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Research Article

Geometric constants and orthogonality in Banach spaces

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Abstract

Based on the parallelogram law and orthogonality, we define a new geometric constant and obtain some of its geometric properties. This constant provides a useful tool for estimating the exact values of Jordan-von Neumann constants in Banach spaces and for studying the orthogonality. In addition, we consider Pythagorean orthogonality and introduce another new constant to investigate a connection between Pythagorean orthogonality and isosceles orthogonality.

Keywords: normed spaces; isosceles orthogonality; uniformly nonsquare.

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1. Introduction

The notion of orthogonality has a long history. Various extensions of orthogonality have been introduced over the last decade. In addition to some common orthogonalities [3], some more special orthogonalities include fuzzy orthogonality [16], Carlsson type orthogonality [14], and ρ_λ-orthogonality [19]. In particular, proposing the notion of orthogonality in normed linear spaces has been the object of extensive research by many mathematicians. Based on the concept of orthogonality, many geometric constants have been introduced, including James constant $J(X)$ (see [6]) and Wu constant $D(X)$ [11]. These constants provide a new geometric perspective for characterizing Banach spaces.

We recall two orthogonality types introduced in normed linear spaces. In 1945, James [8] introduced the so-called isosceles orthogonality as follows: $x \perp_I y$ if and only if $\|x+y\| = \|x-y\|$. Taking into account the classical Pythagorean theorem, one can define the orthogonal relation in a normed space $(X, \|\cdot\|)$ as: $x \perp_P y$ if and only if $\|x-y\|^2 = \|x\|^2 + \|y\|^2$. Some other known orthogonalities in normed linear spaces can be found in [3, 4, 9, 10, 17] and references cited therein.

In this paper, two new geometric constants in a normed linear space are introduced. Some properties of these geometric constants are discussed.

2. Preliminaries

Let X be a normed linear space. Let $S_X = \{x \in X : \|x\| = 1\}$ and $B_X = \{x \in X : \|x\| \le 1\}$ be the unit sphere and unit ball of X, respectively. For convenience, we write $x \not\perp_P y$ to indicate that x and y do not satisfy the relation $x \perp_P y$. Recall that a Banach space X is said to be nonsquare [7] if for any $x, y \in S_X$, one has

$$\min\left\{\left\|\frac{x+y}{2}\right\|, \left\|\frac{x-y}{2}\right\|\right\} < 1.$$

The von Neumann-Jordan constant $C_{NJ}(X)$ was defined in 1937 by Clarkson [5] as

$$C_{NJ}(X) = \sup\left\{\frac{\|x+y\|^2 + \|x-y\|^2}{2\left(\|x\|^2 + \|y\|^2\right)} : x, y \in X, \text{ not both zero}\right\}.$$

We recall some properties about the von Neumann-Jordan constant (see [12, 13]):
(1). $1 \le C_{NJ}(X) \le 2$; X is a Hilbert space if and only if $C_{NJ}(X) = 1$.
(2). X is uniformly nonsquare if and only if $C_{NJ}(X) < 2$.
(3). $C_{NJ}(X) = C_{NJ}(X^*)$.

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3. Main results

The constant $P(X)$

We introduce a new constant based on the parallelogram law and von Neumann-Jordan constant. In the rest of this paper, we consider only Banach spaces of dimension at least 2. We begin by introducing the following key definition:

Definition 3.1. *For a Banach space X, define $P(X)$ as follow:*

$$P(X)=\sup\left\{\frac{\|x\|^2+\|y\|^2-\|x+y\|^2}{\|x\|^2+\|y\|^2-\|x-y\|^2}:x,y\in X,x\not\perp_P y\right\}.$$

The following proposition establishes an alternative form of $P(X)$:

Proposition 3.1. *If X is a Banach space, then*

$$P(X)=\sup\left\{\frac{\|x\|^2+\|y\|^2-\|x+y\|^2}{\|x\|^2+\|y\|^2-\|x-y\|^2}:x,y\in X,\max\{\|x\|,\|y\|\}=1,\min\{\|x\|,\|y\|\}\le 1,x\not\perp_P y\right\}.$$

Proof. If $0\neq\|x\|\geq\|y\|$, then

$$\|x\pm y\|=\|x\|\left\|\frac{x}{\|x\|}\pm\frac{y}{\|x\|}\right\|$$

and hence

$$\frac{\|x\|^2+\|y\|^2-\|x+y\|^2}{\|x\|^2+\|y\|^2-\|x-y\|^2}=\frac{1+\left\|\frac{y}{\|x\|}\right\|^2-\left\|\frac{x}{\|x\|}+\frac{y}{\|x\|}\right\|^2}{1+\left\|\frac{y}{\|x\|}\right\|^2-\left\|\frac{x}{\|x\|}-\frac{y}{\|x\|}\right\|^2},$$

which shows that the supremum in the definition of $P(X)$ can be taken over $x,y\in X$ such that $\|x\|=1$ and $\|y\|\leq 1$. For $\|x\|\leq\|y\|\neq 0$, the proof is similar to the one concerning the case $0\neq\|x\|\geq\|y\|$. □

