# SEMI r-IDEALS OF COMMUTATIVE RINGS 

HANI A. KHASHAN AND ECE YETKIN CELIKEL


#### Abstract

For commutative rings with identity, we introduce and study the concept of semi $r$-ideals which is a kind of generalization of both $r$-ideals and semiprime ideals. A proper ideal $I$ of a commutative ring $R$ is called semi $r$-ideal if whenever $a^{2} \in I$ and $A n n_{R}(a)=0$, then $a \in I$. Several properties and characterizations of this class of ideals are determined. In particular, we investigate semi $r$-ideal under various contexts of constructions such as direct products, localizations, homomorphic images, idealizations and amalagamations rings. We extend semi $r$-ideals of rings to semi $r$-submodules of modules and clarify some of their properties. Moreover, we define submodules satisfying the $D$-annihilator condition and justify when they are semi $r$-submodules.


## 1. Introduction

Throughout, all rings are supposed to be commutative with identity and all modules are unital. Let $R$ be a ring and $M$ an $R$-module. We recall that a proper ideal $I$ of a $R$ is called semiprime if whenever $a \in R$ such that $a^{2} \in I$, then $a \in I$. It is well-known that $I$ is semiprime in $R$ if and only if $I$ is a radical ideal, that is $I=\sqrt{I}$ where $\sqrt{I}=\left\{x \in R: x^{m} \in I\right.$ for some $\left.m \in \mathbb{Z}\right\}$. In 2015, R. Mohamadian [15] introduced the concept of $r$-ideals of commutative rings. A proper ideal $I$ of a ring $R$ is called an $r$-ideal (resp. pr-ideal) if whenever $a, b \in R$ such that $a b \in I$ and $A n n_{R}(a)=0$, then $b \in I$ (resp. $b \in \sqrt{I}$ ) where $A n n_{R}(a)=\{b \in R: a b=0\}$. Prime and $r$-ideals are not comparable in general; but it is verified that every maximal $r$-ideal in a ring is a prime ideal, while every minimal prime ideal is an $r$-ideal. In 2017, Tekir, Koc and Oral [18] introduced the concept of $n$-ideals as a special kind of $r$-ideals by considering the set of nilpotent elements instead of zero divisors. Recently, in [20, Celikel and Khashan generalized $n$-ideals by defining and studying the class of semi $n$-ideals. A proper ideal $I$ of $R$ is called a semi $n$-ideal if for $a \in R$, $a^{2} \in I$ and $a \notin \sqrt{0}$ imply $a \in I$. Later, some other generalizations of semiprime, $n$-ideals and $r$-ideals have been introduced, see for example, [4, [10]-12] and [19].

Motivated by semiprime ideals and semi $n$-ideals, we define a proper ideal $I$ of a ring $R$ to be a semi $r$-ideal if whenever $a \in R$ such that $a^{2} \in I$ and $A n n_{R}(a)=0$, then $a \in I$. It is clear that the class of semi $r$-ideals is a generalization of that of semiprime and $r$-ideals. We start section 2 by giving some examples (see Example 1) to show that this generalization is proper. Next, we determine several equivalent characterizations of semi $r$-ideals (see Theorem (1). Among many other results in this paper, we characterize rings in which every ideal is a semi $r$-ideal (see Theorem (3). We investigate semi $r$-ideals under various contexts of constructions

[^0]such as homomorphic images, quotient rings, localizations and polynomial rings (see Propositions 1 and 3. Corollary 3. Theorem 4). Moreover, we discuss and characterize semi $r$-ideals of cartesian product of rings (see Proposition 5. Theorems 5 and 6, Corollaries 4 and 5). Let $R$ and $S$ be two rings, $J$ be an ideal of $S$ and $f: R \rightarrow S$ be a ring homomorphism. We study some forms of semi $r$-ideals of the amalgamation ring $R \bowtie^{f} J$ of $R$ with $S$ along $J$ with respect to $f$ (see Theorems 7 and 8).

Let $M$ be an $R$-module, $N$ be a submodule of $M$ and $I$ be an ideal of $R$. As usual, we will use the notations $\left(N:_{R} M\right)$ and $\left(N:_{M} I\right)$ for the sets $\{r \in R: r m \in N$ for all $m \in M\}$ and $\{m \in M: I m \subseteq N\}$, respectively. In particular, the annihilator of an element $m \in M$ (resp. $r \in R$ ) denoted by $A n n_{R}(m)$ (resp. $A n n_{M}(r)$ ), is $\left(0:_{R} m\right)$ (resp. ( $0:_{M} r$ ). We recall that the torsion subgroup $T(M)$ of an $R$-module $M$ is defined as $T(M)=\{m \in M$ : there exists $0 \neq r \in R$ such that $r m=0\}$. It is easy to see that $T(M)$ is a submodule of $M$, called the torsion submodule. A module is torsion (resp. torsion-free) if $T(M)=M$ (resp. $T(M)=\{0\})$.

In 2009, the concept of semiprime submodules is presented. A proper submodule is said to be semiprime if whenever $r \in R, m \in M$ and $r^{2} m \in N$, then $r m \in N$, 16. Afterwards, the notions of $r$-submodule and $s r$-submodules are introduced and studied in 13. A proper submodule $N$ is called an $r$-submodule (resp. srsubmodule) of $M$ if whenever $r m \in N$ and $A n n_{M}(r)=0_{M}$ (resp. $\left.A n n_{R}(m)=0\right)$, then $m \in N$ (resp. $r \in\left(N:_{R} M\right)$ ). As a new generalization of above structures, in Section 3, we define a proper submodule $N$ of $M$ to be a semi $r$-submodule if whenever $r \in R, m \in M$ with $r^{2} m \in N, A n n_{M}(r)=0_{M}$ and $A n n_{R}(m)=0$, then $r m \in N$. We illustrate (see Example (4) that this generalization of $r$-submodules is proper. However, it is observed that semi $r$-submodules coincides with semiprime submodules in any torsion-free module. Then, we introduce a new condition for submodules, namely, $D$-annihilator condition as follows: A proper submodule $N$ of an $R$-module $M$ is said to satisfy the $D$-annihilator condition if whenever $K$ is a submodule of $M$ and $r \in R$ such that $r K \subseteq N$ and $A n n_{M}(r)=0_{M}$, then either $K \subseteq$ $N$ or $K \cap T(M)=\left\{0_{M}\right\}$. By using this condition, we totally characterize semi $r$ submodules of finitely generated faithful multiplication $R$-modules (see Proposition 8, Theorems 9 and 10, Corollary (6).

We recall that the idealization of an $R$-module $M$ denoted by $R(+) M$, is the commutative ring $R \times M$ with coordinate-wise addition and multiplication defined as $\left(r_{1}, m_{1}\right)\left(r_{2}, m_{2}\right)=\left(r_{1} r_{2}, r_{1} m_{2}+r_{2} m_{1}\right)$. For an ideal $I$ of $R$ and a submodule $N$ of $M, I(+) N$ is an ideal of $R(+) M$ if and only if $I M \subseteq N$. It is well known from (2) that

$$
z d(R(+) M)=\{(r, m) \mid r \in z d(R) \cup Z(M), m \in M\}
$$

In Proposition 11, we clarify the relation between semi $r$-ideals of the idealization ring $R(+) M$ and those of $R$ which enables us to build some interesting examples of semi $r$-ideals.

Let $f: R_{1} \rightarrow R_{2}$ be a ring homomorphism, $J$ be an ideal of $R_{2}, M_{1}$ be an $R_{1}$ module, $M_{2}$ be an $R_{2}$-module and $\varphi: M_{1} \rightarrow M_{2}$ be an $R_{1}$-module homomorphism. The subring

$$
R_{1} \bowtie^{f} J=\left\{(r, f(r)+j): r \in R_{1}, j \in J\right\}
$$

of $R_{1} \times R_{2}$ is called the amalgamation of $R_{1}$ and $R_{2}$ along $J$ with respect to $f$. In [8], the amalgamation of $M_{1}$ and $M_{2}$ along $J$ with respect to $\varphi$ is defined as

$$
M_{1} \bowtie^{\varphi} J M_{2}=\left\{\left(m_{1}, \varphi\left(m_{1}\right)+m_{2}\right): m_{1} \in M_{1} \text { and } m_{2} \in J M_{2}\right\}
$$

which is an $\left(R_{1} \bowtie^{f} J\right)$-module. The last section is devoted to clarify semi $r$ submodules of the amalgamation of modules.

## 2. Properties of semi $r$-IDEALS

This section deals with many properties of semi $r$-ideals. We justify the relations among the concepts of semiprime ideals, semi $n$-ideals and our new class of ideals. Moreover, several characterizations and examples are presented. In particular, we characterize rings in which every ideal is a semi $r$-ideal.

Definition 1. Let $I$ be a proper ideal of a ring $R$. $I$ is called a semi r-ideal of $R$ if whenever $a \in R$ such that $a^{2} \in I$ and $A n n_{R}(a)=0$, then $a \in I$.

For any non-zero subset $A$ of a ring $R$, we note that $A n n_{R}(A)$ is a semi $r$-ideal of $R$. It is clear that the classes of semiprime ideals, $r$-ideals and semi $n$-ideals are contained in the class of semi $r$-ideals. However, in general these containments are proper as we illustrate in the following examples.
Example 1. Let p and $q$ be prime integers.
(1) Any non-zero semiprime ideal in an integral domain is a semi r-ideal that is not an r-ideal.
(2) In the ring $\mathbb{Z}_{p^{2} q}$, the ideal $\left\langle\overline{p^{2}}\right\rangle$ is a semi $r$-ideal that is not a semi n-ideal.
(3) The zero ideal of a ring $R$ is always a semi r-ideal but it is not a semiprime ideal unless $R$ is a semiprime ring.
(4) Every ideal of a Boolean ring (a ring of which every element is idempotent) is semi r-ideal. Consider the ideal $I=0 \times 0 \times \mathbb{Z}_{2}$ of the Boolean ring $\mathbb{Z}_{2} \times \mathbb{Z}_{2} \times \mathbb{Z}_{2}$. Then $I$ is a semi $r$-ideal that is not prime.
(5) In general pr-ideals and semi r-ideals are not comparable. Let $T$ be a reduced ring with subring $\mathbb{Z}$ and $P$ be a nonzero minimal prime ideal in $T$ with $P \cap \mathbb{Z}=(0)$. From [15, Example 2.17], $J=x^{2} P[x]$ is a pr -ideal of the ring $R=\mathbb{Z}+x T[x]$. Choose an element $0 \neq p \in P$. Then $(x p)^{2} \in J$ and $A n n_{R}(x a)=0$ but $x a \notin J$. Thus, $J$ is not a semi r-ideal. Moreover, any non-zero prime ideal in an integral domain is clearly a semi r-ideal that is not a pr-ideal.

If $I$ and $J$ are semi $r$-ideals of a ring $R$, then $I J$ and $I+J$ need not be so as we can see in the following example.

Example 2. Consider the ideals $I=\langle x\rangle$ and $J=\langle x-4\rangle$ of the ring $R=\mathbb{Z}[x]$. Then $I$ and $J$ are (semi) prime ideals and so are semi r-ideals of $R$. On the other hand, $I+J=\langle x, x-4\rangle=\langle x, 4\rangle$ is not a semi $r$-ideal of $R$. Indeed, $(2+x)^{2} \in I+J$ and $A n n_{R}(2+x)=0$, but $2+x \notin I+J$. Also, $I^{2}=\left\langle x^{2}\right\rangle$ is not a semi r-ideal of $R$ as $x^{2} \in I^{2}$ and $A n n_{R}(x)=0$, but $x \notin I^{2}$.

Next, we give the following characterization of semi $r$-ideals. By $z d(R)$ we denote the set of all zero divisor elements of a ring $R$. Moreover, $\operatorname{reg}(R)$ denotes the set $R \backslash z d(R)$.
Theorem 1. Let $I$ be a proper ideal of $a \operatorname{ring} R$ and $k$ be a positive integer. The following statements are equivalent.
(1) $I$ is a semi $r$-ideal of $R$.
(2) Whenever $a \in R$ with $0 \neq a^{2} \in I$ and $A n n_{R}(a)=0$, then $a \in I$.
(3) Whenever $a \in R$ with $a^{k} \in I$ and $A n n_{R}(a)=0$, then $a \in I$.
(4) $\sqrt{I} \subseteq z d(R) \cup I$.

