{r,s}$ of Γ_1. Let $k = \max\{i \in \mathbb{N}_o : \mathbf{u}_{i,j} \ne 0 \text{ for some } j\}$ and

$\ell = \max\{r \in \mathbb{N}_o : \mathbf{v}_{r,s} \neq 0 \text{ for some } s\}$. Then using (28), Leibnitz rule and part (iv) of Proposition 5.1 it follows that:

$$\dot{E}^{(p+q-k-\ell)/2}\dot{X}_\delta^{k+\ell}(uv) = \binom{k+\ell}{\ell}\binom{\frac{p+q-k-\ell}{2}}{\frac{q-\ell}{2}}\dot{E}^{(p-k)/2}\dot{X}_\delta^{k}(\mathbf{u}_{k,\frac{p-k}{2}})\dot{E}^{(q-\ell)/2}\dot{X}_\delta^{\ell}(\mathbf{v}_{\ell,\frac{q-\ell}{2}}) \neq 0. \tag{29}$$

We point out that the right-hand side of (29) is different from zero because, in view of (iv) of Proposition 5.1, it is a product of two dominant vectors. Also using Leibnitz rule, Proposition 5.1(iv) and (29) it follows that:

$$\dot{X}_\delta^{k+\ell}(uv) = \binom{k+\ell}{\ell}\dot{X}_\delta^{k}(\mathbf{u}_{k,\frac{p-k}{2}})\dot{X}_\delta^{\ell}(\mathbf{v}_{\ell,\frac{q-\ell}{2}}) \neq 0 \tag{30}$$

and

$$\dot{X}_\delta^{k+\ell+1}(uv) = 0. \tag{31}$$

To finish the proof, write

$$uv = \sum_{i,j}\mathbf{b}_{i,j},$$

where $\mathbf{b}_{i,j}$ denote the K-isotypic components of uv corresponding, respectively, to the classes $\gamma_{i,j} \in \Gamma_1$. Then from (29)–(31), we obtain,

$$\dot{X}_\delta^{k+\ell}(uv) = \sum_j \dot{X}_\delta^{k+\ell}(\mathbf{b}_{k+\ell,j})$$

and

$$0 \neq \sum_j \dot{E}^{(p+q-k-\ell)/2}\dot{X}_\delta^{k+\ell}(\mathbf{b}_{k+\ell,j}).$$

Therefore, from Proposition 5.1(iv) it follows that there exists $\mathbf{b}_{k+\ell,j} \neq 0$ such that $(p+q-k-\ell)/2 + k + \ell \leq k + \ell + j$. Thus

$$d(uv) \leq d(u) + d(v) = p + q \leq k + \ell + 2j = d(\mathbf{b}_{k+\ell,j}) \leq d(uv).$$

This completes the proof of the proposition. ■

6 Transversality Results

In this section, we prove several results that will allow us to deal with the congruence modulo $U(\mathfrak{k})\mathfrak{m}^+$ that occur in the equations that define the algebra B (see (4)). In particular, we reduce the congruence modulo $U(\mathfrak{k})\mathfrak{m}^+$ to a congruence modulo $U(\mathfrak{k})\mathfrak{y}$, where $\mathfrak{y} \subset \mathfrak{m}^+$ is the abelian subalgebra defined as follows:

$$\mathfrak{y} = \langle \{X_{\tilde{\epsilon}_3+\tilde{\epsilon}_4}, X_{\tilde{\epsilon}_2+\tilde{\epsilon}_3}, X_{\tilde{\epsilon}_2+\tilde{\epsilon}_4}\}\rangle. \tag{32}$$

Before stating the main results, we introduce the following notation:

$$S_{23} = X_{\tilde{\epsilon}_2+\tilde{\epsilon}_3}, \quad S_{24} = X_{\tilde{\epsilon}_2+\tilde{\epsilon}_4}, \quad \text{and} \quad T_{ij} = X_{\tilde{\epsilon}_i-\tilde{\epsilon}_j} \ (2 \le i \ne j \le 4). \tag{33}$$

Let $\mathfrak{q}^+$ be the linear span of $\{X_\alpha : \alpha \in \Delta^+(\mathfrak{k}, \mathfrak{h}_\mathfrak{k})$ and $\alpha \ne \gamma_1\}$. Since γ_1 is a simple root in $\Delta^+(\mathfrak{k}, \mathfrak{h}_\mathfrak{k})$ (see (24)) it follows that $\mathfrak{q}^+$ is a subalgebra of $\mathfrak{k}^+$. We are interested in considering weight vectors $u \in U(\mathfrak{k})\mathfrak{m}^+$ of weight $\lambda = a(\gamma_4 + \delta) + b\gamma_3$ $(a, b \in \mathbb{Z})$, and such that $\dot{X}(u) \equiv 0 \bmod (U(\mathfrak{k})\mathfrak{y})$ for every $X \in \mathfrak{q}^+$.

Consider the subalgebra $\mathfrak{q} \subset \mathfrak{k}$ defined as follows:

$$\mathfrak{q} = \mathfrak{q}^+ \oplus \mathfrak{h}_\mathfrak{r} \oplus \mathfrak{q}^-, \tag{34}$$

where

$$\mathfrak{h}_\mathfrak{r} = \ker(\gamma_4 + \delta) \cap \ker(\gamma_3) = \langle \{H_{\tilde{\epsilon}_3-\tilde{\epsilon}_4}, H_{\tilde{\epsilon}_4-\tilde{\epsilon}_1}\}\rangle \tag{35}$$

and

$$\mathfrak{q}^- = \langle \{X_{-(\tilde{\epsilon}_3-\tilde{\epsilon}_4)}\}\rangle. \tag{36}$$

Then a simple calculation shows that

$$[\mathfrak{q}, \mathfrak{y}] \subset \mathfrak{y}.$$

Moreover, $\mathfrak{q} = \mathfrak{r} \oplus \mathfrak{u}$ where $\mathfrak{r} = \langle \mathfrak{h}_\mathfrak{r} \cup \{X_{\pm(\tilde{\epsilon}_3-\tilde{\epsilon}_4)}\}\rangle \simeq \mathfrak{gl}(2, \mathbb{C})$, $\mathfrak{h}_\mathfrak{r}$ is a Cartan subalgebra of $\mathfrak{r}$ and $\mathfrak{u}$ is the following nilpotent subalgebra:

$$\mathfrak{u} = \langle \{X_{\tilde{\epsilon}_2\pm\tilde{\epsilon}_j} : 3 \le j \le 4\} \cup \{X_{\tilde{\epsilon}_i\pm\tilde{\epsilon}_1} : 2 \le i \le 4\} \cup \{X_{\gamma_2}, X_{\gamma_3}, X_{\psi_1}, X_{\psi_2}\}\rangle,$$

where

$$\psi_1 = \tfrac{1}{2}(-\tilde{\epsilon}_1 + \tilde{\epsilon}_2 - \tilde{\epsilon}_3 + \tilde{\epsilon}_4), \quad \psi_2 = \tfrac{1}{2}(-\tilde{\epsilon}_1 + \tilde{\epsilon}_2 + \tilde{\epsilon}_3 - \tilde{\epsilon}_4). \tag{37}$$

The proof of the next two lemmas follow from a direct application of Poincaré–Birkhoff–Witt theorem. Let $\mathfrak{g}$ be an arbitrary finite-dimensional complex Lie algebra and let $\mathfrak{l}$ be a subalgebra of $\mathfrak{g}$. If $\{X_1, \ldots, X_p\}$ is an ordered basis of $\mathfrak{l}$ complete it to an ordered basis $\{Y_1, \ldots, Y_q, X_1, \ldots, X_p\}$ of $\mathfrak{g}$. Now, if $I = (i_1, \ldots, i_q) \in \mathbb{N}_o^q$ and $J = (j_1, \ldots, j_p) \in \mathbb{N}_o^p$ define as usual $Y^I X^J = Y_1^{i_1} \cdots Y_q^{i_q} X_1^{j_1} \cdots X_p^{j_p}$ in $U(\mathfrak{g})$. Then we have the following lemma:

Lemma 6.1. Any $u \in U(\mathfrak{g})\mathfrak{l}$ can be written in a unique way as $u = a_1 X_1 + \cdots + a_p X_p$ where

$$a_k = \sum a_{I, j_1, \ldots, j_k} Y^I X_1^{j_1} \cdots X_k^{j_k} \quad \text{for } k = 1, \ldots, p,$$

and the coefficients $a_{I, j_1, \ldots, j_k}$ are complex numbers. □

Lemma 6.2. Let $\mathfrak{g}$ and $\mathfrak{l}$ be as above. Let $u \in U(\mathfrak{g})$ and $X \in \mathfrak{g} - \mathfrak{l}$ be such that $\dot{X}(\mathfrak{l}) \subset \mathfrak{l}$. If $uX^n \equiv 0 \bmod (U(\mathfrak{g})\mathfrak{l})$ for some $n \in \mathbb{N}$, then $u \equiv 0 \bmod (U(\mathfrak{g})\mathfrak{l})$. □

Let $\mathfrak{y}^\perp$ be the orthogonal complement of $\mathfrak{y}$ in $\mathfrak{k}$ with respect to the Killing form of $\mathfrak{k}$. For any $Z \in (\mathfrak{m}^+)^\perp$ consider the linear map $T_Z : \mathfrak{q} \times (\mathfrak{m}^+)^\perp \to \mathfrak{y}^\perp$ given by

$$T_Z(X, Y) = [X, Z] + Y, \quad X \in \mathfrak{q} \text{ and } Y \in (\mathfrak{m}^+)^\perp. \tag{38}$$

Since $[\mathfrak{q}, \mathfrak{y}] \subset \mathfrak{y}$ and $(\mathfrak{m}^+)^\perp \subset \mathfrak{y}^\perp$ it follows that $\mathrm{Im}(T_Z) \subset \mathfrak{y}^\perp$, where $\mathrm{Im}(T_Z)$ denotes the image of the map T_Z. The following proposition will be used in Theorem 6.4 to prove one of the main results of this section.

