Summer 2023

Optimal Experimental Planning of Reliability Experiments Based on Coherent Systems

Yang Yu
yuyang@smu.edu

OPTIMAL EXPERIMENTAL PLANNING

OF RELIABILITY EXPERIMENTS

BASED ON COHERENT SYSTEMS

Approved by:

Dr. Hon Keung Tony Ng
Professor, Department of Mathematical Sciences, Bentley University

Dr. Lynne Stokes
Professor, Department of Statistical Science, Southern Methodist University

Dr. Jing Cao
Professor, Department of Statistical Science, Southern Methodist University

Dr. Chul Moon
Assistant Professor, Department of Statistical Science, Southern Methodist University

OPTIMAL EXPERIMENTAL PLANNING
OF RELIABILITY EXPERIMENTS
BASED ON COHERENT SYSTEMS

A Dissertation Presented to the Graduate Faculty of the

Dedman College

Southern Methodist University

in

Partial Fulfillment of the Requirements

for the degree of

Doctor of Philosophy

with a

Major in Statistical Science

by

Yang Yu

B.S., Mathematics and Applied Mathematics, Nankai University
M.S., Applied Statistics, Bowling Green State University

July 31, 2023

Acknowledgments

I am deeply grateful to my advisor, Dr. Hon Keung Tony Ng, for his unwavering support, expert guidance and invaluable membership. Throughout this dissertation journey, the idea of giving up crossed my mind countless times, but it was his constant encouragement and unwavering belief in my abilities that propelled me to reach this far.

I am also grateful to my committee members, Dr. Stokes, Dr. Cao, and Dr. Moon, for their insightful feedback, constructive criticism, and valuable suggestions. I would also like to extend my appreciation to all the professors and Sheila from the Department of Statistics of SMU for their invaluable support and help.

My deepest appreciation goes to my beloved wife, Yichen Yuan, for her unwavering love, understanding and encouragement. Her presence has not only enriched my life but has also made this journey meaningful and fulfilling. I am blessed to share my life with such a remarkable and supportive partner. I would also like to extend my gratitude to my parents for their unconditional love and sacrifices that have made my educational journey possible. I also want to thank my dear friend Min Wen for always being there to lend a helping hand and for being a great source of inspiration and encouragement.

To all those mentioned above and to those who have supported me in various ways but may not be named here, I am deeply thankful for your presence in my life and for contributing to the successful completion of this thesis. Your support has been invaluable, and I am truly grateful for each and every one of you.

Yu, Yang B.S., Mathematics and Applied Mathematics, Nankai University
M.S., Applied Statistics, Bowling Green State University

Optimal Experimental Planning
of Reliability Experiments
based on Coherent Systems

Advisor: Dr. Hon Keung Tony Ng
Doctor of Philosophy degree conferred July 31, 2023
Dissertation completed July 21, 2023

In industrial engineering and manufacturing, assessing the reliability of a product or system is an important topic. Life-testing and reliability experiments are commonly used reliability assessment methods to gain sound knowledge about product or system lifetime distributions. Usually, a sample of items of interest is subjected to stresses and environmental conditions that characterize the normal operating conditions. During the life-test, successive times to failure are recorded and lifetime data are collected. Life-testing is useful in many industrial environments, including the automobile, materials, telecommunications, and electronics industries.

There are different kinds of life-testing experiments that can be applied for different purposes. For instance, accelerated life tests (ALTs) and censored life tests are commonly used to acquire information in reliability and life-testing experiments with the presence of time and resource limitations. Statistical inference based on the data obtained from a life test and effectively planning a life-testing experiment subject to some constraints are two important problems statisticians are interested in. The experimental design problem for a life test has long been studied; however, the experimental planning considering putting the experimental units into systems for a life-test has not been studied. In this thesis, we study the optimal experimental planning problem in multiple stress levels life-testing experiments and progressively Type-II censored life-testing experiments when the test units can be put into coherent systems for the experiment.

Based on the notion of system signature, a tool in structure reliability to represent the structure of a coherent system, under different experimental settings, models and assumptions, we derive the maximum likelihood estimators of the model parameters and the expected Fisher information matrix. Then, we use the expected Fisher information matrix to obtain the asymptotic variance-covariance matrix of the maximum likelihood estimators when n-component coherent systems are used in the life-testing experiment. Based on different optimality criteria, such as D-optimality, A-optimality and V-optimality, we obtain the optimal experimental plans under different settings. Numerical and Monte Carlo simulation studies are used to demonstrate the advantages and disadvantages of using systems in life-testing experiments.

Keywords. Accelerated life tests; Multi-level stress testing; Coherent systems; System structure; Constant-stress accelerated life tests; Step-stress accelerated life tests; Cumulative exposure model; Type-II censoring; Progressive Type-II censoring; Extreme value distribution; Fisher information; Lognormal distribution; Maximum likelihood estimators; Monte Carlo simulation; Weibull distribution

TABLE OF CONTENTS

APPENDIX

LIST OF FIGURES

LIST OF TABLES

I dedicate this dissertation to my family.

Chapter 1

Introduction

1.1. Problems of Interest

In industrial and system engineering, continuous improvement of product or system reliability is an important topic. A manufacturer of products will be interested in assessing the reliability of the product. To measure reliability and gain a sound knowledge about the failure-time distribution of the product or system, life-testing and reliability experiments are carried out before and while products are put on the market or while the systems are in use. In a life-testing experiment, a sample of the items of interest is usually tested under different stresses and environmental conditions, and successive failure times of the items are recorded. Life testing experiment is useful in many industrial settings and products such as batteries, light bulbs, and automobile tires, etc.

Due to the time and resource limitations, it is almost impossible or it takes an extremely long time period to observe the lifetime of the product or system in their normal operating environment. For this reason, researchers developed accelerated life tests (ALT) to shorten the time requires to obtain useful information about the product reliability within a reasonable period of time. As a summary of the early work of ALT, Yurkowsky et al. (1967) introduced the definition of ALT and its related methods and statistical models. The basic idea of ALT is to subjects the test sample to higher stress levels to stress the product to failure, then suitable statistical model and methods are used to estimate the reliability characteristics of the product at the normal operating condition. Planning an effective ALT experiment, especially with limited resources, to obtain reliability information is an important topic in statistics and reliability engineering. Experimental planning for different kinds of ALTs is the main focus of this thesis.

In many situations, the life testing experiment can be done by using the individual items and/or by using a system that made up of those items. For example, in a reliability study for joints used in case furniture presented in Kłos et al. (2018), the experiment was performed using 10 joints connected in series. The numbers of working cycles until the failures of those 10-component series systems are recorded in order to determine the durability and safety of the joints. On the other hand, putting the test units into a system for a life testing experiment has the advantage of saving the experimental time because of the longest lifetime among n components is always greater than or equal to the lifetime of a n-component system with the same n components. Therefore, it is important to study the experimental planning based on systems as well as based on individual items.

In this chapter, we provide some preliminaries and background knowledge on system structure, censoring methodology, location-scale family of statistical distributions, experimental planning for ALTs. Then, we provide the scope of the thesis.

1.2. System Structure, Coherent System and System Signature

1.2.1. System structure and coherent system

In engineering, a system is a collection of components arranged in some fashion in order to achieve desired functions. Consider a system with n individual components and each component can have two possible states: functioning and failed. The state of the i-th component in the system can be represented by a binary variable β_i, called the *state variable*, defined as

$$\beta_i = \begin{cases} 1, & \text{if the } i\text{-th component is functioning,} \\ 0, & \text{if the } i\text{-th component is failed.} \end{cases}$$

Then, $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_n)$, is the state vector of the system and the state of the system can be described by the *structure function* defined as

$$\phi(\boldsymbol{\beta}) = \phi(\beta_1, \beta_2, \ldots, \beta_n).$$

In Figure 1.1, we illustrate the series and parallel systems with structure functions

$$\phi(\boldsymbol{\beta}) = \beta_1\beta_2\cdots\beta_n = \prod_{i=1}^{n}\beta_i,$$

and

$$\phi(\boldsymbol{\beta}) = 1 - \prod_{i=1}^{n}(1-\beta_i),$$

respectively. The state of the series system is $\phi(\boldsymbol{\beta}) = 1$ (i.e., functioning) if and only if all the n components in the system are functioning (i.e., $\beta_i = 1$ for all $i = 1, 2, \ldots, n$). The state of the parallel system is $\phi(\boldsymbol{\beta}) = 1$ (i.e., functioning) if at least one of the n components in the system is functioning (i.e., $\beta_i = 1$ for at least one $i = 1, 2, \ldots, n$).

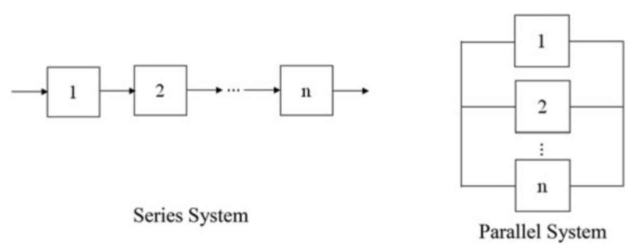

Figure 1.1: Series and parallel n-component systems

A component inside a system is said to be relevant if the status of the system depends on all the components in the system, and a system structure function is said to be monotone if repairing a failed component in the system will not make the system worse. A coherent system is a system that is relevant and its structure function is monotone. For example, a system with a spare component would not be categorized as a coherent system because the status of the system does not depend on the spare component. In addition to the structure function $\phi(\boldsymbol{\beta})$, there are different ways to represent the structure of a system such as minimal path sets and minimal cut sets, etc. (see, for example, Proschan and Barlow, 1965; Rausand and Hoyland, 2004). In this thesis, we consider a n-component coherent system consisting of n independent and identically distributed (i.i.d.) components and the system structure is represented by using system signature and minimal signature.

1.2.2. System signature

Suppose the n-component coherent system consists of n i.i.d. components with lifetimes $X_1, \ldots, X_n$, and we denote the ordered component lifetime as $X_{1:n} < X_{2:n} < \ldots < X_{n:n}$ and the lifetime of the n-component system as T. We also assume that the component lifetimes follow a common absolutely continuous distribution with cumulative distribution function (CDF) $F_X(\cdot)$, probability density function (PDf) $f_X(\cdot)$, and survival function (SF) $\bar{F}_X(\cdot) = 1 - F_X(\cdot)$. We further denote the CDF, PDF and SF of the i-th order statistics by $F_{i:n}(\cdot)$, $f_{i:n}(\cdot)$ and $\bar{F}_{i:n}(\cdot)$, respectively, and the CDF, PDF and SF of the n-component coherent system by $F_T(\cdot)$, $f_T(\cdot)$ and $\bar{F}_T(\cdot)$, respectively.

Let I be a random variable that indicates the corresponding ordered component lifetimes of the system failure, i.e., $I = i$ if the n-component system fails at the i-th ordered component failure. In this case, we have $T = X_{i:n}$. For some systems, the value of I is fixed (i.e., $\Pr(I = i) = 1$). For example, for a l-out-of-n system $\Pr(I = l) = 1$, and $l = 1$ corresponds to the series system and $l = n$ corresponds to the parallel system (see, Figure 1.1). Suppose the random variable I is not observable, then we consider the probability mass function of I as $\Pr(I = i) = \Pr(T = X_{i:n}) = s_i$, $0 \leq s_i \leq 1$, $i = 1, 2, \ldots, n$, with $\sum\limits_{i=1}^{n} s_i = 1$. The n-dimensional probability vector $\boldsymbol{s} = (s_1, s_2, \ldots, s_n)$ is called the system signature, proposed by Samaniego (1985) (see also, Samaniego, 2007), which is a nonparametric representation of the system structure that does not depend on the underlying lifetime distribution of the components. For the series and parallel systems illustrated in Figure 1.1), the system signatures are $\boldsymbol{s} = (1, 0, \ldots, 0)$ and $\boldsymbol{s} = (0, \ldots, 0, 1)$, respectively.

From Samaniego (1985), the PDF and SF of system lifetime T can be expressed in terms of the CDF and PDF of the component lifetime as

$$f_T(t) = \sum_{i=1}^{n} s_i \binom{n}{i} i f_X(t) [F_X(t)]^{i-1} \left[\bar{F}_X(t)\right]^{i-1} \tag{1.1}$$

and

$$\bar{F}_T(t) = \sum_{i=1}^{n} s_i \sum_{j=0}^{i-1} \binom{n}{j} [F_X(t)]^{j} \left[\bar{F}_X(t)\right]^{n-j}. \tag{1.2}$$

Navarro et al. (2007) considered another representation called *minimal signature* of the system by expressing the SF of the system lifetime T as a generalized mixture of the SFs of i-component series system lifetimes $(i = 1, 2, \ldots, n)$, i.e. $\bar{F}_{1:i}(\cdot)$:

$$\bar{F}_T(t) = \sum_{i=1}^{n} a_i \bar{F}_{1:i}(t) = \sum_{i=1}^{n} a_i \left[\bar{F}(t)\right]^{i}, \tag{1.3}$$

for some integers (which can be positive or non-positive) $a_1, a_2, \ldots, a_n$ that do not depend on the underlying lifetime distribution of the components with $\sum_{i=1}^{n} a_i = 1$. The n-dimensional vector $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)$ is called the minimal signature of the system. For example, the minimal signature of a n-component series system is $\boldsymbol{a} = (0, \ldots, 0, 1)$. The minimal signature of a system can be obtained from its system signature and vice versa Navarro et al. (2007).

1.3. Censoring Schemes

In an ideal scenario, researchers would have a complete sample in a life-testing experiment without any censoring, which means that they observe all the failure times of the items in the experiment (Ng, 2010). Nevertheless, this is generally difficult to achieve in practice due to the limitations of time and experimental resources. Therefore, it is important to consider statistical analysis and inference based on censored life-testing experiments. Typically, Type-I and Type-II censoring schemes, also known as time-censoring and item-censoring schemes, respectively, are widely used in life testing experiments. In the past several decades, generalizations of Type-I and Type-II censoring schemes, namely progressive Type-I and Type-II censoring schemes, have been studied extensively. For reviews and the details of progressive censoring, one can refer to the books by Balakrishnan and Aggarwala (2000) and Balakrishnan and Cramer (2014), and the review paper by Balakrishnan (2007).

Suppose m systems are placed on a life testing experiment, we denote the lifetimes of the systems by $T_1, T_2, \ldots, T_m$ and the corresponding order statistics by $T_{1:m} < T_{2:m} < \ldots < T_{m:m}$. Assume that the lifetimes of these m systems follow a distribution with CDF $F_T(\cdot; \boldsymbol{\theta})$, PDF $f_T(\cdot; \boldsymbol{\theta})$ and SF $\bar{F}_T(\cdot; \theta)$, where $\boldsymbol{\theta}$ is an unknown parameter vector. In this subsection, we review the Type-I, Type-II, progressive Type-I and progressive Type-II censoring schemes and provide corresponding expressions of the likelihood functions.

1.3.1. Type-I censoring

Type-I censoring is also called time-censoring. In a life testing experiment starting with m systems, the researchers only observe the failures that occur before a pre-fixed time τ. All failures after time τ would be censored. The number of failures observed in a Type-I censored experiment is a random variable which follows a binomial distribution with m trials and probability of success $F_T(\tau; \boldsymbol{\theta})$, denoted as $Bin(m, F_T(\tau; \boldsymbol{\theta}))$. The likelihood function based on an observed Type-I censored sample with r observed failures $t_{1:m} < t_{2:m} < \ldots < t_{r:m} \leq \tau$ is

$$L(\boldsymbol{\theta}) = \frac{n!}{(n-r)!} \left[\prod_{j=1}^{r} f_T(t_{j:m}; \boldsymbol{\theta}) \right] \left[\bar{F}_T(\tau; \boldsymbol{\theta}) \right]^{m-r}, \quad t_{1:m} < t_{2:m} < \ldots < t_{r:m}.$$

The advantage of this censoring scheme is that the experimenters can control the experimental time. However, one of the drawbacks is that an inappropriate choice of τ may lead to few or an insufficient observed failures.

1.3.2. Type-II censoring

Type-II censoring is also called item-censoring. In contrast to an experiment with Type-I censoring, the number of failed systems r ($\leq m$) instead of the termination time of the experiment is pre-specified. The life testing experiment will be terminated once the r-th failure is observed. In this case, the number of observed failures is fixed while the termination time of the experiment is a random variable $T_{r:m}$. The likelihood function based on an observed Type-II censored sample $t_{1:m} < t_{2:m} < \ldots < t_{r:m}$ is

$$L(\boldsymbol{\theta}) = \frac{n!}{(n-r)!}\left[\prod_{j=1}^{r} f_T(t_{j:m};\boldsymbol{\theta})\right]\left[\bar{F}_T(t_{r:m};\boldsymbol{\theta})\right]^{m-r}, \quad t_{1:m} < t_{2:m} < \ldots < t_{r:m}.$$

The advantage of Type-II censoring is the number of observed failures is fixed. Nevertheless, a large r may take a long experimental time, which is the main disadvantage of this censoring scheme.

1.3.3. Progressive Type-II censoring

Since the conventional Type-I and Type-II censoring schemes described in Sections 1.3.1 and 1.3.2 do not have the flexibility of allowing removal of units at points other than the terminal point of the experiment, a more general censoring scheme called *progressive censoring* was proposed. Suppose we place m systems on a life test and plan to fail r of them. Then a progressive Type-II censored experiment can be described as follows. At the time of the first failure, denoted as $T_{1:r:m}$, R_1 functioning systems are randomly selected and censored or removed from the life test. Then, the life test continues with $n-1-R_1$ systems. When the second failure occurs, denoted as $T_{2:r:m}$, R_2 functioning systems are randomly selected and censored or removed from the life test. The life test continues in this manner. At the time of the r-th system failure, denoted as $T_{r:r:m}$, all the remaining $R_r = n-r-R_1-R_2-\ldots-R_{r-1}$ systems are withdrawn from the test. The pre-fixed values $(R_1, R_2, \ldots, R_r)$ with $\sum_{j=1}^{r} R_j = n-r$ is called the progressive Type-II censoring scheme. The likelihood function based on an observed progressively Type-II censored sample $t_{1:r:m} < t_{2:r:m} < \ldots < t_{r:r:m}$ with censoring scheme $(R_1, R_2, \ldots, R_r)$ is

$$\begin{aligned} L(\boldsymbol{\theta}) &= n(n-R_1-1)\ldots(n-R_1-R_2-\ldots-R_{r-1}-r+1) \\ &\quad \times \prod_{j=1}^{r} f_T(t_{j:r:m};\boldsymbol{\theta}) \left[\bar{F}_T(t_{j:r:m};\boldsymbol{\theta})\right]^{R_j}, \quad t_{1:r:m} < t_{2:r:m} < \ldots < t_{r:r:m}. \end{aligned}$$

The progressive Type-II censoring with censoring scheme $(R_1, R_2, \ldots, R_r) = (0, 0, \ldots, 0, n-r)$ is equivalent to the conventional Type-II censoring scheme described in Section 1.3.2.

1.3.4. Progressive Type-I censoring

The progressive Type-I censoring scheme is a union of Type-I censoring and progressive censoring. A progressive Type-I censored sample, in which the censoring occurs at K different pre-fixed time points $\tau_1 < \tau_2 < \ldots < \tau_K$, is collected as follows. Let m_j be the number of systems failed in the time interval $(\tau_{j-1}, \tau_j]$, $t_{j,l}$ be the l-th ordered failure time of those m_j systems for $l = 1, 2, \ldots, m_j$, c_i be the number of units randomly removed at time τ_j, and M_j be the number of functioning units on the test at the beginning of the j-th time interval $(\tau_{j-1}, \tau_j]$, i.e.,

$$M_j = M - \sum_{j=1}^{k-1} m_j - \sum_{j=1}^{k-1} c_j.$$

Suppose $m = M_1$ systems are placed on a life test at time $\tau_0 = 0$, the failure times of the systems are observed sequentially. At time τ_1, c_1 functioning systems are randomly selected and removed from the experiment. The experiment is continued on $M_2 = m - m_1 - c_1$ remaining units until τ_2, at which point c_2 functioning units are randomly removed from the experiment, and so on. The experiment is terminated at time τ_K and the remaining functioning systems, $c_k = M_k - m_k$, are censored, or at the time that all the systems in the life testing experiment are failed/censored. Note that m_j is a random variable following $Bin(M_j, F_T(\tau_j) - F_T(\tau_{j-1}))$ and hence, the number of functioning systems at time τ_j can be smaller than the pre-fixed value c_j. Therefore, the life testing experiment can be terminated before reaching the planned experimental time c_K, and we define the actual number of systems being removed at time τ_j as $c_j^* = \min(c_j, M_j - m_j)$, $j = 1, 2, \ldots, K-1$.

The observed progressive Type-I censored data includes the actual number of systems being removed at time $\tau_1 < \tau_2 < \ldots < \tau_K$, i.e., $(c_1^*, c_2^*, \ldots, c_K^*)$, the number of system failures in failure times $(\tau_{j-1}, \tau_j]$, $j = 1, 2, \ldots, K$, i.e., $(m_1, m_2, \ldots, m_K)$, and the actual failure times of the systems $(t_{j,1}, t_{j,2}, \ldots, t_{j,m_j})$, $j = 1, 2, \ldots, K$. Based on the observed progressive Type-I censored data, the likelihood function can be expressed as

$$L(\boldsymbol{\theta}) = \mathcal{C} \prod_{j=1}^{K} \left[\prod_{l=1}^{m_j} f_T(t_{j,l}; \boldsymbol{\theta}) \right] \left[\bar{F}_T(\tau_j; \boldsymbol{\theta}) \right]^{c_j^*},$$

where $\mathcal{C}$ is the normalizing constant. When there is no censoring before time τ_K, i.e., $c_1 = c_2 = \ldots = c_{K-1} = 0$, the censoring scheme reduces to the conventional Type-I censoring scheme described in Section 1.3.1.

1.4. Location-scale Family of Distributions and Log-location-scale Family of Distributions

In this thesis, we consider a general model that the lifetimes of the components are i.i.d. with a distribution that belongs to the log-location-scale family of distributions, then the log-transformed component lifetimes, denoted as $U = \ln X$, follow a distribution belong to the location-scale family of distributions. A distribution in the location-scale family of distributions has PDF, CDF and SF, respectively,

$$\begin{aligned} f_U(u;\mu,\sigma) &= \frac{1}{\sigma}f^*\left(\frac{u-\mu}{\sigma}\right) = \frac{1}{\sigma}f^*(z), \\ F_U(u;\mu,\sigma) &= F^*\left(\frac{u-\mu}{\sigma}\right) = F^*(z), \\ \bar{F}_U(u;\mu,\sigma) &= 1 - F^*\left(\frac{u-\mu}{\sigma}\right) = 1 - F^*(z), \end{aligned} \tag{1.4}$$

where $-\infty < \mu < \infty$ is the location parameter and $\sigma > 0$ is the scale parameter, $z = (u-\mu)/\sigma$, $f^*(\cdot)$ and $F^*(\cdot)$ are the PDF and CDF of the standardized distribution (i.e., $\mu = 0$ and $\sigma = 1$) of the corresponding distribution in the location-scale family. The location-scale family contains many commonly used distributions including the normal, smallest extreme-value, largest extreme value, and logistic distributions. These distributions are often used for modeling log-transformed lifetime data. For example, if the component lifetime X follows a lognormal distribution, then $U = \ln X$ follows the normal distribution, which belongs to the location-scale family. If the component lifetime X follows a Weibull distribution, then $U = \ln X$ follows the smallest extreme value (SEV) distribution, which belongs to the location-scale family.

Let $V = \ln T$ be the log-transformed n-component system lifetime with i.i.d. component $X_1, X_2, \ldots, X_n$ and the corresponding log-transformed component lifetime $U_i = \ln X_i$, $i = 1, 2, \ldots, n$. We further denote the PDF, CDF and SF of the log-transformed component lifetime as $f_U(\cdot)$, $F_U(\cdot)$ and $\bar{F}_U(\cdot)$, respectively, and the PDF, CDF and SF of the log-transformed system lifetime as $f_V(\cdot)$, $F_V(\cdot)$ and $\bar{F}_V(\cdot)$, respectively. Based on the system signature $\boldsymbol{s} = (s_1, s_2, \ldots, s_n)$ described in Section 1.2.2, the SF of the log-transformed system lifetime can be expressed in terms of the distribution of the log-transformed component as

$$\bar{F}_V(v) = 1 - F_V(v) = \sum_{i=1}^{n} s_i \sum_{j=0}^{i-1} \binom{n}{j} [F_U(v)]^j [\bar{F}_U(v)]^{n-j}. \tag{1.5}$$

In terms of the minimal signature $\boldsymbol{a}$ instead, the above equation can be rewritten as

$$\bar{F}_V(v) = \sum_{i=1}^{n} a_i \bar{F}_{1:i}^U(v) = \sum_{i=1}^{n} a_i [\bar{F}_U(v)]^i, \tag{1.6}$$

where $\bar{F}_{1:i}^U(\cdot)$ is the SF of the first order statistic $U_{1:i}$ arising out of i log-transformed component lifetimes $U_1, ..., U_i$.

1.5. Statistical Inference Based on System Lifetime Data

In life testing experiments, the test units can be expensive since they are usually products under development. Moreover, the time allowed for the experiment is always a constraint. Observing the lifetimes of systems formed by different numbers of components makes it possible for researchers to save time, save surviving items for future use and improve the accuracy of parameter estimation.

Researchers have discussed and studied using systems in life testing procedures. For example, for progressively censored life testing experiments, Wu and Kuş (2009), Hermanns and Cramer (2017) and Hermanns and Cramer (2018) used the series systems, the parallel systems, and the l-out-of-n systems, respectively, for the life testing experiments. These authors also developed the statistical inference under those experiments with different kinds of systems.

When n-component systems are used in a life testing experiment, there have been several studies on the estimation of parameters of the lifetime distributions of components based on system lifetime data. With known signatures and Type-II censored data, Balakrishnan et al. (2011b) discussed linear inference for the lifetime distributions of components and derived the best linear unbiased estimators (BLUE) for the parameters. Fan and Wang (2011) established a cumulative exposure model for series system with masking and Type-I censoring, and gave inferences on functions of the unknown parameters of the assumed Weibull distribution.

Aslett et al. (2015) studied Bayesian inference for systems and networks based on signature. Zhang et al. (2015) studied the statistical inference and estimation of parameters using the maximum likelihood estimation method and regression-based method. Yang et al. (2016) proposed the stochastic expectation-maximization (SEM) algorithm to estimate the model parameters with complete and censored system lifetimes. Similarly, with unknown system structure, Yang et al. (2019) adopted the expectation-maximization (EM) algorithm to do inference and gave illustration based on a two-parameter Weibull distribution. Hermanns et al. (2020) developed EM algorithms for ordered and censored system lifetime data under a proportional hazard rate model. Fallah et al. (2020) developed the tools for statistical inference of component lifetime distribution based on system lifetime data under a proportional reversed hazard model. Tavangar and Asadi (2020) considered different estimation methods for the reliability of components of the system when progressively censored system failure times are observed.

In addition to parametric inference, nonparametric statistical inference has been developed in the literature. For example, Bhattacharya and Samaniego (2010) considered the nonparametric maximum likelihood estimate of the CDF of component lifetimes based on the observed system failure times and characterized its asymptotic behavior. Balakrishnan et al. (2011a) developed exact nonparametric inference for component lifetime distribution in the form of confidence intervals for population quantiles and tolerance limits, and Coolen and Al-nefaiee (2012) discussed nonparametric predictive inference for system lifetimes with exchangeable components. Hall et al. (2015) and Jin et al. (2017) studied the nonparametric estimation of a common component reliability function using independent samples from coherent systems when the system design is known and unknown, respectively.

1.6. Experimental Planning for Accelerated Life Tests

In experimental planning for ALT, experimental designs are optimized with respect to some statistical criterion. The optimality of an experimental design depends on the statistical model used to analyze the experimental data and it is assessed with respect to a statistical criterion. In this section, we describe two commonly used optimality criteria, D-optimality and A-optimality, and some other optimality criteria will be discussed in the subsequent chapters. Moreover, we review the literature on optimal experimental planning in two different ALTs namely constant-stress ALT (CSALT) and step-stress ALT (SSALT).

1.6.1. Optimality criteria

Suppose we are interested in the estimation of the model parameters and control the time used in the life testing experiment, to discuss the optimal experimental planning, we consider the following optimality criteria:

- D-optimality: Maximization of the determinant of the Fisher information matrix, i.e., maximizing the differential Shannon information contained in the MLEs and minimizing the volume of the Wald-type joint confidence region for the model parameters;
- A-optimality: Minimization of the trace of the variance-covariance matrix of the MLEs, i.e., minimizing the sum of the asymptotic variances of the MLEs.

In addition to these commonly used optimality criteria, other optimality criteria can be considered. For example, the total time of the experiment and the total time on test, which focus on controlling the total time of the experiment and the average time that each experimental unit spent in the experiment, can be considered in optimal planning of life testing experiments.

1.6.2. Constant-stress ALTs

One of the commonly used ALTs is the constant-stress accelerated life test (CSALT). As the name implies, each test unit is tested under a pre-fixed stress level throughout the life-testing experiment. Usually, multiple stress levels are considered in a CSALT in which the products are tested at multiple constant stress levels. Based on the lifetime data obtained from a multi-level CSALT, a statistical model (e.g., a lifetime regression model) that relates the life distribution and the stress factor is used to predict the reliability characteristics of the product at normal operating condition.

There have been numerous studies on optimal experimental planning with the CSALT. One of the optimal experimental planning problems is the optimal allocation of experimental units to the pre-specified stress levels. Ng et al. (2007) formally established this result based on complete sample by providing a mathematical proof. Taking censoring into consideration, Ka et al. (2011), Chan et al. (2016) and Chan et al. (2020) discussed the optimal allocation problems with Type-I and Type-II censored data. In addition, considering that the conventional Type-I and Type-II censoring schemes restrict our ability to observe extreme failure times, Ng et al. (2017) proposed an improved experimental scheme under progressive Type-II extremal censoring. There are also other research works on the optimal design problem based on different settings, different censoring schemes, and/or different lifetime distribution. For example, under progressive Type-II censoring scheme, Huang and Wu (2017) studied the optimal sample size allocation for accelerated life test with competing risks. Monroe and Pan (2008) explored different criteria for effectively planning an accelerated life test in electronic industry. For example, Yue and Shi (2013) discussed the optimal design of multi-level stress experiment under progressive hybrid interval censoring. Pan et al. (2015) proposed an approach to designing accelerated life test plans that are good at selecting the best acceleration model among rival models. El-Raheem (2019) discussed the optimal design of multiple constant-stress testing for generalized half-normal distribution.

1.6.3. Step-stress ALTs

Another widely used ALT is the step-stress accelerated life testing (SSALT) that allows for different stresses at various intermediate stages of the experiment. In a multi-level SSALT with K stress levels, m identical n-component coherent systems are placed on a life test at the initial stress level y_1. At pre-fixed time points $\tau_1 < \tau_2 < \ldots < \tau_{K-1}$, the stress levels are increased to $y_2 < y_3 < \ldots < y_K$, respectively. A simple SSALT is a SSALT that has two stress level , y_1 and y_2 with one stress change point $\tau = \tau_1$. A SSALT can be terminated at a pre-fixed time point (i.e., Type-I censoring), or at the time of observing the rth failure (i.e., Type-II censoring).

To analyze data from a SSALT, a model that relates the lifetime distribution under the step-stress mechanism to the distribution under constant stresses is required. Cumulative exposure model (CE-M), originally proposed by Sediakin (1966), assumes that the residual life of the experimental units depends only on the cumulative exposure of the units without any memory of how the exposure accumulated. For statistical analysis of SSALT data under CE-M, Nelson (1980) described the statistical model and derived the maximum likelihood estimates (MLEs) under the CE-M for Weibull and inverse power law distributions. Balakrishnan et al. (2007) studied the point and interval estimation of a simple step-stress model with Type-II censoring when the experimental units are assumed to follow exponential distribution. Balakrishnan and Xie (2007a,0); Balakrishnan and Han (2008); Balakrishnan (2008) studied the exact inference for step-stress model under different kind of censoring for exponentially distributed experimental units. Alkhalfan (2012) and Alam (2017) studied the statistical inference for multiple step-stress models for gamma and generalized Birnbaum-Saunders distributions, respectively.

For SSALT, the optimal experimental planning problem can be formulated as choosing the optimal time points $\tau_1 < \tau_2 < \ldots < \tau_{K-1}$ for the changing of stresses. There has also been a great development on the optimal experimental planning problem in the past decades. Most of these research articles focus on exponentially distributed experimental units. For example, Miller and Nelson (1983) explored the optimal plans under the A-optimality criterion for a simple SSALT. Bai et al. (1989) extended the results of Miller and Nelson by involving a pre-fixed censoring time. Gouno et al. (2004) investigated the optimal changing time points for a multi-level SSALT with a large progressively Type-I censored sample. To reduce the complexity of the problem, Gouno et al. (2004) adopted a simplified model with equal duration for each testing stage, i.e., $\tau_k = k\tau$ for $k = 1, 2 \ldots, K-1$. Balakrishnan and Han (2009) modified the model studied in Gouno et al. (2004) by determining the censoring rate based on the remaining items at every stress change instead of using a global censoring rate. By relaxing the setting of equal duration for each testing stage, Guan and Tang (2012) obtained the optimum planning for multivariate exponential distribution under Type-I censoring scheme. Lin et al. (2013) and Lin et al. (2020) investigated the optimal choice of the time-changing points for a multi-level SSALT with unequal stress steps for log-location-scale distributions under Type-I and Type-I hybrid censoring schemes, respectively.

In addition to the exponential distribution, other more flexible distributions have also been discussed as the lifetime distribution under different settings. Alhadeed and Yang (2005) obtained the optimal planning for a simple SSALT with lifetimes follow lognormal distribution. Taking the exponential and the Weibull as special cases, Bobotas and Kateri (2019) tackled the optimality problem under interval censoring. Hakamipour (2018) considered a simple bivariate SSALT with log-normally distributed lifetimes. They discussed how to find the two optimal changing points under the D-optimality and A-optimality with using the particle swarm optimization (PSO) algorithm.

In addition to the CE-M, Khamis and Higgins (1998) proposed another model called Khamis-Higgins model (KH-M) for SSALT to relate the lifetime distribution of the experimental units at one stress level to the life distribution of the experimental units at the next stress level. Compared with the CE-M, Khamis and Higgins claimed that the mathematical form of KH-M made it easier to derive the MLE, for which they used the Weibull distribution as an illustration. Under the KH-M, Li and Fard (2007) explored the optimal changing time points for a bivariate step-stress life testing using the maximum likelihood estimation and Fisher information matrix methods. A sensitivity analysis was included in their research to study the effect of using different parameter on the optimal plans.

For a comprehensive review of ALT, one can refer to the books by Nelson (1990) and Bagdonavicius and Nikulin (2002), and the papers by Nelson (2005) and Nelson (2015). For comparisons between the CSALT and SSALT, Han and Ng (2013) and Han and Ng (2014) compared the optimal multi-level CSALT and SSALT with complete and Type-I censored samples under the exponential distribution. They concluded that the SSALT seemed to be more efficient than the constant-stress one under the optimal designs. At the meanwhile, however, the SSALT might require longer termination time than the latter did in the case with more than two stress levels.

1.7. Scope of the Thesis

In this thesis, we develop algorithms to obtain the optimal experimental planning for reliability experiments using coherent systems. In the design of experiments in reliability engineering, minimizing the cost of conducting the experiment and maximizing the information collected from the experiment are usually the major objectives. In practice, using well-designed experiments with censoring is one of the approaches to achieve these objectives. When the experimental units (components) can be put into a system and the experiment is done based on those systems instead of the individual experimental units, some interesting and fundamental research questions are: (i) how to put the experimental units into a system to maximize the information collected from the experiment and minimize the cost of the experiment? (ii) what are the advantages and disadvantages to putting the components into a system for reliability experiments? The purpose of this thesis is to address these important research questions.

Although the idea of using systems instead of components in a life testing experiment has been discussed in the literature, to the best of our knowledge, a thorough study of the experimental planning and evaluation of the efficiency of using systems in a multi-level stress ALT experiment has not been done. In this thesis, we aim to fill this gap by providing a thorough study of the experimental planning based on systems formed by test units. The organization of the thesis is as follows.

In Chapter 2, we discuss the optimal experimental planning problem for multi-level constant-stress testing with Type-II censoring when the test units can be put into coherent systems for the experiment. Based on the notion of system signatures of coherent systems and assuming the lifetime of the test units follows a distribution in a general log-location-scale family of distributions, the maximum likelihood estimators of the model parameters and the Fisher information matrix are derived. For the optimal experimental planning, in addition to some commonly used optimality criteria, such as D-optimality, A-optimality and V-optimality, we also consider the total time of the experiment and the total time on test. Then, motivated by a real-life application in reliability study of furniture joints, we focus the study on using series systems in multi-level stress experiments. The methodology is illustrated by considering lognormal and Weibull distributed test units. Numerical and Monte Carlo simulation studies are used to demonstrate the advantages and disadvantages of using series systems in life-testing experiments. A numerical example based on furniture joints with sensitivity analysis is used to elucidate how the proposed methods can be used in planning a life testing experiment.

In Chapter 3, we investigate the optimal planning of a progressive Type-II censored experiment based on coherent systems formed by components that follow a log-location-scale distribution. We aim to determine the optimal censoring plan $(R_1, \ldots, R_r)$ for a given number of items under different optimality criteria. Specifically, we analyze the expected Fisher information and the asymptotic variance-covariance matrix for the maximum likelihood estimates (MLEs) obtained from a progressively Type-II censored sample with coherent systems. The methodology is illustrated by considering lognormal and Weibull distributed components with different coherent systems. The goal is to gain insights into the effectiveness of different coherent systems and determine which system type yields the most favorable results based on the chosen optimality criteria.

In Chapter 4, we summarize major results in Chapter 2 and Chapter 3 and provide three potential directions for future research. One possible direction is to explore the optimal experimental planning for multi-level step-stress testing with Type-II censoring when systems can be used for the life testing experiment. The other two involve a larger number of components and non location-scale family distributions.

This thesis focuses on the optimal experimental planning problem based on coherent systems. The statistical methods and techniques developed in this thesis will provide engineers and statisticians some efficient ways to perform lab testing that can save both time and cost of the experiment as well as obtain high efficiency of statistical inference, which is important for decision making in the engineering process.

Chapter 2

Optimal Experimental Planning for Constant-Stress Accelerated Life-Testing Experiments based on Coherent Systems

2.1. Introduction

For multi-level stress experiments with Type-II censoring, Elperin and Gertsbakh (1987) discussed the maximum likelihood estimation in a Weibull regression model with Type-I censoring and Paul and Thiagarajah (1996) derived the asymptotic variances and covariances of the maximum likelihood estimates of the model parameters for an extreme value regression model with Type-I censoring. Chan et al. (2008) also studied the point and interval estimation of the model parameters when the extreme-value regression model is used for the data analysis. The optimal experimental planning for multi-level stress experiments has long been studied, as we review in Chapter 1.

This chapter is organized as follows. In Section 2.2, we present the model for multi-level constant-stress experiments with coherent systems and different optimality criteria. Specifically, we review the notation of the system signature and the location-scale family of distributions. Maximum likelihood estimators (MLEs) of the model parameters and the corresponding Fisher information matrix are derived. Different optimality criteria, including some commonly used optimality criteria including the D-optimality, the A-optimality and the V-optimality, and different settings for optimal experimental planning are discussed. Then, in Section 2.3, a special case which series systems are used in the multi-level constant-stress experiments is considered. The computational formulas for lognormal and Weibull distributed components are used to illustrate the methodology developed in Section 2.4. Numerical illustrations with two- and four-stress levels are used to demonstrate the advantages and disadvantages of using series systems in multi-level constant-stress experiments. In Section 2.5, a Monte Carlo simulation study is used to evaluate the validity of the numerical results based on the asymptotic properties of MLEs. An illustrative example based on furniture joints testing is used in Section 2.6 to show how the proposed method can be used to effectively plan an experiment based on series system. Sensitivity analysis for the furniture joints testing example is also presented in Section 2.6. Finally, some concluding remarks and discussions about future research directions are provided in Section 2.7.

2.2. Models and Optimality Criteria

In this chapter, our analysis focuses on the n-component coherent systems under the general model as we describe in Section 1.2.2. Also, we assume that the underlying distribution of the component lifetimes comes from the location-scale family as we described in Section 1.4.

2.2.1. Multi-level constant-stress experiment with Type-II censoring

Consider a multi-level constant stress life testing experiment based on systems with K stress levels denoted as $y_1 < \ldots < y_K$, and m_k n_k-component systems with system signature $\boldsymbol{s}_k = (s_{k1}, s_{k2}, \ldots, s_{kn_k})$ are placed on the life test at the k-th stress level. In other words, we consider that the systems placed at the k-th stress level are assumed to have the same number of components and the same system structure, but systems placed at different stress levels can have different numbers of components and/or different system structures. To obtain a Type-II censored sample at the k-th stress level, we observed the first r_k ($r_k \leq m_k$) system failures out of the m_k systems at the k-th stress level and denote the ordered system lifetimes as $T_{1:m_k} < T_{2:m_k} < \ldots < T_{r_k:m_k}$ and the observed values as $t_{1:m_k} < t_{2:m_k} < \ldots < t_{r_k:m_k}$. We assume that the component that causes the system failure for each failed system is not observable. Hence, $n_k \times m_k$ components are used in the life test at the k-th stress level, the total number of components used in the life test is $N = \sum_{k=1}^{K} n_k m_k$ and the total number of observed system failures is $\sum_{k=1}^{K} r_k$. For the special case when $n_k = 1$, the life testing experiment is done at the component level in which m_k components are placed on the life test at the k-th stress level and the life testing experiment at the k-th stress level is terminated as soon as the r_k-th component failure is observed.