Proof. (1) $\Leftrightarrow(2)$. Suppose (2) holds and let $a \in R$ such that $a^{2} \in I$ and $A n n_{R}(a)=$ 0 . If $a^{2}=0$, then $a=0$ and the result follows obviously. If $a^{2} \neq 0$, then we are also done by (2). The converse part is obvious.
$(1) \Rightarrow(3)$. Suppose $a^{k} \in I$ and $A n n_{R}(a)=0$ for $a \in R$. We use the mathematical induction on $k$. If $k \leq 2$, then the claim is clear. We now assume that (3) holds for all $2<t<k$ and show that it is also true for $k$. Suppose $k$ is even, say, $k=2 m$ for some positive integer $m$. Since $a^{k}=\left(a^{m}\right)^{2} \in I$ and clearly $A n n_{R}\left(a^{m}\right)=0$, then $a^{m} \in I$ as $I$ is a semi $r$-ideal. By the induction hypothesis, we conclude that $a \in I$ as needed. Suppose $k$ is odd, so that $k+1=2 s$ for some $s<k$. Then similarly, we have $\left(a^{s}\right)^{2} \in I$ and $A n n_{R}\left(a^{s}\right)=0$ which imply that $a^{s} \in I$ and again by the induction hypothesis, we conclude $a \in I$.
$(3) \Rightarrow(4)$. Let $a \in \sqrt{I}$. Then $a^{k} \in I$ for some $k \geq 1$ and so by $(3) a \in z d(R)$ or $a \in I$. Thus, $\sqrt{I} \subseteq z d(R) \cup I$.
$(4) \Rightarrow(1)$. Straightforward.
Corollary 1. Let $I$ be a semi r-ideal of a ring $R$ and $k$ be a positive integer. If $J$ is an ideal of $R$ with $J^{k} \subseteq I$ and $J \cap z d(R)=\{0\}$, then $J \subseteq I$.
Proof. Suppose that $J^{k} \subseteq I$ and $J \cap z d(R)=\{0\}$ for some ideal $J$ of $R$. Let $0 \neq a \in J$. From the assumption $J \cap z d(R)=\{0\}$, we have $A n n_{R}(a)=0$. Thus, $a^{k} \in I$ implies that $a \in I$ by Theorem 1 (3).

Corollary 2. Let $I$ and $J$ be proper ideals of a ring $R$ such that $I \cap z d(R)=$ $J \cap z d(R)=\{0\}$.
(1) If $I$ and $J$ are semi $r$-ideals of a ring $R$ with $I^{2}=J^{2}$, then $I=J$.
(2) If $I^{2}$ is a semi $r$-ideal, then $I^{2}=I$.

Proof. (1) Since $I^{2} \subseteq J$ and $J \cap z d(R)=\{0\}$, then we have $I \subseteq J$ by Corollary 1 On the other hand, since $J^{2} \subseteq I$ and $J \cap z d(R)=\{0\}$, we have $J \subseteq I$ again by Corollary 1, so we are done.
(2) A direct consequence of (1).

We note by example 1 that unlike $r$-ideals, if $I$ is a semi $r$-ideal of a ring $R$, then $I$ need not be contained in $z d(R)$. Also, clearly, semi $r$-ideals which contain the zero divisors of a ring $R$ are semiprime.

Next, we present a condition for a semi $r$-ideal to be an $r$-ideal. First, we need the following lemma.

Lemma 1. Let $S$ be a non-empty subset of $R$ where $S \cap z d(R)=\emptyset$. If $I$ is a semi $r$-ideal of $R$ with $S \nsubseteq I$, then $(I: S)$ is a semi r-ideal of $R$.
Proof. Let $a \in R$ such that $a^{2} \in(I: S)$ and $\operatorname{Ann}_{R}(a)=0$. Then $(a s)^{2} \in I$ for all $s \in S$. As $I$ is a semi $r$-ideal of $R$, we have either $a s \in z d(R)$ or $a s \in I$ for all $s \in S$. If as $\in z d(R)$, then $S \cap z d(R)=\emptyset$ implies $a \in z d(R)$, a contradiction. Thus, as $\in I$ for all $s \in S$ and so $a \in(I: S)$ as required.
Theorem 2. If $I$ is maximal among all semi r-ideals of a ring $R$ contained in $z d(R)$, then $I$ is an r-ideal.

Proof. Let $I$ be maximal among all semi $r$-ideals of a ring $R$ contained in $z d(R)$. Suppose that $a b \in I$ and $A n n_{R}(a)=0$. Then $a \notin I \cup z d(R)$ and so $\left(I:_{R} a\right)$ is a semi $r$-ideal of $R$ by Lemma 1 . Since clearly, $\left(I:_{R} a\right) \subseteq z d(R)$ and $I \subseteq\left(I:_{R} a\right)$, then the maximality of $I$ implies, $I=\left(I:_{R} a\right)$. Thus, $b \in I$ and $I$ is an $r$-ideal.

Following [15], we call a ring $R$ a $u z$-ring if $R=U(R) \cup z d(R)$. It is proved in [15] that $R$ is a $u z$-ring if and only if every ideal in $R$ is an $r$-ideal. In particular, a direct product of fields is an example of a $u z$-ring. Next, we generalize this result to semi $r$-ideals.

Theorem 3. The following statements are equivalent for a ring $R$.
(1) $R$ is a $u z$-ring.
(2) Every proper ideal of $R$ is an $r$-ideal.
(3) Every proper ideal of $R$ is a semi $r$-ideal.
(4) Every proper principal ideal of $R$ is a semi $r$-ideal.
(5) Every semi $r$-ideal is an $r$-ideal.

Proof. (1) $\Rightarrow(2)$. Follows by 15, Proposition 3.4].
$(2) \Rightarrow(3) \Rightarrow(4)$. Clear.
$(4) \Rightarrow(1)$. Let $x \in R \backslash z d(R)$. If $\left\langle x^{2}\right\rangle=R$, then $x \in U(R)$. Suppose $\left\langle x^{2}\right\rangle$ is proper in $R$. Since $x^{2} \in\left\langle x^{2}\right\rangle$ and $A n n_{R}(x)=0$, then by assumption, $x \in\left\langle x^{2}\right\rangle$. Thus, $x=r x^{2}$ for some $r \in R$ and so $r x=1$ as $A n n_{R}(x)=0$. Thus, again $x \in U(R)$ and $R=U(R) \cup z d(R)$ as needed.
$(1) \Rightarrow(5)$. Clear by $(1) \Leftrightarrow(2)$.
$(5) \Rightarrow(1)$. Since a maximal ideal of $R$ is clearly a semi $r$-ideal, then by (5), every maximal ideal in $R$ is an $r$-ideal. Let $r \in R$. If $r \notin U(R)$, then $r \in M$ for some maximal ideal $M$ of $R$ and so $r \in z d(R)$ by [15, Remark 2.3(d)]. Therefore, $R=U(R) \cup z d(R)$ and $R$ is a $u z$-ring.

Next, we discuss the behavior of semi $r$-ideals under homomorphisms.
Proposition 1. Let $f: R_{1} \rightarrow R_{2}$ be a ring homomorphism. The following statements hold.
(1) If $f$ is an epimorphism, $I_{1} \subseteq \operatorname{Ker}(f)$ and $I_{1}$ is a semi $r$-ideal of $R_{1}$ such that $I_{1} \cap z d\left(R_{1}\right)=\{0\}$, then $f\left(I_{1}\right)$ is a semi $r$-ideal of $R_{2}$.
(2) If $f$ is an isomorphism and $I_{2}$ is a semi $r$-ideal of $R_{2}$, then $f^{-1}\left(I_{2}\right)$ is a semi $r$-ideal of $R_{1}$.

Proof. (1) Let $a \in R_{2}$ such that $a^{2} \in f\left(I_{1}\right)$ and $a \notin f\left(I_{1}\right)$. Then there exists $x \in R_{1} \backslash I_{1}$ such that $a=f(x)$. Since $f\left(x^{2}\right)=a^{2} \in f\left(I_{1}\right)$, then $x^{2} \in I_{1}$ as $\operatorname{Ker}(f) \subseteq I_{1}$. Now, $I_{1}$ is a semi $r$-ideal of $R_{1}$ implies $x \in z d\left(R_{1}\right)$. If $x=0$, then $a=f(x) \in z d\left(R_{2}\right)$. Suppose $x \neq 0$ and choose $0 \neq y \in R$ such that $x y=0$. Then $f(y) \neq 0$ since otherwise $y \in I_{1} \cap z d\left(R_{1}\right)$, a contradiction. Thus, again $a=f(x) \in z d\left(R_{2}\right)$ and $f\left(I_{1}\right)$ is a semi $r$-ideal of $R_{2}$.
(2) Suppose $I_{2}$ is a semi $r$-ideal of $R_{2}$. Let $x \in R_{1}$ such that $x^{2} \in f^{-1}\left(I_{2}\right)$ and $x \notin f^{-1}\left(I_{2}\right)$. Then $f\left(x^{2}\right)=f(x)^{2} \in I_{2}$ and $f(x) \notin I_{2}$ which imply $f(x) \in z d\left(R_{2}\right)$. Since $f$ is an isomorphism, then clearly $x \in z d\left(R_{1}\right)$ and $f^{-1}\left(I_{2}\right)$ is a semi $r$-ideal of $R_{1}$.

In view of Proposition 1. we have the following result for quotient rings.
Corollary 3. Let $I$ and $J$ be ideals of a ring $R$ with $J \subseteq I$.
(1) If $I$ is a semi $r$-ideal of $R$ and $I \cap z d(R)=\{0\}$, then $I / J$ is a semi $r$-ideal of $R / J$.
(2) If $I / J$ is a semi $r$-ideal of $R / J$ and $J$ is an $r$-ideal of $R$, then $I$ is a semi $r$-ideal of $R$.

Proof. (1). Consider the natural epimorphism $\pi: R \rightarrow R / J$ with $\operatorname{Ker}(\pi)=J$ and apply Proposition 1
(2). Let $a \in R$ such that $a^{2} \in I$ and $a \notin z d(R)$. Then $(a+J)^{2}=a^{2}+J \in I / J$. If $a+J \in z d(R / I)$, then there is $b \notin J$ such that $a b \in J$. Since $J$ is a semi $r$-ideal of $R$, we get $a \in z d(R)$, a contradiction. Thus, $a+J \notin z d(R / I)$ which yields $a+J \in I / J$ as $I / J$ is a semi $n$-ideal of $R / J$ and so $a \in I$.

If $I \cap z d(R) \neq\{0\}$ in Corollary 3(1), then the result need not be true. For example, $4 \mathbb{Z}(+) \mathbb{Z}_{4}$ is a semi $r$-ideal of $\mathbb{Z}(+) \mathbb{Z}_{4}$, see Remark $\mathbb{1 1}$, But $4 \mathbb{Z}(+) \mathbb{Z}_{4} / 0(+) \mathbb{Z}_{4} \cong$ $4 \mathbb{Z}$ is not a semi $r$-ideal of $\mathbb{Z}(+) \mathbb{Z}_{4} / 0(+) \mathbb{Z}_{4} \cong \mathbb{Z}$. We also note that the condition " $J$ is an $r$-ideal" in Corollary (3) is crucial. For example $8 \mathbb{Z} / 16 \mathbb{Z}$ is a semi $r$-ideal of $\mathbb{Z} / 16 \mathbb{Z}$ but $8 \mathbb{Z}$ is not a semi $r$-ideal of $\mathbb{Z}$.

In particular, Corollary 3 holds if $J \subseteq z d(R)$.
Proposition 2. The intersection of any family of semi r-ideals is a semi r-ideal.
Proof. Let $\left\{I_{\alpha}: \alpha \in \Lambda\right\}$ is a family of semi $r$-ideals. Suppose $a^{2} \in \bigcap_{\alpha \in \Lambda} I_{\alpha}$ and $a \notin \bigcap_{\alpha \in \Lambda} I_{\alpha}$. Then $a \notin I_{\gamma}$ for some $\gamma \in \Lambda$. Since $I_{\gamma}$ is a semi $r$-ideal, we have $a \in z d(R)$ and so $\bigcap_{\alpha \in \Lambda} I_{\alpha}$ is a semi $r$-ideal.

Let $I$ be a proper ideal of $R$. In the following we give the relationship between semi $r$-ideals of a ring and those of its localization ring by using the notation $Z_{I}(R)$ which denotes the set $\{r \in R \mid r s \in I$ for some $s \in R \backslash I\}$.
Proposition 3. Let $S$ be a multiplicatively closed subset of a ring $R$ such that $S \cap z d(R)=\emptyset$. Then the following hold.
(1) If $I$ is a semi $r$-ideal of $R$ such that $I \cap S=\emptyset$, then $S^{-1} I$ is a semi $r$-ideal of $S^{-1} R$.
(2) If $S^{-1} I$ is a semi $r$-ideal of $S^{-1} R$ and $S \cap Z_{I}(R)=\emptyset$, then $I$ is a semi $r$-ideal of $R$.
Proof. (1) Suppose for $\frac{a}{s} \in S^{-1} R$ that $\left(\frac{a}{s}\right)^{2} \in S^{-1} I$ and $\left(\frac{a}{s}\right) \notin S^{-1} I$. Then there exits $u \in S$ such that $u a^{2} \in I$ and so $(u a)^{2} \in I$. Since clearly $u a \notin I$ and $I$ is a semi $r$-ideal, we have $u a \in z d(R)$, say, $(u a) b=0$ for some $0 \neq b \in R$. Thus, $\frac{a}{s} \cdot \frac{b}{1}=\frac{u a b}{u s}=0_{S^{-1} R}$ and $\frac{b}{1} \neq 0_{S^{-1} R}$ as $S \cap z d(R)=\emptyset$. Thus, $\frac{a}{s} \in z d\left(S^{-1} R\right)$ and $S^{-1} I$ is a semi $r$-ideal of $S^{-1} R$.
(2) Suppose $a^{2} \in I$ for $a \in R$. Since $S^{-1} I$ is a semi $n$-ideal of $S^{-1} R$ and $\left(\frac{a}{1}\right)^{2} \in S^{-1} I$, we have either $\frac{a}{1} \in S^{-1} I$ or $\frac{a}{1} \in z d\left(S^{-1} R\right)$. If $\frac{a}{1} \in S^{-1} I$, then there exists $u \in S$ such that $u a \in I$. Since $S \cap z d(R)=\emptyset$, we conclude that $a \in I$. If $\frac{a}{1} \in z d\left(S^{-1} R\right)$, then there is $\frac{b}{t} \neq 0_{S^{-1} R}$ such that $\frac{a b}{t}=\frac{a}{1} \cdot \frac{b}{t}=0_{S^{-1} R}$. Hence, $v a b=0$ for some $v \in S$ and so $a b=0$ as $S \cap z d(R)=\emptyset$. Thus, $a \in z d(R)$ as $b \neq 0$ and $I$ is a semi $r$-ideal of $R$.