Proposition 6.3. There exists $Z_o \in (\mathfrak{m}^+)^\perp$ such that $\mathrm{Im}(T_{Z_o}) = \mathfrak{y}^\perp$. □

Proof. Using (15) and the notation introduced in (16), (17) and (33) it is easy to check that

$$\mathfrak{y}^\perp = (\mathfrak{m}^+)^\perp \oplus \langle \{X_{-\delta}, X_{-\delta_1}, X_{-\delta_2}, T_{32}, T_{42}, T_{43}\} \rangle. \tag{39}$$

It is clear, from the definition of T_Z, that $(\mathfrak{m}^+)^\perp \subset \mathrm{Im}(T_Z)$ for every $Z \in (\mathfrak{m}^+)^\perp$. Now, consider the vector,

$$Z_o = X_{-\gamma_4} + X_{-\delta} + X_{-\varphi_2} + X_{-\delta_2} + X_{-\gamma_3} + H_{\tilde{\epsilon}_4 - \tilde{\epsilon}_3}, \tag{40}$$

where $H_{\tilde{\epsilon}_4-\tilde{\epsilon}_3} \in \mathfrak{h}_{\mathfrak{k}}$ is such that $(\tilde{\epsilon}_4 - \tilde{\epsilon}_3)(H_{\tilde{\epsilon}_4-\tilde{\epsilon}_3}) = 2$. Using (15) and (18), it follows that $Z_o \in (\mathfrak{m}^+)^\perp$. In view of (39), to prove that $\mathrm{Im}(T_{Z_o}) = \mathfrak{y}^\perp$ we need to show that $\langle\{X_{-\delta}, X_{-\delta_1}, X_{-\delta_2}, T_{32}, T_{42}, T_{43}\}\rangle$ is contained in $\mathrm{Im}(T_{Z_o})$. In fact, using that $X_{\varphi_1}, X_{\varphi_2}, X_{\psi_1}, X_{\psi_2}, H_{\tilde{\epsilon}_4-\tilde{\epsilon}_1}$, and T_{43} are in $\mathfrak{q}$ (see (16), (17), (33), and (37) for the notation) a simple calculation shows that

$$T_{Z_o}(X_{\varphi_2}, 0) \equiv c_1 T_{42}, \quad T_{Z_o}(X_{\varphi_1}, 0) \equiv c_2 T_{32},$$

$$T_{Z_o}(X_{\psi_2}, 0) \equiv c_3 X_{-\delta_2}, \quad T_{Z_o}(X_{\psi_1}, 0) \equiv c_4 X_{-\delta_1},$$

$$T_{Z_o}(H_{\tilde{\epsilon}_4-\tilde{\epsilon}_1}, 0) \equiv c_5 X_{-\delta}, \quad T_{Z_o}(T_{43}, 0) \equiv c_6 T_{43},$$

where, in all cases, the congruence is modulo the subspace $(\mathfrak{m}^+)^\perp$ and $c_i \neq 0$ for $1 \leq i \leq 6$. This completes the proof of the proposition. ■

Theorem 6.4. Let $u \in U(\mathfrak{k})\mathfrak{m}^+$ be a vector of weight $\lambda = a(\gamma_4 + \delta) + b\gamma_3$, with $a, b \in \mathbb{Z}$, and such that $\dot{X}(u) \equiv 0 \bmod (U(\mathfrak{k})\mathfrak{y})$ for every $X \in \mathfrak{q}^+$. Then $u \equiv 0 \bmod (U(\mathfrak{k})\mathfrak{y})$. □

Proof. Let $U(\mathfrak{k}) = \bigcup_{j\geq 0} U_j(\mathfrak{k})$ be the canonical ascending filtration of $U(\mathfrak{k})$. If $v \in U(\mathfrak{k})$ and $v \neq 0$, define

$$\deg(v) = \min\{j : v \in U_j(\mathfrak{k}) \text{ and } v \notin U_{j-1}(\mathfrak{k})\}, \tag{41}$$

where it is understood that $U_{-1}(\mathfrak{k}) = \{0\}$. Let $\mathcal{S}$ be the set of all $v \in U(\mathfrak{k})\mathfrak{m}^+$ of weight $\lambda = a(\gamma_4 + \delta) + b\gamma_3$ $(a, b \in \mathbb{Z})$, so that $\dot{X}(v) \in U(\mathfrak{k})\mathfrak{y}$ for every $X \in \mathfrak{q}^+$ and $v \notin U(\mathfrak{k})\mathfrak{y}$. The theorem will be proved if we show that $\mathcal{S} = \emptyset$. Assume on the contrary that $\mathcal{S} \neq \emptyset$ and choose $u \in \mathcal{S}$ such that $\deg(u) = \min\{\deg(v) : v \in \mathcal{S}\}$. Set $r = \deg(u)$ and let $p_r : U_r(\mathfrak{k}) \to U_r(\mathfrak{k})/U_{r-1}(\mathfrak{k})$ denote the quotient map. The map p_r intertwines the representations of K on $U_r(\mathfrak{k})$ and on $U_r(\mathfrak{k})/U_{r-1}(\mathfrak{k})$, and since $u \notin U_{r-1}(\mathfrak{k})$ we have $p_r(u) \neq 0$.

Let $S(\mathfrak{k})$ be the symmetric algebra of $\mathfrak{k}$ and let $S(\mathfrak{k}^*)$ denote the algebra of polynomial functions on $\mathfrak{k}$. Let $S_r(\mathfrak{k})$ and $S_r(\mathfrak{k}^*)$ denote the corresponding homogeneous subspaces of $S(\mathfrak{k})$ and $S(\mathfrak{k}^*)$ of degree r. There is an algebra isomorphism between $S(\mathfrak{k})$ and $S(\mathfrak{k}^*)$ defined by the Killing form of $\mathfrak{k}$, this isomorphism maps $S_r(\mathfrak{k})$ onto $S_r(\mathfrak{k}^*)$ and intertwines the canonical representations of K on $S_r(\mathfrak{k})$ and on $S_r(\mathfrak{k}^*)$. Composing this isomorphism with the natural K-isomorphism between $U_r(\mathfrak{k})/U_{r-1}(\mathfrak{k})$ and $S_r(\mathfrak{k})$, we obtain a K-isomorphism,

$$U_r(\mathfrak{k})/U_{r-1}(\mathfrak{k}) \simeq S_r(\mathfrak{k}^*). \tag{42}$$

Hence, we can think of $p_r(u)$ as a homogeneous polynomial function on $\mathfrak{k}$ of degree r, and regard p_r as a K-homomorphism from $U_r(\mathfrak{k})$ to $S_r(\mathfrak{k}^*)$.

Let $(\mathfrak{m}^+)^\perp$ be the orthogonal complement of $\mathfrak{m}^+$ in $\mathfrak{k}$ with respect to the Killing form of $\mathfrak{k}$. Since $u \in U(\mathfrak{k})\mathfrak{m}^+$ and the isomorphism given in (42) is defined by the Killing form of $\mathfrak{k}$ it follows that:

$$p_r(u)(Y) = 0 \quad \text{for every } Y \in (\mathfrak{m}^+)^\perp. \tag{43}$$

Now let $X \in \mathfrak{q}^+$. Since $[\mathfrak{q}^+, \mathfrak{y}] \subset \mathfrak{y}$, we have $\dot{X}^k(U(\mathfrak{k})\mathfrak{y}) \subset U(\mathfrak{k})\mathfrak{y}$ for every $k \in \mathbb{N}$. Then, since by hypothesis $\dot{X}(u) \in U(\mathfrak{k})\mathfrak{y}$, it follows that $\dot{X}^k(u) \in U(\mathfrak{k})\mathfrak{y}$ for any $k \in \mathbb{N}$. Therefore, using that $(\mathfrak{m}^+)^\perp \subset \mathfrak{y}^\perp$ and that p_r is a K-homomorphism it follows by induction on k that

$$X^k(p_r(u))(Y) = p_r(\dot{X}^k(u))(Y) = 0 \quad \text{for } Y \in (\mathfrak{m}^+)^\perp \quad \text{and } X \in \mathfrak{q}^+, \tag{44}$$

where $X(p_r(u))$ denotes the action of X on the polynomial function $p_r(u)$.

Since u is a vector of weight $\lambda = a(\gamma_4 + \delta) + b\gamma_3$, it follows from the definition of $\mathfrak{h}_\mathfrak{r}$ that $\dot{H}(u) = 0$ for every $H \in \mathfrak{h}_\mathfrak{r}$. Then,

$$H^k(p_r(u))(Y) = 0 \quad \text{for } Y \in \mathfrak{k}, \quad H \in \mathfrak{h}_\mathfrak{r} \quad \text{and } k \in \mathbb{N}. \tag{45}$$

Let $0 \neq \bar{u} \in U(\mathfrak{k})/U(\mathfrak{k})\mathfrak{y}$ be the image of u under the quotient map. Normalize $X_{\tilde{\epsilon}_3 - \tilde{\epsilon}_4}$ and $X_{-(\tilde{\epsilon}_3 - \tilde{\epsilon}_4)}$ so that $\{X_{\tilde{\epsilon}_3 - \tilde{\epsilon}_4}, H_{\tilde{\epsilon}_3 - \tilde{\epsilon}_4}, X_{-(\tilde{\epsilon}_3 - \tilde{\epsilon}_4)}\}$ is an $\mathfrak{s}$-triple. Since $X_{\tilde{\epsilon}_3 - \tilde{\epsilon}_4} \in \mathfrak{q}^+$ and $H_{\tilde{\epsilon}_3 - \tilde{\epsilon}_4} \in \mathfrak{h}_\mathfrak{r}$, and by hypothesis $\bar{u}$ is a dominant vector of weight zero with respect to above $\mathfrak{s}$-triple, we obtain that $\dot{X}_{-(\tilde{\epsilon}_3 - \tilde{\epsilon}_4)}(\bar{u}) = 0$. Hence, from (36), we obtain that $\dot{X}(u) \in U(\mathfrak{k})\mathfrak{y}$ for $X \in \mathfrak{q}^-$. Then, since $[\mathfrak{q}^-, \mathfrak{y}] \subset \mathfrak{y}$, it follows that:

$$X^k(p_r(u))(Y) = 0 \quad \text{for } Y \in (\mathfrak{m}^+)^\perp, \quad X \in \mathfrak{q}^-, \quad \text{and } k \in \mathbb{N}. \tag{46}$$

Now recall that for $k \in K$ and $f \in S_r(\mathfrak{k}^*)$ the action of k on f is given by $(kf)(Y) = f(\mathrm{Ad}(k^{-1})Y)$ for every $Y \in \mathfrak{k}$. Then, from (43)–(46) it follows that

$$p_r(u)(\mathrm{Ad}(\exp X)Y) = 0 \quad \text{for } X \in \mathfrak{q}^+ \cup \mathfrak{h}_\mathfrak{r} \cup \mathfrak{q}^- \text{ and } Y \in (\mathfrak{m}^+)^\perp. \tag{47}$$

Let Q be the connected Lie subgroup of K with Lie algebra $\mathfrak{q}$ (see (34)). Since the set $\exp \mathfrak{q}^+ . \exp \mathfrak{h}_\mathfrak{r} . \exp \mathfrak{q}^-$ generates Q, we obtain that

$$p_r(u)(\mathrm{Ad}(g)Y) = 0 \quad \text{for } g \in Q \text{ and } Y \in (\mathfrak{m}^+)^\perp. \tag{48}$$

