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Some Convergent Summation Theorems For Appell's Function F_1 Having Arguments $-1, \frac{1}{2}$

In this paper, we obtain some closed forms of hypergeometric summation theorems for Appell's function of first kind F_1 having the arguments $-1, \frac{1}{2}$ with suitable convergence conditions, by adjustment of parameters and arguments in generalized form of first, second and third summation theorems of Kümmer and others.

Keywords: generalized hypergeometric function, Appell's function of first kind, Kümmer's first, second and third summation theorems.

Introduction

A great interest in the theory of hypergeometric functions (that is, hypergeometric functions of several variables) is motivated essentially by the fact that the solutions to many applied problems involving (for example) partial differential equations are obtainable with the help of such hypergeometric functions (see, for details, [1; 47]; [2] and the references cited therein). For instance, the energy absorbed by some non-ferromagnetic conductor sphere included in an internal magnetic field can be calculated through such functions [3, 4]. Hypergeometric functions of several variables are used in physical and quantum chemical applications as well [5–7].

The extensive development of the theories of hypergeometric functions of a single variable has led to a full-scale investigation of corresponding theories in two or more variables. In 1880, Appell [8–10] considered the product of two Gauss's hypergeometric functions ${}_2F_1$ to obtain four Appell's functions F_1, F_2, F_3, and F_4 in two variables. Later in 1893, Lauricella [11] further generalized the four Appell functions F_i $(i = 1, 2, 3, 4)$ to give the functions $F_A^{(n)}$, $F_B^{(n)}$, $F_C^{(n)}$, and $F_D^{(n)}$ in n-variables. It is noted that $F_A^{(1)} = F_B^{(1)} = F_C^{(1)} = F_D^{(1)} = {}_2F_1$, $F_A^{(2)} = F_2$, $F_B^{(2)} = F_3$, $F_C^{(2)} = F_4$ and $F_D^{(2)} = F_1$.

Over eight decades ago Chaundy [12], Burchnall-Chaundy [13], and recently several others [14–24], systematically, presented a number of expansion and decomposition formulas for some double hypergeometric functions, for example, the Appell's functions F_i, in series of simpler hypergeometric functions. Recently, Khan & Abukhammash [25] introduced and investigated 10 Appell type generalized functions M_i $(i = 1, ..., 10)$ by considering the product of two ${}_3F_2$ functions. Here, motivated by the above-mentioned works, Choi et al. [16] aim to introduce 18 Appell type generalized functions κ_i $(i = 1, ..., 18)$ by considering the product of two ${}_4F_3$ functions.

In the usual notation, let $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers, respectively. Also let

$$\mathbb{N}_0 := \mathbb{N} \cup \{0\} \quad , \qquad \mathbb{N} := \{1, 2, 3, \dots\} = \mathbb{N}_0 \backslash \{0\} \quad ,$$

$$\mathbb{Z}_0^- := \{0, -1, -2, \dots\} = \mathbb{Z}^- \cup \{0\} \quad , \qquad \mathbb{Z}^- := \{-1, -2, -3, \dots\}$$

and $\mathbb{Z} = \mathbb{Z}_0^- \cup \mathbb{N}$ being the sets of integers.

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For definitions of Pochhammer symbol, generalized hypergeometric function ${}_pF_q$ with convergence conditions and other useful results, we refer the monumental work of Abramowitz & Stegun [26], Andrews et al. [27], Erdélyi et al. [28], Prudnikov et al. [29], Rainville [30], and Srivastava & Manocha [31]. *Appell's Function of First Kind* is defined as :

$$F_1[A\ ;\ B,\ C\ ;\ D\ ;\ x,\ y] = \sum_{r,s=0}^{\infty} \frac{(A)_{r+s}(B)_r(C)_s}{(D)_{r+s}} \frac{x^r}{r!} \frac{y^s}{s!}.$$

Convergence conditions of Appell's double series F_1

(a) Appell's series F_1 is convergent when $|x| < 1,\ |y| < 1\ ;\ \ A, B, C, D \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

(b) Appell's series F_1 is absolutely convergent when $|x| = 1,\ |y| = 1\ ;\ \ A, B, C, D \in \mathbb{C}\backslash\mathbb{Z}_0^-$; $\Re(A+B-D) < 0$, $\Re(A+C-D) < 0$ and $\Re(A+B+C-D) < 0$.

