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A subclass of meromorphic Janowski-type multivalent q-starlike functions involving a q-differential operator

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Abstract

Keeping in view the latest trends toward quantum calculus, due to its various applications in physics and applied mathematics, we introduce a new subclass of meromorphic multivalent functions in Janowski domain with the help of the q-differential operator. Furthermore, we investigate some useful geometric and algebraic properties of these functions. We discuss sufficiency criteria, distortion bounds, coefficient estimates, radius of starlikeness, radius of convexity, inclusion property, and convex combinations via some examples and, for some particular cases of the parameters defined, show the credibility of these results.

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1 Introduction and motivation

In the classical calculus, if the limit is replaced by familiarizing the parameter q with limitation $0 < q < 1$, then the study of such notions is called quantum calculus (q-calculus). This area of study has attracted the researchers due to its applications in various branches of mathematics and physics; for details, see [10, 11]. Jackson [19, 20] was the first to give some applications of q-calculus and introduced the q-analogues of the derivative and integral.

Using the notion of q-beta functions, Aral and Gupta [10–12] established a new q-Baskakov–Durrmeyer-type operator. Furthermore, Aral and Anastassiu [7–9] discussed a generalization of complex operators, known as the q-Picard and q-Gauss–Weierstrass singular integral operators. Lately, a q-analogue version of Ruscheweyh-type differential operator was defined by Kanas and Răducanu [21] using the convolution notions and examined some its properties. For more applications of this operator, see [5]. Moreover, Ahuja et al. [2] investigated a q-analogue of Bieberbach–de Branges and Fekete–Szegö theorems for certain families of q-convex and q-close-to-convex functions. Also, Khan et al. [22] studied some families of multivalent q-starlike functions involving higher-order q-derivatives. For more recent work related to q-calculus, we refer the reader to [25, 38, 39].

Let $\mathcal{M}_p$ denote the class of p-valent meromorphic functions f that are regular (analytic) in the punctured disc $\mathbb{D} = \{\zeta \in \mathbb{C} : 0 < |\zeta| < 1\}$ and satisfy the normalization

$$f(\zeta) = \frac{1}{\zeta^p} + \sum_{k=p+1}^{\infty} a_k \zeta^k \quad (\zeta \in \mathbb{D}). \tag{1.1}$$

Also, let $\mathcal{MS}_p^*(\alpha)$ and $\mathcal{MC}_p(\alpha)$ denote the popular classes of meromorphic p-valent starlike and meromorphic p-valent convex functions of order α $(0 \leq \alpha < p)$, respectively.

Definition 1 For two analytic functions f_j $(j = 1, 2)$ in $\mathbb{D}$, the function f_1 is said to be subordinate to the function f_2, written as

$$f_1 \prec f_2 \quad \text{or} \quad f_1(\zeta) \prec f_2(\zeta) \quad (\zeta \in \mathbb{D}),$$

if there is a Schwartz function w, analytic in $\mathbb{D}$, such that

$$w(0) = 0, \qquad \left|w(\zeta)\right| < 1,$$

and

$$f_1(\zeta) = f_2\bigl(w(\zeta)\bigr).$$

Further, if the function f_2 is univalent in $\mathbb{D}$, then we have the following equivalence relation:

$$f_1(z\zeta) \prec f_2(\zeta) \quad (\zeta \in \mathbb{U}) \quad \Longleftrightarrow \quad f_1(0) = f_2(0) \quad \text{and} \quad f_1(\mathbb{D}) \subset f_2(\mathbb{D}).$$

For $q \in (0, 1)$, the q-difference operator or q-derivative of a function f is defined by

$$\partial_q f(\zeta) = \frac{f(\zeta) - f(\zeta q)}{\zeta(1 - q)} \quad (\zeta \neq 0, q \neq 1). \tag{1.2}$$

We can observe that for $k \in \mathbb{N}$ (where $\mathbb{N}$ is the set of natural numbers) and $\zeta \in \mathbb{D}$,

$$\partial_q \left\{ \sum_{k=1}^{\infty} a_k \zeta^k \right\} = \sum_{k=1}^{\infty} [k, q] a_k \zeta^{k-1}, \tag{1.3}$$

where

$$[k, q] = \frac{1 - q^k}{1 - q} = 1 + \sum_{l=1}^{k} q^l \quad \text{and} \quad [0, q] = 0.$$

The q-number shift factorial for any nonnegative integer k is defined as

$$[k, q]! = \begin{cases} 1, & k = 0, \\ [1, q][2, q][3, q] \cdots [k, q], & k \in \mathbb{N}. \end{cases}$$

Furthermore, for $x \in \mathbb{R}$, the q-generalized Pochhammer symbol is defined as

$$[x,q]_n = \begin{cases} [x,q][x+1,q]\cdots[x+k-1,q], & k \in \mathbb{N}, \\ 1, & k = 0. \end{cases}$$

We now recall the differential operator $\mathcal{D}_{\mu,q} : \mathcal{M}_p \to \mathcal{M}_p$ defined by Ahmad et al. [1] by

$$\mathcal{D}_{\mu,q} f(\zeta) = \bigl(1 + [p,q]\mu\bigr) f(\zeta) + \mu q^p \zeta \partial_q f(\zeta), \tag{1.4}$$

where $\mu \geq 0$.