Now consider the map $\Phi : Q \times (\mathfrak{m}^+)^\perp \to \mathfrak{y}^\perp$ defined by $\Phi(g, Y) = \mathrm{Ad}(g)Y$. The fact that the image of Φ is contained in $\mathfrak{y}^\perp$ follows from a simple calculation using that $[\mathfrak{q}, \mathfrak{y}] \subset \mathfrak{y}$ and that $\mathfrak{y} \subset \mathfrak{m}^+$. Let $e \in Q$ be the identity element and $Z \in (\mathfrak{m}^+)^\perp$, then $(d\Phi)_{(e,Z)}$ is the map $T_Z : \mathfrak{q} \times (\mathfrak{m}^+)^\perp \to \mathfrak{y}^\perp$ defined in (38). It follows from Proposition 6.3 that $(d\Phi)_{(e,Z_o)}$ is surjective. This implies that the image of Φ contains an open set of $\mathfrak{y}^\perp$, then in view of (48) we obtain that,

$$p_r(u)(Y) = 0 \quad \text{for every } Y \in \mathfrak{y}^\perp. \tag{49}$$

Recall that $\mathfrak{y} = \langle\{X_2, S_{23}, S_{24}\}\rangle$ (see (32)). Extend the basis of $\mathfrak{y}$ to a basis $\mathcal{B} = \{Z_1, \ldots, Z_q, X_2, S_{23}, S_{24}\}$ of $\mathfrak{k}$, where $q = \dim \mathfrak{k} - 3$. If $I = (i_1, \ldots, i_q) \in \mathbb{N}_o^q$ and $J = (j_1, j_2, j_3) \in \mathbb{N}_o^3$, set $|I| = i_1 + \cdots + i_q$, $|J| = j_1 + j_2 + j_3$ and $Z^I = Z_1^{i_1} \ldots Z_q^{i_q}$ in $S(\mathfrak{k})$. If we regard $p_r(u)$ as an element in $S_r(\mathfrak{k})$, we can write

$$p_r(u) = \sum b_{I,J} Z^I X_2^{j_1} S_{23}^{j_2} S_{24}^{j_3},$$

where $b_{I,J} \in \mathbb{C}$ and the sum extends over all I and J such that $|I| + |J| = r$. Now, identifying $\mathfrak{k}^*$ with $\mathfrak{k}$ via the Killing form of $\mathfrak{k}$ and considering a basis $\tilde{\mathcal{B}}$ of $\mathfrak{k}$ dual to $\mathcal{B}$ it follows from (49) that $b_{I,0} = 0$, for all I such that $|I| = r$. Therefore,

$$p_r(u) = \sum_{|J|>0} b_{I,J} Z^I X_2^{j_1} S_{23}^{j_2} S_{24}^{j_3}, \tag{50}$$

where the sum extends over all I and J such that $|I| + |J| = r$. On the other hand, since p_r is a K-homomorphism from $U_r(\mathfrak{k})$ to $S_r(\mathfrak{k})$ it follows that $p_r(u)$ has weight $\lambda = a(\gamma_4 + \delta) + b\gamma_3$ with respect to $\mathfrak{h}_\mathfrak{k}$. Then, (50) implies that

$$u = \sum_{|J|>0} b_{I,J} Z^I X_2^{j_1} S_{23}^{j_2} S_{24}^{j_3} + u', \tag{51}$$

where the monomials $Z^I X_2^{j_1} S_{23}^{j_2} S_{24}^{j_3}$ are in $U(\mathfrak{k})$, the sum extends over all I and J such that $|I| + |J| = r$ and u' is a vector of weight λ in $U_{r-1}(\mathfrak{k})$. Moreover, since the sum in the first term of (51) is a vector in $U(\mathfrak{k})\mathfrak{y}$ and $\dot{X}(U(\mathfrak{k})\mathfrak{y}) \subset U(\mathfrak{k})\mathfrak{y}$ for $X \in \mathfrak{q}^+$, it follows by hypothesis that $\dot{X}(u') \in U(\mathfrak{k})\mathfrak{y}$ for every $X \in \mathfrak{q}^+$. Also, since $u \in U(\mathfrak{k})\mathfrak{m}^+$ and $u \notin U(\mathfrak{k})\mathfrak{y}$ the same facts hold for u', therefore $u' \in \mathcal{S}$. This is a contradiction since $\deg(u') < \deg(u)$. Then $\mathcal{S} = \emptyset$ and the proof of the theorem is completed. ■

Corollary 6.5. Let $u \in U(\mathfrak{k})\mathfrak{m}^+$ be a $\mathfrak{q}^+$-dominant vector of weight $\lambda = a(\gamma_4 + \delta) + b\gamma_3$ with $a, b \in \mathbb{Z}$. Then $u \in U(\mathfrak{k})\mathfrak{y}$. □

Next theorem will be used in an important way in Section 8. Its proof is similar to that of Theorem 6.4. Consider the following subalgebra of $\mathfrak{k}$,

$$\tilde{\mathfrak{q}} = \{X \in \mathfrak{k} : \dot{X}(V_\gamma^{\mathfrak{k}^+}) = 0 \quad \text{for every } \gamma \in \Gamma_1\}. \tag{52}$$

It is easy to see that,

$$\tilde{\mathfrak{q}} = \mathfrak{k}^+ \oplus \mathfrak{h}_\tau \oplus \langle\{X_{-\tilde{\epsilon}_3+\tilde{\epsilon}_4}, X_{-\tilde{\epsilon}_4+\tilde{\epsilon}_1}, X_{-\tilde{\epsilon}_3+\tilde{\epsilon}_1}\}\rangle,$$

where $\mathfrak{h}_\tau$ is as in (35). Let $\tilde{Q}$ denote the connected Lie subgroup of K with Lie algebra $\tilde{\mathfrak{q}}$.

If $Z \in (\mathfrak{m}^+)^\perp$ consider the linear map $\tilde{T}_Z : \tilde{\mathfrak{q}} \times (\mathfrak{m}^+)^\perp \to \mathfrak{k}$ given by

$$\tilde{T}_Z(X, Y) = [X, Z] + Y, \quad X \in \tilde{\mathfrak{q}} \quad \text{and} \quad Y \in (\mathfrak{m}^+)^\perp. \tag{53}$$

Next proposition is the analog of Proposition 6.3 and will be used in the proof of Theorem 6.7.

Proposition 6.6. If $Z_o \in (\mathfrak{m}^+)^\perp$ is as in (40), it follows that $\mathrm{Im}(\tilde{T}_{Z_o}) = \mathfrak{k}$. □

Proof. Using the definition of $\mathfrak{y}$ (see (32)) it is easy to see that,

$$\mathfrak{k} = \mathfrak{y}^\perp \oplus \langle\{X_{-\tilde{\epsilon}_2-\tilde{\epsilon}_3}, X_{-\tilde{\epsilon}_2-\tilde{\epsilon}_4}, X_{-\tilde{\epsilon}_3-\tilde{\epsilon}_4}\}\rangle. \tag{54}$$

Now, since $\mathfrak{q} \subset \tilde{\mathfrak{q}}$ it follows from Proposition 6.3 that,

$$\tilde{T}_{Z_o}(\mathfrak{q} \times (\mathfrak{m}^+)^\perp) = T_{Z_o}(\mathfrak{q} \times (\mathfrak{m}^+)^\perp) = \mathfrak{y}^\perp.$$

Hence, it follows from (54) that to complete the proof we need to show that $X_{-\tilde{\epsilon}_2-\tilde{\epsilon}_3}$, $X_{-\tilde{\epsilon}_2-\tilde{\epsilon}_4}$, and $X_{-\tilde{\epsilon}_3-\tilde{\epsilon}_4}$ are in the image of $\tilde{T}_{Z_o}$. In fact, a simple calculation shows that,

$$\tilde{T}_{Z_o}(X_{-\tilde{\epsilon}_4+\tilde{\epsilon}_1}, 0) \equiv a_1 X_{-\tilde{\epsilon}_2-\tilde{\epsilon}_4}, \quad \tilde{T}_{Z_o}(X_{\gamma_1}, 0) \equiv a_2 X_{-\tilde{\epsilon}_3-\tilde{\epsilon}_4} \tag{55}$$

and

$$\tilde{T}_{Z_o}(X_{-\tilde{\epsilon}_3+\tilde{\epsilon}_1}, 0) \equiv a_3 X_{-\tilde{\epsilon}_2-\tilde{\epsilon}_3} + a_4 X_{-\tilde{\epsilon}_3-\tilde{\epsilon}_4}, \tag{56}$$

where, in all cases, the congruence is modulo the subspace $\mathfrak{y}^{\perp}$ and the constants a_i are nonzero for $1 \leq i \leq 4$. This completes the proof. ■

Theorem 6.7. Let $u \in U(\mathfrak{k})\mathfrak{m}^+$ be a $\mathfrak{k}^+$-dominant vector of weight $\lambda = a(\gamma_4 + \delta) + b\gamma_3$ with $a, b \in \mathbb{N}_o$. Then $u = 0$. □

Proof. Let $U(\mathfrak{k}) = \bigcup_{j\geq 0} U_j(\mathfrak{k})$ and let $u \in U(\mathfrak{k})\mathfrak{m}^+$ be a $\mathfrak{k}^+$-dominant vector of weight $\lambda = a(\gamma_4 + \delta) + b\gamma_3$ with $a, b \in \mathbb{N}_o$. Assume that $u \neq 0$ and set $r = \deg(u)$ (see (41)). Let $p_r : U_r(\mathfrak{k}) \to S_r(\mathfrak{k}^*)$ be the K-homomorphism defined in the proof of Theorem 6.4. Observe that $p_r(u) \neq 0$ because $u \notin U_{r-1}(\mathfrak{k})$.

Since $u \in U(\mathfrak{k})\mathfrak{m}^+$, and the K-homomorphism $p_r : U_r(\mathfrak{k}) \to S_r(\mathfrak{k}^*)$ is defined via the Killing form of $\mathfrak{k}$, it follows that

$$p_r(u)(Y) = 0 \quad \text{for every } Y \in (\mathfrak{m}^+)^{\perp}. \tag{57}$$

Also, since u is a $\mathfrak{k}^+$-dominant vector of weight $\lambda = a(\gamma_4 + \delta) + b\gamma_3$, it follows from Proposition 5.1 that $u \in V_{\gamma}^{\mathfrak{k}^+}$ for $\gamma \in \Gamma_1$ with highest weight λ. Hence, $\dot{X}(u) = 0$ for every $X \in \tilde{\mathfrak{q}}$. Then since p_r is a K-homomorphism, we have

$$X^k(p_r(u))(Y) = p_r(\dot{X}^k(u))(Y) = 0 \quad \text{for } X \in \tilde{\mathfrak{q}}, \quad Y \in \mathfrak{k} \quad \text{and} \quad k \in \mathbb{N}. \tag{58}$$

Now, since $\{\exp X : X \in \tilde{\mathfrak{q}}\}$ generates $\tilde{Q}$, it follows from (57) and (58) that

$$p_r(u)(\mathrm{Ad}(g)Y) = 0 \quad \text{for } g \in \tilde{Q} \quad \text{and} \quad Y \in (\mathfrak{m}^+)^{\perp}.$$

That is, $p_r(u)$ vanishes on the image of the map $\tilde{\Phi} : \tilde{Q} \times (\mathfrak{m}^+)^{\perp} \to \mathfrak{k}$ defined by $\tilde{\Phi}(g, Y) = \mathrm{Ad}(g)Y$. Now, if $e \in \tilde{Q}$ is the identity element and $Z \in (\mathfrak{m}^+)^{\perp}$, then $(d\tilde{\Phi})_{(e,Z)} = \tilde{T}_Z : \tilde{\mathfrak{q}} \times (\mathfrak{m}^+)^{\perp} \to \mathfrak{k}$. Then it follows from Proposition 6.6 that $(d\tilde{\Phi})_{(e,Z_o)}$ is surjective. This implies that the image of $\tilde{\Phi}$ contains an open set of $\mathfrak{k}$, hence $p_r(u) = 0$ as a polynomial function on $\mathfrak{k}$, which is a contradiction. Therefore, $u = 0$ as we wanted to prove. ■

Before stating the next results we define the following subalgebra of $\mathfrak{k}$,

$$\mathfrak{s} = \mathfrak{k}^- \oplus \mathfrak{h}_{\mathfrak{k}} \oplus \langle \{ X_{\tilde{\epsilon}_3+\tilde{\epsilon}_1}, X_{\tilde{\epsilon}_4+\tilde{\epsilon}_1}, T_{34}, X_1 \} \rangle. \tag{59}$$

The following result is the analog of [3, Proposition 4.9]. Although its proof uses the same idea as that of Proposition 4.9 we include it here because of some technical differences.

