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FINITE SUM REPRESENTATIONS OF ELEMENTS IN $\mathbb{R}$ AND $\mathbb{R}^2$

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Abstract. In February 2017, a number theoretic problem was posed in Mathematics Magazine by Souvik Dey, a master's student in India. The problem asked whether it was possible to represent a real number by a finite sum of elements in an open subset of the real numbers that contained one positive and one negative number. This paper not only provides a solution to the original problem, but proves an analogous statement for elements of $\mathbb{R}^2$.

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1. Introduction

The work in this paper was inspired by a problem proposed by Souvik Dey in the February 2017 Mathematics Magazine [1], namely:

> Let S be an open subset of the set $\mathbb{R}$, such that S contains at least one positive number and one negative number. Then every real number can be written as a finite sum of (not necessarily distinct) elements of S.

In this paper we will also consider an extension of this problem to $\mathbb{R}^2$. When considering the problem in $\mathbb{R}$, we can think of the number line being broken into three distinct parts: $\{a : a < 0\} \cup \{0\} \cup \{a : a > 0\}$. Thinking of the problem posed by Souvik Dey, we can now think of it as saying that we must have an element from each of the sets other than $\{0\}$. Upon considering $\mathbb{R}^2$, we can then think of the coordinate plane as being broken into five distinct parts: $\{(a, b) : a, b > 0\} \cup \{(a, b) : a < 0, b > 0\} \cup \{(a, b) : a = 0 \text{ or } b = 0\} \cup \{(a, b) : a, b < 0\} \cup \{(a, b) : a > 0, b < 0\}$. The four distinct sets, excluding $\{(a, b) : a = 0 \text{ or } b = 0\}$ correspond to the four quadrants in $\mathbb{R}^2$. Throughout this paper, we will use the notation:

$$Q1 = \{(a, b) : a, b \in \mathbb{R} \text{ and } a, b > 0\} \qquad Q3 = \{(a, b) : a, b \in \mathbb{R} \text{ and } a, b < 0\}$$
$$Q2 = \{(a, b) : a, b \in \mathbb{R} \text{ and } a < 0, b > 0\} \qquad Q4 = \{(a, b) : a, b \in \mathbb{R} \text{ and } a > 0, b < 0\}$$

Using this notation, we now propose an analogous statement for $\mathbb{R}^2$:

> Let S be an open subset of the set $\mathbb{R}^2$, such that S contains at least one point in each of $Q1$, $Q2$, $Q3$, and $Q4$. Then every element of $\mathbb{R}^2$ can be written as a finite sum of (not necessarily distinct) elements of S.

In the next section of the paper, we will consider some of the background material that will be necessary for the proof of this main result, seen later as Theorem 3.1.

2. Background Material

To better understand the problem presented by Souvik Dey, as well as the extension to $\mathbb{R}^2$, we will need to understand what an open set looks like in $\mathbb{R}^n$. Specifically, we will need to consider what an open set looks like in $\mathbb{R}^2$ using the standard Euclidean distance. Let us first consider what an open set looks like in $\mathbb{R}^n$, as in the book of Walter Rudin (Definition 2.8 in [2]).

Definition 2.1.

(1) Given $\epsilon \in \mathbb{R}^+$ and $(a, b) \in \mathbb{R}^2$, the set
$$N_{(a,b),\epsilon} = \left\{(x, y) : \sqrt{(x-a)^2 + (y-b)^2} < \epsilon\right\}$$
is called an *open neighborhood* around (a, b).

(2) A set $E \subset \mathbb{R}^n$ is called an *open subset* of $\mathbb{R}^n$ provided that for each $\mathbf{x} \in E$, there exists $\epsilon \in \mathbb{R}^+$ such that $N_{\mathbf{x},\epsilon} \subset E$.

If we consider what this definition means in $\mathbb{R}^2$, this implies that every open set has the form $\cup_i N_{(a_i,b_i),\epsilon_i}$, where each of the $N_{(a_i,b_i),\epsilon_i}$ are open disks. Hence, every open set in $\mathbb{R}^2$ can be thought of as an arbitrary union of open disks.