We recall that if $f=\sum_{i=1}^{m} a_{i} x^{i} \in R[x]$, then the ideal $\left\langle a_{1}, a_{2}, \cdots, a_{m}\right\rangle$ of $R$ generated by the coefficients of $f$ is called the content of $f$ and is denoted by $c(f)$. It is
well known that if $f$ and $g$ are two polynomials in $R[x]$, then the content formula $c(g)^{m+1} c(f)=c(g)^{m} c(f g)$ holds where $m$ is the degree of $f$, [9, Theorem 28.1]. For an ideal $I$ of $R$, it can be easily seen that $I[x]=\{f(x) \in R[x]: c(f) \subseteq I\}$.
Definition 2. $A$ ring $R$ is said to satisfy the property (*) if whenever $f \in \operatorname{reg}(R[x])$, then $c(f) \backslash\{0\} \subseteq \operatorname{reg}(R)$.

Theorem 4. Let $I$ be an ideal of a ring $R$.
(1) If $I[x]$ is a semi $r$-ideal of $R[x]$, then $I$ is a semi $r$-ideal of $R$.
(2) If $R$ satisfies the property (*) and $I$ is a semi $r$-ideal of $R$, then $I[x]$ is a semi $r$-ideal of $R[x]$

Proof. (1) Suppose $I[x]$ is a semi $r$-ideal of $R[x]$. Let $a \in R$ such that $a^{2} \in I$ and $A n n_{R}(a)=0$. Then Clearly, $a^{2} \in I[x]$ and $A n n_{R[x]}(a)=0$. By assumption, $a \in I[x]$ and so $a \in I$ as required.
(2) Suppose $R$ satisfies the property $(*)$ and $I$ is a semi $r$-ideal of $R$. Let $f(x) \in$ $R[x]$ such that $(f(x))^{2} \in I[x]$ and $\operatorname{Ann}_{R[x]}(f(x))=0$. Then $c\left(f^{2}\right) \subseteq I$ and so by the content formula, $(c(f))^{2}=c\left(f^{2}\right) \subseteq I$. Moreover, $c(f) \cap z d(R)=\{0\}$ as $R$ satisfies the property $(*)$ and so $c(f) \subseteq I$ by Corollary 1 It follows that $f(x) \in I[x]$ and we are done.

In general, if $S$ is an overring of a ring $R$, then we may find a semi $r$-ideal $J$ of $S$ where $J \cap R$ is not a semi $r$-ideal in $R$.

Example 3. Let $S=\mathbb{Z} \times \mathbb{Z}$ and consider the ring homomorphism $\varphi: \mathbb{Z} \longrightarrow \mathbb{Z} \times \mathbb{Z}$ defined by $\varphi(x)=(x, 0)$. Then $\varphi$ is a monomorphism and so $R=\varphi(\mathbb{Z})$ is a domain. Now, $J=A n n_{S}((0,1)$ ) is a nonzero (semi) r-ideal in $S$. However, clearly, $R \subseteq J$ and so $J \cap R=R$ is not a semi $r$-ideal in $R$.

Let $S$ be an overring ring of a ring $R$. Following [15, $R$ is said to be essential in $S$ if $J \cap R \neq\{0\}$ for every nonzero ideal $J$ of $S$.
Proposition 4. Let $R \subseteq S$ be rings such that $R$ is essential in $S$. If $J$ is a semi $r$ -ideal of $S$, then $J \cap R$ is a semi $r$-ideal in $R$.
Proof. Let $a \in R$ such that $a^{2} \in J \cap R$ and $A n n_{R}(a)=0$. Then $a \in S$ with $a^{2} \in J$ and $A n n_{S}(a)=0$. Indeed, if $A n n_{S}(a) \neq 0$, then $R$ being essential implies $A n n_{S}(a) \cap R \neq\{0\}$. Thus, there exists $0 \neq r \in R$ such that $r \in A n n_{S}(a)$ and so $r \in A n n_{R}(a)$, a contradiction. Since $J$ is a semi $r$-ideal of $S$, then $a \in J \cap R$ and the result follows.,

The rest of this section is devoted to discuss semi $r$-ideals of cartesian products of rings and their particular subrings: the amalgamation rings.
Proposition 5. Let $R=R_{1} \times R_{2}$ where $R_{1}$ and $R_{2}$ are two rings and $I_{1}, I_{2}$ be proper ideals of $R_{1}$ and $R_{2}$, respectively. Then $I_{1} \times R_{2}$ (resp. $R_{1} \times I_{2}$ ) is a semi $r$-ideal of $R$ if and only if $I_{1}$ is a semi $r$-ideal of $R_{1}$ (resp. $I_{2}$ is a semi r-ideal of $R_{2}$ ).

Proof. Let $I_{1} \times R_{2}$ be a semi $r$-ideal of $R$ and $a \in R_{1}$ with $a^{2} \in I_{1}$ and $\operatorname{Ann}_{R_{1}}(a)=0$. Then $(a, 1)^{2} \in I_{1} \times R_{2}$ and $A n n_{R}(a, 1)=(0,0)$ imply that $(a, 1) \in I_{1} \times R_{2}$ and so $a \in I_{1}$. Thus $I_{1}$ is a semi $r$-ideal of $R_{1}$. Conversely, suppose that $(a, b)^{2} \in I_{1} \times R_{2}$ and $A n n_{R}(a, b)=(0,0)$. Then $a^{2} \in I_{1}$ and clearly $A n n_{R_{1}}(a)=0$ which implies $a \in I_{1}$. Hence, $(a, b) \in I_{1} \times R_{2}$, so we are done. The proof of the case $R_{1} \times I_{2}$ is similar.

The following corollary generalizes Proposition 5 .
Corollary 4. Let $R_{1}, R_{2}, \cdots, R_{n}$ be rings, $R=R_{1} \times R_{2} \times \cdots \times R_{n}$ and $I_{i}$ be a proper ideal of $R_{i}$ for each $i=1,2, \cdots n$. Then for all $j=1,2, \cdots n$, $I=$ $R_{1} \times \cdots \times R_{j-1} \times I_{j} \times R_{j+1} \times \cdots \times R_{n}$ is a semi $r$-ideal of $R$ if and only if $I_{j}$ is a semi $r$-ideal of $R_{j}$.

Theorem 5. Let $R_{1}$ and $R_{2}$ be two rings, $R=R_{1} \times R_{2}$ and $I_{1}, I_{2}$ be proper ideals in $R_{1}$ and $R_{2}$, respectively.
(1) If $I_{1}$ and $I_{2}$ are semi $r$-ideals of $R_{1}$ and $R_{2}$, respectively, then $I=I_{1} \times I_{2}$ is a semi $r$-ideal of $R$.
(2) If $I=I_{1} \times I_{2}$ is a semi $r$-ideal of $R$, then either $I_{1}$ is a semi $r$-ideal of $R_{1}$ or $I_{2}$ is a semi $r$-ideal of $R_{2}$.
(3) If $I=I_{1} \times I_{2}$ is a semi $r$-ideal of $R$ and $I_{2} \nsubseteq z d\left(R_{2}\right)$, then $I_{1}$ is a semi $r$-ideal of $R_{1}$.
(4) If $I=I_{1} \times I_{2}$ is a semi $r$-ideal of $R$ and $I_{1} \nsubseteq z d\left(R_{1}\right)$, then $I_{2}$ is a semi $r$-ideal of $R_{2}$.

Proof. (1) Let $(a, b) \in R$ such that $\left(a^{2}, b^{2}\right)=(a, b)^{2} \in I$ and $A n n_{R}(a, b)=(0,0)$. Then $a^{2} \in I_{1}, b^{2} \in I_{2}$ and clearly $\operatorname{Ann}_{R_{1}}(a)=A n n_{R_{2}}(b)=0$. Therefore, $a \in I_{1}$, $b \in I_{2}$ and so $(a, b) \in I$ as needed.
(2). Suppose $I=I_{1} \times I_{2}$ is a semi $r$-ideal of $R$ but $I_{1}$ and $I_{2}$ are not semi $r$ ideals of $R_{1}$ and $R_{2}$, respectively. Choose $a \in R_{1}$ and $b \in R_{2}$ such that $a^{2} \in I_{1}$, $b^{2} \in I_{2}, A n n_{R 1}(a)=0$ and $A n n_{R_{2}}(b)=0$ but $a \notin I_{1}$ and $b \notin I_{2}$. Then $(a, b)^{2} \in I$ and clearly, $\operatorname{Ann}_{R}(a, b)=(0,0)$. By assumption, we have $(a, b) \in I$ which is a contradiction. Therefore, either $I_{1}$ is a semi $r$-ideal of $R_{1}$ or $I_{2}$ is a semi $r$-ideal of $R_{2}$.
(3) Suppose $a^{2} \in I_{1}$ for some $a \in R_{1}$ with $\operatorname{Ann}_{R_{1}}(a)=0$. Since $I_{2} \nsubseteq Z\left(R_{2}\right)$, we can choose $b \in I_{2} \cap \operatorname{reg}\left(R_{2}\right)$. Then $(a, b)^{2} \in I$ and $A n n_{R}(a, b)=(0,0)$. It follows that $(a, b) \in I$; and hence $a \in I_{1}$.
(4) is similar to (3).

The converse of Theorem 5(1) is not true in general. For example, $4 \mathbb{Z} \times 0$ is a semi $r$-ideal in $\mathbb{Z} \times \mathbb{Z}$ by Proposition 2. On the other hand, the ideal $4 \mathbb{Z}$ is not a semi $r$-ideals of $\mathbb{Z}$.

The following corollary generalizes Theorem 5 to any finite direct product of rings. The proof is similar to that of Theorem [5]

Corollary 5. Let $R_{1}, R_{2}, \cdots, R_{n}$ be rings, $R=R_{1} \times R_{2} \times \cdots \times R_{n}$ and $I_{i}$ be $a$ proper ideal of $R_{i}$ for each $i=1,2, \cdots n$.
(1) If $I_{i}$ is a semi $r$-ideals of $R_{i}$ for each $i=1,2, \cdots n$, then $I=I_{1} \times I_{2} \times \cdots \times I_{n}$ is a semi $r$-ideal of $R$.
(2) If $I=I_{1} \times I_{2} \times \cdots \times I_{n}$ is a semi $r$-ideal of $R$, then $I_{j}$ is a semi $r$-ideal of $R_{j}$ for at least one $j \in\{1,2, \cdots, n\}$.
(3) If $I=I_{1} \times I_{2} \times \cdots \times I_{n}$ is a semi $r$-ideal of $R$ and $I_{j} \nsubseteq Z\left(R_{j}\right)$ for all $j \neq i$, then $I_{i}$ is a semi $r$-ideal of $R_{i}$.

Lemma 2. Let $R=R_{1} \times R_{2} \times \cdots \times R_{n}$ where $R_{i}$ 's are rings and $R_{j}$ is reduced ring for some $j=1, \ldots, n$. If $I_{i}$ is an ideal of $R_{i}$ for all $i \neq j$, then $I=I_{1} \times \cdots \times$ $I_{j-1} \times 0 \times I_{j+1} \times \cdots \times I_{n}$ is a semi $r$-ideal of $R$.

Proof. Let $a=\left(a_{1}, a_{2}, \ldots, a_{n}\right) \in R$ with $a^{2} \in I$. Then $a_{j}^{2}=0$ which implies $a_{j}=0$ as $R_{j}$ is reduced. Since $A n n_{R}(a)=\operatorname{Ann}_{R}\left(a_{1}, \ldots, a_{j-1}, 0, a_{j+1}, \ldots, a_{n}\right) \neq 0, I$ is a semi $r$-ideal of $R$.

Next, we present a characterization for semi $r$-ideals of cartesian products of domains.

Theorem 6. Let $R_{1}, R_{2}, \cdots, R_{n}(n \geq 2)$ be domains, $R=R_{1} \times R_{2} \times \cdots \times R_{n}$ and $I_{i}$ be an ideal of $R_{i}$ for each $i=1,2, \cdots n$. Then $I=I_{1} \times I_{2} \times \cdots \times I_{n}$ is a semi $r$-ideal of $R$ if and only if one of the following statements holds
(1) $I_{j}=\{0\}$ for at least one $j \in\{1,2, \cdots, n\}$.
(2) There exists $j \in\{1,2, \cdots n\}$ such that $I_{i}$ is a semi $r$-ideal of $R_{i}$ for all $i=1, \cdots, j$ and $I_{i}=R_{i}$ for all $i=j+1, \cdots, n$.
(3) $I_{i}$ is a semi $r$-ideals of $R_{i}$ for each $i=1,2, \cdots n$.