Proposition 6.8. Let $u_0, u_1 \in U(\mathfrak{k})$ be such that $\dot{X}_1(u_0) = \dot{X}_1(u_1) = 0$. If $u_0 + u_1 E \equiv 0 \bmod (U(\mathfrak{k})\mathfrak{y})$, then $u_0 \equiv u_1 \equiv 0 \bmod (U(\mathfrak{k})\mathfrak{y})$. □

Proof. Let $\mathfrak{s}$ be the subalgebra of $\mathfrak{k}$ defined in (59). If $\{S_1, \ldots, S_t\}$ is an ordered basis of $\mathfrak{s}$, the following is an ordered basis for $\mathfrak{k}$

$$\{S_1, \ldots, S_t, T_{23}, T_{24}, X_\delta, X_{\psi_2}, X_{\delta_1}, X_{\psi_1}, X_{\delta_2}, X_4, X_3, X_2, S_{23}, S_{24}\}, \tag{60}$$

we refer the reader to (16), (17), (33), and (37) for the notation.

Let $\mathcal{U}_1$ (respectively, $\mathcal{U}_2$) be the subspace of $U(\mathfrak{k})$ spanned by those monomials that, when written in the Poincaré–Birkhoff–Witt bases of $U(\mathfrak{k})$ associated to (60), end with powers of X_2 (respectively, S_{23}) or before. Using that $\dot{X}_1(\mathfrak{s}) \subset \mathfrak{s}$ and taking a close look at the action of $\dot{X}_1$ on the other elements of the basis (60) it follows that $\dot{X}_1(\mathcal{U}_1) \subset \mathcal{U}_1$ and $\dot{X}_1(\mathcal{U}_2) \subset \mathcal{U}_2$.

Since $u_0 + u_1 E \in U(\mathfrak{k})\mathfrak{y}$ in view of Lemma 6.1, we can write

$$u_0 + u_1 E = aX_2 + bS_{23} + cS_{24}, \tag{61}$$

with $a \in \mathcal{U}_1$, $b \in \mathcal{U}_2$ and $c \in U(\mathfrak{k})$. Then applying $\dot{X}_1$, we obtain that,

$$u_1 X_4 = \dot{X}_1(a)X_2 + aE + \dot{X}_1(b)S_{23} + \dot{X}_1(c)S_{24}, \tag{62}$$

and for every $k \geq 2$, we obtain

$$0 = \dot{X}_1^k(a)X_2 + k\dot{X}_1^{k-1}(a)E + \binom{k}{2}\dot{X}_1^{k-2}(a)X_4 + \dot{X}_1^k(b)S_{23} + \dot{X}_1^k(c)S_{24}. \tag{63}$$

Now set $\tilde{\mathfrak{y}} = \langle\{S_{23}, S_{24}\}\rangle$. If n is sufficiently large so that $\dot{X}_1^n(a) = 0$, using Equation (63) and decreasing induction on j it follows that $\dot{X}_1^j(a) = 0$ for every $0 \leq j \leq n$. In particular, $a = 0$. Hence, from (61) and (62), we obtain that $u_1 X_4 \in U(\mathfrak{k})\tilde{\mathfrak{y}}$ and $u_0 + u_1 E \in U(\mathfrak{k})\tilde{\mathfrak{y}}$. Now, using Lemma 6.2 and the fact that $\dot{E}(\mathfrak{y}) = 0$ it follows that $u_0 \equiv u_1 \equiv 0 \bmod (U(\mathfrak{k})\tilde{\mathfrak{y}})$, therefore $u_0 \equiv u_1 \equiv 0 \bmod (U(\mathfrak{k})\mathfrak{y})$ as we wanted to prove. ■

Next proposition will be used in Theorem 8.6 of Section 8.

Proposition 6.9. Let $\{\eta_j : j \in \mathbb{N}_0\}$ be a sequence in $U(\mathfrak{k})$ such that $\eta_j \neq 0$ for a finite number of j's, $\dot{X}_1(\eta_j) = 0$ for every $j \in \mathbb{N}_0$ and $\sum_{j\geq 0} \eta_j E^j \equiv 0 \bmod (U(\mathfrak{k})\mathfrak{y})$. Then

$$\sum_{i\geq 0} \eta_{2i} E^{2i} \equiv 0 \quad \text{and} \quad \sum_{i\geq 0} \eta_{2i+1} E^{2i+1} \equiv 0,$$

where the congruence is $\bmod(U(\mathfrak{k})\mathfrak{y})$. □

Proof. Let $\Delta = 2X_4X_2 - E^2$. Since X_2, X_4, and E commute with each other it follows that $(-1)^j \Delta^j \equiv E^{2j} \bmod(U(\mathfrak{k})\mathfrak{y})$ for every $j \in \mathbb{N}_0$. Also observe that $\dot{X}_1(\Delta) = 0$. From now on the proof follows in the same way as that of [3, Proposition 4.11], simply changing the congruence $\bmod(U(\mathfrak{k})X_2)$ for a congruence $\bmod(U(\mathfrak{k})\mathfrak{y})$. ■

7 An Estimate on the Kostant Degree

In this section, we introduce the *degree property* and show that every $b \in P(U(\mathfrak{g})^K)$ has the degree property. This result is used in Proposition 7.11. We also show that to prove Theorem 2.7, and therefore our main result Theorem 1.1, it is enough to prove Theorem 7.12.

Definition 7.1. Let $b = b_m \otimes Z^m + \cdots + b_0 \in U(\mathfrak{k})^M \otimes U(\mathfrak{a})$ with $b_m \neq 0$. We say that b has the *degree property* if $d(b_{m-j}) \leq m + 2j$ for every $0 \leq j \leq m$. □

We begin by recalling a few facts about $\mathfrak{s}$-triples in $\mathfrak{g}$. Recall that an $\mathfrak{s}$-triple is a set of three linearly independent elements $\{x, e, f\}$ in $\mathfrak{g}$ such that $[x, e] = 2e$, $[x, f] = -2f$, and $[e, f] = x$. The $\mathfrak{s}$-triple $\{x, e, f\}$ is called *normal* if $e, f \in \mathfrak{p}$ and $x \in \mathfrak{k}$. A normal $\mathfrak{s}$-triple $\{x, e, f\}$ is called *principal* if e (and hence f) is a regular element in $\mathfrak{p}$. Theorem 3 of [10] guarantees that principal normal $\mathfrak{s}$-triples exist, and in Theorem 6 of the same paper

it is proved that any two principal normal $\mathfrak{s}$-triples are K_θ-conjugate, where K_θ is the subgroup of all elements in G that commute with θ.

Fix a principal normal $\mathfrak{s}$-triple $\{x, e, f\}$ in $\mathfrak{g}$ and set $z = x/2$. In [14, Proposition 1], it is proved that the map $\mathrm{ad}(z) : \mathfrak{p} \to \mathfrak{p}$ is diagonalizable with eigenvalues 1, -1, and possibly 0. Since in our case $\mathfrak{g}$ is the complexification of the Lie algebra of F_4^{-20}, the eigenvalues of $\mathrm{ad}(z)$ in $\mathfrak{g}$ are -2, -1, 0, 1, and 2 (see the [14, proof of Proposition 1]), then the next result follows.

Lemma 7.2. The map $\mathrm{ad}(z) : \mathfrak{k} \to \mathfrak{k}$ is diagonalizable and its highest eigenvalue is 2. □

In [14, Corollary 9], it is shown that if $\mathfrak{g}_o$ is a semisimple Lie algebra over $\mathbb{R}$, different from $\mathfrak{sl}(2, \mathbb{R})$, and V_γ is an irreducible K-module of type $\gamma \in \Gamma$, then $d(\gamma)$ is the highest eigenvalue of z in V_γ. From this result, we have the following lemma:

Lemma 7.3. Let V be a finite-dimensional K-module and let n be the highest eigenvalue of z in V. If $u \in V^M$ and $u \neq 0$, then $d(u) \leq n$. □

As an application of Lemmas 7.2 and 7.3, we obtain the following result that will be useful in what follows.

Lemma 7.4. If $u \in U_m(\mathfrak{k})^M$ and $u \neq 0$, then $d(u) \leq 2m$. □

Recall that $P : U(\mathfrak{g}) \longrightarrow U(\mathfrak{k}) \otimes U(\mathfrak{a})$ is the projection on the first summand of the direct sum $U(\mathfrak{g}) = (U(\mathfrak{k}) \otimes U(\mathfrak{a})) \oplus U(\mathfrak{g})\mathfrak{n}$, associated to an Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ adapted to $\mathfrak{k}$. The proof of the following result follows easily by choosing an appropriate Poincaré–Birkhoff–Witt bases of $U(\mathfrak{g})$.

Lemma 7.5.

$$P(U_m(\mathfrak{g})) = \sum_{0 \leq \ell \leq m} U_{m-\ell}(\mathfrak{k}) \otimes Z^\ell \quad \text{for every } m \geq 0.$$

□

Let $\sigma : S(\mathfrak{g}) \longrightarrow U(\mathfrak{g})$ be the symmetrization mapping. It is known that σ is a K-linear isomorphism. Let $\varphi : U(\mathfrak{k}) \otimes S(\mathfrak{p}) \longrightarrow U(\mathfrak{g})$ be the K-linear isomorphism defined by $\varphi(u \otimes p) = u\sigma(p)$. Then we have,

$$U(\mathfrak{g})^K = \sum_{m \geq 0} (U(\mathfrak{k})\sigma(S_m(\mathfrak{p})))^K.$$

Theorem 7.6. Let $u \in (U(\mathfrak{k})\sigma(S_m(\mathfrak{p})))^K$ where m is the smallest possible. Then $P(u) = b_m \otimes Z^m + \cdots + b_0 \in U(\mathfrak{k})^M \otimes U(\mathfrak{a})$, $b_m \neq 0$ and $d(b_{m-j}) \leq m + 2j$ for $0 \leq j \leq m$. □

Proof. Let $\tilde{u} \in (U(\mathfrak{k}) \otimes S_m(\mathfrak{p}))^K$ be such that $\varphi(\tilde{u}) = u$. Write $S_m(\mathfrak{p}) = \sum W_\tau$ where the sum runs over a finite set $J \subset \Gamma$. Then by Schur's Lemma we have,

$$(U(\mathfrak{k}) \otimes S_m(\mathfrak{p}))^K = \sum_{\tau \in J} (U(\mathfrak{k})_{\tau^*} \otimes W_\tau)^K, \tag{64}$$

where τ^* is the contragradient representation of τ, and $U(\mathfrak{k})_{\tau^*}$ denotes the τ^*-isotypic component of $U(\mathfrak{k})$.