(c) Appell's series F_1 is conditionally convergent when $|x| = 1,\ |y| = 1\ ;\ x \neq 1, y \neq 1$; $A, B, C, D \in \mathbb{C}\backslash\mathbb{Z}_0^-$; $\Re(A+B-D) < 1$, $\Re(A+C-D) < 1$ and $\Re(A+B+C-D) < 2$.

(d) Appell's series F_1 is a polynomial if A is a negative integer; $B, C, D \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

(e) Appell's series F_1 is a polynomial if B and C are negative integers; $A, D \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

For absolutely and conditionally convergence (b,c) of Appell's function F_1, interested readers may consult the paper of Hài et al. [32] related to the convergence of multiple hypergeometric functions of Kampé de Fériet.

A result of Appell and Kampé de Fériet[8], see also [31; 55, Equation 1.6(15)]:

$$F_1[a\ ;\ b,\ c\ ;\ d\ ;\ 1, 1] = \frac{\Gamma(d)\Gamma(d-a-b-c)}{\Gamma(d-a)\Gamma(d-b-c)}\ , \tag{1}$$

$$\left(\Re(d-a-b-c) > 0\ ;\ d \in \mathbb{C}\backslash\mathbb{Z}_0^-\right)\ .$$

Motivated by the work in equation (1) of Appell and Kampé de Fériet , we obtain some summation theorems for Appell's function of first kind F_1 having equal argument other than unity, in section 1, by suitable adjustment of numerator and denominator parameters.

When the values of parameters leading to the results which do not make sense are tacitly excluded, then using series iteration technique, the Appell's function F_1 with equal argument can also be written as [8; 23, Equation (25)]

$$F_1[A; B, C; D; x, x] = {}_2F_1\left[\begin{array}{r} A, B+C; \\ D; \end{array} x\right], \qquad \left(|x| < 1\ ;\ A, B, C, D \in \mathbb{C}\backslash\mathbb{Z}_0^-\right)\ . \tag{2}$$

1 Some new Summations using the function $F_1[A; B, C; D; x, x]$

Further by putting $x = 1$ in equation (2) and applying Gauss classical summation theorem [31; 30, Equation 1.2(7)], we get a known result (1) of Appell and Kampé de Fériet.

In equation (2), by putting $A = a, B = b,\ C = c,\ D = 1 + a - b - c - m$ and $x = -1$, using a summation theorem [33; 1524, Equation (2.3)], we get

$$F_1[a; b, c; 1+a-b-c-m; -1, -1] = \frac{\Gamma(1+a-b-c-m)}{2\Gamma(a)} \sum_{r=0}^{m} \left\{ \binom{m}{r} \frac{\Gamma(\frac{r+a}{2})}{\Gamma(\frac{r+a+2-2b-2c-2m}{2})} \right\}\ ,$$

$$(\Re(b+c)<\frac{2-m}{2}\ ;\Re(2b+c)<2-m,\Re(2c+b)<2-m,a,b,c,b+c,$$

$$1+a-b-c-m\in\mathbb{C}\backslash\mathbb{Z}_0^-\ ;\ m\in\mathbb{N}_0).$$

In equation (2), by putting $A=a,B=b$, $C=c$, $D=1+a-b-c+m$ and $x=-1$, using another summation theorem [33; 1523, Equation (2.2)], we obtain

$$F_1[a;b,c;1+a-b-c+m;-1,-1]=\frac{\Gamma(1+a-b-c+m)}{2\Gamma(a)(1-b-c)_m}\sum_{r=0}^{m}\left\{\binom{m}{r}\frac{(-1)^r\Gamma(\frac{r+a}{2})}{\Gamma(\frac{r+a+2-2b-2c}{2})}\right\}\ ,$$

$$(\Re(b+c)<\frac{2+m}{2},\Re(2b+c)<2+m\ ,\Re(2c+b)<2+m;\ a,b,c,b+c,\ 1+a-b-c+m,$$

$$1-b-c\in\mathbb{C}\backslash\mathbb{Z}_0^-\ ;\ m\in\mathbb{N}_0).$$

In equation (2), by putting $A=a,B=b$, $C=c$, $D=a-b-c-m$ and $x=-1$, using the summation theorem [34; 14, Equation (3.1)], we find