Another property that will be integral to the paper is the *density* of the rationals in the real numbers. This can seen in the following theorem (Theorem 1.20 in [2]).

Theorem 2.1. *If $x, y \in \mathbb{R}$ and $x < y$, then there exists $q \in \mathbb{Q}$ such that $x < q < y$.*

Essentially, the fact that the rationals are dense in the real numbers implies that between any two distinct real numbers, you can find a rational number. Moreover, this can be extended to the following theorem about the density of the $\mathbb{Q}^2$ in $\mathbb{R}^2$.

Corollary 2.1. *Fix $\epsilon \in \mathbb{R}^+$, and let $(a, b), (c, d) \in \mathbb{R}^2$ such that $a \neq c$ and $b \neq d$. Then there exists $(p, q) \in \mathbb{Q}^2$ such that $(p, q) \in \{(x, y) : a < x < c$ and $b < y < d\}$.*

Proof. The result follows immediately by projecting the points (a, b) and (c, d) first onto the x-axis and secondly onto the y-axis, and then applying Theorem 2.1. ∎

This result says that for any rectangular region in $\mathbb{R}^2$, we are guaranteed to be able to find a point in the interior of this region whose x- and y-coordinates are both rational.

3. The Main Result

We begin by considering an open set, S, in $\mathbb{R}^2$ that contains an element of each of the four quadrants. By considering Definition 2.1, it immediately follows that any open set $S \subset \mathbb{R}^2$ that contains a point from each of the four quadrants must also contain a set of the form:
$$N_{(a_1,b_1),\epsilon_1} \cup N_{(a_2,b_2),\epsilon_2} \cup N_{(a_3,b_3),\epsilon_3} \cup N_{(a_4,b_4),\epsilon_4}$$
where $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 \in \mathbb{R}^+$ and $(a_1, b_1) \in Q1$, $(a_2, b_2) \in Q2$, $(a_3, b_3) \in Q3$, and $(a_4, b_4) \in Q4$. Furthermore, by setting $\epsilon = \min\{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\}$, we get that
$$N_{(a_1,b_1),\epsilon} \cup N_{(a_2,b_2),\epsilon} \cup N_{(a_3,b_3),\epsilon} \cup N_{(a_4,b_4),\epsilon} \subset S.$$

In order to prove the main result, we will first show that $(0, 0)$ can be written as a linear combination of elements from the open set S.

Lemma 3.1. *Let S be an open set in $\mathbb{R}^2$ containing the points $(a_1, b_1) \in Q1$, $(a_2, b_2) \in Q2$, $(a_3, b_3) \in Q3$, and $(a_4, b_4) \in Q4$. Then there exist $n, m, s, t \in \mathbb{N}$ such that*

$$n(p_1, q_1) + m(p_2, q_2) + s(p_3, q_3) + t(p_4, q_4) = (0, 0)$$

where $(p_i, q_i) \in N_{(a_i,b_i),\epsilon}$ with $(p_i, q_i) \in \mathbb{Q}^2$ for $i \in \{1, 2, 3, 4\}$ and some $\epsilon \in \mathbb{R}^+$.

Proof. Let S be an open set in $\mathbb{R}^2$ containing the points $(a_1, b_1) \in Q1$, $(a_2, b_2) \in Q2$, $(a_3, b_3) \in Q3$, and $(a_4, b_4) \in Q4$. Then, by Definition 2.1, there exists $\epsilon \in \mathbb{R}^+$ such that

$$N_{(a_1,b_1),\epsilon} \cup N_{(a_2,b_2),\epsilon} \cup N_{(a_3,b_3),\epsilon} \cup N_{(a_4,b_4),\epsilon} \subset S$$

where $N_{(a_i,b_i),\epsilon} \subset Qi$ for $i \in \{1, 2, 3, 4\}$. Furthermore, by Corollary 2.1 we may find $(p_i, q_i) \in \mathbb{Q}^2$ such that $(p_i, q_i) \in N_{(a_i,b_i),\epsilon}$ for $i \in \{1, 2, 3, 4\}$. Furthermore:

$$(p_1, q_1) = \left(\frac{n_{11}}{n_{12}}, \frac{m_{11}}{m_{12}}\right) \qquad (p_3, q_3) = \left(\frac{-n_{31}}{n_{32}}, \frac{-m_{31}}{m_{32}}\right)$$