Based on the multi-level constant-stress life testing experiment with Type-II censoring, the observed system-level lifetime data are $(t_{1:m_k} < t_{2:m_k} < \ldots < t_{r_k:m_k})$, $k = 1, 2, \ldots, K$. We denote the PDF, CDF and SF of the log-lifetimes of components in the systems under the k-th stress level as $f_{U_k}(\cdot, \boldsymbol{\theta}_k)$, $F_{U_k}(\cdot, \boldsymbol{\theta}_k)$ and $\bar{F}_{U_k}(\cdot, \boldsymbol{\theta}_k)$, respectively, where $\boldsymbol{\theta}_k$ is the parameter vector depending on the stress level y_k, and the PDF, CDF and SF of the log-transformed system lifetimes under the k-th stress level as $f_{V_k}(\cdot, \boldsymbol{\theta}_k)$, $F_{V_k}(\cdot, \boldsymbol{\theta}_k)$ and $\bar{F}_{V_k}(\cdot, \boldsymbol{\theta}_k)$, respectively. Assume that the log-lifetimes of components in the systems under the k-th stress level follow a location-scale family of distribution described in Eq. (1.4) with location parameter μ_k and scale parameter σ, and a linear relationship between the location parameter μ_k and the stress level y_k, i.e.,

$$\mu_k = \beta_0 + \beta_1 y_k, k = 1, \ldots, K, \tag{2.1}$$

then the model parameter vector can be expressed as $\boldsymbol{\theta} = (\beta_0, \beta_1, \sigma)$ and $\boldsymbol{\theta}_k = (\mu_k, \sigma) = (\beta_0 + \beta_1 y_k, \sigma)$, $k = 1, 2, \ldots, K$. Then, the CDF and the PDF of the log-transformed system lifetime at the k-th stress level can be written as

$$\begin{aligned}
F_{V_k}(v;\boldsymbol{\theta}) &= 1-\sum_{i=1}^{n_k} s_{ki}\sum_{j=0}^{i-1}\binom{n_k}{j}[F_{U_k}(v;\boldsymbol{\theta}_k)]^j[\bar{F}_{U_k}(v;\boldsymbol{\theta}_k)]^{n_k-j} \\
&= 1-\sum_{i=1}^{n_k} s_{ki}\sum_{j=0}^{i-1}\binom{n_k}{j}\left[F^*\left(\frac{v-\mu_k}{\sigma}\right)\right]^j\left[\bar{F}^*\left(\frac{v-\mu_k}{\sigma}\right)\right]^{n_k-j} \\
&= 1-\sum_{i=1}^{n_k} s_{ki}\sum_{j=0}^{i-1}\binom{n_k}{j}\left[F^*\left(\frac{v-(\beta_0+\beta_1 y_k)}{\sigma}\right)\right]^j\left[\bar{F}^*\left(\frac{v-(\beta_0+\beta_1 y_k)}{\sigma}\right)\right]^{n_k-j} \\
&= 1-\sum_{i=1}^{n_k} s_{ki}\sum_{j=0}^{i-1}\binom{n_k}{j}\,[F^*(z_k)]^j\,[\bar{F}^*(z_k)]^{n_k-j}, k=1,...,K, \qquad (2.2)
\end{aligned}$$

and

$$\begin{aligned}
f_{V_k}(v;\boldsymbol{\theta}) &= \frac{1}{\sigma}f^*\left(\frac{v-\mu_k}{\sigma}\right)\sum_{i=1}^{n_k} s_{ki}\binom{n_k}{i}i\left[F^*\left(\frac{v-\mu_k}{\sigma}\right)\right]^{i-1}\left[\bar{F}^*\left(\frac{v-\mu_k}{\sigma}\right)\right]^{n_k-i} \\
&= \frac{1}{\sigma}f^*(z_k)\sum_{i=1}^{n_k} s_{ki}\binom{n_k}{i}i[F^*(z_k)]^{i-1}[\bar{F}^*(z_k)]^{n_k-i}, k=1,...,K, \qquad (2.3)
\end{aligned}$$

where

$$z_k=\frac{v-\mu_k}{\sigma}=\frac{v-(\beta_0+\beta_1 y_k)}{\sigma}.$$

Note that the distribution of the system lifetime at the k-th stress level depends on the system signature of the systems used in the k-th stress level ($\boldsymbol{s}_k$) and the number of components in each system n_k. Here, for simplicity, we suppress the notations $\boldsymbol{s}_k$ and n_k in the CDF and PDF of the lifetime distributions.

2.2.2. Parameter estimation

2.2.2.1. Maximum likelihood estimation

Based on the log-transformed observed system-level lifetime data $\{(v_{1:m_k} < v_{2:m_k} < \ldots < v_{r_k:m_k}),\ k = 1, 2, \ldots, K\}$ obtained from a multi-level constant-stress life testing experiment with Type-II censoring and assuming the log-lifetime of the components follow a location-scale distribution, the likelihood function can be expressed as

$$L(\boldsymbol{\theta}) = L(\beta_0, \beta_1, \sigma) = \prod_{k=1}^{K} \frac{m_k!}{(m_k - r_k)!} \left\{ \prod_{\ell=1}^{r_k} f_{V_k}(v_{\ell:m_k}; \boldsymbol{\theta}) \right\} \left[\bar{F}_{V_k}(v_{r_k:m_k}; \boldsymbol{\theta}) \right]^{m_k - r_k} \tag{2.4}$$

and the log-likelihood function is

$$l(\boldsymbol{\theta}) = \ln L(\beta_0, \beta_1, \sigma) = constant + \sum_{k=1}^{K} \sum_{\ell=1}^{r_k} \ln f_{V_k}(v_{\ell:m_k}; \boldsymbol{\theta}) + \sum_{k=1}^{K} (m_k - r_k) \ln \bar{F}_{V_k}(v_{r_k:m_k}; \boldsymbol{\theta}). \tag{2.5}$$

Taking the first partial derivatives of the log-likelihood function, we can obtain the likelihood equations

$$\begin{aligned} \frac{\partial l(\boldsymbol{\theta})}{\partial \beta_0} &= \sum_{k=1}^{K} \sum_{\ell=1}^{r_k} \frac{1}{f_{V_k}(v_{\ell:m_k}; \boldsymbol{\theta})} \frac{\partial f_{V_k}(v_{\ell:m_k}; \boldsymbol{\theta})}{\partial \beta_0} \\ &\quad + \sum_{k=1}^{K} \frac{m_k - r_k}{\bar{F}_{V_k}(v_{r_k:m_k}; \boldsymbol{\theta})} \frac{\partial \bar{F}_{V_k}(v_{r_k:m_k}; \boldsymbol{\theta})}{\partial \beta_0} = 0, \end{aligned} \tag{2.6}$$

$$\begin{aligned} \frac{\partial l(\boldsymbol{\theta})}{\partial \beta_1} &= \sum_{k=1}^{K} \sum_{\ell=1}^{r_k} \frac{1}{f_{V_k}(v_{\ell:m_k}; \boldsymbol{\theta})} \frac{\partial f_{V_k}(v_{\ell:m_k}; \boldsymbol{\theta})}{\partial \beta_1} \\ &\quad + \sum_{k=1}^{K} \frac{m_k - r_k}{\bar{F}_{V_k}(v_{r_k:m_k}; \boldsymbol{\theta})} \frac{\partial \bar{F}_{V_k}(v_{r_k:m_k}; \boldsymbol{\theta})}{\partial \beta_1} = 0, \end{aligned} \tag{2.7}$$

$$\begin{aligned} \frac{\partial l(\boldsymbol{\theta})}{\partial \sigma} &= \sum_{k=1}^{K} \sum_{\ell=1}^{r_k} \frac{1}{f_{V_k}(v_{\ell:m_k}; \boldsymbol{\theta})} \frac{\partial f_{V_k}(v_{\ell:m_k}; \boldsymbol{\theta})}{\partial \sigma} \\ &\quad + \sum_{k=1}^{K} \frac{m_k - r_k}{\bar{F}_{V_k}(v_{r_k:m_k}; \boldsymbol{\theta})} \frac{\partial \bar{F}_{V_k}(v_{r_k:m_k}; \boldsymbol{\theta})}{\partial \sigma} = 0. \end{aligned} \tag{2.8}$$

The MLEs of the parameters β_0, β_1 and σ, denoted as $\hat{\beta}_0$, $\hat{\beta}_1$ and $\hat{\sigma}$, can be obtained by solving Eqs. (2.6)–(2.8) simultaneously. Using the relations in Eqs. (2.2) and (2.3), we can express the first derivatives of the SF of the system lifetime with respect to the parameters in terms of the component lifetime distributions as

$$\begin{aligned}
\frac{\partial \bar{F}_{V_k}(v_{\ell:m_k};\boldsymbol{\theta})}{\partial \beta_0} &= \sum_{i=1}^{n_k} s_{ki} \sum_{j=0}^{i-1} \binom{n_k}{j} \Bigg[j\,[F^*(z_{\ell:m_k})]^{j-1} \left(f^*(z_{\ell:m_k}) \frac{\partial z_{\ell:m_k}}{\partial \beta_0} \right) [\bar{F}^*(z_{\ell:m_k})]^{n_k-j} \\
&\quad + [F^*(z_{\ell:m_k})]^{j} \, (n_k-j) \left[\bar{F}^*(z_{\ell:m_k})\right]^{n_k-j-1} \left(-f^*(z_{\ell:m_k}) \frac{\partial z_{\ell:m_k}}{\partial \beta_0} \right) \Bigg] \\
&= \left[-\frac{1}{\sigma} f^*(z_{\ell:m_k}) \right] \sum_{i=1}^{n_k} s_{ki} \sum_{j}^{i-1} \binom{n_k}{j} \Bigg[j\,[F^*(z_{\ell:m_k})]^{j-1} \, [\bar{F}^*(z_{\ell:m_k})]^{n_k-j} \\
&\quad - [F^*(z_{\ell:m_k})]^{j} \, (n_k-j) [\bar{F}^*(z_{\ell:m_k})]^{n_k-j-1} \Bigg], \qquad (2.9) \\
\frac{\partial \bar{F}_{V_k}(v_{\ell:m_k};\boldsymbol{\theta})}{\partial \beta_1} &= \left[-\frac{y_k}{\sigma} f^*(z_{\ell:m_k}) \right] \sum_{i=1}^{n_k} s_{ki} \sum_{j}^{i-1} \binom{n_k}{j} \Bigg[j\,[F^*(z_{\ell:m_k})]^{j-1} \, [\bar{F}^*(z_{\ell:m_k})]^{n_k-j} \\
&\quad - [F^*(z_{\ell:m_k})]^{j} \, (n_k-j) [\bar{F}^*(z_{\ell:m_k})]^{n_k-j-1} \Bigg], \qquad (2.10) \\
\frac{\partial \bar{F}_{V_k}(v_{\ell:m_k};\boldsymbol{\theta})}{\partial \sigma} &= \left[-\frac{z_{\ell:m_k}}{\sigma} f^*(z_{\ell:m_k}) \right] \sum_{i=1}^{n_k} s_{ki} \sum_{j}^{i-1} \binom{n_k}{j} \Bigg[j\,[F^*(z_{\ell:m_k})]^{j-1} \, [\bar{F}^*(z_{\ell:m_k})]^{n_k-j} \\
&\quad - [F^*(z_{\ell:m_k})]^{j} \, (n_k-j) [\bar{F}^*(z_{\ell:m_k})]^{n_k-j-1} \Bigg]. \qquad (2.11)
\end{aligned}$$

Similarly, using the relations in Eqs. (2.2) and (2.3), we can express the first partial derivatives of the PDF of the system lifetime with respect to the parameters in terms of the component lifetime distributions as

$$\begin{aligned}
\frac{\partial f_{V_k}(v_{\ell:m_k};\boldsymbol{\theta})}{\partial \beta_0} &= \frac{1}{\sigma} \frac{\partial f^*(z_{\ell:m_k})}{\partial \beta_0} \sum_{i=1}^{n_k} s_{ki} \binom{n_k}{i} i [F^*(z_{\ell:m_k})]^{i-1} [\bar{F}^*(z_{\ell:m_k})]^{n_k-i} \\
&\quad - \frac{1}{\sigma^2} \left(f^*(z_{\ell:m_k}) \right)^2 \sum_{i=1}^{n_k} s_{ki} \binom{n_k}{i} i \Bigg[(i-1) [F^*(z_{\ell:m_k})]^{i-2} [\bar{F}^*(z_{\ell:m_k})]^{n_k-i} \\
&\quad - [F^*(z_{\ell:m_k})]^{i-1} (n_k-i) [\bar{F}^*(z_{\ell:m_k})]^{n_k-i-1} \Bigg], \qquad (2.12)
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial f_{V_k}(v_{\ell:m_k};\boldsymbol{\theta})}{\partial \beta_1} &= \frac{1}{\sigma}\frac{\partial f^*(z_{\ell:m_k})}{\partial \beta_1}\sum_{i=1}^{n_k} s_{ki}\binom{n_k}{i} i[F^*(z_{\ell:m_k})]^{i-1}[\bar{F}^*(z_{\ell:m_k})]^{n_k-i} \\
&\quad -\frac{y_k}{\sigma^2}\left(f^*(z_{\ell:m_k})\right)^2 \sum_{i=1}^{n_k} s_{ki}\binom{n_k}{i} i\Big[(i-1)[F^*(z_{\ell:m_k})]^{i-2}[\bar{F}^*(z_{\ell:m_k})]^{n_k-i} \\
&\quad -[F^*(z_{\ell:m_k})]^{i-1}(n_k-i)[\bar{F}^*(z_{\ell:m_k})]^{n_k-i-1}\Big], \qquad (2.13) \\
\frac{\partial f_{V_k}(v_{\ell:m_k};\boldsymbol{\theta})}{\partial \sigma} &= \left[-\frac{1}{\sigma^2}f^*(z_{\ell:m_k}) + \frac{1}{\sigma}\frac{\partial f^*(z_{\ell:m_k})}{\partial \sigma}\right] \\
&\quad \times \sum_{i=1}^{n_k} s_{ki}\binom{n_k}{i} i[F^*(z_{\ell:m_k})]^{i-1}[\bar{F}^*(z_{\ell:m_k})]^{n_k-i} \\
&\quad -\frac{z_{\ell:m_k}}{\sigma^2}\left(f^*(z_{\ell:m_k})\right)^2 \sum_{i=1}^{n_k} s_{ki}\binom{n_k}{i} i\Big[(i-1)[F^*(z_{\ell:m_k})]^{i-2}[\bar{F}^*(z_{\ell:m_k})]^{n_k-i} \\
&\quad -[F^*(z_{\ell:m_k})]^{i-1}(n_k-i)[\bar{F}^*(z_{\ell:m_k})]^{n_k-i-1}\Big]. \qquad (2.14)
\end{aligned}
$$

2.2.2.2. Fisher information and asymptotic variance-covariance matrices

To measure the amount of information in the observed lifetime data and to qualify the uncertainty in the MLEs, we consider the expected Fisher information matrix of $\boldsymbol{\theta} = (\beta_0, \beta_1, \sigma)$, denoted as $\boldsymbol{I}(\beta_0, \beta_1, \sigma)$, which is defined as

$$
\boldsymbol{I}(\beta_0,\beta_1,\sigma) = -\begin{bmatrix}
E\left(\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \beta_0^2}\right) & E\left(\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \beta_0 \partial \beta_1}\right) & E\left(\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \beta_0 \partial \sigma}\right) \\
E\left(\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \beta_0 \partial \beta_1}\right) & E\left(\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \beta_1^2}\right) & E\left(\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \beta_1 \partial \sigma}\right) \\
E\left(\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \beta_0 \partial \sigma}\right) & E\left(\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \beta_1 \partial \sigma}\right) & E\left(\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \sigma^2}\right)
\end{bmatrix}. \qquad (2.15)
$$

The second partial derivatives of the log-likelihood function involved in the Fisher information matrix can be expressed in terms of the PDF and CDF of the component lifetimes as described in Section 2.2.2.1 and these expressions are presented in Appendix A. The asymptotic variance-covariance matrix of the MLEs can be obtained as the inverse of the Fisher information matrix as

$$\boldsymbol{V}(\beta_0, \beta_1, \sigma) = \boldsymbol{I}^{-1}(\beta_0, \beta_1, \sigma) = \begin{bmatrix} Var(\hat{\beta}_0) & Cov(\hat{\beta}_0, \hat{\beta}_1) & Cov(\hat{\beta}_0, \hat{\sigma}) \\ & Var(\hat{\beta}_1) & Cov(\hat{\beta}_1, \hat{\sigma}) \\ & & Var(\hat{\sigma}) \end{bmatrix}. \quad (2.16)$$

2.2.3. Optimality criteria

Suppose we are interested in the estimation of the model parameters and controlling the time used in the life testing experiment. To discuss the optimal experimental planning, we consider the following optimality criteria:

[C1] D-optimality: Maximization of the determinant of the Fisher information matrix, $\mathbf{I}$, i.e., maximizing the differential Shannon information contained in the MLEs and minimizing the volume of the Wald-type joint confidence region for the model parameter $(\beta_0, \beta_1, \sigma)$;

[C2] A-optimality: Minimization of the trace of the variance-covariance matrix of the MLEs, $V(\beta_0, \beta_1, \sigma)$, i.e., minimizing the sum of the asymptotic variances of the MLEs:

$$Var(\hat{\beta}_0) + Var(\hat{\beta}_1) + Var(\hat{\sigma});$$

[C3] V-optimality: Minimization of the variance of the estimator of β_1, $Var(\hat{\beta}_1)$. With the linear relationship between the mean lifetime and the stress level, it is important in estimating the slope parameter β_1 accurately. Therefore, this criterion minimizes the variance of the estimate of the slope parameter;

[C4] Minimization of the expected total time of the experiment, denoted as TTE, which is defined as

$$E(TTE) = E\left\{\max_k \left[\exp\left(V_{r_k:m_k}\right)\right]\right\} = \max_k\{E\left[\exp\left(V_{r_k:m_k}\right)\right]\}.$$

Using the relationship between $Z_{\ell:m_k}$ and $V_{\ell:m_k}$:

$$Z_{\ell:m_k} = \frac{V_{\ell:m_k} - \mu_k}{\sigma} = \frac{V_{\ell:m_k} - (\beta_0 + \beta_1 y_k)}{\sigma}.$$

The TTE can be expressed in terms of $Z_{\ell:m_k}$, which is the ordered standardized log-transformed failure time, as

$$E(TTE) \quad = \quad \max_k \left(\exp(\beta_0 + \beta_1 y_k) E\left[\exp(\sigma Z_{r_k:m_k})\right]\right). \tag{2.17}$$

Based on the multi-level constant-stress experiment with Type-II censoring described in Section 2.2.1, the experiment terminates at the last observed failure. This criterion aims to minimize the termination time of the experiment.

[C5] Minimization of the expected total time for all the components spending on the experiment, denoted as TTT, which is defined as

$$\begin{aligned} E(TTT) &= nE\left\{\sum_{k=1}^{K}\left[\sum_{\ell=1}^{r_k} \exp\left(V_{\ell:m_k}\right) + (m_k - r_k)\exp\left(V_{r_k:m_k}\right)\right]\right\} \\ &= n\sum_{k=1}^{K}\left\{\sum_{\ell=1}^{r_k} E\left[\exp\left(V_{\ell:m_k}\right)\right] + (m_k - r_k)E\left[\exp\left(V_{r_k:m_k}\right)\right]\right\} \\ &= n\sum_{k=1}^{K}\exp(\beta_0 + \beta_1 y_k)\left\{\sum_{\ell=1}^{r_k} E\left[\exp(\sigma Z_{\ell:m_k})\right] + (m_k - r_k)E\left[\exp(\sigma Z_{r_k:m_k})\right]\right\}. \end{aligned}$$

Since the components that not fail in the life testing experiment can be used in the future for other purposes, the shorter the time those components spent in the experiment, the longer they can be used for other purposes. Therefore, this criterion aims to minimize the total time the components spent in the experiment.

2.2.4. Settings for optimal experimental planning

In this section, we consider different settings to plan the multi-level constant-stress experiment with Type-II censoring described in Section 2.2.1. Suppose there are N components available for a multi-level constant-stress experiment with K stress levels $y_1 < y_2 < \ldots < y_K$. We are interested in finding the optimal way to perform the experiment based on the optimality criteria presented in Section 2.2.3 when putting the components together as a system for the life testing experiment is possible. To simplify the problem, we consider the case that the system structures of the systems assigned to different stress levels are the same, i.e., the N components are put into $M = N/n$ n-component systems with system signature $\boldsymbol{s} = (s_1, \ldots, s_n)$. Then, m_k systems are assigned to stress level y_k, and the first r_k system failures are observed. We have $\sum\limits_{k=1}^{K} m_k = M$ and the expected number of failed components in the failed systems (denoted as R) as

$$E(R) = \sum_{k=1}^{K} \left[\sum_{i=1}^{n} i s_i \right] r_k.$$

It is noteworthy that, except for the series system, some of the components in the systems being censored may still be failed. In that case, the expected number of failed components in a censored system depends on the termination time of the experiment in a Type-II censored experiment, which also depends on the underlying lifetime distribution of the components. We consider the expected number of failed components in the failed systems only because this expected value is independent of the component lifetime distribution and it depends on the system structure (i.e., the system signature $\boldsymbol{s}$) and the experimental planning (i.e., the values of r_k, $k = 1, 2, \ldots, K$) only. For instance, for l-out-of-n system, R is fixed as $R = l \sum_{k=1}^{K} r_k$.

Suppose that N, K, $(y_1, ..., y_K)$ and $E(R)$ are pre-fixed, since the number of components in a system n, the number of systems/components assigned to the K stress levels $(m_1, m_2, \ldots, m_K)$ and the number of observed components/system failures at the K stress levels $(r_1, r_2, \ldots, r_K)$ can be adjusted, the following settings are considered:

[M1] Fix $m_1 = m_2 = \ldots = m_k$ and $r_1 = r_2 = \ldots = r_k$ for $k = 1, ..., K$, and determine the values of n and $\boldsymbol{s}$, the number of components in the system and the system structure that optimize the specific objective function. In this setting, we assign the same number of systems and fix the same number of observed system failures at all the K stress level.

[M2] Fix $m_1 = m_2 = \ldots = m_k$ for $k = 1, \ldots, K$ and fix $E(R)$, and then determine the values of r_k (number of observed system failures at stress level k), the values of n (number of components in the system) and $\boldsymbol{s}$ (system structure) that optimize the specific objective function. In this setting, we assign the same number of systems to each level and fix the expected number of failed components in those failed systems, and then we allow the censoring proportion at each stress level to be different.

[M3] No specific constraints on m_k and r_k for $k = 1, \ldots, K$ with fixed $\sum_{k=1}^{K} m_k = M$ and $E(R)$, then determine the values of $(m_1, \ldots, m_K)$ and $(r_1, \ldots, r_K)$ that optimize the specific objective function.

Since the optimal allocation cannot be found in a closed-form and there are a finite number of possible experimental plans, a discrete search method can be used to obtain the optimal plans of the experiment under different settings for different optimality criteria.

2.3. Using n-component Series Systems for Experiments

Motivated by the reliability study for joints used in case furniture presented in Kłos et al. (2018) in which n furniture joints are connected in series and the breakdown time of the weakest link of the n-component system is recorded, we consider a special situation that series systems are used in a multi-level constant-stress life testing experiment with Type-II censoring. In other words, we fixed the system signature as $\boldsymbol{s} = (1, 0, \ldots, 0)$ and $n_1 = \ldots = n_k = n$. As mentioned in Section 2.1, using series system in life testing experiments with censoring has been considered in the literature. For example, the progressive first-failure censoring scheme proposed by Wu and Kuş (2009) is equivalent to using series systems in a progressive censored life testing experiment. Based on the multi-level constant-stress life testing experiment with Type-II censoring described in Section 2.2.1 with series systems, the expected number of failed components in the failed systems is $R = \sum\limits_{k=1}^{K} r_k$.

For a n-component series system, the SF and PDF of the log-transformed system failure times at the k-th stress level are, respectively,

$$\begin{aligned} \bar{F}_{V_k}(v; \boldsymbol{\theta}) &= [\bar{F}^*(z_k)]^n, \\ \text{and } f_{V_k}(v; \boldsymbol{\theta}) &= \frac{n}{\sigma} f^*(z_k)[\bar{F}^*(z_k)]^{n-1}, \end{aligned}$$

$k = 1, \ldots, K$. The log-likelihood functions in Eq. (2.5) can be simplified as

$$l(\boldsymbol{\theta}) = \text{constant} + \sum_{k=1}^{K} \sum_{\ell=1}^{r_k} \ln f_{V_k}(v_{\ell:m_k}; \boldsymbol{\theta}) + \sum_{k=1}^{K} (m_k - r_k) \ln \bar{F}_{V_k}(v_{r_k:m_k}; \boldsymbol{\theta}),$$

where

$$\begin{aligned} \ln f_{V_k}(v_{\ell:m_k}; \boldsymbol{\theta}) &= \ln \left\{ \frac{n}{\sigma} f^*(z_{\ell:m_k}) \left[\bar{F}^*(z_{\ell:m_k})\right]^{n-1} \right\} \\ &= \ln n - \ln \sigma + \ln f^*(z_{\ell:m_k}) + (n-1) \ln \bar{F}^*(z_{\ell:m_k}) \\ \text{and } \ln \bar{F}_{V_k}(v_{r_k:m_k}; \boldsymbol{\theta}) &= \ln \left\{ \left[\bar{F}^*(z_{r_k:m_k})\right]^n \right\} = n \ln \bar{F}^*(z_{r_k:m_k}). \end{aligned}$$

For notation simplicity, we define the following notations for the first and second derivatives of the PDF and CDF of standardized distribution:

$$\begin{aligned}
P_{10}(z) &= \frac{\partial f^*(z)}{\partial \beta_0}, \quad P_{20}(z) = \frac{\partial f^*(z)}{\partial \beta_1}, \quad P_{30}(z) = \frac{\partial f^*(z)}{\partial \sigma}, \\
P_{11}(z) &= \frac{\partial^2 f^*(z)}{\partial \beta_0^2}, \quad P_{22}(z) = \frac{\partial^2 f^*(z)}{\partial \beta_1^2}, \quad P_{33}(z) = \frac{\partial^2 f^*(z)}{\partial \sigma^2}, \\
P_{12}(z) &= \frac{\partial^2 f^*(z)}{\partial \beta_0 \partial \beta_1}, \quad P_{13}(z) = \frac{\partial^2 f^*(z)}{\partial \beta_0 \partial \sigma}, \quad P_{23}(z) = \frac{\partial^2 f^*(z)}{\partial \beta_1 \partial \sigma}, \\
C_{10}(z) &= \frac{\partial F^*(z)}{\partial \beta_0}, \quad C_{20}(z) = \frac{\partial F^*(z)}{\partial \beta_1}, \quad C_{30}(z) = \frac{\partial F^*(z)}{\partial \sigma}, \\
C_{11}(z) &= \frac{\partial^2 F^*(z)}{\partial \beta_0^2}, \quad C_{22}(z) = \frac{\partial^2 F^*(z)}{\partial \beta_1^2}, \quad C_{33}(z) = \frac{\partial^2 F^*(z)}{\partial \sigma^2}, \\
C_{12}(z) &= \frac{\partial^2 F^*(z)}{\partial \beta_0 \partial \beta_1}, \quad C_{13}(z) = \frac{\partial^2 F^*(z)}{\partial \beta_0 \partial \sigma}, \quad C_{23}(z) = \frac{\partial^2 F^*(z)}{\partial \beta_1 \partial \sigma}.
\end{aligned}$$

Based on the assumption that the log-transformed component lifetimes follow a distribution in the location-scale family, from the relations in Eq. (1.4), we can obtain

$$\begin{aligned}
\frac{\partial z}{\partial \beta_0} &= -\frac{1}{\sigma}, \quad \frac{\partial z}{\partial \beta_1} = -\frac{y}{\sigma}, \quad \frac{\partial z}{\partial \sigma} = -\frac{z}{\sigma}, \\
\frac{\partial \bar{F}^*(z)}{\partial z} &= -\frac{\partial F^*(z)}{\partial z} = -f^*(z).
\end{aligned}$$

Then, the partial derivatives of $l(\boldsymbol{\theta})$ with respect to the parameters β_0, β_1, and σ in Eqs. (2.6), (2.7), (2.8) can be simplified as

$$\begin{aligned}
\frac{\partial l(\boldsymbol{\theta})}{\partial \beta_0} &= \sum_{k=1}^{K}\sum_{\ell=1}^{r_k}\left[\frac{P_{10}(z_{\ell:m_k})}{f^*(z_{\ell:m_k})} + (n-1)\frac{-C_{10}(z_{\ell:m_k})}{\bar{F}^*(z_{\ell:m_k})}\right] \\
&\quad + \sum_{k=1}^{K}(m_k - r_k)\left[\frac{n(-C_{10}(z_{r_k:m_k}))}{\bar{F}^*(z_{r_k:m_k})}\right], \qquad (2.18)\\
\frac{\partial l(\boldsymbol{\theta})}{\partial \beta_1} &= \sum_{k=1}^{K}\sum_{\ell=1}^{r_k}\left[\frac{P_{20}(z_{\ell:m_k})}{f^*(z_{\ell:m_k})} + (n-1)\frac{-C_{20}(z_{\ell:m_k})}{\bar{F}^*(z_{\ell:m_k})}\right] \\
&\quad + \sum_{k=1}^{K}(m_k - r_k)\left[\frac{n(-C_{20}(z_{r_k:m_k}))}{\bar{F}^*(z_{r_k:m_k})}\right], \qquad (2.19)\\
\frac{\partial l(\boldsymbol{\theta})}{\partial \sigma} &= \sum_{k=1}^{K}\sum_{\ell=1}^{r_k}\left[-\frac{1}{\sigma} + \frac{P_{30}(z_{\ell:m_k})}{f^*(z_{\ell:m_k})} + (n-1)\frac{-C_{30}(z_{\ell:m_k})}{\bar{F}^*(z_{\ell:m_k})}\right] \\
&\quad + \sum_{k=1}^{K}(m_k - r_k)\left[\frac{n(-C_{30}(z_{r_k:m_k}))}{\bar{F}^*(z_{r_k:m_k})}\right], \qquad (2.20)
\end{aligned}$$

respectively. Similarly, we can obtain the second partial derivatives of $l(\boldsymbol{\theta})$ as

$$\begin{aligned}
\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \beta_0^2} &= \sum_{k=1}^{K}\sum_{\ell=1}^{r_k}\left\{-\left(\frac{P_{10}(z_{\ell:m_k})}{f^*(z_{\ell:m_k})}\right)^2+\left(\frac{P_{11}(z_{\ell:m_k})}{f^*(z_{\ell:m_k})}\right)\right. \\
&\qquad \left.+(n-1)\left[-\left(\frac{C_{10}(z_{\ell:m_k})}{\bar{F}^*(z_{\ell:m_k})}\right)^2-\left(\frac{C_{11}(z_{\ell:m_k})}{\bar{F}^*(z_{\ell:m_k})}\right)\right]\right\} \\
&\quad +\sum_{k=1}^{K} n(m_k-r_k)\left[-\left(\frac{C_{10}(z_{r_k:m_k})}{\bar{F}^*(z_{r_k:m_k})}\right)^2-\left(\frac{C_{11}(z_{r_k:m_k})}{\bar{F}^*(z_{r_k:m_k})}\right)\right], \qquad (2.21)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \beta_1^2} &= \sum_{k=1}^{K}\sum_{\ell=1}^{r_k}\left\{-\left(\frac{P_{20}(z_{\ell:m_k})}{f^*(z_{\ell:m_k})}\right)^2+\left(\frac{P_{22}(z_{\ell:m_k})}{f^*(z_{\ell:m_k})}\right)\right. \\
&\qquad \left.+(n-1)\left[-\left(\frac{C_{20}(z_{\ell:m_k})}{\bar{F}^*(z_{\ell:m_k})}\right)^2-\left(\frac{C_{22}(z_{\ell:m_k})}{\bar{F}^*(z_{\ell:m_k})}\right)\right]\right\} \\
&\quad +\sum_{k=1}^{K} n(m_k-r_k)\left[-\left(\frac{C_{20}(z_{r_k:m_k})}{\bar{F}^*(z_{r_k:m_k})}\right)^2-\left(\frac{C_{22}(z_{r_k:m_k})}{\bar{F}^*(z_{r_k:m_k})}\right)\right], \qquad (2.22)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \sigma^2} &= \sum_{k=1}^{K}\sum_{\ell=1}^{r_k}\left\{\frac{1}{\sigma^2}-\left(\frac{P_{30}(z_{\ell:m_k})}{f^*(z_{\ell:m_k})}\right)^2+\left(\frac{P_{33}(z_{\ell:m_k})}{f^*(z_{\ell:m_k})}\right)\right. \\
&\qquad \left.+(n-1)\left[-\left(\frac{C_{30}(z_{\ell:m_k})}{\bar{F}^*(z_{\ell:m_k})}\right)^2-\left(\frac{C_{33}(z_{\ell:m_k})}{\bar{F}^*(z_{\ell:m_k})}\right)\right]\right\} \\
&\quad +\sum_{k=1}^{K} n(m_k-r_k)\left[-\left(\frac{C_{30}(z_{r_k:m_k})}{\bar{F}^*(z_{r_k:m_k})}\right)^2-\left(\frac{C_{33}(z_{r_k:m_k})}{\bar{F}^*(z_{r_k:m_k})}\right)\right], \qquad (2.23)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \beta_0 \partial \beta_1} &= \sum_{k=1}^{K}\sum_{\ell=1}^{r_k}\left\{-\left(\frac{P_{10}(z_{\ell:m_k})P_{20}(z_{\ell:m_k})}{(f^*(z_{\ell:m_k}))^2}\right)+\left(\frac{P_{12}(z_{\ell:m_k})}{f^*(z_{\ell:m_k})}\right)\right. \\
&\qquad \left.+(n-1)\left[-\left(\frac{C_{10}(z_{\ell:m_k})C_{20}(z_{\ell:m_k})}{(\bar{F}^*(z_{\ell:m_k}))^2}\right)-\left(\frac{C_{12}(z_{\ell:m_k})}{\bar{F}^*(z_{\ell:m_k})}\right)\right]\right\} \\
&\quad +\sum_{k=1}^{K} n(m_k-r_k)\left[-\left(\frac{C_{10}(z_{r_k:m_k})C_{20}(z_{r_k:m_k})}{(\bar{F}^*(z_{r_k:m_k}))^2}\right)-\left(\frac{C_{12}(z_{r_k:m_k})}{\bar{F}^*(z_{r_k:m_k})}\right)\right], (2.24)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \beta_0 \partial \sigma} &= \sum_{k=1}^{K}\sum_{\ell=1}^{r_k}\left\{-\left(\frac{P_{10}(z_{\ell:m_k})P_{30}(z_{\ell:m_k})}{(f^*(z_{\ell:m_k}))^2}\right)+\left(\frac{P_{13}(z_{\ell:m_k})}{f^*(z_{\ell:m_k})}\right)\right. \\
&\qquad \left.+(n-1)\left[-\left(\frac{C_{10}(z_{\ell:m_k})C_{30}(z_{\ell:m_k})}{(\bar{F}^*(z_{\ell:m_k}))^2}\right)-\left(\frac{C_{13}(z_{\ell:m_k})}{\bar{F}^*(z_{\ell:m_k})}\right)\right]\right\} \\
&\quad +\sum_{k=1}^{K} n(m_k-r_k)\left[-\left(\frac{C_{10}(z_{r_k:m_k})C_{30}(z_{r_k:m_k})}{(\bar{F}^*(z_{r_k:m_k}))^2}\right)-\left(\frac{C_{13}(z_{r_k:m_k})}{\bar{F}^*(z_{r_k:m_k})}\right)\right], (2.25)
\end{aligned}$$

$$\frac{\partial^2 l(\boldsymbol{\theta})}{\partial\beta_1\partial\sigma} = \sum_{k=1}^{K}\sum_{\ell=1}^{r_k}\Bigg\{-\left(\frac{P_{20}(z_{\ell:m_k})P_{30}(z_{\ell:m_k})}{(f^*(z_{\ell:m_k}))^2}\right)+\left(\frac{P_{23}(z_{\ell:m_k})}{f^*(z_{\ell:m_k})}\right)$$
$$+(n-1)\left[-\left(\frac{C_{20}(z_{\ell:m_k})C_{30}(z_{\ell:m_k})}{(\bar{F}^*(z_{\ell:m_k}))^2}\right)-\left(\frac{C_{23}(z_{\ell:m_k})}{\bar{F}^*(z_{\ell:m_k})}\right)\right]\Bigg\}$$
$$+\sum_{k=1}^{K} n(m_k-r_k)\left[-\left(\frac{C_{20}(z_{r_k:m_k})C_{30}(z_{r_k:m_k})}{(\bar{F}^*(z_{r_k:m_k}))^2}\right)-\left(\frac{C_{23}(z_{r_k:m_k})}{\bar{F}^*(z_{r_k:m_k})}\right)\right]. \quad (2.26)$$

Then, the Fisher information matrix and the asymptotic variance-covariance matrix in Eqs. (2.15) and (2.16) can be computed based on the second derivatives in Eqs. (2.21) – (2.26). In the following subsections, we provide the computational formulas when the components are distributed as lognormal and Weibull.

2.3.1. Lognormal distributed components

Consider the lifetimes of the components follow a two-parameter lognormal distribution with CDF and PDF

$$F_{LN}(x;\mu,\sigma) = \Phi_{nor}\left[\frac{\ln(x)-\mu}{\sigma}\right], x>0,$$
$$f_{LN}(x;\mu,\sigma) = \frac{1}{\sigma x}\phi_{nor}\left(\frac{\ln(x)-\mu}{\sigma}\right) = \frac{1}{\sqrt{2\pi}\sigma x}\exp\left[-\frac{1}{2}\left(\frac{\ln(x)-\mu}{\sigma}\right)^2\right] x>0,$$

respectively, where $\exp(\mu)$ is the scale parameter, $\sigma>0$ is the shape parameter and $\Phi_{nor}(z)$ and $\phi_{nor}(z)$ are the CDF, SF and PDF for the standard normal distribution:

$$\Phi_{nor}(z) = \int_{-\infty}^{z}\phi_{nor}(w)dw,\ \bar{\Phi}_{nor}(z) = 1-\Phi_{nor}(z),$$
$$\phi_{nor}(z) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{z^2}{2}\right), -\infty<z<\infty.$$

Then, the log-transformed component lifetimes follow a normal distribution with location parameter μ and scale parameter σ with CDF, SF and PDF

$$
\begin{aligned}
F_U(u;\mu,\sigma) &= \Phi_{nor}\left(\frac{u-\mu}{\sigma}\right), \\
\bar{F}_U(u;\mu,\sigma) &= 1-\Phi_{nor}\left(\frac{u-\mu}{\sigma}\right) = \bar{\Phi}_{nor}\left(\frac{u-\mu}{\sigma}\right), \\
\text{and } f_U(u;\mu,\sigma) &= \frac{1}{\sigma}\phi_{nor}\left(\frac{u-\mu}{\sigma}\right), \quad -\infty < u < \infty,
\end{aligned}
$$

respectively. The first and second derivatives of the PDF and CDF of standard normal distribution are presented in Appendix B. Then, the second derivatives in Eqs. (2.21) – (2.26) can be obtained as

$$
\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \beta_0^2} = \frac{1}{\sigma^2}\left\{\sum_{k=1}^{K}\sum_{\ell=1}^{r_k}\left[(n-1)B(z_{\ell:m_k})-1\right]+\sum_{k=1}^{K} n(m_k-r_k)B(z_{r_k:m_k})\right\}, \tag{2.27}
$$

$$
\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \beta_1^2} = \frac{1}{\sigma^2}\left\{\sum_{k=1}^{K}\sum_{\ell=1}^{r_k} y_k^2\left[(n-1)B(z_{\ell:m_k})-1\right]+\sum_{k=1}^{K} n(m_k-r_k)y_k^2 B(z_{r_k:m_k})\right\}, \tag{2.28}
$$

$$
\begin{aligned}
\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \sigma^2} = \frac{1}{\sigma^2}\Bigg\{ & \sum_{k=1}^{K}\sum_{\ell=1}^{r_k}\left[(n-1)z_{\ell:m_k}^2 B(z_{\ell:m_k}) - 2z_{\ell:m_k}^2 - (n-1)z_{\ell:m_k}\frac{\phi_{nor}(z_{\ell:m_k})}{\bar{\Phi}_{nor}(z_{\ell:m_k})}\right] \\
& + \sum_{k=1}^{K} n(m_k-r_k)z_{r_k:m_k}\left[z_{r_k:m_k}B(z_{r_k:m_k}) - \frac{\phi_{nor}(z_{r_k:m_k})}{\bar{\Phi}_{nor}(z_{r_k:m_k})}\right]\Bigg\},
\end{aligned} \tag{2.29}
$$

$$
\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \beta_0 \partial \beta_1} = \frac{1}{\sigma^2}\left\{\sum_{k=1}^{K}\sum_{\ell=1}^{r_k} y_k\left[(n-1)B(z_{\ell:m_k})-1\right]+\sum_{k=1}^{K} n y_k(m_k-r_k)B(z_{r_k:m_k})\right\}, \tag{2.30}
$$

$$
\begin{aligned}
\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \beta_0 \partial \sigma} = \frac{1}{\sigma^2}\Bigg\{ & \sum_{k=1}^{K}\sum_{\ell=1}^{r_k} z_{\ell:m_k}\left[(n-1)B(z_{\ell:m_k})-1\right] \\
& + \sum_{k=1}^{K} n z_{r_k:m_k}(m_k-r_k)B(z_{r_k:m_k})\Bigg\},
\end{aligned} \tag{2.31}
$$

$$
\begin{aligned}
\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \beta_1 \partial \sigma} = \frac{1}{\sigma^2}\Bigg\{ & \sum_{k=1}^{K}\sum_{\ell=1}^{r_k} y_k z_{\ell:m_k}\left[(n-1)B(z_{\ell:m_k})-1\right] \\
& + \sum_{k=1}^{K} n(m_k-r_k)y_k z_{r_k:m_k} B(z_{r_k:m_k})\Bigg\},
\end{aligned} \tag{2.32}
$$

where $B(z) = \phi_{nor}(z)\left[z\bar{\Phi}_{nor}(z) - \phi_{nor}(z)\right] / \left[\bar{\Phi}_{nor}(z)\right]^2$. In order to compute the expected Fisher information matrix, we need to compute the expectations $E(Z^2_{\ell:m_k})$, $E(Z_{\ell:m_k} f^*(Z_{\ell:m_k})/\bar{\Phi}_{nor}(Z_{\ell:m_k}))$, $E(B(Z_{\ell:m_k}))$, $E(Z_{\ell:m_k} B(Z_{\ell:m_k}))$, and $E(Z^2_{\ell:m_k} B(Z_{\ell:m_k}))$, $\ell = 1, ..., r_k$, $k = 1, ..., K$.