$$F_1[a\ ;\ b,\ c\ ;\ a-b-c-m\ ;-1,-1]$$

$$=\frac{\Gamma(a-b-c-m)}{2\Gamma(a)}\sum_{r=0}^{m}\left\{\binom{m}{r}\left[\frac{\Gamma(\frac{r+a}{2})}{\Gamma(\frac{r+a-2b-2c-2m}{2})}+\frac{\Gamma(\frac{r+a+1}{2})}{\Gamma(\frac{r+a-2b-2c-2m+1}{2})}\right]\right\}\ ,$$

$$(\Re(b+c)<\frac{1-m}{2},\Re(2b+c)<1-m,\Re(2c+b)<1-m;\ a,b,c,b+c,$$

$$a-b-c-m\in\mathbb{C}\backslash\mathbb{Z}_0^-\ ;\ m\in\mathbb{N}_0)\ .$$

In equation (2), by putting $A=a,B=b$, $C=c$, $D=a-b-c+m$ and $x=-1$, using another summation theorem [34; 14, Equation (3.2)], we have

$$F_1[a\ ;\ b,\ c\ ;\ a-b-c+m\ ;-1,-1]$$

$$=\frac{\Gamma(a-b-c+m)}{2\Gamma(a)(-b-c)_m}\sum_{r=0}^{m}\left\{\binom{m}{r}\left[\frac{(-1)^r\Gamma(\frac{r+a}{2})}{\Gamma(\frac{r+a-2b-2c}{2})}+\frac{(-1)^r\Gamma(\frac{r+a+1}{2})}{\Gamma(\frac{r+a-2b-2c+1}{2})}\right]\right\}\ ,$$

$$(\Re(b+c)<\frac{1+m}{2},\Re(2b+c)<1+m,\Re(2c+b)<1+m\ ;\ a,b,c,b+c,$$

$$-b-c,\ a-b-c+m\in\mathbb{C}\backslash\mathbb{Z}_0^-\ ;\ m\in\mathbb{N}_0)\ .$$

In equation (2), by putting $A=n$, $B=C=\frac{a}{2}$, $D=-a-m$ and $x=-1$, using the summation theorem [34; 14, Equation (3.3)], we get

$$F_1\left[n;\frac{a}{2},\frac{a}{2};-a-m;-1,-1\right]=\frac{\Gamma(-m-a)}{2\Gamma(n)}\sum_{r=0}^{m+n+1}\left\{\frac{(-1)^r(-m-n-1)_r\Gamma(\frac{r+n}{2})}{r!\Gamma(\frac{r-n-2a-2m}{2})}\right\}\ ,$$

$$\left(\Re(a)<\tfrac{2}{3}(1-m-n);\quad n,\ a,\ -m-a\in\mathbb{C}\backslash\mathbb{Z}_0^-;\quad m+n\in\mathbb{N}_0\cup\{-1\}\right)\ .$$

In equation (2), $A = n$, $B = C = \frac{a}{2}$, $D = -a + m$ and $x = -1$, using another summation theorem [34; 14, Equation (3.4)], we have

$$F_1\left[n; \frac{a}{2}, \frac{a}{2}; -a+m; -1, -1\right] = \frac{\Gamma(1-a)\Gamma(m-a)}{2\Gamma(n)\Gamma(m-a-n)} \sum_{r=0}^{m-n-1} \left\{ \frac{(1+n-m)_r \Gamma(\frac{r+n}{2})}{r!\Gamma(\frac{n+r+2-2a}{2})} \right\} ,$$

$$\left(\Re(a) < \left(\tfrac{1+m-n}{2}\right);\ n, a,\ m-a-n,\ m-a \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ \ m-n \in \mathbb{N}\right) .$$