$$(p_2, q_2) = \left(\frac{-n_{21}}{n_{22}}, \frac{m_{21}}{m_{22}}\right) \qquad (p_4, q_4) = \left(\frac{n_{41}}{n_{42}}, \frac{-m_{41}}{m_{42}}\right)$$

where $n_{ij}, m_{ij} \in \mathbb{N}$ for $i \in \{1, 2, 3, 4\}$ and $j \in \{1, 2\}$. Set $n_0 = n_{12}n_{21}$ and $m_0 = n_{22}n_{11}$. Then it follows that the linear combination

$$n_0(p_1, q_1) + m_0(p_2, q_2) = \left(0, \frac{n_{12}n_{21}m_{11}m_{22} + n_{11}n_{22}m_{12}m_{21}}{m_{12}m_{22}}\right)$$

Letting $c_1 = n_{12}n_{21}m_{11}m_{22} + n_{11}n_{22}m_{12}m_{21}$, we get

$$n_0(p_1, q_1) + m_0(p_2, q_2) = \left(0, \frac{c_1}{m_{12}m_{22}}\right). \tag{3.1}$$

Similarly, set $s_0 = n_{32}n_{41}$ and $t_0 = n_{31}n_{42}$. Then it follows that the linear combination

$$s_0(p_3, q_3) + t_0(p_4, q_4) = \left(0, \frac{-(n_{32}n_{41}m_{31}m_{42} + n_{31}n_{42}m_{32}m_{41})}{m_{32}m_{42}}\right).$$

Letting $c_2 = n_{32}n_{41}m_{31}m_{42} + n_{31}n_{42}m_{32}m_{41}$ we get:

$$s_0(p_3, q_3) + t_0(p_4, q_4) = \left(0, \frac{-c_2}{m_{32}m_{42}}\right). \tag{3.2}$$

Now consider $u_0 = m_{12}m_{22}c_2$ and $v_0 = m_{32}m_{42}c_1$. Then it follows that:

$$u_0\left(0, \frac{c_1}{m_{12}m_{22}}\right) + v_0\left(0, \frac{-c_2}{m_{32}m_{42}}\right) = (0, 0). \tag{3.3}$$

Combining together Equations 3.1, 3.2, and 3.3 provides

$$u_0n_0(p_1, q_1) + u_0m_0(p_2, q_2) + v_0s_0(p_3, q_3) + v_0t_0(p_4, q_4) = (0, 0). \tag{3.4}$$

Since $n_{ij}, m_{ij} > 0$ for all $i \in \{1,2,3,4\}$ and $j \in \{1,2\}$ it follows that $u_0n_0, u_0m_0, v_0s_0, v_0t_0 \in \mathbb{N}$. Setting $n = u_0n_0$, $m = u_0m_0$, $s = v_0s_0$, and $t = v_0t_0$ the result follows immediately from Equation 3.4. ∎

It will also be essential that we can scale a point in $\mathbb{R}^2$ to be arbitrarily close to $(0,0)$. To do this, we will need the following two functions.

Definition 3.1. Let $x \in \mathbb{R}$.

(1) The *floor function*, denoted $\lfloor x \rfloor$, assigns to the input x the greatest integer that is less than or equal to x.
(2) The *ceiling function*, denoted $\lceil x \rceil$, assigns to the input x the least integer that is greater than or equal to x.

These two functions can now be used to perform the necessary scaling.

Lemma 3.2. *Let $(z_1, z_2) \in \mathbb{R}^2$, and let $\epsilon \in \mathbb{R}^+$. Then there exists $\beta \in \mathbb{N}$ such that*

$$\frac{1}{\beta\lfloor |z_x|+1\rfloor\lfloor |z_y|+1\rfloor}(z_1, z_2) \in N_{(0,0),\epsilon}.$$