Since these expectations cannot be obtained analytically, we consider computing these expectations by using numerical integration or Monte Carlo integration. For instance, since the PDF of the ℓ-th order statistic from a sample of size m_k, $z_{\ell:m_k}$, $\ell = 1, \ldots, r_k$, $k = 1, \ldots, K$ is given by

$$f_{\ell:m_k}(z) = \frac{m_k!}{(\ell-1)!\,(m_k-\ell)!}\frac{n}{\sigma}\phi_{nor}(z)\left[\Phi_{nor}(z)\right]^{n(\ell-1)}\left[\bar{\Phi}_{nor}(z)\right]^{m_k-\ell+n-1}, \quad -\infty < z < \infty, \tag{2.33}$$

the expectation of a function of $Z_{\ell:m_k}$ (denoted as $g(Z_{\ell:m_k})$), $E\left[g(Z_{\ell:m_k})\right]$ can be approximated by Monte Carlo integration method as

$$E\left[g(Z_{\ell:m_k})\right] = \int_{-\infty}^{\infty} g(Z_{\ell:m_k}) f_{\ell:m_k}(z)dz \approx \frac{1}{S}\sum_{i=1}^{S} g(z_s), \tag{2.34}$$

where z_s is a random variate generated from the PDF in Eq. (2.33). By the law of large numbers, when $S \to \infty$, $\frac{1}{S}\sum\limits_{i=1}^{S} g(z_s)$ converges almost surely to $E\left[g(Z_{\ell:m_k})\right]$.

2.3.2. Weibull distributed components

Suppose the lifetimes of the components follow a two-parameter Weibull distribution with CDF and PDF

$$
\begin{aligned}
F_{WE}(x;\eta,\beta) &= 1-\exp\left[-\left(\frac{x}{\eta}\right)^{\beta}\right], x>0,\\
f_{WE}(x;\eta,\beta) &= \frac{\beta}{\eta}\left(\frac{x}{\eta}\right)^{\beta-1}\exp\left[-\left(\frac{x}{\eta}\right)^{\beta}\right], x>0,
\end{aligned}
$$

respectively, where $\eta > 0$ is the scale parameter and $\beta > 0$ is the shape parameter. Then, the log-transformed component lifetimes follow a SEV distribution (also known as Gumbel Type I distribution) with location parameter μ and scale parameter σ, i.e.,

$$
\begin{aligned}
F_U(u;\mu,\sigma) &= \Phi_{sev}\left(\frac{u-\mu}{\sigma}\right), -\infty<u<\infty,\\
f_U(u;\mu,\sigma) &= \frac{1}{\sigma}\phi_{sev}\left(\frac{u-\mu}{\sigma}\right), -\infty<u<\infty,
\end{aligned}
$$

where $\sigma = 1/\beta$, $\mu = \ln(\eta)$, and $\Phi_{sev}(z) = 1-\exp[-\exp(z)]$ and $\phi_{sev}(z) = \exp[z-\exp(z)]$ are the CDF and PDF for the standard SEV distribution. The first and second derivatives of the PDF and CDF of standard SEV distribution are presented in Appendix C. Then, the second derivatives in Eqs. (2.21) – (2.26) can be obtained as

$$
\frac{\partial^2 l(\boldsymbol{\theta})}{\partial\beta_0^2} = -\frac{n}{\sigma^2}\Big\{\sum_{k=1}^{K}\sum_{\ell=1}^{r_k}[\exp(z_{\ell:m_k})]+\sum_{k=1}^{K}(m_k-r_k)[\exp(z_{r_k:m_k})]\Big\}, \tag{2.35}
$$

$$
\frac{\partial^2 l(\boldsymbol{\theta})}{\partial\beta_1^2} = -\frac{1}{\sigma^2}\Big\{\sum_{k=1}^{K}\sum_{\ell=1}^{r_k}[ny_l^2\exp(z_{\ell:m_k})]+\sum_{k=1}^{K}ny_l^2(m_k-r_k)[\exp(z_{r_k:m_k})]\Big\}, \tag{2.36}
$$

$$\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \sigma^2} = -\frac{1}{\sigma^2}\left\{\sum_{k=1}^{K}\sum_{\ell=1}^{r_k}[n z_{\ell:m_k}\exp(z_{\ell:m_k})(z_{\ell:m_k}+1) - z_{\ell:m_k}] + \sum_{k=1}^{K}[n z_{r_k:m_k}\exp(z_{r_k:m_k})(m_k - r_k)(z_{r_k:m_k}+1)]\right\}, \quad (2.37)$$

$$\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \beta_0 \partial \beta_1} = -\frac{n}{\sigma^2}\Big\{\sum_{k=1}^{K}\sum_{\ell=1}^{r_k}[y_l \exp(z_{\ell:m_k})] + \sum_{k=1}^{K}[y_k(m_k - r_k)\exp(z_{r_k:m_k})]\Big\}, \quad (2.38)$$

$$\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \beta_0 \partial \sigma} = -\frac{n}{\sigma^2}\Big\{\sum_{k=1}^{K}\sum_{\ell=1}^{r_k}[z_{\ell:m_k}\exp(z_{\ell:m_k})] + \sum_{k=1}^{K} z_{r_k:m_k}(m_k - r_k)[\exp(z_{r_k:m_k})]\Big\}, \quad (2.39)$$

$$\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \beta_1 \partial \sigma} = -\frac{n}{\sigma^2}\Big\{\sum_{k=1}^{K}\sum_{\ell=1}^{r_k}[y_l z_{\ell:m_k}\exp(z_{\ell:m_k})] + \sum_{k=1}^{K} y_k z_{r_k:m_k}(m_k - r_k)[\exp(z_{r_k:m_k})]\Big\}. \quad (2.40)$$

In order to compute the expected Fisher information matrix, we need to compute the expectations $E(\exp(Z_{\ell:m_k}))$ and $E(Z^d_{\ell:m_k}\exp(Z_{\ell:m_k}))$, $d = 0, 1, 2$. Consider the PDF of the ℓ-th order statistic of a sample of size m_k, $\ell = 1, \ldots, r_k$, $k = 1, \ldots, K$ of the minimum among n random variables from the standard SEV distribution is given by

$$\begin{aligned} f_{\ell:m_k}(z) &= \frac{m_k!}{(\ell-1)!\,(m_k-\ell)!}[\Phi_{sev}(z+\ln n)]^{\ell-1}[1-\Phi_{sev}(z+\ln n)]^{m_k-\ell}\phi_{sev}(z+\ln n) \\ &= \frac{m_k!}{(\ell-1)!\,(m_k-\ell)!}[1-e^{-ne^z}]^{\ell-1}[e^{-ne^z}]^{m_k-\ell}(ne^{z-ne^z}). \end{aligned}$$

Then, the expectations $E(\exp(Z_{\ell:m_k}))$ and $E(Z^d_{\ell:m_k}\exp(Z_{\ell:m_k}))$ can be expressed as

$$E(Z_{\ell:m_k}) = \frac{n(m_k!)}{(\ell-1)!\,(m_k-\ell)!}\sum_{p=0}^{m_k-\ell}(-1)^p\binom{m_k-\ell}{p}\int_{-\infty}^{\infty} z e^{z-n(p+\ell)e^z}\,dz$$

and

$$E[Z^d_{\ell:m_k}\exp(Z_{\ell:m_k})] = \frac{n(m_k!)}{(\ell-1)!\,(m_k-\ell)!}\sum_{p=0}^{m_k-\ell}(-1)^p\binom{m_k-\ell}{p}\int_{-\infty}^{\infty} z^d e^{2z-n(p+\ell)e^z}\,dz,$$

respectively, which can be computed by evaluating the integrals

$$h(c) = \int_{-\infty}^{\infty} z e^{z-ce^z} dz \text{ and } g_d(c) = \int_{-\infty}^{\infty} z^d e^{2z-ce^z} dz,$$

for $d = 0, 1, 2$ as

$$h(c) = -\frac{\gamma + \log c}{c}$$

and

$$\begin{aligned}
g_0(c) &= \frac{1}{c^2}, \\
g_1(c) &= \frac{1 - \gamma - \log c}{c^2}, \\
g_2(c) &= \frac{\log c^2 - 2(1 - \gamma \log c + \gamma^2 - 2\gamma + \pi^2/6)}{c^2},
\end{aligned}$$

where $\gamma = -\Gamma'(1) = 0.57721566...$ is Euler's constant.

Note that the results for life testing experiment based on n-component series system with Weibull distributed components can be obtained from the results for life testing experiment based on individual components due to the fact that the first order statistic of n i.i.d. Weibull random variables is also Weibull distributed. Specifically, if the lifetimes of components in an n-component series system are Weibull distributed with scale parameter η and shape parameter β, then the lifetime of the n-component series system follows a Weibull distribution with scale parameter $n^{-1/\beta}\eta$ and shape parameter β. It is noteworthy that although the results for life testing experiment based on n-component series system with Weibull distributed components can be obtained from the results for life testing experiment based on individual components, comparing optimal allocations with series systems with different number of components is one of the main focuses in this paper.

2.4. Numerical Illustrations for Series Systems

In this section, we illustrate how the methodologies developed in Sections 2.2 and 2.3 can be used to plan a multi-level constant-stress experiment with Type-II censoring when series systems can be used in the experiment in addition to using individual components. We compare the experimental schemes based on series systems with different number of components by fixing the number of failed components in the experiment and demonstrate the possible advantages of using systems instead of individual components in a life testing experiment.

We consider that there are $N = 40$ and 60 components available for a multi-level constant-stress experiment with $K = 2$ or 4 stress levels. We can perform the experiment with individual components (i.e., $n = 1$) or with n-component series systems with $n \geq 2$. For an n-component series system, we set the possible values of n to be 1, 2, and 4 for $N = 40$ and n to be 1, 2, and 3 for $N = 60$. To provide a fair ground for comparison, we consider fixing the number of failed components R in obtaining the optimal experimental schemes. That is, for a fixed value of R, we obtain the optimal experimental schemes based on different optimality criteria with $n = 1$, 2 and 4. On the other hand, in this study, for a fixed value of n, we can also compare the results with different R. For illustrative purposes, we consider the true values of the model parameters as $\beta_0 = 0$, $\beta_1 = 1$ and $\sigma = 1$.

2.4.1. Two-stress-level case

For the case of $K = 2$ stress levels, we set the stress levels to be $y = (-0.5, 0.5)$. Respectively, we assign (m_1, m_2) systems to the first and second stress levels, and plan to fail (r_1, r_2) systems of those (m_1, m_2) systems. In Tables 2.1 and 2.2, the optimal allocations for $N = 40$ with respect to the five optimality criteria described in Section 2.2.3 and under different settings **[M1]**, **[M2]** and **[M3]** described in Section 2.2.4 when the components are assumed to follow a lognormal and Weibull distributions, respectively. In addition to $N = 40$, we also explore the optimal allocations for a larger N ($N = 60$) with larger numbers of observed failures $R = 17(1)20$. In Table 2.3 and Table 2.4, we present the numerical results of optimal allocations for $N = 60$ with total number of failures $R = 17(1)20$ for $n = 1$, 2 and 3 for different optimality criteria under different settings **[M1]**, **[M2]** and **[M3]**:

[M1] We set $m_1 = m_2 = N/(2n)$ and $r_1 = r_2 = R/2$ and then obtain the optimal experimental plans under different optimality criteria.

[M2] We set $m_1 = m_2 = N/(2n)$, and then determine the values of (r_1, r_2) for the optimal experimental plans under different optimality criteria.

[M3] With no specific constraints on (m_1, m_2) and (r_1, r_2), we determine the values of (m_1, m_2) and (r_1, r_2) for the optimal experimental plans under different optimality criteria.

Table 2.1: Optimal allocations for n-component series systems with lognormal distributed components when $N = 40$, $K = 2$, $(y_1, y_2) = (-0.5, 0.5)$ by fixing $R = 7, 8, 9$ and 10 with $n = 1, 2$ and 4

R	n	Setting	(m_1, m_2)	(r_1, r_2)	**[C1]** $\|\mathbf{I}\|$	**[C2]** $\text{tr}(\mathbf{I}^{-1})$	**[C3]** $var(\hat{\beta}_1)$	**[C4]** $E(TTE)$	**[C5]** $E(TTT)$
10	1	**[M1]**	(20, 20)	(5, 5)	**2142.29**	**0.3306**	**0.1717**	**6.06**	**152.68**
		[M2]	(20, 20)	(5, 5)	**2142.29**	**0.3306**	**0.1717**	6.06	152.68
			(20, 20)	(2, 8)	1602.47	0.4277	0.2506	**3.41**	120.83
			(20, 20)	(1, 9)	1048.50	0.5895	0.3797	3.87	**105.80**
		[M3]	(20, 20)	(5, 5)	**2142.29**	**0.3306**	**0.1717**	6.06	152.68
			(12 28)	(1, 9)	1007.19	0.5989	0.3750	**2.78**	101.77
			(2, 38)	(1, 9)	503.55	1.0127	0.7368	9.65	**94.72**
	2	**[M1]**	(10, 10)	(5, 5)	**2136.00**	**0.3287**	**0.1763**	**6.71**	**151.65**
		[M2]	(10, 10)	(5, 5)	**2136.00**	**0.3287**	**0.1763**	6.71	151.65
			(10, 10)	(3, 7)	1939.33	0.3629	0.2059	**4.33**	131.97
			(10, 10)	(1, 9)	1099.03	0.5748	0.3806	6.20	**106.48**
		[M3]	(10, 10)	(5, 5)	**2136.00**	**0.3287**	**0.1763**	6.71	151.65
			(5, 15)	(1, 9)	991.51	0.6061	0.3887	**3.07**	100.07
			(1, 19)	(1, 9)	513.89	1.0071	0.7362	9.65	**94.12**
	4	**[M1]**	(5, 5)	(5, 5)	**2134.91**	**0.3273**	**0.1902**	**10.50**	**150.32**
		[M2]	(5, 5)	(5, 5)	**2134.91**	**0.3273**	**0.1902**	**10.50**	**150.32**
		[M3]	(5, 5)	(5, 5)	**2134.91**	**0.3273**	**0.1902**	10.50	150.32
			(1, 9)	(1, 9)	768.80	0.7498	0.5282	**5.50**	**94.75**
9	1	**[M1]**	No optimal allocation can be obtained due to r_1 cannot be equal to r_2						
		[M2]	(20, 20)	(5, 4)	**1705.27**	**0.3666**	**0.1841**	6.06	147.09
			(20, 20)	(4, 5)	**1705.27**	**0.3666**	**0.1841**	5.11	137.48
			(20, 20)	(2, 7)	1344.75	0.4497	0.2510	**3.22**	115.36
			(20, 20)	(1, 8)	887.40	0.6134	0.3811	3.41	**100.26**
		[M3]	(20, 20)	(5, 4)	**1705.27**	0.3666	0.1841	6.06	147.09
			(20, 20)	(4, 5)	**1705.27**	0.3666	0.1841	5.11	137.48
			(21, 19)	(5, 4)	1703.89	**0.3664**	**0.1837**	5.84	148.28
			(19, 21)	(4, 5)	1703.89	**0.3664**	**0.1837**	5.30	136.38
			(13, 27)	(1, 8)	864.18	0.6168	0.3722	**2.66**	96.91
			(2, 38)	(1, 8)	424.98	1.0372	0.7465	9.65	**89.19**
	2	**[M1]**	No optimal allocation can be obtained due to r_1 cannot be equal to r_2						
		[M2]	(10, 10)	(5, 4)	**1702.64**	**0.3639**	**0.1881**	6.71	146.11
			(10, 10)	(4, 5)	**1702.64**	**0.3639**	**0.1881**	5.45	136.59
			(10, 10)	(2, 7)	1368.42	0.4434	0.2552	**3.71**	114.99
			(10, 10)	(1, 8)	920.18	0.6004	0.3829	4.66	**100.33**
		[M3]	(10, 10)	(5, 4)	**1702.64**	**0.3639**	0.1881	6.71	146.11
			(10, 10)	(4, 5)	**1702.64**	**0.3639**	0.1881	5.45	136.59
			(11, 9)	(5, 4)	1694.38	0.3642	**0.1881**	6.14	148.44
			(9, 11)	(4, 5)	1694.38	0.3642	**0.1881**	5.95	134.38
			(6, 14)	(1, 8)	871.34	0.6137	0.3778	**2.89**	95.89
			(1, 19)	(1, 8)	432.19	1.0314	0.7455	9.65	**88.65**
	4	**[M1]**	No optimal allocation can be obtained due to r_1 cannot be equal to r_2						
		[M2]	(5, 5)	(5, 4)	**1707.86**	**0.3594**	**0.1998**	10.50	144.68
			(5, 5)	(4, 5)	**1707.86**	**0.3594**	**0.1998**	**6.67**	**135.00**
		[M3]	(5, 5)	(5, 4)	**1707.86**	**0.3594**	**0.1998**	10.50	144.68
			(5, 5)	(4, 5)	**1707.86**	**0.3594**	**0.1998**	6.67	135.00
			(2, 8)	(1, 8)	837.09	0.6366	0.4154	**4.48**	92.63
			(1, 9)	(1, 8)	633.68	0.7715	0.5305	5.50	**89.00**
8	1	**[M1]**	(20, 20)	(4, 4)	**1340.87**	**0.4063**	**0.1943**	**5.11**	**131.89**
		[M2]	(20, 20)	(4, 4)	**1340.87**	**0.4063**	**0.1943**	5.11	131.89
			(20, 20)	(1, 7)	735.84	0.6435	0.3822	**2.98**	**94.79**
		[M3]	(20, 20)	(4, 4)	**1340.87**	**0.4063**	**0.1943**	5.11	131.89
			(15, 25)	(1, 7)	730.68	0.6407	0.3702	**2.47**	92.52
			(2, 38)	(1, 7)	351.21	1.0695	0.7600	9.65	**83.57**
	2	**[M1]**	(10, 10)	(4, 4)	**1338.39**	**0.4031**	**0.1978**	**5.45**	**131.06**
		[M2]	(10, 10)	(4, 4)	**1338.39**	**0.4031**	**0.1978**	5.45	131.06
			(10, 10)	(2, 6)	1114.86	0.4724	0.2557	**3.27**	109.45
			(10, 10)	(1, 7)	756.39	0.6317	0.3847	3.71	**94.59**
		[M3]	(10, 10)	(4, 4)	**1338.39**	**0.4031**	**0.1978**	5.45	131.06
			(7, 13)	(1, 7)	738.80	0.6345	0.3733	**2.70**	91.60
			(1, 19)	(1, 7)	355.99	1.0635	0.7585	9.65	**83.11**
	4	**[M1]**	(5, 5)	(4, 4)	**1336.68**	**0.3970**	**0.2072**	6.67	129.36
		[M2]	(5, 5)	(4, 4)	**1336.68**	**0.3970**	**0.2072**	6.67	129.36
			(5, 5)	(3, 5)	1303.88	0.4096	0.2206	**4.77**	**119.75**
		[M3]	(5, 5)	(4, 4)	**1336.68**	**0.3970**	**0.2072**	6.67	129.36
			(2, 8)	(1, 7)	675.07	0.6659	0.4160	**3.50**	86.92
			(1, 9)	(1, 7)	514.53	0.8000	0.5349	5.50	**83.53**
7	1	**[M1]**	No optimal allocation can be obtained due to r_1 cannot be equal to r_2						
		[M2]	(20, 20)	(4, 3)	**995.35**	**0.4671**	**0.2143**	5.11	126.06
			(20, 20)	(3, 4)	**995.35**	**0.4671**	**0.2143**	4.18	116.04
			(20, 20)	(1, 6)	594.28	0.6826	0.3833	**2.60**	**89.34**
		[M3]	(20, 20)	(4, 3)	**995.35**	0.4671	0.2143	5.11	126.06
			(20, 20)	(3, 4)	**995.35**	0.4671	0.2143	4.18	116.04
			(21, 19)	(4, 3)	994.78	**0.4666**	**0.2136**	4.94	127.21
			(19, 21)	(3, 4)	994.78	**0.4666**	**0.2136**	4.32	115.04
			(17, 23)	(1, 6)	595.53	0.6781	0.3728	**2.32**	88.03
			(2, 38)	(1, 6)	282.53	1.1133	0.7792	9.65	**77.81**
	2	**[M1]**	No optimal allocation can be obtained due to r_1 cannot be equal to r_2						
		[M2]	(10, 10)	(4, 3)	**994.79**	0.4632	0.2174	5.45	125.31
			(10, 10)	(3, 4)	**994.79**	**0.4632**	**0.2174**	4.33	115.42
			(10, 10)	(1, 6)	606.40	0.6720	0.3861	**3.01**	**89.06**
		[M3]	(10, 10)	(4, 3)	**994.79**	0.4632	0.2174	5.45	125.31
			(10, 10)	(3, 4)	**994.79**	0.4632	0.2174	4.33	115.42
			(11, 9)	(4, 3)	990.73	**0.4631**	**0.2168**	5.06	127.59
			(9, 11)	(3, 4)	990.73	**0.4631**	**0.2168**	4.67	113.39
			(8, 12)	(1, 6)	602.98	0.6697	0.3739	**2.49**	87.19
			(1, 19)	(1, 6)	285.48	1.1072	0.7774	9.65	**77.43**
	4	**[M1]**	No optimal allocation can be obtained due to r_1 cannot be equal to r_2						
		[M2]	(5, 5)	(4, 3)	**996.67**	**0.4549**	**0.2253**	6.67	123.75
			(5, 5)	(3, 4)	**996.67**	**0.4549**	**0.2253**	4.77	114.11
			(5, 5)	(2, 5)	906.86	0.4971	0.2659	**3.86**	**103.24**
		[M3]	(5, 5)	(4, 3)	**996.67**	**0.4549**	**0.2253**	6.67	123.75
			(5, 5)	(3, 4)	**996.67**	**0.4549**	**0.2253**	4.77	114.11
			(3, 7)	(1, 6)	600.60	0.6631	0.3841	**2.88**	84.25
			(1, 9)	(1, 6)	407.96	0.8388	0.5424	5.50	**78.04**

Table 2.2: Optimal allocations for n-component series systems with Weibull distributed components when $N = 40$, $K = 2$, $(y_1, y_2) = (-0.5, 0.5)$ by fixing $R = 7, 8, 9$ and 10 with $n = 1, 2$ and 4

					[C1]	[C2]	[C3]	[C4]	[C5]
R	n	Setting	(m_1, m_2)	(r_1, r_2)	$\|\mathbf{I}\|$	$\text{tr}(\mathbf{I}^{-1})$	$var(\hat{\beta}_1)$	$E(TTE)$	$E(TTT)$
10	1	[M1]	(20, 20)	(5, 5)	**309.61**	**0.7040**	**0.4000**	**3.41**	**83.32**
		[M2]	(20, 20)	(5, 5)	**309.61**	**0.7040**	**0.4000**	3.41	83.32
			(20, 20)	(3, 7)	262.50	0.8649	0.5296	**1.93**	67.92
			(20, 20)	(1, 9)	114.76	1.9650	1.4079	2.59	**52.52**
		[M3]	(35, 5)	(5, 5)	**357.12**	0.9561	0.7519	10.23	83.32
			(5, 35)	(5, 5)	**357.12**	0.9561	0.7519	27.82	83.32
			(20, 20)	(5, 5)	309.61	**0.7040**	**0.4000**	3.41	83.32
			(9, 31)	(1, 9)	110.41	1.6760	1.1469	**1.51**	52.52
			(3, 37)	(1, 9)	109.29	1.5838	1.1433	4.06	**52.52**
	2	[M1]	(10, 10)	(5, 5)	**328.89**	**0.6801**	**0.4000**	**3.93**	**83.32**
		[M2]	(10, 10)	(5, 5)	**328.89**	**0.6801**	**0.4000**	3.93	83.32
			(10, 10)	(3, 7)	285.99	0.8400	0.5358	**2.46**	67.92
			(10, 10)	(1, 9)	136.75	1.9027	1.4093	4.32	**52.52**
		[M3]	(15, 5)	(5, 5)	**363.29**	0.7558	0.5255	5.12	83.32
			(5, 15)	(5, 5)	**363.29**	0.7558	0.5255	13.91	83.32
			(10, 10)	(5, 5)	328.89	**0.6801**	**0.4000**	3.93	83.32
			(4, 16)	(1, 9)	118.49	1.6320	1.1381	**1.77**	52.52
			(2, 18)	(1, 9)	116.34	1.5610	1.1154	3.05	**52.52**
	4	[M1]	(5, 5)	(5, 5)	**411.23**	**0.6172**	**0.4000**	**6.95**	**83.32**
		[M2]	(5, 5)	(5, 5)	**411.23**	**0.6172**	**0.4000**	**6.95**	**83.32**
		[M3]	(5, 5)	(5, 5)	**411.23**	**0.6172**	**0.4000**	6.95	83.32
			(1, 9)	(1, 9)	148.04	1.5061	1.1111	**3.17**	**52.52**
9	1	[M1]	No optimal allocation can be obtained due to r_1 cannot be equal to r_2						
		[M2]	(20, 20)	(5, 4)	**224.88**	**0.8158**	**0.4541**	3.41	78.84
			(20, 20)	(4, 5)	**224.88**	**0.8158**	**0.4541**	2.64	71.14
			(20, 20)	(2, 7)	159.37	1.2024	0.7619	**1.87**	55.74
			(20, 20)	(1, 8)	91.67	2.0236	1.4134	2.22	**48.04**
		[M3]	(35, 5)	(4, 5)	**264.02**	1.1416	0.9046	10.23	71.14
			(5, 35)	(5, 4)	**264.02**	1.1416	0.9046	27.82	78.84
			(23, 17)	(5, 4)	224.72	**0.8101**	0.4510	2.91	78.84
			(17, 23)	(4, 5)	224.72	**0.8101**	0.451	3.16	71.14
			(22, 18)	(5, 4)	224.71	0.8108	**0.4500**	3.06	78.84
			(18, 22)	(4, 5)	224.71	0.8108	**0.4500**	2.97	71.14
			(9, 31)	(1, 8)	88.81	1.7264	1.1515	**1.35**	48.04
			(4, 36)	(1, 8)	88.17	1.6393	1.1416	3.05	**48.04**
	2	[M1]	No optimal allocation can be obtained due to r_1 cannot be equal to r_2						
		[M2]	(10, 10)	(5, 4)	**236.80**	**0.7914**	**0.4546**	3.93	78.84
			(10, 10)	(4, 5)	**236.80**	**0.7914**	**0.4546**	2.92	71.14
			(10, 10)	(3, 6)	216.64	0.8968	0.5428	**2.05**	63.44
			(10, 10)	(1, 8)	104.30	1.9725	1.4201	3.20	**48.04**
		[M3]	(15, 5)	(4, 5)	**266.82**	0.9057	0.6348	5.12	71.14
			(5, 15)	(5, 4)	**266.82**	0.9057	0.6348	13.91	78.84
			(11, 9)	(5, 4)	236.28	**0.7864**	**0.4500**	3.47	78.84
			(9, 11)	(4, 5)	236.28	**0.7864**	**0.4500**	3.32	71.14
			(4, 16)	(1, 8)	94.04	1.6824	1.1431	**1.52**	48.04
			(1, 19)	(1, 8)	92.14	1.6620	1.2340	6.09	**48.04**
	4	[M1]	No optimal allocation can be obtained due to r_1 cannot be equal to r_2						
		[M2]	(5, 5)	(5, 4)	**283.13**	0.7291	0.4571	6.95	78.84
			(5, 5)	(4, 5)	**283.13**	0.7291	0.4571	3.91	71.14
		[M3]	(5, 5)	(5, 4)	**283.13**	0.7291	0.4571	6.95	78.84
			(5, 5)	(4, 5)	**283.13**	0.7291	0.4571	3.91	71.14
			(4, 6)	(4, 5)	280.02	**0.7254**	**0.4552**	6.35	71.14
			(6, 4)	(5, 4)	280.02	**0.7254**	**0.4552**	4.42	78.84
			(3, 7)	(2, 7)	203.81	0.9891	0.6594	**2.91**	55.74
			(1, 9)	(1, 8)	109.15	1.5644	1.1277	3.05	**48.04**
8	1	[M1]	(20, 20)	(4, 4)	**161.67**	**0.9319**	**0.5000**	**2.64**	**66.66**
		[M2]	(20, 20)	(4, 4)	**161.67**	**0.9319**	**0.5000**	2.64	66.66
			(20, 20)	(2, 6)	122.14	1.2692	0.7661	**1.55**	51.26
			(20, 20)	(1, 7)	71.48	2.0965	1.4183	1.87	**43.55**
		[M3]	(36, 4)	(4, 4)	**184.85**	1.3010	1.0318	9.34	66.66
			(4, 36)	(4, 4)	**184.85**	1.3010	1.0318	25.38	66.66
			(20, 20)	(4, 4)	161.67	**0.9319**	**0.5000**	2.64	66.66
			(11, 29)	(1, 7)	69.91	1.8427	1.1891	**1.21**	43.55
			(6, 34)	(1, 7)	69.44	1.7325	1.1438	2.03	**43.55**
	2	[M1]	(10, 10)	(4, 4)	**168.39**	0.9072	0.5000	2.92	66.66
		[M2]	(10, 10)	(4, 4)	**168.39**	0.9072	0.5000	2.92	66.66
			(10, 10)	(2, 6)	130.23	1.2419	0.7740	**1.89**	51.26
			(10, 10)	(1, 7)	78.64	2.0534	1.4280	2.46	**43.55**
		[M3]	(16, 4)	(4, 4)	**186.87**	1.0421	0.7298	4.67	66.66
			(4, 16)	(4, 4)	**186.87**	1.0421	0.7298	12.69	66.66
			(10, 10)	(4, 4)	168.39	**0.9072**	**0.5000**	2.92	66.66
			(5, 15)	(1, 7)	73.44	1.7957	1.1785	**1.35**	43.55
			(2, 18)	(1, 7)	72.10	1.6949	1.1681	3.05	**43.55**
	4	[M1]	(5, 5)	(4, 4)	189.82	**0.8475**	**0.5000**	**3.91**	**66.66**
		[M2]	(5, 5)	(4, 4)	189.82	**0.8475**	**0.5000**	3.91	66.66
			(5, 5)	(3, 5)	187.68	0.9131	0.5672	**2.56**	**58.96**
		[M3]	(6, 4)	(4, 4)	**195.72**	0.8562	0.5244	2.89	66.66
			(4, 6)	(4, 4)	**195.72**	0.8562	0.5244	6.35	66.66
			(5, 5)	(4, 4)	189.82	**0.8475**	**0.5000**	3.91	66.66
			(2, 8)	(1, 7)	84.73	1.6960	1.1620	**1.92**	43.55
			(1, 9)	(1, 7)	81.00	1.6417	1.1550	3.05	**43.55**

Table 2.2: (Continue)

R	n	Setting	(m_1, m_2)	(r_1, r_2)	**[C1]** $\lvert\mathbf{I}\rvert$	**[C2]** $\mathrm{tr}(\mathbf{I}^{-1})$	**[C3]** $var(\hat{\beta}_1)$	**[C4]** $E(TTE)$	**[C5]** $E(TTT)$
7	1	**[M1]**	No optimal allocation can be obtained due to r_1 cannot be equal to r_2						
		[M2]	(20,20)	(4,3)	**107.82**	**1.1266**	**0.5911**	2.64	62.18
			(20,20)	(3,4)	**107.82**	**1.1266**	**0.5911**	1.93	54.47
			(20,20)	(2,5)	90.08	1.3629	0.7755	**1.25**	46.77
			(20,20)	(1,6)	54.04	2.1907	1.4228	1.55	**39.07**
		[M3]	(36,4)	(3,4)	**125.70**	1.6245	1.2979	9.34	54.47
			(4,36)	(4,3)	**125.70**	1.6245	1.2979	25.38	62.18
			(24,16)	(4,3)	107.72	**1.1147**	0.5860	2.17	62.18
			(16,24)	(3,4)	107.72	**1.1147**	0.5860	2.44	54.47
			(23,17)	(4,3)	107.72	1.1156	**0.5836**	2.27	62.18
			(17,23)	(3,4)	107.72	1.1156	**0.5836**	2.29	54.47
			(12,28)	(1,6)	53.19	1.9591	1.2144	**1.06**	39.07
			(1,39)	(1,6)	52.64	2.1646	1.7083	12.18	**39.07**
	2	**[M1]**	No optimal allocation can be obtained due to r_1 cannot be equal to r_2						
		[M2]	(10,10)	(4,3)	**111.49**	**1.1018**	**0.5918**	2.92	62.18
			(10,10)	(3,4)	**111.49**	**1.1018**	**0.5918**	2.05	54.47
			(10,10)	(2,5)	94.24	1.3370	0.7814	**1.45**	46.77
			(10,10)	(1,6)	57.91	2.1538	1.4339	1.89	**39.07**
		[M3]	(16,4)	(3,4)	**126.42**	1.311	0.9282	4.67	54.47
			(4,16)	(4,3)	**126.42**	1.311	0.9282	12.69	62.18
			(12,8)	(4,3)	111.24	**1.0901**	0.5860	2.35	62.18
			(8,12)	(3,4)	111.24	**1.0901**	0.5860	2.65	54.47
			(11,9)	(4,3)	111.27	1.0932	**0.5838**	2.6	62.18
			(9,11)	(3,4)	111.27	1.0932	**0.5838**	2.31	54.47
			(6,14)	(1,6)	55.49	1.9362	1.2206	**1.20**	39.07
			(1,19)	(1,6)	54.21	1.8992	1.3903	6.09	**39.07**
	4	**[M1]**	No optimal allocation can be obtained due to r_1 cannot be equal to r_2						
		[M2]	(5,5)	(4,3)	**122.63**	**1.0428**	**0.5942**	3.91	62.18
			(5,5)	(3,4)	**122.63**	**1.0428**	**0.5942**	**2.39**	54.47
			(5,5)	(2,5)	111.41	1.2631	0.7993	2.56	**46.77**
		[M3]	(6,4)	(3,4)	**129.22**	1.0706	0.6518	2.33	54.47
			(4,6)	(4,3)	**129.22**	1.0706	0.6518	6.35	62.18
			(6,4)	(4,3)	121.48	**1.0322**	**0.5860**	2.89	62.18
			(4,6)	(3,4)	121.48	**1.0322**	**0.5860**	3.3	54.47
			(2,8)	(1,6)	60.73	1.7926	1.1741	**1.52**	39.07
			(3,7)	(1,6)	63.46	1.8747	1.2368	1.78	**39.07**

Table 2.3: Optimal allocations for n-component series systems with lognormal distributed components when $N = 60$, $K = 2$, $(y_1, y_2) = (-0.5, 0.5)$ by fixing $R = 17(1)20$ with $n = 1, 2$ and 3

R	n	Setting	(m_1, m_2)	(r_1, r_2)	[C1] $\lvert\mathbf{I}\rvert$	[C2] tr($\mathbf{I}^{-1}$)	[C3] $var(\hat{\beta}_1)$	[C4] $E(TTE)$	[C5] $E(TTT)$
20	1	**[M1]**	(30,30)	(10,10)	**13004.44**	**0.1724**	**0.0982**	**7.81**	**279.78**
		[M2]	(30,30)	(10,10)	**13004.44**	**0.1724**	**0.0982**	7.81	279.78
			(30,30)	(5,15)	11090.24	0.2044	0.1251	**4.54**	232.29
			(30,30)	(1,19)	4289.25	0.4399	0.3152	6.18	**179.57**
		[M3]	(30,30)	(10,10)	**13004.44**	**0.1724**	**0.0982**	7.81	279.78
			(9,51)	(1,19)	3484.38	0.5219	0.3704	**3.26**	168.03
			(2,58)	(1,19)	1811.17	0.9228	0.7037	9.65	**162.93**
	2	**[M1]**	(15,15)	(10,10)	**12958.68**	**0.1727**	**0.1026**	**9.60**	**278.23**
		[M2]	(15,15)	(10,10)	**12958.68**	**0.1727**	**0.1026**	9.60	278.23
			(15,15)	(7,13)	12487.40	0.1822	0.1113	**6.37**	251.12
			(15,15)	(5,15)	11498.83	0.2019	0.1291	11.28	**235.04**
		[M3]	(15,15)	(10,10)	**12958.68**	**0.1727**	0**0.1026**	9.60	278.23
			(4,26)	(1,19)	3499.47	0.5342	0.3859	**4.15**	166.77
			(1,29)	(1,19)	1882.89	0.9195	0.7041	9.65	**161.98**
	3	**[M1]-[M2]**	(10,10)	(10,10)	**12979.91**	**0.1749**	**0.1094**	**17.15**	**280.14**
		[M3]	(10,10)	(10,10)	**12979.91**	**0.1749**	**0.1094**	17.15	280.14
			(1,19)	(1,19)	2467.00	0.7580	0.5759	**7.63**	**163.62**
19	1	**[M1]**	No optimal allocation can be obtained due to r_1 cannot be equal to r_2						
		[M2]	(30,30)	(10,9)	**11690.40**	**0.1802**	**0.1009**	7.81	274.37
			(30,30)	(9,10)	**11690.40**	**0.1802**	**0.1009**	7.11	265.06
			(30,30)	(4,15)	9256.89	0.2283	0.1410	**4.41**	216.11
			(30,30)	(1,18)	3956.17	0.4466	0.3165	5.67	**173.45**
		[M3]	(30,30)	(10,9)	**11690.40**	0.1802	0.1009	7.81	274.37
			(30,30)	(9,10)	**11690.40**	0.1802	0.1009	7.11	265.06
			(31,29)	(10,9)	11684.5	**0.1802**	**0.1008**	7.58	275.57
			(29,31)	(9,10)	11684.5	**0.1802**	**0.1008**	7.32	263.86
			(10,50)	(1,18)	3323.45	0.5134	0.3593	**3.11**	163.28
			(2,58)	(1,18)	1668.95	0.9277	0.7052	9.65	**157.55**
	2	**[M1]**	No optimal allocation can be obtained due to r_1 cannot be equal to r_2						
		[M2]	(15,15)	(10,9)	**11660.14**	**0.1801**	**0.1051**	9.60	272.75
			(15,15)	(9,10)	**11660.14**	**0.1801**	**0.1051**	8.35	263.34
			(15,15)	(6,13)	10967.88	0.1952	0.1188	**5.81**	236.16
			(15,15)	(4,15)	9675.77	0.2241	0.1443	11.28	**219.09**
		[M3]	(15,15)	(10,9)	**11660.14**	**0.1801**	**0.1051**	9.60	272.75
			(15,15)	(9,10)	**11660.14**	**0.1801**	**0.1051**	8.35	263.34
			(4,26)	(1,18)	3213.6	0.539	0.3859	**3.80**	161.21
			(1,29)	(1,18)	1729.95	0.9242	0.7055	9.65	**156.54**
	3	**[M1]**	No optimal allocation can be obtained due to r_1 cannot be equal to r_2						
		[M2]	(10,10)	(10,9)	**11684.60**	**0.1814**	**0.1112**	17.15	273.94
			(10,10)	(9,10)	**11684.60**	**0.1814**	**0.1112**	**11.54**	**263.31**
		[M3]	(10,10)	(10,9)	**11684.60**	**0.1814**	**0.1112**	17.15	273.94
			(10,10)	(9,10)	**11684.60**	**0.1814**	**0.1112**	11.54	263.31
			(1,19)	(1,18)	2243.75	0.7624	0.5763	**6.83**	**157.05**
18	1	**[M1]**	(30,30)	(9,9)	**10475.36**	**0.1884**	**0.1033**	**7.11**	**259.65**
		[M2]	(30,30)	(9,9)	**10475.36**	**0.1884**	**0.1033**	7.11	259.65
			(30,30)	(4,14)	8449.40	0.2338	0.141	**4.06**	210.47
			(30,30)	(1,17)	3633.06	0.454	0.3179	5.21	**167.48**
		[M3]	(30,30)	(9,9)	**10475.36**	**0.1884**	**0.1033**	7.11	259.65
			(11,49)	(1,17)	3140.16	0.5082	0.3503	**3.00**	158.50
			(2,58)	(1,17)	1530.90	0.9334	0.7071	9.65	**152.17**
	2	**[M1]**	(15,15)	(9,9)	**10450.14**	**0.1879**	**0.1073**	8.35	257.86
		[M2]	(15,15)	(9,9)	**10450.14**	**0.1879**	**0.1073**	8.35	257.86
			(15,15)	(6,12)	9912.41	0.2008	0.1190	**5.54**	230.14
			(15,15)	(3,15)	7754.43	0.2574	0.1679	11.28	**202.07**
		[M3]	(15,15)	(9,9)	**10450.14**	**0.1879**	**0.1073**	8.35	257.86
			(4,26)	(1,17)	2938.34	0.5444	0.3859	**3.50**	155.71
			(1,29)	(1,17)	1582.40	0.9297	0.7072	9.65	**151.13**
	3	**[M1]**	(10,10)	(9,9)	**10454.59**	**0.1884**	**0.1128**	11.54	257.11
		[M2]	(10,10)	(9,9)	**10454.59**	**0.1884**	**0.1128**	11.54	257.11
			(10,10)	(8,10)	10443.25	0.1897	0.1142	**8.99**	**247.99**
		[M3]	(10,10)	(9,9)	**10454.59**	**0.1884**	**0.1128**	11.54	257.11
			(2,18)	(1,17)	2783.72	0.5885	0.4288	**5.33**	154.09
			(1,19)	(1,17)	2037.02	0.7674	0.5769	6.83	**151.13**
17	1	**[M1]**	No optimal allocation can be obtained due to r_1 cannot be equal to r_2						
		[M2]	(30,30)	(9,8)	**9290.12**	**0.1984**	**0.1067**	7.11	254.22
			(30,30)	(8,9)	**9290.12**	**0.1984**	**0.1067**	6.44	244.88
			(30,30)	(4,13)	7667.29	0.2400	0.1411	**3.91**	204.92
			(30,30)	(1,16)	3320.04	0.4623	0.3192	4.80	**161.64**
		[M3]	(30,30)	(9,8)	**9290.12**	0.1984	0.1067	7.11	254.22
			(30,30)	(8,9)	**9290.12**	0.1984	0.1067	6.44	244.88
			(31,29)	(9,8)	9285.81	**0.1984**	**0.1066**	6.91	255.43
			(29,31)	(8,9)	9285.81	**0.1984**	**0.1066**	6.61	243.69
			(12,48)	(1,16)	2940.21	0.5058	0.3429	**2.89**	153.70
			(2,58)	(1,16)	1397.15	0.9399	0.7093	9.65	**146.78**
	2	**[M1]**	No optimal allocation can be obtained due to r_1 cannot be equal to r_2						
		[M2]	(15,15)	(9,8)	**9275.69**	**0.1974**	**0.1104**	8.35	252.43
			(15,15)	(8,9)	**9275.69**	**0.1974**	**0.1104**	7.30	243.09
			(15,15)	(5,12)	8495.03	0.2187	0.1293	**4.81**	214.82
			(15,15)	(2,15)	5662.93	0.3150	0.2103	11.28	**183.07**
		[M3]	(15,15)	(9,8)	**9275.69**	**0.1974**	**0.1104**	8.35	252.43
			(15,15)	(8,9)	**9275.69**	**0.1974**	**0.1104**	7.30	243.09
			(5,25)	(1,16)	2886.46	0.5203	0.3609	**3.39**	151.67
			(1,29)	(1,16)	1440.22	0.9360	0.7092	9.65	**145.73**
	3	**[M1]**	No optimal allocation can be obtained due to r_1 cannot be equal to r_2						
		[M2]	(10,10)	(9,8)	**9286.76**	**0.1973**	**0.1154**	11.54	251.48
			(10,10)	(8,9)	**9286.76**	**0.1973**	**0.1154**	8.99	241.80
			(10,10)	(7,10)	9236.26	0.2002	0.1185	**7.34**	**233.11**
		[M3]	(10,10)	(9,8)	**9286.76**	**0.1973**	**0.1154**	11.54	251.48
			(10,10)	(8,9)	**9286.76**	**0.1973**	**0.1154**	8.99	241.80
			(2,18)	(1,16)	2513.77	0.5945	0.4289	**4.35**	148.21
			(1,19)	(1,16)	1842.66	0.7732	0.5778	6.83	**145.48**