Proposition 3.2. *If X is a Banach space, then $P(X)\geq -1$.*

Proof. For a Banach space X, let $x\in X$, $\|x\|=1$, $y=\frac{1}{2}x$, then x and y do not satisfy the relation $x\perp_P y$. Thus,

$$\begin{aligned}P(X)&\geq\frac{\|x\|^2+\|y\|^2-\|x+y\|^2}{\|x\|^2+\|y\|^2-\|x-y\|^2}\\&=\frac{\|x\|^2+\frac{1}{4}\|x\|^2-\frac{9}{4}\|x\|^2}{\|x\|^2+\frac{1}{4}\|x\|^2-\frac{1}{4}\|x\|^2}\\&=-1.\end{aligned}$$

□

Proposition 3.3. *A normed space X is a Hilbert space if and only if $P(X)=-1$.*

Proof. If X is a Hilbert space, then we get $2\|x\|^2+2\|y\|^2=\|x+y\|^2+\|x-y\|^2$. Thus $P(X)=-1$. Conversely, assume that $P(X)=-1$. Then, we have

$$\frac{\|x\|^2+\|y\|^2-\|x+y\|^2}{\|x\|^2+\|y\|^2-\|x-y\|^2}\leq -1.$$

Case 1. If $\|x\|^2+\|y\|^2-\|x-y\|^2>0$, then we have $\|x+y\|^2+\|x-y\|^2\geq 2\|x\|^2+2\|y\|^2$.

Case 2. If $\|x\|^2+\|y\|^2-\|x-y\|^2<0$, then we have $\|x+y\|^2+\|x-y\|^2\leq 2\|x\|^2+2\|y\|^2$.

Combining Case 1 and Case 2, we get

$$\|x+y\|^2+\|x-y\|^2=2\|x\|^2+2\|y\|^2.$$

According to the properties of Hilbert space, i-orthogonality leads to p-orthogonality. This is equivalent to the proposition that non-p-orthogonality implies non-i-orthogonality. However, if $\|x+y\|=\|x-y\|$, then

$$\|x+y\|^2+\|x-y\|^2\neq 2\|x\|^2+2\|y\|^2,$$

which is not possible. This completes the proof. □

Proposition 3.4. *If X is a finite-dimensional Banach space and $P(X)=1$, then there exist $x,y\in X$ such that $x\perp_I y$.*

Proof. Since $P(X) = 1$, we have $\|u_n\| = 1$ and $\|v_n\| \leq 1$ such that

$$\frac{\|u_n\|^2 + \|v_n\|^2 - \|u_n + v_n\|^2}{\|u_n\|^2 + \|v_n\|^2 - \|u_n - v_n\|^2} \to 1.$$

By the compactness of the closed unit ball of X, there exist $u_0, v_0 \in X$ such that $u_{n_k} \to u_0$ and $v_{n_k} \to v_0$, and thereby

$$\frac{\|u_0\|^2 + \|v_0\|^2 - \|u_0 + v_0\|^2}{\|u_0\|^2 + \|v_0\|^2 - \|u_0 - v_0\|^2} = 1.$$

Since $\|u_0\|^2 + \|v_0\|^2 - \|u_0 - v_0\|^2 \neq 0$, replacing u_0 by x and v_0 by $-y$ gives $x \perp_I y$. □

Proposition 3.5. *If X is a finite-dimensional Banach space and if $P(X) = 0$, then there exist $x, y \in X$ such that $x \perp_P y$.*

Proof. Since $P(X) = 0$, we have $\|u_n\| = 1$and $\|v_n\| \leq 1$ such that

$$\frac{\|u_n\|^2 + \|v_n\|^2 - \|u_n + v_n\|^2}{\|u_n\|^2 + \|v_n\|^2 - \|u_n - v_n\|^2} \to 0.$$

By the compactness of the closed unit ball of X, there exist $u_0, v_0 \in X$ such that $u_{n_k} \to u_0$ and $v_{n_k} \to v_0$, and so

$$\frac{\|u_0\|^2 + \|v_0\|^2 - \|u_0 + v_0\|^2}{\|u_0\|^2 + \|v_0\|^2 - \|u_0 - v_0\|^2} = 0.$$

Since $\|u_0\|^2 + \|v_0\|^2 - \|u_0 - v_0\|^2 \neq 0$, replacing u_0 by x and v_0 by $-y$ yields $x \perp_P y$. □

Theorem 3.1. *Let X be a Banach space and $P(X) < 1$. If $x, y \in X$ such that $x \perp_I y$, then $x \perp_P y$.*