Proof. Let $(z_1, z_2) \in \mathbb{R}^2$, and let $\epsilon \in \mathbb{R}^+$. Set $\Delta = \dfrac{|z_x|+|z_y|}{\epsilon\lfloor |z_x|+1\rfloor\lfloor |z_y|+1\rfloor}$. Then

$$\begin{aligned}
\sqrt{\left(\frac{z_x}{\Delta\lfloor |z_x|+1\rfloor\lfloor |z_y|+1\rfloor}\right)^2+\left(\frac{z_y}{\Delta\lfloor |z_x|+1\rfloor\lfloor |z_y|+1\rfloor}\right)^2} &\le \sqrt{\left(\frac{z_x}{\Delta\lfloor |z_x|+1\rfloor\lfloor |z_y|+1\rfloor}\right)^2}+\sqrt{\left(\frac{z_y}{\Delta\lfloor |z_x|+1\rfloor\lfloor |z_y|+1\rfloor}\right)^2}\\
&= \frac{|z_x|+|z_y|}{\Delta\lfloor |z_x|+1\rfloor\lfloor |z_y|+1\rfloor}\\
&= \epsilon
\end{aligned}$$

Choosing $\beta = \lceil \Delta \rceil$ produces the desired result. ∎

Now, we are ready to prove the main result of the paper, which is an extension of the original problem posed in the February 2017 edition of Mathematics Magazine (see [1]).

Theorem 3.1. *Let S be an open set in $\mathbb{R}^2$ containing the points $(a_1, b_1) \in Q1$, $(a_2, b_2) \in Q2$, $(a_3, b_3) \in Q3$, and $(a_4, b_4) \in Q4$. Furthermore, let $(z_1, z_2) \in \mathbb{R}^2$. Then there exist $n, m, s, t \in \mathbb{N}$ such that*

$$n\overline{\mathbf{x}_1} + m\overline{\mathbf{x}_2} + s\overline{\mathbf{x}_3} + t\overline{\mathbf{x}_4} = (z_1, z_2)$$

where $\overline{\mathbf{x}_i} \in N_{(a_i,b_i),\epsilon} \subset S$ for $i \in \{1,2,3,4\}$ and some $\epsilon \in \mathbb{R}^+$.

Proof. Let S be an open set in $\mathbb{R}^2$ containing the points $(a_1, b_1) \in Q1$, $(a_2, b_2) \in Q2$, $(a_3, b_3) \in Q3$, and $(a_4, b_4) \in Q4$. Then from Lemma 3.1, there exists $n_0, m_0, s_0, t_0 \in \mathbb{N}$ and $\epsilon \in \mathbb{R}^+$ such that

$$n_0(p_1, q_1) + m_0(p_2, q_2) + s_0(p_3, q_3) + t_0(p_4, q_4) = (0,0) \tag{3.5}$$

where $(p_i, q_i) \in N_{(a_i,b_i),\epsilon} \subset S$ with $(p_i, q_i) \in \mathbb{Q}^2$ for $i \in \{1,2,3,4\}$. Moreover, Lemma 3.2 provides a $\beta \in \mathbb{N}$, for any $\epsilon \in \mathbb{R}^+$, such that

$$\frac{1}{\beta\lfloor |z_x|+1\rfloor\lfloor |z_y|+1\rfloor}(z_1, z_2) \in N_{(0,0),\epsilon}.$$

Thus, we can find a $\beta \in \mathbb{N}$ such that

$$\frac{1}{\Delta}(z_1, z_2) + (p_i, q_i) \in N_{(p_i,q_i),\epsilon'} \subset N_{(a_i,b_i),\epsilon} \subset S$$

where $\epsilon' \in \mathbb{R}^+$ and $\Delta = \beta\lfloor |z_x| + 1\rfloor \lfloor |z_y| + 1\rfloor$. It follows that $\Delta \in \mathbb{N}$ and

$$\overline{\mathbf{x_1}} = \frac{1}{4n_0\Delta}(z_1, z_2) + (p_1, q_1) \in S \qquad \overline{\mathbf{x_3}} = \frac{1}{4s_0\Delta}(z_1, z_2) + (p_3, q_3) \in S$$

$$\overline{\mathbf{x_2}} = \frac{1}{4m_0\Delta}(z_1, z_2) + (p_2, q_2) \in S \qquad \overline{\mathbf{x_4}} = \frac{1}{4t_0\Delta}(z_1, z_2) + (p_4, q_4) \in S.$$