Table 2.4: Optimal allocations for n-component series systems with Weibull distributed components when $N = 60$, $K = 2$, $(y_1, y_2) = (-0.5, 0.5)$ by fixing $R = 17(1)20$ with $n = 1, 2$ and 3

R	n	Setting	(m_1, m_2)	(r_1, r_2)	**[C1]** $\|\mathbf{I}\|$	**[C2]** $\mathrm{tr}(\mathbf{I}^{-1})$	**[C3]** $var(\hat{\beta}_1)$	**[C4]** $E(TTE)$	**[C5]** $E(TTT)$
20	1	**[M1]**	(30,30)	(10,10)	**2369.53**	**0.3316**	**0.2000**	**4.84**	**166.64**
		[M2]	(30,30)	(10,10)	**2369.53**	**0.3316**	**0.2000**	4.84	166.64
			(30,30)	(6,14)	2027.41	0.4164	0.2705	**2.75**	135.84
			(30,30)	(1,19)	490.66	1.7449	1.3427	4.37	**97.33**
		[M3]	(49,11)	(9,11)	**2815.88**	0.4432	0.3454	13.53	158.94
			(11,49)	(11,9)	**2815.88**	0.4432	0.3454	36.79	174.34
			(30,30)	(10,10)	2369.53	**0.3316**	**0.2000**	4.84	166.64
			(7,53)	(1,19)	447.68	1.4370	1.0689	**1.97**	**97.29**
	2	**[M1]**	(15,15)	(10,10)	**2648.27**	**0.3155**	**0.2000**	**6.30**	**166.64**
		[M2]	(15,15)	(10,10)	**2648.27**	**0.3155**	**0.2000**	6.30	166.64
			(15,15)	(7,13)	2526.63	0.3600	0.2406	**4.07**	143.54
			(15,15)	(5,15)	2323.56	0.4533	0.3255	7.44	**128.14**
		[M3]	(19,11)	(9,11)	**2898.93**	0.3403	0.2356	6.77	158.94
			(11,19)	(11,9)	**2898.93**	0.3403	0.2356	18.39	174.34
			(15,15)	(10,10)	2648.27	**0.3155**	**0.2000**	6.30	166.64
			(3,27)	(1,19)	505.46	1.4098	1.0648	**2.63**	**97.33**
	3	**[M1]-[M2]**	(10,10)	(10,10)	**3289.87**	**0.2943**	**0.2000**	**11.89**	**166.64**
		[M3]	(10,10)	(10,10)	**3289.87**	**0.2943**	**0.2000**	11.89	166.64
			(1,19)	(1,19)	625.07	1.3601	1.0526	**5.30**	**97.33**
19	1	**[M1]**	No optimal allocation can be obtained due to r_1 cannot be equal to r_2						
		[M2]	(30,30)	(10,9)	**2024.74**	**0.3552**	**0.2117**	4.84	162.16
			(30,30)	(9,10)	**2024.74**	**0.3552**	**0.2117**	4.26	154.46
			(30,30)	(6,13)	1779.49	0.4279	0.2717	**2.67**	131.36
			(30,30)	(1,18)	436.97	1.7657	1.3494	4.00	**92.85**
		[M3]	(49,11)	(8,11)	**2406.26**	0.4886	0.3828	13.53	146.75
			(11,49)	(11,8)	**2406.26**	0.4886	0.3828	36.79	169.86
			(28,32)	(9,10)	2024.15	**0.3546**	**0.2112**	4.62	154.46
			(32,28)	(10,9)	2024.15	**0.3546**	**0.2112**	4.48	162.16
			(7,53)	(1,18)	402.03	1.4474	1.0696	**1.84**	**92.81**
	2	**[M1]**	No optimal allocation can be obtained due to r_1 cannot be equal to r_2						
		[M2]	(15,15)	(10,9)	**2242.67**	**0.3386**	**0.2118**	6.30	162.16
			(15,15)	(9,10)	**2242.67**	**0.3386**	**0.2118**	5.29	154.46
			(15,15)	(7,12)	2154.94	0.3728	0.2425	**3.66**	139.06
			(15,15)	(4,15)	1787.42	0.5495	0.3994	7.44	**115.96**
		[M3]	(20,10)	(9,10)	**2480.03**	0.3632	0.2511	6.56	154.46
			(10,20)	(10,9)	**2480.03**	0.3632	0.2511	17.84	162.16
			(16,14)	(10,9)	2240.14	**0.3380**	**0.2112**	5.67	162.16
			(14,16)	(9,10)	2240.14	**0.3380**	**0.2112**	5.90	154.46
			(3,27)	(1,18)	448.67	1.4198	1.0656	**2.38**	**92.85**
	3	**[M1]**	No optimal allocation can be obtained due to r_1 cannot be equal to r_2						
		[M2]	(10,10)	(10,9)	**2699.88**	**0.3176**	**0.2121**	11.89	162.16
			(10,10)	(9,10)	**2699.88**	**0.3176**	**0.2121**	**7.83**	**154.46**
		[M3]	(10,10)	(10,9)	**2699.88**	0.3176	0.2121	11.89	162.16
			(10,10)	(9,10)	**2699.88**	0.3176	0.2121	7.83	**154.46**
			(11,9)	(10,9)	2692.16	**0.3173**	**0.2120**	8.20	162.16
			(9,11)	(9,10)	2692.16	**0.3173**	**0.2120**	11.49	154.46
			(1,19)	(1,18)	529.55	1.3717	1.0559	**4.06**	**92.85**
18	1	**[M1]**	(30,30)	(9,9)	**1725.21**	**0.3791**	**0.2222**	4.26	149.98
		[M2]	(30,30)	(9,9)	**1725.21**	**0.3791**	**0.2222**	4.26	149.98
			(30,30)	(5,13)	1410.74	0.4994	0.3209	**2.49**	119.17
			(30,30)	(1,17)	387.33	1.7881	1.3561	3.65	**88.37**
		[M3]	(50,10)	(8,10)	**2048.98**	0.5188	0.4067	13.13	142.27
			(10,50)	(10,8)	**2048.98**	0.5188	0.4067	35.68	157.68
			(30,30)	(9,9)	1725.21	**0.3791**	**0.2222**	4.26	149.98
			(7,53)	(1,17)	359.10	1.4594	1.0706	**1.74**	**88.33**
	2	**[M1]**	(15,15)	(9,9)	**1891.71**	**0.3620**	**0.2222**	**5.29**	**149.98**
		[M2]	(15,15)	(9,9)	**1891.71**	**0.3620**	**0.2222**	5.29	149.98
			(15,15)	(6,12)	1753.49	0.424	0.2771	**3.33**	126.88
			(15,15)	(3,15)	1288.66	0.7025	0.5181	7.44	**103.77**
		[M3]	(20,10)	(8,10)	**2101.03**	0.4003	0.2778	6.56	142.28
			(10,20)	(10,8)	**2101.03**	0.4003	0.2778	17.84	157.68
			(15,15)	(9,9)	1891.71	**0.3620**	**0.2222**	5.29	149.98
			(3,27)	(1,17)	396.50	1.4314	1.0667	**2.16**	**88.37**
	3	**[M1]**	(10,10)	(9,9)	2194.98	**0.3413**	**0.2222**	**7.83**	**149.98**
		[M2]	(10,10)	(10,8)	**2227.37**	0.3471	0.2292	11.89	157.68
			(10,10)	(8,10)	**2227.37**	0.3471	0.2292	**5.80**	**142.28**
			(10,10)	(9,9)	2194.98	**0.3413**	**0.2222**	7.83	149.98
		[M3]	(11,9)	(9,9)	**2242.05**	0.3425	0.2258	6.17	149.98
			(9,11)	(9,9)	**2242.05**	0.3425	0.2258	11.49	149.98
			(10,10)	(9,9)	2194.98	**0.3413**	**0.2222**	7.83	149.98
			(3,17)	(2,16)	884.11	0.7790	0.5678	**3.64**	96.07
			(2,18)	(1,17)	472.72	1.4135	1.0713	3.73	**88.34**
17	1	**[M1]**	No optimal allocation can be obtained due to r_1 cannot be equal to r_2						
		[M2]	(30,30)	(9,8)	**1449.08**	**0.4097**	**0.2368**	4.26	145.5
			(30,30)	(8,9)	**1449.08**	**0.4097**	**0.2368**	3.71	137.80
			(30,30)	(5,12)	1225.00	0.5142	0.3221	**2.24**	114.69
			(30,30)	(1,16)	341.46	1.8127	1.3627	3.33	**83.89**
		[M3]	(51,9)	(8,9)	**1730.98**	0.5525	0.4334	12.68	137.79
			(9,51)	(9,8)	**1730.98**	0.5525	0.4334	34.46	145.49
			(32,28)	(9,8)	1448.61	**0.4088**	**0.2361**	3.95	145.50
			(28,32)	(8,9)	1448.61	**0.4088**	**0.2361**	4.01	137.80
			(8,52)	(1,16)	319.50	1.4885	1.0798	**1.63**	**83.89**
	2	**[M1]**	No optimal allocation can be obtained due to r_1 cannot be equal to r_2						
		[M2]	(15,15)	(9,8)	**1576.50**	**0.3922**	**0.2370**	5.29	145.50
			(15,15)	(8,9)	**1576.50**	**0.3922**	**0.2370**	4.42	137.80
			(15,15)	(6,11)	1483.97	0.4404	0.2792	**2.98**	122.39
			(15,15)	(2,15)	824.99	0.9870	0.7412	7.44	**91.59**
		[M3]	(21,9)	(8,9)	**1768.76**	0.4296	0.2979	6.34	137.80
			(9,21)	(9,8)	**1768.76**	0.4296	0.2979	17.23	145.50
			(16,14)	(9,8)	1574.77	**0.3912**	**0.2361**	4.8	145.50
			(14,16)	(8,9)	1574.77	**0.3912**	**0.2361**	4.88	137.80
			(3,27)	(1,16)	348.60	1.4450	1.0683	**2.03**	**83.89**
	3	**[M1]**	No optimal allocation can be obtained due to r_1 cannot be equal to r_2						
		[M2]	(10,10)	(10,7)	**1822.73**	0.3855	0.253	11.89	153.20
			(10,10)	(7,10)	**1822.73**	0.3855	0.2530	**4.45**	**130.09**
			(10,10)	(9,8)	1795.83	**0.3713**	**0.2372**	7.83	145.50
			(10,10)	(8,9)	1795.83	**0.3713**	**0.2372**	5.8	137.80
		[M3]	(11,9)	(8,9)	**1858.76**	0.3744	0.2446	4.82	137.81
			(9,11)	(9,8)	**1858.76**	0.3744	0.2446	11.49	145.50
			(11,9)	(9,8)	1791.29	**0.3705**	**0.2367**	6.17	145.51
			(9,11)	(8,9)	1791.29	**0.3705**	**0.2367**	7.43	137.80
			(2,18)	(1,16)	402.60	1.4276	1.0721	**2.98**	**83.86**

2.4.2. Four-stress-level case

In the case of $K = 4$, we set the stress levels to be $y = (-1.0, -0.5, 0.5, 1.0)$. Similarly, we assign m_k systems to the k-th stress level y_k and fail r_k systems out of those m_k systems ($k = 1, 2, 3, 4$). The optimal experimental plans for $N = 40$ with respect to the five optimality criteria for lognormal and Weibull distributed components are presented in Table 2.5 and Table 2.6. The following settings are considered:

[M1] We set $m_k = N/(4n)$ and $r_k = R/4$, $k = 1, 2, 3, 4$ and then obtain the optimal experimental plans under different optimality criteria.

[M2] We set $m_k = N/(4n)$, $k = 1, 2, 3, 4$, and then to determine the values of (r_1, r_2, r_3, r_4) for the optimal experimental plans under different optimality criteria.

[M3] With no specific constraints on m_k and r_k for $k = 1, 2, 3, 4$, we need to determine the values of (m_1, m_2, m_3, m_4) and (r_1, r_2, r_3, r_4) for the optimal experimental plans under different optimality criteria.

Table 2.5: Optimal allocations for n-component series systems with lognormal distributed components when $N = 40$, $K = 4$, $(y_1, y_2, y_3, y_4) = (-1.0, -0.5, 0.5, 1.0)$ by fixing $R = 7, 8, 9$ and 10 with $n = 1, 2$ and 4

R	n	Setting	(m_1, m_2, m_3, m_4)	(r_1, r_2, r_3, r_4)	**[C1]** $\lvert I\mathbf{I}\rvert$	**[C2]** $\mathrm{tr}(\mathbf{I}^{-1})$	**[C3]** $var(\hat{\beta}_1)$	**[C4]** $E(TTE)$	**[C5]** $E(TTT)$
10	1	**[M1]**	No optimal allocation can be obtained due to r_k cannot be equal, $k = 1, 2, 3, 4$						
		[M2]	(10,10,10,10)	(4,1,1,4)	**6447.47**	0.1983	**0.0623**	14.93	188.65
			(10,10,10,10)	(5,0,1,4)	5434.85	**0.1970**	0.0635	19.17	183.39
			(10,10,10,10)	(0,1,3,6)	1643.18	0.4519	0.2155	**3.32**	78.51
			(10,10,10,10)	(0,0,1,9)	147.20	3.7772	2.0061	8.28	**49.15**
		[M3]	(20,0,0,20)	(5,0,0,5)	**8569.17**	0.2018	**0.0429**	9.99	208.93
			(20,4,0,16)	(2,4,0,4)	7557.39	**0.1889**	0.0593	44.45	204.87
			(16,0,4,20)	(4,0,4,2)	7557.39	**0.1889**	0.0593	16.35	190.84
			(0,0,7,33)	(0,0,1,9)	216.76	3.4099	1.6924	**1.46**	53.39
			(25,0,2,13)	(0,0,1,9)	78.63	5.9117	3.0474	4.27	**43.53**
	2	**[M1]**	No optimal allocation can be obtained due to r_k cannot be equal, $k = 1, 2, 3, 4$						
		[M2]	(5,5,5,5)	(4,1,1,4)	**6373.78**	0.1962	**0.0651**	19.18	188.99
			(5,5,5,5)	(5,0,1,4)	5351.35	**0.1954**	0.0676	33.94	187.80
			(5,5,5,5)	(0,1,4,5)	1659.10	0.4338	0.2180	**4.59**	81.12
			(5,5,5,5)	(0,0,5,5)	268.90	1.8173	1.0805	7.57	**56.99**
		[M3]	(10,0,0,10)	(5,0,0,5)	**8543.99**	0.1965	**0.0441**	11.06	207.52
			(7,0,8,5)	(4,0,1,5)	6985.74	**0.1895**	0.0601	12.46	175.84
			(5,8,0,7)	(5,1,0,4)	6985.74	**0.1895**	0.0601	33.94	216.21
			(0,0,3,17)	(0,0,1,9)	210.05	3.5750	1.7886	**1.62**	52.93
			(9,0,1,10)	(0,0,1,9)	100.38	5.8106	2.9793	3.76	**45.73**
	4	**[M1]**	No optimal allocation can be obtained due to m_k cannot be equal, $k = 1, 2, 3, 4$						
		[M2]	No optimal allocation can be obtained due to m_k cannot be equal, $k = 1, 2, 3, 4$						
		[M3]	(5,0,0,5)	(5,0,0,5)	**8539.63**	**0.1847**	**0.0475**	17.32	205.70
			(0,0,2,8)	(0,0,2,8)	341.63	2.2882	1.1885	**2.76**	55.40
			(0,0,1,9)	(0,0,1,9)	192.20	4.1563	2.1127	2.81	**52.22**

Table 2.5: (Continue)

R	n	Setting	(m_1, m_2, m_3, m_4)	(r_1, r_2, r_3, r_4)	**[C1]** $\lvert\mathbf{I}\rvert$	**[C2]** $\mathrm{tr}(\mathbf{I}^{-1})$	**[C3]** $var(\hat{\beta}_1)$	**[C4]** $E(TTE)$	**[C5]** $E(TTT)$
9	1	**[M1]**	No optimal allocation can be obtained due to r_k cannot be equal, $k = 1, 2, 3, 4$						
		[M2]	(10,10,10,10)	(4,1,1,3)	**5095.97**	0.2239	0.067	14.93	185.28
			(10,10,10,10)	(3,1,1,4)	**5095.97**	0.2239	0.0670	11.37	163.72
			(10,10,10,10)	(5,0,0,4)	3541.68	**0.2176**	**0.0661**	19.17	172.09
			(10,10,10,10)	(4,0,0,5)	3541.68	**0.2176**	**0.0661**	14.93	150.08
			(10,10,10,10)	(0,1,3,5)	1374.72	0.4692	0.2161	**3.07**	74.90
			(10,10,10,10)	(0,0,1,8)	124.45	3.8240	2.0173	5.73	**44.06**
		[M3]	(20,1,0,19)	(4,1,0,4)	**6880.30**	0.2196	0.0486	20.15	200.23
			(19,0,1,20)	(4,0,1,4)	**6880.30**	0.2196	0.0486	8.74	184.96
			(20,4,0,16)	(2,4,0,3)	6070.41	**0.2080**	0.0645	44.45	201.44
			(16,0,4,20)	(3,0,4,2)	6070.41	**0.2080**	0.0645	16.35	165.50
			(21,0,0,19)	(5,0,0,4)	6815.56	0.2286	**0.0459**	9.63	208.19
			(19,0,0,21)	(4,0,0,5)	6815.56	0.2286	**0.0459**	8.74	181.35
			(0,0,8,32)	(0,0,1,8)	191.20	3.3102	1.6316	**1.36**	50.19
			(26,0,2,12)	(0,0,1,8)	64.19	5.9587	3.0791	3.97	**39.54**
	2	**[M1]**	No optimal allocation can be obtained due to r_k cannot be equal, $k = 1, 2, 3, 4$						
		[M2]	(5,5,5,5)	(4,1,1,3)	**5060.51**	0.2209	**0.0693**	19.18	185.52
			(5,5,5,5)	(3,1,1,4)	**5060.51**	0.2209	**0.0693**	12.76	163.33
			(5,5,5,5)	(5,0,0,4)	3499.74	**0.2149**	0.0704	33.94	176.50
			(5,5,5,5)	(4,0,0,5)	3499.74	**0.2149**	0.0704	19.18	150.96
			(5,5,5,5)	(0,1,4,4)	1381.77	0.4499	0.2191	**4.28**	77.12
			(5,5,5,5)	(0,0,4,5)	218.73	1.9319	1.1256	4.59	**50.40**
		[M3]	(10,0,0,10)	(5,0,0,4)	**6810.55**	0.2228	0.0470	11.06	204.16
			(10,0,0,10)	(4,0,0,5)	**6810.55**	0.2228	0.0470	8.99	182.70
			(6,0,10,4)	(4,0,1,4)	5264.36	**0.2087**	0.0683	14.84	167.48
			(4,10,0,6)	(4,1,0,4)	5264.36	**0.2087**	0.0683	30.99	188.20
			(11,0,0,9)	(5,0,0,4)	6777.51	0.2231	**0.0470**	10.13	209.36
			(9,0,0,11)	(4,0,0,5)	6777.51	0.2231	**0.0470**	9.82	177.65
			(0,0,4,16)	(0,0,1,8)	195.11	3.3132	1.6399	**1.53**	49.88
			(10,0,1,9)	(0,0,1,8)	80.77	5.8577	3.0118	3.57	**41.49**
	4	**[M1]**	No optimal allocation can be obtained due to m_k cannot be equal, $k = 1, 2, 3, 4$						
		[M2]	No optimal allocation can be obtained due to m_k cannot be equal, $k = 1, 2, 3, 4$						
		[M3]	(5,0,0,5)	(5,0,0,4)	**6831.44**	0.2096	**0.0499**	17.32	202.28
			(5,0,0,5)	(4,0,0,5)	**6831.44**	0.2096	**0.0499**	11.00	180.44
			(5,1,0,4)	(5,0,0,4)	6148.35	**0.2065**	0.0535	17.32	200.8
			(5,0,1,4)	(5,0,0,4)	6148.35	**0.2065**	0.0535	17.32	200.8
			(4,1,0,5)	(4,0,0,5)	6148.35	**0.2065**	0.0535	16.05	169.48
			(4,0,1,5)	(4,0,0,5)	6148.35	**0.2065**	0.0535	16.05	169.48
			(0,0,1,9)	(0,0,1,8)	158.42	4.1757	2.1221	**2.02**	48.73
			(1,0,1,8)	(0,0,1,8)	153.75	4.1993	2.1391	2.72	**47.32**
8	1	**[M1]**	(10,10,10,10)	(2,2,2,2)	**3400.67**	**0.2782**	**0.0805**	**8.20**	**154.90**
		[M2]	(10,10,10,10)	(3,1,1,3)	3975.26	0.2538	0.0709	11.37	160.35
			(10,10,10,10)	(4,0,0,4)	2823.42	**0.2416**	**0.0691**	14.93	146.63
			(10,10,10,10)	(0,0,3,5)	170.21	2.0956	1.1673	**2.59**	44.18
			(10,10,10,10)	(0,0,1,7)	103.33	3.8715	2.0262	4.29	**39.73**
		[M3]	(20,0,1,19)	(4,0,1,3)	**5371.66**	0.2480	0.0528	8.43	183.99
			(19,1,0,20)	(3,1,0,4)	**5371.66**	0.2480	0.0528	20.15	171.78
			(21,3,0,16)	(2,3,0,3)	4976.53	**0.2319**	0.065	38.27	183.68
			(16,0,3,21)	(3,0,3,2)	4976.53	**0.2319**	0.065	14.08	158.42
			(20,0,0,20)	(4,0,0,4)	5363.49	0.2606	**0.0486**	8.43	180.48
			(0,0,9,31)	(0,0,1,7)	163.85	3.2471	1.5876	**1.26**	46.95
			(28,0,2,10)	(0,0,1,7)	48.93	6.0601	3.1511	4.29	**35.53**
	2	**[M1]**	(5,5,5,5)	(2,2,2,2)	**3387.11**	**0.2756**	**0.0814**	**8.55**	**154.29**
		[M2]	(5,5,5,5)	(3,1,1,3)	**3949.94**	0.2508	0.0727	12.76	159.86
			(5,5,5,5)	(4,0,0,4)	2787.44	**0.2378**	**0.0725**	19.18	146.97
			(5,5,5,5)	(0,1,3,4)	1138.36	0.4844	0.2215	**3.07**	71.40
			(5,5,5,5)	(0,0,3,5)	170.14	2.1647	1.2254	4.59	**44.67**
		[M3]	(10,0,0,10)	(4,0,0,4)	**5353.57**	0.2547	**0.0495**	8.99	179.34
			(6,0,10,4)	(3,0,1,4)	4112.80	**0.2362**	0.0741	10.68	142.52
			(4,10,0,6)	(4,1,0,3)	4112.80	**0.2362**	0.0741	30.99	184.82
			(0,0,4,16)	(0,0,1,7)	160.69	3.3352	1.6453	**1.34**	46.57
			(12,0,1,7)	(0,0,1,7)	60.24	5.9609	3.0868	5.23	**37.22**
	4	**[M1]**	No optimal allocation can be obtained due to m_k cannot be equal, $k = 1, 2, 3, 4$						
		[M2]	No optimal allocation can be obtained due to m_k cannot be equal, $k = 1, 2, 3, 4$						
		[M3]	(5,0,0,5)	(4,0,0,4)	**5346.71**	0.2415	**0.0518**	11.00	177.02
			(4,2,0,4)	(4,0,0,4)	4371.97	**0.2309**	0.0594	16.05	164.57
			(4,1,1,4)	(4,0,0,4)	4371.97	**0.2309**	0.0594	16.05	164.57
			(4,0,2,4)	(4,0,0,4)	4371.97	**0.2309**	0.0594	16.05	164.57
			(0,0,3,7)	(0,0,2,6)	263.94	2.0668	1.0571	**1.81**	49.46
			(2,0,1,7)	(0,0,1,7)	119.58	4.2542	2.1730	2.61	**42.41**
7	1	**[M1]**	No optimal allocation can be obtained due to r_k cannot be equal, $k = 1, 2, 3, 4$						
		[M2]	(10,10,10,10)	(3,1,1,2)	**2886.94**	0.3000	0.0796	11.37	156.91
			(10,10,10,10)	(4,0,0,3)	2134.70	**0.2776**	**0.0751**	14.93	143.26
			(10,10,10,10)	(0,0,3,4)	133.42	2.1428	1.2012	**2.54**	40.73
			(10,10,10,10)	(0,0,1,6)	83.73	3.9215	2.0331	3.32	**35.84**
		[M3]	(20,1,0,19)	(3,1,0,3)	**4084.21**	0.2842	0.0575	20.15	170.59
			(19,0,1,20)	(3,0,1,3)	**4084.21**	0.2842	0.0575	7.41	155.50
			(20,2,1,17)	(2,2,1,2)	3710.06	**0.2631**	0.0732	30.62	165.31
			(17,1,2,20)	(2,1,2,2)	3710.06	**0.2631**	0.0732	20.15	146.31
			(21,0,0,19)	(4,0,0,3)	3979.13	0.3065	**0.0534**	8.14	179.48
			(19,0,0,21)	(3,0,0,4)	3979.13	0.3065	**0.0534**	7.12	152.02
			(22,0,0,18)	(4,0,0,3)	3966.76	0.3065	**0.0534**	7.89	181.99
			(18,0,0,22)	(3,0,0,4)	3966.76	0.3065	**0.0534**	7.38	149.66
			(0,0,10,30)	(0,0,1,6)	135.85	3.2150	1.5570	**1.15**	43.66
			(29,0,2,9)	(0,0,1,6)	37.71	6.1407	3.2064	3.90	**31.52**
	2	**[M1]**	No optimal allocation can be obtained due to r_k cannot be equal, $k = 1, 2, 3, 4$						
		[M2]	(5,5,5,5)	(3,1,1,2)	**2877.61**	0.2966	0.0810	12.76	156.44
			(5,5,5,5)	(4,0,0,3)	2118.57	**0.2722**	**0.0780**	19.18	143.50
			(5,5,5,5)	(0,0,3,4)	132.41	2.2018	1.2476	**2.85**	40.68
			(5,5,5,5)	(0,0,2,5)	121.11	2.6524	1.4459	4.59	**39.04**
		[M3]	(10,0,0,10)	(4,0,0,3)	**3979.15**	0.3002	0.0543	8.99	175.86
			(10,0,0,10)	(3,0,0,4)	**3979.15**	0.3002	0.0543	7.15	153.56
			(5,0,12,3)	(3,0,1,3)	2894.89	**0.2673**	0.0863	12.76	134.02
			(3,12,0,5)	(3,1,0,3)	2894.89	**0.2673**	0.0863	27.4	156.54
			(11,0,0,9)	(4,0,0,3)	3962.90	0.3005	**0.0542**	8.34	180.89
			(9,0,0,11)	(3,0,0,4)	3962.90	0.3005	**0.0542**	7.70	148.86
			(0,0,5,15)	(0,0,1,6)	137.64	3.2174	1.5643	**1.24**	43.43
			(13,0,1,6)	(0,0,1,6)	45.18	6.0523	3.1511	4.93	**32.92**
	4	**[M1]**	No optimal allocation can be obtained due to m_k cannot be equal, $k = 1, 2, 3, 4$						
		[M2]	No optimal allocation can be obtained due to m_k cannot be equal, $k = 1, 2, 3, 4$						
		[M3]	(5,0,0,5)	(4,0,0,3)	**3986.69**	0.2859	**0.0563**	11.00	173.62
			(5,0,0,5)	(3,0,0,4)	**3986.69**	0.2859	**0.0563**	7.86	151.88
			(4,3,0,3)	(4,0,0,3)	2868.98	**0.2667**	0.0693	16.05	159.67
			(4,2,1,3)	(4,0,0,3)	2868.98	**0.2667**	0.0693	16.05	159.67
			(0,0,2,8)	(0,0,1,6)	133.37	3.3701	1.6705	**1.44**	42.72
			(3,0,1,6)	(0,0,1,6)	89.68	4.3268	2.2183	2.49	**37.51**

Table 2.6: Optimal allocations for n-component series systems with Weibull distributed components when $N = 40$, $K = 4$, $(y_1, y_2, y_3, y_4) = (-1.0, -0.5, 0.5, 1.0)$ by fixing $R = 7, 8, 9$ and 10 with $n = 1, 2$ and 4

					[C1]	[C2]	[C3]	[C4]	[C5]
R	n	Setting	(m_1, m_2, m_3, m_4)	(r_1, r_2, r_3, r_4)	$\|I\|$	$\text{tr}(I^{-1})$	$var(\hat{\beta}_1)$	$E(TTE)$	$E(TTT)$
10	1	[M1]	No optimal allocation can be obtained due to r_k cannot be equal, $k = 1, 2, 3, 4$						
		[M2]	(10,10,10,10)	(5,0,1,4)	**1364.58**	0.3138	0.1108	12.97	115.78
			(10,10,10,10)	(4,1,0,5)	**1364.58**	0.3138	0.1108	9.62	106.12
			(10,10,10,10)	(5,0,0,5)	1315.57	**0.2933**	**0.1000**	12.97	114.02
			(10,10,10,10)	(1,1,3,5)	630.07	0.6135	0.2656	**2.01**	59.30
			(10,10,10,10)	(0,0,1,9)	34.19	10.8714	5.6371	5.24	**28.95**
		[M3]	(4,32,0,4)	(4,2,0,4)	**2460.89**	0.2578	0.1218	41.84	115.58
			(4,0,32,4)	(4,0,2,4)	**2460.89**	0.2578	0.1218	41.84	100.18
			(5,0,31,4)	(5,0,1,4)	2460.83	**0.2479**	0.1100	45.86	115.78
			(4,31,0,5)	(4,1,0,5)	2460.83	**0.2479**	0.1100	41.84	106.12
			(15,10,0,15)	(5,0,0,5)	1261.36	0.3475	**0.1000**	7.82	114.02
			(15,9,1,15)	(5,0,0,5)	1261.36	0.3475	**0.1000**	7.82	114.02
			(0,0,6,34)	(0,0,1,9)	27.45	8.8370	4.4527	**0.82**	**28.95**
	2	[M1]	No optimal allocation can be obtained due to r_k cannot be equal, $k = 1, 2, 3, 4$						
		[M2]	(5,5,5,5)	(5,0,0,5)	**1644.93**	**0.2652**	**0.1000**	22.93	114.02
			(5,5,5,5)	(1,2,3,4)	745.70	0.5033	0.2206	**2.74**	68.77
			(5,5,5,5)	(0,0,5,5)	102.81	2.6652	1.6000	5.12	**36.00**
		[M3]	(5,0,11,4)	(5,0,1,4)	**1977.44**	0.2688	0.1094	22.93	115.78
			(4,11,0,5)	(4,1,0,5)	**1977.44**	0.2688	0.1094	20.92	106.12
			(5,10,0,5)	(5,0,0,5)	1644.93	**0.2652**	**0.1000**	22.93	114.02
			(0,0,3,17)	(0,0,1,9)	29.33	8.8470	4.4616	**0.98**	**28.95**
	4	[M1]	No optimal allocation can be obtained due to r_k cannot be equal, $k = 1, 2, 3, 4$						
		[M2]	No optimal allocation can be obtained due to m_k cannot be equal, $k = 1, 2, 3, 4$						
		[M3]	(5,0,0,5)	(5,0,0,5)	**1644.93**	**0.3172**	**0.1000**	11.47	114.02
			(0,0,2,8)	(0,0,2,8)	65.80	4.7422	2.5000	**1.85**	30.71
			(0,0,1,9)	(0,0,1,9)	37.01	8.6728	4.4444	1.92	**28.95**
9	1	[M1]	No optimal allocation can be obtained due to r_k cannot be equal, $k = 1, 2, 3, 4$						
		[M2]	(10,10,10,10)	(4,1,0,4)	**988.04**	0.3607	0.1219	9.62	103.40
			(10,10,10,10)	(4,0,1,4)	**988.04**	0.3607	0.1219	9.62	95.70
			(10,10,10,10)	(5,0,0,4)	947.19	**0.3404**	**0.1136**	12.97	111.3
			(10,10,10,10)	(4,0,0,5)	947.19	**0.3404**	**0.1136**	9.62	93.93
			(10,10,10,10)	(0,1,3,5)	220.60	1.1562	0.5731	**1.76**	39.22
			(10,10,10,10)	(0,0,1,8)	26.08	11.0138	5.6804	3.88	**26.23**
		[M3]	(4,32,0,4)	(4,1,0,4)	**1864.25**	**0.2743**	0.1230	41.84	103.40
			(4,0,32,4)	(4,0,1,4)	**1864.25**	**0.2743**	0.1230	41.84	95.70
			(5,31,0,4)	(5,0,0,4)	1184.35	0.3046	**0.1125**	45.86	111.30
			(5,30,1,4)	(5,0,0,4)	1184.35	0.3046	**0.1125**	45.86	111.30
			(0,0,7,33)	(0,0,1,8)	22.13	9.0096	4.5182	**0.74**	**26.23**
	2	[M1]	No optimal allocation can be obtained due to r_k cannot be equal, $k = 1, 2, 3, 4$						
		[M2]	(5,5,5,5)	(5,0,0,4)	**1132.53**	**0.3104**	**0.1143**	22.93	111.30
			(5,5,5,5)	(4,0,0,5)	**1132.53**	**0.3104**	**0.1143**	12.89	93.93
			(5,5,5,5)	(1,1,3,4)	515.98	0.6318	0.2748	**2.01**	56.59
			(5,5,5,5)	(0,0,4,5)	70.78	3.1567	1.8284	3.10	**31.52**
		[M3]	(4,12,0,4)	(4,1,0,4)	**1490.71**	0.2981	0.1225	20.92	103.40
			(4,0,12,4)	(4,0,1,4)	**1490.71**	0.2981	0.1225	20.92	95.70
			(5,11,0,4)	(5,0,0,4)	1184.35	**0.2975**	**0.1125**	22.93	111.30
			(5,10,1,4)	(5,0,0,4)	1184.35	**0.2975**	**0.1125**	22.93	111.30
			(0,0,3,17)	(0,0,1,8)	23.33	8.9006	4.5037	**0.83**	**26.23**
	4	[M1]	No optimal allocation can be obtained due to r_k cannot be equal, $k = 1, 2, 3, 4$						
		[M2]	No optimal allocation can be obtained due to m_k cannot be equal, $k = 1, 2, 3, 4$						
		[M3]	(5,1,0,4)	(5,0,0,4)	**1184.35**	**0.3553**	**0.1125**	11.47	111.30
			(5,0,1,4)	(5,0,0,4)	**1184.35**	**0.3553**	**0.1125**	11.47	111.30
			(0,0,1,9)	(0,0,1,8)	27.29	8.7527	4.5107	**1.24**	**26.23**
8	1	[M1]	(10,10,10,10)	(2,2,2,2)	**455.33**	**0.5737**	**0.2000**	**4.24**	**78.94**
		[M2]	(10,10,10,10)	(3,1,0,4)	**675.68**	0.4348	0.1425	6.75	83.31
			(10,10,10,10)	(4,0,1,3)	**675.68**	0.4348	0.1425	9.62	92.98
			(10,10,10,10)	(4,0,0,4)	673.54	**0.3930**	**0.1250**	9.62	91.22
			(10,10,10,10)	(0,1,3,4)	164.36	1.1984	0.5857	**1.51**	36.50
			(10,10,10,10)	(0,0,1,7)	19.66	11.1424	5.7119	2.98	**23.51**
		[M3]	(4,0,33,3)	(4,0,1,3)	**1346.79**	**0.3102**	0.1417	41.84	92.98
			(3,33,0,4)	(3,1,0,4)	**1346.79**	**0.3102**	0.1417	36.82	83.31
			(20,0,0,20)	(4,0,0,4)	646.66	0.5569	**0.1250**	4.36	91.22
			(0,0,7,33)	(0,0,1,7)	17.38	9.0806	4.5748	**0.64**	**23.51**
	2	[M1]	(5,5,5,5)	(2,2,2,2)	**466.63**	**0.5605**	**0.2000**	**4.52**	**78.94**
		[M2]	(5,5,5,5)	(4,0,0,4)	**759.29**	**0.3634**	**0.1250**	12.89	91.22
			(5,5,5,5)	(0,1,3,4)	178.22	1.1756	0.5930	**1.76**	36.50
			(5,5,5,5)	(0,0,3,5)	46.92	4.0792	2.2686	3.10	**27.04**
		[M3]	(4,0,13,3)	(4,0,1,3)	**1075.75**	0.3394	0.1407	20.92	92.98
			(3,13,0,4)	(3,1,0,4)	**1075.75**	0.3394	0.1407	18.41	83.31
			(4,12,0,4)	(4,0,0,4)	842.21	**0.3315**	**0.1250**	20.92	91.22
			(0,0,3,17)	(0,0,1,7)	18.12	8.9820	4.5721	**0.75**	**23.51**
	4	[M1]	No optimal allocation can be obtained due to m_k cannot be equal, $k = 1, 2, 3, 4$						
		[M2]	No optimal allocation can be obtained due to m_k cannot be equal, $k = 1, 2, 3, 4$						
		[M3]	(4,2,0,4)	(4,0,0,4)	**842.21**	**0.3965**	**0.1250**	10.46	91.22
			(4,1,1,4)	(4,0,0,4)	**842.21**	**0.3965**	**0.1250**	10.46	91.22
			(4,0,2,4)	(4,0,0,4)	**842.21**	**0.3965**	**0.1250**	10.46	91.22
			(0,0,3,7)	(0,0,2,6)	35.93	5.1816	2.6881	**1.08**	25.27
			(0,0,1,9)	(0,0,1,7)	20.25	8.8793	4.6202	1.12	**23.51**
7	1	[M1]	No optimal allocation can be obtained due to r_k cannot be equal, $k = 1, 2, 3, 4$						
		[M2]	(10,10,10,10)	(3,1,0,3)	**450.50**	0.5213	0.1613	6.75	80.59
			(10,10,10,10)	(3,0,1,3)	**450.50**	0.5213	0.1613	6.75	72.89
			(10,10,10,10)	(4,0,0,3)	445.96	**0.4793**	**0.1480**	9.62	88.50
			(10,10,10,10)	(3,0,0,4)	445.96	**0.4793**	**0.1480**	6.75	71.13
			(10,10,10,10)	(0,1,2,4)	127.94	1.3295	0.6065	**1.30**	32.02
			(10,10,10,10)	(0,0,1,6)	14.48	11.2650	5.7357	2.30	**20.79**
		[M3]	(3,34,0,3)	(3,1,0,3)	**948.55**	**0.3534**	0.1636	36.82	80.59
			(3,0,34,3)	(3,0,1,3)	**948.55**	**0.3534**	0.1636	36.82	72.89
			(4,33,0,3)	(4,0,0,3)	552.70	0.3940	**0.1458**	41.84	88.50
			(0,0,8,32)	(0,0,1,6)	13.24	9.3034	4.6717	**0.56**	**20.79**
	2	[M1]	No optimal allocation can be obtained due to r_k cannot be equal, $k = 1, 2, 3, 4$						
		[M2]	(5,5,5,5)	(4,0,0,3)	**490.54**	**0.4479**	**0.1486**	12.89	88.50
			(5,5,5,5)	(3,0,0,4)	**490.54**	**0.4479**	**0.1486**	7.87	71.13
			(5,5,5,5)	(0,1,2,4)	138.08	1.3065	0.6162	**1.74**	32.02
			(5,5,5,5)	(0,0,2,5)	27.85	5.9924	3.1970	3.10	**22.55**
		[M3]	(3,14,0,3)	(3,1,0,3)	**751.61**	0.3877	0.1628	18.41	80.59
			(3,0,14,3)	(3,0,1,3)	**751.61**	0.3877	0.1628	18.41	72.89
			(4,13,0,3)	(4,0,0,3)	552.70	**0.3849**	**0.1458**	20.92	88.50
			(0,0,4,16)	(0,0,1,6)	13.71	9.3108	4.6768	**0.61**	**20.79**
	4	[M1]	No optimal allocation can be obtained due to r_k cannot be equal, $k = 1, 2, 3, 4$						
		[M2]	No optimal allocation can be obtained due to m_k cannot be equal, $k = 1, 2, 3, 4$						
		[M3]	(3,4,0,3)	(3,1,0,3)	**570.07**	0.4833	0.1616	9.21	80.59
			(3,0,4,3)	(3,0,1,3)	**570.07**	0.4833	0.1616	9.21	72.89
			(4,3,0,3)	(4,0,0,3)	552.70	**0.4591**	**0.1458**	10.46	88.50
			(0,0,2,8)	(0,0,1,6)	15.18	9.3306	4.6965	**0.83**	**20.79**

2.4.3. Discussion

The results presented in this section provide the necessary information to select an experimental plan for a fixed values of N and R when series systems are used in the experiment. For example, there are 40 components that assumed to follow a lognormal distribution available for the multi-level constant-stress experiment with Type-II censoring, and we can only afford to fail 10 of the 40 components in the experiment. Based on the numerical results presented in Table 2.1, for optimal criterion **[C1]** (D-optimality) that maximizes the determinant of the expected Fisher information matrix and for optimal criterion **[C3]** (V-optimality) that minimizes the asymptotic variance of the MLE $\hat{\beta}_1$, we would choose to perform the life testing experiment based on individual components by allocating $m_1 = m_2 = 20$ components to the two stress levels and terminate the experiment at each of the two stress levels when the 5-th failure occurs. The values of the optimal objective functions for **[C1]** and **[C3]** when $n = 1$, $n = 2$ and $n = 4$ are ($|I|= 2142.29$, $var(\hat{\beta}_1) = 0.1717$), ($|I|= 2136.00$, $var(\hat{\beta}_1) = 0.1763$) and ($|I|= 2134.91$, $var(\hat{\beta}_1) = 0.1902$), respectively.

