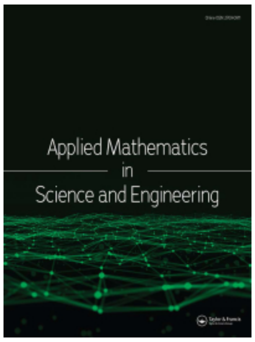

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A new family of Apostol–Genocchi polynomials associated with their certain identities

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ABSTRACT

In this paper, we provide a generating function for mix type Apostol–Genocchi polynomials of order η associated with Bell polynomials. We also derive certain important identities of Apostol Genocchi polynomials of order η based on Bell polynomials, such as the correlation formula, the implicit summation formula, the derivative formula, some correlation with Stirling numbers and their special instances. Moreover, we discover some symmetric identities and their related known results.

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1. Introduction and preliminaries

Polynomials and numbers are important in many fields of science, including mathematics, applied science, physics, and engineering sciences, as well as some related research fields such as fluid dynamics, number theory, quantum mechanics, differential equations, and mathematical physics (see [1–3]). Recently, Duran et al. [4] studied the Bell-based Bernoulli polynomials and their application, Husain et al. [5] studied the Bell-based Apostol–Bernoulli polynomials and their properties and Khan et al. [6] studied the Bell-based Euler polynomials and their application. Motivated by above mention work, in this paper we study Apostol–Genocchi polynomials of order η associated with Bell polynomials and certain properties such as correlation formula, derivative formula, implicit summation formula, relation with Stirling numbers and their special cases. Moreover, we define some symmetric identities and their related known results.

The symbol $\mathbb{R}$ is used all over the paper to denote the set of all real numbers, $\mathbb{N}$ has been used to denote the set of natural numbers, $\mathbb{C}$ can be used to denote the set of complex numbers, $\mathbb{Z}$ is used to denote the set of integers numbers and $\mathbb{N}_0$ is being used to denote the set of all positive integers. Many authors have recently (see [7–10]) studied the bivariate

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Bell polynomials, classical Bell polynomials, Bernoulli polynomials and Euler polynomials, Genocchi polynomials and Apostol type Euler, Bernoulli, Genocchi polynomials define as follows.

The following generating function (see [4]) defines the Bell polynomials of two variables (i.e. bivariate Bell polynomials):

$$\sum_{k\geq 0} \mathcal{B}_k(u,v)\frac{l^k}{k!} = e^{ul+v(e^l-1)}. \tag{1}$$

When $u = 0$, $\mathcal{B}_k(0;v) = \mathcal{B}_k(v)$ are known as classical Bell polynomials (or exponential polynomials) and are described by the following generating function (see [11–14]):

$$\sum_{k\geq 0} \mathcal{B}_k(v)\frac{l^k}{k!} = e^{v(e^l-1)}. \tag{2}$$

If $v = 1$ in (2) i.e. $\mathcal{B}_k(0;1) = \mathcal{B}_k(1) = \mathcal{B}_k$ are known as Bell numbers described by the following generating function (see [11]):

$$\sum_{k\geq 0} \mathcal{B}_k\frac{l^k}{k!} = e^{(e^l-1)}. \tag{3}$$

The following generating function defines Euler polynomials $\mathcal{E}_k(u)$ and Euler numbers $\mathcal{E}_k(0)$ (see [15,16]):

$$e^{ul}\left(\frac{2}{e^l+1}\right) = \sum_{k\geq 0}\frac{l^k}{k!}\mathcal{E}_k(u), \quad (|l| < \pi). \tag{4}$$

If $u = 0$ then the Euler numbers $\mathcal{E}_k(0) := \mathcal{E}_k$ are described by the following generating function:

$$\left(\frac{2}{e^l+1}\right) = \sum_{k\geq 0}\frac{l^k}{k!}\mathcal{E}_k, \quad (|l| < \pi). \tag{5}$$

Dattoli et al. [17] introduced the Bernoulli polynomials and Bernoulli numbers, which are defined by the following generating function

$$e^{ul}\left(\frac{l}{e^l-1}\right) = \sum_{k\geq 0}\frac{l^k}{k!}\mathfrak{B}_k(u), \quad (|l| < 2\pi). \tag{6}$$

If $u = 0$ then the Bernoulli numbers $\mathfrak{B}_k(0) := \mathfrak{B}_k$ are described by the following generating function:

$$\left(\frac{l}{e^l-1}\right) = \sum_{k\geq 0}\frac{l^k}{k!}\mathfrak{B}_k, \quad (|l| < 2\pi). \tag{7}$$

The following generating function defines Genocchi polynomials $\mathcal{G}_k(u)$ and Genocchi numbers $\mathcal{G}_k(0)$(see [18,19]):

$$e^{ul}\left(\frac{2l}{e^l+1}\right) = \sum_{k\geq 0}\frac{l^k}{k!}\mathcal{G}_k(u), \quad (|l| < \pi). \tag{8}$$

If $u = 0$ then the Genocchi numbers $\mathcal{G}_k(0) := \mathcal{G}_k$ are described by the following generating function:

$$\left(\frac{2l}{e^l+1}\right) = \sum_{k\geq 0} \frac{l^k}{k!}\mathcal{G}_k, \quad (|l| < \pi). \tag{9}$$

The Euler polynomials, Bernoulli polynomials and Genocchi polynomials of order $\eta \in \mathbb{C}$ (see [20–22]) are defined by the following generating function as follows:

$$\sum_{k\geq 0} \mathcal{E}_k^{(\eta)}(u)\frac{l^k}{k!} = e^{ul}\left(\frac{2}{e^l+1}\right)^{\eta} \quad (|l| < \pi, 1^{\eta} =: 1), \tag{10}$$

$$\sum_{k\geq 0} \mathfrak{B}_k^{(\eta)}(u)\frac{l^k}{k!} = e^{ul}\left(\frac{l}{e^l-1}\right)^{\eta} \quad (|l| < 2\pi, 1^{\eta} =: 1), \tag{11}$$

$$\sum_{k\geq 0} \mathcal{G}_k^{(\eta)}(u)\frac{l^k}{k!} = e^{ul}\left(\frac{2l}{e^l+1}\right)^{\eta} \quad (|l| < \pi, 1^{\eta} =: 1). \tag{12}$$

If we take $u = 0$ in (10), (11) and (12) i.e. $\mathcal{E}_k^{(\eta)}(0) = \mathcal{E}_k^{(\eta)}$, $\mathfrak{B}_k^{(\eta)}(0) = \mathfrak{B}_k^{(\eta)}$ and $\mathcal{G}_k^{(\eta)}(0) = \mathcal{G}_k^{(\eta)}$ are called Euler numbers, Bernoulli numbers and Genocchi numbers of order η are defined as follows:

$$\sum_{k\geq 0} \mathcal{E}_k^{(\eta)}\frac{l^k}{k!} = \left(\frac{2}{e^l+1}\right)^{\eta}, \tag{13}$$

$$\sum_{k\geq 0} \mathfrak{B}_k^{(\eta)}\frac{l^k}{k!} = \left(\frac{l}{e^l-1}\right)^{\eta}, \tag{14}$$

$$\sum_{k\geq 0} \mathcal{G}_k^{(\eta)}\frac{l^k}{k!} = \left(\frac{2l}{e^l+1}\right)^{\eta}. \tag{15}$$

The Apostol–Bernoulli polynomials $\mathfrak{B}_k^{(\eta)}(u;\mu)$ of order η (see [10,23]) are defined by the generating function as:

$$e^{ul}\left(\frac{l}{\mu e^l-1}\right)^{\eta} = \sum_{k\geq 0} \frac{l^k}{k!}\mathfrak{B}_k^{(\eta)}(u;\mu),$$

$$(|l| < 2\pi \text{ when } \mu = 1; |l| < |\log(\mu)| \text{ when } \mu \neq 1; 1^{\eta} =: 1) \tag{16}$$

with

$$\mathfrak{B}_k^{(\eta)}(u;1) := \mathfrak{B}_k^{(\eta)}(u),$$

and

$$\mathfrak{B}_k^{(\eta)}(0;\mu) := \mathfrak{B}_k^{(\eta)}(\mu),$$

where $\mathfrak{B}_k^{(\eta)}(0;\mu)$ are known as Apostol–Bernoulli numbers of order η.