Proof. Suppose $I=I_{1} \times I_{2} \times \cdots \times I_{n}$ is a semi $r$-ideal of $R$. Suppose that all $I_{i}$ 's are nonzero. If for all $i \in\{1,2, \cdots n\}, I_{i}$ is proper in $R_{i}$, then $I_{i}$ is a semi $r$-ideals of $R_{i}$ by Corollary 5(3). Without loss of generality assume that $I_{1}, \ldots, I_{j}$ are proper in $R_{1}, \cdots, R_{j}$, respectively and $I_{i}=R_{i}$ for all $i \in\{j+1, \ldots, n\}$. For each $i \in\{2, \ldots, j\}$, choose a nonzero element $b_{i} \in I_{i}$. Let $a \in R_{1}$ such that $a^{2} \in I_{1}$. Since $\left(a, b_{2}, b_{3}, \ldots b_{j}, 1_{R_{j+1}}, \ldots, 1_{R_{n}}\right)^{2} \in I$ and $A n n_{R}\left(a, b_{2}, b_{3}, \ldots b_{j}, 1_{R_{j+1}}, \ldots, 1_{R_{n}}\right)=0$, we have $\left(a, b_{2}, b_{3}, \ldots b_{j}, 1_{R_{j+1}}, \ldots, 1_{R_{n}}\right) \in I$ and so $a \in I_{1}$. Therefore, $I_{1}$ is a semi $r$-ideal of $R_{1}$. Similarly, $I_{i}$ is a semi $r$-ideals of $R_{i}$ for all $i \in\{1, \ldots, j\}$.

Conversely, if (1) holds, then $I$ is clearly a semi $r$-ideal of $R$. Suppose that $I_{1}, \ldots, I_{j}$ are semi $r$-ideals and $I_{k}=R_{k}$ for all $k \in\{j+1, \ldots, n\}$. Let $a=$ $\left(a_{1}, a_{2}, \ldots, a_{n}\right) \in R$ with $a^{2} \in I$ and $A n n_{R}(a)=0$. Then for each $i \in\{1, \ldots, j\}$, $a_{i}^{2} \in I$ and $A n n_{R_{i}}\left(a_{i}\right)=0$ as $R_{i}$ 's are domain. Thus, $a_{i} \in I_{i}$ and so $a \in I$. Finally, if (3) holds, then $I=I_{1} \times I_{2} \times \cdots \times I_{n}$ is a semi $r$-ideal of $R$ by Corollary [5).

Let $R$ and $S$ be two rings, $J$ be an ideal of $S$ and $f: R \rightarrow S$ be a ring homomorphism. As a subring of $R \times S$, the amalgamation of $R$ and $S$ along $J$ with respect to $f$ is defined by $\left.R \bowtie^{f} J=(a, f(a)+j): a \in R, j \in J\right\}$. If $f$ is the identity homomorphism on $R$, then we get the amalgamated duplication of $R$ along an ideal $J, R \bowtie J=\{(a, a+j): a \in R, j \in J\}$. For more related definitions and several properties of this kind of rings, one can see [6. If $I$ is an ideal of $R$ and $K$ is an ideal of $f(R)+J$, then $I \bowtie^{f} J=\{(i, f(i)+j): i \in I, j \in J\}$ and $\bar{K}^{f}=\{(a, f(a)+j): a \in R, j \in J, f(a)+j \in K\}$ are ideals of $R \bowtie^{f} J, 7$.
Lemma 3. 3] Let $R, S, J$ and $f$ be as above. Let $A=\{(r, f(r)+j) \mid r \in z d(R)\}$ and $B=\left\{(r, f(r)+j) \mid j^{\prime}(f(r)+j)=0\right.$ for some $\left.j^{\prime} \in J \backslash\{0\}\right\}$. Then $z d\left(R \bowtie^{f} J\right) \subseteq A \cup B$.

Next, we determine conditions under which $I \bowtie^{f} J$ and $\bar{K}^{f}$ are semi $r$-ideals of $R \bowtie^{f} J$.

Theorem 7. Let $R, S, J$ and $f$ be as above. If $I$ is a semi r-ideal of $R$, then $I \bowtie^{f} J$ is a semi r-ideal of $R \bowtie^{f} J$. The converse is true if $f(r e g(R)) \cap Z(J)=\emptyset$

Proof. Suppose $I$ is a semi $r$-ideal of $R$. Let $(a, f(a)+j) \in R \bowtie^{f} J$ such that $(a, f(a)+j)^{2}=\left(a^{2}, f\left(a^{2}\right)+2 j f(a)+j^{2}\right) \in I \bowtie^{f} J$ and $(a, f(a)+j) \notin z d\left(R \bowtie^{f} J\right)$. Then $a^{2} \in I$ and $a \notin z d(R)$ by Lemma 3. Therefore, $a \in I$ and so $(a, f(a)+j) \in$ $I \bowtie^{f} J$ as needed. Now, suppose $f(r e g(R)) \cap Z(J)=\emptyset$ and $I \bowtie^{f} J$ is a semi $r$-ideal of $R \bowtie^{f} J$. Let $a^{2} \in I$ for $a \in R$ and $a \notin z d(R)$. Then $(a, f(a)) \in R \bowtie^{f} J$
with $(a, f(a))^{2}=\left(a^{2}, f\left(a^{2}\right)\right) \in I \bowtie^{f} J$. If $(a, f(a)) \in z d\left(R \bowtie^{f} J\right)$, then Lemma 3 implies $f(a) \in Z(J)$ which is a contradiction. Therefore, $(a, f(a)) \notin z d\left(R \bowtie^{f} J\right)$ and so $(a, f(a)) \in I \bowtie^{f} J$ as $I \bowtie^{f} J$ is a semi $r$-ideal of $R \bowtie^{f} J$. Thus, $a \in I$ as required.

Theorem 8. Let $f: R \rightarrow S$ be a ring homomorphism and $J, K$ be ideals of $S$. If $K$ is a semi r-ideal of $f(R)+J$, then $\bar{K}^{f}$ is a semi r-ideal of $R \bowtie^{f} J$.
(1) If $K$ is a semi $r$-ideal of $f(R)+J$ and $z d(f(R)+J)=Z(J)$, then $\bar{K}^{f}$ is a semi $r$-ideal of $R \bowtie^{f} J$.
(2) If $\bar{K}^{f}$ is a semi $r$-ideal of $R \bowtie^{f} J, f(z d(R)) \subseteq z d(f(R)+J)$ and $f(z d(R)) J=$ 0 , then $K$ is a semi $r$-ideal of $f(R)+J$.

Proof. (1) Suppose $K$ is a semi $r$-ideal of $f(R)+J$. Let $(a, f(a)+j) \in R \bowtie^{f} J$ such that $(a, f(a)+j)^{2}=\left(a^{2},(f(a)+j)^{2}\right) \in \bar{K}^{f}$ and $(a, f(a)+j) \notin z d\left(R \bowtie^{f} J\right)$. Then $(f(a)+j)^{2} \in K$ and by Lemma 3, $f(a)+j \notin Z(J)=z d(f(R)+J)$. Therefore, $f(a)+j \in K$ and $(a, f(a)+j) \in \bar{K}^{f}$ as needed.
(2) Suppose $\bar{K}^{f}$ is a semi $r$-ideal of $R \bowtie^{f} J$ and $f(z d(R)) J=0$. Let $f(a)+j \in$ $f(R)+J$ such that $(f(a)+j)^{2} \in K$ and $f(a)+j \notin z d(f(R)+J)$. Then $(a, f(a)+j) \in$ $R \bowtie^{f} J$ with $(a, f(a)+j)^{2} \in \bar{K}^{f}$. Suppose $(a, f(a)+j) \in z d\left(R \bowtie^{f} J\right)$. Then as $Z(J) \subseteq z d(f(R)+J)$ and by Lemma 3, we conclude that $a \in z d(R)$. Since $f(a) \in$ $z d(f(R)+J)$, then $f(a) f(b)=0$ for some $0 \neq f(b) \in f(R)$. Thus, $(f(a)+j) f(b)=0$ as $f(z d(R)) J=0$ which contradicts that $f(a)+j \notin z d(f(R)+J)$. Therefore, $(a, f(a)+j) \notin z d\left(R \bowtie^{f} J\right)$ and so $(a, f(a)+j) \in \bar{K}^{f}$. It follows that $f(a)+j \in K$ and $K$ is a semi $r$-ideal of $f(R)+J$.

## 3. SEMI $r$-SUBMODULES OF MODULES OVER COMMUTATIVE RINGS

The aim of this section is to extend semi $r$-ideals of commutative rings to semi $r$-submodules of modules over commutative rings. Recall that a module $M$ is said to be faithful if $A n n_{R}(M)=\left(0:_{R} M\right)=0_{R}$.

Definition 3. Let $M$ be an $R$-module and $N$ a proper submodule of $M$.
(1) $N$ is called a semiprime submodule if whenever $r^{2} m \in N$, then $r m \in N$. 16
(2) $N$ is called a $r$-submodule if whenever $r m \in N$ and $A n n_{M}(r)=0_{M}$, then $m \in N .13$
(3) $N$ is called a $s r$-submodule if whenever $r m \in N$ and $A n n_{R}(m)=0$, then $m \in N$. 13

Definition 4. Let $M$ be an $R$-module and $N$ a proper submodule of $M$. We call $N$ a semi $r$-submodule if whenever $r \in R, m \in M$ with $r^{2} m \in N, \operatorname{Ann}_{M}(r)=0_{M}$ and $A n n_{R}(m)=0$, then $r m \in N$.

The reader clearly observe that any semi $r$-submodule of an $R$-module $R$ is a semi $r$-ideal of $R$. The zero submodule is always a semi $r$-submodule of $M$. Also, see the implications:
$r$-submodule
$s r$-submodule $\quad \begin{aligned} & \searrow \\ & \\ & \\ & \\ & \\ & \end{aligned}$ semi $r$-submodule
semiprime submodule

However, the next examples show that these arrows are irreversible.

## Example 4.

(1) Consider the submodule $N=6 \mathbb{Z} \times\langle 0\rangle$ of the $\mathbb{Z}$-module $M=\mathbb{Z} \times \mathbb{Z}$. Let $r \in \mathbb{Z}$ and $m=\left(m_{1}, m_{2}\right) \in M$ such that $r^{2} \cdot\left(m_{1}, m_{2}\right) \in N$. Then $r^{2} m_{1} \in 6 \mathbb{Z}$, $r^{2} m_{2}=0$ and $A n n_{\mathbb{Z}}(r)=A n n_{\mathbb{Z}}\left(m_{1}\right)=A n n_{\mathbb{Z}}\left(m_{2}\right)=0$ as $\mathbb{Z}$ is a domain. Since $6 \mathbb{Z}$ and $\langle 0\rangle$ are semi $r$-ideals of $\mathbb{Z}$, then $r \cdot\left(m_{1}, m_{2}\right) \in N$ and so $N$ is a semi $r$-submodule of $M$. On the other hand, we have $2 \cdot(3,0) \in N$ with $A n n_{M}(2)=0_{M}$ and $A n n_{\mathbb{Z}}((3,0))=0$ but $(3,0) \notin N$ and so $N$ is neither $r$-submodule nor $s r$-submodule of $M$.
(2) Consider the submodule $N=\langle\overline{4}\rangle \times\langle 0\rangle$ of the $\mathbb{Z}$-module $M=\mathbb{Z}_{8} \times \mathbb{Z}$. Let $r \in \mathbb{Z}$ and $m=\left(m_{1}, m_{2}\right) \in M$ such that $r^{2} \cdot\left(m_{1}, m_{2}\right) \in N$. Then it is clear to observe that $A n n_{\mathbb{Z}}(r)=A n n_{\mathbb{Z}}\left(m_{1}\right)=A n n_{\mathbb{Z}}\left(m_{2}\right)=0$. Since again $N$ is a semi $r$-submodule of $M$ as $\langle\overline{4}\rangle$ is a semi $r$-ideal of $\mathbb{Z}_{8}$ and $\langle 0\rangle$ is a semi $r$-ideals of $\mathbb{Z}$. However, $2^{2} \cdot(\overline{1}, 0) \in N$ but $2 \cdot(\overline{1}, 0) \notin N$ and so $N$ is not a semiprime submodule of $M$.
Proposition 6. Let $M$ be an $R$-module, $N$ a proper submodule of $M$ and $k$ any positive integer. Then $N$ is a semi r-submodule of $M$ if and only if whenever $r \in R$, $m \in M$ with $r^{k} m \in N, A n n_{M}(r)=0_{M}$ and $A n n_{R}(m)=0$, then $r m \in N$.
Proof. The proof follows by mathematical induction on $k$ in a similar way to that of Theorem (3).

We recall that a module $M$ is torsion (resp. torsion-free) if $T(M)=M$ (resp. $T(M)=\{0\}$ ) where $T(M)=\{m \in M$ : there exists $0 \neq r \in R$ such that $r m=0\}$. It is clear that any torsion-free module is faithful.

Proposition 7. Semi r-submodules and semiprime submodules are coincide in any torsion-free module.

Proof. Since every semiprime submodule is semi $r$-submodule, we need to show the converse. Let $N$ be a semi $r$-submodule of an $R$-module $M, r \in R, m \in M$ with $r^{2} m \in N$. Keeping in mind that $M$ is torsion-free, we have $A n n_{R}(m)=0$. Now, suppose that $m^{\prime} \in A n n_{M}(r)$. Then $r m^{\prime}=0$ and if $r=0$, then clearly $r m \in N$. If $r \neq 0$, then $m^{\prime}=0$ again as $M$ is torsion-free. Since $N$ is a semi $r$-submodule, we conclude $r m \in N$, as required.