In equation (2), by putting $A = a, B = b$, $C = c$, $D = \frac{1+a+b+c-m}{2}$ and $x = \frac{1}{2}$, using the summation theorem [29; 491, Entry (7.3.7.2)], we obtain

$$F_1\left[a; b, c; \frac{1+a+b+c-m}{2}; \frac{1}{2}, \frac{1}{2}\right] = \frac{2^{a-1}\Gamma(\frac{1+a+b+c-m}{2})}{\Gamma(a)} \sum_{r=0}^{m} \left\{ \binom{m}{r} \frac{\Gamma(\frac{r+a}{2})}{\Gamma(\frac{1+b+c+r-m}{2})} \right\} ,$$

$$\left(a, b, c,\ \tfrac{1+a+b+c-m}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-\ ;\ m \in \mathbb{N}_0\right) .$$

In equation (2), by putting $A = a, B = b$, $C = c$, $D = \frac{1+a+b+c+m}{2}$ and $x = \frac{1}{2}$, using the summation theorem [35; 827, Theorems (1)], we find

$$F_1\left[a; b, c; \frac{1+a+b+c+m}{2}; \frac{1}{2}, \frac{1}{2}\right] = \frac{2^{a-1}\Gamma(\frac{1+a+b+c+m}{2})}{\Gamma(a)(\frac{1-a+b+c-m}{2})_m} \sum_{r=0}^{m} \left\{ \binom{m}{r} \frac{(-1)^r\Gamma(\frac{r+a}{2})}{\Gamma(\frac{1+b+c+r-m}{2})} \right\} ,$$

$$\left(a, b, c,\ \tfrac{1+a+b+c+m}{2},\ \tfrac{1-a+b+c-m}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-\ ;\ m \in \mathbb{N}_0\right) .$$

In the equation (2), by putting $A = a, B = b, C = c, D = \frac{a+b+c-m}{2}$ and $x = \frac{1}{2}$, using the summation theorem [36; 48, Equation (3.1)], we have

$$F_1\left[a; b, c; \frac{a+b+c-m}{2}; \frac{1}{2}, \frac{1}{2}\right]$$

$$= \frac{2^{a-1}\Gamma(\frac{a+b+c-m}{2})}{\Gamma(a)} \sum_{r=0}^{m} \left\{ \binom{m}{r} \left[\frac{\Gamma(\frac{r+a}{2})}{\Gamma(\frac{b+c+r-m}{2})} + \frac{\Gamma(\frac{r+a+1}{2})}{\Gamma(\frac{b+c+r-m+1}{2})} \right] \right\},$$

$$\left(a, b, c, \tfrac{a+b+c-m}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-;\quad m \in \mathbb{N}_0\right) .$$

In the equation (2), by putting $A = a, B = b$, $C = c$, $D = \frac{a+b+c+m}{2}$ and $x = \frac{1}{2}$, using another summation theorem [36; 48, Equation (3.3)], we get

$$F_1\left[a; b, c; \frac{a+b+c+m}{2}; \frac{1}{2}, \frac{1}{2}\right]$$

$$= \frac{2^{a-1}\Gamma(\frac{a+b+c+m}{2})}{\Gamma(a)(\frac{b+c-a-m}{2})_m} \sum_{r=0}^{m} \left\{ \binom{m}{r} \left[\frac{(-1)^r\Gamma(\frac{r+a}{2})}{\Gamma(\frac{b+c+r-m}{2})} + \frac{(-1)^r\Gamma(\frac{r+a+1}{2})}{\Gamma(\frac{b+c+r-m+1}{2})} \right] \right\},$$

$$\left(a, b, c, \tfrac{a+b+c+m}{2}, \tfrac{b+c-a-m}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-;\quad m \in \mathbb{N}_0\right).$$

In equation (2), by putting $A=a, B=b, C=1-a-b-m, D=d$ and $x=\frac{1}{2}$, using the summation theorem [35; 828, Theorem (6)], we find

$$F_1\left[a;b,1-a-b-m;d;\frac{1}{2},\frac{1}{2}\right]=\frac{\Gamma(d)}{2^{a+m}\Gamma(d-a)}\sum_{r=0}^{m}\left\{\binom{m}{r}\frac{\Gamma(\frac{d-a+r}{2})}{\Gamma(\frac{d+a+r}{2})}\right\},$$

$$\left(a,b,1-a-b-m,d,\ d-a\in\mathbb{C}\backslash\mathbb{Z}_0^-\ ;\ m\in\mathbb{N}_0\right).$$