The Apostol–Euler polynomials $\mathcal{E}_k^{(\eta)}(u,\mu)$ of order η (see [8]) are defined by the generating function as:

$$e^{ul}\left(\frac{2}{\mu e^l+1}\right)^\eta=\sum_{k\geq 0}\frac{l^k}{k!}\mathcal{E}_k^{(\eta)}(u;\mu),$$
$$(|l|<\pi \text{ when } \mu=1; |l|<|\log(-\mu)| \text{ when } \mu\neq 1; 1^\eta=:1) \tag{17}$$

with

$$\mathcal{E}_k^{(\eta)}(u;1):=\mathcal{E}_k^{(\eta)}(u),$$

and

$$\mathcal{E}_k^{(\eta)}(0;\mu):=\mathcal{E}_k^{(\eta)}(\mu),$$

where $\mathcal{E}_k^{(\eta)}(0;\mu)$ are known as Apostol–Euler numbers of order η.

The Apostol–Genocchi polynomials $\mathcal{G}_k^{(\eta)}(u,\mu)$ of order η (see [9]) are defined by the generating function as:

$$e^{ul}\left(\frac{2l}{\mu e^l+1}\right)^\eta=\sum_{k\geq 0}\frac{l^k}{k!}\mathcal{G}_k^{(\eta)}(u;\mu),$$
$$(|l|<\pi \text{ when } \mu=1; |l|<|\log(-\mu)| \text{ when } \mu\neq 1; 1^\eta=:1) \tag{18}$$

with

$$\mathcal{G}_k^{(\eta)}(u;1):=\mathcal{G}_k^{(\eta)}(u),$$

and

$$\mathcal{G}_k^{(\eta)}(0;\mu):=\mathcal{G}_k^{(\eta)}(\mu),$$

where $\mathcal{G}_k^{(\eta)}(0;\mu)$ are known as Apostol–Genocchi numbers of order η.

For each integer $(m\geq 0)$, $\mathcal{M}_m(k)=\sum_{j=0}^{k}(-1)^j j^m$ is called alternative integer powers. The exponential generating function for $\mathcal{M}_m(k)$ is

$$\sum_{m=0}^{\infty}\mathcal{M}_m(k)\frac{t^m}{m!}=1-e^t+e^{2t}+\cdots+(-1)^k e^{kt}=\frac{1-(-e^t)^{k+1}}{e^t+1}. \tag{19}$$

For an arbitrary real and complex parameter μ, the generalized sum of alternative integer power $\mathcal{M}_m(k;\mu)$ is defined by the following generating function (see [24,25]):

$$\sum_{m\geq 0}\mathcal{M}_m(k;\mu)\frac{l^m}{m!}=\frac{1-\mu(-e^t)^{k+1}}{\mu e^t+1}. \tag{20}$$

The generating function of second kind Stirling polynomials $\mathcal{S}_2(k,m;u)$ and Stirling number $\mathcal{S}_2(k,m)$ are defined as (see [11,12]):

$$\sum_{k\geq 0}\mathcal{S}_2(k,m;u)\frac{l^k}{k!}=\frac{(e^l-1)^m}{m!}e^{lu}. \tag{21}$$

When $u = 0$ in (21) i.e. $\mathcal{S}_2(k, m; 0) = \mathcal{S}_2(k, m)$ are called Stirling number of the second kind and defined by the following exponential generating function (see [11,12]):

$$\sum_{k\geq 0} \mathcal{S}_2(k, m)\frac{l^k}{k!} = \frac{(e^l - 1)^m}{m!}. \tag{22}$$

Inspired by earlier research (see [4–6,23,26]) and its significance and applications in various disciplines of science and engineering. The present paper deals with Apostol Genocchi polynomials of order η associated with Bell polynomials.

The paper is organized as follows: In Section 2, we defined Apostol–Genocchi polynomials of order η associated with Bell polynomials (AGPBP) and studies its particular cases. Section 3, deals with their explicit summarization formula. In Section 4, we described their implicit summation formulae. In Section 5, we discuss their derivative formula and finally in Section 6, we define some symmetric identities of Apostol–Genocchi polynomials of η based on Bell polynomials.

2. Apostol–Genocchi polynomials based on Bell polynomials (AGPBP)

In this part, we present the Apostol–Genocchi polynomials of order η associated with Bell polynomials (AGPBP) and look at their many relationships, such as correlation formula, implicit summation formula and derivative formula. The following is the definition of the generating function for Apostol–Genocchi polynomials of order η based on Bell polynomials:

Definition 2.1: For any $\eta \in \mathbb{C}$ Apostol–Genocchi polynomial of order η based on Bell polynomials is defined by:

$$\sum_{k\geq 0} {}_{\mathcal{B}}\mathcal{G}_k^{(\eta)}(u, v; \mu)\frac{l^k}{k!} = \left(\frac{2l}{\mu e^l + 1}\right)^\eta e^{ul+v(e^l-1)},$$
$$(|l| < \pi \text{ when } \mu = 1; |l| < |-\log(\mu)| \text{ when } \mu \neq 1; 1^\eta =: 1). \tag{23}$$

If $u = 0$ and $v = 1$ in (23) then we get an Apostol–Genocchi number of order η based on Bell number, which is defined as follows:

$$\sum_{k\geq 0} {}_{\mathcal{B}}\mathcal{G}_k^{(\eta)}(\mu)\frac{l^k}{k!} = \left(\frac{2l}{\mu e^l + 1}\right)^\eta e^{(e^l-1)}. \tag{24}$$

Now, we define some remarks related to Apostol–Genocchi polynomials of order η based on Bell polynomials, which are obtained by putting a particular value in (23) and defined as follows:

Remark 2.1: If $\eta = 0$ in (23), Apostol–Genocchi polynomials of order η based on Bell polynomials must be reduced to bivariate Bell polynomials defined in (1) as follows:

$$\sum_{k\geq 0} {}_{\mathcal{B}}\mathcal{G}_k^{(0)}(u, v)\frac{l^k}{k!} = e^{ut+v(e^l-1)} = \sum_{k\geq 0} \mathcal{B}_k(u, v)\frac{l^k}{k!}.$$

Remark 2.2: If $v = 0$ in (23) the Apostol–Genocchi polynomials of order η based on Bell polynomials reduce to well-known Genocchi polynomials $\mathcal{G}_k^{(\eta)}(u)$ of order η defined in (10)

$$\sum_{k\geq 0} {}_{\mathcal{B}}\mathcal{G}_k^{(\eta)}(u)\frac{l^k}{k!} = \left(\frac{2l}{e^l+1}\right)^{\eta} e^{ul} = \sum_{k\geq 0}\mathcal{G}_k^{(\eta)}(u)\frac{l^k}{k!}.$$

