2004

Some Interesting Multiples of Nine: Use Your Digits to Get the Digits!

Kevin Hurley

Electronic Proceedings of Undergraduate Mathematics Day
2004, No. 1, http://www.udayton.edu/∽mathdept/EPUMD/EPMUD.htm

Some Interesting Multiples of Nine:

Use Your Digits to Get the Digits!

Kevin Hurley

Maine South High School

khurley@maine207south.k12.il.us

Abstract: We have unraveled two neat and powerful algorithms for calculating certain multiples of nine. These discussions might make for an interesting introduction to a number theory course, or a supplemental project in calculus or advanced algebra. The mathematics involved is within a student's grasp, and the results are quite startling.

An interesting problem came up in my Algebra 2 class after viewing the movie "Stand and Deliver". I showed the movie in class and gave some problems that were loosely related to scenes that took place in the movie, one of which dealt with multiples of nine.

In the scene, the teacher shows a student how to multiply by nine using fingers. For example:

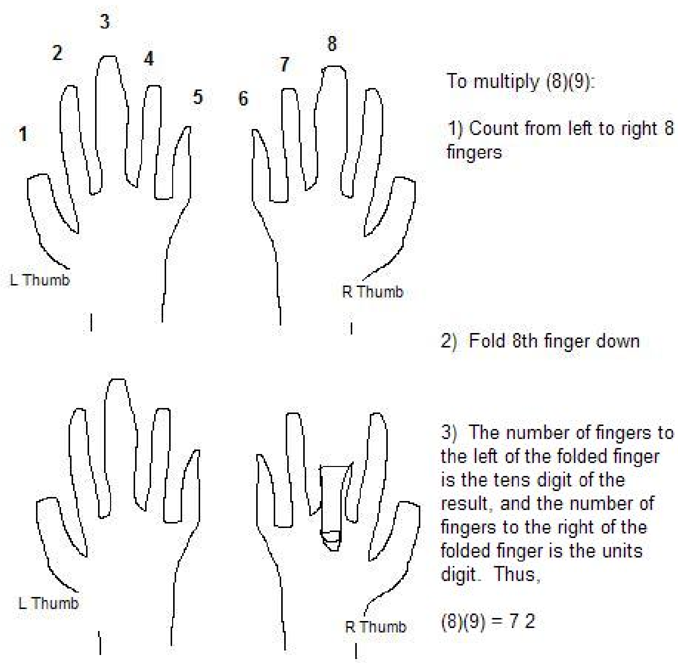

Figure 1

You can express this trick algebraically like this. Suppose we have a single-digit number *M* being multiplied by nine. We can write the result as:

$$(M)(9) = (M)(10-1) = 10M - M = 10M - 10 + 10 - M = (M-1)10 + (10-M)$$

The result is a two-digit number with $(M-1)$ as the tens place, and $(10-M)$ as the ones place. This explains the phenomenon; if you fold down the *Mth* finger looking from left to right, you always have $(M-1)$ fingers to the left of that finger, and $(10-M)$ fingers to the right.

The question I asked of the students is, "*Does this method work with larger multiples of nine?*" The answer in general is of course no, the easiest counterexample being (9)(11) = 99. But one student did find *some* numbers that did work, and adapted the method to suit them. Alexandra Cameruci, a junior at Chaminade-Julienne Catholic High School, showed that certain two-digit numbers could be multiplied by nine very quickly by using a variation of the trick shown in the movie. An example of her method:

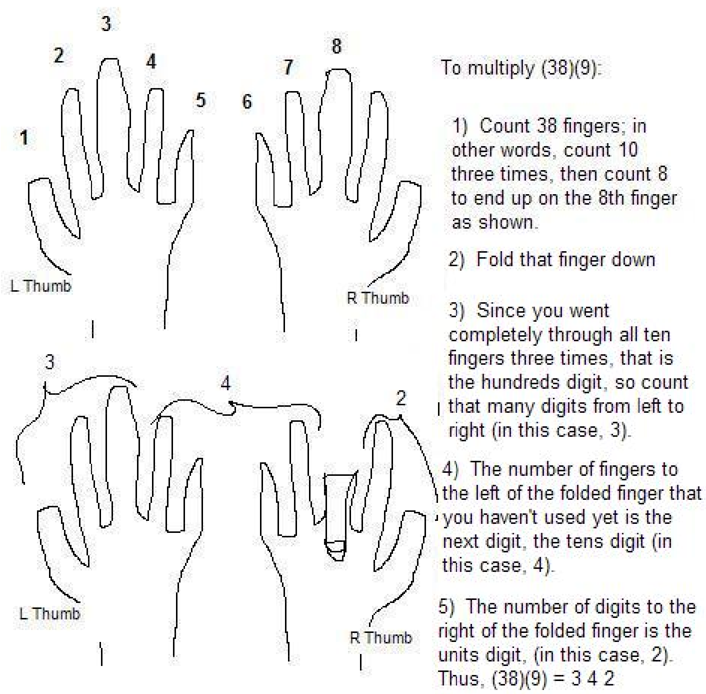

Figure 2

Unfortunately, this only works for certain numbers, since this method does not work for (42)(9) = 378. The reason is simple: We have only ten fingers on our hands, thus the digits must add to a number less than or equal to ten in order to count them. But since multiples of nine have digit sums that are also multiples of nine, the only possible digit sum we can have to utilize this method would be nine, so we cannot expect this method to work unless the digit sum of the product is nine.

The next question then is: When do multiples of nine have digit-sums of nine? To answer this question for two-digit numbers, we looked at spreadsheet listings of multiples of nine, and it appeared that the digit sum would be nine if the number being multiplied by nine has digits in ascending order; in other words, the tens digit is less than the units digit. Some specific cases: (9)(23) = 207, (9)(46) = 414.

This can be shown in general for all ascending two-digit numbers *M* by writing *M* in expanded form as $M = \sum_{j=1}^{2} m_j 10^{j-1} = m_2 10^1 + m_1$, where $0 < m_2 < m_1 < 10$, and each m_j is an integer. Multiplying *M* by nine yields

$$
\begin{aligned}
(9)(M) &= (10-1)\left(m_2 10^1 + m_1\right) \\
&= m_2 10^2 + m_1 10^1 - m_2 10^1 - m_1 \\
&= m_2 10^2 + \left(m_1 - m_2\right) 10^1 - m_1 \\
&= m_2 10^2 + \left(m_1 - m_2 - 1\right) 10^1 + (10 - m_1)
\end{aligned}
$$

Note that each coefficient of the result must be non-negative since $0 < m_2 < m_1 < 10$. So the result has a hundreds digit of $[m_2]$, a tens digit of $[m_1 - m_2 - 1]$, and a units digit of $[10 - m_1]$. This "proves" that the method illustrated in Figure 2 will work for all ascending two-digit multiples of nine, and adding these digits produces $[m_2] + [m_1 - m_2 - 1] + [10 - m_1] = 9$.

Note also that the Figure 2 finger technique would fail if the tens digit of M, m_2, was equal to the units digit, m_1. This is because the tens digit of the result, $[m_1 - m_2 - 1]$, would be a negative number.

The algebra is starting to reveal an even easier formula for the product (9)(M) than using fingers. To illustrate this point, consider a three-digit number $M = \sum_{j=1}^{3} m_j 10^{j-1} = m_3 10^2 + m_2 10^1 + m_1$. Multiplying M by nine yields

$$
\begin{aligned}
(9)(M) &= (10-1)\left(m_3 10^2 + m_2 10^1 + m_1\right) \\
&= m_3 10^3 + m_2 10^2 + m_1 10^1 - m_3 10^2 - m_2 10^1 - m_1 \\
&= m_3 10^3 + \left(m_2 - m_3\right)10^2 + \left(m_1 - m_2\right)10^1 - m_1 \\
&= m_3 10^3 + \left(m_2 - m_3\right)10^2 + \left(m_1 - m_2 - 1\right)10^1 + (10 - m_1)
\end{aligned}
$$

Note that each coefficient will be non-negative, as long as $0 < m_3 \le m_2 < m_1 < 10$. Thus, the result is a number with a thousands digit of $[m_3]$, a hundreds digit of $[m_2 - m_3]$, a tens digit of $[m_1 - m_2 - 1]$, and a units digit of $[10-m_1]$. Adding the digits of this result produces

$[m_3]+[m_2-m_3]+[m_1-m_2-1]+[10-m_1]=9$.