In equation (2), by putting $A=a, B=b, C=1-a-b+m, D=d$ and $x=\frac{1}{2}$, using another summation theorem [35; 828, Theorem (5)], we have

$$F_1\left[a;b,1-a-b+m;d;\frac{1}{2},\frac{1}{2}\right]=\frac{\Gamma(d)\Gamma(a-m)}{2^{a-m}\Gamma(a)\Gamma(d-a)}\sum_{r=0}^{m}\left\{\binom{m}{r}\frac{(-1)^r\Gamma(\frac{d-a+r}{2})}{\Gamma(\frac{d+a+r-2m}{2})}\right\},$$

$$\left(a,b,1-a-b+m,a-m,\ d-a,\ d\in\mathbb{C}\backslash\mathbb{Z}_0^-\ ;\ m\in\mathbb{N}_0\right).$$

In equation (2), by putting $A=a, B=b, C=-a-b-m,\ D=d$ and $x=\frac{1}{2}$, using the summation theorem [37; 144, Equation (3.3)], we get

$$F_1[a;b,-a-b-m;d;\frac{1}{2},\frac{1}{2}]=\frac{\Gamma(d)2^{-a-m-1}}{\Gamma(d-a)}\sum_{r=0}^{m}\left\{\binom{m}{r}\left[\frac{\Gamma(\frac{d-a+r}{2})}{\Gamma(\frac{d+a+r}{2})}+\frac{\Gamma(\frac{d-a+r+1}{2})}{\Gamma(\frac{d+a+r+1}{2})}\right]\right\},$$

$$\left(a,b,-a-b-m,d,\ d-a\in\mathbb{C}\backslash\mathbb{Z}_0^-;\quad m\in\mathbb{N}_0\right).$$

In equation (2), by putting $A=a, B=b, C=-a-b+m, D=d$ and $x=\frac{1}{2}$, using another summation theorem [37; 144, Equation (3.5)], we obtain

$$F_1\left[a;b,-a-b+m;d;\frac{1}{2},\frac{1}{2}\right]=\frac{2^{-a+m-1}\Gamma(d)\Gamma(a-m)}{\Gamma(a)\Gamma(d-a)}\times$$

$$\times\sum_{r=0}^{m}\left\{\binom{m}{r}(-1)^r\left[\frac{\Gamma(\frac{d-a+r}{2})}{\Gamma(\frac{d+a+r-2m}{2})}+\frac{\Gamma(\frac{d-a+r+1}{2})}{\Gamma(\frac{d+a+r+1-2m}{2})}\right]\right\},$$

$$\left(a,b,-a-b+m,\ d,\ a-m,\ d-a\in\mathbb{C}\backslash\mathbb{Z}_0^-;\quad m\in\mathbb{N}_0\right).$$

Remark

By the theory of analytic continuation some convergence conditions associated with each result can be relaxed.

Conclusion

We conclude our present analysis by observing that several interesting summation theorems for Appell function of first kind can be derived in an analogous manner. Moreover, presented summation theorems should be beneficial to those who are interested in the field of applied mathematics and applied physics.

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$-1, \frac{1}{2}$ аргументтері бар F_1 Апелль функциясына арналған кейбір конвергентті қосындылар теоремалары

Жұмыста параметрлер мен аргументтерді Куммердің бірінші, екінші және үшінші жиынтық теоремаларының жалпыланған түрінде сәйкес келтіру арқылы $-1, \frac{1}{2}$ аргументтері бар F_1 бірінші текті Аппель функциясы үшін гипергеометриялық қосындылар теоремаларының кейбір жабық формалары алынған.

Кілт сөздер: жалпыланған гипергеометриялық функция, бірінші текті Апелл функциясы, Куммердің бірінші, екінші және үшінші жиынтық теоремалары.

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Некоторые теоремы о сходящемся суммировании для функции Аппеля F_1 с аргументами $-1, \frac{1}{2}$

В статье мы получаем некоторые замкнутые формы гипергеометрических теорем суммирования для функции Аппеля первого рода F_1 с аргументами $-1, \frac{1}{2}$ с подходящими условиями сходимости путем подгонки параметров и аргументов в обобщенной форме первой, второй и третьей суммирующих теорем Кюммера и других.

Ключевые слова: обобщенная гипергеометрическая функция, функция Аппеля первого рода, первая, вторая и третья теоремы суммирования Кюммера.