Definition 5. A proper submodule $N$ of an $R$-module $M$ is said to satisfy the $D$-annihilator condition if whenever $K$ is a submodule of $M$ and $r \in R$ such that $r K \subseteq N$ and $A n n_{M}(r)=0_{M}$, then either $K \subseteq N$ or $K \cap T(M)=\left\{0_{M}\right\}$.

Obviously, any $r$-submodule satisfies the $D$-annihilator condition. The converse is not true in general. For example the submodule $N=6 \mathbb{Z} \times\langle 0\rangle$ of the $\mathbb{Z}$-module $M=\mathbb{Z} \times \mathbb{Z}$ clearly satisfies the $D$-annihilator condition. On the other hand, $N$ is not an $r$-submodule of $M$, (see Example 4(1)). It is clear that any proper submodule of a torsion-free module satisfies the $D$-annihilator condition. However, we may find a submodule satisfying the $D$-annihilator condition in a torsion module. For example, for any positive integer $n$, every proper submodule of the $\mathbb{Z}$-module $\mathbb{Z}_{n}$ satisfies the $D$-annihilator condition. Indeed, suppose that $r m \in\langle\bar{d}\rangle$ for some integer $d$ dividing $n$. Put $n=c d$ then $c r \bar{m}=0$. Since $A n n_{M}(r)=0_{M}$, we get $c \bar{m}=0$ and so $\bar{m} \in\langle\bar{d}\rangle$.

Proposition 8. Let $N$ be a proper submodule of an $R$-module $M$ satisfying the $D$-annihilator condition. Then the following are equivalent.
(1) $N$ is a semi $r$-submodule of $M$.
(2) For $r \in R$ and a submodule $K$ of $M$ with $r^{2} K \subseteq N$ and $A n n_{M}(r)=0_{M}$, then $r K \subseteq N$.

Proof. (1) $\Rightarrow$ (2). Suppose that $r^{2} K \subseteq N$ and $A n n_{M}(r)=0_{M}=A n n_{M}\left(r^{2}\right)$. If $K \subseteq N$, then we are done. If $K \nsubseteq N$, then $A n n_{R}(k)=0_{R}$ for each $k \in K$ since by assumption $K \cap T(M)=\left\{0_{M}\right\}$. Since $N$ is a semi $r$-submodule, we conclude that $r k \in N$. Therefore, $r k \in N$ for all $k \in K$ and the result follows.
$(2) \Rightarrow(1)$. is straightforward.
Recall that an $R$-module $M$ is called a multiplication module if every submodule $N$ of $M$ has the form $I M$ for some ideal $I$ of $R$. Moreover, we have $N=\left(N:_{R}\right.$ $M) M$. Next, we conclude a useful characterization for semi $r$-submodules. First, recall the following lemmas.
Lemma 4. 17] Let $N$ be a submodule of a finitely generated faithful multiplication $R$-module $M$. For an ideal $I$ of $R,\left(I N:_{R} M\right)=I\left(N:_{R} M\right)$, and in particular, $\left(I M:_{R} M\right)=I$.
Lemma 5. [1] Let $N$ is a submodule of faithful multiplication $R$-module $M$. If $I$ is a finitely generated faithful multiplication ideal of $R$, then
(1) $N=\left(I N:_{M} I\right)$.
(2) If $N \subseteq I M$, then $\left(J N:_{M} I\right)=J\left(N:_{M} I\right)$ for any ideal $J$ of $R$.

Theorem 9. Let $M$ be a finitely generated faithful multiplication $R$-module. Then a submodule $N=I M$ satisfying the $D$-annihilator condition is a semi $r$-submodule of $M$ if and only if $I$ is a semi r-ideal of $R$.

Proof. Suppose $N=I M$ is a semi $r$-submodule of $M$ and let $r \in R$ such that $r^{2} \in I$ with $A n n_{R}(r)=0$. We claim that $\operatorname{Ann}_{M}(r)=0_{M}$. Indeed, if there is $0_{M} \neq m \in M$ such that $r m=0_{M}$, then $\langle r\rangle\left(\langle m\rangle:_{R} M\right)=\left(\langle r m\rangle:_{R} M\right)=\left(0_{M}:_{R} M\right)=0$ by Lemma4. Thus, $\left(\langle m\rangle:_{R} M\right)=0$ as $A n n_{R}(r)=0$ and then $\langle m\rangle=\left(\langle m\rangle:_{R} M\right) M=$ $0_{M}$, a contradiction. Since $N$ satisfies the $D$-annihilator condition and $r^{2} M \subseteq I M$, then $r M \subseteq I M$ by Proposition 8. Thus, $r \in\left(r M:_{R} M\right) \subseteq\left(I M:_{R} M\right)=I$, as needed.

Conversely, suppose that $I$ is a semi $r$-ideal of $R$. Let $r \in R$ and $K=J M$ be a submodule of $M$ such that $r^{2} J M=r^{2} K \subseteq I M$ and $A n n_{M}(r)=0_{M}$. Take $A=r J$ and note that $A^{2} \subseteq r^{2} J M: M \subseteq\left(I M:_{R} M\right)=I$ by Lemma 4. Now, we claim that $A \cap z d(R)=\{0\}$. Suppose on contrary that there exists $0 \neq a=r j \in A$ such that $A n n_{R}(a) \neq 0$. Choose $0 \neq b \in R$ with $a b=r j b=0$. Then $r j b M=0_{M}$ and so $j b M=0_{M}$ as $A n n_{M}(r)=0_{M}$. Since $b \neq 0, j M \subseteq K$ and $N$ satisfies the $D$-annihilator condition, then $j M=0$ and we conclude $j=0$ as $M$ is faithful, which is a contradiction. Therefore, $A \cap z d(R)=\{0\}$ and $A \subseteq I$ by Corollary 1 Thus, $r K=r J M=A M \subseteq I M=N$ as needed.

In view of Theorem 9 we give the following characterization.
Corollary 6. Let $R$ be a ring and $M$ be a finitely generated faithful multiplication $R$-module. For a submodule $N$ of $M$ satisfying the $D$-annihilator condition, the following statements are equivalent.
(1) $N$ is a semi $r$-submodule of $M$.
(2) $\left(N:_{R} M\right)$ is semi $r$-ideal of $R$.
(3) $N=I M$ for some semi $r$-ideal $I$ of $R$.

Let $N$ be a submodule of an $R$-module $M$ and $I$ be an ideal of $R$. The residual of $N$ by $I$ is the set $\left(N:_{M} I\right)=\{m \in M: I m \subseteq N\}$. It is clear that $\left(N:_{M} I\right)$ is a submodule of $M$ containing $N$. More generally, for any subset $S \subseteq R,\left(N:_{M} S\right)$ is a submodule of $M$ containing $N$. We recall that $M-\operatorname{rad}(N)$ denotes the intersection of all prime submodules of $M$ containing $N$. Moreover, if $M$ is finitely generated faithful multiplication, then $M-\operatorname{rad}(N)=\sqrt{\left(N:_{R} M\right)} M$, 17.

Proposition 9. Let $M$ be a finitely generated multiplication $R$-module and $N$ be a semi r-submodule of $M$ satisfying the $D$-annihilator condition.
(1) For any ideal $I$ of $R$ with $\left(N:_{M} I\right) \neq M,\left(N:_{M} I\right)$ is a semi $r$-submodule of $M$.
(2) If $M$ is faithful, then $\left(M-\operatorname{rad}(N):_{R} M\right) \subseteq z d(R) \cup \sqrt{\left(N:_{R} M\right)}$.

Proof. (1) First, we show that $\left(N:_{M} I\right)$ satisfies the $D$-annihilator condition. Let $K$ be a submodule of $M$ and $r \in R$ such that $r K \subseteq\left(N:_{M} I\right), K \nsubseteq\left(N:_{M} I\right)$ and $A n n_{M}(r)=0_{M}$. Then $r I K \subseteq N$ and so $I K \cap T(M)=\left\{0_{M}\right\}$. It follows clearly that $K \cap T(M)=\left\{0_{M}\right\}$ as needed. Suppose $N$ is a semi $r$-submodule of $M$. Let $K$ be a submodule of $M$ such that $r^{2} K \subseteq\left(N:_{M} I\right)$ and $A n n_{M}(r)=0_{M}$. Then $r^{2} I K \subseteq N$ which implies that $r I K \subseteq N$ by Proposition 8 and thus, $r K \subseteq\left(N:_{M} I\right)$. Therefore, $\left(N:_{M} I\right)$ is a semi $r$-submodule of $M$ again by Proposition 8 .
(2) Since $N$ be a semi $r$-submodule, $\left(N:_{R} M\right)$ is a semi $r$-ideal of $R$ by Corollary 6. Then the claim follows as $M-\operatorname{rad}(N)=\sqrt{\left(N:_{R} M\right)} M$ and by using Theorem 1(4).

Next, we discuss when $I N$ is a semi $r$-submodule of a finitely generated multiplication module $M$ where $I$ is an ideal of $R$ and $N$ is a submodule of $M$. Recall that a submodule $N$ of an $R$-module $M$ is said to be pure if $J N=J M \cap N$ for every ideal $J$ of $R$.

Theorem 10. Let $I$ be an ideal of a ring $R, M$ be a finitely generated faithful multiplication $R$-module and $N$ be a submodule of $M$ such that $I N$ satisfies the $D$-annihilator condition.
(1) If $I$ is a semi $r$-ideal of $R$ and $N$ is a pure semi $r$-submodule of $M$, then $I N$ is a semi $r$-submodule of $M$.
(2) Let $I$ be a finitely generated faithful multiplication ideal of $R$. If $I N$ is semi $r$-submodule of $M$, then either $I$ is a semi $r$-ideal of $R$ or $N$ is a semi $r$-submodule of $M$.

Proof. (1) Suppose that $r^{2} K \subseteq I N$ and $A n n_{M}(r)=0_{M}$ for some $r \in R$ and a submodule $K=J M$ of $M$. If we take $A=r J$, then $A^{2} \subseteq r^{2} J M: M \subseteq(I N: M)=$ $I(N: M) \subseteq I \cap(N: M)$. By Theorem $9\left(N:_{R} M\right)$ is a semi $r$-ideal. We show that $A \cap z d(R)=\{0\}$. Let $x \in A \cap z d(R)$, say, $x=r y$ for some $y \in J$. Choose a nonzero $z \in R$ such that $x z=r y z=0$. Then $r y z M=0_{M}$ and since $A n n_{M}(r)=0_{M}$, we have $y z M=0_{M}$. Since $M$ is faithful and $z \neq 0$, we conclude that $y M=0_{M}$ and so $y=0$. Thus $x=0$, as required. Since $\left(N:_{R} M\right)$ is a semi $r$-ideal, then $A \subseteq\left(N:_{R} M\right)$ by Corollary 1 Therefore, $r K=A M \subseteq\left(N:_{R} M\right) M=N$. On the
other hand, since $I$ is also a semi $r$-ideal, we have $A \subseteq I$ and so $r K=A M \subseteq I M$. Since $N$ is pure, we conclude that $r K \subseteq I M \cap N=I N$ and we are done.
(2) First, by using Lemma 5e note clearly that $N$ satisfies the $D$-annihilator condition. We have two cases.

Case I. Let $N=M$. Then $I=I\left(N:_{R} M\right)=\left(I N:_{R} M\right)$ is a semi $r$-ideal of $R$ by Corollary 6

Case II. Let $N$ be proper. Observe that by Lemma 5, we have the equality $\left(N:_{R} M\right)=\left(\left(I N:_{M} I\right):_{R} M\right)=\left(I\left(N:_{R} M\right):_{M} I\right)$. Suppose that $r^{2} \in\left(N:_{R} M\right)$ and $r \notin z d(R)$. Then $(r I)^{2} \subseteq r^{2} I \subseteq I\left(N:_{R} M\right)=\left(I N:_{R} M\right)$ by Lemma 4. Here, similar to the proof of Theorem 9, it can be easily verify that $r I \cap z d(R)=\{0\}$. Since $\left(I N:_{R} M\right)$ is a semi $r$-ideal, $r I \subseteq\left(I N:_{R} M\right)=I\left(N:_{R} M\right)$ which means $r \in\left(I\left(N:_{R} M\right):_{M} I\right)=\left(N:_{R} M\right)$ by Lemma 5. Thus, $\left(N:_{R} M\right)$ is a semi $r$-ideal of $R$ and Corollary 6 implies that $N$ is a semi $r$-submodule of $M$.

Next, we study the behavior of the semi $r$-submodule property under module homomorphisms.

Proposition 10. Let $M$ and $M^{\prime}$ be $R$-modules and $f: M \rightarrow M^{\prime}$ be an $R$-module homomorphism.
(1) If $f$ is an epimorphism and $N$ is a semi $r$-submodule of $M$ such that $\operatorname{Ker}(f) \subseteq N$ and $N \cap T(M)=\left\{0_{M}\right\}$, then $f(N)$ is a semi $r$-submodule of $M^{\prime}$.
(2) If $f$ is an isomorphism and $N^{\prime}$ is a semi $r$-submodule of $M^{\prime}$, then $f^{-1}\left(N^{\prime}\right)$ is a semi $r$-submodule of $M$.