In the three-digit case, we can use a less-than-or-equal-to sign in-between m_3 and m_2. This is because if $m_3 = m_2$, then you get zero for the hundreds place in the result, which is certainly plausible.

Let us now use our formula for a specific example. Consider the product of (123)(9). Since $M = 123$, then $m_3 = 1, m_2 = 2, m_3 = 3$, and from before, the digits of the result are $[m_3]=1$ (thousands digit), $[m_2-m_3]=[2-1]=1$ (hundreds digit), $[m_1-m_2-1]=[3-2-1]=0$ (tens digit), and $[10-m_1]=[10-3]=7$ (units digit). Thus, (123)(9) = 1107.

To simplify the notation, let the result be written as (thousands digit) | (hundreds) | (tens) | (units). So the number (9)(*M)* is therefore:

$$[m_3]\,|\,[m_2-m_3]\,|\,[m_1-m_2-1]\,|\,[10-m_1]$$

To illustrate:

a) (137)(9) = 1|3-1|7-3-1|10-7 = 1|2|3|3 = 1233

b) (448)(9) = 4|4-4|8-4-1|10-8 = 4|0|3|2 = 4032

Naturally, the next question is: Can this algebraic technique be extended for multiplication of nine by a *k*-digit number? The answer is also yes, provided that the following two rules are satisfied:

1) The digits of *M* are in non-decreasing order. *Any* number that has any part of it in decreasing order will fail. For example, 12345638 fails: (9)(12345638) = 111110742, which is a number whose digits add to 18. It fails

because the digit in the hundreds place (6), is followed by a lower number in the tens place (3); i.e. $m_1 - m_2 - 1$ is negative.

2) Repeats are not OK *in the last two digits*. For example, (1222223)(9) = 11000007 works, but (1222222)(9) = 10999998 fails.

To formalize this result:

Theorem: Let $(9)(M) = P$, where *M* is a positive, *k*-digit integer such that $M = \sum_{j=1}^{k} m_j 10^{j-1} = m_k 10^{k-1} + m_{k-1}10^{k-2} + ... + m_3 10^2 + m_2 10^1 + m_1$,

with $0 < m_k \le m_{k-1} \le m_{k-2} \le ... \le m_2 < m_1 < 10$, and each m_j is an integer.

Then,

a) the number 9*M* (using the notation from before) is then

$[m_k] | [m_{k-1} - m_k] | [m_{k-2} - m_{k-1}] | ... | [m_2 - m_3] | [m_1 - m_2 - 1] | [10 - m_1]$

b) 9*M* has digits that add to 9.

Proof: If $M = \sum_{j=1}^{k} m_j 10^{j-1} = m_k 10^{k-1} + m_{k-1}10^{k-2} + ... + m_3 10^2 + m_2 10^1 + m_1$

then $10M = 10\sum_{j=1}^{k} m_j 10^{j-1} = \sum_{j=1}^{k} m_j 10^j$. Thus, 9*M* = 10*M* – *M* =

$$\begin{aligned}
&\sum_{j=1}^{k} m_j 10^j - \sum_{j=1}^{k} m_j 10^{j-1} = \sum_{j=1}^{k} m_j 10^j - \sum_{j=0}^{k-1} m_{j+1} 10^j \\
&= m_k 10^k + \left[\sum_{j=1}^{k-1} m_j 10^j - \sum_{j=1}^{k-1} m_{j+1} 10^j \right] - m_1 \\
&= m_k 10^k + \left[\sum_{j=1}^{k-1} 10^j (m_j - m_{j+1}) \right] - m_1 \\
&= m_k 10^k + (m_{k-1} - m_k)10^{k-1} + ... + (m_2 - m_3)10^2 + (m_1 - m_2)10^1 - m_1
\end{aligned}$$