Now using (1.1), we get

$$\mathcal{D}_{\mu,q} f(\zeta) = \frac{1}{\zeta^p} + \sum_{k=p+1}^{\infty} \bigl(1 + [p,q]\mu + \mu q^p [k,q]\bigr) a_k \zeta^k.$$

We define this operator in such a way that

$$\mathcal{D}^0_{\mu,q} f(\zeta) = f(\zeta)$$

and

$$\mathcal{D}^2_{\mu,q} f(\zeta) = \mathcal{D}_{\mu,q}\bigl(\mathcal{D}_{\mu,q} f(\zeta)\bigr) = \frac{1}{\zeta^p} + \sum_{k=p+1}^{\infty} \bigl(1 + [p,q]\mu + \mu q^p [k,q]\bigr)^2 a_k \zeta^k.$$

In the identical way, for $m \in N$, we get

$$\mathcal{D}^m_{\mu,q} f(\zeta) = \frac{1}{\zeta^p} + \sum_{k=p+1}^{\infty} \bigl(1 + [p,q]\mu + \mu q^p [k,q]\bigr)^m a_k \zeta^k. \tag{1.5}$$

From (1.4) and (1.5) after some simplification, we get the identity

$$\mathcal{D}^{m+1}_{\mu,q} f(\zeta) = \mu q^p \zeta \partial_q \mathcal{D}^m_{\mu,q} f(\zeta) + \bigl(1 + [p,q]\mu\bigr) \mathcal{D}^m_{\mu,q} f(\zeta). \tag{1.6}$$

Now as of $q \to 1-$, the q-differential operator defined in (1.4) reduces to the well-known differential operator defined in [28]. For details on q-analogues of differential operators, we refer the reader to [3, 4, 27, 32].

Definition 2 ([18]) A function $f \in \mathcal{A}$ belongs to the functions class $\mathcal{S}^*_q$ if

$$f(0) = f'(0) - 1 = 0 \tag{1.7}$$

and

$$\left| \frac{z}{f(z)} (D_q f)(z) - \frac{1}{1-q} \right| \leq \frac{1}{1-q}. \tag{1.8}$$

Note that by the last inequality it is obvious that in the limit as $q \to 1-$, we have

$$\left| w - \frac{1}{1-q} \right| \leq \frac{1}{1-q}.$$

This closed disk is merely in the right-half planem and the class $\mathcal{S}_q^*$ of q-starlike functions turns into the prominent class $\mathcal{S}^*$.

Inspired by the above-mentioned works and [14–17, 23, 29, 31, 34–37, 42–44], we now define the subfamily $\mathcal{M}_{\mu,q}(p, m, \mathcal{O}_1, \mathcal{O}_2)$ of $\mathcal{M}_p$ using the idea of the operator $\mathcal{D}_{\mu,q}^m$ as follows.

Definition 3 Under conditions $-1 \leq \mathcal{O}_2 < \mathcal{O}_1 \leq 1$ and $q \in (0, 1)$, we define $f \in \mathcal{M}_p$ to be in the set $\mathcal{M}_{\mu,q}(p, m, \mathcal{O}_1, \mathcal{O}_2)$ if it satisfies

$$\frac{-q^p \zeta \partial_q \mathcal{D}_{\mu,q}^m f(\zeta)}{[p,q] \mathcal{D}_{\mu,q}^m f(\zeta)} \prec \frac{1 + \mathcal{O}_1 \zeta}{1 + \mathcal{O}_2 \zeta}, \tag{1.9}$$

where the notation "$\prec$" stands for the familiar notion of subordination. Equivalently, we can write condition (1.9) as

$$\left| \frac{\frac{q^p \zeta \partial_q \mathcal{D}_{\mu,q}^m f(\zeta)}{[p,q] \mathcal{D}_{\mu,q}^m f(\zeta)} + 1}{\mathcal{O}_1 + \mathcal{O}_2 \frac{q^p \zeta \partial_q \mathcal{D}_{\mu,q}^m f(\zeta)}{[p,q] \mathcal{D}_{\mu,q}^m f(\zeta)}} \right| < 1. \tag{1.10}$$

Remark 1 First of all, it is easy to see that

$$\lim_{q \to 1-} \mathcal{M}_{\mu,q}(1, 0, \mathcal{O}_1, \mathcal{O}_2) = \mathcal{MS}^*[\mathcal{O}_1, \mathcal{O}_2],$$

where $\mathcal{MS}^*[\mathcal{O}_1, \mathcal{O}_2]$ is the function class introduced and studied by Ali et al. [6]. Secondly, we have

$$\mathcal{M}_{\mu,q}(p, 0, 1, -1) = \mathcal{MS}_{p,q}^*,$$

where $\mathcal{MS}_{p,q}^*$ is the class of meromorphic p-valent q-starlike functions. Thirdly, we have

$$\lim_{q \to 1-} \mathcal{M}_{\mu,q}(p, 0, 1, -1) = \mathcal{MS}_{p,q}^*,$$

where $\mathcal{MS}_p^*$ is the well-known class of meromorphic p-valent starlike functions. Fourthly, we have

$$\lim_{q \to 1-} \mathcal{M}_{\mu,q}(1, 0, 1, -1) = \mathcal{MS}^*,$$

where $\mathcal{MS}^*$ is the class of meromorphic starlike functions. The class $\mathcal{MS}^*$ and other similar classes have been studied by Pommerenke [30] and Clunie and Miller in [13, 26], respectively, and by many others.