Proof. (1). Let $N$ be a semi $r$-submodule of $M$ and $r \in R, m^{\prime}:=f(m) \in M^{\prime}$ $(m \in M)$ such that $r^{2} m^{\prime} \in f(N), A n n_{M^{\prime}}(r)=0_{M}$, and $A n n_{R}(f(m))=0_{M}$, . Then $r^{2} m \in N$ as $\operatorname{Ker}(f) \subseteq N$. We show that $A n n_{M}(r)=0_{M}$. If $r=0$, then the claim is obvious. Suppose $r \neq 0$ and there is $m_{1} \in M$ such that $r m_{1}=$ $0_{M}$. Then $r f\left(m_{1}\right)=0_{M^{\prime}}$ and so $f\left(m_{1}\right)=0_{M^{\prime}}$ as $A n n_{M^{\prime \prime}}(r)=0_{M^{\prime}}$. Thus, $m_{1} \in \operatorname{Ker}(f) \cap T(M) \subseteq N \cap T(M)=\left\{0_{M}\right\}$ as needed. Also, it is clear that $A n n_{R}(m)=0_{M}$. Therefore, $r m \in N$ and so $r m^{\prime} \in f(N)$ as required.
(2). Let $N^{\prime}$ is a semi $r$-submodule of $M^{\prime}$. Suppose that $r^{2} m \in f^{-1}\left(N^{\prime}\right)$, $A n n_{M}(r)=0_{M}$ and $A n n_{R}(m)=0$ for some $r \in R$ and $m \in M$. Then $r^{2} f(m)=$ $f\left(r^{2} m\right) \in N^{\prime}, A n n_{M^{\prime}}(r)=0_{M^{\prime}}$ and $A n n_{R}(f(m))=0$. Indeed, if $r m^{\prime}=0$ for some $0 \neq m^{\prime}=f\left(m_{1}\right) \in M^{\prime}$, then $r m_{1} \in K \operatorname{erf}=\left\{0_{M}\right\}$ and clearly $0 \neq m_{1} \in M$, a contradiction. Similarly, if there exists $0 \neq c \in R$ such that $c f(m)=0_{M^{\prime}}$, then $\mathrm{cm}=0_{M}$ which is also a contradiction. Since $N^{\prime}$ is a semi $R$-submodule, then $r f(m) \in N^{\prime}$ and so $r m \in f^{-1}\left(N^{\prime}\right)$. Thus, $f^{-1}\left(N^{\prime}\right)$ is a semi $r$-submodule of $M$.

In the following, we discuss semi $r$-submodules of localizations of modules. Here, the notation $Z_{N}(R)$ denotes the set $\{r \in R: r m \in N$ for some $m \in M \backslash N\}$.

Theorem 11. Let $S$ be a multiplicatively closed subset of $a$ ring $R$ and $M$ be an $R$-module such that $S \cap Z(M)=\emptyset$.
(1) If $N$ is a semi $r$-submodule of $M$ such that $\left(N:_{R} M\right) \cap S=\emptyset$, then $S^{-1} N$ is a semi $r$-submodule of $S^{-1} M$.
(2) If $S^{-1} N$ is a semi $r$-submodule of $S^{-1} R$ and $S \cap Z_{N}(R)=\emptyset$, then $N$ is a semi $r$-submodule of $M$.

Proof. (1) Let $\left(\frac{r}{s}\right)^{2}\left(\frac{m}{t}\right) \in S^{-1} N$ with $A n n_{S^{-1} M}\left(\frac{r}{s}\right)=0_{S^{-1} M}$ and $A n n_{S^{-1} R}\left(\frac{m}{t}\right)=$ $0_{S^{-1} R}$ for some $\frac{r}{s} \in S^{-1} R$ and $\frac{m}{t} \in S^{-1} M$. Choose $u \in S$ such that $r^{2}(u m) \in N$. We show that $\operatorname{Ann}_{M}(r)=0_{M}$ and $A n n_{R}(u m)=0$. First, assume that $r m^{\prime}=0_{M}$ for some $m^{\prime} \in M$. Then $\left(\frac{r}{s}\right)\left(\frac{m^{\prime}}{1}\right)=0_{S^{-1} M}$ and so $\frac{m^{\prime}}{1}=0_{S^{-1} M}$ as $A n n_{S^{-1} M}\left(\frac{r}{s}\right)=$ $0_{S^{-1} M}$. Hence, there exists $v \in S$ such that $v m^{\prime}=0_{M}$. Since $S \cap Z(M)=\emptyset$, then $m^{\prime}=0_{M}$ and so $\operatorname{Ann}_{M}(r)=0_{M}$. Secondly, assume that $r^{\prime} u m=0$ for some $r^{\prime} \in R$. Then $\frac{r^{\prime} u}{1} \frac{m}{t}=0_{S^{-1} M}$ and $A n n_{S^{-1} R}\left(\frac{m}{t}\right)=0_{S^{-1} R}$ imply that $r^{\prime} u s=0$ for some $s \in S$. But, clearly, $u m \neq 0_{M}$ and so $u s \in S \cap Z(M)=\emptyset$, a contradiction. Hence, $A n n_{R}(u m)=0$. Therefore, $r^{2}(u m) \in N$ implies that rum $\in N$ and so $\frac{r}{s} \frac{m}{t}=\frac{r u m}{s u t} \in S^{-1} N$.
(2) Suppose that $r^{2} m \in N$ with $\operatorname{Ann}_{M}(r)=0_{M}$ and $A n n_{R}(m)=0$ for some $r \in R$ and $m \in M$. Now, $\left(\frac{r}{1}\right)^{2} \frac{m}{1} \in S^{-1} N$. If $A n n_{S^{-1} M}\left(\frac{r}{1}\right) \neq 0_{S^{-1} M}$, then there exists $0_{S^{-1} M} \neq \frac{m^{\prime}}{t} \in S^{-1} M$ such that $\frac{r}{1} \frac{m^{\prime}}{t}=0_{S^{-1} M}$ which implies $u r m^{\prime}=0_{M}$ for some $u \in S$. Since $A n n_{M}(r)=0_{M}$, we have $u m^{\prime}=0_{M}$ and $\frac{m^{\prime}}{t}=\frac{u m^{\prime}}{u t}=0_{S^{-1} M}$, a contradiction. Now, assume that $A n n_{S^{-1} R}\left(\frac{m}{1}\right) \neq 0_{S^{-1} R}$. Then $\frac{r^{\prime}}{s^{\prime}} \frac{m}{1}=0_{S^{-1} M}$ for some $0_{S^{-1} R} \neq \frac{r^{\prime}}{s^{\prime}} \in S^{-1} R$. Thus, $r^{\prime} v m=0$ for some $v \in S$ and clearly $r^{\prime} m \neq 0_{M}$. Hence, again $v \in S \cap Z(M)=\emptyset$, a contradiction. Thus, $A n n_{S^{-1} M}\left(\frac{r}{1}\right)=0_{S^{-1} M}$ and $A n n_{S^{-1} R}\left(\frac{m}{1}\right)=0_{S^{-1} R}$ imply that $\frac{r}{1} \frac{m}{1} \in S^{-1} N$ and so $w r m \in N$ for some $w \in S$. Since $S \cap Z_{N}(M)=\emptyset$, we conclude that $r m \in N$, as desired.

We recall from [2] that for an $R$-module $M$, we have

$$
z d(R(+) M)=\{(r, m) \mid r \in z d(R) \cup Z(M), m \in M\}
$$

where $Z(M)=\left\{r \in R: r m=0\right.$ for some $\left.0_{M} \neq m \in M\right\}$. In the following proposition, we justify the relation between semi $r$-ideals of $R$ and those of the idealization ring $R(+) M$.
Proposition 11. Let $M$ be an $R$-module and $I$ be a proper ideal of $R$.
(1) If $I$ is a semi $r$-ideal of $R$, then $I(+) M$ is a semi $r$-ideal of $R(+) M$. Moreover, the converse is true if $Z(M) \subseteq z d(R)$.
(2) If $I$ is a semi r-ideal of $R$ and $N$ is an r-submodule of $M$, then $I(+) N$ is a semi r-ideal of $R(+) M$. Moreover, the converse is true if $Z(M) \subseteq z d(R)$.
Proof. (1). Suppose that $(a, m)^{2} \in I(+) M$ and $(a, m) \notin z d(R(+) M)$. Then $a^{2} \in I$ and $a \notin z d(R)$. Since $I$ is a semi $r$-ideal, we conclude that $a \in I$ and so $(a, m) \in$ $I(+) M$. Now, assume that $Z(M) \subseteq z d(R)$ and $I(+) M$ is a semi $r$-ideal of $R(+) M$. Let $a \in R$ such that $a^{2} \in I$ but $a \notin I$. Then $(a, 0)^{2} \in I(+) M$ and $(a, 0) \notin I(+) M$ which imply that $(a, 0) \in z d(R(+) M)$. Since $Z(M) \subseteq z d(R)$, we conclude that $a \in z d(R)$ and we are done.
(2). Suppose that $(a, m)^{2} \in I(+) N$ and $(a, m) \notin z d(R(+) M)$. Then $a \in I$ as in (1). Moreover, $a . m \in N$ as $I M \subseteq N$. Since also, $a \notin Z(M)$, then $A n n_{M}(a)=0$. Therefore, $m \in N$ as $N$ is an $r$-submodule of $M$ and $(a, m) \in I(+) N$ as needed. If $Z(M) \subseteq z d(R)$, then similar to the proof of (1), the converse holds.

Remark 1. In general, if $Z(M) \nsubseteq z d(R)$, then the converse of Proposition 11 need not be true. For example, consider the idealization ring $R=\mathbb{Z}(+) \mathbb{Z}_{4}$ and the ideal $4 \mathbb{Z}(+) \mathbb{Z}_{4}$ of $R$. Let $(a, m)^{2} \in 4 \mathbb{Z}(+) \mathbb{Z}_{4}$ for $(a, m) \in R$. Then $a^{2} \in 4 \mathbb{Z}$ and so $(a, m) \in 2 \mathbb{Z} \times \mathbb{Z}_{4}=z d(R)$. Thus, $4 \mathbb{Z}(+) \mathbb{Z}_{4}$ is a (semi) r-ideal of $R$. On the other hand, $4 \mathbb{Z}$ is not a semi $r$-ideal of $\mathbb{Z}$.

## 4. SEMI $r$-SUBMODULES OF AMALGAMATED MODULES

Let $R$ be a ring, $J$ an ideal of $R$ and $M$ an $R$-module. Recently, in [5], the duplication of the $R$-module $M$ along the ideal $J$ (denoted by $M \bowtie J$ ) is defined as

$$
M \bowtie J=\left\{\left(m, m^{\prime}\right) \in M \times M: m-m^{\prime} \in J M\right\}
$$

which is an $(R \bowtie J)$-module with scaler multiplication defined by $(r, r+j) \cdot\left(m, m^{\prime}\right)=$ $\left(r m,(r+j) m^{\prime}\right)$ for $r \in R, j \in J$ and $\left(m, m^{\prime}\right) \in M \bowtie J$. For various properties and results concerning this kind of modules, one may see [5].

Let $J$ be an ideal of a ring $R$ and $N$ be a submodule of an $R$-module $M$. Then

$$
N \bowtie J=\{(n, m) \in N \times M: n-m \in J M\}
$$

and

$$
\bar{N}=\{(m, n) \in M \times N: m-n \in J M\}
$$

are clearly submodules of $M \bowtie J$. Moreover,

$$
\left.A n n_{R \bowtie J}(M \bowtie J)=(r, r+j) \in R \bowtie I \mid r \in A n n_{R}(M) \text { and } j \in A n n_{R}(M) \cap J\right\}
$$

and so $M \bowtie J$ is a faithful $R \bowtie J$-module if and only if $M$ is a faithful $R$-module, [5, Lemma 3.6].

In general, let $f: R_{1} \rightarrow R_{2}$ be a ring homomorphism, $J$ be an ideal of $R_{2}, M_{1}$ be an $R_{1}$-module, $M_{2}$ be an $R_{2}$-module (which is an $R_{1}$-module induced naturally by $f$ ) and $\varphi: M_{1} \rightarrow M_{2}$ be an $R_{1}$-module homomorphism. The subring

$$
R_{1} \bowtie^{f} J=\left\{(r, f(r)+j): r \in R_{1}, j \in J\right\}
$$

of $R_{1} \times R_{2}$ is called the amalgamation of $R_{1}$ and $R_{2}$ along $J$ with respect to $f$. In [8], the amalgamation of $M_{1}$ and $M_{2}$ along $J$ with respect to $\varphi$ is defined as

$$
M_{1} \bowtie^{\varphi} J M_{2}=\left\{\left(m_{1}, \varphi\left(m_{1}\right)+m_{2}\right): m_{1} \in M_{1} \text { and } m_{2} \in J M_{2}\right\}
$$

which is an $\left(R_{1} \bowtie^{f} J\right)$-module with the scaler product defined as

$$
(r, f(r)+j)\left(m_{1}, \varphi\left(m_{1}\right)+m_{2}\right)=\left(r m_{1}, \varphi\left(r m_{1}\right)+f(r) m_{2}+j \varphi\left(m_{1}\right)+j m_{2}\right)
$$

For submodules $N_{1}$ and $N_{2}$ of $M_{1}$ and $M_{2}$, respectively, one can easily justify that the sets

$$
N_{1} \bowtie^{\varphi} J M_{2}=\left\{\left(m_{1}, \varphi\left(m_{1}\right)+m_{2}\right) \in M_{1} \bowtie^{\varphi} J M_{2}: m_{1} \in N_{1}\right\}
$$

and

$$
{\overline{N_{2}}}^{\varphi}=\left\{\left(m_{1}, \varphi\left(m_{1}\right)+m_{2}\right) \in M_{1} \bowtie^{\varphi} J M_{2}: \varphi\left(m_{1}\right)+m_{2} \in N_{2}\right\}
$$

are submodules of $M_{1} \bowtie^{\varphi} J M_{2}$.
Note that if $R=R_{1}=R_{2}, M=M_{1}=M_{2}, f=I d_{R}$ and $\varphi=I d_{M}$, then the amalgamation of $M_{1}$ and $M_{2}$ along $J$ with respect to $\varphi$ is exactly the duplication of the $R$-module $M$ along the ideal $J$. Moreover, in this case, we have $N_{1} \bowtie^{\varphi} J M_{2}=$ $N \bowtie J$ and ${\overline{N_{2}}}^{\varphi}=\bar{N}$.