Remark 2.3: In case $v = 0$, $\mu = 1$ and η=1 in (23) the Apostol–Genocchi polynomials of order η based on Bell polynomials ${}_{\mathcal{B}}\mathcal{G}_k^{(\eta)}(u;v)$ reduces to usual Genocchi polynomials $\mathcal{G}_k(u)$ defined as:

$$\sum_{k\geq 0} {}_{\mathcal{B}}\mathcal{G}_k^{(1)}(u)\frac{l^k}{k!} = \left(\frac{2l}{e^l+1}\right) e^{ul} = \sum_{k\geq 0}\mathcal{G}_k(u)\frac{l^k}{k!}.$$

3. Explicit summation formulas of AGPBP

This section deals with various relations of Apostol Genocchi polynomials of order η based on Bell polynomials (AGPBP) in the following theorems:

Theorem 3.1: *The Apostol–Genocchi polynomials of order η based on Bell polynomials have the following relation for $\eta \in \mathbb{C}$ and $k \in \mathbb{N}$;*

$${}_{\mathcal{B}}\mathcal{G}_k^{(\eta)}(u,v;\mu) = \sum_{m=0}^{k}\binom{k}{m}\mathcal{G}_m^{(\eta)}(u;\mu)\mathcal{B}_{k-m}(v). \tag{25}$$

Proof: Using relation (23), we obtain

$$\sum_{k\geq 0} {}_{\mathcal{B}}\mathcal{G}_k^{(\eta)}(u,v;\mu)\frac{l^k}{k!} = \left(\frac{2l}{\mu e^l+1}\right)^{\eta} e^{ul+v(e^l-1)}$$

$$= \left\{\left(\frac{2l}{\mu e^l+1}\right)^{\eta} e^{ul}\right\}\left\{e^{v(e^l-1)}\right\}$$

$$= \left\{\sum_{m\geq 0}\mathcal{G}_m^{(\eta)}(u;\mu)\frac{l^m}{m!}\right\}\left\{\sum_{k\geq 0}\mathcal{B}_k(v)\frac{l^k}{k!}\right\}.$$

By making use of the series rearrangement algorithm, we have

$$\sum_{k\geq 0} {}_{\mathcal{B}}\mathcal{G}_k^{(\eta)}(u,v;\mu)\frac{l^k}{k!} = \sum_{k\geq 0}\left\{\sum_{m=0}^{k}\binom{k}{m}\mathcal{G}_m^{(\eta)}(u;\mu)\mathcal{B}_{k-m}(v)\right\}\frac{l^k}{k!}.$$

Thus, we obtain the desired result (25) by equating both sides. ■

Theorem 3.2: *The Apostol–Genocchi polynomials of order η based on Bell polynomials have the following relation for $\eta \in \mathbb{C}$ and $k \in \mathbb{N}$;*

$${}_{\mathcal{B}}\mathcal{G}_k^{(\eta)}(u,v;\mu) = \sum_{m=0}^{k}\binom{k}{m}\mathcal{G}_m^{(\eta)}(\mu)\mathcal{B}_{k-m}(u;v). \tag{26}$$

Proof: Using the generating function (23), we have

$$\sum_{k\geq 0} {}_{\mathcal{B}}\mathcal{G}_k^{(\eta)}(u,v;\mu)\frac{l^k}{k!} = \left(\frac{2l}{\mu e^l+1}\right)^{\eta} e^{ul+v(e^l-1)}$$

$$= \left\{\left(\frac{2l}{\mu e^l+1}\right)^{\eta}\right\}\left\{e^{ul+v(e^l-1)}\right\}$$

$$= \left\{\sum_{m\geq 0} \mathcal{G}_m^{(\eta)}(\mu)\frac{l^m}{m!}\right\}\left\{\sum_{k\geq 0} \mathcal{B}_k(u;v)\frac{l^k}{k!}\right\}.$$

Thus, we obtain the desired result (26) by using the series rearrangement algorithm. ■

Theorem 3.3: *If $\eta \in \mathbb{C}$ and $k \in \mathbb{N}$, then the Apostol–Genocchi polynomials of order η based on Bell polynomials satisfy the following relation;*

$$ {}_{\mathcal{B}}\mathcal{G}_k^{(\eta)}(u,v;\mu) = \sum_{m=0}^{k}\binom{k}{m}\mathcal{G}_m^{(\eta)}(v;\mu)\,u^{k-m}. \tag{27}$$

Proof: With the help of relation (23), we have

$$\sum_{k\geq 0} {}_{\mathcal{B}}\mathcal{G}_k^{(\eta)}(u,v;\mu)\frac{l^k}{k!} = \left(\frac{2l}{\mu e^l+1}\right)^{\eta} e^{ul+v(e^l-1)}$$

$$= \left\{\left(\frac{2l}{\mu e^l+1}\right)^{\eta} e^{v(e^l-1)}\right\}\left\{e^{ul}\right\}$$

$$= \left\{\sum_{m\geq 0} {}_{\mathcal{B}}\mathcal{G}_m^{(\eta)}(v;\mu)\frac{l^m}{m!}\right\}\left\{\sum_{k\geq 0}\frac{(ul)^k}{k!}\right\}$$

$$= \left\{\sum_{k\geq 0}\sum_{m\geq 0} {}_{\mathcal{B}}\mathcal{G}_m^{(\eta)}(v;\mu)\,\frac{u^k}{k!}\frac{l^{k+m}}{m!}\right\}.$$

By using the series rearrangement algorithm, we have

$$\sum_{k\geq 0} {}_{\mathcal{B}}\mathcal{G}_k^{(\eta)}(u,v;\mu)\frac{l^k}{k!} = \sum_{k\geq 0}\left\{\sum_{m=0}^{k}\binom{k}{m} {}_{\mathcal{B}}\mathcal{G}_m^{(\eta)}(v;\mu)\,u^{k-m}\right\}\frac{l^k}{k!}.$$

Thus, we obtain the desired result (27) by equating both sides. ■

Theorem 3.4: *The Apostol–Genocchi polynomials of order η based on Bell polynomials have the following relation for $\eta \in \mathbb{C}$ and $k \in \mathbb{N}$;*

$$ {}_{\mathcal{B}}\mathcal{G}_k^{(\eta)}(u+v,z;\mu) = \sum_{m=0}^{k}\binom{k}{m}\mathcal{G}_m^{(\eta)}(u;\mu)\mathcal{B}_{k-m}(v,z). \tag{28}$$

Proof: Using generating function (23), we have

$$\sum_{k\geq 0} {}_{\mathcal{B}}\mathcal{G}_k^{(\eta)}(u+v,z;\mu)\frac{l^k}{k!} = \left(\frac{2l}{\lambda e^l+1}\right)^{\eta} e^{(u+v)l+z(e^l-1)}$$
$$= \left\{\left(\frac{2l}{\mu e^l+1}\right)^{\eta} e^{ul}\right\}\left\{e^{vl+z(e^l-1)}\right\}$$
$$= \left\{\sum_{m\geq 0}\mathcal{G}_m^{(\eta)}(u;\mu)\frac{l^m}{m!}\right\}\left\{\sum_{k\geq 0}\mathcal{B}_k(v,z)\frac{l^k}{k!}\right\}.$$

We obtain the desired result (28) by using the series rearrangement method. ■

4. Implicit summation formulas of AGPBP

In this part, we investigate new and interesting identities, such as the implicit summation formula for the Apostol–Genocchi polynomials of order η based on Bell polynomials (AGPBP), specified in the following theorems:

Theorem 4.1: *If $\eta_1,\eta_2\in\mathbb{C}$ and $k\in\mathbb{N}$, then the Apostol–Genocchi polynomials of order η based on Bell polynomials satisfy the following relation;*

$${}_{\mathcal{B}}\mathcal{G}_k^{(\eta_1+\eta_2)}(u_1+u_2,v_1+v_2;\mu) = \sum_{m=0}^{k}\binom{k}{m}{}_{\mathcal{B}}\mathcal{G}_m^{(\eta_1)}(u_1,v_1;\mu)\,{}_{\mathcal{B}}\mathcal{G}_{k-m}^{(\eta_2)}(u_2,v_2;\mu). \quad (29)$$

Proof: We know that

$$\left(\frac{2l}{\mu e^l+1}\right)^{\eta_1+\eta_2} e^{(u_1+u_2)t+(v_1+v_2)(e^l-1)}$$
$$= \left\{\left(\frac{2l}{\mu e^l+1}\right)^{\eta_1} e^{u_1t+v_1(e^l-1)}\right\}\left\{\left(\frac{2l}{\mu e^l+1}\right)^{\eta_2} e^{u_2t+v_2(e^l-1)}\right\}.$$

Using the above identity in the generating function (23), we obtain

$$\sum_{k\geq 0}{}_{\mathcal{B}}\mathcal{G}_k^{(\eta_1+\eta_2)}(u_1+u_2,v_1+v_2;\lambda)\frac{l^k}{k!}$$
$$= \left(\frac{2l}{\mu e^l+1}\right)^{\eta_1+\eta_2} e^{(u_1+u_2)l+(v_1+v_2)(e^l-1)}$$
$$= \left\{\left(\frac{2l}{\mu e^l+1}\right)^{\eta_1} e^{u_1l+v_1(e^l-1)}\right\}\left\{\left(\frac{2l}{\mu e^l+1}\right)^{\eta_2} e^{u_2l+v_2(e^l-1)}\right\}$$
$$= \left\{\sum_{m\geq 0}{}_{\mathcal{B}}\mathcal{G}_m^{(\eta_1)}(u_1,v_1;\mu)\frac{l^m}{m!}\right\}\left\{\sum_{k\geq 0}{}_{\mathcal{B}}\mathcal{G}_k^{(\eta_2)}(u_2,v_2;\mu)\frac{l^k}{k!}\right\}$$
$$= \left\{\sum_{k\geq 0}\sum_{m\geq 0}{}_{\mathcal{B}}\mathcal{G}_m^{(\eta_1)}(u_1,v_1;\mu)\,{}_{\mathcal{B}}\mathcal{G}_k^{(\eta_2)}(u_2,v_2;\mu)\frac{l^{k+m}}{k!m!}\right\},$$

using the series rearrangement algorithm, we get

$$\sum_{k\geq 0} {}_{\mathcal{B}}\mathcal{G}_k^{(\eta_1+\eta_2)}(u_1+u_2, v_1+v_2; \mu)\frac{l^k}{k!}$$
$$= \sum_{k\geq 0}\left\{\sum_{m=0}^{k}\binom{k}{m}{}_{\mathcal{B}}\mathcal{G}_m^{(\eta_1)}(u_1, v_1; \mu)\, {}_{\mathcal{B}}\mathcal{G}_{k-m}^{(\eta_2)}(u_2, v_2; \mu)\right\}\frac{l^k}{k!}.$$

We achieved the desired result (29) by equating both sides. ■

Remark 4.1: If $\eta_1 = \eta$, $\eta_2 = 0$, $u_1 = u$, $u_2 = 1$, $v_1 = v$ and $v_2 = 0$ in (29), we get

$$ {}_{\mathcal{B}}\mathcal{G}_k^{(\eta)}(u+1, v; \mu) = \sum_{m=0}^{k}\binom{k}{m}{}_{\mathcal{B}}\mathcal{G}_m^{(\eta)}(u, v; \mu). \tag{30}$$

It is an extension of the Genocchi polynomials known as

$$\mathcal{G}_k(u+1) = \sum_{m=0}^{k}\binom{k}{m}\mathcal{G}_m(x). \tag{31}$$

Theorem 4.2: *For $\eta \in \mathbb{C}$ and $k \in \mathbb{N}$, the Apostol–Genocchi polynomials of order η based on Bell polynomials satisfy the following relation;*

$$ {}_{\mathcal{B}}\mathcal{G}_{m+l}^{(\eta)}(u, v; \mu) = \sum_{k,j=0}^{m,n}\binom{m}{j}\binom{n}{j}(u-x)^{k+j}\,{}_{\mathcal{B}}\mathcal{B}_{m+n-k-j}^{(\eta)}(x, v; \mu). \tag{32}$$

Proof: We are familiar with a well-known series manipulation formula

$$\sum_{J=0}^{\infty} g(J)\frac{(x+w)^J}{J!} = \sum_{k,j=0}^{\infty} g(k+j)\frac{x^k}{k!}\frac{w^j}{j!}. \tag{33}$$

In (23) the place of l putting $l+w$, we get

$$\left(\frac{2(l+w)}{\mu e^{l+w}-1}\right)^{\eta} e^{v(e^{l+w}-1)} = e^{-u(l+w)}\sum_{m,n\geq 0} {}_{\mathcal{B}}\mathcal{G}_{m+n}^{(\eta)}(u, v; \mu)\frac{l^m}{m!}\frac{w^n}{n!}. \tag{34}$$

In (34) the place of u putting x, we have

$$\left(\frac{2(l+w)}{\mu e^{l+w}-1}\right)^{\eta} e^{v(e^{l+w}-1)} = e^{-x(l+w)}\sum_{m,n\geq 0} {}_{\mathcal{B}}\mathcal{G}_{m+l}^{(\eta)}(x, v; \mu)\frac{l^m}{m!}\frac{w^n}{n!}. \tag{35}$$

With the help of Equations (34) and (35), we obtain

$$\sum_{m,n\geq 0} {}_{\mathcal{B}}\mathcal{G}_{m+n}^{(\eta)}(u, v; \mu)\frac{l^m}{m!}\frac{w^n}{n!} = e^{(u-x)(l+w)}\sum_{m,n\geq 0} {}_{\mathcal{B}}\mathcal{G}_{m+n}^{(\eta)}(x, v; \mu)\frac{l^m}{m!}\frac{w^n}{n!}.$$

It may also be written as

$$\sum_{m,n\geq 0} {}_{\mathcal{B}}\mathcal{G}_{m+n}^{(\eta)}(u,v;\mu)\frac{l^m}{m!}\frac{w^n}{n!} = \sum_{k,j=0}^{\infty}(u-x)^{k+j}\frac{l^k}{k!}\frac{w^j}{j!}\sum_{m,n\geq 0} {}_{\mathcal{B}}\mathcal{G}_{m+n}^{(\eta)}(x,v;\mu)\frac{l^m}{m!}\frac{w^n}{n!}.$$

We achieved the desired result (32), by using the series rearrangement algorithm. ■

Theorem 4.3: *If $\eta \in \mathbb{C}$ and $k \in \mathbb{N}$, then the Apostol–Genocchi polynomials of order η based on Bell polynomials satisfy the following summation formula;*

$$ {}_{\mathcal{B}}\mathcal{G}_{k+1}^{(\eta)}(u+1,v;\mu) - {}_{\mathcal{B}}\mathcal{G}_{k+1}^{(\eta)}(u,v;\mu) = \sum_{m=0}^{k}\binom{k+1}{m} {}_{\mathcal{B}}\mathcal{G}_{m}^{(\eta)}(u,v;\mu). \tag{36}$$