In this paper, with the help of a certain q-differential operator, we introduce a new subclass of meromorphic multivalent functions involving the Janowski functions. Furthermore, we investigate some useful geometric and algebraic properties of these functions. We discuss sufficiency criteria, distortion bounds, coefficient estimates, radius of starlikness, radius of convexity, inclusion property, and convex combinations via some examples, and for some particular cases of the parameters defined, we show the credibility of these results.

2 A set of lemmas

In our main results, we use the following important lemmas.

Lemma 1 ([24]) *Let* $-1 \leq \mathcal{O}_4 \leq \mathcal{O}_2 < \mathcal{O}_1 \leq \mathcal{O}_3 \leq 1$. *Then*

$$\frac{1+\mathcal{O}_1\zeta}{1+\mathcal{O}_2\zeta} \prec \frac{1+\mathcal{O}_3\zeta}{1+\mathcal{O}_4\zeta}.$$

Lemma 2 ([33]) *Let* $h(\zeta)$ *be a regular function in* $\mathbb{D}$ *of the form*

$$h(\zeta) = 1 + \sum_{k=1}^{\infty} d_k \zeta^k,$$

and let $k(\zeta)$ *be a regular convex function in* $\mathbb{D}$ *of the form*

$$k(\zeta) = 1 + \sum_{k=1}^{\infty} k_k \zeta^k.$$

So if $h(\zeta) \prec k(\zeta)$, *then* $|d_k| \leq |k_1|$ *for all* $k \in \mathbb{N} = \{1, 2, \dots\}$.

3 Main results

Theorem 1 *A function* $f \in \mathfrak{A}_p$ *of the form* (1.1) *is in the class* $\mathcal{M}_{\mu,q}(p, m, \mathcal{O}_1, \mathcal{O}_2)$ *if and only if*

$$\sum_{k=p+1}^{\infty} \Lambda_k |a_k| \leq [p,q](\mathcal{O}_1 - \mathcal{O}_2), \tag{3.1}$$

where

$$\Lambda_k = \bigl(q^p[k,q](1+\mathcal{O}_2) + (1+\mathcal{O}_1)[p,q]\bigr)\bigl(1+[p,q]\mu + \mu q^p[k,q]\bigr)^m.$$

Proof For f to be in the class $\mathcal{M}_{\mu,q}(p, m, \mathcal{O}_1, \mathcal{O}_2)$, we need to show inequality (1.10). For this, consider

$$\left| \frac{\frac{q^p\zeta\partial_q\mathcal{D}^m_{\mu,q}f(\zeta)}{[p,q]\mathcal{D}^m_{\mu,q}f(\zeta)} + 1}{\mathcal{O}_1 + \mathcal{O}_2\frac{q^p\zeta\partial_q\mathcal{D}^m_{\mu,q}f(\zeta)}{[p,q]\mathcal{D}^m_{\mu,q}f(\zeta)}} \right| = \left| \frac{q^p\zeta\partial_q\mathcal{D}^m_{\mu,q}f(\zeta) + [p,q]\mathcal{D}^m_{\mu,q}f(\zeta)}{\mathcal{O}_1[p,q]\mathcal{D}^m_{\mu,q}f(\zeta) + \mathcal{O}_2 q^p\zeta\partial_q\mathcal{D}^m_{\mu,q}f(\zeta)} \right|.$$

Using (1.4), after simplification, by (1.2) and (1.5) we get that it is equal to

$$\begin{aligned}
&\left|\frac{\sum_{k=p+1}^{\infty}(1+[p,q]\mu+\mu q^p[k,q])^m(q^p[k,q]+[p,q])a_k\zeta^k}{\frac{(\mathcal{O}_1-\mathcal{O}_2)[p,q]}{\zeta^p}+\sum_{k=p+1}^{\infty}\vartheta_q a_k\zeta^k}\right| \\
&\quad=\left|\frac{\sum_{k=p+1}^{\infty}(1+[p,q]\mu+\mu q^p[k,q])^m(q^p[k,q]+[p,q])a_k\zeta^{k+p}}{(\mathcal{O}_1-\mathcal{O}_2)[p,q]+\sum_{n=p+1}^{\infty}\vartheta_q a_k\zeta^{k+p}}\right| \\
&\quad\leq\frac{\sum_{k=p+1}^{\infty}(1+[p,q]\mu+\mu q^p[k,q])^m(q^p[k,q]+[p,q])|a_k|}{(\mathcal{O}_1-\mathcal{O}_2)[p,q]-\vartheta_q|a_k|}<1,
\end{aligned}$$

where

$$\vartheta_q=\left(1+[p,q]\mu+\mu q^p[k,q]\right)^m\left(\mathcal{O}_1[p,q]+\mathcal{O}_2 q^p[k,q]\right).$$

Using inequality (3.1), we can get the direct part of the proof.