Theorem 12. Consider the $\left(R_{1} \bowtie^{f} J\right)$-module $M_{1} \bowtie^{\varphi} J M_{2}$ defined as above. Assume $J M_{2}=\left\{0_{M_{2}}\right\}$ and let $N_{1}$ be submodule of $M_{1}$. Then
(1) $N_{1}$ is an $r$-submodule of $M_{1}$ if and only if $N_{1} \bowtie^{\varphi} J M_{2}$ is an $r$-submodule of $M_{1} \bowtie^{\varphi} J M_{2}$.
(2) If $N_{1}$ is a semi $r$-submodule of $M_{1}$, then $N_{1} \bowtie^{\varphi} J M_{2}$ is a semi $r$-submodule of $M_{1} \bowtie^{\varphi} J M_{2}$.
(3) If $M_{2}$ is faithful and $N_{1} \bowtie^{\varphi} J M_{2}$ is a semi $r$-submodule of $M_{1} \bowtie^{\varphi} J M_{2}$, then $N_{1}$ is a semi $r$-submodule of $M_{1}$.

Proof. (1) Let $N_{1}$ be an $r$-submodule of $M_{1}$ and let $\left(r_{1}, f\left(r_{1}\right)+j\right) \in R_{1} \bowtie^{f}$ $J,\left(m_{1}, \varphi\left(m_{1}\right)\right) \in M_{1} \bowtie^{\varphi} J M_{2}$ such that $\left(r_{1}, f\left(r_{1}\right)+j\right)\left(m_{1}, \varphi\left(m_{1}\right)\right) \in N_{1} \bowtie^{\varphi}$ $J M_{2}$ and $A n n_{M_{1} \bowtie \varphi J M_{2}}\left(\left(r_{1}, f\left(r_{1}\right)+j\right)\right)=0_{M_{1} \bowtie \varphi J M_{2}}$. Then $r_{1} m_{1} \in N_{1}$ and we prove that $A n n_{M_{1}}\left(r_{1}\right)=0_{M_{1}}$. Suppose $r_{1} m_{1}^{\prime}=0_{M_{1}}$ for some $m_{1}^{\prime} \in M_{1}$. Then $\left(r_{1}, f\left(r_{1}\right)+j\right)\left(m_{1}^{\prime}, \varphi\left(m_{1}^{\prime}\right)\right)=\left(0_{M_{1}}, j \varphi\left(m_{1}^{\prime}\right)\right)=\left(0_{M_{1}}, 0_{M_{2}}\right)$ as $J M_{2}=\left\{0_{M_{2}}\right\}$. Thus, $\left(m_{1}^{\prime}, \varphi\left(m_{1}^{\prime}\right)\right) \in \operatorname{Ann}_{M_{1} \bowtie^{\varphi} J M_{2}}\left(\left(r_{1}, f\left(r_{1}\right)+j\right)\right)=0_{M_{1} \bowtie^{\varphi} J M_{2}}$. Hence, $m_{1}^{\prime}=0_{M_{1}}$ and $A n n_{M_{1}}\left(r_{1}\right)=0_{M_{1}}$. By assumption, $m_{1} \in N_{1}$ and then $\left(m_{1}, \varphi\left(m_{1}\right)\right) \in N_{1} \bowtie^{\varphi} J M_{2}$, as needed.

Conversely, let $r_{1} \in R_{1}$ and $m_{1} \in M_{1}$ such that $r_{1} m_{1} \in N_{1}$ and $A n n_{M_{1}}\left(r_{1}\right)=$ $0_{M_{1}}$. Then $\left(r_{1}, f\left(r_{1}\right)\right) \in R_{1} \bowtie^{f} J,\left(m_{1}, \varphi\left(m_{1}\right)\right) \in M_{1} \bowtie^{\varphi} J M_{2}$ and $\left(r_{1}, f\left(r_{1}\right)\right)\left(m_{1}, \varphi\left(m_{1}\right)\right)=$ $\left(r_{1} m_{1}, \varphi\left(r_{1} m_{1}\right)\right) \in N_{1} \bowtie^{\varphi} J M_{2}$. Moreover, $A_{n} n_{M_{1} \bowtie^{\varphi} J M_{2}}\left(\left(r_{1}, f\left(r_{1}\right)\right)\right)=0_{M_{1} \bowtie^{\varphi} J M_{2}}$. Indeed, suppose that there $\left(m_{1}^{\prime}, \varphi\left(m_{1}^{\prime}\right)\right) \in M_{1} \bowtie^{\varphi} J M_{2}$ such that $\left(r_{1}, f\left(r_{1}\right)\right)\left(m_{1}^{\prime}, \varphi\left(m_{1}^{\prime}\right)\right)=$ $0_{M_{1} \bowtie \varphi}{ }^{\bowtie} M_{2}$. Then $\left(m_{1}^{\prime}, \varphi\left(m_{1}^{\prime}\right)\right)=\left(0_{M_{1}}, 0_{M_{2}}\right)$ as $A n n_{M_{1}}\left(r_{1}\right)=0_{M_{1}}$. Since $N_{1} \bowtie^{\varphi}$ $J M_{2}$ is an $r$-submodule of $M_{1} \bowtie^{\varphi} J M_{2}$, then $\left(m_{1}, \varphi\left(m_{1}\right)\right) \in N_{1} \bowtie^{\varphi} J M_{2}$ so that $m_{1} \in N_{1}$ and we are done.
(2) Let $\left(r_{1}, f\left(r_{1}\right)+j\right) \in R_{1} \bowtie^{f} J$ and $\left(m_{1}, \varphi\left(m_{1}\right)\right) \in M_{1} \bowtie^{\varphi} J M_{2}$ such that $\left(r_{1}, f\left(r_{1}\right)+j\right)^{2}\left(m_{1}, \varphi\left(m_{1}\right)\right) \in N_{1} \bowtie^{\varphi} J M_{2}, A n n_{M_{1} \bowtie \varphi J M_{2}}\left(\left(r_{1}, f\left(r_{1}\right)+j\right)\right)=$ $0_{M_{1} \bowtie^{\varphi} J M_{2}}$ and $\operatorname{Ann}_{R_{1} \bowtie^{f} J}\left(\left(m_{1}, \varphi\left(m_{1}\right)\right)\right)=0_{R_{1} \bowtie^{f} J}$. Then $r_{1}^{2} m_{1} \in N_{1}$ and similar to the proof of (1), we have $\operatorname{Ann}_{M_{1}}\left(r_{1}\right)=0_{M_{1}}$. We show that $\operatorname{Ann}_{R_{1}}\left(m_{1}\right)=0_{R_{1}}$. Assume on the contrary that there is nonzero element $r_{1} \in R_{1}$ such that $r_{1} m_{1}=0_{R_{1}}$. Then, $\left(r_{1}, f\left(r_{1}\right)\right)\left(m_{1}, \varphi\left(m_{1}\right)\right)=0_{M_{1} \bowtie \varphi M_{2}}$, but our assumption $\operatorname{Ann}_{R_{1} \bowtie f J}\left(\left(m_{1}, \varphi\left(m_{1}\right)\right)\right)=$ $0_{R_{1} \bowtie_{J}}$ implies that $\left(r_{1}, f\left(r_{1}\right)\right)=0_{R_{1} \bowtie_{J} J}$; i.e. $r_{1}=0_{R_{1}}$, a contradiction. Thus $A n n_{R_{1}}\left(m_{1}\right)=0_{R_{1}}$, and it follows that $r_{1} m_{1} \in N_{1}$ and so $\left(r_{1}, f\left(r_{1}\right)+j\right)\left(m_{1}, \varphi\left(m_{1}\right)+\right.$ $\left.m_{2}\right) \in N_{1} \bowtie^{\varphi} J M_{2}$.
(3) Since $M_{2}$ is faithful, then clearly $J=\left\{0_{R_{2}}\right\}$. Let $r_{1} \in R_{1}$ and $m_{1} \in$ $M_{1}$ such that $r_{1}^{2} m_{1} \in N_{1}, A n n_{M_{1}}\left(r_{1}\right)=0_{M_{1}}$ and $\operatorname{Ann}_{R_{1}}\left(m_{1}\right)=0_{R_{1}}$. Then $\left(r_{1}, f\left(r_{1}\right)\right)^{2}\left(m_{1}, \varphi\left(m_{1}\right)\right) \in N_{1} \bowtie^{\varphi} J M_{2}$ where $\left(r_{1}, f\left(r_{1}\right)\right) \in R_{1} \bowtie^{f} J$ and $\left(m_{1}, \varphi\left(m_{1}\right)\right) \in$ $M_{1} \bowtie^{\varphi} J M_{2}$. Again, similar to the proof of (1), we have $A n n_{M_{1} \bowtie M_{2}}\left(\left(r_{1}, f\left(r_{1}\right)\right)\right)=$ $0_{M_{1} \bowtie \varphi J M_{2}}$. Moreover, suppose there is $\left(r_{1}^{\prime}, f\left(r_{1}^{\prime}\right)\right) \in R_{1} \bowtie^{f} J$ such that $\left(r_{1}^{\prime} m_{1}, \varphi\left(r_{1}^{\prime} m_{1}\right)\right)=$ $\left(r_{1}^{\prime}, f\left(r_{1}^{\prime}\right)+j\right)\left(m_{1}, \varphi\left(m_{1}\right)\right)=0_{M_{1} \bowtie^{\varphi} J M_{2}}$. Then $\left(r_{1}^{\prime}, f\left(r_{1}^{\prime}\right)\right)=\left(0_{R_{1}}, 0_{R_{2}}\right)$ as $\operatorname{Ann}_{R_{1}}\left(m_{1}\right)=$ $0_{R_{1}}$ and so $A n n_{R_{1} \bowtie f J}\left(\left(m_{1}, \varphi\left(m_{1}\right)\right)\right)=0_{M_{1} \bowtie^{\varphi} J M_{2}}$. By assumption, $\left(r_{1}, f\left(r_{1}\right)\right)\left(m_{1}, \varphi\left(m_{1}\right)\right) \in$ $N_{1} \bowtie^{\varphi} J M_{2}$. It follows that $r_{1} m_{1} \in N_{1}$ and $N_{1}$ is a semi $r$-submodule of $M_{1}$.

Corollary 7. Let $N$ be a submodule of an $R$-module $M$ and $J$ be an ideal of $R$. Then
(1) If $N \bowtie J$ is an $r$-submodule of $M \bowtie J$, then $N$ is an $r$-submodule of $M$. The converse is true if $J M=0_{M}$.
(2) If $N \bowtie J$ is a semi $r$-submodule of $M \bowtie J$, then $N$ is a semi $r$-submodule of $M$. The converse is true if $J M=0_{M}$.

Proof. (1) Let $r \in R$ and $m \in M$ such that $r m \in N$ and $A n n_{M}(r)=0_{M}$. Then $(r, r)(m, m) \in N \bowtie J$ and clearly, $A n n_{M \bowtie J}((r, r))=0_{M \bowtie J}$. Thus, $(m, m) \in N \bowtie J$ and so $m \in N$ as needed. Conversely, suppose $J M=0_{M}$ and let $(r, r+j) \in R \bowtie J$, $\left(m, m+m^{\prime}\right) \in M \bowtie J$ such that $(r, r+j)\left(m, m+m^{\prime}\right) \in N \bowtie J$ and $A n n_{M \bowtie J}((r, r+$
$j))=0_{M \bowtie J}$. If $r m^{\prime \prime}=0_{M}$ for some $m^{\prime \prime} \in M$, then $(r, r+j)\left(m^{\prime \prime}, m^{\prime \prime}\right)=\left(0, j m^{\prime \prime}\right)=$ $\left(0_{M}, 0_{M}\right)$ as $J M=0_{M}$. Thus, $m^{\prime \prime}=0_{M}$ and $A n n_{M}(r)=0_{M}$. Since $r m \in N$, then $m \in N$ and so $\left(m, m+m^{\prime}\right) \in N \bowtie J$.
(2) Let $r \in R$ and $m \in M$ such that $r^{2} m \in N, A n n_{M}(r)=0_{M}$ and $A n n_{R}(m)=$ $0_{R}$. Then $(r, r)^{2}(m, m) \in N \bowtie J$. If there exists an element $\left(m^{\prime}, m^{\prime \prime}\right)$ of $M \bowtie$ $J,(r, r)\left(m^{\prime}, m^{\prime \prime}\right)=\left(0_{M}, 0_{M}\right)$, then clearly $\left(m^{\prime}, m^{\prime \prime}\right)=\left(0_{M}, 0_{M}\right)$ as $\operatorname{Ann}_{M}(r)=$ $0_{M}$; and so $A n n_{M \bowtie J}((r, r))=0_{M \bowtie J}$. Also, if for $\left(r^{\prime}, r^{\prime}+j\right) \in R \bowtie J,\left(r^{\prime}, r^{\prime}+\right.$ $j)(m, m)=\left(0_{M}, 0_{M}\right)$, then $\left(r^{\prime}, r^{\prime}+j\right)=\left(0_{R}, 0_{R}\right)$ and $A n n_{R \bowtie J}((m, m))=0_{R \bowtie J}$. By assumption, $(r, r)(m, m) \in N \bowtie J$ and so $r m \in N$. The proof of the converse part is similar to that of the converse of (1).