If we force $0 < m_k \le m_{k-1} \le m_{k-2} \le \ldots \le m_2 < m_1 < 10$, we've obtained all non-negative integers except for the units place. We then "borrow" from the tens place to yield:

$$9M = m_k 10^k + (m_{k-1} - m_k)10^{k-1} + \ldots + (m_2 - m_3)10^2 + (m_1 - m_2 - 1)10^1 - (10 - m_1)$$

which is the *k*-digit number

$$[m_k]\,|\,[m_{k-1} - m_k]\,|\,[m_{k-2} - m_{k-1}]\,|\ldots|\,[m_2 - m_3]\,|\,[m_1 - m_2 - 1]\,|\,[10 - m_1]$$

so we have thus satisfied a), with all digits non-negative since $m_1 - m_2 - 1 \ge 0$.

Adding these digits yields:

$$[m_k] + [m_{k-1} - m_k] + [m_{k-2} - m_{k-1}] + \ldots + [m_2 - m_3] + [m_1 - m_2 - 1] + [10 - m_1] = -1 + 10 = 9$$

thus we have proven b). QED

This theorem proves to be quite useful when calculating large multiples of 9 that fit the criteria given. For example:

(345568)(9) = 3|4-3|5-4|5-5|6-5|8-6-1|10-8 = 3110112

(12334556)(9) = 1|2-1|3-2|3-3|4-3|5-4|5-5|6-5-1|10-6 = 111011004

(2455667889)(9) = 2|4-2|5-4|5-5|6-5|6-6|7-6|8-7|8-8|9-8-1|10-9

= 22101011001

Naturally, the next question is, what is the pattern for non-increasing digits? If we modify our assumption in the proof above to be $0 < m_1 \le m_2 \le \ldots \le m_{k-1} \le m_k < 10$ with each m_j an integer, we still arrive at $9M = m_k 10^k + (m_{k-1} - m_k)10^{k-1} + \ldots + (m_2 - m_3)10^2 + (m_1 - m_2)10^1 - m_1$, but in this case we would like to modify this to force our new assumption to yield non-negative coefficients (since currently, all but the $m_k 10^k$ term is non-positive.)

The trick here is to borrow on *each* term, not just the last. Borrowing from the $m_k 10^k$ term we obtain:

$$9M = (m_k - 1)10^k + (10 + m_{k-1} - m_k)10^{k-1} + ... + (m_2 - m_3)10^2 + (m_1 - m_2)10^1 - m_1.$$

We then borrow from the (*k*-1)th term to make the next one positive, and so on down the line. When finished we obtain:

$$9M = (m_k - 1)10^k + (9 + m_{k-1} - m_k)10^{k-1} + ... + (9 + m_2 - m_3)10^2 + (9 + m_1 - m_2)10^1 + (10 - m_1).$$

which is the *k*-digit number

$$[m_k - 1] | [9 + m_{k-1} - m_k] | [9 + m_{k-2} - m_{k-1}] | ... | [9 + m_2 - m_3] | [9 + m_1 - m_2] | [10 - m_1]$$

These digits are all nonnegative, since by assumption the digits cannot differ by more than 9. Note that m_2 need not be *strictly* greater than m_1; they may be equal, and the formula still holds. Furthermore, adding these digits produces 9*k*, the details of which are left to the reader.

Now we may multiply nine by numbers with non-increasing digits with ease as well:

(9)(432) = 4-1|9 + 3 – 4|9 + 2 – 3|10 – 2| = 3888

(9)(87532) = 8 – 1|9 + 7 – 8|9 + 5 – 7|9 + 3 – 5|9 + 2 – 3|10 – 2| = 787788

We have unraveled two neat and powerful algorithms for calculating certain multiples of nine. This might make for an interesting introduction for a number theory course, or a supplemental project in a PreCalculus or Advanced Algebra class. The methods of mathematics involved are within a student's grasp, and the results are quite startling. This result has been shown to numerous students at varying levels of mathematical competency, and there has been almost universal interest in it.