For the converse part, let $f\in\mathcal{M}_{\mu,q}(p,m,\mathcal{O}_1,\mathcal{O}_2)$ be given by (1.1). Then from (1.10), for $\zeta\in\mathbb{D}$, we have

$$\begin{aligned}
&\left|\frac{\frac{q^p\zeta\partial_q\mathcal{D}^m_{\mu,q}f(\zeta)}{[p,q]\mathcal{D}^m_{\mu,q}f(\zeta)}+1}{\mathcal{O}_1+\mathcal{O}_2\frac{q^p\zeta\partial_q\mathcal{D}^m_{\mu,q}f(\zeta)}{[p,q]\mathcal{D}^m_{\mu,q}f(\zeta)}}\right| \\
&\quad=\left|\frac{\sum_{k=p+1}^{\infty}(1+[p,q]\mu+\mu q^p[k,q])^m(q^p[k,q]+[p,q])a_k\zeta^{k+p}}{(\mathcal{O}_1-\mathcal{O}_2)[p,q]+\sum_{k=p+1}^{\infty}\vartheta_q a_k\zeta^{k+p}}\right|.
\end{aligned}$$

Since $\Re(\zeta)\leq|\zeta|$, we have

$$\Re\left\{\frac{\sum_{k=p+1}^{\infty}(1+[p,q]\mu+\mu q^p[k,q])^m(q^p[k,q]+[p,q])a_k\zeta^{k+p}}{(\mathcal{O}_1-\mathcal{O}_2)[p,q]+\sum_{k=p+1}^{\infty}\vartheta_q a_k\zeta^{k+p}}\right\}<1. \tag{3.2}$$

Now choose values of ζ on the real axis such that

$$\frac{q^p\zeta\partial_q\mathcal{D}^m_{\mu,q}f(\zeta)}{[p,q]\mathcal{D}^m_{\mu,q}f(\zeta)}$$

is real. Clearing the denominator in (3.2) and letting $\zeta\to 1^-$ through real values, we obtain (3.1). □

Example 2 For the function

$$f(\zeta)=\frac{1}{\zeta^p}+\sum_{k=p+1}^{\infty}\frac{[p,q](\mathcal{O}_1-\mathcal{O}_2)}{\Lambda_k}x_k\zeta^k \quad (\zeta\in\mathbb{D})$$

such that

$$\sum_{k=p+1}^{\infty}|x_k|=1,$$

we have

$$\sum_{k=p+1}^{\infty} \Lambda_k |a_k| = \sum_{k=p+1}^{\infty} [p,q](\mathcal{O}_1 - \mathcal{O}_2)|x_k|$$
$$= [p,q](\mathcal{O}_1 - \mathcal{O}_2) \sum_{k=p+1}^{\infty} |x_k| = [p,q](\mathcal{O}_1 - \mathcal{O}_2).$$

Thus $f \in \mathcal{M}_{\mu,q}(p,m,\mathcal{O}_1,\mathcal{O}_2)$, and inequality (3.1) is sharp for this function.

Corollary 1 ([6]) *If f is in the class $\mathcal{MS}^*[\mathcal{O}_1,\mathcal{O}_2]$ and has the form (1.1) in univalent form, then*

$$\sum_{n=2}^{\infty} \bigl(k(1+\mathcal{O}_2) + 1 + \mathcal{O}_1\bigr)|a_k| \leq (\mathcal{O}_1 - \mathcal{O}_2).$$

The result is sharp for function given by

$$f(\zeta) = \frac{1}{\zeta} + \frac{(\mathcal{O}_1 - \mathcal{O}_2)}{(k(1+\mathcal{O}_2) + 1 + \mathcal{O}_1)} t_k \zeta^k, \quad \text{where} \sum_{k=p+1}^{\infty} |t_k| = 1. \tag{3.3}$$

In the following, we discuss the growth and distortion theorems for our new class of functions.

Theorem 3 *Let $f \in \mathcal{M}_{\mu,q}(p,m,\mathcal{O}_1,\mathcal{O}_2)$ be of the form (1.1). Then for $|\zeta| = r$, we have*

$$\frac{1}{r^p} - \tau_1 r^p \leq \bigl|f(\zeta)\bigr| \leq \frac{1}{r^p} + \tau_1 r^p,$$

where

$$\tau_1 = \frac{[p,q](\mathcal{O}_1 - \mathcal{O}_2)}{\Lambda_{p+1}}.$$

The result is sharp for the function given in (3.3) with $k = p + 1$.