Theorem 13. Consider the $\left(R_{1} \bowtie^{f} J\right)$-module $M_{1} \bowtie^{\varphi} J M_{2}$ defined as in Theorem 12 and let $N_{2}$ be a submodule of $M_{2}$.
(1) If $N_{2}$ is an $r$-submodule of $M_{2}, J M_{2} \neq\left\{0_{M_{2}}\right\}$ and $T\left(M_{2}\right) \subseteq J M_{2}$, then ${\overline{N_{2}}}^{\varphi}$ is an $r$-submodule of $M_{1} \bowtie^{\varphi} J M_{2}$. Moreover, if $f$ is an epimorphism and $\varphi$ is an isomorphism, then the converse holds.
(2) If $f$ and $\varphi$ are isomorphisms and ${\overline{N_{2}}}^{\varphi}$ is a semi $r$-submodule of $M_{1} \bowtie^{\varphi} J M_{2}$, then $N_{2}$ is a semi $r$-submodule of $M_{2}$.

Proof. (1). Suppose $N_{2}$ is an $r$-submodule of $M_{2}$. Let $\left(r_{1}, f\left(r_{1}\right)+j\right) \in R_{1} \bowtie^{f} J$ and $\left(m_{1}, \varphi\left(m_{1}\right)+m_{2}\right) \in M_{1} \bowtie J M_{2}$ such that $\left(r_{1}, f\left(r_{1}\right)+j\right)\left(m_{1}, \varphi\left(m_{1}\right)+m_{2}\right) \in \bar{N}_{2}^{\varphi}$ and $A n n_{M_{1} \bowtie M_{2}}\left(\left(r_{1}, f\left(r_{1}\right)+j\right)\right)=0_{M_{1} \bowtie^{\varphi} J M_{2}}$. Then $\left(f\left(r_{1}\right)+j\right)\left(\varphi\left(m_{1}\right)+m_{2}\right) \in N_{2}$ and $A n n_{M_{2}}\left(\left(f\left(r_{1}\right)+j\right)\right)=0_{M_{2}}$. Indeed, suppose $\left(f\left(r_{1}\right)+j\right) m_{2}^{\prime}=0_{M_{2}}$ for some $0_{M_{2}} \neq$ $m_{2}^{\prime} \in M_{2}$. If $m_{2}^{\prime} \in J M_{2}$, then $\left(r_{1}, f\left(r_{1}\right)+j\right)\left(0_{M_{1}}, 0_{M_{2}}+m_{2}^{\prime}\right)=0_{M_{1} \bowtie J M_{2}}$ where $\left(0_{M_{1}}, 0_{M_{2}}+m_{2}^{\prime}\right) \neq 0_{M_{1} \bowtie J M_{2}}$, a contradiction. If $m_{2}^{\prime} \notin J M_{2}$, then $m_{2}^{\prime} \notin T\left(M_{2}\right)$ and so $\left(f\left(r_{1}\right)+j\right)=0_{R_{2}}$. If we choose $0 \neq m_{2}^{\prime \prime} \in J M_{2}$, then $\left(r_{1}, f\left(r_{1}\right)+j\right)\left(0_{M_{1}}, m_{2}^{\prime \prime}\right)=$ $0_{M_{1} \bowtie J M_{2}}$ which is also a contradiction. By assumption, $\left.\varphi\left(m_{1}\right)+m_{2}\right) \in N_{2}$ and so $\left(m_{1}, \varphi\left(m_{1}\right)+m_{2}\right) \in{\overline{N_{2}}}^{\varphi}$.

Conversely, suppose $\varphi$ is an isomorphism and ${\overline{N_{2}}}^{\varphi}$ is an $r$-submodule of $M_{1} \bowtie^{\varphi}$ $J M_{2}$. Let $r_{2}=f\left(r_{1}\right) \in R_{2}$ and $m_{2}=\varphi\left(m_{1}\right) \in M_{2}$ such that $r_{2} m_{2} \in N_{2}$ and $A n n_{M_{2}}\left(r_{2}\right)=0_{M_{2}}$. Then $\left(r_{1}, r_{2}\right) \in R_{1} \bowtie^{f} J,\left(m_{1}, m_{2}\right) \in M_{1} \bowtie^{\varphi} J M_{2}$ and $\left(r_{1}, r_{2}\right)\left(m_{1}, m_{2}\right) \in{\overline{N_{2}}}^{\varphi}$. Suppose on contrary that there is $\left(m_{1}^{\prime}, \varphi\left(m_{1}^{\prime}\right)+m_{2}^{\prime}\right) \neq$ $0_{M_{1} \bowtie \varphi J M_{2}}$ such that $\left(r_{1}, r_{2}\right)\left(m_{1}^{\prime}, \varphi\left(m_{1}^{\prime}\right)+m_{2}^{\prime}\right)=0_{M_{1} \bowtie \varphi J M_{2}}$. If $\varphi\left(m_{1}^{\prime}\right)+m_{2}^{\prime} \neq 0_{M_{2}}$, we get a contradiction. If $\varphi\left(m_{1}^{\prime}\right)+m_{2}^{\prime}=0_{M_{2}}$ (and so $m_{1}^{\prime} \neq 0_{M_{1}}$ ), then clearly $r_{2} m_{2}^{\prime}=0_{M_{2}}$ and then $m_{2}^{\prime}=0_{M_{2}}$. It follows that $\varphi\left(m_{1}^{\prime}\right)=0_{M_{2}}$ and so $m_{1}^{\prime}=0_{M_{1}}$, a contradiction. Since ${\overline{N_{2}}}^{\varphi}$ is an $r$-submodule of $M_{1} \bowtie^{\varphi} J M_{2}$, then $\left(m_{1}, m_{2}\right) \in{\overline{N_{2}}}^{\varphi}$ and so $m_{2} \in N_{2}$ as required.
(3) Let $r_{2}=f\left(r_{1}\right) \in R_{2}$ and $m_{2}=\varphi\left(m_{1}\right) \in M_{2}$ such that $r_{2}^{2} m_{2} \in N_{2}$, $\operatorname{Ann}_{M_{2}}\left(r_{2}\right)=0_{M_{2}}$ and $\operatorname{Ann}_{R_{2}}\left(m_{2}\right)=0_{R_{2}}$. Then $\left.\left(r_{1}, r_{2}\right)\right)^{2}\left(m_{1}, m_{2}\right) \in{\overline{N_{2}}}^{\varphi}$ where $\left(r_{1}, f\left(r_{1}\right)\right) \in R_{1} \bowtie^{f} J$ and $\left(m_{1}, \varphi\left(m_{1}\right)\right) \in M_{1} \bowtie^{\varphi} J M_{2}$. Similar to the proof of the converse part of (1), we have $\operatorname{Ann}_{M_{1} \bowtie^{\varphi} J M_{2}}\left(\left(r_{1}, r_{2}\right)\right)=0_{M_{1} \bowtie \varphi J M_{2}}$. We prove that $A n n_{R_{1} \bowtie{ }^{f} J}\left(\left(m_{1}, m_{2}\right)\right)=0_{R_{1} \bowtie^{f} J}$. Let $\left(r_{1}^{\prime}, f\left(r_{1}^{\prime}\right)+j^{\prime}\right) \in R_{1} \bowtie^{f} J$ such that $\left(r_{1}^{\prime}, f\left(r_{1}^{\prime}\right)+j^{\prime}\right)\left(m_{1}, m_{2}\right)=0_{M_{1} \bowtie \varphi J M_{2}}$. Then $f\left(r_{1}^{\prime}\right)+j^{\prime}=0_{R_{2}}$ and $r_{1}^{\prime} m_{1}=0_{M_{1}}$. Thus, $f\left(r_{1}^{\prime}\right) m_{2}=0$ and so $f\left(r_{1}^{\prime}\right)=0_{R_{2}}$. Since $f$ is one to one, then $r_{1}^{\prime}=0_{R_{1}}$ and so $\left(r_{1}^{\prime}, f\left(r_{1}^{\prime}\right)+j^{\prime}\right)=0_{R_{1} \bowtie{ }^{f} J}$ as needed. By assumption, $\left.\left(r_{1}, r_{2}\right)\right)\left(m_{1}, m_{2}\right) \in{\overline{N_{2}}}^{\varphi}$ and so $r_{2} m_{2} \in N_{2}$.

Corollary 8. Let $N$ be a submodule of an $R$-module $M$ and $J$ be an ideal of $R$. Then
(1) If $\bar{N}$ is an $r$-submodule of $M \bowtie J$, then $N$ is an $r$-submodule of $M$. The converse is true if $J M=0_{M}$.
(2) If $\bar{N}$ is a semi $r$-submodule of $M \bowtie J$, then $N$ is a semi $r$-submodule of $M$. The converse is true if $J M=0_{M}$.

Proof. The proof is similar to that of Corollary 7 and left to the reader.

## References

## Statements \& Declarations

The authors declare that no funds, grants, or other support were received during the preparation of this manuscript. The authors have no relevant financial or nonfinancial interests to disclose. All authors read and approved the final manuscript., References
[1] M.M. Ali, Residual submodules of multiplication modules, Beitr"age zur Algebra und Geometrie, 46 (2005), 405-422.
[2] D. D. Anderson, M. Winders, Idealization of a module, J. Commut. Algebra, 1 (1) (2009), 3-56.
[3] Y. Azimi, P. Sahandi and N. Shirmohammadi, Prüfer conditions under the amalgamated construction, Commun. Algebra 47(5) (2019), 2251-2261.
[4] A. Badawi, On weakly semiprime ideals of commutative rings, Beitr. Algebra Geom. 57 (2016) 589-597.
[5] E. M. Bouba, N. Mahdou, and M. Tamekkante, Duplication of a module along an ideal, Acta Math. Hungar., 154 (1) (2018), 29-42.
[6] M. D'Anna and M. Fontana, An amalgamated duplication of a ring along an ideal: the basic properties, J. Algebra Appl. 6 (2007), no. 3, 443-459.
[7] M. D'Anna, C.A. Finocchiaro, and M. Fontana, Properties of chains of prime ideals in an amalgamated algebra along an ideal, J. Pure Appl. Algebra 214 (2010), 1633-1641.
[8] R. El Khalfaoui, N. Mahdou, P. Sahandi and N. Shirmohammadi, Amalgamated modules along an ideal, Commun. Korean Math. Soc., 36 (1), (2021) 1-10.
[9] R. Gilmer, Multiplicative Ideal Theory. New York, NY, USA: Marcel Dekker, 1972.
[10] H. A. Khashan, A. B. Bani-Ata, J-ideals of commutative rings, International Electronic Journal of Algebra, 29 (2021), 148-164.
[11] H. A. Khashan, E. Yetkin Celikel, , Weakly J-ideals of commutative rings, Filomat, 36(2), (2022), 485-495. https://doi.org/10.2298/FIL2202485K.
[12] H. A. Khashan, E. Yetkin Celikel, Quasi $J$-ideals of commutative rings, Ricerche di Matematica, (2022) (Published online) https://doi.org/10.1007/s11587-022-00716-2.
[13] S. Koc, U. Tekir, $r$-Submodules and $s r$-Submodules, Turkish Journal of Mathematics, 42(4) (2018),1863-1876.
[14] T. K. Lee and Y. Zhou, Reduced modules, Rings, Modules, Algebras and Abelian Groups, 236 (2004),365-377.
[15] R. Mohamadian, $r$-ideals in commutative rings, Turkish Journal of Mathematics, 39 (2015), 733-749.
[16] B. Saraç, On semiprime submodules, Communications in Algebra, 37(7), (2009), 24852495.
[17] P. Smith, Some remarks on multiplication modules, Arch. Math., 50 (1988), 223-235.
[18] U. Tekir, S. Koc and K. H. Oral, n-ideals of commutative rings, Filomat, 31 (10) (2017), 2933-2941.
[19] E. Yetkin Celikel, Generalizations of $n$-ideals of Commutative Rings . Erzincan Universitesi Fen Bilimleri Enstitüsü Dergisi, 12 (2) , (2019) 650-657. DOI: 10.18185/erzifbed. 471609.
[20] E. Yetkin Celikel, H. A. Khashan, Semi $n$-ideals of commutative rings, Czechoslovak Mathematical Journal, in press.

Department of Mathematics, Faculty of Science, Al al-Bayt University, Al Mafraq, Jordan.

Email address: hakhashan@aabu.edu.jo
Department of Basic Sciences, Faculty of Engineering, Hasan Kalyoncu University, Gaziantep, Turkey.

Email address: ece.celikel@hku.edu.tr, yetkinece@gmail.com


[^0]:    2010 Mathematics Subject Classification. Primary 13A15, 16P40, Secondary 16D60.
    Key words and phrases. Semiprime ideal, semiprime submodule, semi $r$-ideal, semi $n$-ideal, semi $r$-submodule.

    This paper is in final form and no version of it will be submitted for publication elsewhere.