Let $\mathfrak{q}$ be a subspace of $\mathfrak{p}$ such that $\mathfrak{p} = \mathfrak{a} \oplus \mathfrak{q}$ and let $\{X_1, \ldots, X_r\}$ be an ordered bases of $\mathfrak{q}$. If $a = (a_1, \ldots, a_r)$ with $a_i \in \mathbb{N}_0$, and $X^a = X_1^{a_1} \cdots X_r^{a_r}$ in $S(\mathfrak{p})$, it follows that $\{Z^\ell X^a : 0 \leq \ell + |a| \leq m\}$ is a bases of $S_m(\mathfrak{p})$, where $|a| = a_1 + \cdots + a_r$. Then, in view of (64), we can write

$$\tilde{u} = \sum_{0 \leq \ell + |a| \leq m} u_{\ell,a} \otimes Z^\ell X^a,$$

where $u_{\ell,a}$ belongs to the K-module $V = \sum_{\tau \in J} U(\mathfrak{k})^M_{\tau^*}$ for every pair (ℓ, a). Then,

$$P(u) = \sum_{0 \leq \ell + |a| \leq m} P(u_{\ell,a}\sigma(Z^\ell X^a)) = \sum_{0 \leq \ell + |a| \leq m} u_{\ell,a} P(\sigma(Z^\ell X^a)). \tag{65}$$

Now, since $\sigma(Z^\ell X^a) \in U_{\ell+|a|}(\mathfrak{g})$, it follows from Lemma 7.5 that

$$P(\sigma(Z^\ell X^a)) = \sum_{0 \leq j \leq \ell + |a|} v_{\ell,a,j} \otimes Z^j,$$

with $v_{\ell,a,j} \in U_{\ell+|a|-j}(\mathfrak{k})$. Hence from (65) we have,

$$P(u) = \sum_{0 \leq j \leq m} \left(\sum_{j \leq \ell + |a| \leq m} u_{\ell,a} v_{\ell,a,j} \right) \otimes Z^j.$$

Then from the uniqueness of the coefficients b_j it follows that

$$b_j = \sum_{j \leq \ell + |a| \leq m} u_{\ell,a} v_{\ell,a,j} \quad \text{for } 0 \leq j \leq m, \tag{66}$$

where $v_{\ell,a,j} \in U_{\ell+|a|-j}(\mathfrak{k}) \subset U_{m-j}(\mathfrak{k})$ for every pair (ℓ, a). Hence from (66), we obtain that,

$$b_j \in \langle V \cdot U_{m-j}(\mathfrak{k})\rangle^M \subset U(\mathfrak{k})^M \quad \text{for } 0 \le j \le m. \tag{67}$$

Recall that $\langle S\rangle$ denotes the linear space spanned by the set S. Observe that in this case $\langle V \cdot U_{m-j}(\mathfrak{k})\rangle$ is a K-module.

Now, since the highest eigenvalue of z in $\mathfrak{p}$ is 1, it follows that the highest eigenvalue of z in $S_m(\mathfrak{p})$ is m. Then $d(\tau) \le m$ for every $\tau \in J$, and therefore $d(\tau^*) \le m$ for every $\tau \in J$. This implies that the highest eigenvalue of z in V is less or equal to m. On the other hand, we know that the highest eigenvalue of z in $U_{m-j}(\mathfrak{k})$ is less or equal to $2(m-j)$, hence the highest eigenvalue of z in $\langle V \cdot U_{m-j}(\mathfrak{k})\rangle$ is less or equal to $m + 2(m-j)$. Then, from Lemma 7.3 and (67) it follows that $d(b_j) \le m + 2(m-j)$ for $0 \le j \le m$, and therefore $d(b_{m-j}) \le m + 2j$ for $0 \le j \le m$, as we wanted to prove. ■

Theorem 7.7. Let $b \in P(U(\mathfrak{g})^K)$ be such that $b = b_m \otimes Z^m + \cdots + b_0$ with $b_m \neq 0$, then $d(b_{m-j}) \le m + 2j$ for every $0 \le j \le m$. □

Proof. Let $u \in U(\mathfrak{g})^K$ be such that $P(u) = b$. Since $b_m \neq 0$, it follows from [11, Corollary 7.3] that $u \in (U(\mathfrak{k})\sigma(S_m(\mathfrak{p})))^K$ and m is the smallest possible. Hence the result follows from Theorem 7.6. ■

Our next goal is to show that Theorem 2.7 follows from Theorem 7.12. In the next lemma, we single out a particular element $\omega \in B$. This element is a scalar multiple of $P(\Omega)$, where Ω is the Casimir of $\mathfrak{g}$.

Lemma 7.8. There exist $\omega = \omega_2 \otimes Z^2 + \omega_1 \otimes Z + \omega_0 \in P(U(\mathfrak{g})^K) \subset B$ such that $\omega_2 = 1$, ω_1 is a nonzero scalar, ω_0 is a scalar multiple of the Casimir element of $\mathfrak{m}$ and $d(\omega_0) \le 4$. □

Proposition 7.9. For any $b \in U(\mathfrak{k})^M \otimes U(\mathfrak{a})$, there exist $n \in \mathbb{N}_0$ such that $b\omega^n$ has the degree property. □

Proof. Let $b = b_m \otimes Z^m + \cdots + b_0 \in U(\mathfrak{k})^M \otimes U(\mathfrak{a})$. Fix $n \in \mathbb{N}_0$ sufficiently large so that $d(b_{m-j}) \le m + 2n + 2j$ for every $0 \le j \le m$. A simple calculation shows that

$$\omega^n = \sum_{k=0}^{2n} \tilde{\omega}_{k,n} \otimes Z^{2n-k}, \tag{68}$$

where $\tilde{\omega}_{k,n} = \sum_{i=0}^{[k/2]} \binom{n}{k-i}\binom{k-i}{i}\omega_1^{k-2i}\omega_0^i$ for $0 \le k \le 2n$, and that

$$b\omega^n = \sum_{j=0}^{m+2n}\left(\sum_{k=0}^{\min\{j,2n\}} b_{m+k-j}\tilde{\omega}_{k,n}\right)\otimes Z^{m+2n-j}. \tag{69}$$

Then if $(b\omega^n)_\ell$ denotes the coefficient of Z^ℓ in $b\omega^n$, we have

$$\begin{aligned} d((b\omega^n)_{m+2n-j}) &\le \max\{d(b_{m+k-j}\tilde{\omega}_{k,n}) : 0 \le k \le j\} \\ &= \max\{d(b_{m+k-j}) + d(\tilde{\omega}_{k,n}) : 0 \le k \le j\} \\ &\le \max\{m + 2n + 2(j-k) + 2k : 0 \le k \le j\} \\ &= m + 2n + 2j, \end{aligned}$$

for every $0 \le j \le m + 2n$. Hence $b\omega^n$ has the degree property. ■

It is now convenient to introduce the following notation, for any $m \in \mathbb{N}_0$ and $0 \le r \le m$ define d_r as follows:

$$d_r = \left[\frac{3m - 2r + 2}{2}\right]. \tag{70}$$

In the next lemma, we obtain an upper bound on the Kostant degree of the coefficients b_r of certain $b \in U(\mathfrak{k})^M \otimes U(\mathfrak{a})$.

Lemma 7.10. Let $b = b_m \otimes Z^m + \cdots + b_0 \in U(\mathfrak{k})^M \otimes U(\mathfrak{a})$ with $b_m \neq 0$. If $b\omega$ has the degree property, then $d(b_r) \le 2d_r$ for every $0 \le r \le m$. □

Proof. Let $(b\omega)_\ell$ denote the coefficient of Z^ℓ in $b\omega$. It follows from (68) and (69), or directly by computing $b\omega$, that

$$b_{m-j} = (b\omega)_{m+2-j} - b_{m-j+1}\omega_1 - b_{m-j+2}\omega_0 \tag{71}$$

for $0 \le j \le m$, with the understanding that $b_{m+1} = b_{m+2} = 0$. Then, since ω_1 is a scalar and $d(\omega_0) \le 4$, from (71) we obtain that

$$d(b_{m-j}) \le \max\{d((b\omega)_{m+2-j}), d(b_{m-j+1}), d(b_{m-j+2}) + 4\}. \tag{72}$$

Hence, using (72) and the fact that $b\omega$ has the degree property, it follows by induction on j that $d(b_{m-j}) \leq m+2+2j$ for every $0 \leq j \leq m$. Now, since the Kostant degree of any element of $U(\mathfrak{k})^M$ is even (see (ii) and (iii) of Proposition 5.1), it follows that $d(b_r) \leq 2d_r$ for every $0 \leq r \leq m$. ■

Let $b = b_m \otimes Z^m + \cdots + b_0 \in B$ be such that $d(b_r) \leq 2d_r$ for $0 \leq r \leq m$, where d_r is as in (70). Using Proposition 5.1 and the above bound on $d(b_r)$ we can decompose the coefficients b_r of b as follows:

$$b_r = \sum_{t=0}^{2d_r} \sum_{\max\{0,t-d_r\} \leq i \leq [t/2]} b^r_{2i,t-2i} \quad \text{for } 0 \leq r \leq m, \tag{73}$$

where $b^r_{2i,t-2i}$ is the component of b_r in the isotypic component of $U(\mathfrak{k})^M$ of type $(2i, t-2i)$. Consider now the following linear subspace of B:

$$\tilde{B} = \{b \in B : \ b^{2k}_{2i,j} = 0 \quad \text{if } i+j \leq k \quad \text{and } 0 \leq 2k \leq \deg(b)\}. \tag{74}$$

That is, $\tilde{B}$ consists of the elements $b \in B$ such that the K-types $b^{2k}_{2i,j}$ that occur in the coefficient b_{2k} of b, have Kostant degree $> 2k$ for all k such that $0 \leq 2k \leq \deg(b)$.

Proposition 7.11. Let $b = b_m \otimes Z^m + \cdots + b_0 \in B$, $b_m \neq 0$, and $d(b_r) \leq 2d_r$ for $0 \leq r \leq m$. Then there exist $\tilde{b} \in \tilde{B}$ such that $d(\tilde{b}_r) \leq 2d_r$ for $0 \leq r \leq m$, $\tilde{b}_m = b_m$ if m is odd, and $d(b_m - \tilde{b}_m) \leq m$ if m is even. Moreover $\tilde{b}^r_{2i,j} = b^r_{2i,j}$ if $i+j = d_r$ for every $0 \leq r \leq m$. □

Proof. Let $b = b_m \otimes Z^m + \cdots + b_0 \in B$ be such that $b_m \neq 0$ and $d(b_r) \leq 2d_r$ for $0 \leq r \leq m$. Set $p = 2[m/2]$ and using (73) define,

$$c_p = \sum_{t=0}^{p} \sum_{\max\{0,t-\frac{p}{2}\} \leq i \leq [t/2]} b^p_{2i,t-2i}.$$

That is, c_p contains all the K-types of b_p of Kostant degree smaller or equal to p. Hence, $c_p \in U(\mathfrak{k})^M$ and $d(c_p) \leq p$. Since p is even $c_p \otimes Z^p \in (U(\mathfrak{k})^M \otimes U(\mathfrak{a}))^W$. Then from Proposition 2.6 it follows that $c_p \otimes Z^p$ is the leading term of an element $c^{(p)} = c_p \otimes Z^p + \cdots \in P(\mathrm{U}(\mathfrak{g})^K)$. Now define $b^{(p)} = b - c^{(p)} \in B$. All the K-types that occur in the p-coefficient of $b^{(p)}$ have Kostant degree $> p$ and, since $c^{(p)} \in P(\mathrm{U}(\mathfrak{g})^K)$, it follows from Theorem 7.7 that

$d(b_r^{(p)}) \leq 2d_r$ for $0 \leq r \leq m$. Moreover, the K-types of Kostant degree $2d_r$ of $b_r^{(p)}$ and b_r are the same for $0 \leq r \leq m$.

Considering now the $(p-2)$-coefficient of $b^{(p)}$ we construct, in a similar way, elements $c^{(p-2)} \in P(\mathrm{U}(\mathfrak{g})^K)$ and $b^{(p-2)} = b^{(p)} - c^{(p-2)} \in B$, such that the coefficients of $b^{(p-2)}$ corresponding to degrees $> p-2$ are the same as those of $b^{(p)}$, and all the K-types that occur in the $(p-2)$-coefficient of $b^{(p-2)}$ have Kostant degree $> p-2$. Moreover, since $c^{(p-2)} \in P(\mathrm{U}(\mathfrak{g})^K)$, Theorem 7.7 implies that $d(b_r^{(p-2)}) \leq 2d_r$ for $0 \leq r \leq m$, and that the K-types of Kostant degree $2d_r$ of $b_r^{(p-2)}$ and b_r are the same for every $0 \leq r \leq m$.