Proof. The inequality $P(X) < 1$ implies that

$$\frac{\|x\|^2 + \|y\|^2 - \|x + y\|^2}{\|x\|^2 + \|y\|^2 - \|x - y\|^2} < 1$$

for $x, y \in X$ and $x \not\perp_P y$. On the other hand, since $x \perp_I y$, we get

$$\frac{\|x\|^2 + \|y\|^2 - \|x + y\|^2}{\|x\|^2 + \|y\|^2 - \|x - y\|^2} = \frac{\|x\|^2 + \|y\|^2 - \|x + y\|^2}{\|x\|^2 + \|y\|^2 - \|x + y\|^2} = 1.$$

Since it contradicts that $P(X) < 1$, which enforces that $\|x\|^2 + \|y\|^2 - \|x + y\|^2 = 0$, and hence $x \perp_P y$. □

Observation 3.1. *Let $X = \mathbb{R}^2$ and*

$$\|\cdot\| = \max\left\{\|\cdot\|_\infty, \frac{1}{\sqrt{2}}\|\cdot\|_1\right\}.$$

Then

$$P(X) \geq \frac{1}{5 - 4\sqrt{2}} \approx -1.52439.$$

Proof. Take $x = (\sqrt{2} - 1, 1)$ and $y = (1 - \sqrt{2}, 1)$. Then, $\|x\| = \|y\| = 1$, $\|x + y\| = 2$, and $\|x - y\| = 2\sqrt{2} - 2$, which yield

$$P(X) \geq \frac{1 + 1 - 4}{1 + 1 - (2\sqrt{2} - 2)^2} = \frac{1}{5 - 4\sqrt{2}} \approx -1.52439.$$

□

Observation 3.2. *Let X be the $l_2 - l_1$ space, i.e., $\mathbb{R}^2$ with the norm*

$$\|(x, y)\| = \begin{cases} |x| + |y|, & \text{if } xy \leq 0 \\ \sqrt{|x|^2 + |y|^2}, & \text{if } xy > 0. \end{cases}$$

Then $P(X) \geq 1$.

Proof. Take $x = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $y = \left(-\frac{1}{2}, \frac{1}{2}\right)$. Then $\|x\| = \|y\| = 1$ and $\|x \pm y\| = \sqrt{\frac{3}{2}}$, which yield

$$P(X) \geq \frac{1 + 1 - \frac{3}{2}}{1 + 1 - \frac{3}{2}} = 1.$$

□

Observation 3.3. *Let $X = \mathbb{R}^2$ and $\|(x_1, x_2)\| = \max\{|x_1|, |x_2|\}$. Then, for $x = (1,0), y = (0,1)$, we have $\|x+y\| = \|x-y\| = 1$ and thus $P(X) \geq 1$.*

Theorem 3.2. *For any real Banach space X, there exists an equivalence norm $\||\cdot\||$, such that $P((X, \||\cdot\||)) \geq 1$.*

Proof. For $f \in X^*$ with $f \neq 0$, we define $M = \{x \in X : f(x) = 0\}$. Then $X = R \bigoplus M$. Let

$$\||x\|| = \|(r,m)\| = \max\{|r|, \|m\|\},$$

then $\||\cdot\||$ is an equivalence norm on X. Take $x = (1,m)$ and $y = (-1,m)$ for $\|m\| = 1, m \in M$. We have $\||x\|| = \||y\|| = 1$ and $\||x+y\|| = \||x-y\|| = 2$. Thus, $P((X, \||\cdot\||)) \geq 1$. □

The following theorem is due to Maurey:

Theorem 3.3. [15] *The separable Banach space X contains l_1 copy if and only if its second conjugate space contains a non-zero element g, such that for all $x \in X$,*

$$\|g+x\| = \|g-x\|.$$

By using Theorem 3.3, we get the following result for $P(X)$ in the second conjugate space.

Theorem 3.4. *Let X be a separable Banach space and contains l_1 copy, then $P(X) \geq 1$.*

Proof. We just need to show that there exists a point $x \in S_X$ such that $\|g\|^2 + 1 \neq \|g-x\|^2$. Arguing by contradiction, we suppose that for any $x \in S_X$, we have $\|g\|^2 + 1 = \|g-x\|^2$, this means the distance between a fixed point and any point on the unit sphere is constant, which is impossible. □

Theorem 3.5. *Let X be a Banach space. If X is not nonsquare, then $P(X) \geq 1$.*

Proof. If X is not nonsquare, then there exist $x, y \in S(X)$ such that $\|x+y\| = \|x-y\| = 2$. Thus,

$$P(X) \geq \frac{\|x\|^2 + \|y\|^2 - \|x+y\|^2}{\|x\|^2 + \|y\|^2 - \|x-y\|^2} = \frac{1+1-4}{1+1-4} = 1.$$

□

The following theorem gives the relationship between $P(X)$ and $C_{NJ}(X)$.