If one consider the optimal criterion **[C2]** (A-optimality) that minimizes the sum of asymptotic variances of the MLEs, since the values of the optimal objective function (i.e., $\text{tr}(\mathbf{I}^{-1})$) for **[C2]** when $n = 1$, $n = 2$ and $n = 4$ are 0.3306, 0.3287 and 0.3273, respectively, we would perform the experiment by putting the 40 components into 10 series 4-component systems and then allocate these 10 systems equally to the two stress levels and observe the lifetimes of all these 10 systems. Note that the expected total time on test for the experiment based on 4-component series systems is the shortest compared to $n = 1$ and 2, while the expected total experimental time based on 4-component systems is the longest compared to $n = 1$ and 2.

In the above illustration, we can see that putting the components into 4-component series systems provides a slight decrease (about 1%) in the total estimation variances of the MLEs compared to using individual components or 2-component systems. However, in some cases, using series systems can give a higher degree of advantage over using individual components, especially for A-optimality **[C2]**. For instance, in Table 2.4, when $K = 2$, $N = 60$, $R = 20$ and the component lifetimes follow a Weibull distribution, for D-optimality and A-optimality, the optimal experimental plan based on 3-component series systems give $|\mathbf{I}| = 3289.87$ and $\text{tr}(\mathbf{I}^{-1}) = 0.2943$, while the optimal experimental plan based on individual components give $|\mathbf{I}| = 2815.88$ and $\text{tr}(\mathbf{I}^{-1}) = 0.3316$. In other words, using 3-component series systems for the multi-level constant-stress experiment can give 16.83% and 11.25% improvements in terms of optimality criteria **[C1]** and **[C2]**.

From the numerical results, we observe that the advantages and disadvantages of using systems in a multi-level constant-stress experiment does not show a specific pattern and it depends on the value of R, the number of stress levels K and the underlying component distributions. Therefore, it is important to evaluate the performance of the optimal experimental plans under different values of n in order to determine the optimal experimental planning for a specific situation, and the methodologies proposed in here are useful for this purpose. Although using systems in multi-level constant-stress experiment with Type-II censoring does not always give advantage over using individual components, when using systems is superior to using individual components, it always gains efficiency in estimation and gives shorter expected TTT.

2.5. Monte Carlo Simulation Study

Since the optimal experimental plans discussed in the previous sections are based on asymptotic properties of the MLEs and the accuracy of the asymptotic approximation depends on the sample sizes, we use a Monte Carlo simulation study to evaluate the validity of the numerical results based on expected Fisher information matrix and asymptotic variance-covariance matrix.

For each optimal experimental plans presented in Section 2.4, we simulate the failure times based on the multi-level constant-stress experiment with Type-II censoring and compute the corresponding MLEs of the model parameters $(\beta_0, \beta_1, \sigma)$ (denoted as $(\hat{\beta}_0, \hat{\beta}_1, \hat{\sigma})$) and the observed Fisher information matrix (denoted as $\mathcal{I}$) defined as

$$\mathcal{I}(\hat{\beta}_0, \hat{\beta}_1, \hat{\sigma}) = -\begin{bmatrix} \frac{\partial^2 l(\boldsymbol{\theta})}{\partial \beta_0^2} & \frac{\partial^2 l(\boldsymbol{\theta})}{\partial \beta_0 \partial \beta_1} & \frac{\partial^2 l(\boldsymbol{\theta})}{\partial \beta_0 \partial \sigma} \\ \frac{\partial^2 l(\boldsymbol{\theta})}{\partial \beta_0 \partial \beta_1} & \frac{\partial^2 l(\boldsymbol{\theta})}{\partial \beta_1^2} & \frac{\partial^2 l(\boldsymbol{\theta})}{\partial \beta_1 \partial \sigma} \\ \frac{\partial^2 l(\boldsymbol{\theta})}{\partial \beta_0 \partial \sigma} & \frac{\partial^2 l(\boldsymbol{\theta})}{\partial \beta_1 \partial \sigma} & \frac{\partial^2 l(\boldsymbol{\theta})}{\partial \sigma^2} \end{bmatrix}_{(\beta_0, \beta_1, \sigma) = (\hat{\beta}_0, \hat{\beta}_1, \hat{\sigma})}. \tag{2.41}$$

We also record the total time of the experiment (TTE) and total time on test (TTT) in each simulation. Based on 100,000 simulations, we compute the average of the 100,000 observed Fisher information matrices (denoted as $\bar{\mathcal{I}}$), the variances of the 100,000 MLEs $\hat{\beta}_0$, $\hat{\beta}_1$ and $\hat{\sigma}$ (denoted as V_{β_0}, V_{β_1}, V_σ), the average of the 100,000 TTE (denoted as $\overline{TTE}$), and the average of the 100,000 TTT (denoted as $\overline{TTT}$). We only present the simulation results for setting [**M3**] since it is the setting with the fewest restrictions. The simulated values of $\bar{\mathcal{I}}$, $V_{\beta_0} + V_{\beta_1} + V_\sigma$, V_{β_1}, $\overline{TTE}$ and $\overline{TTT}$ for the settings corresponding to Tables 2.1, 2.2, 2.5 and 2.6 are presented in Tables 2.7, 2.8, 2.9 and 2.10, respectively.

Table 2.7: Simulation results for the optimal allocations for n-component series systems with lognormal distributed components when $N = 40$, $K = 2$, $(y_1, y_2) = (-0.5, 0.5)$ by fixing $R = 7, 8, 9$ and 10 with $n = 1, 2$ and 4 under setting **[M3]** (referring to Table 2.1)

					[C1]	**[C2]**	**[C3]**	**[C4]**	**[C5]**
R	n	Setting	(m_1, m_2)	(r_1, r_2)	$\|\bar{\mathcal{I}}\|$	$V_{\beta_0} + V_{\beta_1} + V_\sigma$	V_{β_1}	$\overline{TTE}$	$\overline{TTT}$
10	1	**[M3]**	(20,20)	(5,5)	1780.47	0.3279	0.1718	6.06	152.53
			(12,28)	(1,9)	807.47	0.6432	0.4104	3.41	101.64
			(2,38)	(1,9)	425.69	1.0158	0.7399	9.69	94.61
	2	**[M3]**	(10,10)	(5,5)	1798.24	0.3267	0.1773	6.71	151.59
			(5,15)	(1,9)	812.23	0.6468	0.4214	3.80	99.98
			(1,19)	(1,9)	437.79	1.0212	0.7489	9.73	94.07
	4	**[M3]**	(5,5)	(5,5)	1855.42	0.3275	0.1911	10.58	150.25
			(1,9)	(1,9)	666.86	0.7747	0.5473	6.62	94.70
9	1	**[M3]**	(20, 20)	(5, 4)	1369.57	0.3658	0.1882	6.06	146.98
			(20, 20)	(4, 5)	1369.87	0.3642	0.1858	5.12	137.40
			(21, 19)	(5, 4)	1367.56	0.3643	0.1845	5.84	148.30
			(19, 21)	(4, 5)	1365.56	0.3622	0.1842	5.32	136.40
			(13, 27)	(1, 8)	664.36	0.6567	0.4046	3.22	96.87
			(2, 38)	(1, 8)	350.11	1.0468	0.7550	9.67	89.09
	2	**[M3]**	(10, 10)	(5, 4)	1390.65	0.3628	0.1898	6.69	145.77
			(10, 10)	(4, 5)	1388.74	0.3611	0.1888	5.48	136.50
			(11, 9)	(5, 4)	1380.46	0.3618	0.1888	6.14	148.21
			(9, 11)	(4, 5)	1383.33	0.3629	0.1900	5.95	134.14
			(6, 14)	(1, 8)	688.35	0.6567	0.4124	3.50	95.81
			(1, 19)	(1, 8)	359.70	1.0388	0.7518	9.71	88.58
	4	**[M3]**	(5, 5)	(5, 4)	1437.90	0.3586	0.2014	10.48	144.41
			(5, 5)	(4, 5)	1437.91	0.3586	0.2010	6.93	134.89
			(2, 8)	(1, 8)	694.47	0.6779	0.4489	5.13	92.60
			(1, 9)	(1, 8)	533.72	0.7937	0.5487	5.91	88.87
8	1	**[M3]**	(20, 20)	(4, 4)	1029.73	0.4005	0.1960	5.12	131.90
			(15, 25)	(1, 7)	533.90	0.6812	0.4067	3.02	92.38
			(2, 38)	(1, 7)	278.43	1.0659	0.7604	9.68	83.49
	2	**[M3]**	(10, 10)	(4, 4)	1048.87	0.3997	0.1993	5.45	130.89
			(7, 13)	(1, 7)	552.20	0.6753	0.4093	3.24	91.64
			(1, 19)	(1, 7)	285.57	1.0616	0.7592	9.68	83.03
	4	**[M3]**	(5, 5)	(4, 4)	1077.64	0.3958	0.2107	6.69	129.29
			(2, 8)	(1, 7)	536.55	0.7035	0.4485	4.14	86.75
			(1, 9)	(1, 7)	416.41	0.8106	0.5451	5.70	83.43
7	1	**[M3]**	(20, 20)	(4, 3)	718.24	0.4591	0.2170	5.11	125.95
			(20, 20)	(3, 4)	717.51	0.4625	0.2185	4.20	116.01
			(21, 19)	(4, 3)	715.96	0.4588	0.2163	4.94	127.07
			(19, 21)	(3, 4)	718.43	0.4592	0.2169	4.33	114.88
			(17, 23)	(1, 6)	403.68	0.7185	0.4130	2.84	87.85
			(2, 38)	(1, 6)	212.89	1.1138	0.7841	9.66	77.76
	2	**[M3]**	(10, 10)	(3, 4)	729.73	0.4581	0.2209	4.37	115.30
			(10, 10)	(4, 3)	730.00	0.4585	0.2216	5.45	125.19
			(11, 9)	(4, 3)	724.32	0.4548	0.2181	5.06	127.57
			(9, 11)	(3, 4)	729.36	0.4579	0.2198	4.68	113.27
			(8, 12)	(1, 6)	422.49	0.7094	0.4115	3.00	87.14
			(1, 19)	(1, 6)	217.94	1.0937	0.7685	9.71	77.50
	4	**[M3]**	(5, 5)	(3, 4)	758.82	0.4528	0.2281	4.85	114.01
			(5, 5)	(4, 3)	758.67	0.4516	0.2285	6.67	123.63
			(3, 7)	(1, 6)	443.77	0.6994	0.4144	3.52	84.12
			(1, 9)	(1, 6)	313.72	0.8549	0.5585	5.62	78.02

Table 2.8: Simulation results for the optimal allocations for n-component series systems with Weibull distributed components when $N = 40$, $K = 2$, $(y_1, y_2) = (-0.5, 0.5)$ by fixing $R = 7, 8, 9$ and 10 with $n = 1, 2$ and 4 under setting **[M3]** (referring to Table 2.2)

R	n	Setting	(m_1, m_2)	(r_1, r_2)	**[C1]** $\lvert\tilde{\mathcal{I}}\rvert$	**[C2]** $V_{\beta_0} + V_{\beta_1} + V_\sigma$	**[C3]** V_{β_1}	**[C4]** $\overline{TTE}$	**[C5]** $\overline{TTT}$
10	1	**[M3]**	(35,5)	(5,5)	285.51	1.0062	0.7845	10.24	83.28
			(5,35)	(5,5)	285.98	1.0113	0.7886	27.85	83.35
			(20,20)	(5,5)	206.13	0.7704	0.4441	3.43	83.26
			(9,31)	(1,9)	71.43	2.5819	1.8470	1.98	52.51
			(3,37)	(1,9)	78.56	2.4193	1.7791	4.26	52.57
	2	**[M3]**	(15,5)	(5,5)	279.61	0.8232	0.5719	5.27	83.37
			(5,15)	(5,5)	279.27	0.8186	0.5679	13.91	83.29
			(10,10)	(5,5)	230.81	0.7456	0.4440	3.97	83.30
			(4,16)	(1,9)	81.25	2.5073	1.8172	2.28	52.51
			(2,18)	(1,9)	84.43	2.3894	1.7525	3.39	52.56
	4	**[M3]**	(5,5)	(5,5)	332.58	0.6865	0.4472	7.07	83.31
			(1,9)	(1,9)	119.72	2.3710	1.7815	4.35	52.53
9	1	**[M3]**	(35, 5)	(4, 5)	201.90	1.2131	0.9515	10.23	71.17
			(5, 35)	(5, 4)	202.80	1.2169	0.9561	27.86	78.89
			(23, 17)	(5, 4)	137.10	0.8818	0.5009	2.95	78.74
			(17, 23)	(4, 5)	137.45	0.8932	0.5081	3.19	71.14
			(22, 18)	(5, 4)	137.03	0.8872	0.5041	3.09	78.83
			(18, 22)	(4, 5)	137.18	0.8931	0.5072	3.00	70.96
			(9, 31)	(1, 8)	52.46	2.6202	1.8394	1.85	48.03
			(4, 36)	(1, 8)	57.49	2.4936	1.7940	3.26	48.13
	2	**[M3]**	(15, 5)	(4, 5)	194.83	0.9892	0.6913	5.18	71.14
			(5, 15)	(5, 4)	195.63	0.9868	0.6900	13.93	78.81
			(11, 9)	(5, 4)	153.39	0.8682	0.5057	3.51	78.90
			(9, 11)	(4, 5)	152.50	0.8672	0.5083	3.38	71.12
			(4, 16)	(1, 8)	59.69	2.5683	1.8279	2.10	48.03
			(1, 19)	(1, 8)	66.74	2.4508	1.8334	6.19	48.03
	4	**[M3]**	(5, 5)	(5, 4)	211.48	0.8179	0.5196	6.97	78.85
			(5, 5)	(4, 5)	211.97	0.8199	0.5216	4.32	71.33
			(4, 6)	(4, 5)	209.15	0.8060	0.5099	6.39	71.21
			(6, 4)	(5, 4)	208.12	0.8098	0.5159	4.64	78.76
			(3, 7)	(2, 7)	150.76	1.2181	0.8290	3.61	55.68
			(1, 9)	(1, 8)	80.95	2.4334	1.8008	3.65	47.99
8	1	**[M3]**	(36, 4)	(4, 4)	134.98	1.3666	1.0711	9.36	66.75
			(4, 36)	(4, 4)	133.62	1.3733	1.0767	25.26	66.48
			(20, 20)	(4, 4)	88.12	1.0245	0.5682	2.68	66.62
			(11, 29)	(1, 7)	36.24	2.7423	1.8876	1.61	43.53
			(6, 34)	(1, 7)	38.92	2.5938	1.8018	2.26	43.52
	2	**[M3]**	(16, 4)	(4, 4)	127.41	1.1295	0.7876	4.74	66.66
			(4, 16)	(4, 4)	128.61	1.1323	0.7905	12.7	66.67
			(10, 10)	(4, 4)	98.04	1.0051	0.5673	2.97	66.74
			(5, 15)	(1, 7)	41.27	2.6815	1.8575	1.79	43.66
			(2, 18)	(1, 7)	44.39	2.5193	1.7974	3.22	43.51
	4	**[M3]**	(6, 4)	(4, 4)	133.62	0.9566	0.5945	3.41	66.61
			(4, 6)	(4, 4)	134.32	0.9552	0.5918	6.35	66.59
			(5, 5)	(4, 4)	126.45	0.9562	0.5762	3.98	66.67
			(2, 8)	(1, 7)	55.59	2.6241	1.8771	2.41	43.58
			(1, 9)	(1, 7)	55.47	2.4936	1.8072	3.40	43.57
7	1	**[M3]**	(36, 4)	(3, 4)	85.38	1.7376	1.3734	9.34	54.47
			(4, 36)	(4, 3)	85.38	1.7366	1.3722	25.42	62.27
			(24, 16)	(4, 3)	50.19	1.2325	0.6734	2.23	62.22
			(16, 24)	(3, 4)	50.31	1.2366	0.6786	2.48	54.39
			(23, 17)	(4, 3)	50.00	1.2374	0.6776	2.32	62.17
			(17, 23)	(3, 4)	50.21	1.2485	0.6852	2.34	54.42
			(12, 28)	(1, 6)	23.54	2.8628	1.9125	1.45	39.06
			(1, 39)	(1, 6)	35.26	2.8192	2.1767	12.14	39.02
	2	**[M3]**	(16, 4)	(3, 4)	79.62	1.4346	1.0133	4.70	54.44
			(4, 16)	(4, 3)	79.14	1.4372	1.0154	12.65	62.01
			(12, 8)	(4, 3)	55.46	1.2153	0.6770	2.42	62.39
			(8, 12)	(3, 4)	55.17	1.2176	0.6800	2.68	54.39
			(11, 9)	(4, 3)	55.44	1.2163	0.6741	2.63	62.21
			(9, 11)	(3, 4)	54.99	1.2245	0.6834	2.38	54.48
			(6, 14)	(1, 6)	26.03	2.8711	1.9466	1.54	39.05
			(1, 19)	(1, 6)	33.47	2.6725	1.9665	6.15	39.01
	4	**[M3]**	(6, 4)	(3, 4)	79.00	1.2110	0.7490	2.78	54.46
			(4, 6)	(4, 3)	78.89	1.2145	0.7547	6.35	62.18
			(6, 4)	(4, 3)	69.84	1.1725	0.6879	2.99	62.18
			(4, 6)	(3, 4)	69.81	1.1709	0.6863	3.35	54.41
			(2, 8)	(1, 6)	34.26	2.6998	1.8703	2.03	39.07
			(3, 7)	(1, 6)	35.80	2.7980	1.9513	2.00	39.02

Table 2.9: Simulation results for the optimal allocations for n-component series systems with lognormal distributed components when $N = 40$, $K = 4$, $(y_1, y_2, y_3, y_4) = (-1.0, -0.5, 0.5, 1.0)$ by fixing $R = 7, 8, 9$ and 10 with $n = 1, 2$ and 4 under setting **[M3]**

					[C1]	**[C2]**	**[C3]**	**[C4]**	**[C5]**
R	n	Setting	(m_1, m_2, m_3, m_4)	(r_1, r_2, r_3, r_4)	$\lvert\bar{\mathcal{I}}\rvert$	$V_{\beta_0} + V_{\beta_1} + V_\sigma$	V_{β_1}	$\overline{TTE}$	$\overline{TTT}$
10	1	**[M3]**	(20,0,0,20)	(5,0,0,5)	7121.88	0.1991	0.0429	9.99	208.80
			(20,4,0,16)	(2,4,0,4)	6543.39	0.1903	0.0606	44.44	204.86
			(16,0,4,20)	(4,0,4,2)	6532.32	0.1906	0.0607	17.89	190.64
			(0,0,7,33)	(0,0,1,9)	177.40	3.6664	1.8145	1.79	53.34
			(25,0,2,13)	(0,0,1,9)	69.76	5.9997	3.0930	5.26	43.47
	2	**[M3]**	(10,0,0,10)	(5,0,0,5)	7192.97	0.1937	0.0443	11.05	207.45
			(7,0,8,5)	(4,0,1,5)	6142.45	0.1898	0.0606	12.62	175.83
			(5,8,0,7)	(5,1,0,4)	6150.96	0.1903	0.0603	33.95	216.18
			(0,0,3,17)	(0,0,1,9)	174.52	3.8024	1.8974	1.99	52.89
			(9,0,1,10)	(0,0,1,9)	88.93	5.8525	2.9994	4.90	45.68
	4	**[M3]**	(5,0,0,5)	(5,0,0,5)	7421.68	0.1841	0.0478	17.32	205.59
			(0,0,2,8)	(0,0,2,8)	296.43	2.3416	1.2139	3.45	55.38
			(0,0,1,9)	(0,0,1,9)	166.71	4.3198	2.1891	3.19	52.20
9	1	**[M3]**	(20,1,0,19)	(4,1,0,4)	5611.65	0.2170	0.0492	21.56	200.05
			(19,0,1,20)	(4,0,1,4)	5627.87	0.2180	0.0492	11.35	185.03
			(20,4,0,16)	(2,4,0,3)	5148.16	0.2100	0.0658	44.30	201.09
			(16,0,4,20)	(3,0,4,2)	5152.46	0.2104	0.0657	17.17	165.40
			(21,0,0,19)	(5,0,0,4)	5475.98	0.2252	0.0460	9.62	208.01
			(19,0,0,21)	(4,0,0,5)	5480.75	0.2263	0.0467	8.72	180.97
			(0,0,8,32)	(0,0,1,8)	150.07	3.5369	1.7391	1.65	50.19
			(26,0,2,12)	(0,0,1,8)	55.77	5.9677	3.0837	5.01	39.39
	2	**[M3]**	(10,0,0,10)	(5,0,0,4)	5562.78	0.2204	0.0475	11.04	203.85
			(10,0,0,10)	(4,0,0,5)	5549.53	0.2202	0.0474	8.98	182.58
			(6,0,10,4)	(4,0,1,4)	4575.62	0.2102	0.0695	14.92	167.38
			(4,10,0,6)	(4,1,0,4)	4575.93	0.2088	0.0688	30.86	187.54
			(11,0,0,9)	(5,0,0,4)	5533.39	0.2208	0.0473	10.13	209.39
			(9,0,0,11)	(4,0,0,5)	5520.90	0.2195	0.0470	9.81	177.44
			(0,0,4,16)	(0,0,1,8)	155.88	3.5269	1.7413	1.76	49.85
			(10,0,1,9)	(0,0,1,8)	70.47	5.8952	3.0303	4.78	41.44
	4	**[M3]**	(5,0,0,5)	(5,0,0,4)	5739.78	0.2086	0.0507	17.32	202.39
			(5,0,0,5)	(4,0,0,5)	5751.14	0.2081	0.0503	10.99	180.25
			(5,1,0,4)	(5,0,0,4)	5240.31	0.2050	0.0541	17.27	200.49
			(5,0,1,4)	(5,0,0,4)	5249.63	0.2066	0.0543	17.30	200.82
			(4,1,0,5)	(4,0,0,5)	5254.04	0.2053	0.0542	16.08	169.50
			(4,0,1,5)	(4,0,0,5)	5250.63	0.2052	0.0538	16.02	169.19
			(0,0,1,9)	(0,0,1,8)	133.41	4.3014	2.1785	2.55	48.67
			(1,0,1,8)	(0,0,1,8)	130.97	4.3486	2.2110	3.12	47.25
8	1	**[M3]**	(20,0,1,19)	(4,0,1,3)	4237.09	0.2477	0.0540	11.11	183.96
			(19,1,0,20)	(3,1,0,4)	4232.73	0.2465	0.0535	21.02	171.57
			(21,3,0,16)	(2,3,0,3)	4079.36	0.2352	0.0672	38.10	183.22
			(16,0,3,21)	(3,0,3,2)	4085.45	0.2353	0.0669	15.37	158.19
			(20,0,0,20)	(4,0,0,4)	4131.50	0.2541	0.0488	8.41	180.18
			(0,0,9,31)	(0,0,1,7)	123.01	3.4826	1.7009	1.53	46.84
			(28,0,2,10)	(0,0,1,7)	41.90	6.0843	3.1620	5.29	35.50
	2	**[M3]**	(10,0,0,10)	(4,0,0,4)	4181.87	0.2490	0.0499	8.97	179.10
			(6,0,10,4)	(3,0,1,4)	3485.05	0.2384	0.0744	10.88	142.38
			(4,10,0,6)	(4,1,0,3)	3486.35	0.2385	0.0753	30.91	184.60
			(0,0,4,16)	(0,0,1,7)	122.62	3.5702	1.7598	1.64	46.50
			(12,0,1,7)	(0,0,1,7)	52.10	6.0012	3.1072	6.10	37.18
	4	**[M3]**	(5,0,0,5)	(4,0,0,4)	4308.26	0.2386	0.0523	10.99	176.77
			(4,2,0,4)	(4,0,0,4)	3644.47	0.2290	0.0602	16.08	164.61
			(4,1,1,4)	(4,0,0,4)	3659.17	0.2290	0.0603	16.07	164.52
			(4,0,2,4)	(4,0,0,4)	3654.69	0.2289	0.0598	16.01	164.22
			(0,0,3,7)	(0,0,2,6)	212.31	2.1125	1.0811	2.17	49.41
			(2,0,1,7)	(0,0,1,7)	99.83	4.4513	2.2707	3.05	42.41
7	1	**[M3]**	(20,1,0,19)	(3,1,0,3)	3069.44	0.2826	0.0589	20.89	169.96
			(19,0,1,20)	(3,0,1,3)	3074.34	0.2854	0.0590	10.21	155.44
			(20,2,1,17)	(2,2,1,2)	2942.29	0.2697	0.0759	31.88	165.06
			(17,1,2,20)	(2,1,2,2)	2929.31	0.2668	0.0757	24.14	146.16
			(21,0,0,19)	(4,0,0,3)	2873.22	0.2991	0.0551	8.14	179.33
			(19,0,0,21)	(3,0,0,4)	2866.77	0.2967	0.0543	7.13	152.16
			(22,0,0,18)	(4,0,0,3)	2865.85	0.2973	0.0542	7.88	181.87
			(18,0,0,22)	(3,0,0,4)	2856.40	0.2965	0.0545	7.37	149.51
			(0,0,10,30)	(0,0,1,6)	95.61	3.4454	1.6689	1.44	43.67
			(29,0,2,9)	(0,0,1,6)	31.57	6.1923	3.2329	5.01	31.43
	2	**[M3]**	(10,0,0,10)	(4,0,0,3)	2909.51	0.2920	0.0555	8.97	175.59
			(10,0,0,10)	(3,0,0,4)	2919.66	0.2939	0.0551	7.14	153.52
			(5,0,12,3)	(3,0,1,3)	2404.89	0.2706	0.0874	12.89	134.13
			(3,12,0,5)	(3,1,0,3)	2402.25	0.2694	0.0876	27.37	156.40
			(11,0,0,9)	(4,0,0,3)	2915.46	0.2931	0.0546	8.32	180.58
			(9,0,0,11)	(3,0,0,4)	2906.54	0.2924	0.0550	7.69	148.74
			(0,0,5,15)	(0,0,1,6)	98.42	3.4776	1.6929	1.49	43.45
			(13,0,1,6)	(0,0,1,6)	38.20	6.1344	3.1911	5.88	32.90
	4	**[M3]**	(5,0,0,5)	(4,0,0,3)	3026.28	0.2796	0.0568	10.99	173.43
			(5,0,0,5)	(3,0,0,4)	3035.58	0.2799	0.0572	7.84	151.54
			(4,3,0,3)	(4,0,0,3)	2327.77	0.2648	0.0702	16.01	159.32
			(4,2,1,3)	(4,0,0,3)	2328.97	0.2643	0.0701	16.06	159.67
			(0,0,2,8)	(0,0,1,6)	99.91	3.5983	1.7776	1.71	42.69
			(3,0,1,6)	(0,0,1,6)	72.69	4.4804	2.2929	2.96	37.43

Table 2.10: Simulation results for the optimal allocations for n-component series systems with Weibull distributed components when $N = 40$, $K = 4$, $(y_1, y_2, y_3, y_4) = (-1.0, -0.5, 0.5, 1.0)$ by fixing $R = 7, 8, 9$ and 10 with $n = 1, 2$ and 4 under setting **[M3]** (referring to Table 2.6)

R	n	Setting	(m_1, m_2, m_3, m_4)	(r_1, r_2, r_3, r_4)	**[C1]** $\|\mathcal{I}\|$	**[C2]** $V_{\beta_0} + V_{\beta_1} + V_\sigma$	**[C3]** V_{β_1}	**[C4]** $\overline{TTE}$	**[C5]** $\overline{TTT}$
10	1	**[M3]**	(4,32,0,4)	(4,2,0,4)	2361.27	0.2940	0.1393	41.87	115.52
			(4,0,32,4)	(4,0,2,4)	2360.58	0.2941	0.1391	41.87	100.13
			(5,0,31,4)	(5,0,1,4)	2378.05	0.2807	0.1250	45.92	115.83
			(4,31,0,5)	(4,1,0,5)	2377.35	0.2810	0.1252	41.87	106.06
			(15,10,0,15)	(5,0,0,5)	910.09	0.3786	0.1112	7.82	114.03
			(15,9,1,15)	(5,0,0,5)	910.09	0.3786	0.1112	7.82	114.03
			(0,0,6,34)	(0,0,1,9)	18.34	14.3116	7.1323	1.23	28.95
	2	**[M3]**	(5,0,11,4)	(5,0,1,4)	1778.92	0.3060	0.1243	22.94	115.70
			(4,11,0,5)	(4,1,0,5)	1780.97	0.3054	0.1241	20.95	106.10
			(5,10,0,5)	(5,0,0,5)	1464.14	0.2921	0.1119	22.94	113.95
			(0,0,3,17)	(0,0,1,9)	20.55	14.2585	7.1234	1.30	28.95
	4	**[M3]**	(5,0,0,5)	(5,0,0,5)	1330.32	0.3510	0.1118	11.47	113.96
			(0,0,2,8)	(0,0,2,8)	53.06	6.0225	3.1281	2.35	30.71
			(0,0,1,9)	(0,0,1,9)	166.71	4.3198	2.1891	3.19	52.20
9	1	**[M3]**	(4,32,0,4)	(4,1,0,4)	1795.06	0.3147	0.1413	41.88	103.40
			(4,0,32,4)	(4,0,1,4)	1809.15	0.3143	0.1406	42.00	96.02
			(5,31,0,4)	(5,0,0,4)	1147.36	0.3295	0.1285	45.86	111.36
			(5,30,1,4)	(5,0,0,4)	1143.91	0.3279	0.1289	45.83	111.22
			(0,0,7,33)	(0,0,1,8)	13.50	14.4535	7.1779	1.16	26.25
	2	**[M3]**	(4,12,0,4)	(4,1,0,4)	1328.27	0.3397	0.1391	20.94	103.42
			(4,0,12,4)	(4,0,1,4)	1336.11	0.3429	0.1408	20.95	95.84
			(5,11,0,4)	(5,0,0,4)	1038.27	0.3298	0.1272	22.91	111.07
			(5,10,1,4)	(5,0,0,4)	1034.93	0.3276	0.1266	22.96	111.46
			(0,0,3,17)	(0,0,1,8)	15.18	14.2266	7.1072	1.24	26.25
	4	**[M3]**	(5,1,0,4)	(5,0,0,4)	923.57	0.3962	0.1281	11.47	111.36
			(5,0,1,4)	(5,0,0,4)	925.65	0.3956	0.1284	11.47	111.30
			(0,0,1,9)	(0,0,1,8)	20.37	14.1392	7.1510	1.69	26.23
8	1	**[M3]**	(4,0,33,3)	(4,0,1,3)	1284.79	0.3652	0.1673	41.88	93.01
			(3,33,0,4)	(3,1,0,4)	1286.80	0.3630	0.1661	36.97	83.45
			(20,0,0,20)	(4,0,0,4)	352.47	0.5970	0.1409	4.37	91.35
			(0,0,7,33)	(0,0,1,7)	9.54	14.5605	7.2431	1.15	23.49
	2	**[M3]**	(4,0,13,3)	(4,0,1,3)	944.89	0.3987	0.1651	21.00	93.14
			(3,13,0,4)	(3,1,0,4)	939.40	0.4001	0.1648	18.39	83.02
			(4,12,0,4)	(4,0,0,4)	726.19	0.3714	0.1436	20.91	91.13
			(0,0,3,17)	(0,0,1,7)	10.64	14.3125	7.1578	1.21	23.53
	4	**[M3]**	(4,2,0,4)	(4,0,0,4)	635.37	0.4457	0.1434	10.45	91.10
			(4,1,1,4)	(4,0,0,4)	640.17	0.4473	0.1434	10.49	91.40
			(4,0,2,4)	(4,0,0,4)	639.17	0.4443	0.1434	10.48	91.40
			(0,0,3,7)	(0,0,2,6)	23.75	6.5525	3.3617	1.41	25.27
			(0,0,1,9)	(0,0,1,7)	13.79	14.1451	7.1799	1.53	23.49
7	1	**[M3]**	(3,34,0,3)	(3,1,0,3)	897.69	0.4223	0.1946	36.83	80.50
			(3,0,34,3)	(3,0,1,3)	905.95	0.4249	0.1961	36.96	72.99
			(4,33,0,3)	(4,0,0,3)	529.51	0.4324	0.1708	41.81	88.38
			(0,0,8,32)	(0,0,1,6)	6.18	14.8476	7.3706	1.10	20.75
	2	**[M3]**	(3,14,0,3)	(3,1,0,3)	654.99	0.4668	0.1927	18.42	80.60
			(3,0,14,3)	(3,0,1,3)	651.45	0.4663	0.1943	18.42	72.91
			(4,13,0,3)	(4,0,0,3)	465.88	0.4396	0.1719	20.87	88.26
			(0,0,4,16)	(0,0,1,6)	6.69	15.0237	7.4531	1.10	20.79
	4	**[M3]**	(3,4,0,3)	(3,1,0,3)	419.63	0.5675	0.1922	9.23	80.68
			(3,0,4,3)	(3,0,1,3)	419.95	0.5657	0.1918	9.21	72.77
			(4,3,0,3)	(4,0,0,3)	396.19	0.5243	0.1716	10.48	88.61
			(0,0,2,8)	(0,0,1,6)	8.57	14.9864	7.4590	1.16	20.79

From the simulation results presented in Tables 2.7 – 2.10, we observe that the simulated values corresponding to the objective functions of optimality criteria **[C2]**, **[C3]**, **[C4]**, **[C5]** are close to the corresponding values presented in Tables 2.1, 2.2, 2.5 and 2.6, which verify the validity of the numerical results obtained by the methodologies developed in Sections 2.2

and 2.3. Although the simulated values of the determinant of average observed Fisher information and the expected Fisher information are not close (say, within 10%), experimental schemes that give similar determinants of the expected Fisher information matrices also give similar determinants of average observed Fisher information matrices. For example, in Table 2.6, when $K = 4$, $R = 7$ and $n = 2$, the allocations $(m_1, m_2, m_3, m_4) = (3, 14, 0, 3)$ with $(r_1, r_2, r_3, r_4) = (3, 1, 0, 3)$ and $(m_1, m_2, m_3, m_4) = (3, 0, 14, 3)$ with $(r_1, r_2, r_3, r_4) = (3, 0, 1, 3)$ give $|\mathbf{I}|= 751.61$, while in Table 2.10, these two allocations give $|\bar{\mathcal{I}}| = 654.99$ and 651.45, respectively. On the other hand, from this simulation study, we notice that Monte Carlo simulation is a effective tool to verify the optimal experimental schemes derived from the asymptotic theory of MLEs.

2.6. Illustrative Example

In this section, we use the motivating example of furniture joints testing presented in Kłos et al. (2018) to illustrate the methodologies developed here for planning a future experiment. In Kłos et al. (2018), different furniture joints have rupture force $P_{max} = 134$ Newton, 116 Newton, and 60 Newton. Therefore, for a future reliability test of furniture joints, we consider three stress levels at 60, 116, and 134 (Newton), which yields the standardized stress levels as $y_1 = -1.12, y_2 = 0.33$ and $y_3 = 0.79$. Suppose the total number of furniture joints available for the future reliability test is $N = 180$ and we consider putting the furniture joints into an n-component series system (i.e., n joints connected in series) with $n = 12$, 10, 5, and 3 (i.e., 15, 18, 36 and 60 systems, respectively). Based on assuming the lifetimes of the joints are Weibull distributed (Kłos et al., 2018) and consider the optimality criteria and settings described in Section 1.2, we determine the optimal experimental plans for different optimality criteria under different settings. These optimal experimental plans are presented in Table 2.11.

Based on the results in Table 2.11, if optimality criteria **[C1]** and **[C2]** are considered and setting **[M3]** is used, the experimental plans based on $n = 3$ are preferred over $n = 12, 10$

and 5. For example, when $R = 15$, the optimal values of $\text{tr}(\mathbf{I}^{-1})$ based on setting [**M3**] for $n = 12$, 10, 5 and 3 are 0.3537, 0.3246, 0.2338 and 0.1950, respectively. In practice, one of the issues of considering settings [**M3**] or [**M2**] is that those optimal experimental schemes may assign none or very few items/systems in the non-extreme stress levels, which makes it hard, if not impossible, to check the validity of the link function μ_k in Eq. (2.1) based on the observed experimental data. Therefore, the setting [**M1**] (i.e., assigning equal number of systems to each stress levels and fixed number of censored systems) can be considered instead of setting [**M3**] or [**M2**] in case that the validation of the link function is needed in a practical situation. If setting [**M1**] is considered, based on the results presented in Table 2.11, then the optimal experimental plans based on $n = 12$ are preferred over $n = 10, 5$ and 3.