Continuing in this way, we obtain a sequence $b^{(p)}, b^{(p-2)}, \ldots, b^{(0)}$ of elements in B of degree at most m. If we set $\tilde{b} = b^{(0)}$, it is clear that $\tilde{b} \in \tilde{B}$ and that $\tilde{b}$ has all the required properties. ■

Finally, in Proposition 7.14, we show that next theorem implies Theorem 2.7 (and therefore Theorem 1.1). The proof of Theorem 7.12 will be done in the next section.

Theorem 7.12. Let $b = b_m \otimes Z^m + \cdots + b_0 \in \tilde{B}$ be such that $d(b_r) \leq 2d_r$ for every $0 \leq r \leq m$. Then $b = 0$. □

If we assume that Theorem 7.12 holds, we obtain the following corollary.

Corollary 7.13. Let $b = b_m \otimes Z^m + \cdots + b_0 \in B$ be such that $b_m \neq 0$ and $b\omega$ has the degree property. Then m is even and b has the degree property. □

Proof. Since $b\omega$ has the degree property, it follows from Lemma 7.10 that $d(b_r) \leq 2d_r$ for $0 \leq r \leq m$. Then Proposition 7.11 implies that there exist $\tilde{b} \in \tilde{B}$ such that $d(\tilde{b}_r) \leq 2d_r$ for $0 \leq r \leq m$, $\tilde{b}_m = b_m$ if m is odd and $\tilde{b}^r_{2i,j} = b^r_{2i,j}$ if $i + j = d_r$ for $0 \leq r \leq m$. On the other hand, Theorem 7.12 implies that $\tilde{b} = 0$. Hence, m must be even and $b^r_{2i,j} = 0$ if $i + j = d_r$ for $0 \leq r \leq m$, which implies that b has the degree property. ■

Proposition 7.14. Let $b = b_m \otimes Z^m + \cdots + b_0 \in B$ with $b_m \neq 0$. Then, m is even and b has the degree property. In particular $d(b_m) \leq m$, and therefore Theorem 2.7 holds. □

Proof. Let $b = b_m \otimes Z^m + \cdots + b_0 \in B$ be as in the statement of the theorem. It follows from Proposition 7.9 that there exist $n \in \mathbb{N}_0$ such that $b\omega^n$ has the degree property. Now, since $b\omega^{n-1} = b_m \otimes Z^{m+2(n-1)} + \cdots \in B$ and $b_m \neq 0$, it follows from Corollary 7.13 that $m + 2(n-1)$ is even and $b\omega^{n-1}$ has the degree property. Hence m is even, and from

Corollary 7.13 and induction on k it follows that $b\omega^{n-k}$ has the degree property for every $0 \leq k \leq n$. In particular, b has the degree property, as we wanted to prove. ■

8 Proof of Theorem 7.12

Our goal in this section is to prove Theorem 7.12. To do this, given any $b = b_m \otimes Z^m + \cdots + b_0 \in B$ such that $d(b_r) \leq 2d_r$ for $0 \leq r \leq m$, we will construct a linear system of equations in $U(\mathfrak{k})$ where the unknowns are $\mathfrak{k}^+$-dominant vectors associated to certain K-types of the coefficients of b (see Theorem 8.6). This system will allow us to carry out a decreasing induction process that, when applied to $b \in \tilde{B}$, will lead to the proof of Theorem 7.12.

Let $b = b_m \otimes Z^m + \cdots + b_0 \in B$ be such that $d(b_r) \leq 2d_r$ for $0 \leq r \leq m$. As indicated in (73), we can decompose the coefficient b_r of b as follows:

$$b_r = \sum_{t=0}^{2d_r} \sum_{\max\{0,t-d_r\} \leq i \leq [t/2]} b^r_{2i,t-2i} \quad \text{for } 0 \leq r \leq m. \tag{75}$$

We find it very convenient to keep in mind the following array of the K-types that occur in b_r.

$$\begin{aligned} b_r = {} & b^r_{2d_r,0} + b^r_{2d_r-2,1} + b^r_{2d_r-4,2} + b^r_{2d_r-6,3} + \cdots + b^r_{0,d_r} \\ & + b^r_{2d_r-2,0} + b^r_{2d_r-4,1} + b^r_{2d_r-6,2} + \cdots + b^r_{0,d_r-1} \\ & + b^r_{2d_r-4,0} + b^r_{2d_r-6,1} + \cdots + b^r_{0,d_r-2} \\ & + b^r_{2d_r-6,0} + \cdots + b^r_{0,d_r-3} + \cdots + b^r_{0,0}. \end{aligned} \tag{76}$$

Observe that the parameter t used in (75) may be regarded as a label for the skew diagonals of the array (76). In fact, for $0 \leq t \leq 2d_r$ we shall refer to the set $\{b^r_{2i,t-2i} : \max\{0, t-d_r\} \leq i \leq [t/2]\}$ as the skew diagonal associated to t. Also observe that the Kostant degree is constant along the rows of the array (76), it takes the values $2d_r, 2d_r - 2, \ldots, 0$ from the top to the bottom row of the array corresponding to b_r.

Let $T \in \mathbb{N}_o$ denote the label of the skew diagonals in the array corresponding to b_0. We will use T as a parameter for a decreasing induction. For $m \leq T \leq 2d_0$ if m is even, and $m-1 \leq T \leq 2d_0$ if m is odd, consider the following propositional function associated to b:

$$P(T) : b_r = \sum_{t=0}^{\min\{T-r,2d_r\}} \sum_{\max\{0,t-d_r\} \leq i \leq [t/2]} b^r_{2i,t-2i}, \quad 0 \leq r \leq m. \tag{77}$$

Observe that $P(T)$ holds if and only if $b^r_{2i,t-2i} = 0$ for $t > \min\{T - r, 2d_r\}$ for every $0 \le r \le m$. Also, in view of (73), it follows that $P(2d_0)$ holds. This will be the starting point of our inductive argument.

Let E, X_δ, H, Y, and $\tilde{Y}$ be as in Section 4. Recall that $\dot{E}(H) = -\frac{1}{2}E$, $\dot{X}_\delta(\tilde{Y}) = X_\delta$ and $\dot{X}_\delta(H) = \dot{E}(X_\delta) = 0$. In the following lemma, we state some properties of the derivations $\dot{E}$ and $\dot{X}_\delta$, we refer to [3, Lemma 6.1] for their proof.

Lemma 8.1.

(i) $\dot{E}^k(H^k) = k!(-\frac{1}{2}E)^k$ and $\dot{E}^k(H^j) = 0$ if $k > j$.
(ii) $\dot{E}^k\varphi_k(H) = (-\frac{1}{2}E)^k$, where φ_k is as in (9).
(iii) $\dot{X}^k_\delta((-\tilde{Y})^k) = k!(-X_\delta)^k$ and $\dot{X}^k_\delta((-\tilde{Y})^j) = 0$ if $k > j$.
(iv) $\dot{X}^k_\delta\varphi_k(a - \tilde{Y}) = (-X_\delta)^k$ for any $a \in \mathbb{C}$. □

The following proposition is the analog of [3, Proposition 6.2]. Its proof is the same as that of Proposition 6.2 and it is obtained by applying $\dot{X}_\delta^{T-n-\ell}$ to $\epsilon(\ell, n)$ of Theorem 3.6, and using Lemma 3.3 and Lemma 8.1. Also observe that the derivation $\dot{X}_\delta$ preserves the ideal $U(\mathfrak{k})\mathfrak{m}^+$.

Proposition 8.2. Let $b = b_m \otimes Z^m + \cdots + b_0 \in B$ be such that $d(b_r) \le 2d_r$ for $0 \le r \le m$, and assume that $P(T)$ holds for $m \le T \le 2d_0$. Then for every (ℓ, n) such that $0 \le \ell, n$ and $\ell + n \le T$ we have

$$(-1)^n\Sigma_1 E^n - (-1)^\ell \Sigma_2 E^\ell \equiv 0 \bmod (U(\mathfrak{k})\mathfrak{m}^+), \tag{78}$$

where

$$\begin{aligned}\Sigma_1 &= \sum_{(i,r)\in I_1} A_{i,r}(T, n, \ell)\dot{X}_\delta^{T-\ell-i}\dot{E}^{\ell+i-r}(b_r)E^{r-i}X_\delta^{i-n},\\ \Sigma_2 &= \sum_{(i,r)\in I_2} A_{i,r}(T, \ell, n)\dot{X}_\delta^{T-n-i}\dot{E}^{n+i-r}(b_r)E^{r-i}X_\delta^{i-\ell},\end{aligned} \tag{79}$$

and

$$A_{i,r}(T, n, \ell) = \left(-\frac{1}{2}\right)^{r-i}(-1)^{i-n}r!\binom{T-n-\ell}{i-n}\binom{\ell}{r-i,},$$

$$I_1 = \{(i,r) \in N_0^2 : n \le i \le \min\{m, T-\ell\}, i \le r \le \min\{m, i+\ell\}\},$$

$$I_2 = \{(i,r) \in N_0^2 : \ell \le i \le \min\{m, T-n\}, i \le r \le \min\{m, i+n\}\}.$$

□

Next proposition is the analog of [3, Proposition 6.3] and its proof is the same as that of Proposition 6.3. It is obtained by replacing b_r in (79) by its decomposition in K-types given in (77), then one uses Proposition 5.1(iv) to simplify the sums Σ_1 and Σ_2, and finally one multiplies both sums on the right by X_δ^T and then changes in each term a certain number of X_δ's by the same number of X_4's so that Σ_1 and Σ_2 become weight vectors with respect to $\mathfrak{h}_\mathfrak{k}$. Here, we use that $X_\delta \equiv X_4 \bmod (U(\mathfrak{k})\mathfrak{m}^+)$ and that the derivation $\dot{X}_\delta$ preserves the ideal $U(\mathfrak{k})\mathfrak{m}^+$.

Proposition 8.3. Let $b = b_m \otimes Z^m + \cdots + b_0 \in B$ be such that $d(b_r) \le 2d_r$ for $0 \le r \le m$, and assume that $P(T)$ holds for $m \le T \le 2d_0$. Then for every (ℓ, n) such that $0 \le \ell, n$ and $\ell + n \le T$ we have

$$(-1)^n \Sigma_1 E^n - (-1)^\ell \Sigma_2 E^\ell \equiv 0 \bmod (U(\mathfrak{k})\mathfrak{m}^+), \tag{80}$$

where

$$\Sigma_1 = \sum_{\substack{(i,r)\in I_1 \\ \max\{0,T-r-d_r\}\le k\le[\frac{T-r}{2}]}} A_{i,r}(T,n,\ell)\dot{X}_\delta^{T-\ell-i}\dot{E}^{\ell+i-r}(b^r_{2k,T-r-2k}) \times E^{r-i}X_\delta^{T-k}X_4^{k+i-n},$$

$$\Sigma_2 = \sum_{\substack{(i,r)\in I_2 \\ \max\{0,T-r-d_r\}\le k\le[\frac{T-r}{2}]}} A_{i,r}(T,\ell,n)\dot{X}_\delta^{T-n-i}\dot{E}^{n+i-r}(b^r_{2k,T-r-2k}) \times E^{r-i}X_\delta^{T-k}X_4^{k+i-\ell},$$

with the understanding that the K-types $b^r_{2k,T-r-2k}$ that do not occur in b_r are assumed to be zero. Moreover, in Equation (80) all the terms of the left-hand side are weight vectors of weight $(2T-\ell-n)\gamma_1 + T(\gamma_2+\delta)$. □

Equation (80) may be regarded as a system of linear equations where the unknowns, $\dot{X}_\delta^{T-j-i}\dot{E}^{j+i-r}(b^r_{2k,T-r-2k})$, are derivatives of the K-types that occur in the $T-r$ skew diagonal of the coefficient b_r of b (see (76)). Since the unknowns in this system are, in general, not $\mathfrak{k}^+$-dominant, we are going to replace the system by an equivalent one where all the unknowns become $\mathfrak{k}^+$-dominant vectors associated to the K-types $b^r_{2k,T-r-2k}$.