Theorem 3.6. *If X is a Banach space, then $P(X) \geq 1 - C_{NJ}(X)$.*

Proof. Since

$$P(X) = \sup\left\{\frac{\|x\|^2 + \|y\|^2 - \|x+y\|^2}{\|x\|^2 + \|y\|^2 - \|x-y\|^2} : x, y \in X, x \not\perp_P y\right\},$$

we have

$$\begin{aligned} P(X) &\geq \frac{\|x\|^2 + \|y\|^2 - \|x+y\|^2}{\|x\|^2 + \|y\|^2 - \|x-y\|^2} \\ &\geq \frac{\|x\|^2 + \|y\|^2 - \|x+y\|^2}{\|x\|^2 + \|y\|^2} \\ &\geq 1 - \frac{\|x+y\|^2}{\|x\|^2 + \|y\|^2} \end{aligned}$$

and

$$\begin{aligned} P(X) &\geq \frac{\|x\|^2 + \|-y\|^2 - \|x-y\|^2}{\|x\|^2 + \|-y\|^2 - \|x+y\|^2} \\ &\geq \frac{\|x\|^2 + \|-y\|^2 - \|x-y\|^2}{\|x\|^2 + \|-y\|^2} \\ &\geq 1 - \frac{\|x-y\|^2}{\|x\|^2 + \|y\|^2}. \end{aligned}$$

Thus,

$$2P(X) \geq 2 - \frac{\|x+y\|^2 + \|x-y\|^2}{\|x\|^2 + \|y\|^2}$$

and so

$$P(X) \geq 1 - \frac{\|x+y\|^2 + \|x-y\|^2}{2\left(\|x\|^2 + \|y\|^2\right)} \geq 1 - C_{NJ}(X).$$

□

Observation 3.4. *Let* $X = \mathbb{R}^2$ *and* $\|(x_1, x_2)\| = \max\{|x_1| + (\sqrt{2}-1)|x_2|, |x_2| + (\sqrt{2}-1)|x_1|\}$. *It is already known that* $C_{NJ}(X) = 4 - 2\sqrt{2}$ (see [2]). *Thus,* $P(X) \geq -3 + 2\sqrt{2}$.

Definition 3.2. *For* $p \geq 1, l^p(X)$ *denotes the set of sequences space as follows:*

$$l^p(X) = \left\{ x = \{x_n\} : x_n \in X, n \in \mathbb{N}, \textit{such that} \|x\|_p = \left\{ \sum_{n=1}^{\infty} \|x_n\|^p \right\}^{\frac{1}{p}} < \infty \right\}.$$

It is well known and easy to prove that $l^p(X)$ and $l^\infty(X)$ both are Banach spaces under the norms $\|x\|_p$ and $\|x\|_\infty$, respectively. They play an important role in functional analysis.

Theorem 3.7. *For any Banach space* X*, the inequality* $P(l^p(X)) \geq 1$ *holds.*

Proof. We take $x_1 \in S_X$. Set $x = (x_1, 0, 0, \ldots)$ and $y = (0, -x_1, 0, \ldots)$. Then, $x + y = (x_1, -x_1, 0, \ldots)$, $x - y = (x_1, x_1, 0, \ldots)$, $\|x\|_p = \|y\|_p = 1$, $\|x+y\|_p = \|x-y\|_p = 2^{\frac{1}{p}}$, and hence $P(l^p(X)) \geq 1$. □

Theorem 3.8. *For any Banach space* X*,* $P(l^\infty(X)) \geq 1$.

Proof. Take $x_1 \in S_X$. Put $x = (x_1, 0, 0, \ldots)$ and $y = (0, x_1, 0, \ldots)$. Then, $x + y = (x_1, x_1, 0, \ldots)$, $x - y = (x_1, -x_1, 0, \ldots)$, $\|x\|_\infty = \|y\|_\infty = 1$, $\|x+y\|_\infty = \|x-y\|_\infty = 1$, and hence $P(l^\infty(X)) \geq 1$. □

The constant $I(X)$

Inspired by the isosceles orthogonality and the polarization identity for inner product spaces, we introduce a new constant estimating the distance between two unit vectors x and y satisfying $x \perp_P y$.

Definition 3.3.

$$I(X) = \sup_{x,y\neq 0} \left\{ \frac{\|x+y\|^2 - \|x-y\|^2}{\|x\|\|y\|} : x, y \in B_X, x \perp_P y \right\}$$

Proposition 3.6. *If* X *is a Banach space, then* $0 \leq I(X) \leq 2$.

Proof. Since there exist $x, y \in S_X$ such that $\|x+y\| = \|x-y\| = \sqrt{2}$, we can find x and y satisfying $x \perp_P y$ and

$$\|x+y\| = \|x-y\| = \sqrt{2}.$$

Therefore,

$$\frac{\|x+y\|^2 - \|x-y\|^2}{\|x\|\|y\|} = 0.$$

According to the following inequality:

$$\frac{\|x+y\|^2 - \|x-y\|^2}{\|x\|\|y\|} \leq \frac{(\|x\| + \|y\|)^2 - \|x\|^2 - \|y\|^2}{\|x\|\|y\|} = 2,$$

we arrive at the desired result. □

In the next result, we note that the constant $I(X)$ can be reformulated.

Proposition 3.7. *If* X *is a Banach space, then* $I(X) = \sup\{\|x+y\|^2 - \|x-y\|^2 : \|x\|\|y\| = 1, x \perp_P y\}$.