Proof: Using (23), we obtain

$$\begin{aligned}
&\sum_{k\geq 0} {}_{\mathcal{B}}\mathcal{G}_{k}^{(\eta)}(u+1,v;\mu)\frac{l^k}{k!} - \sum_{k\geq 0} {}_{\mathcal{B}}\mathcal{G}_{k}^{(\eta)}(u,v;\mu)\frac{l^k}{k!}\\
&= \left(\frac{2l}{\mu e^l+1}\right)^{\eta} e^{(u+1)l+v(e^l-1)} - \left(\frac{2l}{\mu e^l+1}\right)^{\eta} e^{ul+v(e^l-1)}\\
&= \left(\frac{2l}{\mu e^l+1}\right)^{\eta} e^{ul+v(e^l-1)}\,(e^l-1)\\
&= \left\{\sum_{m\geq 0} {}_{\mathcal{B}}\mathcal{G}_{m}^{(\eta)}(u,v;\mu)\frac{l^m}{m!}\right\}\left\{\sum_{k\geq 0}\frac{l^{k+1}}{(k+1)!}\right\}.
\end{aligned}$$

We achieved the desired result (36), by using the series rearrangement algorithm. ■

Theorem 4.4: *The Apostol–Genocchi polynomials of order η based on Bell polynomials have the following relation for $\eta = 1$ and $k \in \mathbb{N}$;*

$$\mathcal{B}_k(u;v) = \frac{\mu\, {}_{\mathcal{B}}\mathcal{G}_{k+1}(u+1;v) + {}_{\mathcal{B}}\mathcal{G}_{k+1}(u;v)}{2(k+1)}. \tag{37}$$

Proof: Using Bell polynomials of two variables and the generating function (23) for $\eta = 1$, we obtain

$$\begin{aligned}
\sum_{k\geq 0}\mathcal{B}_k(u;v)\frac{l^k}{k!} &= e^{ul+v(e^l-1)}\\
&= \frac{\mu e^l+1}{2l}\left\{\sum_{m=0}^{\infty} {}_{\mathcal{B}}\mathcal{G}_m(u;v)\right\}\\
&= \frac{\mu e^l+1}{2l}\left\{\left(\frac{2l}{e^l+1}\right)e^{ul+v(e^l-1)}\right\}\\
&= \frac{1}{2l}\left\{\mu\left(\frac{2l}{e^l+1}\right)e^{(u+1)l+v(e^l-1)} + \left(\frac{2l}{e^l+1}\right)e^{ul+v(e^l-1)}\right\}
\end{aligned}$$

$$= \frac{1}{2l}\left\{\mu\sum_{k\geq 0}{}_{\mathcal{B}}\mathcal{G}_k(u+1;v)\frac{l^k}{k!}+\sum_{k\geq 0}{}_{\mathcal{B}}\mathcal{G}_k(u;v)\frac{l^k}{k!}\right\}$$

$$= \frac{1}{2}\left\{\mu\sum_{k\geq 0}{}_{\mathcal{B}}\mathcal{G}_k(u+1;v)\frac{l^{k-1}}{k!}+\sum_{k\geq 0}{}_{\mathcal{B}}\mathcal{G}_k(u;v)\frac{l^{k-1}}{k!}\right\}$$

$$= \frac{1}{2(k+1)}\left\{\mu\sum_{k\geq 0}{}_{\mathcal{B}}\mathcal{G}_{k+1}(u+1;v)\frac{l^k}{k!}+\sum_{k\geq 0}{}_{\mathcal{B}}\mathcal{G}_{k+1}(u;v)\frac{l^k}{k!}\right\}.$$

We achieved the desired result (37) by equating the both sides. ■

Theorem 4.5: *For $k \geq 0$ and $\eta \in \mathbb{C}$, the Apostol–Genocchi polynomials of order η based on Bell polynomials satisfy the following relation;*

$$_{\mathcal{B}}\mathcal{G}_k^{(\eta)}(u,v;\mu) = \sum_{i=0}^{k}\sum_{m\geq 0}\binom{k}{i}(u)_m\,\mathcal{S}_2(i,m)\,{}_{\mathcal{B}}\mathcal{G}_k^{(\eta)}(v;\mu). \tag{38}$$

Proof: By using the relation (23), we have

$$\sum_{k\geq 0}{}_{\mathcal{B}}\mathcal{G}_k^{(\eta)}(u,v;\mu)\frac{l^k}{k!} = \left(\frac{2l}{\mu e^l+1}\right)^{\eta} e^{ul+v(e^l-1)}$$

$$= \left(\frac{2l}{\mu e^l+1}\right)^{\eta} e^{v(e^l-1)}\,e^{ul}$$

$$= \left(\frac{2l}{\mu e^l+1}\right)^{\eta} e^{v(e^l-1)}\,(1+e^l-1)^u$$

$$= \left\{\sum_{k\geq 0}{}_{\mathcal{B}}\mathcal{G}_k^{(\eta)}(v;\mu)\frac{l^k}{k!}\right\}\left\{\sum_{m\geq 0}(u)_m\frac{(e^l-1)^m}{m!}\right\}$$

$$\sum_{k\geq 0}{}_{\mathcal{B}}\mathcal{G}_k^{(\eta)}(u,v;\mu)\frac{l^k}{k!} = \left\{\sum_{k\geq 0}{}_{\mathcal{B}}\mathcal{G}_k^{(\eta)}(v;\mu)\frac{l^k}{k!}\right\}\left\{\sum_{m\geq 0}(u)_m\sum_{i\geq 0}\mathcal{S}_2(i,m)\frac{l^i}{i!}\right\}.$$

We obtained the required result (38) by applying the series rearrangement algorithm above. ■

5. Partial derivative formulae of AGPBP

Theorem 5.1: *The differential operator formula for the Apostol–Genocchi polynomials of order η based on Bell polynomials with respect to the variable u are given by*

$$\frac{\partial}{\partial u}\,{}_{\mathcal{B}}\mathcal{G}_k^{(\eta)}(u,v;\mu) = k\,{}_{\mathcal{B}}\mathcal{G}_{k-1}^{(\eta)}(u,v;\mu), \tag{39}$$

which hold for all $k \in \mathbb{N}$.

Proof: Since

$$\frac{\partial}{\partial u}\, e^{ul+v(e^l-1)} = l\, e^{ul+v(e^l-1)}. \tag{40}$$

By using the definition (23) in (40), we achieved the desired result (39). ■

Theorem 5.2: *The difference operator formula for the Apostol–Genocchi polynomials of order η based on Bell polynomials with respect to the variable v is given by*

$$\frac{\partial}{\partial v}\, {}_{\mathcal{B}}\mathcal{G}_k^{(\eta)}(u,v;\mu) = \left\{ {}_{\mathcal{B}}\mathcal{G}_k^{(\eta)}(u+1,v;\mu) - {}_{\mathcal{B}}\mathcal{G}_k^{(\eta)}(u,v;\mu) \right\}, \tag{41}$$

which hold for all $k \in \mathbb{N}$.