Proof We have

$$\bigl|f(\zeta)\bigr| = \left| \frac{1}{\zeta^p} + \sum_{k=p+1}^{\infty} a_k \zeta^k \right|$$
$$\leq \frac{1}{|\zeta^p|} + \sum_{k=p+1}^{\infty} |a_k||\zeta|^k = \frac{1}{r^p} + \sum_{k=p+1}^{\infty} |a_k| r^k.$$

Since $r^k < r^p$ for $r < 1$ and $k \geq p + 1$, for $|\zeta| = r < 1$, we have

$$\bigl|f(\zeta)\bigr| \leq \frac{1}{r^p} + r^p \sum_{k=p+1}^{\infty} |a_k|. \tag{3.4}$$

Similarly, we have

$$\left|f(\zeta)\right| \geq \frac{1}{r^p} - r^p \sum_{k=p+1}^{\infty} |a_k|. \tag{3.5}$$

Now (3.1) implies that

$$\sum_{k=p+1}^{\infty} \Lambda_k |a_k| \leq [p,q](\mathcal{O}_1 - \mathcal{O}_2).$$

Since

$$\Lambda_{p+1} \sum_{k=p+1}^{\infty} |a_k| \leq \sum_{k=p+1}^{\infty} \Lambda_k |a_k|,$$

we have

$$\sum_{k=p+1}^{\infty} \Lambda_{p+1} |a_k| \leq [p,q](\mathcal{O}_1 - \mathcal{O}_2),$$

which also can be written as

$$\sum_{k=p+1}^{\infty} |a_k| \leq \frac{[p,q](\mathcal{O}_1 - \mathcal{O}_2)}{\Lambda_{p+1}}.$$

Now by putting this value into (3.4) and (3.5), we get the required result. □

Theorem 4 *Let $f \in \mathcal{M}_{\mu,q}(p,m,\mathcal{O}_1,\mathcal{O}_2)$ be of the form (1.1). Then for $|\zeta| = r$,*

$$\frac{[p,q]_m}{q^{mp+\delta} r^{m+p}} - \tau_2 r^p \leq \left|\partial_q^m f(\zeta)\right| \leq \frac{[p,q]_m}{q^{mp+\delta} r^{m+p}} + \tau_2 r^p,$$

where

$$\tau_2 = \frac{(\mathcal{O}_1 - \mathcal{O}_2)[p,q][p+1,q]}{\alpha_{p+1}} \quad \textit{and} \quad \delta = \sum_{k=1}^{m} k.$$

Proof By (1.2) and (1.3) we can write

$$\partial_q^m f(\zeta) = \frac{(-1)^m [p,q]_m}{q^{mp+\delta} \zeta^{p+m}} + \sum_{k=p+1}^{\infty} \left[k - (m-1), q\right]_{m+1} a_k \zeta^{k-m}.$$

Since $r^{k-m} \leq r^p$ for $m \leq k$ and $k \geq p+1$, for $|\zeta| = r < 1$, we have

$$\left|\partial_q^m f(\zeta)\right| \leq \frac{[p,q]_m}{q^{mp+\delta} r^{m+p}} + r^p \sum_{n=p+1}^{\infty} \left[k - (m-1), q\right]_{m+1} |a_k|. \tag{3.6}$$

Similarly,

$$\left|\partial_q^m f(\zeta)\right| \geq \frac{[p,q]_m}{q^{mp+\delta} r^{m+p}} - r^p \sum_{k=p+1}^{\infty} \big[k-(m-1),q\big]_{m+1} |a_k|. \tag{3.7}$$

Now by (3.1) we get the inequality

$$\frac{\alpha_{p+1}}{[p+1,q]} \sum_{k=p+1}^{\infty} [k,q]|a_k| \leq (\mathcal{O}_1 - \mathcal{O}_2)[p,q],$$

so that

$$\sum_{k=p+1}^{\infty} [k,q]|a_k| \leq \frac{(\mathcal{O}_1 - \mathcal{O}_2)[p,q][p+1,q]}{\Lambda_{p+1}}.$$

We easily observe that

$$\sum_{k=p+1}^{\infty} \big[k-(m-1),q\big]|a_k| \leq \sum_{k=p+1}^{\infty} [k,q]|a_k|,$$

which implies

$$\sum_{k=p+1}^{\infty} \big[k-(m-1),q\big]|a_k| \leq \frac{(\mathcal{O}_1 - \mathcal{O}_2)[p,q][p+1,q]}{\Lambda_{p+1}}.$$

Now using this inequality in (3.6) and (3.7), we obtain the required result. □

Corollary 2 *If $f \in \mathcal{MS}_p^*$ is of the form (1.1), then*

$$\frac{1}{r^p} - \frac{2p(p+1)r^p}{(k(1+\mathcal{O}_2)+(p+1)(1+\mathcal{O}_1))} \leq \left|f'(\zeta)\right| \leq \frac{1}{r^p} + \frac{2p(p+1)r^p}{(k(1+\mathcal{O}_2)+(p+1)(1+\mathcal{O}_1))}.$$

In the next two theorems, we discuss the radii problems for the functions of the class $\mathcal{M}_{\mu,q}(p,m,\mathcal{O}_1,\mathcal{O}_2)$.