Proof. First, assume that $\|x\| \geq \|y\|$. We note that

$$\begin{aligned}
\frac{\|x+y\|^2 - \|x-y\|^2}{\|x\|\|y\|} &= \frac{\|x+y\|^2}{\|x\|\|y\|} - \frac{\|x-y\|^2}{\|x\|\|y\|} \\
&= \left\| \frac{x+y}{\sqrt{\|x\|\|y\|}} \right\|^2 - \left\| \frac{x-y}{\sqrt{\|x\|\|y\|}} \right\|^2 \\
&= \left\| \frac{x}{\sqrt{\|x\|\|y\|}} + \frac{y}{\sqrt{\|x\|\|y\|}} \right\|^2 - \left\| \frac{x}{\sqrt{\|x\|\|y\|}} - \frac{y}{\sqrt{\|x\|\|y\|}} \right\|^2.
\end{aligned}$$

Here,

$$\left\| \frac{x}{\sqrt{\|x\|\|y\|}} \right\| \geq 1, \left\| \frac{y}{\sqrt{\|x\|\|y\|}} \right\| \leq 1 \quad \text{and} \quad \left\| \frac{x}{\sqrt{\|x\|\|y\|}} \right\| \left\| \frac{y}{\sqrt{\|x\|\|y\|}} \right\| = 1.$$

On the other, $\frac{x}{\sqrt{\|x\|\|y\|}}$ and $\frac{y}{\sqrt{\|x\|\|y\|}}$ also meet the Pythagorean orthogonal conditions, that is

$$\left\|\frac{x-y}{\sqrt{\|x\|\|y\|}}\right\|^2 = \left\|\frac{x}{\sqrt{\|x\|\|y\|}}\right\|^2 + \left\|\frac{y}{\sqrt{\|x\|\|y\|}}\right\|^2,$$

which shows that the supremum in the definition of $I(X)$ can be taken over $x, y \in X$ such that $\|x\|\|y\| = 1$. For $\|x\| \leq \|y\|$, the proof is similar to the one concerning the case when $\|x\| \geq \|y\|$. □

Lemma 3.1. *A normed linear space X is an inner product space if and only if*

$$x, y \in X, \quad x \perp_P y \Rightarrow \|x+y\|^2 + \|x-y\|^2 \sim 2\|x\|^2 + 2\|y\|^2,$$

where "$\sim$" denotes either "$\leq$" or "$\geq$".

We call the relation "$\sim$", used in Lemma 3.1, as "allowing diagonals" if for any $x, y \neq 0$, there exists $\alpha > 0$ such that

$$(x + \alpha y) \sim (x - \alpha y).$$

Lemma 3.2. *A normed linear space X is an inner product space if and only if*

$$x, y \in X, \quad x \approx y \Rightarrow \|x+y\|^2 + \|x-y\|^2 \sim 2\|x\|^2 + 2\|y\|^2,$$

where "$\sim$" denotes either "$\leq$" or "$\geq$".

Proposition 3.8. *For a Banach space X, the equation $I(X) = 0$ holds if and only if X is a Hilbert space.*

Proof. If $I(X) = 0$, then we have

$$\frac{\|x+y\|^2 - \|x-y\|^2}{\|x\|\|y\|} \leq 0,$$

where $x \perp_P y$. That is, for $x \perp_P y$, we have $\|x+y\|^2 \leq \|x-y\|^2 = \|x\|^2 + \|y\|^2$. It follows that

$$\|x+y\|^2 + \|x-y\|^2 \leq 2\|x\|^2 + 2\|y\|^2,$$

where $x \perp_P y$. From Lemma 3.1, we conclude that X is a Hilbert space.

If X is a Hilbert space, then the Pythagorean orthogonality and isosceles orthogonality are equivalent. Since $x \perp_P y$, we have $x \perp_I y$, and hence $\|x+y\|^2 - \|x-y\|^2 = 0$. Therefore, $I(X) = 0$. □

Observation 3.5. *Let $X = (\mathbb{R}^2, \|\cdot\|_\infty)$. Then $I(X) = 2$.*

Proof. Take $x = \left(1, \frac{\sqrt{2}}{2}\right)$ and $y = \left(1, -\frac{\sqrt{2}}{2}\right)$. Then $x, y \in B_X$ and $x \perp_P y$. Moreover, we have $\|x\| = \|y\| = 1$, $\|x-y\| = \sqrt{2}$, and $\|x+y\| = 2$. Thus, we obtain

$$\frac{\|x+y\|^2 - \|x-y\|^2}{\|x\|\|y\|} = \frac{4-2}{1} = 2$$

and hence $I(X) = 2$. □

Theorem 3.9. *Let X be a Banach space. The upper bound 2 of $I(X)$ is attained by a pair of points $x, y \in B_X$ if and only if the pair of points satisfies the equation $\|x+y\| = \|x\| + \|y\|$.*