Proof: Now, applying the derivative properties in the definition (23), we get

$$\begin{aligned}
&\frac{\partial}{\partial v}\left\{ \sum_{k\geq 0} {}_{\mathcal{B}}\mathcal{G}_k^{(\eta)}(u,v;\mu)\frac{l^k}{k!} \right\} \\
&\quad = \frac{\partial}{\partial v}\left\{ \left(\frac{2l}{\mu e^l+1}\right)^{\eta} e^{ul+v(e^l-1)} \right\} \\
&\quad = \left\{ \left(\frac{2l}{\mu e^l+1}\right)^{\eta} e^{ul+y(e^l-1)} \right\} (e^l-1) \\
&\quad = \left\{ \left(\frac{2l}{\mu e^l+1}\right)^{\eta} e^{(u+1)l+y(e^l-1)} \right\} - \left\{ \left(\frac{2l}{\mu e^l+1}\right)^{\eta} e^{ul+v(e^l-1)} \right\} \\
&\quad = \left\{ \sum_{k\geq 0} {}_{\mathcal{B}}\mathcal{G}_k^{(\eta)}(u+1,v;\mu)\frac{l^k}{k!} \right\} - \left\{ \sum_{k\geq 0} {}_{\mathcal{B}}\mathcal{G}_k^{(\eta)}(u,v;\mu)\frac{l^k}{k!} \right\} \\
&\quad = \left\{ \sum_{k\geq 0} {}_{\mathcal{B}}\mathcal{G}_k^{(\eta)}(u+1,v;\mu) - \sum_{k\geq 0} {}_{\mathcal{B}}\mathcal{G}_k^{(\eta)}(u,v;\mu) \right\} \frac{l^k}{k!}.
\end{aligned}$$

We achieved the desired result (41) by equating both sides. ■

6. Some symmetric identities related to AGPBP

In this part, we investigate many symmetric identities of Apostol Genocchi polynomials of order η associated with Bell polynomials (AGPBP) using the generating functions (20) and (23). We discuss in the following theorems and corollary.

Theorem 6.1: *For a, b, c, d > 0, $n \geq 0$, $\mu \in \mathbb{C}$ and $\eta \geq 1$, Apostol–Genocchi polynomials of order η based on Bell polynomials satisfy the following relation:*

$$\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^{k+1}{}_{\mathcal{B}}\mathcal{G}_{n-k}^{(\eta)}(bu,cv;\mu)\sum_{i=0}^{k}\binom{k}{i}\mathcal{M}_i(a-1;\mu){}_{\mathcal{B}}\mathcal{G}_{k-i}^{(\eta)}(aw,\mathrm{d}x;\mu)$$
$$=\sum_{k=0}^{n}\binom{n}{k}a^{k+1}b^{n-k}{}_{\mathcal{B}}\mathcal{G}_{n-k}^{(\eta)}(aw,\mathrm{d}x;\mu)\sum_{i=0}^{k}\binom{k}{i}\mathcal{M}_i(b-1;\mu){}_{\mathcal{B}}\mathcal{G}_{k-i}^{(\eta)}(bu,cv;\mu). \quad (42)$$

Proof: We use (20) and (23) in the following relation to get a result

$$h(l)=\frac{1}{a^{\eta}b^{\eta-1}}\left(\frac{2al}{\mu e^{al}+1}\right)^{\eta}e^{abul+cv(e^{al}-1)}\left(\frac{1-\mu(-e^{bl})^{a}}{\mu e^{al}+1}\right)\left(\frac{2bl}{\mu e^{bl}+1}\right)^{\eta}e^{abwl+\mathrm{d}x(e^{bl}-1)}$$
$$=\frac{1}{a^{\eta}b^{\eta-1}}\left(\sum_{n\geq 0}{}_{\mathcal{B}}\mathcal{G}_{n}^{(\eta)}(bu,cv;\mu)\frac{(al)^n}{n!}\right)\left(\sum_{i\geq 0}\mathcal{M}_i(a-1;\mu)\frac{(bl)^i}{i!}\right)$$
$$\times\left(\sum_{k\geq 0}{}_{\mathcal{B}}\mathcal{G}_{k}^{(\eta)}(aw,\mathrm{d}x;\mu)\frac{(bl)^k}{k!}\right)$$
$$=\frac{1}{a^{\eta}b^{\eta-1}}\left(\sum_{n\geq 0}{}_{\mathcal{B}}\mathcal{G}_{n}^{(\eta)}(bu,cv;\mu)\frac{(al)^n}{n!}\right)$$
$$\times\left(\sum_{k\geq 0}\sum_{i\geq 0}b^{k+i}\mathcal{M}_i(a-1;\mu){}_{\mathcal{B}}\mathcal{G}_{k}^{(\eta)}(aw,\mathrm{d}x;\mu)\frac{l^{k+i}}{i!k!}\right)$$
$$=\frac{1}{a^{\eta}b^{\eta-1}}\left(\sum_{n\geq 0}{}_{\mathcal{B}}\mathcal{G}_{n}^{(\eta)}(bu,cv;\mu)\frac{(al)^n}{n!}\right)$$
$$\times\left(\sum_{k\geq 0}\sum_{i=0}^{k}\binom{k}{i}b^{k}\mathcal{M}_i(a-1;\mu){}_{\mathcal{B}}\mathcal{G}_{k-i}^{(\eta)}(aw,\mathrm{d}x;\mu)\frac{l^{k}}{k!}\right)$$
$$=\frac{1}{a^{\eta}b^{\eta}}$$
$$\times\left(\sum_{n\geq 0}\sum_{k\geq 0}a^{n}b^{k+1}{}_{\mathcal{B}}\mathcal{G}_{n}^{(\eta)}(bu,cv;\mu)\sum_{i=0}^{k}\binom{k}{i}b^{k}\mathcal{M}_i(a-1;\mu){}_{\mathcal{B}}\mathcal{G}_{k-i}^{(\eta)}(aw,\mathrm{d}x;\mu)\frac{l^{n+k}}{n!k!}\right), \quad (43)$$

using the series rearrangement algorithm, we get

$$h(l)=\frac{1}{a^{\eta}b^{\eta}}\times\sum_{n\geq 0}\left(\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^{k+1}{}_{\mathcal{B}}\mathcal{G}_{n-k}^{(\eta)}(bu,cv;\mu)\right.$$
$$\left.\times\sum_{i=0}^{k}\binom{k}{i}\mathcal{M}_i(a-1;\mu){}_{\mathcal{B}}\mathcal{G}_{k-i}^{(\eta)}(aw,\mathrm{d}x;\mu)\right)\frac{l^n}{n!}. \quad (44)$$

Using a similar algorithm, we obtain

$$h(l)\frac{1}{a^\eta b^\eta} = \times \sum_{n\geq 0}\left(\sum_{k=0}^{n}\binom{n}{k} b^{n-k}a^{k+1}{}_{\mathcal{B}}\mathcal{G}_{n-k}^{(\eta)}(aw,\mathrm{d}x;\mu)\right.$$
$$\left.\times \sum_{i=0}^{k}\binom{k}{i}\mathcal{M}_i(b-1;\mu){}_{\mathcal{B}}\mathcal{G}_{k-i}^{(\eta)}(bu,cv;\mu)\right)\frac{l^n}{n!}. \tag{45}$$

By equating Equations (44) and (45), we have achieved the desired result (42). ■

By substituting $v = x = 0$ in Theorem (6.1), we may reduce (42) to the known result obtained by Khan et al. [26] as follows:

Corollary 6.1: *For a, $b > 0$, $n \geq 0$, $\eta \geq 1$ and $\mu \in \mathbb{C}$ the following relation holds true:*

$$\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^{k+1}\mathcal{G}_{n-k}^{(\eta)}(bu;\mu)\sum_{i=0}^{k}\binom{k}{i}\mathcal{M}_i(a-1;\mu)\mathcal{G}_{k-i}^{(\eta)}(aw;\mu)$$
$$=\sum_{k=0}^{n}\binom{n}{k}a^{k+1}b^{n-k}\mathcal{G}_{n-k}^{(\eta)}(aw;\mu)\sum_{i=0}^{k}\binom{k}{i}\mathcal{M}_i(b-1;\mu)\mathcal{G}_{k-i}^{(\eta)}(bu;\mu). \tag{46}$$