Table 2.11: Optimal allocations for n-component series systems with Weibull distributed components when $N = 180$, $K = 3$, $(y_1, y_2, y_3) = (-1.12, 0.33, 0.79)$ by fixing $R = 15, 12$ and 9 with $n = 12, 10, 5$ and 3

R	n	Setting	(m_1, m_2, m_3)	(r_1, r_2, r_3)	**[C1]** $\lvert\mathbf{I}\rvert$	**[C2]** $\mathrm{tr}(\mathbf{I}^{-1})$	**[C3]** $var(\hat{\beta}_1)$	**[C4]** $E(TTE)$	**[C5]** $E(TTT)$
15	12	**[M1-2]**	(5,5,5)	(5,5,5)	**3678**	**0.3802**	**0.1006**	**9235**	**431790**
		[M3]	(8,0,7)	(8,0,7)	**5041**	0.3568	**0.0734**	10487	397226
			(7,0,8)	(7,0,8)	**5041**	**0.3537**	**0.0734**	10992	438572
			(14,1,0)	(14,1,0)	726	1.3228	0.5096	**2553**	**131253**
	10	**[M1]**	(6,6,6)	(5,5,5)	**3319**	**0.4200**	**0.1006**	**7037**	**431790**
		[M2]	(6,6,6)	(6,3,6)	**4429**	**0.3584**	**0.0870**	11891	426233
			(6,6,6)	(6,5,4)	3562	0.4120	0.0997	**4611**	390443
			(6,6,6)	(6,6,3)	3641	0.3894	0.1059	7506	**372549**
		[M3]	(7,4,7)	(7,1,7)	**5154**	0.3310	0.0770	12584	420677
			(8,3,7)	(8,0,7)	5041	0.3277	**0.0734**	12584	397226
			(7,3,8)	(7,0,8)	5041	**0.3246**	**0.0734**	13191	438572
			(16,2,0)	(14,1,0)	650	1.4352	0.5144	**1532**	**131253**
	5	**[M1]**	(12,12,12)	(5,5,5)	**2877**	**0.4913**	**0.1006**	**4954**	**431790**
		[M2]	(12,12,12)	(7,1,7)	**4446**	0.3730	0.0771	7958	420677
			(12,12,12)	(8,0,7)	4066	0.3681	**0.0738**	7958	397226
			(12,12,12)	(7,0,8)	4066	0.3599	**0.0738**	9900	438572
			(12,12,12)	(6,0,9)	3978	**0.3589**	0.0793	12326	479918
			(12,12,12)	(10,3,2)	2912	0.5854	0.1466	**2304**	260848
			(12,12,12)	(12,3,0)	1796	0.7435	0.2499	4460	**178156**
		[M3]	(7,23,6)	(7,2,6)	**6592**	0.2484	0.0822	23781	402782
			(8,21,7)	(8,0,7)	5041	0.2415	**0.0734**	25168	397226
			(7,21,8)	(7,0,8)	5041	0.2384	**0.0734**	26381	438572
			(6,22,8)	(6,1,8)	6347	**0.2338**	0.0794	26381	462023
			(29,7,0)	(14,1,0)	550	1.6416	0.5332	**925**	131253
			(34,2,0)	(14,1,0)	538	1.3865	0.5154	3064	**131252**
	3	**[M1]**	(20,20,20)	(5,5,5)	**2769**	**0.5144**	**0.1006**	**4522**	**431790**
		[M2]	(20,20,20)	(7,1,7)	**4164**	0.4012	0.0771	6756	420677
			(20,20,20)	(7,0,8)	3755	**0.3944**	**0.0737**	8000	438572
			(20,20,20)	(8,0,7)	3755	0.4026	**0.0737**	6756	397226
			(20,20,20)	(10,3,2)	2637	0.6318	0.1442	**1660**	260848
			(20,20,20)	(14,1,0)	594	1.7820	0.6527	2750	**131253**
		[M3]	(7,47,6)	(7,2,6)	**7907**	0.2067	0.0827	39635	402782
			(7,45,8)	(7,0,8)	5041	0.1999	**0.0734**	43969	438572
			(8,45,7)	(8,0,7)	5041	0.2030	**0.0734**	41946	397226
			(6,46,8)	(6,1,8)	7240	**0.1950**	0.0795	43969	462023
			(47,13,0)	(14,1,0)	517	1.7114	0.5390	**839**	131488
			(55,5,0)	(14,1,0)	182	2.1579	0.5195	2043	**128475**
12	12	**[M1]**	(5,5,5)	(4,4,4)	**1698**	**0.5833**	**0.1258**	**5190**	**345432**
		[M2]	(5,5,5)	(5,2,5)	**2416**	**0.4752**	**0.1053**	9235	339875
			(5,5,5)	(5,4,3)	1839	0.5715	0.1257	**3277**	304086
			(5,5,5)	(5,5,2)	1882	0.5265	0.1353	5830	**286191**
		[M3]	(6,4,5)	(6,1,5)	**2641**	0.4558	0.0977	9235	316424
			(6,3,6)	(6,0,6)	2592	**0.4433**	**0.0914**	9909	334319
			(12,3,0)	(11,1,0)	424	1.6715	0.5423	**1260**	**109693**
	10	**[M1]**	(6,6,6)	(4,4,4)	**1618**	**0.6152**	**0.1258**	**4611**	**345432**
		[M2]	(6,6,6)	(5,1,6)	**2622**	0.4405	0.0993	11891	357770
			(6,6,6)	(6,0,6)	2592	**0.4069**	**0.0914**	11891	334319
			(6,6,6)	(6,4,2)	1794	0.6064	0.1397	**2911**	262739
			(6,6,6)	(6,6,0)	1494	0.4963	0.1585	7506	**226949**
		[M3]	(6,7,5)	(6,1,5)	**2845**	0.4127	0.0979	11082	316424
			(6,6,6)	(6,0,6)	2592	0.4069	**0.0914**	11891	334319
			(5,7,6)	(5,1,6)	2807	**0.4065**	0.0992	11891	357770
			(14,4,0)	(11,1,0)	393	1.7504	0.5476	**1019**	**109693**
	5	**[M1]**	(12,12,12)	(4,4,4)	**1484**	**0.6810**	**0.1258**	**3740**	**345432**
		[M2]	(12,12,12)	(5,1,6)	**2175**	0.5351	0.0991	6340	357770
			(12,12,12)	(6,0,6)	2038	0.5251	**0.0914**	6340	334319
			(12,12,12)	(5,0,7)	1994	**0.5198**	0.0964	7958	375665
			(12,12,12)	(8,3,1)	1329	0.8195	0.1904	**1680**	197942
			(12,12,12)	(11,1,0)	424	1.8546	0.6606	3023	**109693**
		[M3]	(5,26,5)	(5,2,5)	**3580**	0.3112	0.1068	22163	339875
			(18,0,18)	(6,0,6)	1949	0.6825	**0.0914**	3804	334319
			(17,2,17)	(6,0,6)	1959	0.6590	**0.0914**	4074	334319
			(5,25,6)	(5,1,6)	3527	**0.2907**	0.0995	23781	357770
			(27,9,0)	(11,1,0)	344	1.8883	0.5526	**734**	**109693**
	3	**[M1]**	(20,20,20)	(4,4,4)	**1446**	**0.7035**	**0.1258**	**3511**	**345432**
		[M2]	(20,20,20)	(5,1,6)	**2081**	0.5635	0.0991	5600	357770
			(20,20,20)	(6,0,6)	1934	0.5598	**0.0914**	5600	334319
			(20,20,20)	(5,0,7)	1885	**0.5560**	0.0961	6756	375665
			(20,20,20)	(9,2,1)	1107	0.9892	0.2334	**1384**	174490
			(20,20,20)	(11,1,0)	359	1.9805	0.6636	1842	**109693**
		[M3]	(5,50,5)	(5,2,5)	**4318**	0.2604	0.1073	36939	339875
			(30,0,30)	(6,0,6)	1891	0.7099	**0.0914**	3543	334319
			(24,12,24)	(6,0,6)	1912	0.6245	**0.0914**	4544	334319
			(5,49,6)	(5,1,6)	4051	**0.2434**	0.0996	39635	357770
			(44,16,0)	(11,1,0)	331	1.9457	0.5569	**680**	109693
			(57,3,0)	(11,1,0)	327	1.4919	0.5447	3404	**109685**

Table 2.11: (Continue)

R	n	Setting	(m_1, m_2, m_3)	(r_1, r_2, r_3)	**[C1]** $\lvert\mathbf{I}\rvert$	**[C2]** $\mathrm{tr}(\mathbf{I}^{-1})$	**[C3]** $var(\hat{\beta}_1)$	**[C4]** $E(TTE)$	**[C5]** $E(TTT)$
9	12	**[M1]**	(5,5,5)	(3,3,3)	**690**	**0.9286**	**0.1677**	**3168**	**259074**
		[M2]	(5,5,5)	(5,0,4)	**1033**	0.6731	**0.1253**	5190	230066
			(5,5,5)	(4,0,5)	**1033**	**0.6472**	**0.1253**	9235	271412
			(5,5,5)	(5,2,2)	825	0.9206	0.1802	**1820**	194276
			(5,5,5)	(5,4,0)	595	0.8235	0.2174	3277	**158486**
		[M3]	(4,7,4)	(4,1,4)	**1222**	0.5922	0.1338	8426	253517
			(5,6,4)	(5,0,4)	1080	0.5983	**0.1234**	8426	230066
			(4,6,5)	(4,0,5)	1080	**0.5897**	**0.1234**	9235	271412
			(11,4,0)	(8,1,0)	213	2.0787	0.5668	**711**	**88132**
	10	**[M1]**	(6,6,6)	(3,3,3)	**674**	**0.9550**	**0.1677**	**2993**	**259074**
		[M2]	(6,6,6)	(4,1,4)	**966**	0.7794	0.1335	4611	253517
			(6,6,6)	(5,0,4)	954	0.7306	**0.1249**	4611	230066
			(6,6,6)	(4,0,5)	954	0.7053	**0.1249**	7037	271412
			(6,6,6)	(3,0,6)	923	**0.6922**	0.1514	11891	312759
			(6,6,6)	(6,2,1)	685	1.0665	0.2452	**1761**	152930
			(6,6,6)	(6,3,0)	532	1.0013	0.2627	1889	**135035**
		[M3]	(4,10,4)	(4,1,4)	**1311**	**0.5370**	0.1340	10111	253517
			(4,9,5)	(4,0,5)	1080	0.5411	**0.1234**	11082	271412
			(5,9,4)	(5,0,4)	1080	0.5498	**0.1234**	10111	230066
			(13,5,0)	(8,1,0)	205	2.1376	0.5690	**645**	**88132**
	5	**[M1]**	(12,12,12)	(3,3,3)	**643**	**1.0131**	**0.1677**	**2662**	**259074**
		[M2]	(12,12,12)	(4,1,4)	**890**	0.8571	0.1334	3740	253517
			(12,12,12)	(5,0,4)	848	0.8367	**0.1246**	3740	230066
			(12,12,12)	(4,0,5)	848	**0.8119**	**0.1246**	4954	271412
			(12,12,12)	(6,2,1)	583	1.2105	0.2383	**1068**	152930
			(12,12,12)	(8,1,0)	208	2.2062	0.6747	1466	**88132**
		[M3]	(4,28,4)	(4,1,4)	**1649**	0.3905	0.1346	20222	253517
			(5,27,4)	(5,0,4)	1080	0.4062	**0.1234**	20222	230066
			(4,27,5)	(4,0,5)	1080	**0.3976**	**0.1234**	22163	271412
			(4,28,4)	(4,1,4)	1649	0.3905	0.1346	**20222**	253517
			(25,11,0)	(8,1,0)	189	2.2724	0.5752	557	**88132**
	3	**[M1]**	(20,20,20)	(3,3,3)	**633**	**1.0338**	**0.1677**	**2559**	**259074**
		[M2]	(20,20,20)	(4,1,4)	**868**	0.8836	0.1334	3511	253517
			(20,20,20)	(5,0,4)	820	0.8716	**0.1245**	3511	230066
			(20,20,20)	(4,0,5)	820	**0.8469**	**0.1245**	4522	271412
			(20,20,20)	(7,1,1)	484	1.4215	0.3019	**1000**	129479
			(20,20,20)	(8,1,0)	193	2.2889	0.6723	1185	**88132**
		[M3]	(4,53,3)	(4,2,3)	**1937**	0.3745	0.1533	29659	235622
			(5,51,4)	(5,0,4)	1080	0.3419	**0.1234**	33703	230066
			(4,51,5)	(4,0,5)	1080	0.3333	**0.1234**	36939	271412
			(4,52,4)	(4,1,4)	1912	**0.3290**	0.1350	33703	253517
			(40,20,0)	(8,1,0)	185	2.3499	0.5848	**527**	88132
			(59,1,0)	(8,1,0)	182	1.6026	0.7731	10213	**88132**

Since the expected Fisher information matrix depends on the underlying lifetime distribution of the components, consequently the optimal experimental scheme also depends on the unknown underlying lifetime distribution. For this reason, it is desired to have an experimental design that is not sensitive to the model specification. It would then assure the practitioners that misspecification of the assumed lifetime distribution would not result in unacceptable variation in the results of the experiment. For this reason, in Table 2.12, we presented the optimal experimental plans for different optimality criteria under different settings when lognormal distribution is assumed to be the lifetime distribution of the joints. To facilitate a sensitivity analysis on the model misspecification, we define the relative efficiency (RE) as follows. Suppose O_A is the value of the objective function corresponding to optimality criteria **[C*]** (where $* = 1, 2, 3, 4, 5$) based on the optimal plan obtained under the assumed model, and O_T is the value of the objective function corresponding to optimality criteria **[C*]** based on the optimal plan obtained under the true model. The RE of the optimal experimental scheme under the assumed model relative to the optimal experimental scheme under the true model is

$$\text{RE}_{[C*]}(Assumed, True) = \frac{O_A}{O_T}.$$

Under the optimal criterion [**C1**], the values of RE will be larger than or equal to 1. Under the optimality criteria [**C2**]–[**C5**], the values of RE will be smaller than or equal to 1. The REs when the assumed model is Weibull and the true model is lognormal under setting [**M2**] are presented in Table 2.13, which provide the information about the deviation in the values of the objective functions if we use the optimal experimental scheme obtained under the assumed Weibull distribution when the true underlying lifetime distribution of the joints is lognormal. From the results in Table 2.13, except for optimal criterion [**C4**], misspecifying the lifetime distribution as Weibull when the true underlying distribution is lognormal does not affect the efficiency of the optimal experimental plans. In fact, the optimal experimental plans under the Weibull and the lognormal distributions are the same in most of the settings considered here.

Table 2.12: Optimal allocations for n-component series systems with lognormal distributed components when $N = 180$, $K = 3$, $(y_1, y_2, y_3) = (-1.12, 0.33, 0.79)$ by fixing $R = 15, 12,$ and 9 with $n = 12, 10, 5$ and 3

R	n	Setting	(m_1, m_2, m_3)	(r_1, r_2, r_3)	**[C1]** $\lvert\mathbf{I}\rvert$	**[C2]** $\mathrm{tr}(\mathbf{I}^{-1})$	**[C3]** $var(\hat{\beta}_1)$	**[C4]** $E(TTE)$	**[C5]** $E(TTT)$
15	12	**[M1-M2]**	(5,5,5)	(5,5,5)	**48088**	**0.1545**	**0.0298**	**18720**	**1181152**
		[M3]	(7,1,7)	(7,1,7)	**62901**	**0.1479**	0.0228	20343	1150699
			(8,0,7)	(8,0,7)	49947	0.1900	**0.0208**	20343	1111600
			(7,0,8)	(7,0,8)	49947	0.1853	**0.0208**	20987	1369048
			(12,3,0)	(12,3,0)	3594	0.9702	0.0633	**10265**	**557811**
	10	**[M1]**	(6,6,6)	(5,5,5)	47345	0.1629	0.0290	15901	1196040
		[M2]	(6,6,6)	(6,3,6)	**53441**	**0.1511**	**0.0273**	22075	1175813
			(6,6,6)	(6,5,4)	48433	0.1613	0.0290	**12515**	1137450
			(6,6,6)	(6,6,3)	47734	0.1570	0.0306	13936	**1099837**
		[M3]	(8,3,7)	(7,1,7)	**61401**	0.1477	0.0233	22946	1125703
			(9,0,9)	(7,0,8)	27546	0.3347	**0.0190**	18352	1346640
			(9,0,9)	(8,0,7)	27546	0.3380	**0.0190**	15176	1292532
			(7,3,8)	(7,0,8)	57303	**0.1412**	0.0234	23702	1105259
			(14,4,0)	(14,1,0)	2210	1.3146	0.1748	**3981**	469090
			(14,1,3)	(14,1,0)	1375	1.0146	0.1641	7726	**389384**
	5	**[M1]**	(12,12,12)	(5,5,5)	**46391**	**0.1792**	**0.0276**	**13063**	**1226938**
		[M2]	(12,12,12)	(7,1,7)	**57153**	0.1501	0.0242	17176	1143462
			(12,12,12)	(8,0,7)	46879	0.1518	**0.0240**	17176	953726
			(12,12,12)	(7,0,8)	46879	0.1499	**0.0240**	19661	1006638
			(12,12,12)	(5,1,9)	54887	**0.1474**	0.0261	22673	1246102
			(12,12,12)	(11,3,1)	35813	0.2039	0.0439	**5941**	809904
			(12,12,12)	(12,3,0)	21405	0.2294	0.0621	6613	**503778**
		[M3]	(17,2,17)	(7,2,6)	**61750**	0.1667	0.0217	15936	1163835
			(19,3,14)	(1,0,14)	2758	3.1160	**0.0159**	46463	1571340
			(14,3,19)	(14,0,1)	2758	3.1102	**0.0159**	18503	1951930
			(7,21,8)	(7,0,8)	32077	**0.1279**	0.0315	39995	846371
			(21,15,0)	(14,1,0)	8827	0.5135	0.1184	**3035**	467656
			(14,1,21)	(14,1,0)	4624	0.5429	0.2183	11834	**253310**
	3	**[M1]**	(20,20,20)	(5,5,5)	**46165**	**0.1847**	**0.0272**	**12440**	**1237644**
		[M2]	(20,20,20)	(7,1,7)	**56827**	0.1565	0.0236	15601	1155285
			(20,20,20)	(7,0,8)	46486	0.1581	**0.0234**	17251	1020698
			(20,20,20)	(8,0,7)	46486	0.1600	**0.0234**	15601	965880
			(20,20,20)	(6,1,8)	56256	**0.1544**	0.0240	17251	1209718
			(20,20,20)	(14,1,0)	9943	0.3964	0.1225	**4478**	**385037**
		[M3]	(28,4,28)	(6,4,5)	**63933**	0.1509	0.0236	31559	1164100
			(22,9,29)	(13,1,1)	6593	1.4205	**0.0162**	18562	1924941
			(7,45,8)	(7,0,8)	19887	**0.1279**	0.0402	63647	737328
			(31,28,1)	(14,1,0)	8442	0.5320	0.1225	**2904**	471229
			(14,1,45)	(14,1,0)	2866	0.6580	0.2788	17185	**220678**

Table 2.12: (Continue)

R	n	Setting	(m_1, m_2, m_3)	(r_1, r_2, r_3)	[C1] $\|\mathbf{I}\|$	[C2] $\mathrm{tr}(\mathbf{I}^{-1})$	[C3] $var(\hat{\beta}_1)$	[C4] $E(TTE)$	[C5] $E(TTT)$
12	12	[M1]	(5,5,5)	(4,4,4)	**28525**	**0.2146**	**0.0332**	13336	**1066190**
			(5,5,5)	(5,2,5)	**33435**	**0.1923**	**0.0307**	18720	1037157
			(5,5,5)	(5,4,3)	29271	0.2118	0.0335	**10244**	1001541
			(5,5,5)	(5,5,2)	28475	0.2028	0.0361	11818	**953052**
		[M3]	(7,1,7)	(6,0,6)	**37335**	0.2012	0.0249	15135	995093
			(7,0,8)	(6,0,6)	26816	0.2916	**0.0228**	13041	1230957
			(8,0,7)	(6,0,6)	26816	0.2969	**0.0228**	15135	1030020
			(6,3,6)	(6,0,6)	33899	**0.1837**	0.0271	19600	914479
			(11,4,0)	(11,1,0)	2357	1.1810	0.1570	**3633**	439306
			(11,1,3)	(11,1,0)	1489	0.9334	0.1546	6987	**348771**
	10	[M1]	(6,6,6)	(4,4,4)	28302	0.2218	0.0327	12515	1075520
		[M2]	(6,6,6)	(5,1,6)	**35053**	**0.1814**	**0.0294**	**22075**	**1032379**
			(6,6,6)	(6,0,6)	**29468**	**0.1769**	**0.0291**	22075	842588
			(6,6,6)	(6,4,2)	28370	0.2177	0.0359	**7900**	927093
			(6,6,6)	(6,6,0)	16983	0.2054	0.0505	13936	**571983**
		[M3]	(8,2,8)	(6,1,5)	**36224**	0.2156	0.0257	12095	1017607
			(9,0,9)	(6,0,6)	9872	0.7804	**0.0210**	12912	1205135
			(5,7,6)	(5,1,6)	33835	**0.1743**	0.0313	22075	1042741
			(12,6,0)	(11,1,0)	2482	1.2006	0.1665	**3316**	447678
			(11,1,6)	(11,1,0)	1579	0.8003	0.1651	7726	**316352**
	5	[M1]	(12,12,12)	(4,4,4)	**27934**	**0.2373**	**0.0318**	**11223**	**1095931**
		[M2]	(12,12,12)	(5,1,6)	**34182**	0.2014	0.0278	15018	1058994
			(12,12,12)	(6,0,6)	28543	0.2032	**0.0270**	15018	870191
			(12,12,12)	(4,1,7)	32812	**0.2001**	0.0298	17176	1112841
			(12,12,12)	(10,2,0)	11312	0.3250	0.0799	**4743**	426156
			(12,12,12)	(11,1,0)	7236	0.4290	0.1241	4812	**356346**
		[M3]	(17,2,17)	(6,2,4)	**37516**	0.2135	0.0259	15936	992151
			(10,8,18)	(10,1,1)	1518	4.5221	**0.0203**	18539	1964138
			(6,24,6)	(6,0,6)	16496	**0.1604**	0.0391	36776	645235
			(17,17,2)	(11,1,0)	8692	0.4588	0.1126	**2891**	428499
			(11,1,24)	(11,1,0)	2906	0.5642	0.2222	11834	**211718**
	3	[M1]	(20,20,20)	(4,4,4)	**27834**	**0.2427**	**0.0314**	**10867**	**1103235**
		[M2]	(20,20,20)	(5,1,6)	**33988**	0.2079	0.0273	14007	1067984
			(20,20,20)	(6,0,6)	28370	0.2116	**0.0265**	14007	879490
			(20,20,20)	(4,1,7)	32494	**0.2072**	0.0293	15601	1123746
			(20,20,20)	(11,1,0)	6806	0.4563	0.1244	**3380**	**358944**
		[M3]	(28,4,28)	(5,4,3)	**38929**	0.1812	0.0292	31559	972695
			(31,17,12)	(1,1,10)	1575	4.0862	**0.0197**	39269	1284788
			(6,47,7)	(5,0,7)	10921	**0.1599**	0.0482	60907	634143
			(24,28,8)	(11,1,0)	7924	0.4716	0.1176	**2904**	416429
			(11,1,48)	(11,1,0)	1801	0.6764	0.2839	17185	**184447**
9	12	[M1]	(5,5,5)	(3,3,3)	**14602**	**0.3166**	**0.0394**	**10244**	**934131**
		[M2]	(5,5,5)	(4,1,4)	**17801**	0.2735	0.0349	13336	886861
			(5,5,5)	(5,0,4)	15283	0.2612	**0.0346**	13336	697494
			(5,5,5)	(4,0,5)	15283	**0.2562**	**0.0346**	18720	752535
			(5,5,5)	(5,3,1)	13673	0.3100	0.0473	**6467**	744834
			(5,5,5)	(5,4,0)	8808	0.3003	0.0600	8419	**476560**
		[M3]	(7,1,7)	(5,0,4)	**18876**	0.3086	0.0296	10157	840023
			(7,1,7)	(4,0,5)	**18876**	0.3033	0.0296	12278	905079
			(8,0,7)	(4,0,5)	12485	0.4905	**0.0264**	12278	938546
			(7,0,8)	(5,0,4)	12485	0.4853	**0.0264**	9308	1066030
			(4,7,4)	(4,1,4)	17257	**0.2408**	0.0389	17645	853695
			(9,6,0)	(8,1,0)	2800	0.9542	0.1383	**3084**	415152
			(8,1,6)	(8,1,0)	1656	0.7144	0.1574	6987	**266201**
	10	[M1]	(6,6,6)	(3,3,3)	**14544**	**0.3228**	**0.0391**	**9962**	**939245**
		[M2]	(6,6,6)	(4,1,4)	**17681**	0.2818	0.0343	12515	892881
			(6,6,6)	(5,0,4)	15068	0.2741	**0.0339**	12515	703976
			(6,6,6)	(4,0,5)	15068	0.2689	**0.0339**	15901	761678
			(6,6,6)	(3,0,6)	14402	**0.2628**	0.0380	22075	812075
			(6,6,6)	(6,2,1)	13124	0.3317	0.0520	**5252**	699575
			(6,6,6)	(6,3,0)	8300	0.3322	0.0659	6289	**441903**
		[M3]	(8,2,8)	(4,1,4)	**18334**	0.3167	0.0309	10172	918954
			(11,0,7)	(4,0,5)	5382	1.0649	**0.0258**	13550	894305
			(7,0,11)	(5,0,4)	5382	1.0559	**0.0258**	8421	1222202
			(4,9,5)	(4,0,5)	12277	**0.2355**	0.0393	21047	684082
			(9,8,1)	(8,1,0)	2840	0.9186	0.1467	**2960**	411674
			(8,1,9)	(8,1,0)	2831	0.5228	0.1703	7726	**222151**
	5	[M1]	(12,12,12)	(3,3,3)	**14437**	**0.3365**	**0.0384**	**9412**	**950914**
		[M2]	(12,12,12)	(4,1,4)	**17472**	0.2994	0.0333	11223	906050
			(12,12,12)	(5,0,4)	14780	0.2994	**0.0326**	11223	718019
			(12,12,12)	(4,0,5)	14780	0.2940	**0.0326**	13063	780742
			(12,12,12)	(2,6,1)	11279	**0.2914**	0.0560	9481	858693
			(12,12,12)	(8,1,0)	4318	0.5330	0.1263	**3316**	**329957**
		[M3]	(17,2,17)	(4,2,3)	**19986**	0.2808	0.0323	15936	874647
			(22,0,14)	(4,0,5)	5767	0.9779	**0.0267**	11816	891555
			(14,0,22)	(5,0,4)	5767	0.9840	**0.0267**	8093	1098548
			(4,27,5)	(4,0,5)	6873	**0.2133**	0.0528	34776	523861
			(13,17,6)	(8,1,0)	4857	0.5546	0.1234	**2891**	381650
			(8,1,27)	(8,1,0)	1585	0.6001	0.2292	11834	**170124**
	3	[M1]	(20,20,20)	(3,3,3)	**14405**	**0.3415**	**0.0382**	**9235**	**955212**
		[M2]	(20,20,20)	(4,1,4)	**17412**	0.3055	0.0330	10867	910763
			(20,20,20)	(5,0,4)	14709	0.3080	**0.0322**	10867	723014
			(20,20,20)	(4,0,5)	14709	0.3026	**0.0322**	12440	787366
			(20,20,20)	(2,6,1)	11095	**0.2991**	0.0558	8842	863956
			(20,20,20)	(8,1,0)	4185	0.5536	0.1259	**3316**	**331813**
		[M3]	(28,4,28)	(3,4,2)	**19704**	0.2239	0.0402	31559	835090
			(35,0,25)	(4,0,5)	5776	0.9828	**0.0270**	10940	913131
			(25,0,35)	(5,0,4)	5776	0.9890	**0.0270**	8159	1030463
			(26,6,28)	(2,6,1)	14109	**0.2094**	0.0578	36494	771256
			(18,28,14)	(8,1,0)	4421	0.5679	0.1306	**2904**	370922
			(8,1,51)	(8,1,0)	982	0.7074	0.2928	17185	**148214**

Table 2.13: Relative efficiency of the optimal experimental plans obtained when the assumed distribution is Weibull and the true distribution is lognormal for n-component series systems with $N = 180$, $K = 3$, $(y_1, y_2, y_3) = (-1.12, 0.33, 0.79)$ by fixing $R = 15, 12, 9$ and 42 with $n = 12, 10, 5$ and 3 under setting **[M2]** (referring to Table 2.11)

R	n	Optimal Criteria	Under the assumed model (Weibull) (m_1, m_2, m_3)	(r_1, r_2, r_3)	O_A	Under the assumed model (Lognormal) (m_1, m_2, m_3)	(r_1, r_2, r_3)	O_T	RE
15	12	C1	(5,5,5)	(5,5,5)	48088	(5,5,5)	(5,5,5)	48088	1.0000
		C2	(5,5,5)	(5,5,5)	0.1545	(5,5,5)	(5,5,5)	0.1545	1.0000
		C3	(5,5,5)	(5,5,5)	0.0298	(5,5,5)	(5,5,5)	0.0298	1.0000
		C4	(5,5,5)	(5,5,5)	18720	(5,5,5)	(5,5,5)	18720	1.0000
		C5	(5,5,5)	(5,5,5)	1181152	(5,5,5)	(5,5,5)	1181152	1.0000
	10	C1	(6,6,6)	(6,3,6)	53441	(6,6,6)	(6,3,6)	53441	1.0000
		C2	(6,6,6)	(6,3,6)	0.1511	(6,6,6)	(6,3,6)	0.1511	1.0000
		C3	(6,6,6)	(6,3,6)	0.0273	(6,6,6)	(6,3,6)	0.0273	1.0000
		C4	(6,6,6)	(6,5,4)	12515	(6,6,6)	(6,5,4)	12515	1.0000
		C5	(6,6,6)	(6,6,3)	1099837	(6,6,6)	(6,6,3)	1099837	1.0000
	5	C1	(12,12,12)	(7,1,7)	57153	(12,12,12)	(7,1,7)	57153	1.0000
		C2	(12,12,12)	(6,0,9)	0.1490	(12,12,12)	(5,1,9)	0.1474	1.0110
		C3	(12,12,12)	(7,0,8)	0.0240	(12,12,12)	(7,0,8)	0.0240	1.0000
		C4	(12,12,12)	(10,3,2)	7514	(12,12,12)	(11,3,1)	5941	1.2646
		C5	(12,12,12)	(12,3,0)	503778	(12,12,12)	(12,3,0)	503778	1.0000
	3	C1	(20,20,20)	(7,1,7)	56827	(20,20,20)	(7,1,7)	56827	1.0000
		C2	(20,20,20)	(7,0,8)	0.1581	(20,20,20)	(6,1,8)	0.1544	1.0239
		C3	(20,20,20)	(7,0,8)	0.0234	(20,20,20)	(7,0,8)	0.0234	1.0000
		C4	(20,20,20)	(10,3,2)	7452	(20,20,20)	(14,1,0)	4478	1.6640
		C5	(20,20,20)	(14,1,0)	385037	(20,20,20)	(14,1,0)	385037	1.0000
12	12	C1	(5,5,5)	(5,2,5)	33435	(5,5,5)	(5,2,5)	33435	1.0000
		C2	(5,5,5)	(5,2,5)	0.1923	(5,5,5)	(5,2,5)	0.1923	1.0000
		C3	(5,5,5)	(5,2,5)	0.0307	(5,5,5)	(5,2,5)	0.0307	1.0000
		C4	(5,5,5)	(5,4,3)	10244	(5,5,5)	(5,4,3)	10244	1.0000
		C5	(5,5,5)	(5,5,2)	953052	(5,5,5)	(5,5,2)	953052	1.0000
	10	C1	(6,6,6)	(5,1,6)	35053	(6,6,6)	(5,1,6)	35053	1.0000
		C2	(6,6,6)	(6,0,6)	0.1769	(6,6,6)	(6,0,6)	0.1769	1.0000
		C3	(6,6,6)	(6,0,6)	0.0291	(6,6,6)	(6,0,6)	0.0291	1.0000
		C4	(6,6,6)	(6,4,2)	7900	(6,6,6)	(6,4,2)	7900	1.0000
		C5	(6,6,6)	(6,6,0)	571983	(6,6,6)	(6,6,0)	571983	1.0000
	5	C1	(12,12,12)	(5,1,6)	34182	(12,12,12)	(5,1,6)	34182	1.0000
		C2	(12,12,12)	(5,0,7)	0.2014	(12,12,12)	(4,1,7)	0.2001	1.0066
		C3	(12,12,12)	(6,0,6)	0.0270	(12,12,12)	(6,0,6)	0.0270	1.0000
		C4	(12,12,12)	(8,3,1)	5941	(12,12,12)	(10,2,0)	4743	1.2526
		C5	(12,12,12)	(11,1,0)	356346	(12,12,12)	(11,1,0)	356346	1.0000
	3	C1	(20,20,20)	(5,1,6)	33988	(20,20,20)	(5,1,6)	33988	1.0000
		C2	(20,20,20)	(5,0,7)	0.2102	(20,20,20)	(4,1,7)	0.2072	1.0144
		C3	(20,20,20)	(6,0,6)	0.0265	(20,20,20)	(6,0,6)	0.0265	1.0000
		C4	(20,20,20)	(9,2,1)	5252	(20,20,20)	(11,1,0)	3380	1.5538
		C5	(20,20,20)	(11,1,0)	358944	(20,20,20)	(11,1,0)	358944	1.0000
9	12	C1	(5,5,5)	(5,0,4)	15283	(5,5,5)	(4,1,4)	17801	0.8585
		C2	(5,5,5)	(4,0,5)	0.2562	(5,5,5)	(4,0,5)	0.2562	1.0000
		C3	(5,5,5)	(4,0,5)	0.0346	(5,5,5)	(5,0,4)	0.0346	1.0000
		C4	(5,5,5)	(5,2,2)	7764	(5,5,5)	(5,3,1)	6467	1.2006
		C5	(5,5,5)	(5,4,0)	476560	(5,5,5)	(5,4,0)	476560	1.0000
	10	C1	(6,6,6)	(4,1,4)	17681	(6,6,6)	(4,1,4)	17681	1.0000
		C2	(6,6,6)	(3,0,6)	0.2628	(6,6,6)	(3,0,6)	0.2628	1.0000
		C3	(6,6,6)	(4,0,5)	0.0339	(6,6,6)	(4,0,5)	0.0339	1.0000
		C4	(6,6,6)	(6,2,1)	5252	(6,6,6)	(6,2,1)	5252	1.0000
		C5	(6,6,6)	(6,3,0)	441903	(6,6,6)	(6,3,0)	441903	1.0000
	5	C1	(12,12,12)	(4,1,4)	17472	(12,12,12)	(4,1,4)	17472	1.0000
		C2	(12,12,12)	(4,0,5)	0.2940	(12,12,12)	(2,6,1)	0.2914	1.0091
		C3	(12,12,12)	(5,0,4)	0.0326	(12,12,12)	(4,0,5)	0.0326	1.0000
		C4	(12,12,12)	(6,2,1)	5252	(12,12,12)	(8,1,0)	3316	1.5841
		C5	(12,12,12)	(8,1,0)	329957	(12,12,12)	(8,1,0)	329957	1.0000
	3	C1	(20,20,20)	(4,1,4)	17412	(20,20,20)	(4,1,4)	17412	1.0000
		C2	(20,20,20)	(4,0,5)	0.3026	(20,20,20)	(2,6,1)	0.2991	1.0115
		C3	(20,20,20)	(5,0,4)	0.0322	(20,20,20)	(4,0,5)	0.0322	1.0000
		C4	(20,20,20)	(7,1,1)	5252	(20,20,20)	(8,1,0)	3316	1.5841
		C5	(20,20,20)	(8,1,0)	331813	(20,20,20)	(8,1,0)	331813	1.0000

2.7. Concluding Remarks

In this article, we discuss the optimal experimental planning for multi-level stress experiments with Type-II censored when the experimental units can be put into coherent systems for the life testing experiments. The component lifetime distributions are assumed to be in the log-location-scale family of distributions, a general class of distributions that includes some commonly used lifetime distributions, such as the Weibull and the lognormal distributions. The formulations required for optimal experimental planning are presented in terms of system signatures for different optimality criteria. Motivated by the furniture joints experiment, the proposed methodologies are illustrated by the special case that the experimental units are put into series systems for the life testing experiment. The numerical results for two- and four-stress levels experimental planning are provided and these results are verified by a Monte Carlo simulation study.

We have shown that considering the use of systems formed by the test units in a multi-level stress experiment gives another level of flexibility in planning an experiment with Type-II censoring and gains in efficiency in some situations. To demonstrate the usefulness of the proposed methodologies, the proposed methods are applied to plan a reliability test of furniture joints, and a sensitivity analysis for misspecification of the underlying lifetime distribution is provided.

Overall, although using systems in multi-level constant-stress experiments with Type-II censoring does not always give advantages over using individual components when using systems is superior to using individual components, it always gains efficiency in estimation and gives shorter expected total time for all the components spent on the experiment. For future studies, considering some different optimality criteria that take into account the time of the experiment, efficiency in the estimation of the model parameters, and the cost of failing a component simultaneously, will be of interest. On the other hand, in addition to series systems, the formulations presented in Section 2.2 can be applied to experimental planning for different kinds of coherent systems. It will be an interesting research direction to compare the performance of optimal experimental planning based on different coherent systems. We study these research topics and hope to report the results in a future paper.

Chapter 3

Optimal Experimental Planning for Progressively Type-II Censored Experiments based on Coherent Systems

3.1. Introduction

As mentioned in Chapter 1, the idea of optimal experimental planning of life testing procedures involving systems has been discussed in the literature. This chapter considers the progressively Type-II right-censored life testing experiment, which can be described as follows. Consider a life testing experiment in which m units are placed on the test simultaneously. At the time of the first failure, R_1 experimental units are randomly removed from the remaining $(m-1)$ surviving units. At the second failure, R_2 experimental units are randomly removed from the remaining $(m-R_1-1)$ surviving units. The life testing experiment continues similarly until the r-th failure, at which time, all remaining $(m-R_1-R_2-\ldots-R_{r-1}-1)$ surviving units are removed. The progressive censoring scheme $(R_1, R_2, \ldots, R_r)$ are fixed prior to the study. For progressively Type-II censored life testing experiments, Wu and Kuş (2009), Hermanns and Cramer (2017) and Hermanns and Cramer (2018) used the series systems, the parallel systems, and the l-out-of-n systems, respectively, for the censored life testing experiments and developed the related statistical inference. The problem of optimal progressive censoring has been studied in the literature after people realized that progressive Type-II censoring is a versatile censoring scheme in which there is flexibility in the choice of $(R_1, R_2, \ldots, R_r)$. For example, Balakrishnan and Aggarwala (2000) defined the objective functions to find the optimal progressive censoring plans based on the variances and covariance of best linear unbiased estimators (BLUEs) of the parameters in location-scale distribution, and implemented analytical and computational methods to calculate these functions for the location-scale family of distributions where moments of the usual order statistics are available. Ng et al. (2004) studied the optimal progressive censoring plans when the experimental units follow the Weibull distribution. Burkschat et al. (2006) determined the optimal progressive Type-II censoring schemes for exponential, uniform, and Pareto distributions. They used the variance of the BLUE for the one-parameter models. For the two-parameter models, two optimality criteria, minimizing the trace and the determinant of the variance-covariance matrix of the BLUEs, are utilized. Burkschat et al. (2007) obtained optimal censoring schemes by applying the p-criteria from experimental design to the variance-covariance matrix of the BLUEs. They also discussed the monotonicity properties of the trace and determinant of the variance-covariance matrix of the maximum likelihood estimators (MLEs) with respect to the sample size and the initial number of experimental units. Dahmen et al. (2012) established a representation of the Fisher information matrix of the MLEs in terms of the hazard rate of the baseline distribution and identified the A- and D-optimal progressively Type-II censoring plans when the experimental units follow various distributions. Salemi et al. (2019) explored the relationship between the missing information matrix and the Fisher information matrix, and presented a straightforward expression and method for determining the optimal progressive censoring plan from a specific class of one-step censoring plans under the A-optimality and D-optimality criteria for models with multiple parameters. In Bhattacharya and Balakrishnan (2023), the authors proposed a probabilistic approach to determine the optimal progressive censoring schemes by defining a probability structure on the set of feasible solutions, allowing for the computation of an updated solution. There have also been studies involving optimal censoring schemes based on non-parametric statistical inference. For instance, Balakrishnan and Han (2007) investigated the optimal censoring schemes for non-parametric confidence intervals of population quantiles in the context of progressive Type-II right censoring. The optimization process for optimal censoring schemes is independent of the observed sample values.

Although statistical inference for system-level progressively censored data and optimal progressive censoring schemes have been studied in the literature, a comprehensive study of the optimal experimental planning for progressively Type-II censored experiments based on coherent systems has not been done to our knowledge. Therefore, this chapter aims to develop the algorithm to obtain the optimal progressive Type-II censoring schemes when the components can be put into coherent systems for the life testing experiment.

In this chapter, we study the problem of optimal experimental planning for progressively Type-II censored experiments when coherent systems are used. We compute the expected Fisher information and the asymptotic variance-covariance matrix of the maximum likelihood estimates (MLEs) based on a progressively Type-II censored sample with coherent systems when the component lifetimes follow a statistical distribution in the log-location-scale family of distributions. Based on different optimality criteria, we use the expected Fisher information and the asymptotic variance–covariance matrix to determine the optimal progressive censoring plans. Optimal progressive censoring schemes based on different coherent systems are compared.

This chapter is organized as follows. In Section 3.2, we describe the model for progressively Type-II censored experiments with coherent systems and different optimality criteria. Specifically, we review the notation of the system signature and the location-scale family of distributions. Then, in Section 3.3, MLEs of the model parameters and the corresponding Fisher information matrix for the location-scale family of distributions are derived. Different optimality criteria, including some commonly used optimality criteria including the D-optimality, the A-optimality, the V-optimality, and a comprehensive criterion, and different settings for optimal experimental planning are discussed in Section 3.4. In Section 3.5, we consider multiple cases with various types of systems being used in the experiments with progressively censored samples. In Section 3.6, the computational formulas for lognormal and Weibull distributed components are used to illustrate the methodology developed in this chapter. Numerical illustrations with n up to 4 or 5 are used to demonstrate the advantages and disadvantages. In Section 3.7, a practical example of insulating fluids testing is used to illustrate the methodologies for planning a future progressively Type-II censored experiment. Finally, some concluding remarks and discussions about future research directions are provided in Section 3.8.

3.2. Progressively Type-II Censored Experiments with Coherent Systems

Suppose we have $N = m \times n$ components available for a progressively Type-II censored experiment, and these components can be placed as m coherent n-component systems with system signature $\boldsymbol{s} = (s_1, s_2, \ldots, s_n)$ for the life testing experiment, we plan to fail $r \leq m$ systems (i.e., r is the effective sample size and m is the total sample size in terms of systems) with progressive censoring scheme $\boldsymbol{R} = (R_1, R_2, \ldots, R_r)$. Following the notations defined in Chapter 1, we use $X_1, X_2, \ldots, X_n$ to denote the independent and identically distributed (i.i.d.) component lifetimes, and $T_1, T_2, \ldots, T_m$ to denote the m system lifetimes. For the special case when $n = 1$, the life testing experiment is done at the component level in which $m = N$ components are placed on the life test, and the test is terminated as soon as the r-th component failure with progressive censoring scheme $\boldsymbol{R} = (R_1, R_2, \ldots, R_r)$ is observed.