Furthermore,

$$n_0\Delta\overline{\mathbf{x_1}}+m_0\Delta\overline{\mathbf{x_2}}+s_0\Delta\overline{\mathbf{x_3}}+t_0\Delta\overline{\mathbf{x_4}} = (z_1, z_2)+\Delta\left[n_0(p_1, q_1) + m_0(p_2, q_2) + s_0(p_3, q_3) + t_0(p_4, q_4)\right]$$

Then, by Equation 3.5

$$n_0\Delta\overline{\mathbf{x_1}} + m_0\Delta\overline{\mathbf{x_2}} + s_0\Delta\overline{\mathbf{x_3}} + t_0\Delta\overline{\mathbf{x_4}} = (z_1, z_2) + \Delta(0, 0) = (z_1, z_2).$$

The claim follows by setting $n = n_0\Delta$, $m = m_0\Delta$, $s = s_0\Delta$, and $t = t_0\Delta$. ∎

We can further use this main result to return to the original problem posed by Souvik Dey.

Corollary 3.1. *Let S be an open subset of the set $\mathbb{R}$, such that S contains at least one positive number and one negative number. Then every real number can be written as a finite sum of (not necessarily distinct) elements of S.*

Proof. Let S be an open subset of $\mathbb{R}$ such that there exists $a, b \in S$ with $a < 0$ and $b < 0$. Moreover, let $r \in \mathbb{R}$. Then $(r, 0) \in \mathbb{R}^2$. Since S is an open subset of $\mathbb{R}$, it follows that $S\times S = \{(x, y) : x, y \in S\}$ is an open subset of $\mathbb{R}^2$. Moreover, $(a, a), (a, b), (b, a), (b, b) \in S\times S$ with $(b, b) \in Q1$, $(a, b) \in Q2$, $(a, a) \in Q3$, and $(b, a) \in Q4$. By Theorem 3.1, there exist $n, m, s, t \in \mathbb{N}$ and $\overline{\mathbf{x_1}}, \overline{\mathbf{x_2}}, \overline{\mathbf{x_3}}, \overline{\mathbf{x_4}} \in S \times S$ such that

$$n\overline{\mathbf{x_1}} + m\overline{\mathbf{x_2}} + s\overline{\mathbf{x_3}} + t\overline{\mathbf{x_4}} = (r, 0). \tag{3.6}$$

Let $(\overline{\mathbf{x_i}})_1$ denote the first coordinate of $\overline{\mathbf{x_i}}$ for $i \in \{1, 2, 3, 4\}$. Then from Equation 3.6, it follows that:

$$n\,(\overline{\mathbf{x_1}})_1 + m\,(\overline{\mathbf{x_2}})_1 + s\,(\overline{\mathbf{x_3}})_1 + t\,(\overline{\mathbf{x_4}})_1 = r.$$

Since $\overline{\mathbf{x_i}} \in S \times S$ for $i \in \{1, 2, 3, 4\}$, $(\overline{\mathbf{x_i}})_1 \in S$ for $i \in \{1, 2, 3, 4\}$ and the result follows. ∎

4. Conclusion

From the initial problem posed by Souvik Dey (see [1]), we were able to generalize the problem to $\mathbb{R}^2$. There is still the possibility to extend this problem to $\mathbb{R}^n$. We end with the following conjecture regarding this general problem.

Conjecture 1. Let S be an open set in $\mathbb{R}^n$ containing the points $(x_{1i}, x_{2i}, \ldots, x_{ni})$ where $i \in \{1, 2, \ldots, 2^n\}$ such that each of these points lies in one of the 2^n distinct regions of $\mathbb{R}^n$ determined by the $(n-1)$-dimensional hyperplanes partitioning $\mathbb{R}^n$. Furthermore, let $(z_1, z_2, \ldots, z_n) \in \mathbb{R}^n$. Then there exist $c_1, c_2, \ldots, c_{2^n} \in \mathbb{N}$ such that

$$\sum_{i=1}^{2^n} c_i(x_{1i}, x_{2i}, \ldots, x_{ni}) = (z_1, z_2, \ldots, z_n)'$$

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