Proof. Assume that $I(X) = 2$. Then, there exist $x, y \in B_X$ such that

$$\frac{\|x+y\|^2 - \|x-y\|^2}{\|x\|\|y\|} = 2,$$

which yields

$$\begin{aligned}\|x\|^2 + \|y\|^2 + 2\|x\|\|y\| &= \|x+y\|^2 \\ &\leq (\|x\| + \|y\|)^2 \\ &= \|x\|^2 + \|y\|^2 + 2\|x\|\|y\|.\end{aligned}$$

Consequently, we have $\|x+y\| = \|x\| + \|y\|$. Conversely, suppose $\|x+y\| = \|x\| + \|y\|$. Then, we have

$$\begin{aligned}\frac{\|x+y\|^2 - \|x-y\|^2}{\|x\|\|y\|} &= \frac{(\|x\| + \|y\|)^2 - \|x\|^2 - \|y\|^2}{\|x\|\|y\|} \\ &= \frac{2\|x\|\|y\|}{\|x\|\|y\|} \\ &= 2.\end{aligned}$$

Thus, the upper bound 2 of $I(X)$ is attained by x and y. □

As a consequence of Theorem 3.9, we have the following result.

Corollary 3.1. *Let X be a Banach space. If the upper bound 2 of $I(X)$ is attained, then X is not strictly convex.*

Proof. Since the upper bound 2 of $I(X)$ is attained, we have $\|x+y\| = \|x\| + \|y\|$. If $x = \lambda y$, then x and y do not meet the Pythagorean orthogonal condition, and hence $x \neq \lambda y$, which implies that X is not strictly convex. □

In the next proposition, we see that $I(X)$ can be defined in another way.

Proposition 3.9. *For a Banach space X, we have*

$$I(X) = \sup\left\{t\|x+y\|^2 - t\|x-y\|^2 : x \perp_P y,\ \max\{\|x\|, \|y\|\} = 1,\ t = \frac{1}{\min\{\|x\|, \|y\|\}} \in [1, \infty)\right\}.$$

Proof. First, assume that $1 \geq \|x\| \geq \|y\| \neq 0$. Then

$$\|x \pm y\| = \|x\| \left\| \frac{x}{\|x\|} \pm \frac{y}{\|x\|} \right\|.$$

Take $t = \frac{\|x\|}{\|y\|}$ and observe that

$$\begin{aligned}\frac{\|x+y\|^2 - \|x-y\|^2}{\|x\|\|y\|} &= \frac{\|x\|^2 \left\| \frac{x}{\|x\|} + \frac{y}{\|x\|} \right\|^2 - \|x\|^2 \left\| \frac{x}{\|x\|} - \frac{y}{\|x\|} \right\|^2}{\|x\|\|y\|} \\ &= \frac{\|x\|}{\|y\|} \left\| \frac{x}{\|x\|} + \frac{y}{\|x\|} \right\|^2 - \frac{\|x\|}{\|y\|} \left\| \frac{x}{\|x\|} - \frac{y}{\|x\|} \right\|^2,\end{aligned}$$

which shows that the supremum in the definition of $I(X)$ can be taken over $x, y \in X$ such that $\|x\| = 1$ and $\|y\| \leq 1$. For $0 \neq \|x\| \leq \|y\| \leq 1$, the proof is similar to the one concerning the case when $1 \geq \|x\| \geq \|y\| \neq 0$. □

Definition 3.4. [1] *Let X be a Banach space. A function $\delta_X : [0,2] \to [0,1]$ is said to be the modulus of convexity of X if*

$$\delta_X(\varepsilon) = \inf\left\{1 - \left\| \frac{x+y}{2} \right\| : \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \varepsilon\right\}.$$

It is easy to see that $\delta_X(0) = 0$ and $\delta_X(t) \geq 0$ for all $t \geq 0$.

Remark 3.1. *If we restrict x and y to the unit sphere, then we can get a better estimate. Consider the constant*

$$\begin{aligned}I'(X) &= \sup_{x,y\neq 0} \left\{ \frac{\|x+y\|^2 - \|x-y\|^2}{\|x\|\|y\|} : x, y \in S_X, x \perp_P y \right\} \\ &= \sup_{x,y\neq 0} \left\{ \|x+y\|^2 - 2 : x, y \in S_x, x \perp_P y \right\}\end{aligned}$$

and take $K = 4(1-\delta(\sqrt{2}))^2 - 2$. For $x, y \in S_X$, $\|x-y\| = \sqrt{2}$ and thus we have $\delta_X(\sqrt{2}) \leq 1 - \frac{\|x+y\|}{2}$. Also,

$$\begin{aligned}\frac{\|x+y\|^2 - \|x-y\|^2}{\|x\|\|y\|} &\leq \frac{4\left(1 - \delta_X(\sqrt{2})\right)^2 - 2}{1} \\ &= 4\left(1 - \delta_X(\sqrt{2})\right)^2 - 2 \\ &= K,\end{aligned}$$

from which it follows that $I'(X) \leq K$.