Again, by substituting $v = x = 0$ and $\eta = 1$ in Theorem (6.1), we have to get the following relation:

Corollary 6.2: *For a, $b > 0$, $n \geq 0$ and $\mu \in \mathbb{C}$ the following relation holds true:*

$$\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^{k+1}\mathcal{G}_{n-k}(bu;\mu)\sum_{i=0}^{k}\binom{k}{i}\mathcal{M}_i(a-1)\mathcal{G}_{k-i}(aw;\mu)$$
$$=\sum_{k=0}^{n}\binom{n}{k}a^{k+1}b^{n-k}\mathcal{G}_{n-k}(aw;\mu)\sum_{i=0}^{k}\binom{k}{i}\mathcal{M}_i(b-1)\mathcal{G}_{k-i}(bu;\mu). \tag{47}$$

If we put $\mu = 1$ in (47) then we have to get the following relation:

Corollary 6.3: *For a, $b > 0$ and $n \geq 0$ the following relation holds true:*

$$\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^{k+1}\mathcal{G}_{n-k}(bu)\sum_{i=0}^{k}\binom{k}{i}\mathcal{M}_i(a-1)\mathcal{G}_{k-i}(aw)$$
$$=\sum_{k=0}^{n}\binom{n}{k}a^{k+1}b^{n-k}\mathcal{G}_{n-k}(aw)\sum_{i=0}^{k}\binom{k}{i}\mathcal{M}_i(b-1)\mathcal{G}_{k-i}(bu). \tag{48}$$

Theorem 6.2: *For a, b, c, d > 0, n ≥ 0, μ ∈ ℂ and η ≥ 1, Apostol–Genocchi polynomials η based on Bell polynomials satisfy the following relation:*

$$\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}(\mu)^{i+j}a^{k}b^{n-k}{}_{\mathcal{B}}\mathcal{G}_{k}^{(\eta)}\left(bu+\frac{b}{a}i,cv;\mu\right){}_{\mathcal{B}}\mathcal{G}_{n-k}^{(\eta)}\left(aw+\frac{a}{b}j,cv;\mu\right)$$
$$=\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{b-1}\sum_{j=0}^{a-1}(\mu)^{i+j}a^{n-k}b^{k}{}_{\mathcal{B}}\mathcal{G}_{k}^{(\eta)}\left(au+\frac{a}{b}i,cv;\mu\right){}_{\mathcal{B}}\mathcal{G}_{n-k}^{(\eta)}\left(bw+\frac{b}{a}j,cv;\mu\right). \tag{49}$$

Proof: Let $h(l)$ be symmetric in a and b, and we can prove the theorem by expanding $h(l)$ into a series in two different ways.

$$h(l)=\frac{(2al)^{\eta}(2bl)^{\eta}e^{ab(u+w)l}e^{cv(e^{al}+e^{bl}-2)}(\mu^{a}e^{abl}+1)(\mu^{b}e^{abl}+1)}{(\mu e^{al}+1)^{\eta+1}(\mu e^{bl}+1)^{\eta+1}}$$

$$h(l)=\left(\frac{2al}{\mu e^{al}+1}\right)^{\eta}e^{abul+cv(e^{al}-1)}\left(\frac{\mu^{a}e^{abl}+1}{\mu e^{bl}+1}\right)\left(\frac{2bl}{\mu e^{bl}+1}\right)^{\eta}e^{abwl+cv(e^{bl}-1)}$$
$$\times\left(\frac{\mu^{b}e^{abl}+1}{\mu e^{al}+1}\right)$$
$$=\left(\frac{2al}{\mu e^{al}+1}\right)^{\eta}e^{abul+cv(e^{al}-1)}\sum_{i=0}^{a-1}\mu^{i}e^{bli}\left(\frac{2bl}{\mu e^{bl}+1}\right)^{\eta}e^{abwl+cv(e^{bl}-1)}\sum_{j=0}^{b-1}\mu^{j}e^{alj}$$
$$=\sum_{i=0}^{a-1}\mu^{i}\left(\frac{2al}{\mu e^{al}+1}\right)^{\eta}e^{\left(bu+\frac{b}{a}i\right)al+cv(e^{al}-1)}\sum_{j=0}^{b-1}\mu^{j}\left(\frac{2bl}{\mu e^{al}+1}\right)^{\eta}e^{\left(aw+\frac{a}{b}j\right)bl+cv(e^{bl}-1)}$$
$$=\left(\sum_{i=0}^{a-1}\mu^{i}\sum_{k\geq0}{}_{\mathcal{B}}\mathcal{G}_{k}^{(\eta)}\left(bu+\frac{b}{a}i,cv;\mu\right)\frac{(al)^{k}}{k!}\right)\left(\sum_{j=0}^{b-1}\mu^{j}\sum_{n\geq0}{}_{\mathcal{B}}\mathcal{G}_{n}^{(\eta)}\left(aw+\frac{a}{b}j,cv;\mu\right)\frac{(bl)^{n}}{n!}\right)$$
$$=\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}\sum_{n\geq0}\sum_{k\geq0}\mu^{i+j}{}_{\mathcal{B}}\mathcal{G}_{k}^{(\eta)}\left(bu+\frac{b}{a}i,cv;\mu\right){}_{\mathcal{B}}\mathcal{G}_{n}^{(\eta)}\left(aw+\frac{a}{b}j,cv;\mu\right)a^{k}b^{n}\frac{l^{n+k}}{n!k!}. \tag{50}$$

By using the series rearrangement algorithm, we get

$$h(l)=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}(\mu)^{i+j}a^{k}b^{n-k}{}_{\mathcal{B}}\mathcal{G}_{k}^{(\eta)}\left(bu+\frac{b}{a}i,cv;\mu\right){}_{\mathcal{B}}\mathcal{G}_{n-k}^{(\eta)}\right.$$
$$\left.\times\left(aw+\frac{a}{b}j,cv;\mu\right)\right)\frac{l^{n}}{n!}. \tag{51}$$

If the same arguments apply to different ways, we see that

$$h(l) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{b-1} \sum_{j=0}^{a-1} (\mu)^{i+j} a^{n-k} b^{k} {}_{\mathcal{B}}\mathcal{G}_{k}^{(\eta)} \left(au + \frac{a}{b} i, cv; \mu \right) {}_{\mathcal{B}}\mathcal{G}_{n-k}^{(\eta)} \right.$$
$$\left. \times \left(bw + \frac{b}{a} j, cv; \mu \right) \right) \frac{l^n}{n!}. \tag{52}$$

By equating (51) and (52), we obtain the desired result (49). ■

By assuming the value $v = 0$ in Theorem (6.2), we have to get the known result given by Khan et al. (see [26, Eq. 3.11]) as follows:

Corollary 6.4: *For a, b, c, d > 0, n ≥ 0, μ ∈ ℂ and η ≥ 1 Apostol–Genocchi polynomials of order η based on Bell polynomials satisfy the following relation:*

$$\sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} (\mu)^{i+j} a^{k} b^{n-k} {}_{\mathcal{B}}\mathcal{G}_{k}^{(\eta)} \left(bu + \frac{b}{a} i; \mu \right) {}_{\mathcal{B}}\mathcal{G}_{n-k}^{(\eta)} \left(aw + \frac{a}{b} j; \mu \right)$$
$$= \sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{b-1} \sum_{j=0}^{a-1} (\mu)^{i+j} a^{n-k} b^{k} {}_{\mathcal{B}}\mathcal{G}_{k}^{(\eta)} \left(au + \frac{a}{b} i; \mu \right) {}_{\mathcal{B}}\mathcal{G}_{n-k}^{(\eta)} \left(bw + \frac{b}{a} j; \mu \right). \tag{53}$$