Theorem 5 *Let $f \in \mathcal{M}_{\mu,q}(p,m,\mathcal{O}_1,\mathcal{O}_2)$. Then $f \in \mathcal{MC}_p(\alpha)$ for $|\zeta| < r_1$, where*

$$r_1 = \left(\frac{p(p-\alpha)\alpha_{p+n}}{(p+n)(n+p+\alpha)(\mathcal{O}_1-\mathcal{O}_2)[p,q]}\right)^{\frac{1}{k+2p}}.$$

Proof Let $f \in \mathcal{M}_{\mu,q}(p,m,\mathcal{O}_1,\mathcal{O}_2)$. To prove $f \in \mathcal{MC}_p(\alpha)$, we only need to show

$$\left|\frac{\zeta f''(\zeta)+(p+1)f'(\zeta)}{\zeta f''(\zeta)+(1+2\alpha-p)f'(\zeta)}\right| \leq 1.$$

Using (1.1), after some simple computation, we get

$$\sum_{n=1}^{\infty} \frac{(p+n)(p+n+\alpha)}{p(p-\alpha)} |a_{k+p}||\zeta|^{k+2p} \leq 1. \tag{3.8}$$

From (3.1) we can easily obtain that

$$\sum_{k=p+1}^{\infty} \Lambda_k |a_{k+p}| \leq [p,q](\mathcal{O}_1 - \mathcal{O}_2) \sum_{k=p+1}^{\infty} \frac{\Lambda_k}{(\mathcal{O}_1 - \mathcal{O}_2)[p,q]} |a_k| < 1.$$

Equivalently, we have

$$\sum_{k=1}^{\infty} \frac{\Lambda_{p+k}}{(\mathcal{O}_1 - \mathcal{O}_2)[p,q]} |a_{k+p}| < 1.$$

Now inequality (3.8) will hold if

$$\sum_{k=1}^{\infty} \frac{(p+k)(k+p+\alpha)}{p(p-\alpha)} |a_{k+p}||\zeta|^{k+2p} < \sum_{k=1}^{\infty} \frac{\Lambda_{p+k}}{(\mathcal{O}_1 - \mathcal{O}_2)[p,q]} |a_{k+p}|,$$

which implies that

$$|\zeta|^{k+2p} < \frac{p(p-\alpha)\Lambda_{p+k}}{(p+k)(k+p+\alpha)(\mathcal{O}_1 - \mathcal{O}_2)[p,q]}$$

and thus

$$|\zeta| < \left(\frac{p(p-\alpha)\alpha_{p+k}}{(p+k)(k+p+\alpha)(A-B)[p,q]} \right)^{\frac{1}{k+2p}} = r_1,$$

from which we get the desired condition. □

Corollary 3 *If $f \in \mathcal{MS}_p^*$ is of the form (1.1), then $f \in \mathcal{MC}_p(\alpha)$ for $|\zeta| < r_1'$, where*

$$r_1' = \left(\frac{p(p-\alpha)(n(1+\mathcal{O}_2) + (p+k)(1+\mathcal{O}_1))}{(p+k)(k+p+\alpha)(\mathcal{O}_1 - \mathcal{O}_2)p} \right)^{\frac{1}{k+2p}}.$$

Theorem 6 *Let $f \in \mathcal{M}_{\mu,q}(p,m,\mathcal{O}_1,\mathcal{O}_2)$. Then $f \in \mathcal{MS}_p^*(\alpha)$ for $|\zeta| < r_2$, where*

$$r_2 = \left(\frac{(p-\alpha)\Lambda_{p+k}}{(k+p+\alpha)(\mathcal{O}_1 - \mathcal{O}_2)[p,q]} \right)^{\frac{1}{k+2p}}.$$

Proof We know that $f \in \mathcal{MS}_p^*(\alpha)$ if and only if

$$\left| \frac{\zeta f'(\zeta) + pf(\zeta)}{\zeta f'(\zeta) - (p-2\alpha)f(\zeta)} \right| \leq 1.$$

Using (1.1), after simplification, we get

$$\sum_{k=1}^{\infty} \left(\frac{k+p+\alpha}{p-\alpha} \right) |a_{k+p}||\zeta|^{k+2p} \leq 1. \tag{3.9}$$

Now from (3.1) we easily obtain

$$\sum_{k=1}^{\infty} \frac{\Lambda_{p+k}}{(\mathcal{O}_1 - \mathcal{O}_2)[p,q]} |a_{k+p}| < 1.$$

For inequality (3.9) to be true, it suffices that

$$\sum_{k=1}^{\infty} \left(\frac{k+p+\alpha}{p-\alpha} \right) |a_{k+p}| |\zeta|^{k+2p} < \sum_{k=1}^{\infty} \frac{\Lambda_{p+k}}{(\mathcal{O}_1 - \mathcal{O}_2)[p,q]} |a_{k+p}|.$$

This gives

$$|\zeta|^{k+2p} < \frac{(p-\alpha)\Lambda_{p+k}}{(k+p+\alpha)(\mathcal{O}_1 - \mathcal{O}_2)[p,q]},$$

and hence

$$|\zeta| < \left(\frac{(p-\alpha)\Lambda_{p+k}}{(k+p+\alpha)(\mathcal{O}_1 - \mathcal{O}_2)[p,q]} \right)^{\frac{1}{k+2p}} = r_2.$$