On the other hand for any $\mu \geq 0$ there exist $x, y \in S_X$ such that $\|x-y\| = \sqrt{2}$ and $1 - \frac{\|x+y\|}{2} \leq \delta_X(\sqrt{2}) + \mu$. Hence

$$\begin{aligned}I'(X) &\geq \frac{\|x+y\|^2 - \|x-y\|^2}{\|x\|\|y\|} \\ &\geq \frac{\left(2 - 2\delta_X(\sqrt{2}) - 2\mu\right)^2 - 2}{1} \\ &= \left(2 - 2\delta_X(\sqrt{2}) - 2\mu\right)^2 - 2.\end{aligned}$$

Since μ is arbitrary, we obtain $I'(X) \geq 4\left(1 - \delta_X(\sqrt{2})\right)^2 - 2 = K$, which shows that $I'(X) = 4\left(1 - \delta_X(\sqrt{2})\right)^2 - 2 = K$.

The next theorem gives a characterization of the case when $I(X) = 2$ is attained at points of S_X.

Theorem 3.10. *Let X be a normed linear space and consider $x, y \in S_X$. The following properties are equivalent:*

(i). $x \perp_P y$ *and* $\frac{\|x+y\|^2-\|x-y\|^2}{\|x\|\|y\|} = 2$.

(ii). *The segments $[x, y]$ is contained in S_X and the point $\frac{x-y}{\sqrt{2}}$ is contained in S_X.*

Proof. **(i) ⇒ (ii).** Take $x, y \in S_X$ such that $x \perp_P y$ and

$$\frac{\|x+y\|^2 - \|x-y\|^2}{\|x\|\|y\|} = 2.$$

Since $\|x+y\| \le 2$ and $\|x-y\|^2 = 2$, we have that $\|x+y\| = 2$ and $\|x-y\| = \sqrt{2}$. Therefore $x, y, \frac{1}{2}(x+y) \in S_X$, which implies that $[x, y] \subset S_X$ and the point $\frac{x-y}{\sqrt{2}}$ is contained in S_X.

(ii) ⇒ (i). Take $x, y \in S_X$ such that $[x, y] \subset S_X$ and the point $\frac{x-y}{\sqrt{2}}$ is contained in S_X. It is clear that $\|x+y\| = 2$ and $\|x-y\| = \sqrt{2}$. Hence, $x \perp_P y$ and

$$\frac{\|x+y\|^2 - \|x-y\|^2}{\|x\|\|y\|} = 2.$$

□

Remark 3.2. *In [18], the constant $\mathcal{R}(X)$ defined below was considered*

$$\mathcal{R}(X) := \sup\left\{\|x-y\| : \operatorname{conv}\{x, y\} \subset S_X\right\}.$$

It is easily to see that if $\mathcal{R}(X) \ge \sqrt{2}$, then $I(X) = 2$.

In fact, for any point $\frac{x}{\|x\|} \in S_x$, there always exists a point w in S_X such that $\frac{x}{\|x\|} \perp_P w$. Fix a nonzero $x \in X$ and $t > 0$, and define a function $f : S_t \to \mathbb{R}$, where $S_t := \{z \in X : \|z\| = t\}$, by the formula

$$f(y) := \left\| \frac{x}{\|x\|} - \frac{y}{t} \right\| - \sqrt{2} \text{ for all } y \in S_t.$$

Then, f is continuous and

$$f\left(\frac{xt}{\|x\|}\right) = -\sqrt{2} < 0 \quad \text{and} \quad f\left(-\frac{xt}{\|x\|}\right) = 2 - \sqrt{2} > 0.$$

So, there exists a $y_0 \in S_t$ such that $f(y_0) = 0$, i.e., $\frac{x}{\|x\|} \perp_P \frac{y_0}{\|y_0\|}$. The claim is proved by taking $w = \frac{y_0}{\|y_0\|}$.

Finally, we establish a relation between $I(X)$ and the modulus of convexity $\delta_X(\varepsilon)$, where

$$\delta_X(\varepsilon) := \inf\left\{1 - \frac{\|x+y\|}{2} : x, y \in B_X, \|x-y\| \ge \varepsilon\right\} \quad \text{for each } \varepsilon \in [0, 2].$$

Theorem 3.11. *If X is a Banach space with $\delta_X(\sqrt{2}) < \frac{2-\sqrt{2}}{2}$, then*

$$4\left(1 - \delta_X(\sqrt{2})\right)^2 - 2 \le I(X) \le 8\left(1 - \delta_X(\sqrt{2})\right)^2 - 4.$$