Let $\tilde{\epsilon}(\ell, n)$ be the left-hand side of Equation (80). For $0 \le n \le \min\{2d_m, T\}$ and $0 \le L \le \min\{2m, T\} - n$ consider the following linear combination:

$$\mathcal{E}_L(n) = \sum_{\ell=0}^{L} (-2)^\ell \binom{L}{\ell} \tilde{\epsilon}(\ell, n) E^{L-\ell} X_4^{\ell+n}. \tag{81}$$

Under the hypothesis of Proposition 8.3, we have $\mathcal{E}_L(n) \equiv 0 \bmod (U(\mathfrak{k})\mathfrak{m}^+)$. Also set,

$$\mathcal{E}_L^1(n) = \sum_{\ell=0}^{L} (-2)^\ell \binom{L}{\ell} \Sigma_1 E^{L-\ell} X_4^{\ell+n} \quad \text{and} \quad \mathcal{E}_L^2(n) = \sum_{\ell=0}^{L} 2^\ell \binom{L}{\ell} \Sigma_2 X_4^{\ell+n}.$$

Then it follows that

$$\mathcal{E}_L(n) = (-1)^n \mathcal{E}_L^1(n) E^n - \mathcal{E}_L^2(n) E^L. \tag{82}$$

The following lemma is the analog of [3, Lemma 6.5]. For the symplectic group $\mathrm{Sp}(n,1)$ the vectors $D_k(b_{2i,j})$ are $\mathfrak{k}^+$-dominant, however, in F_4 this property does not hold.

Lemma 8.4. Let $b_{2i,j} \in U(\mathfrak{k})^M$ be an M-invariant element of type $(2i, j)$. For $0 \le k \le 2i$ define,

$$D_k(b_{2i,j}) = \Sigma_{\ell=0}^{k} (-2)^\ell \binom{k}{\ell} \binom{j+\ell}{\ell}^{-1} \dot{X}_\delta^{2i-\ell} \dot{E}^{j+\ell}(b_{2i,j}) E^{k-\ell} X_4^\ell. \tag{83}$$

Then $D_k(b_{2i,j})$ is a vector of weight $i(\gamma_4 + \delta) + (j+k)\gamma_3$ with respect to $\mathfrak{h}_\mathfrak{k}$, $\dot{X}(D_k(b_{2i,j})) \equiv 0 \bmod (U(\mathfrak{k})\mathfrak{y})$ for every $X \in \mathfrak{q}^+$ and $\dot{X}_1(D_k(b_{2i,j})) = 0$. □

Proof. Recall that $\mathfrak{q}^+$ is the linear span of $\{X_\alpha : \alpha \in \Delta^+(\mathfrak{k}, \mathfrak{h}_\mathfrak{k}) - \{\gamma_1\}\}$. Since γ_1 is a simple root in $\Delta^+(\mathfrak{k}, \mathfrak{h}_\mathfrak{k})$, if α is a positive root it follows that $\alpha - \gamma_1$ is either a positive root different from γ_1 or it is not a root. Hence if $u \in U(\mathfrak{k})$ is a $\mathfrak{k}^+$-dominant vector, we have

$$\dot{X}_\alpha(\dot{X}_{-1}^\ell(u)) = 0 \quad \text{for every } \alpha \in \Delta^+(\mathfrak{k}, \mathfrak{h}_\mathfrak{k}) - \{\gamma_1\} \text{ and } \ell \in \mathbb{N}_0.$$

Then, in view of (22), it follows that

$$\dot{X}(\dot{X}_\delta^{2i-\ell} \dot{E}^{j+\ell}(b_{2i,j})) = 0 \quad \text{for every } X \in \mathfrak{q}^+. \tag{84}$$

On the other hand, since $\dot{E}(\mathfrak{y}) = \dot{X}_4(\mathfrak{y}) = 0$ and $[\mathfrak{q}^+, \mathfrak{y}] \subset \mathfrak{y}$ it follows that

$$\dot{X}(E^n) \equiv 0 \quad \text{and} \quad \dot{X}(X_4^n) \equiv 0 \bmod (U(\mathfrak{k})\mathfrak{y}) \ \text{ for every } X \in \mathfrak{q}^+. \tag{85}$$

Hence, from (83) to (85), we obtain that

$$\dot{X}(D_k(b_{2i,j})) \equiv 0 \bmod (U(\mathfrak{k})\mathfrak{y}) \quad \text{for every } X \in \mathfrak{q}^+. \tag{86}$$

Now, since $\dot{X}_1(E) = X_4$ and $\dot{X}_1(X_4) = 0$, using (20) it follows that $\dot{X}_1(D_k(b_{2i,j})) = 0$. The details of this calculation can be found in [3, proof of Lemma 6.5]. Finally, it is easy to check that each term of $D_k(b_{2i,j})$ is a vector of weight $i(\gamma_4 + \delta) + (j + k)\gamma_3$ with respect to $\mathfrak{h}_{\mathfrak{k}}$. ■

As indicated at the beginning of the section, we are interested in proving that $P(T)$ implies $P(T-1)$ for $m \leq T \leq 2d_0$. To do this, we need to show that the K-types $b^r_{2i,T-r-2i}$ that occur in the $T-r$ skew diagonal of b_r are equal to zero for $0 \leq r \leq m$. That is,

$$b^r_{2i,T-r-2i} = 0 \quad \text{if } 0 \leq T - r - 2i \leq \min\{T, 2d_0 - T\} - r,$$

for $0 \leq r \leq m$. For this purpose, we introduce another propositional function $Q(n)$ defined for $0 \leq n \leq \min\{T, 2d_0 - T\} + 1$ as follows:

$$Q(n): \ b^r_{2i,T-r-2i} = 0 \quad \text{if } 0 \leq T - r - 2i < n \ \text{ for } 0 \leq r \leq m. \tag{87}$$

Clearly, $Q(0)$ is true. Also, since we have that $d(b_r) \leq 2d_r$ for $0 \leq r \leq m$, we obtain that (87) holds if $T - r - 2i > \min\{T, 2d_0 - T\} - r$.

Next theorem is the analog of [3, Theorem 6.6] and its proof is the same as that of Theorem 6.6, it consist in rewriting the sum $\mathcal{E}^1_L(n)$ in terms of the vectors $D_k(b_{2i,j})$ defined in Lemma 8.4, and the sum $\mathcal{E}^2_L(n)$ in terms of $\mathfrak{k}^+$-dominant vectors. We refer the reader to [3, Section 6] for the details.

Theorem 8.5. Let $b = b_m \otimes Z^m + \cdots + b_0 \in B$ be such that $d(b_r) \leq 2d_r$ for $0 \leq r \leq m$, let $m \leq T \leq 2d_0$ and $0 \leq n \leq \min\{T, 2d_0 - T\}$. Then if $P(T)$ and $Q(n)$ are true, we have,

$$\sum_{\substack{r,k \\ T-L\leq 2k+r\leq T-n}} B_{r,k}(T,n,L) D_{L+2k+r-T}(b^r_{2k,T-r-2k})(X_\delta X_4)^{T-k} E^n$$

$$- \sum_{\substack{r,\ell \\ r\equiv T-n}} (-2)^\ell \binom{L}{\ell} \binom{T-n-\ell}{r-\ell} u^r_{T-r-n,n} (X_\delta X_4)^{(T+r+n)/2} E^L \equiv 0, \tag{88}$$

for all L such that $0 \leq L \leq \min\{2m, T\} - n$. Here, the congruence is module the left ideal $U(\mathfrak{k})\mathfrak{m}^+$, $u^r_{T-r-n,n} = r!(-1)^r \dot{X}_\delta^{T-n-r} \dot{E}^n (b^r_{T-n-r,n})$ and

$$B_{r,k}(T,n,L) = r!(-1)^T 2^{T-r-2k} \binom{L}{T-r-2k} \binom{T-L-n}{r-n}.$$

Moreover, the left-hand side of Equation (88) is a weight vector of weight $T(\gamma_4 + \delta) + (n+L)\gamma_3$. □

We are now in a good position to obtain the system of equations that we are looking for. Using the notation introduced in (33) define

$$U = X_\delta X_4 - T_{23} S_{23} + T_{24} S_{24}. \tag{89}$$

Then U is a $\mathfrak{k}^+$-dominant vector of weight $\gamma_4 + \delta$ with respect to $\mathfrak{h}_\mathfrak{k}$ and $U \equiv X_\delta X_4$ mod $(U(\mathfrak{k})\mathfrak{y})$. For any T and n such that $m \leq T \leq 2d_0$ and $0 \leq n \leq \min\{T, 2d_0 - T\}$ consider the following sets:

$$L(T,n) = \{L \in \mathbb{N}_0 : 0 \leq L \leq \min\{2m, T\} - n, \quad L \not\equiv n\},$$

$$R_F(T,n) = \{r \in \mathbb{N}_0 : 0 \leq r \leq \min\{m, \min\{T, 2d_0 - T\} - n\}, \ r \equiv T - n\},$$

the congruence is mod(2) and the subindex F stands for F_4^{-20}. Let $|L(T,n)|$ and $|R_F(T,n)|$ denote the cardinality of these sets. The set $L(T,n)$ was also considered for the symplectic group Sp(*n*,1) while $R_F(T,n)$ is the analog of the set $R(T,n)$ defined in [3, Section 6].

Next theorem gives a system of linear equations where the unknowns, $u^r_{T-r-n,n}$, are $\mathfrak{k}^+$-dominant vectors associated to the K-types that occur in the $T-r$ skew diagonal of the coefficient b_r of b for $0 \leq r \leq m$ (see (76)).