We further assume the component lifetimes follow a log-location-scale distribution; hence, the log-transformed component lifetimes follow a location-scale family distribution with location parameter $\mu \in (-\infty, \infty)$ and scale parameter $\sigma \in (0, \infty)$. Suppose $f_U(\cdot; \boldsymbol{\theta})$, $F_U(\cdot; \boldsymbol{\theta})$ and $\bar{F}_U(\cdot; \boldsymbol{\theta})$ are the probability density function (PDF), cumulative distribution function (CDF) and survival function (SF) of the log-lifetimes of components in the systems (i.e.,

$U = \ln X$) with parameter vector $\boldsymbol{\theta} = (\mu, \sigma)$, then we have

$$
\begin{aligned}
f_U(u; \mu, \sigma) &= \frac{1}{\sigma} f^* \left(\frac{u - \mu}{\sigma} \right), \\
F_U(u; \mu, \sigma) &= F^* \left(\frac{u - \mu}{\sigma} \right), \\
\bar{F}_U(u; \mu, \sigma) &= 1 - F^* \left(\frac{u - \mu}{\sigma} \right),
\end{aligned}
$$

where $-\infty < \mu < \infty$ is the location parameter and $\sigma > 0$ is the scale parameter, $f^*(\cdot)$ and $F^*(\cdot)$ are the PDF and CDF of the standardized distribution (i.e., $\mu = 0$ and $\sigma = 1$) of the corresponding distribution in the location-scale family. The PDF and SF of the log-transformed system lifetimes (i.e., $V = \ln T$) can be expressed in terms of the PDF, CDF, and SF of the log-transformed component lifetimes as

$$
\begin{aligned}
f_V(v; \boldsymbol{\theta}) &= \sum_{i=1}^{n} s_i \binom{n}{i} i f_U(v; \boldsymbol{\theta}) [F_U(v; \boldsymbol{\theta})]^{i-1} \left[\bar{F}_U(v; \boldsymbol{\theta}) \right]^{i-1} \\
&= \frac{1}{\sigma} f^* \left(\frac{v - \mu}{\sigma} \right) \sum_{i=1}^{n} s_i \binom{n}{i} i \left[F^* \left(\frac{v - \mu}{\sigma} \right) \right]^{i-1} \left[\bar{F}^* \left(\frac{v - \mu}{\sigma} \right) \right]^{n-i} \\
&= \frac{1}{\sigma} f^*(z) \sum_{i=1}^{n} s_i \binom{n}{i} i [F^*(z)]^{i-1} \left[\bar{F}^*(z) \right]^{n-i}, \qquad (3.1)
\end{aligned}
$$

and

$$
\begin{aligned}
\bar{F}_V(v; \boldsymbol{\theta}) &= \sum_{i=1}^{n} s_i \sum_{j=0}^{i-1} \binom{n}{j} [F_U(v; \boldsymbol{\theta})]^j \left[\bar{F}_U(v; \boldsymbol{\theta}) \right]^{n-j} \\
&= \sum_{i=1}^{n} s_i \sum_{j=0}^{i-1} \binom{n}{j} \left[F^* \left(\frac{v - \mu}{\sigma} \right) \right]^j \left[\bar{F}^* \left(\frac{v - \mu}{\sigma} \right) \right]^{n-j} \\
&= \sum_{i=1}^{n} s_i \sum_{j=0}^{i-1} \binom{n}{j} [F^*(z)]^j \left[\bar{F}^*(z) \right]^{n-j}, \qquad (3.2)
\end{aligned}
$$

where

$$
z = \frac{v - \mu}{\sigma}.
$$

3.3. Maximum Likelihood Estimation

3.3.1. Maximum likelihood estimation of model parameters

To obtain a progressively Type-II censored sample with the progressive censoring scheme $\boldsymbol{R} = (R_1, R_2, \ldots, R_r)$, R_j operating systems are randomly removed from the life test when the j-th failure is observed, $\ell = 1, 2, \ldots, r$. Thus, the pre-fixed censoring scheme $(R_1, \ldots, R_r)$ and the pre-fixed number of failures r satisfy the constraint $\sum\limits_{\ell=1}^{r} R_\ell + r = m$, and the log-transformed observed progressively Type-II censored sample with coherent systems is denoted as $v_{1:r:m} < v_{2:r:m} < \ldots < v_{r:r:m}$. The likelihood function and the log-likelihood function based on the observed system-level progressively Type-II censored sample can be expressed as

$$\begin{aligned} L(\boldsymbol{\theta}) &= m(m - R_1 - 1) \ldots (m - R_1 - R_2 - \ldots - R_{r-1} - r + 1) \\ &\quad \times \prod_{\ell=1}^{r} f_V(v_{\ell:r:m}; \boldsymbol{\theta}) \left[\bar{F}_V(v_{\ell:r:m}; \boldsymbol{\theta})\right]^{R_\ell}, \quad v_{1:r:m} < v_{2:r:m} < \ldots < v_{r:r:m}, \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} l(\boldsymbol{\theta}) &= C + \sum_{\ell=1}^{r} \ln f_V(v_{\ell:r:m}; \boldsymbol{\theta}) \\ &\quad + \sum_{\ell=1}^{r} R_\ell \ln \left[\bar{F}_V(v_{\ell:r:m}; \boldsymbol{\theta})\right], \quad v_{1:r:m} < v_{2:r:m} < \ldots < v_{r:r:m}, \end{aligned} \tag{3.4}$$

respectively, where $C = \ln[m(m - R_1 - 1) \ldots (m - R_1 - R_2 - \ldots - R_{r-1} - r + 1)]$ is a constant that does not depend on the parameter vector $\boldsymbol{\theta}$. We can obtain the likelihood equations by taking the first partial derivatives of the log-likelihood function and setting them to zero, i.e.,

$$\begin{aligned}\frac{\partial l(\boldsymbol{\theta})}{\partial \mu} &= \sum_{\ell=1}^{r} \frac{1}{f_V(v_{\ell:r:m};\boldsymbol{\theta})} \frac{\partial f_V(v_{\ell:r:m};\boldsymbol{\theta})}{\partial \mu} \\ &\quad + \sum_{\ell=1}^{r} \frac{R_\ell}{\bar{F}_V(v_{\ell:r:m};\boldsymbol{\theta})} \frac{\partial \bar{F}_V(v_{\ell:r:m};\boldsymbol{\theta})}{\partial \mu} = 0, \qquad (3.5)\\ \frac{\partial l(\boldsymbol{\theta})}{\partial \sigma} &= \sum_{\ell=1}^{r} \frac{1}{f_V(v_{\ell:r:m};\boldsymbol{\theta})} \frac{\partial f_V(v_{\ell:r:m};\boldsymbol{\theta})}{\partial \sigma} \\ &\quad + \sum_{\ell=1}^{r} \frac{R_\ell}{\bar{F}_V(v_{\ell:r:m};\boldsymbol{\theta})} \frac{\partial \bar{F}_V(v_{\ell:r:m};\boldsymbol{\theta})}{\partial \sigma} = 0. \qquad (3.6)\end{aligned}$$

The MLEs of the parameters μ and σ, denoted as $\hat{\mu}$ and $\hat{\sigma}$, respectively, can be obtained by solving Eqs. (3.5) and (3.6) simultaneously.

Following Eqs. (2.9)–(2.14), the first derivatives of the SF and the PDF of the log-transformed system lifetime with respect to the parameters μ and σ can be expressed in terms of the log-transformed component lifetime distributions as

$$\begin{aligned}\frac{\partial \bar{F}_V(v_{\ell:r:m};\boldsymbol{\theta})}{\partial \mu} &= \left[-\frac{1}{\sigma} f^*(z_{\ell:r:m})\right] \sum_{i=1}^{n} s_i \sum_{j}^{i-1} \binom{n}{j} \Big[j\,[F^*(z_{\ell:r:m})]^{j-1} \,[\bar{F}^*(z_{\ell:r:m})]^{n-j} \\ &\quad - [F^*(z_{\ell:r:m})]^{j}\,(n-j)[\bar{F}^*(z_{\ell:r:m})]^{n-j-1}\Big], \qquad (3.7)\\ \frac{\partial \bar{F}_V(v_{\ell:r:m};\boldsymbol{\theta})}{\partial \sigma} &= \left[-\frac{z_{\ell:r:m}}{\sigma} f^*(z_{\ell:r:m})\right] \sum_{i=1}^{n} s_i \sum_{j}^{i-1} \binom{n}{j} \Big[j\,[F^*(z_{\ell:r:m})]^{j-1} \,[\bar{F}^*(z_{\ell:r:m})]^{n-j} \\ &\quad - [F^*(z_{\ell:r:m})]^{j}\,(n-j)[\bar{F}^*(z_{\ell:r:m})]^{n-j-1}\Big], \qquad (3.8)\end{aligned}$$

and

$$
\begin{aligned}
\frac{\partial f_V(v_{\ell:r:m};\boldsymbol{\theta})}{\partial \mu} &= \frac{1}{\sigma}\frac{\partial f^*(z_{\ell:r:m})}{\partial \mu}\sum_{i=1}^{n} s_i\binom{n}{i} i[F^*(z_{\ell:r:m})]^{i-1}[\bar{F}^*(z_{\ell:r:m})]^{n-i} \\
&\quad -\frac{1}{\sigma^2}\left(f^*(z_{\ell:r:m})\right)^2\sum_{i=1}^{n} s_i\binom{n}{i} i\Big[(i-1)[F^*(z_{\ell:r:m})]^{i-2}[\bar{F}^*(z_{\ell:r:m})]^{n-i} \\
&\quad -[F^*(z_{\ell:r:m})]^{i-1}(n-i)[\bar{F}^*(z_{\ell:r:m})]^{n-i-1}\Big], \qquad (3.9)\\
\frac{\partial f_V(v_{\ell:r:m};\boldsymbol{\theta})}{\partial \sigma} &= \left[-\frac{1}{\sigma^2}f^*(z_{\ell:r:m}) + \frac{1}{\sigma}\frac{\partial f^*(z_{\ell:r:m})}{\partial \sigma}\right] \\
&\quad \times \sum_{i=1}^{k} s_i\binom{n}{i} i[F^*(z_{\ell:r:m})]^{i-1}[\bar{F}^*(z_{\ell:r:m})]^{n-i} \\
&\quad -\frac{z_{\ell:r:m}}{\sigma^2}\left(f^*(z_{\ell:r:m})\right)^2\sum_{i=1}^{n} s_i\binom{n}{i} i\Big[(i-1)[F^*(z_{\ell:r:m})]^{i-2}[\bar{F}^*(z_{\ell:r:m})]^{n-i} \\
&\quad -[F^*(z_{\ell:r:m})]^{i-1}(n-i)[\bar{F}^*(z_{\ell:r:m})]^{n-i-1}\Big]. \qquad (3.10)
\end{aligned}
$$

3.3.2. Fisher information and asymptotic variance-covariance matrices

The expected Fisher information matrix of the parameter vector $\boldsymbol{\theta} = (\mu, \sigma)$ can be expressed as

$$
\boldsymbol{I}(\mu,\sigma) = -\begin{bmatrix} E\left(\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \mu^2}\right) & E\left(\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \mu \partial \sigma}\right) \\ E\left(\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \sigma \partial \mu}\right) & E\left(\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \sigma^2}\right) \end{bmatrix}. \qquad (3.11)
$$

The second partial derivatives of the log-likelihood function involved in the Fisher information matrix in Eq. (3.11) can be expressed in terms of the PDF and CDF of the log-transformed component lifetimes as described in Section 3.3.1. Then, the asymptotic variance-covariance matrix of the MLEs of μ and σ can be obtained as the inverse of the expected Fisher information matrix as

$$\boldsymbol{V}(\mu,\sigma) \quad = \quad \boldsymbol{I}^{-1}(\mu,\sigma) = \begin{bmatrix} Var(\hat{\mu}) & Cov(\hat{\mu},\hat{\sigma}) \\ & Var(\hat{\sigma}) \end{bmatrix}. \tag{3.12}$$

For a particular standardized location-scale distribution, $f^*(\cdot)$, $F^*(\cdot)$, and $\bar{F}^*(\cdot)$ are specified and the normal equations in Eqs. (3.5) and (3.6), the expected Fisher information matrix in Eq. (3.11) and the asymptotic variance-covariance matrix in Eq. (3.12) can be obtained.

3.4. Experimental Planning and Optimality Criteria

3.4.1. Settings for optimal experimental planning

This subsection considers different settings and scenarios to plan the progressively Type-II censored experiments described in Section 3.1. Suppose there are N components available for a progressively Type-II censored experiment and assume that the components can be placed on a life test individually or systems formed by these components can be placed on a life test. We consider the case the system structures of the systems used in the progressively Type-II censored experiments are the same, i.e., the N components are put into $M = N/n$ n-component systems with system signature $\boldsymbol{s} = (s_1, \ldots, s_n)$ and these M systems are placed on a life test with progressive censoring scheme $\boldsymbol{R} = (R_1, R_2, \ldots, R_r)$ such that r failures are observed. In order to compare progressively Type-II censored experiments with different kinds of systems, the following two scenarios are considered:

Scenario I. Specifying the expected number of failed components in the failed systems: In this scenario, we consider fixing the expected number of failed components in the failed systems, denoted as

$$E(R) = \left[\sum_{i=1}^{n} i s_i\right] r.$$

For example, if we have $N = 20$ available for the experiment, we used ten 2-component parallel systems (i.e., $\boldsymbol{s} = (0, 1)$) for the progressively Type-II censored experiment with an effective sample size $r = 4$, then $E(R) = 2 \times 4 = 8$. Note that $E(R)$ is not the total number of failed components in the progressively censored experiment since the systems being censored can contain different numbers of failed components, except for the series systems. In the aforementioned example, with ten 2-component systems and $r = 4$, the six censored systems can contain 0 or 1 failed component.

Scenario II. Specifying the maximum number of failed components in both failed and censored systems: To take into account the component failures in the censored systems, we consider the maximum possible number of failed components in both failed and censored systems, denoted as $R_{\max}$. In the aforementioned example, with ten 2-component systems and $r = 4$, the four failed systems contain two failed components in each system, and the six censored systems can contain at most one failed component in each system. Therefore, the maximum possible number of failed components in both failed and censored systems is

$$R_{\max} = (2 \times 4) + (1 \times 6) = 14.$$

Note that this scenario considers the worst-case scenario.

When the values of $N = n \times m$ (i.e., the number of components available for the experiment) and $E(R)$ or $R_{\max}$ are fixed in advance according to the availability of resources, we aim to determine the optimal experimental plan with progressive Type-II censoring. In other words, we aim to determine the number of components in each system n, the system signature $\boldsymbol{s} = (s_1, s_2, \ldots, s_n)$, and the progressive censoring scheme $\boldsymbol{R} = (R_1, R_2, \ldots, R_r)$.

To obtain the optimal progressive censoring schemes, we consider the following ways to settings when N and r ($E(R)$ in Scenario I or $R_{\max}$ in Scenario II) are specified.

[M1] Fix $R_\ell = m - r$, $R_{\ell'} = 0$, for $\ell, \ell' = 1, 2, \ldots, r$ and $\ell' \neq \ell$, then we determine the optimal number of components in the system (denoted as $n^{(\ell)}$) and the system signature (denoted as $\boldsymbol{s}^{(\ell)}$) that optimize the specific objective function. The progressive censoring scheme that censored all $m - r$ systems at a certain observed failure is known as the progressive Type-II extremal censoring scheme (see, for example, Ng et al., 2017). In this setting, for each progressive Type-II extremal censoring scheme $(0, \ldots, 0, R_\ell = m - r, 0, \ldots, 0)$ for $\ell = 1, 2, \ldots, r$, we search for the optimal values of n and $\boldsymbol{s}$.

[M2] Fix $\sum_{\ell=1}^{r} R_\ell = m - r$, and then determine the values of R_ℓ (number of censored systems at each observed failure), the values of n (number of components in the system) and $\boldsymbol{s}$ (system structure) that optimize the specific objective function. In this setting, we either fix the expected number of failed components in those failed systems (i.e., Scenario I) or the maximum possible number of failed components in the experiment (i.e., Scenario II). We do not put any restriction on the censoring scheme $(R_1, R_2, \ldots, R_r)$.

3.4.2. Optimality criteria

Suppose we are interested in estimating the model parameters or function of the model parameters based on the settings presented in Section 3.4.1, or controlling the experimental time and cost, we plan the optimal progressively Type-II censored experiment by considering the following optimality criteria:

[C1] D-optimality: Maximizing the determinant of the expected Fisher information matrix, **I**, i.e., maximizing the differential Shannon information contained in the MLEs and minimizing the volume of the Wald-type joint confidence region for the model parameter (μ, σ). Specifically, we aim to maximize

$$\det[\boldsymbol{I}(\mu,\sigma)] = \left| E\left(\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \mu^2}\right) E\left(\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \sigma^2}\right) - \left[E\left(\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \mu \partial \sigma}\right)\right]^2 \right|.$$

[C2] A-optimality: Minimizing the trace of the asymptotic variance-covariance matrix of the MLEs, $V(\mu, \sigma)$, i.e., minimizing the sum of the asymptotic variances of the MLEs of μ and σ. Specifically, we aim to minimize

$$Var(\hat{\mu}) + Var(\hat{\sigma}).$$

For notational convenience, we denote $V_\mu = Var(\hat{\mu})$ and $V_\sigma = Var(\hat{\sigma})$.

[C3] V-optimality: Minimizing the variance of the estimator of μ, $Var(\hat{\mu})$. This criterion minimizes the variance of the estimate of the slope parameter.

[C4] Minimization of the expected total time of the experiment, denoted as TTE, which is defined as

$$E(TTE) = E\left\{[\exp(V_{r:r:m})]\right\}.$$

Using the relationship between $Z_{\ell:r:m}$ and $V_{\ell:r:m}$:

$$Z_{\ell:r:m} = \frac{V_{\ell:r:m} - \mu}{\sigma},$$

the TTE can be expressed in terms of $Z_{\ell:r:m}$, which is the ℓ-th ordered standardized log-transformed failure time, as

$$E(TTE) = \exp(\mu) E\left[\exp(\sigma Z_{r:r:m})\right]. \tag{3.13}$$

Based on the progressive Type-II censored experiments described in Section 3.2, the experiment terminates at the last observed failure. This criterion aims to minimize the termination time of the experiment.

[C5] Minimization of the expected maximum total time for all the components spending on the experiment, denoted as TTT, which is defined as

$$\begin{aligned} E(TTT) &= nE\left\{\sum_{\ell=1}^{r}(R_\ell + 1)\exp\left(V_{\ell:r:m}\right)\right\} \\ &= n\sum_{\ell=1}^{r}(R_\ell + 1)\exp(\mu)E\left[\exp(\sigma Z_{\ell:r:m})\right]. \end{aligned}$$

Since the components that did not fail in the life testing experiment can be used in the future for other purposes, the shorter the time those components spent in the experiment, the longer they can be used for other purposes. Therefore, this criterion aims to minimize the maximum expected total time the components spent in the experiment.

[C6] Minimization of a cost function: For experimental planning purposes, we consider the optimal progressive censoring scheme by including the cost in the life testing experiment due to failures and time and the cost in the inaccuracy of parameter estimation. Specifically, for specific values of N, m, n, and system signature $\boldsymbol{s}$, we minimize the objective function

$$\psi(\mathbf{R}) = c_1 E(R) + c_2 E(TTE) + c_3(Var(\hat{\mu}) + Var(\hat{\sigma})), \tag{3.14}$$

where c_1, c_2 and c_3 are the cost per unit of failed components, the cost per unit of time for the duration of the experiment, and the cost of variance of the estimates of the unknown model parameters, respectively.

3.5. Specific Coherent Systems Being Considered

Many coherent systems can be considered for the progressively Type-II censored experiment involving systems, especially when N and n are large. In practice, it may be more feasible to use coherent systems with a small number of components since using systems with a large number of components reduce the flexibility of the progressively censored experiments. For example, with $N = 120$, if we consider using 3-component coherent systems (i.e., $n = 3$), we have 40 systems for the progressively Type-II censored experiment. However, if we consider using 10-component coherent systems (i.e., $n = 10$), we only have 12 systems for the progressively Type-II censored experiment. Therefore, to simplify the optimal experimental planning problem and for illustrative purposes, we restrict our discussion to $n \leq 3$. Note that the methodologies proposed in this chapter can be extended to $n > 3$.

For $n = 1$, the experiment reduces to a component-wise progressively Type-II censored experiment, in which the optimal experimental schemes have been discussed under different optimal criteria in the literature. For $n = 2$, two possible coherent systems can be considered: the series system with system signature $\boldsymbol{s} = (1, 0)$ and the parallel system with system signature $\boldsymbol{s} = (0, 1)$. For $n = 3$, five possible coherent systems can be considered, including the series system with system signature $\boldsymbol{s} = (1, 0, 0)$, parallel system, 2-out-of-3 system with system signature $\boldsymbol{s} = (0, 1, 0)$, parallel-series system with system signature $\boldsymbol{s} = (0, 2/3, 1/3)$, and series-parallel system with system signature $\boldsymbol{s} = (1/3, 2/3, 0)$. Among those different types of systems, series systems and parallel systems are just special cases of l-out-of-n systems for $\ell = 1$ and $\ell = n$, respectively; hence, we consider l-out-on-n, parallel-series, and series-parallel systems for $n \leq 3$.

The following subsections provide the formulas required for obtaining the normal equations in Eqs. (3.5) and (3.6) for specific n-component coherent systems with $n \leq 3$.

3.5.1. l-out-of-n systems

For a l-out-of-n system, the system signature is $\boldsymbol{s} = (0, \ldots, 0, R_\ell = 1, 0, \ldots, 0)$. Thus, the SF and PDF of the log-transformed system lifetimes can be expressed as

$$\begin{aligned}
\bar{F}_V(v;\boldsymbol{\theta}) &= \sum_{i=1}^{n} s_i \sum_{j=0}^{i-1} \binom{n}{j} [F^*(z)]^j \left[\bar{F}^*(z)\right]^{n-j} \\
&= \sum_{j=0}^{l-1} \binom{n}{j} [F^*(z)]^j \left[\bar{F}^*(z)\right]^{n-j}, \\
\text{and } f_V(v;\boldsymbol{\theta}) &= \frac{1}{\sigma} f^*(z) \sum_{i=1}^{n} s_i \binom{n}{i} i [F^*(z)]^{i-1} \left[\bar{F}^*(z)\right]^{n-i} \\
&= \frac{1}{\sigma} f^*(z) \binom{n}{l} l [F^*(z)]^{l-1} \left[\bar{F}^*(z)\right]^{n-l}.
\end{aligned}$$

The log-likelihood functions in Eq. (3.4) can be simplified as

$$l(\boldsymbol{\theta}) = C + \sum_{\ell=1}^{r} \ln f_V(v_{\ell:r:m};\boldsymbol{\theta}) + \sum_{\ell=1}^{r} R_\ell \ln \left[\bar{F}_V(v_{\ell:r:m};\boldsymbol{\theta})\right], \quad v_{1:r:m} < v_{2:r:m} < \ldots < v_{r:r:m},$$

where

$$\begin{aligned}
\ln f_V(v_{\ell:r:m};\boldsymbol{\theta}) &= \ln \left\{ \frac{1}{\sigma} f^*(z_{\ell:r:m}) \binom{n}{l} l [F^*(z_{\ell:r:m})]^{l-1} \left[\bar{F}^*(z_{\ell:r:m})\right]^{n-l} \right\} \\
\text{and } \ln \bar{F}_V(v_{\ell:r:m};\boldsymbol{\theta}) &= \ln \left\{ \sum_{j=0}^{l-1} \binom{n}{j} [F^*(z_{\ell:r:m})]^j \left[\bar{F}^*(z_{\ell:r:m})\right]^{n-j} \right\}.
\end{aligned}$$

Then, we can obtain the first partial derivatives of the log-likelihood function with respect to μ and σ. respectively, as

$$\begin{aligned}
\frac{\partial l(\boldsymbol{\theta})}{\partial \mu} &= \sum_{\ell=1}^{r}\left[\frac{\partial \ln f_V(v_{\ell:r:m};\boldsymbol{\theta})}{\partial \mu} + R_\ell \frac{\partial \ln \bar{F}_V(v_{\ell:r:m};\boldsymbol{\theta})}{\partial \mu}\right] \\
\text{and } \frac{\partial l(\boldsymbol{\theta})}{\partial \sigma} &= \sum_{\ell=1}^{r}\left[\frac{\partial \ln f_V(v_{\ell:r:m};\boldsymbol{\theta})}{\partial \sigma} + R_\ell \frac{\partial \ln \bar{F}_V(v_{\ell:r:m};\boldsymbol{\theta})}{\partial \sigma}\right].
\end{aligned}$$

3.5.1.1. Series system: $l = 1$

For $l = 1$, we have the series system. Thus, the PDF and SF of the log-transformed system lifetimes can be further simplified as

$$\begin{aligned}
f_V(v;\boldsymbol{\theta}) &= \frac{n}{\sigma} f^*(z)\left[\bar{F}^*(z)\right]^{n-1}, \\
\text{and } \bar{F}_V(v;\boldsymbol{\theta}) &= \left[\bar{F}^*(z)\right]^{n}, \quad n = 1, 2, 3.
\end{aligned}$$

Then, we have

$$\begin{aligned}
\ln f_V(v_{\ell:r:m};\boldsymbol{\theta}) &= \ln\left\{\frac{n}{\sigma} f^*(z_{\ell:r:m})\left[\bar{F}^*(z_{\ell:r:m})\right]^{n-1}\right\} \\
&= \ln n - \ln \sigma + \ln f^*(z_{\ell:r:m}) + (n-1)\ln \bar{F}^*(z_{\ell:r:m}) \\
\text{and } \ln \bar{F}_V(v_{\ell:r:m};\boldsymbol{\theta}) &= \ln\left\{\left[\bar{F}^*(z_{\ell:r:m})\right]^{n}\right\} = n \ln \bar{F}^*(z_{\ell:r:m}).
\end{aligned}$$

Thus, we can obtain the first partial derivatives of the PDF and SF as

$$\begin{aligned}
\frac{\partial \ln f_V(v_{\ell:r:m};\boldsymbol{\theta})}{\partial \mu} &= \frac{\partial \ln f^*(z_{\ell:r:m})}{\partial \mu} + (n-1)\frac{\partial \ln \bar{F}^*(z_{\ell:r:m})}{\partial \mu} \\
\frac{\partial \ln f_V(v_{\ell:r:m};\boldsymbol{\theta})}{\partial \sigma} &= -\frac{1}{\sigma} + \frac{\partial \ln f^*(z_{\ell:r:m})}{\partial \sigma} + (n-1)\frac{\partial \ln \bar{F}^*(z_{\ell:r:m})}{\partial \sigma},
\end{aligned}$$

and

$$
\begin{aligned}
\frac{\partial \ln \bar{F}_V(v_{\ell:r:m};\boldsymbol{\theta})}{\partial \mu} &= n\left(\frac{\partial \ln \bar{F}^*(z_{\ell:r:m})}{\partial \mu}\right) \\
\frac{\partial \ln \bar{F}_V(v_{\ell:r:m};\boldsymbol{\theta})}{\partial \sigma} &= n\left(\frac{\partial \ln \bar{F}^*(z_{\ell:r:m})}{\partial \sigma}\right).
\end{aligned}
$$

Hence, the normal equations can be expressed as

$$
\begin{aligned}
\frac{\partial l(\boldsymbol{\theta})}{\partial \mu} &= \sum_{\ell=1}^{r}\left[\frac{\partial \ln f^*(z_{\ell:r:m})}{\partial \mu} + (R_\ell n + n - 1)\frac{\partial \ln \bar{F}^*(z_{\ell:r:m})}{\partial \mu}\right] \\
&= \sum_{\ell=1}^{r}\left[\frac{1}{f^*(z_{\ell:r:m})}\frac{\partial f^*(z_{\ell:r:m})}{\partial \mu} + (R_\ell n + n - 1)\frac{1}{\bar{F}^*(z_{\ell:r:m})}\frac{\partial \bar{F}^*(z_{\ell:r:m})}{\partial \mu}\right] = 0 \qquad (3.15) \\
\frac{\partial l(\boldsymbol{\theta})}{\partial \sigma} &= \sum_{\ell=1}^{r}\left[-\frac{1}{\sigma} + \frac{\partial \ln f^*(z_{\ell:r:m})}{\partial \sigma} + (R_\ell n + n - 1)\frac{\partial \ln \bar{F}^*(z_{\ell:r:m})}{\partial \sigma}\right] \\
&= \sum_{\ell=1}^{r}\left[-\frac{1}{\sigma} + \frac{1}{f^*(z_{\ell:r:m})}\frac{\partial f^*(z_{\ell:r:m})}{\partial \sigma} + (R_\ell n + n - 1)\frac{1}{\bar{F}^*(z_{\ell:r:m})}\frac{\partial \bar{F}^*(z_{\ell:r:m})}{\partial \sigma}\right] = 0. \qquad (3.16)
\end{aligned}
$$

3.5.1.2. Parallel system: $l = n$

For $l = n$, we have the parallel system with system signature $\boldsymbol{s} = (0, 0, \ldots, 1)$. The PDF and SF of the log-transformed system lifetimes can be expressed as

$$
\begin{aligned}
f_V(v;\boldsymbol{\theta}) &= \frac{n}{\sigma} f^*(z)\left[F^*(z)\right]^{n-1}, \\
\text{and } \bar{F}_V(v;\boldsymbol{\theta}) &= \sum_{j=0}^{n-1}\binom{n}{j}\left[\bar{F}^*(z)\right]^{j}\left[\bar{F}^*(z)\right]^{n-j} \\
&= 1 - \left[F^*(z)\right]^{n},
\end{aligned}
$$

respectively. Then, we have

$$\begin{aligned}
\ln f_V(v_{\ell:r:m};\boldsymbol{\theta}) &= \ln\left\{\frac{n}{\sigma}f^*(z)\,[F^*(z)]^{n-1}\right\} \\
&= \ln n - \ln\sigma + \ln f^*(z_{\ell:r:m}) + (n-1)\ln F^*(z_{\ell:r:m}) \\
\text{and}\ \ \ln \bar{F}_V(v_{\ell:r:m};\boldsymbol{\theta}) &= \ln\left\{1-[F^*(z_{\ell:r:m})]^n\right\}.
\end{aligned}$$

Thus, we can obtain the first partial derivatives of the PDF and SF as

$$\begin{aligned}
\frac{\partial \ln f_V(v_{\ell:r:m};\boldsymbol{\theta})}{\partial\mu} &= \frac{\partial \ln f^*(z_{\ell:r:m})}{\partial\mu} + (n-1)\frac{\partial \ln F^*(z_{\ell:r:m})}{\partial\mu} \\
&= \frac{1}{f^*(z_{\ell:r:m})}\frac{\partial f^*(z_{\ell:r:m})}{\partial\mu} + \frac{n-1}{F^*(z_{\ell:r:m})}\frac{\partial F^*(z_{\ell:r:m})}{\partial\mu} \qquad (3.17)\\
\frac{\partial \ln f_V(v_{\ell:r:m};\boldsymbol{\theta})}{\partial\sigma} &= -\frac{1}{\sigma} + \frac{\partial \ln f^*(z_{\ell:r:m})}{\partial\sigma} + (n-1)\frac{\partial \ln F^*(z_{\ell:r:m})}{\partial\sigma} \\
&= -\frac{1}{\sigma} + \frac{1}{f^*(z_{\ell:r:m})}\frac{\partial f^*(z_{\ell:r:m})}{\partial\sigma} + \frac{n-1}{F^*(z_{\ell:r:m})}\frac{\partial F^*(z_{\ell:r:m})}{\partial\sigma}, \qquad (3.18)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial \ln \bar{F}_V(v_{\ell:r:m};\boldsymbol{\theta})}{\partial\mu} &= -\frac{n\,[F^*(z_{\ell:r:m})]^{n-1}}{1-[F^*(z_{\ell:r:m})]^n}\frac{\partial F^*(z_{\ell:r:m})}{\partial\mu} \\
\frac{\partial \ln \bar{F}_V(v_{\ell:r:m};\boldsymbol{\theta})}{\partial\sigma} &= -\frac{n\,[F^*(z_{\ell:r:m})]^{n-1}}{1-[F^*(z_{\ell:r:m})]^n}\frac{\partial F^*(z_{\ell:r:m})}{\partial\sigma}.
\end{aligned}$$

The normal equations can be obtained as

$$\begin{aligned}
\frac{\partial l(\boldsymbol{\theta})}{\partial\mu} &= \sum_{\ell=1}^{r}\left[\frac{1}{f^*(z_{\ell:r:m})}\frac{\partial f^*(z_{\ell:r:m})}{\partial\mu}\right.\\
&\qquad\left.+\frac{(n-1)-(R_\ell n+n-1)\,[F^*(z_{\ell:r:m})]^n}{F^*(z_{\ell:r:m})\,(1-[F^*(z_{\ell:r:m})]^n)}\frac{\partial F^*(z_{\ell:r:m})}{\partial\mu}\right] = 0 \\
\frac{\partial l(\boldsymbol{\theta})}{\partial\sigma} &= \sum_{\ell=1}^{r}\left[-\frac{1}{\sigma} + \frac{1}{f^*(z_{\ell:r:m})}\frac{\partial f^*(z_{\ell:r:m})}{\partial\sigma}\right.\\
&\qquad\left.+\frac{(n-1)-(R_\ell n+n-1)\,[F^*(z_{\ell:r:m})]^n}{F^*(z_{\ell:r:m})\,(1-[F^*(z_{\ell:r:m})]^n)}\frac{\partial F^*(z_{\ell:r:m})}{\partial\sigma}\right] = 0.
\end{aligned}$$

3.5.1.3. 2-out-of-n system

For 2-out-of-n ($n > 2$) system, the system signature is $\boldsymbol{s} = (0, 1, 0, \ldots, 0)$, and the PDF and SF of the log-transformed system lifetimes are

$$\begin{aligned} f_V(v; \boldsymbol{\theta}) &= \frac{1}{\sigma} f^*(z) \left\{ n(n-1)[F^*(z)] \left[\bar{F}^*(z)\right]^{n-2} \right\} \\ \text{and } \bar{F}_V(v; \boldsymbol{\theta}) &= \left[\bar{F}^*(z)\right]^n + n[F^*(z)] \left[\bar{F}^*(z)\right]^{n-1}. \end{aligned}$$

Then, we have

$$\begin{aligned} \ln f_V(v_{\ell:r:m}; \boldsymbol{\theta}) &= \ln n(n-1) - \ln \sigma + \ln f^*(z_{\ell:r:m}) + \ln \left\{ [F^*(z_{\ell:r:m})] \left[\bar{F}^*(z_{\ell:r:m})\right]^{n-2} \right\} \\ \text{and } \ln \bar{F}_V(v_{\ell:r:m}; \boldsymbol{\theta}) &= \ln \left\{ \left[\bar{F}^*(z_{\ell:r:m})\right]^n + n[F^*(z_{\ell:r:m})] \left[\bar{F}^*(z_{\ell:r:m})\right]^{n-1} \right\}. \end{aligned}$$

Thus, we can obtain the first partial derivatives of the PDF and SF with respect to μ and σ as

$$\begin{aligned} \frac{\partial \ln f_V(v_{\ell:r:m}; \boldsymbol{\theta})}{\partial \mu} &= \frac{\partial \ln f^*(z_{\ell:r:m})}{\partial \mu} + \frac{\partial \ln \left\{ [F^*(z_{\ell:r:m})] \left[\bar{F}^*(z_{\ell:r:m})\right]^{n-2} \right\}}{\partial \mu} \\ \frac{\partial \ln f_V(v_{\ell:r:m}; \boldsymbol{\theta})}{\partial \sigma} &= -\frac{1}{\sigma} + \frac{\partial \ln f^*(z_{\ell:r:m})}{\partial \sigma} + \frac{\partial \ln \left\{ [F^*(z_{\ell:r:m})] \left[\bar{F}^*(z_{\ell:r:m})\right]^{n-2} \right\}}{\partial \sigma} \end{aligned}$$

and

$$\begin{aligned} \frac{\partial \ln \bar{F}_V(v_{\ell:r:m}; \boldsymbol{\theta})}{\partial \mu} &= \frac{\partial \ln \left\{ \left[\bar{F}^*(z_{\ell:r:m})\right]^n + n[F^*(z_{\ell:r:m})] \left[\bar{F}^*(z_{\ell:r:m})\right]^{n-1} \right\}}{\partial \mu} \\ \frac{\partial \ln \bar{F}_V(v_{\ell:r:m}; \boldsymbol{\theta})}{\partial \sigma} &= \frac{\partial \ln \left\{ \left[\bar{F}^*(z_{\ell:r:m})\right]^n + n[F^*(z_{\ell:r:m})] \left[\bar{F}^*(z_{\ell:r:m})\right]^{n-1} \right\}}{\partial \sigma}. \end{aligned}$$

The normal equations can be obtained as

$$\begin{aligned}
\frac{\partial l(\boldsymbol{\theta})}{\partial \mu} &= \sum_{\ell=1}^{r}\Bigg[\frac{1}{f^*(z_{\ell:r:m})}\frac{\partial f^*(z_{\ell:r:m})}{\partial \mu} + \frac{\partial \ln\left\{[F^*(z_{\ell:r:m})]\left[\bar{F}^*(z_{\ell:r:m})\right]^{n-2}\right\}}{\partial \mu} \\
&\quad + R_\ell \frac{\partial \ln\left\{\left[\bar{F}^*(z_{\ell:r:m})\right]^{n} + n[F^*(z_{\ell:r:m})]\left[\bar{F}^*(z_{\ell:r:m})\right]^{n-1}\right\}}{\partial \mu}\Bigg] = 0, \qquad (3.19)\\
\frac{\partial l(\boldsymbol{\theta})}{\partial \sigma} &= \sum_{\ell=1}^{r}\Bigg[-\frac{1}{\sigma} + \frac{1}{f^*(z_{\ell:r:m})}\frac{\partial f^*(z_{\ell:r:m})}{\partial \sigma} + \frac{\partial \ln\left\{[F^*(z_{\ell:r:m})]\left[\bar{F}^*(z_{\ell:r:m})\right]^{n-2}\right\}}{\partial \sigma} \\
&\quad + R_\ell \frac{\partial \ln\left\{\left[\bar{F}^*(z_{\ell:r:m})\right]^{n} + n[F^*(z_{\ell:r:m})]\left[\bar{F}^*(z_{\ell:r:m})\right]^{n-1}\right\}}{\partial \sigma}\Bigg] = 0. \qquad (3.20)
\end{aligned}$$

3.5.1.4. 3-out-of-n

For 3-out-of-n ($n > 3$) system, the system signature is $\boldsymbol{s} = (0, 0, 1, 0, \ldots, 0)$, and the PDF and SF of the log-transformed system lifetimes are

$$\begin{aligned}
f_V(v;\boldsymbol{\theta}) &= \frac{1}{\sigma} f^*(z)\left\{\frac{n(n-1)(n-2)}{2}[F^*(z)]^2\left[\bar{F}^*(z)\right]^{n-3}\right\} \\
\text{and } \bar{F}_V(v;\boldsymbol{\theta}) &= \left[\bar{F}^*(z)\right]^{n} + n[F^*(z)]\left[\bar{F}^*(z)\right]^{n-1} + \frac{n(n-1)}{2}[F^*(z)]^2\left[\bar{F}^*(z)\right]^{n-2}.
\end{aligned}$$

Then, we have

$$\begin{aligned}
&\ln f_V(v_{\ell:r:m};\boldsymbol{\theta}) \\
&= \ln\left[\frac{n(n-1)(n-2)}{2}\right] - \ln\sigma + \ln f^*(z_{\ell:r:m}) + \ln\left\{[F^*(z_{\ell:r:m})]^2\left[\bar{F}^*(z_{\ell:r:m})\right]^{n-3}\right\} \\
\text{and } &\ln \bar{F}_V(v_{\ell:r:m};\boldsymbol{\theta}) \\
&= \ln\left\{\left[\bar{F}^*(z_{\ell:r:m})\right]^{n} + n[F^*(z_{\ell:r:m})]\left[\bar{F}^*(z_{\ell:r:m})\right]^{n-1} + \frac{n(n-1)}{2}[F^*(z)]^2\left[\bar{F}^*(z)\right]^{n-2}\right\}.
\end{aligned}$$

Thus, we can obtain the first partial derivatives of the PDF and SF with respect to μ and σ as

$$\begin{aligned}
\frac{\partial \ln f_V(v_{\ell:r:m};\boldsymbol{\theta})}{\partial \mu} &= \frac{\partial \ln f^*(z_{\ell:r:m})}{\partial \mu} + \frac{\partial \ln \left\{[F^*(z_{\ell:r:m})]^2 \left[\bar{F}^*(z_{\ell:r:m})\right]^{n-3}\right\}}{\partial \mu} \\
\frac{\partial \ln f_V(v_{\ell:r:m};\boldsymbol{\theta})}{\partial \sigma} &= -\frac{1}{\sigma} + \frac{\partial \ln f^*(z_{\ell:r:m})}{\partial \sigma} + \frac{\partial \ln \left\{[F^*(z_{\ell:r:m})]^2 \left[\bar{F}^*(z_{\ell:r:m})\right]^{n-3}\right\}}{\partial \sigma}
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\partial \ln \bar{F}_V(v_{\ell:r:m};\boldsymbol{\theta})}{\partial \mu} \\
&= \frac{\partial \ln \left\{ \left[\bar{F}^*(z_{\ell:r:m})\right]^n + n[F^*(z_{\ell:r:m})] \left[\bar{F}^*(z_{\ell:r:m})\right]^{n-1} + \frac{n(n-1)}{2}[F^*(z)]^2 \left[\bar{F}^*(z)\right]^{n-2} \right\}}{\partial \mu} \\
&\frac{\partial \ln \bar{F}_V(v_{\ell:r:m};\boldsymbol{\theta})}{\partial \sigma} \\
&= \frac{\partial \ln \left\{ \left[\bar{F}^*(z_{\ell:r:m})\right]^n + n[F^*(z_{\ell:r:m})] \left[\bar{F}^*(z_{\ell:r:m})\right]^{n-1} + \frac{n(n-1)}{2}[F^*(z)]^2 \left[\bar{F}^*(z)\right]^{n-2} \right\}}{\partial \sigma}.
\end{aligned}$$

The normal equations can be obtained as

$$\begin{aligned}
\frac{\partial l(\boldsymbol{\theta})}{\partial \mu} &= \sum_{\ell=1}^{r} \left[\frac{1}{f^*(z_{\ell:r:m})} \frac{\partial f^*(z_{\ell:r:m})}{\partial \mu} + \frac{\partial \ln \left\{[F^*(z_{\ell:r:m})]^2 \left[\bar{F}^*(z_{\ell:r:m})\right]^{n-3}\right\}}{\partial \mu} \right. \\
&\left. + \frac{\partial \ln \left\{ \left[\bar{F}^*(z_{\ell:r:m})\right]^n + n[F^*(z_{\ell:r:m})] \left[\bar{F}^*(z_{\ell:r:m})\right]^{n-1} + \frac{n(n-1)}{2}[F^*(z)]^2 \left[\bar{F}^*(z)\right]^{n-2} \right\}}{\partial \mu} \right] \\
&= 0, \qquad (3.21)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial l(\boldsymbol{\theta})}{\partial \sigma} &= \sum_{\ell=1}^{r} \left[-\frac{1}{\sigma} + \frac{1}{f^*(z_{\ell:r:m})} \frac{\partial f^*(z_{\ell:r:m})}{\partial \sigma} + \frac{\partial \ln \left\{[F^*(z_{\ell:r:m})]^2 \left[\bar{F}^*(z_{\ell:r:m})\right]^{n-3}\right\}}{\partial \sigma} \right. \\
&\left. + \frac{\partial \ln \left\{ \left[\bar{F}^*(z_{\ell:r:m})\right]^n + n[F^*(z_{\ell:r:m})] \left[\bar{F}^*(z_{\ell:r:m})\right]^{n-1} + \frac{n(n-1)}{2}[F^*(z)]^2 \left[\bar{F}^*(z)\right]^{n-2} \right\}}{\partial \sigma} \right] \\
&= 0. \qquad (3.22)
\end{aligned}$$