Proof. For $x, y \in B_X$, we have

$$(\|x\| - \|y\|)^2 = \|x\|^2 + \|y\|^2 - 2\|x\|\|y\| \le 1,$$

which gives

$$2\|x\|\|y\| \ge \|x\|^2 + \|y\|^2 - 1.$$

Since $\|x-y\| \ge \sqrt{2}$, we have $\|x-y\|^2 = \|x\|^2 + \|y\|^2 \ge 2$, and thus $\|x\|\|y\| \ge \frac{1}{2}$. Hence,

$$\begin{aligned}
\frac{\|x+y\|^2 - \|x-y\|^2}{\|x\|\|y\|} &\le \frac{4\left(1 - \delta_X(\sqrt{2})\right)^2 - 2}{\|x\|\|y\|} \\
&\le \frac{4\left(1 - \delta_X(\sqrt{2})\right)^2 - 2}{\frac{1}{2}} \\
&= 8\left(1 - \delta_X(\sqrt{2})\right)^2 - 4.
\end{aligned}$$

On the other hand, for any $\mu \ge 0$ there exist $x, y \in B_X$ such that $\|x-y\| = \sqrt{2}$ and

$$1 - \frac{\|x+y\|}{2} \le \delta_X(\sqrt{2}) + \mu.$$

Hence,

$$I(X) \geq \frac{\|x+y\|^2 - \|x-y\|^2}{\|x\|\|y\|}$$
$$\geq \frac{\left(2 - 2\delta_X(\sqrt{2}) - 2\mu\right)^2 - 2}{1}.$$

Since μ is arbitrary, we obtain $I(X) \geq 4\left(1 - \delta_X(\sqrt{2})\right)^2 - 2$. □

The constant $I^{(p)}(X)$

Motivated by the Pythagorean orthogonality, we define the generalized p-Pythagorean orthogonality as follows:

Definition 3.5. *In a normed linear space X, a vector x is said to be p-Pythagorean orthogonal to a vector y if*

$$\|x-y\|^p = \|x\|^p + \|y\|^p.$$

We write $x \perp_P^p y$ to indicate that x is p-Pythagorean orthogonal to y.

Definition 3.6. *For $1 < p < \infty$, define*

$$I^{(p)}(X) = \sup_{x,y \neq 0} \left\{ \frac{\|x+y\|^p - \|x-y\|^p}{2^{p-1} - 1} : x, y \in S_X, x \perp_P^p y \right\}.$$

Lemma 3.3. *Let $\|\cdot\|$ be a norm. Then $\|a+b\|^p \leq 2^{p-1}\left(\|a\|^p + \|b\|^p\right)$ for $a, b \in \mathbb{R}$ and $p > 1$.*

Proof. For $f(x) = x^p$, we have

$$f\left(\frac{\|a\| + \|b\|}{2}\right) \leqslant \frac{f(\|a\|) + f(\|b\|)}{2},$$

which gives

$$\left(\frac{\|a\| + \|b\|}{2}\right)^p \leqslant \frac{\|a\|^p + \|b\|^p}{2}.$$

Thus,

$$\left\|\frac{a+b}{2}\right\|^p \leq \left(\frac{\|a\| + \|b\|}{2}\right)^p \leq \frac{\|a\|^p + \|b\|^p}{2},$$

which implies that $\|a+b\|^p \leq 2^{p-1}\left(\|a\|^p + \|b\|^p\right)$. □

Proposition 3.10. *If* X *is a Banach space, then $0 \leq I^{(p)}(X) \leq 2$.*

Proof. It is clear that $I^{(p)}(X) \geq 0$. Since

$$\|x+y\|^p - \|x-y\|^p \leq 2^{p-1}\left(\|x\|^p + \|y\|^p\right) - \left(\|x\|^p + \|y\|^p\right) \leq 2\left(2^{p-1} - 1\right),$$

we have

$$I^{(p)}(X) \leq \frac{2\left(2^{p-1} - 1\right)}{2^{p-1} - 1} = 2.$$

□

For some special spaces, we observe that $I^{(p)}(X)$ is small.

Observation 3.6. *Let X be the classical Lebesgue space L_p with $p \geq 2$. Then*

$$I^{(p)}(X) \leq \frac{2^p - 4}{2^{p-1} - 1}.$$

Proof. By using Clarkson's inequalities, for $\|x\|_p \leq 1$ and $\|y\|_p \leq 1$, we have

$$\|x+y\|_p^p + \|x-y\|_p^p \leq 2^p.$$

Thus,

$$\frac{\|x+y\|^p - \|x-y\|^p}{2^{p-1} - 1} = \frac{\|x+y\|^p + \|x-y\|^p - 2\|x-y\|^p}{2^{p-1} - 1}$$
$$\leq \frac{2^p - 4}{2^{p-1} - 1}.$$

Certainly,

$$\frac{2^p - 4}{2^{p-1} - 1} \leq 2 \quad \text{for } p \geq 2.$$

□

Theorem 3.12. *Let X be a Banach space. If the upper bound 2 of $I^{(p)}(X)$ is attained by a pair of points of S_X, then X is not a strictly convex space.*

Proof. Since $I^{(p)}(X) = 2$, there exist $x, y \in S_X$ such that

$$\frac{\|x+y\|^p - \|x-y\|^p}{2^{p-1}-1} = 2.$$

Thus, $\|x+y\|^p = 2^p$, which implies that $\|x+y\| = 2$. Therefore, X is not a strictly convex space. □

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