Thus we obtain the required result. □

Theorem 7 *Let $f \in \mathcal{M}_{\mu,q}(p,m,\mathcal{O}_1,\mathcal{O}_2)$ be of the form (1.1). Then*

$$|a_{p+1}| \leq \frac{[p,q](\mathcal{O}_1 - \mathcal{O}_2)l(0)}{(q^p[p+1,q] + [p,q])l(1)},$$
$$|a_{p+2}| \leq \frac{[p,q](\mathcal{O}_1 - \mathcal{O}_2)l(0)}{(q^p[p+2,q] + [p,q])l(2)} \left(1 + \frac{[p,q](\mathcal{O}_1 - \mathcal{O}_2)}{q^p[p+1,q] + [p,q]} \right),$$

and

$$|a_{p+3}| \leq \frac{[p,q](\mathcal{O}_1 - \mathcal{O}_2)l(0)}{(q^p[p+3,q] + [p,q])l(3)} \left(1 + \frac{[p,q](\mathcal{O}_1 - \mathcal{O}_2)}{q^p[p+1,q] + [p,q]} + \frac{[p,q](\mathcal{O}_1 - \mathcal{O}_2)}{q^p[p+2,q] + [p,q]} + \frac{([p,q](\mathcal{O}_1 - \mathcal{O}_2))^2}{(q^p[p+2,q] + [p,q])(q^p[p+1,q] + [p,q])} \right),$$

where

$$l(k) = \big(1 + [p,q]\mu + \mu q^p[p+k,q]\big)^m. \tag{3.10}$$

Proof If $f \in \mathfrak{A}$ is in the class $\mathcal{M}_{\mu,q}(p,m,\mathcal{O}_1,\mathcal{O}_2)$, then it satisfies

$$\frac{-q^p \zeta \partial_q \mathcal{D}^m_{\mu,q} f(\zeta)}{[p,q]\mathcal{D}^m_{\mu,q} f(\zeta)} \prec \frac{1 + \mathcal{O}_1 \zeta}{1 + \mathcal{O}_2 \zeta}.$$

The right-hand side

$$h(\zeta) = \frac{-q^p \zeta \partial_q \mathcal{D}^m_{\mu,q} f(\zeta)}{[p,q]\mathcal{D}^m_{\mu,q} f(\zeta)} \tag{3.11}$$

is of the form

$$h(\zeta) = 1 + \sum_{k=1}^{\infty} d_k \zeta^k,$$

which implies that

$$h(\zeta) \prec \frac{1 + \mathcal{O}_1 \zeta}{1 + \mathcal{O}_2 \zeta}.$$

However,

$$\frac{1 + A\zeta}{1 + B\zeta} = 1 + (\mathcal{O}_1 - \mathcal{O}_2)\zeta + \cdots .$$

Now using Lemma 2, we obtain

$$|d_k| \leq (\mathcal{O}_1 - \mathcal{O}_2). \tag{3.12}$$

Putting the series expansions of $h(\zeta)$ and $f(\zeta)$ into (3.11), simplifying, and comparing the coefficients at ζ^{k+p} on both sides, we get

$$\begin{aligned} &-q^p\big(1 + [p,q]\mu + \mu q^p[k+p,q]\big)^m [p+k,q] a_{p+k} \\ &\quad = [p,q]\big(1 + [p,q]\mu + \mu q^p[k+p,q]\big)^m a_{p+k} \\ &\qquad + [p,q] \sum_{i=0}^{k-1} \big(1 + [p,q]\mu + \mu q^p[p+i,q]\big)^m a_{p+i} d_{k-i}, \end{aligned}$$

and hence

$$\begin{aligned} &-\big(1 + [p,q]\mu + \mu q^p[k+p,q]\big)^m \big(q^p[p+k,q] + [p,q]\big) a_{p+k} \\ &\quad = [p,q] \sum_{i=1}^{k-1} \big(1 + [p,q]\mu + \mu q^p[p+i,q]\big)^m a_{p+i} d_{k-i}. \end{aligned}$$

Now by taking the absolute values of both sides, using the triangle inequality, and then using (3.12), we obtain

$$\begin{aligned} &\big(1 + [p,q]\mu + \mu q^p[p+k,q]\big)^m \big(q^p[p+k,q] + [p,q]\big) |a_{p+k}| \\ &\quad \leq [p,q](\mathcal{O}_1 - \mathcal{O}_2) \sum_{i=1}^{k-1} \big(1 + [p,q]\mu + \mu q^p[p+i,q]\big)^m |a_{p+i}|. \end{aligned}$$

Notation (3.10) implies that

$$|a_{p+k}| \leq \frac{[p,q](\mathcal{O}_1 - \mathcal{O}_2)}{l(k)(q^p[p+k,q] + [p,q])} \sum_{i=0}^{k-1} l(i) |a_{p+i}|.$$

Now for $k = 1, 2$, and 3, using the fact that $|a_p| = 1$, we get the required result. □

Using the notion of subordination, we get the next result on inclusion property of this class.