Theorem 8.6. Let $b = b_m \otimes Z^m + \cdots + b_0 \in B$ be such that $d(b_r) \leq 2d_r$ for $0 \leq r \leq m$, and let $m \leq T \leq 2d_0$ and $0 \leq n \leq \min\{T, 2d_0 - T\}$. Then if $P(T)$ and $Q(n)$ are true we have,

$$\sum_{r \in R_F(T,n)} \left(\sum_{\ell} (-2)^{\ell} \binom{L}{\ell} \binom{T-n-\ell}{r-\ell} \right) u^r_{T-r-n,n} U^{(T+r+n)/2} = 0, \tag{90}$$

for every $L \in L(T, n)$. Here, $u^r_{T-r-n,n} = r!(-1)^r \dot{X}_\delta^{T-n-r} \dot{E}^n(b^r_{T-n-r,n})$. □

Proof. Let u denote the left-hand side of Equation (88). Then, in view of Theorem 8.5, u is a vector in $U(\mathfrak{k})\mathfrak{m}^+$ of weight $\lambda = T(\gamma_4 + \delta) + (n + L)\gamma_3$ with respect to $\mathfrak{h}_\mathfrak{k}$. On the other hand, using that $\dot{X}(X_\delta) \equiv 0 \bmod (U(\mathfrak{k})\mathfrak{y})$, for every $X \in \mathfrak{q}^+$, together with (85), (86), and the fact that E, X_4, and X_δ commute with $\mathfrak{y}$ and that $[\mathfrak{q}^+, \mathfrak{y}] \subset \mathfrak{y}$, it follows that $\dot{X}(u) \equiv 0 \bmod (U(\mathfrak{k})\mathfrak{y})$ for every $X \in \mathfrak{q}^+$. Then applying Theorem 6.4, we obtain that $u \equiv 0 \bmod (U(\mathfrak{k})\mathfrak{y})$, that is,

$$\sum_{\substack{r,k \\ T-L \leq 2k+r \leq T-n}} B_{r,k}(T, n, L) D_{L+2k+r-T}(b^r_{2k,T-r-2k})(X_\delta X_4)^{T-k} E^n$$

$$- \sum_{\substack{r,\ell \\ r \equiv T-n}} (-2)^{\ell} \binom{L}{\ell} \binom{T-n-\ell}{r-\ell} u^r_{T-r-n,n} (X_\delta X_4)^{(T+r+n)/2} E^L \equiv 0. \tag{91}$$

Since $U \equiv X_\delta X_4 \bmod (U(\mathfrak{k})\mathfrak{y})$ (see (89)), we replace $X_\delta X_4$ by U in (91). Also, recall that $\dot{X}_1(X_\delta) = \dot{X}_1(X_4) = 0$ and $\dot{X}_1(D_k(b_{2i,j})) = 0$ for $b_{2i,j} \in U(\mathfrak{k})^M$ of type $(2i, j)$ and $0 \leq k \leq 2i$ (see Lemma 8.4). Hence, since $L \not\equiv n \bmod (2)$, it follows from Proposition 6.9 and Lemma 6.2 that

$$\sum_{r \in R_F(T,n)} \left(\sum_{\ell} (-2)^{\ell} \binom{L}{\ell} \binom{T-n-\ell}{r-\ell} \right) u^r_{T-r-n,n} U^{(T+r+n)/2} \equiv 0, \tag{92}$$

module the left ideal $U(\mathfrak{k})\mathfrak{y}$. Now, since the left-hand side of Equation (92) is a $\mathfrak{k}^+$-dominant vector of weight $T(\gamma_4 + \delta) + n\gamma_3$, applying Theorem 6.7 we can replace the congruence mod$(U(\mathfrak{k})\mathfrak{y})$ by an equality. This completes the proof of the theorem. ■

For T and n fixed, Theorem 8.6 gives a system of $|L(T, n)|$ linear equations in the $|R_F(T, n)|$ unknowns $u^r_{T-r-n,n}$. This system is the analog of the one given in [3, Theorem 6.7]. The main advantage of this system is that the unknowns are all

$\mathfrak{k}^+$-dominant vectors. Let $A(T,n)$ denote the coefficient matrix of this system. In [3, Section 6], a very thorough study of this matrix is being carried out (see Section 6.2). This is done by considering a $(k+1)\times(k+1)$ matrix $A(s)$ with polynomial entries $A_{ij}(s)\in\mathbb{C}[s]$ that generalizes $A(T,n)$. This matrix is defined as follows:

$$A_{ij}(s)=\sum_{0\le\ell\le\min\{L_i,2j+\delta\}}(-2)^\ell\binom{L_i}{\ell}\binom{s-\ell}{2j+\delta-\ell},$$

where $0\le L_0<\cdots<L_k$ is a sequence of integers and $\delta\in\{0,1\}$. In [3, Theorem 6.15], we obtained an explicit formula for $\det A(s)$ as a product of polynomials of degree 1 in the variable s. Hence, we know the exact values of s for which $A(s)$ is singular. Moreover, from the proof of Theorem 6.15 it follows that whenever $A(s)$ is singular the reason is that it has several pairs of equal rows. In this case, the strategy consist in replacing one equation in each one of these pairs by a new equation obtained from Theorem 8.5. We refer the reader to [3, Section 6.3] for the details.

Since our goal in this section is to prove Theorem 7.12, we need to restate Theorem 8.6 for elements $b\in\tilde{B}$. If $b=\sum_{r=0}^m b_r\otimes Z^r\in\tilde{B}$, it follows from (74) that for r even we have $b^r_{2i,j}=0$ if $d(b^r_{2i,j})=2(i+j)\le r$. Hence, when $T-n\equiv 0$ and $r\in R_F(T,n)$ is such that $d(b^r_{T-r-n,n})=T-r+n\le r$, we have $u^r_{T-r-n,n}=0$ in Equation (90). Then we may consider a new index set defined as follows:

$$\tilde{R}_F(T,n)=\begin{cases}\left\{r\in R_F(T,n):r<\dfrac{T+n}{2}\right\} & \text{if } T-n\equiv 0,\\ R_F(T,n) & \text{if } T-n\equiv 1,\end{cases}\tag{93}$$

where the congruence is $\bmod(2)$. For $b\in\tilde{B}$, we restate Theorem 8.6 as follows. This theorem is the analog of [3, Theorem 6.19] and it will be our main tool in the proof of Theorem 7.12.

Theorem 8.7. Let $b=b_m\otimes Z^m+\cdots+b_0\in\tilde{B}$ be such that $d(b_r)\le 2d_r$ for $0\le r\le m$, and let $m\le T\le 2d_0$ and $0\le n\le\min\{T,2d_0-T\}$. Then if $P(T)$ and $Q(n)$ are true, we have,

$$\sum_{r\in\tilde{R}_F(T,n)}\left(\sum_\ell(-2)^\ell\binom{L}{\ell}\binom{T-n-\ell}{r-\ell}\right)u^r_{T-r-n,n}U^{(T+r+n)/2}=0,$$

for every $L\in L(T,n)$. Here, $u^r_{T-r-n,n}=r!(-1)^r\dot{X}_\delta^{T-n-r}\dot{E}^n(b^r_{T-n-r,n})$. □

Now we recall the definition of the sets $R(T, n)$ and $\tilde{R}(T, n)$ used in the case of the group Sp(n,1) (see [3, Section 6]). Let $b = b_m \otimes Z^m + \cdots + b_0 \in \tilde{B}$ with $b_m \neq 0$. For positive integers T and n such that $m \leq T \leq 4m$ and $0 \leq n \leq \min\{T, 4m - T\}$ consider the following set:

$$R(T, n) = \{r \in \mathbb{N}_0 : 0 \leq r \leq \min\{m, \min\{T, 4m - T\} - n\}, r \equiv T - n\},$$

where the congruence is mod(2). The set $\tilde{R}(T, n)$ is defined as in (93) replacing $R_F(T, n)$ by $R(T, n)$ (see [3, (116)]). Next we will show that Theorem 7.12 follows from [3, Proposition 6.21 and Proposition 6.22].

Proof of Theorem 7.12. Let $b = b_m \otimes Z^m + \cdots + b_0 \in \tilde{B}$ be such that $d(b_r) \leq 2d_r$ for $0 \leq r \leq m$. We need to show that $b = 0$. Assume on the contrary that $b \neq 0$ and that $m = \deg(b)$, that is $b_m \neq 0$. We will obtain a contradiction by showing that $b_m = 0$. In view of the definition of $\tilde{B}$ (see (74)) to do this it is enough to show that $P(\frac{3m}{2})$ holds if m is even and that $P(m - 1)$ is true if m is odd. Since $P(2d_0)$ holds (see (73) and (77)) this will follow from the fact that $P(T)$ implies $P(T - 1)$ for any $m \leq T \leq 2d_0$.

Consider first $m \geq 1$. Let $m \leq T \leq 2d_0$ and $0 \leq n \leq \min\{T, 2d_0 - T\}$, and assume that $P(T)$ and $Q(n)$ hold. Since $2d_0 \leq 4m$, it follows that $\min\{T, 2d_0 - T\} \leq \min\{T, 4m - T\}$ and a simple calculation shows that

$$\min\{m, \min\{T, 2d_0 - T\} - n\} \leq \min\{m, \min\{T, 4m - T\} - n\}.$$

Hence, $R_F(T, n) \subset R(T, n)$ and therefore $\tilde{R}_F(T, n) \subset \tilde{R}(T, n)$.

Now set, $u^r_{T-r-n,n} = 0$ if $r \in \tilde{R}(T, n)$ and $r \notin \tilde{R}_F(T, n)$ and $u^r_{T-r-n,n} = r!(-1)^r \dot{X}_\delta^{T-n-r} \dot{E}^n(b^r_{T-n-r,n})$ if $r \in \tilde{R}_F(T, n)$. Then from Theorem 8.7, we obtain for every $L \in L(T, n)$ that

$$\sum_{r \in \tilde{R}(T,n)} \left(\sum_\ell (-2)^\ell \binom{L}{\ell} \binom{T - n - \ell}{r - \ell} \right) u^r_{T-r-n,n} U^{(T+r+n)/2} = 0. \tag{94}$$

Observe that, except for the fact that the vector $X_\delta X_4$ is replaced by U, the system of equations given by (94) is the same as that of [3, Theorem 6.19], in particular, their coefficient matrices are exactly the same. Then that $P(T)$ implies $P(T - 1)$ for any $m \leq T \leq 2d_0$ follows from [3, Proposition 6.21 and Proposition 6.22]. We point out that the proof of these propositions are based on a very thorough study of the coefficient matrix of these

system. We refer the reader to [3, Theorem 6.15, Corollary 6.16 and Proposition 6.20] for the details.

Consider now $m = 0$. Assume that $b = b_0 \in \tilde{B}$, $b \neq 0$, and that $d(b) = d(b_0) \leq 2d_0 = 2$. From the definition of $\tilde{B}$ (see (74)) we have $b = b_0 = b^0_{2,0} + b^0_{0,1}$, therefore $b^0_{2,0} \neq 0$ or $b^0_{0,1} \neq 0$, in particular, $d(b) = 2$. Consider the element $b^2\omega = b^2 \otimes Z^2 + \omega_1 b^2 \otimes Z + b^2\omega_0 \in B$, where $\omega = 1 \otimes Z^2 + \omega_1 \otimes Z + \omega_0$ is the element in $P(U(\mathfrak{g}))^K$ defined in Lemma 7.8.

From Proposition 5.3, we have $d(b^2) = 4$, hence the component of Kostant degree 4 of b^2 is nonzero. Now, as in Proposition 7.11, we can remove the components of Kostant degree less or equal to two from b^2 and the components of Kostant degree less or equal to zero from $b^2\omega_0$. This procedure defines an element $\tilde{b} = \tilde{b}_2 \otimes Z^2 + \tilde{b}_1 \otimes Z + \tilde{b}_0 \in \tilde{B}$ with $d(\tilde{b}_r) \leq 2d_r$ for $0 \leq r \leq 2$, and such that the component of Kostant degree 4 of $\tilde{b}_2$ is the same as that of b^2. Then $\tilde{b} \neq 0$, which contradicts the first part of the proof. Therefore, $b = 0$, as we wanted to prove. ■

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