Theorem 6.3: *For a, b, c, d > 0, n ≥ 0, μ ∈ ℂ and η ≥ 1, Apostol–Genocchi polynomials of order η based on Bell polynomials satisfy the following relation:*

$$\sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} (\mu)^{i+j} a^{k} b^{n-k} {}_{\mathcal{B}}\mathcal{G}_{k}^{(\eta)} \left(bu + \frac{b}{a} i + j, cv; \mu \right) {}_{\mathcal{B}}\mathcal{G}_{n-k}^{(\eta)} (aw, cv; \mu)$$
$$= \sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{b-1} \sum_{j=0}^{a-1} (\mu)^{i+j} a^{n-k} b^{k} {}_{\mathcal{B}}\mathcal{G}_{k}^{(\eta)} \left(au + \frac{a}{b} i + j, cv; \mu \right) {}_{\mathcal{B}}\mathcal{G}_{n-k}^{(\eta)} (bw, cy; \mu). \tag{54}$$

Proof: Let $h(l)$ be symmetric in a and b, and we can prove the theorem by expanding $h(l)$ into a series in two different ways.

$$h(l) = \frac{(2al)^{\eta} (2bl)^{\eta} e^{ab(u+w)l} e^{cv(e^{al}+e^{bl}-2)} (\mu^{a} e^{abl} + 1)(\mu^{b} e^{abl} + 1)}{(\mu e^{al} + 1)^{\eta+1} (\mu e^{bl} + 1)^{\eta+1}}$$

$$h(l) = \left(\frac{2al}{\mu e^{al} + 1} \right)^{\eta} e^{abul + cv(e^{al}-1)} \left(\frac{\mu^{a} e^{abl} + 1}{\mu e^{bl} + 1} \right) \left(\frac{2bl}{\mu e^{bl} + 1} \right)^{\eta} e^{abwl + cv(e^{bl}-1)}$$

$$\times\left(\frac{\mu^b e^{abl}+1}{\mu e^{al}+1}\right)$$

$$= \left(\frac{2al}{\mu e^{al}+1}\right)^{\eta} e^{abul+cv(e^{al}-1)} \sum_{i=0}^{a-1}\mu^i e^{bli}\left(\frac{2bl}{\mu e^{bl}+1}\right)^{\eta} e^{abwl+cv(e^{bl}-1)}\sum_{j=0}^{b-1}\mu^j e^{alj}$$

$$= \left(\frac{2al}{\mu e^{al}+1}\right)^{\eta} e^{abul+cv(e^{al}-1)} \left(\frac{\mu^a e^{abl}+1}{\mu e^{bl}+1}\right)\left(\frac{2bl}{\mu e^{bl}+1}\right)^{\eta} e^{abwl+cv(e^{bl}-1)}$$

$$\times\left(\frac{\mu^b e^{abl}+1}{\mu e^{al}+1}\right)$$

$$= \left(\frac{2al}{\mu e^{al}+1}\right)^{\eta} e^{abul+cv(e^{al}-1)} \sum_{i=0}^{a-1}\mu^i e^{bli}\left(\frac{2bl}{\mu e^{bl}+1}\right)^{\eta} e^{abwl+cv(e^{bl}-1)}\sum_{j=0}^{b-1}\mu^j e^{alj}$$

$$= \sum_{i=0}^{a-1}\sum_{j=0}^{b-1}\mu^{i+j}\left(\frac{2al}{\mu e^{al}+1}\right)^{\eta} e^{abul+bli+alj+cv(e^{al}-1)}\left(\frac{2bl}{\mu e^{bl}+1}\right)^{\eta} e^{abwl+cv(e^{bl}-1)}$$

$$= \sum_{i=0}^{a-1}\sum_{j=0}^{b-1}\mu^{i+j}\left(\frac{2al}{\mu e^{al}+1}\right)^{\eta} e^{\left(bu+\frac{b}{a}i+j\right)al+cv(e^{al}-1)}\left(\frac{2bl}{\mu e^{bl}+1}\right)^{\eta} e^{abwl+cv(e^{bl}-1)}$$

$$= \sum_{i=0}^{a-1}\sum_{j=0}^{b-1}\mu^{i+j}a^k b^{n-k}\left(\sum_{k\geq 0}{}_{\mathcal{B}}\mathcal{G}_k^{(\eta)}\left(bu+\frac{b}{a}i+j,cv;\mu\right)\right)\left(\sum_{n\geq 0}{}_{\mathcal{B}}\mathcal{G}_n^{(\eta)}(aw,cv;\mu)\right)\frac{l^{n+k}}{n!k!}. \tag{55}$$

By using the series rearrangement algorithm, we have to obtain

$$h(l) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}(\mu)^{i+j}a^k b^{n-k}{}_{\mathcal{B}}\mathcal{G}_k^{(\eta)}\left(bu+\frac{b}{a}i+j,cv;\mu\right){}_{\mathcal{B}}\mathcal{G}_{n-k}^{(\eta)}(aw,cv;\mu)\right)\frac{l^n}{n!}. \tag{56}$$

If the same arguments apply to different ways, we see that

$$h(l) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{b-1}\sum_{j=0}^{a-1}(\mu)^{i+j}a^{n-k} b^{k}{}_{\mathcal{B}}\mathcal{G}_k^{(\eta)}\left(au+\frac{a}{b}i+j,cv;\mu\right){}_{\mathcal{B}}\mathcal{G}_{n-k}^{(\eta)}(bw,cv;\mu)\right)\frac{l^n}{n!}. \tag{57}$$

By equating the results obtained in (56) and (57); we have to get a desired result (58). ■

By assuming the value $v = 0$ in Theorem (6.3), we have to get the known result given by Khan et al. (see [26, Eq. 3.18]) as follows:

Corollary 6.5: *For $a, b, c, d > 0$, $n \geq 0$, $\mu \in \mathbb{C}$ and $\eta \geq 1$ Apostol–Genocchi polynomials of η based on Bell polynomials satisfy the following relation:*

$$\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}(\mu)^{i+j}a^{k}b^{n-k}{}_{\mathcal{B}}\mathcal{G}_{k}^{(\eta)}\left(bu+\frac{b}{a}i+j;\mu\right){}_{\mathcal{B}}\mathcal{G}_{n-k}^{(\eta)}(aw;\mu)$$

$$=\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{b-1}\sum_{j=0}^{a-1}(\mu)^{i+j}a^{n-k}b^{k}{}_{\mathcal{B}}\mathcal{G}_{k}^{(\eta)}\left(au+\frac{a}{b}i+j;\mu\right){}_{\mathcal{B}}\mathcal{G}_{n-k}^{(\eta)}(bw;\mu). \tag{58}$$

7. Conclusions

Motivated by multiple applications in the diverse field of Mathematical sciences such as Combinatorial analysis, Number theory, etc., in this research, we have presented a mix type Apostol–Genocchi polynomials of order η based on Bell polynomials (AGPBP) and investigated their different important identities such as correlation formulas, implicit summation formulas, derivative formulas and some symmetric identities related to AGPBP. The result obtained in this paper is specialized to yield a large number of new and known identities involving basic and unified polynomials presented by other authors. Motivated by the above, we are able to construct various Bell-based unified polynomials and study their identities and properties.

Disclosure statement

No potential conflict of interest was reported by the author(s).

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