Theorem 8 *Let* $-1 \leq \mathcal{O}_4 \leq \mathcal{O}_2 < \mathcal{O}_1 \leq \mathcal{O}_3 \leq 1$, *let* $\mathcal{D}^m_{\mu,q} f(\zeta) \neq 0$ *in* $\mathbb{D}$, *and let*

$$\frac{1}{\mu[p,q]}\left(\left(1+[p,q]\mu\right) - \frac{\mathcal{D}^m_{\mu,q} f(\zeta)}{\mathcal{D}^m_{\mu,q} f(\zeta)}\right) \prec \frac{1+\mathcal{O}_1\zeta}{1+\mathcal{O}_2\zeta}. \tag{3.13}$$

Then $f \in \mathcal{M}_{\mu,q}(p,m,\mathcal{O}_3,\mathcal{O}_4)$.

Proof For $\mathcal{D}^m_{\mu,q} f(\zeta) \neq 0$ in $\mathbb{D}$, we define the function $p(\zeta)$ by

$$\frac{-q^p\zeta\,\partial_q\mathcal{D}^m_{\mu,q} f(\zeta)}{[p,q]\mathcal{D}^m_{\mu,q} f(\zeta)} = p(\zeta) \quad (\zeta \in \mathbb{D}).$$

Using identity (1.6), we easily obtain

$$\frac{1}{\mu[p,q]}\left(\left(1+[p,q]\mu\right) - \frac{\mathcal{D}^m_{\mu,q} f(\zeta)}{\mathcal{D}^m_{\mu,q} f(\zeta)}\right) = p(\zeta).$$

Therefore, using (3.13), we have

$$\frac{-q^p\zeta\,\partial_q\mathcal{D}^m_{\mu,q} f(\zeta)}{[p,q]\mathcal{D}^m_{\mu,q} f(\zeta)} = p(\zeta) \prec \frac{1+\mathcal{O}_1\zeta}{1+\mathcal{O}_2\zeta},$$

and by Lemma 1 we get

$$\frac{1+A_1\zeta}{1+B_1\zeta} \prec \frac{1+\mathcal{O}_3\zeta}{1+\mathcal{O}_4\zeta},$$

so that $f \in \mathcal{M}_{\mu,q}(p,m,\mathcal{O}_3,\mathcal{O}_4)$. □

Theorem 9 *The class* $\mathcal{M}_{\mu,q}(p,m,\mathcal{O}_1,\mathcal{O}_2)$ *is closed under convex combination.*

Proof Let $f_k(\zeta) \in \mathcal{M}_{\mu,q}(p,m,\mathcal{O}_1,\mathcal{O}_2)$ be such that

$$f_k(\zeta) = \frac{1}{\zeta^p} + \sum_{k=p+1}^{\infty} a_{k,i}\zeta^k \quad \text{for } i = 1,2 \text{ and } \zeta \in \mathbb{D}. \tag{3.14}$$

We have to show that $F(\zeta) = tf_1(\zeta) + (1-t)f_2(\zeta) \in \mathcal{M}_{\mu,q}(p,m,\mathcal{O}_1,\mathcal{O}_2)$. We have

$$\begin{aligned} F(\zeta) &= tf_1(\zeta) + (1-t)f_2(\zeta) \\ &= \frac{1}{\zeta^p} + \sum_{k=p+1}^{\infty} \left(ta_{1,i} + (1-t)a_{2,i}\right)\zeta^k. \end{aligned}$$

Consider

$$\begin{aligned}\sum_{k=p+1}^{\infty}\alpha_k\big(ta_{1,i}+(1-t)a_{2,i}\big)&=t\sum_{k=p+1}^{\infty}a_{1,i}+(1-t)\sum_{k=p+1}^{\infty}a_{2,i}\\&\le t[p,q](\mathcal{O}_1-\mathcal{O}_2)+(1-t)[p,q](\mathcal{O}_1-\mathcal{O}_2)\\&=[p,q](\mathcal{O}_1-\mathcal{O}_2).\end{aligned}$$

Hence $F(\zeta)\in\mathcal{M}_{\mu,q}(p,m,\mathcal{O}_1,\mathcal{O}_2)$, which is the desired result. □

4 Conclusions

In this paper, we introduced a subclass of meromorphic multivalent functions in Janowski domain using the idea of q-calculus. Then we characterized these functions with the help of some useful their properties like sufficiency criteria, distortion bounds, coefficient estimates, radius of starlikness, radius of convexity, inclusion property, and convex combinations. These results were supported by some sharp examples and corollaries in particular cases.

We recall the attention of curious readers to the prospect influenced by Srivastava's [40] newly published survey-cum-expository review paper that the $(\mathfrak{p},q)$-extension would be a relatively minor and unimportant change, as the new parameter $\mathfrak{p}$ is redundant (for details, see Srivastava [40, p. 340]). Furthermore, in light of Srivastava's recent result [41], the interested reader's attention is brought to further investigation of the (k,s)-extension of the Riemann–Liouville fractional integral.

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