3.5.2. Series-parallel and parallel-series systems for $n = 3$

3.5.2.1. Series-parallel systems for $n = 3$

For $n = 3$, the system signature for series-parallel systems is $\boldsymbol{s} = (1/3, 2/3, 0)$, and the PDF and SF of the log-transformed system lifetimes are

$$\begin{aligned} f_V(v;\boldsymbol{\theta}) &= \frac{1}{\sigma} f^*(z)\left\{\left[\bar{F}^*(z)\right]^2 + 2[F^*(z)]\left[\bar{F}^*(z)\right]\right\} \\ &= \frac{1}{\sigma} f^*(z)\left\{1 - [F^*(z)]^2\right\} \\ \text{and } \bar{F}_V(v;\boldsymbol{\theta}) &= \left[\bar{F}^*(z)\right]^3 + 2[F^*(z)]\left[\bar{F}^*(z)\right]^2 \\ &= [F^*(z)]^3 - [F^*(z)]^2 - F^*(z) + 1, \end{aligned}$$

respectively. Then, we have

$$\begin{aligned} \ln f_V(v_{\ell:r:m};\boldsymbol{\theta}) &= -\ln\sigma + \ln f^*(z_{\ell:r:m}) + \ln\left\{1 - [F^*(z_{\ell:r:m})]^2\right\} \\ \text{and } \ln \bar{F}_V(v_{\ell:r:m};\boldsymbol{\theta}) &= \ln\left\{[F^*(z)]^3 - [F^*(z)]^2 - F^*(z) + 1\right\}. \end{aligned}$$

Thus, we can obtain the first partial derivatives of the PDF and SF with respect to μ and σ as

$$\begin{aligned} \frac{\partial \ln f_V(v_{\ell:r:m};\boldsymbol{\theta})}{\partial\mu} &= \frac{\partial \ln f^*(z_{\ell:r:m})}{\partial\mu} + \frac{\partial \ln\left\{1 - [F^*(z_{\ell:r:m})]^2\right\}}{\partial\mu} \\ \frac{\partial \ln f_V(v_{\ell:r:m};\boldsymbol{\theta})}{\partial\sigma} &= -\frac{1}{\sigma} + \frac{\partial \ln f^*(z_{\ell:r:m})}{\partial\sigma} + \frac{\partial \ln\left\{1 - [F^*(z_{\ell:r:m})]^2\right\}}{\partial\sigma} \end{aligned}$$

and

$$\begin{aligned} \frac{\partial \ln \bar{F}_V(v_{\ell:r:m};\boldsymbol{\theta})}{\partial\mu} &= \frac{\partial \ln\left\{[F^*(z)]^3 - [F^*(z)]^2 - F^*(z) + 1\right\}}{\partial\mu} \\ \frac{\partial \ln \bar{F}_V(v_{\ell:r:m};\boldsymbol{\theta})}{\partial\sigma} &= \frac{\partial \ln\left\{[F^*(z)]^3 - [F^*(z)]^2 - F^*(z) + 1\right\}}{\partial\sigma}. \end{aligned}$$

The normal equations can be obtained as

$$\begin{aligned}
\frac{\partial l(\boldsymbol{\theta})}{\partial \mu} &= \sum_{\ell=1}^{r}\left[\frac{1}{f^*(z_{\ell:r:m})}\frac{\partial f^*(z_{\ell:r:m})}{\partial \mu} + \frac{\partial \ln\left\{1-[F^*(z_{\ell:r:m})]^2\right\}}{\partial \mu}\right.\\
&\quad \left. + \frac{\partial \ln\left\{[F^*(z)]^3-[F^*(z)]^2-F^*(z)+1\right\}}{\partial \mu}\right] = 0, \qquad (3.23)\\
\frac{\partial l(\boldsymbol{\theta})}{\partial \sigma} &= \sum_{\ell=1}^{r}\left[-\frac{1}{\sigma} + \frac{1}{f^*(z_{\ell:r:m})}\frac{\partial f^*(z_{\ell:r:m})}{\partial \sigma} + \frac{\partial \ln\left\{1-[F^*(z_{\ell:r:m})]^2\right\}}{\partial \sigma}\right.\\
&\quad \left. + \frac{\partial \ln\left\{[F^*(z)]^3-[F^*(z)]^2-F^*(z)+1\right\}}{\partial \sigma}\right] = 0. \qquad (3.24)
\end{aligned}$$

3.5.2.2. Parallel-series systems for $n = 3$

For $n = 3$, the system signature for parallel-series systems is $\boldsymbol{s} = (0, 2/3, 1/3)$. Thus, the PDF and SF of the log-transformed system lifetimes are,

$$\begin{aligned}
f_V(v;\boldsymbol{\theta}) &= \frac{1}{\sigma}f^*(z)\left\{2[F^*(z)]\left[\bar{F}^*(z)\right] + \frac{1}{3}[F^*(z)]^2\right\}\\
&= \frac{1}{\sigma}f^*(z)\left\{2[F^*(z)] - \frac{5}{3}[F^*(z)]^2\right\}\\
\text{and } \bar{F}_V(v;\boldsymbol{\theta}) &= \left[\bar{F}^*(z)\right]^3 + 3[F^*(z)]\left[\bar{F}^*(z)\right]^2 + [F^*(z)]^2\left[\bar{F}^*(z)\right],\\
&= [F^*(z)]^3 - 2[F^*(z)]^2 + 1,
\end{aligned}$$

respectively. Thus, we have

$$\begin{aligned}
\ln f_V(v_{\ell:r:m};\boldsymbol{\theta}) &= -\ln\sigma + \ln f^*(z_{\ell:r:m}) + \ln\left\{2[F^*(z)] - \frac{5}{3}\left[F^*(z)\right]^2\right\}\\
\text{and } \ln \bar{F}_V(v_{\ell:r:m};\boldsymbol{\theta}) &= \ln\left\{[F^*(z)]^3 - 2\left[F^*(z)\right]^2 + 1\right\}.
\end{aligned}$$

Thus, we can obtain the first partial derivatives of the PDF and SF with respect to μ and σ as

$$\begin{aligned}\frac{\partial \ln f_V(v_{\ell:r:m};\boldsymbol{\theta})}{\partial \mu} &= \frac{\partial \ln f^*(z_{\ell:r:m})}{\partial \mu} + \frac{\partial \ln \left\{2\left[F^*(z_{\ell:r:m})\right] - \frac{5}{3}\left[F^*(z_{\ell:r:m})\right]^2\right\}}{\partial \mu}\\ \frac{\partial \ln f_V(v_{\ell:r:m};\boldsymbol{\theta})}{\partial \sigma} &= -\frac{1}{\sigma} + \frac{\partial \ln f^*(z_{\ell:r:m})}{\partial \sigma} + \frac{\partial \ln \left\{2\left[F^*(z_{\ell:r:m})\right] - \frac{5}{3}\left[F^*(z_{\ell:r:m})\right]^2\right\}}{\partial \sigma},\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \ln \bar{F}_V(v_{\ell:r:m};\boldsymbol{\theta})}{\partial \mu} &= \frac{\partial \ln \left\{\left[F^*(z_{\ell:r:m})\right]^3 - 2\left[F^*(z_{\ell:r:m})\right]^2 + 1\right\}}{\partial \mu}\\ \frac{\partial \ln \bar{F}_V(v_{\ell:r:m};\boldsymbol{\theta})}{\partial \sigma} &= \frac{\partial \ln \left\{\left[F^*(z_{\ell:r:m})\right]^3 - 2\left[F^*(z_{\ell:r:m})\right]^2 + 1\right\}}{\partial \sigma}.\end{aligned}$$

The normal equations can be obtained as

$$\begin{aligned}\frac{\partial l(\boldsymbol{\theta})}{\partial \mu} &= \sum_{\ell=1}^{r}\left[\frac{1}{f^*(z_{\ell:r:m})}\frac{\partial f^*(z_{\ell:r:m})}{\partial \mu} + \frac{\partial \ln \left\{2\left[F^*(z_{\ell:r:m})\right] - \frac{5}{3}\left[F^*(z_{\ell:r:m})\right]^2\right\}}{\partial \mu}\right.\\ &\quad\left. + \frac{\partial \ln \left\{\left[F^*(z_{\ell:r:m})\right]^3 - 2\left[F^*(z_{\ell:r:m})\right]^2 + 1\right\}}{\partial \mu}\right] = 0, \qquad (3.25)\\ \frac{\partial l(\boldsymbol{\theta})}{\partial \sigma} &= \sum_{\ell=1}^{r}\left[-\frac{1}{\sigma} + \frac{1}{f^*(z_{\ell:r:m})}\frac{\partial f^*(z_{\ell:r:m})}{\partial \sigma} + \frac{\partial \ln \left\{2\left[F^*(z_{\ell:r:m})\right] - \frac{5}{3}\left[F^*(z_{\ell:r:m})\right]^2\right\}}{\partial \sigma}\right.\\ &\quad\left. + \frac{\partial \ln \left\{\left[F^*(z_{\ell:r:m})\right]^3 - 2\left[F^*(z_{\ell:r:m})\right]^2 + 1\right\}}{\partial \sigma}\right] = 0. \qquad (3.26)\end{aligned}$$

3.6. Numerical Illustrations

3.6.1. Lognormal distributed components

Consider the lifetimes of the components follow a two-parameter lognormal distribution with CDF and PDF

$$\begin{aligned} F_{LN}(x;\mu,\sigma) &= \Phi_{nor}\left[\frac{\ln(x)-\mu}{\sigma}\right], x>0, \\ f_{LN}(x;\mu,\sigma) &= \frac{1}{\sigma x}\phi_{nor}\left(\frac{\ln(x)-\mu}{\sigma}\right) = \frac{1}{\sqrt{2\pi}\sigma x}\exp\left[-\frac{1}{2}\left(\frac{\ln(x)-\mu}{\sigma}\right)^2\right] x>0, \end{aligned}$$

respectively, where $\exp(\mu)$ is the scale parameter, $\sigma>0$ is the shape parameter, and $\Phi_{nor}(z)$ and $\phi_{nor}(z)$ are the CDF and PDF for the standard normal distribution:

$$\begin{aligned} \Phi_{nor}(z) = \int_{-\infty}^{z}\phi_{nor}(w)dw,\ \bar{\Phi}_{nor}(z) = 1-\Phi_{nor}(z), \\ \phi_{nor}(z) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{z^2}{2}\right), -\infty<z<\infty. \end{aligned}$$

Then, the log-transformed component lifetimes follow a normal distribution with location parameter μ and scale parameter σ with CDF, SF and PDF

$$\begin{aligned} F_U(u;\mu,\sigma) &= \Phi_{nor}\left(\frac{u-\mu}{\sigma}\right), \\ \bar{F}_U(u;\mu,\sigma) &= 1-\Phi_{nor}\left(\frac{u-\mu}{\sigma}\right) = \bar{\Phi}_{nor}\left(\frac{u-\mu}{\sigma}\right), \\ \text{and } f_U(u;\mu,\sigma) &= \frac{1}{\sigma}\phi_{nor}\left(\frac{u-\mu}{\sigma}\right), \quad -\infty<u<\infty, \end{aligned}$$

respectively.

We consider the experimental planning for a fixed value of N and r when various types of coherent systems are used in progressively Type-II censored experiments. In Tables 3.1 and

3.2, we present the optimal progressive censoring schemes for n-component coherent systems with lognormal distributed components under settings [**M1**] and [**M2**]. In Table 3.3, we present the optimal progressive censoring schemes for n-component coherent systems with lognormal distributed components ($n = 1, 2$, and 3) by fixing $R_{\max} = 8$ under setting [**M1**] and [**M2**].

Table 3.1: Optimal progressive censoring schemes for n-component systems with lognormal distributed components when $N = 30$, by fixing $n = 1, 2,$ and 3 under setting **[M1]** and **[M2]**

System Type	n	Setting	m	r	$E(R)$	$R_{\max}$	R	[C1] $\|I\|$	[C2] $V_\mu + V_\sigma$	[C3] V_μ	[C4] $\overline{TTE}$	[C5] $\overline{TTT}$	[C6] $(1,1,10)$	$(1,0,10)$	$(1,1,0)$	$(0,1,10)$	$(0,1,5)$	$(0,1,20)$	$(0,1,30)$
Series	1	**[M1]**	30	6	6	6	(0,24,0,0,0,0)	**194.48**	0.1984	0.1420	27.91	92.08	35.90	7.98	33.91	29.90	28.91	31.88	33.87
							(24,0,0,0,0,0)	172.23	**0.1943**	**0.1419**	30.56	89.61	33.5	**2.94**	31.56	32.50	31.53	34.45	36.39
							(0,0,0,0,0,24)	143.63	0.2905	0.1818	**3.13**	**87.90**	**12.04**	8.91	**9.13**	**6.04**	4.58	**8.94**	**11.85**
		[M2]	30	6	6	6	(0,24,0,0,0,0)	**194.48**	0.1984	0.1420	27.91	92.08	35.90	7.98	33.91	29.90	28.91	31.88	33.87
							(23,0,0,0,0,1)	171.21	**0.1933**	0.1369	16.60	84.33	24.54	**7.93**	22.60	18.54	17.57	20.47	22.40
							(20,0,0,0,0,4)	166.61	0.2000	**0.1330**	8.35	**81.47**	16.35	8.00	14.35	10.35	9.35	12.35	14.35
							(0,0,0,0,0,24)	143.63	0.2905	0.1818	**3.13**	87.90	**12.04**	8.90	**9.13**	**6.04**	4.58	8.94	11.85
							(5,0,0,0,0,19)	147.64	0.2678	0.1671	3.52	86.20	12.19	8.68	9.52	6.19	4.85	**8.87**	11.55
							(6,0,0,0,0,18)	148.27	0.2640	0.1650	3.61	85.72	12.25	8.64	9.61	6.25	4.93	8.89	**11.53**
	2	**[M1]**	15	6	6	6	(0,9,0,0,0,0)	**183.59**	0.207	0.1404	12.13	86.15	20.20	8.07	18.13	14.20	13.17	16.27	18.34
							(9,0,0,0,0,0)	171.43	**0.1985**	**0.1348**	12.84	**83.33**	20.83	**7.99**	18.84	14.83	13.83	16.81	18.80
							(0,0,0,0,0,9)	146.57	0.2796	0.1752	**3.36**	87.25	**12.16**	8.80	**9.36**	**6.16**	4.76	**8.95**	**11.75**
		[M2]	15	6	6	6	(0,9,0,0,0,0)	**183.59**	0.207	0.1404	12.13	86.15	20.20	8.07	18.13	14.20	13.17	16.27	18.34
							(9,0,0,0,0,0)	171.43	**0.1985**	**0.1348**	12.84	83.33	20.83	**7.99**	18.84	14.83	13.83	16.81	18.80
							(0,0,0,0,0,9)	146.57	0.2796	0.1752	**3.36**	87.25	**12.16**	8.80	**9.36**	**6.16**	4.76	**8.95**	11.75
							(8,0,0,0,0,1)	167.13	0.2058	0.1357	8.17	**82.30**	16.23	8.06	14.17	10.23	9.20	12.29	14.34
							(3,0,0,0,0,6)	152.09	0.2521	0.1581	4.03	85.17	12.55	8.52	10.03	6.55	5.29	9.07	11.59
	3	**[M1]**	10	6	6	6	(0,4,0,0,0,0)	**174.29**	0.2189	0.1443	8.28	85.23	16.47	8.19	14.28	10.47	9.37	12.66	14.85
							(4,0,0,0,0,0)	168.06	**0.2113**	**0.1385**	8.57	**83.25**	16.68	**8.11**	14.57	10.68	9.63	12.8	14.91
							(0,0,0,0,0,4)	149.97	0.2682	0.1684	**3.70**	86.61	**12.38**	8.68	**9.7**	**6.38**	5.04	**9.06**	**11.75**
		[M2]	10	6	6	6	(0,4,0,0,0,0)	**174.29**	0.2189	0.1443	8.28	85.23	16.47	8.19	14.28	10.47	9.37	12.66	14.85
							(4,0,0,0,0,0)	168.06	**0.2113**	**0.1385**	8.57	**83.25**	16.68	**8.11**	14.57	10.68	9.63	12.80	14.91
							(0,0,0,0,0,4)	149.97	0.2682	0.1684	**3.70**	86.61	**12.38**	8.68	**9.70**	**6.38**	5.04	**9.06**	11.75
							(1,0,0,0,0,3)	153.23	0.254	0.1598	4.11	85.58	12.65	8.54	10.11	6.65	5.38	9.19	**11.73**
Parallel	2	**[M1]**	15	3	6	18	(0,12,0)	**70.43**	0.2595	0.1078	20.73	177.68	29.33	8.60	26.73	23.33	22.03	25.92	28.52
							(12,0,0)	69.79	**0.2465**	0.1261	28.01	**169.73**	36.47	**8.47**	34.01	30.47	29.24	32.94	35.40
							(0,0,12)	61.01	0.3010	**0.0992**	**6.23**	178.90	**15.24**	9.01	**12.23**	**9.24**	7.73	**12.25**	**15.26**
		[M2]	15	3	6	18	(0,12,0)	**70.43**	0.2595	0.1078	20.73	177.68	29.33	8.60	26.73	23.33	22.03	25.92	28.52
							(12,0,0)	69.79	**0.2465**	0.1261	28.01	169.73	36.47	**8.47**	34.01	30.47	29.24	32.94	35.40
							(1,0,11)	61.39	0.2971	**0.0991**	6.40	177.85	15.37	8.97	12.40	9.37	7.89	12.34	15.31
							(0,0,12)	61.01	0.3010	0.0992	**6.23**	178.90	**15.24**	9.01	**12.23**	**9.24**	**7.73**	**12.25**	**15.26**
							(11,0,1)	69.22	0.2481	0.1164	15.96	**165.62**	24.44	8.48	21.96	18.44	17.20	20.92	23.40
	3	**[M1]**	10	2	6	22	(8,0)	**39.33**	**0.3295**	0.1258	25.04	**240.98**	34.33	**9.30**	31.04	28.33	26.69	31.63	34.92
							(0,8)	36.03	0.3845	**0.0980**	**8.65**	252.04	**18.50**	9.84	**14.65**	**12.50**	**10.57**	**16.34**	**20.19**
		[M2]	10	2	6	22	(8,0)	**39.33**	**0.3295**	0.1258	25.04	240.98	34.33	**9.30**	31.04	28.33	26.69	31.63	34.92
							(0,8)	36.03	0.3845	**0.0980**	**8.65**	252.04	**18.50**	9.84	**14.65**	**12.50**	**10.57**	**16.34**	**20.19**
							(7,1)	38.86	0.3398	0.1176	15.30	**239.28**	24.70	9.40	21.30	18.70	17.00	22.10	25.50
2-out-of-n	3	**[M1]**	10	3	6	13	(7,0,0)	**80.81**	**0.2513**	0.1232	13.28	**127.92**	21.79	**8.51**	19.28	15.79	14.54	18.31	20.82
							(0,7,0)	80.04	0.2783	**0.1227**	10.71	135.80	19.49	8.78	16.71	13.49	12.1	16.28	19.06
							(0,0,7)	69.76	0.3273	0.1271	**4.96**	139.41	**14.23**	9.27	**10.96**	**8.23**	**6.60**	**11.51**	14.78
		[M2]	10	3	6	13	(7,0,0)	**80.81**	**0.2513**	0.1232	13.28	**85.28**	21.79	**8.51**	19.28	15.79	14.54	18.31	20.82
							(5,0,2)	76.09	0.2753	**0.1181**	7.24	131.18	15.99	8.75	13.24	9.99	8.62	12.75	15.50
							(0,0,7)	69.76	0.3273	0.1271	**4.96**	139.41	**14.23**	9.27	**10.96**	**8.23**	**6.60**	**11.51**	14.78
							(2,0,5)	71.53	0.3092	0.1227	5.46	136.50	14.55	9.09	11.46	8.55	7.01	11.64	**14.74**
Series-parallel	3	**[M1]**	10	4	6 2/3	14	(0,6,0,0)	**79.86**	0.2573	**0.1458**	11.93	115.16	21.17	9.24	18.60	14.50	13.22	17.08	19.65
							(6,0,0,0)	77.68	**0.2471**	0.1467	13.32	**110.36**	22.46	**9.14**	19.99	15.79	14.56	18.26	20.73
							(0,0,0,6)	65.04	0.3143	0.1515	**4.48**	118.08	**14.29**	9.81	**11.15**	**7.62**	**6.05**	**10.77**	**13.91**
		[M2]	10	4	6 2/3	14	(0,6,0,0)	**79.86**	0.2573	0.1458	11.93	115.16	21.17	9.24	18.60	14.50	13.22	17.08	19.65
							(6,0,0,0)	77.68	**0.2471**	0.1467	13.32	**110.36**	22.46	**9.14**	19.99	15.79	14.56	18.26	20.73
							(4,0,0,2)	72.46	0.2644	**0.1381**	6.83	112.25	16.14	9.31	13.50	9.47	8.15	12.12	14.76
							(0,0,0,6)	65.04	0.3143	0.1515	**4.48**	118.08	**14.29**	9.81	**11.15**	**7.62**	**6.05**	**10.77**	13.91
							(1,0,0,5)	66.59	0.3014	0.1465	4.81	116.84	14.49	14.68	16.48	12.82	6.32	10.84	**13.85**
Parallel-series	3	**[M1]**	10	3	7	23	(0,7,0)	**63.56**	0.2836	0.1226	16.37	172.26	26.21	9.84	23.37	19.21	17.79	22.04	24.88
							(7,0,0)	62.22	**0.2706**	0.1364	21.42	169.30	31.13	**9.71**	28.42	24.13	22.77	26.83	29.54
							(0,0,7)	58.26	0.3128	**0.1146**	**6.04**	**169.05**	**16.17**	10.13	**13.04**	**9.17**	**7.60**	**12.30**	**15.42**
		[M2]	10	3	7	23	(0,7,0)	**63.56**	0.2836	0.1226	16.37	172.26	26.21	9.84	23.37	19.21	17.79	22.04	24.88
							(7,0,0)	62.22	**0.2706**	0.1364	21.42	169.30	31.13	**9.71**	28.42	24.13	22.77	26.83	29.54
							(1,0,6)	58.73	0.3066	**0.1144**	6.35	167.77	16.42	10.07	13.35	9.42	7.88	12.48	15.55
							(0,0,7)	58.26	0.3128	0.1146	**6.04**	169.05	**16.17**	10.13	**13.04**	**9.17**	**7.60**	**12.30**	**15.42**
							(5,0,2)	61.72	0.2779	0.1197	9.46	**162.73**	19.24	9.78	16.46	12.24	10.85	15.02	17.80

For criterion **[C6]**, the objective function is $\psi(\mathbf{R}) = c_1 E(R) + c_2 E(TTE) + c_3(Var(\hat{\mu}) + Var(\hat{\sigma}))$.

Table 3.2: Optimal progressive censoring schemes for n-component systems with lognormal distributed components when $N = 24$, by fixing $n = 1, 2, 3$, and 4 under setting **[M1]** and **[M2]**

								[C1]	[C2]	[C3]	[C4]	[C5]			[C6]				
System Type	n	Setting	m	r	$E(R)$	$R_{\max}$	R	$\|\boldsymbol{I}\|$	$V_\mu + V_\sigma$	V_μ	TTE	TTT	(1, 1, 10)	(1, 0, 10)	(1, 1, 0)	(0, 1, 10)	(0, 1, 5)	(0, 1, 20)	(0, 1, 30)
Series	1	**[M1]**	24	4	4	4	(0,20,0,0)	**94.62**	0.3162	0.2241	19.61	64.56	26.77	7.16	23.61	22.77	21.19	25.93	29.10
							(20,0,0,0)	88.84	**0.2904**	0.2128	23.75	62.94	30.65	**6.90**	27.75	26.65	25.20	29.56	32.46
							(0,0,0,20)	70.15	0.4529	0.2935	**2.74**	**62.68**	**11.27**	8.53	**6.74**	**7.27**	**5.00**	**11.80**	**16.33**
		[M2]	24	4	4	4	(0,20,0,0)	**94.62**	0.3162	0.2241	19.61	64.56	26.77	7.16	23.61	22.77	21.19	25.93	29.10
							(20,0,0,0)	88.84	**0.2904**	0.2128	23.75	62.94	30.65	**6.90**	27.75	26.65	25.20	29.56	32.46
							(18,0,0,2)	85.78	0.2985	**0.2067**	8.94	58.34	15.93	6.99	12.94	11.93	10.43	14.91	17.90
							(0,0,0,20)	70.15	0.4529	0.2935	**2.74**	62.68	11.27	8.53	**6.74**	7.27	**5.00**	11.80	16.33
							(17,0,0,3)	84.24	0.3063	0.2084	7.23	**58.13**	14.29	7.06	11.23	10.29	8.76	13.36	16.42
							(2,0,0,18)	71.00	0.4385	0.2839	2.86	62.15	**11.25**	8.39	6.86	**7.25**	5.05	11.63	16.02
							(10,0,0,10)	76.06	0.3718	0.2415	3.79	59.81	11.51	7.72	7.79	7.51	5.65	**11.23**	14.94
							(12,0,0,8)	77.77	0.3535	0.2308	4.25	59.19	11.79	7.54	8.25	7.79	6.02	11.32	**14.86**
	2	**[M1]**	12	4	4	4	(0,8,0,0)	**88.64**	0.3322	0.2276	9.22	61.35	16.54	7.32	13.22	12.54	10.88	15.86	19.19
							(8,0,0,0)	86.23	**0.3016**	**0.2089**	10.48	**58.94**	17.50	**7.02**	14.48	13.50	11.99	16.51	19.53
							(0,0,0,8)	71.24	0.4401	0.2855	**2.87**	62.27	**11.27**	8.40	**6.87**	**7.27**	**5.07**	**11.67**	**16.07**
		[M2]	12	4	4	4	(0,8,0,0)	**88.64**	0.3322	0.2276	9.22	61.35	16.54	7.32	13.22	12.54	10.88	15.86	19.19
							(8,0,0,0)	86.23	**0.3016**	**0.2089**	10.48	58.94	17.50	**7.02**	14.48	13.50	11.99	16.51	19.53
							(0,0,0,8)	71.24	0.4401	0.2855	**2.87**	62.27	**11.27**	8.40	**6.87**	**7.27**	**5.07**	11.67	16.07
							(7,0,0,1)	83.02	0.3180	0.2135	6.50	**58.49**	13.68	7.18	10.50	9.68	8.09	12.86	16.04
							(4,0,0,4)	76.13	0.3745	0.2435	3.82	60.01	11.57	7.75	7.82	7.57	5.69	**11.31**	15.06
							(5,0,0,3)	78.03	0.3559	0.2326	4.31	59.42	11.87	7.56	8.31	7.87	6.09	11.43	**14.99**
	3	**[M1]**	8	4	4	4	(0,4,0,0)	**84.55**	0.3469	0.2331	6.59	60.83	14.06	7.47	10.59	10.06	8.32	13.53	17.00
							(4,0,0,0)	83.54	**0.3200**	**0.2154**	**7.20**	58.86	14.40	**7.20**	11.20	10.40	8.80	13.60	16.80
							(0,0,0,4)	72.09	0.4277	0.2772	**3.05**	61.90	**11.32**	8.28	**7.05**	**7.32**	**5.19**	**11.60**	**15.88**
		[M2]	8	4	4	4	(0,4,0,0)	**84.55**	0.3469	0.2331	6.59	60.83	14.06	7.47	10.59	10.06	8.32	13.53	17.00
							(4,0,0,0)	83.54	**0.3200**	**0.2154**	**7.20**	58.86	14.40	**7.20**	11.20	10.40	8.80	13.60	16.80
							(0,0,0,4)	72.09	0.4277	0.2772	**3.05**	61.90	**11.32**	8.28	**7.05**	**7.32**	**5.19**	11.60	15.88
							(2,0,0,2)	76.23	0.3768	0.2451	3.88	60.21	11.65	7.77	7.88	7.65	5.76	**11.42**	**15.18**
	4	**[M1]**	6	4	4	4	(0,2,0,0)	**81.60**	0.3597	0.2389	5.40	60.69	12.99	7.60	9.40	8.99	7.20	12.59	16.19
							(2,0,0,0)	81.25	**0.3400**	**0.2255**	5.70	**59.33**	13.10	**7.40**	9.70	9.10	7.40	12.50	15.90
							(0,0,0,2)	73.48	0.4131	0.2682	**3.28**	61.48	**11.42**	8.13	**7.28**	**7.42**	**5.35**	**11.54**	**15.67**
		[M2]	6	4	4	4	(0,2,0,0)	**81.60**	0.3597	0.2389	5.40	60.69	12.99	7.60	9.40	8.99	7.20	12.59	16.19
							(2,0,0,0)	81.25	**0.3400**	**0.2255**	5.70	**59.33**	13.10	**7.40**	9.70	9.10	7.40	12.50	15.90
							(0,0,0,2)	73.48	0.4131	0.2682	**3.28**	61.48	**11.42**	8.13	**7.28**	**7.42**	**5.35**	11.54	15.67
							(1,0,0,1)	76.56	0.3782	0.2465	3.97	60.36	11.75	7.78	7.97	7.75	5.86	**11.53**	**15.32**
Parallel	2	**[M1]**	12	2	4	14	(10,0)	**34.81**	**0.3636**	0.1716	19.86	**124.24**	27.49	**7.64**	23.86	23.49	21.67	27.13	30.76
							(0,10)	30.16	0.4430	**0.1526**	**5.55**	129.76	**13.98**	8.43	**9.55**	**9.98**	**7.76**	**14.41**	**18.84**
		[M2]	12	2	4	14	(10,0)	**34.81**	**0.3636**	0.1716	19.86	124.24	27.49	**7.64**	23.86	23.49	21.67	27.13	30.76
							(4,6)	31.09	0.4194	**0.1513**	6.32	126.98	14.52	8.19	10.32	10.52	8.42	14.71	18.91
							(0,10)	30.16	0.4430	0.1526	**5.55**	129.76	**13.98**	8.43	**9.55**	**9.98**	**7.76**	**14.41**	**18.84**
							(9,1)	33.98	0.3723	0.1604	11.41	**122.52**	19.14	7.72	15.41	15.14	13.27	18.86	22.58
	3	**[M1]**	8	2	6	18	(6,0)	**34.54**	**0.3530**	0.1418	25.55	**217.32**	35.08	**9.53**	31.55	29.08	27.32	32.61	36.14
							(0,6)	32.18	0.3996	**0.1151**	**9.68**	223.38	**19.68**	10.00	**15.68**	**13.68**	**11.68**	**17.67**	**21.67**
		[M2]	8	2	6	18	(6,0)	**34.54**	**0.3530**	0.1418	25.55	217.32	35.08	**9.53**	31.55	29.08	27.32	32.61	36.14
							(0,6)	32.18	0.3996	**0.1151**	**9.68**	223.38	**19.68**	10.00	**15.68**	**13.68**	**11.68**	**17.67**	**21.67**
							(5,1)	33.99	0.3651	0.1333	15.73	**214.94**	25.38	9.65	21.73	19.38	17.55	23.03	26.68
	4	**[M1]**	6	2	8	20	(4,0)	**33.70**	0.4021	0.1786	30.37	**311.70**	42.39	**12.02**	38.37	34.39	32.38	38.41	42.43
							(0,4)	32.66	**0.4477**	**0.1684**	**13.93**	316.65	**26.40**	12.48	**21.93**	**18.40**	**16.17**	**22.88**	**27.36**
		[M2]	6	2	8	20	(4,0)	**33.70**	0.4021	0.1786	30.37	311.70	42.39	**12.02**	38.37	34.39	32.38	38.41	42.43
							(0,4)	32.66	**0.4477**	**0.1684**	**13.93**	316.65	**26.40**	12.48	**21.93**	**18.40**	**16.17**	**22.88**	**27.36**
							(3,1)	33.36	0.4207	0.1764	19.58	**308.97**	31.79	12.21	27.58	23.79	21.69	28.00	32.20
2-out-of-n	3	**[M1]**	8	2	6	13	(6,0)	**39.83**	**0.3859**	**0.1855**	10.13	**95.15**	17.99	**7.86**	14.13	13.99	12.06	17.85	21.71
							(0,6)	34.38	0.4909	0.2015	**4.41**	101.76	**13.31**	8.91	**8.41**	**9.31**	**6.86**	**14.22**	**19.13**
		[M2]	8	2	6	13	(6,0)	**39.83**	**0.3859**	0.1855	10.13	**95.15**	17.99	**7.86**	14.13	13.99	12.06	17.85	21.71
							(5,1)	37.90	0.4118	**0.1840**	6.80	96.33	14.92	8.12	10.80	10.92	8.86	15.04	19.16
							(0,6)	34.38	0.4909	0.2015	**4.41**	101.76	**13.31**	8.91	**8.41**	**9.31**	**6.86**	14.22	19.13
							(2,4)	35.23	0.4664	0.1940	4.82	100.04	13.48	8.66	8.82	9.48	7.15	**14.15**	18.81
							(3,3)	35.85	0.4510	0.1900	5.17	99.02	13.68	8.51	9.17	9.68	7.42	14.19	**18.70**
Series-parallel	3	**[M1]**	8	4	6 2/3	14	(6,0)	**25.54**	**0.5022**	**0.3085**	8.34	**36.74**	16.69	**8.36**	11.67	13.36	10.85	18.38	23.41
							(0,6)	20.69	0.6750	0.3748	**3.09**	39.33	**13.18**	10.08	**6.43**	**9.84**	**6.47**	**16.59**	**23.34**
		[M2]	8	4	6 2/3	14	(6,0)	**25.54**	**0.5022**	**0.3085**	8.34	**36.74**	16.69	**8.36**	11.67	13.36	10.85	18.38	23.41
							(0,6)	20.69	0.6750	0.3748	**3.09**	39.33	13.18	10.08	**6.43**	9.84	**6.47**	16.59	23.34
							(2,4)	21.51	0.6304	0.3509	3.46	38.65	**13.10**	9.64	6.80	**9.77**	6.62	16.07	22.37
							(4,2)	22.91	0.5721	0.3242	4.28	37.72	13.33	9.05	7.61	10.00	7.14	**15.72**	21.44
							(5,1)	24.07	0.5360	0.3115	5.27	37.16	13.97	8.69	8.61	10.63	7.95	15.99	**21.35**
Parallel-series	3	**[M1]**	10	3	7	23	(6,0)	**31.18**	**0.4026**	0.1923	15.35	**121.77**	24.04	**8.69**	20.02	19.38	17.36	23.40	27.43
							(0,6)	28.89	0.4646	**0.1771**	**5.31**	122.37	**14.63**	9.31	**9.98**	**9.96**	**7.64**	**14.61**	**19.25**
		[M2]	10	3	7	23	(6,0)	**31.18**	**0.4026**	0.1923	15.35	121.77	24.04	**8.69**	20.02	19.38	17.36	23.40	27.43
							(3,3)	29.83	0.4356	**0.1753**	6.38	119.85	15.41	9.02	11.05	10.74	8.56	15.10	19.45
							(0,6)	28.89	0.4646	0.1771	**5.31**	122.37	**14.63**	9.31	**9.98**	**9.96**	**7.64**	**14.61**	19.25
							(5,1)	30.89	0.4108	0.1802	8.95	**118.56**	17.72	8.77	13.61	13.05	11.00	17.16	21.27
							(1,5)	29.10	0.4566	0.1762	5.55	121.56	14.79	9.23	10.22	10.12	7.84	14.68	**19.25**

For criterion **[C6]**, the objective function is $\psi(\mathbf{R}) = c_1E(R) + c_2E(TTE) + c_3(Var(\hat{\mu}) + Var(\hat{\sigma}))$.

Table 3.3: Optimal progressive censoring schemes for n-component systems with lognormal distributed components when $N = 18$, by fixing $n = 1, 2$, and 3, $R_{\max} = 8$ under setting **[M1]** and **[M2]**

								[C1]	**[C2]**	**[C3]**	**[C4]**	**[C5]**			**[C6]**				
System Type	n	Setting	m	r	$E(R)$	$R_{\max}$	R	$\lvert \boldsymbol{I} \rvert$	$V_\mu + V_\sigma$	V_μ	$\overline{TTE}$	$\overline{TTT}$	$(1,1,10)$	$(1,0,10)$	$(1,1,0)$	$(0,1,10)$	$(0,1,5)$	$(0,1,20)$	$(0,1,30)$
Series	1	**[M1]**	18	8	8	8	(0,0,10,0,0,0,0,0)	**217.5**	0.1552	0.1022	32.96	104.27	42.51	9.55	40.96	34.51	33.74	36.06	37.62
							(0,10,0,0,0,0,0,0)	214.69	**0.1551**	0.105	34.81	105.17	44.36	**9.55**	42.81	36.36	35.59	37.91	39.46
							(0,0,0,0,0,0,0,10)	182.19	0.1725	**0.0965**	**6.27**	**92.55**	**16.00**	9.73	**14.27**	**8.00**	**7.13**	**9.72**	**11.45**
		[M2]	18	8	8	8	(0,0,10,0,0,0,0,0)	**217.5**	0.1552	0.1022	32.96	104.27	42.51	9.55	40.96	34.51	33.74	36.06	37.62
							(0,8,2,0,0,0,0,0)	215.11	**0.155**	0.1038	34.06	104.67	43.61	9.55	42.06	35.61	34.84	37.16	38.71
							(3,0,0,0,0,0,0,7)	186.33	0.1649	**0.0956**	7.67	**92.22**	17.32	9.65	15.67	9.32	8.49	10.97	12.62
							(0,0,0,0,0,0,0,10)	182.19	0.1725	0.0965	**6.27**	92.55	**16.00**	9.73	**14.27**	**8.00**	**7.13**	**9.72**	**11.45**
							(0,9,0,1,0,0,0,0)	214.45	0.155	0.1037	33.78	104.34	43.33	**9.55**	41.78	35.33	34.56	36.88	38.43
	2	**[M1] and [M2]**	9	8	8	8	(0,1,0,0,0,0,0,0)	**200.68**	0.1590	0.0976	14.76	95.39	24.35	9.59	22.76	16.35	15.56	17.94	19.53
							(1,0,0,0,0,0,0,0)	198.19	**0.1589**	0.0976	14.85	95.11	24.44	**9.59**	22.85	16.44	15.65	18.03	19.62
							(0,0,0,0,0,0,0,1)	192.95	0.1633	**0.0963**	**9.71**	**93.42**	**19.34**	9.63	**17.71**	**11.34**	**10.52**	**12.97**	**14.61**
2-out-of-3	3	**[M1]**	6	2	4	8	(4,0)	**33.82**	**0.3933**	0.1814	10.40	**83.06**	22.33	**11.93**	18.40	14.33	12.36	18.26	22.20
							(0,4)	30.33	0.4657	**0.1812**	**5.12**	87.15	**17.78**	12.66	**13.12**	**9.78**	**7.45**	**14.43**	**19.09**
		[M2]	6	2	4	8	(4,0)	**33.82**	**0.3933**	0.1814	10.40	**83.06**	22.33	**11.93**	18.40	14.33	12.36	18.26	22.20
							(3,1)	32.30	0.4171	**0.1767**	7.09	84.05	19.26	12.17	15.09	11.26	9.17	15.43	19.60
							(1,3)	30.82	0.4520	0.1788	5.46	86.28	17.98	12.52	13.46	9.98	7.72	14.50	**19.02**
							(0,4)	30.33	0.4657	0.1812	**5.12**	87.15	**17.78**	12.66	**13.12**	**9.78**	**7.45**	**14.43**	19.09
Series-parallel	3	**[M1]**	6	2	3 1/3	8	(4,0)	**21.76**	**0.5036**	**0.2939**	8.61	**60.01**	21.64	**13.04**	16.61	13.64	11.12	18.68	23.72
							(0,4)	18.47	0.6191	0.3185	**3.73**	62.86	**17.93**	14.19	**11.73**	**9.93**	**6.83**	**16.12**	**22.31**
		[M2]	6	2	3 1/3	8	(4,0)	**21.76**	**0.5036**	0.2939	8.61	**60.01**	21.64	**13.04**	16.61	13.64	11.12	18.68	23.72
							(3,1)	20.37	0.5376	**0.2927**	5.53	60.52	18.90	13.38	13.53	10.90	8.22	16.28	21.66
							(2,2)	19.47	0.5697	0.3004	4.54	61.38	18.24	13.70	12.54	10.24	7.39	**15.93**	**21.63**
							(0,4)	18.47	0.6191	0.3185	**3.73**	62.86	**17.93**	14.19	**11.73**	**9.93**	**6.83**	16.12	22.31

For criterion **[C6]**, the objective function is $\psi(\mathbf{R}) = c_1 R_{\max} + c_2 E(TTE) + c_3(Var(\hat{\mu}) + Var(\hat{\sigma}))$.

3.6.2. Weibull distributed components

Suppose the lifetimes of the components follow a two-parameter Weibull distribution with CDF and PDF

$$
\begin{aligned}
F_{WE}(x;\eta,\beta) &= 1-\exp\left[-\left(\frac{x}{\eta}\right)^{\beta}\right],\ x>0,\\
f_{WE}(x;\eta,\beta) &= \frac{\beta}{\eta}\left(\frac{x}{\eta}\right)^{\beta-1}\exp\left[-\left(\frac{x}{\eta}\right)^{\beta}\right],\ x>0,
\end{aligned}
$$

respectively, where $\eta > 0$ is the scale parameter and $\beta > 0$ is the shape parameter. Then, the log-transformed component lifetimes follow a SEV distribution (also known as Gumbel Type I distribution) with location parameter μ and scale parameter σ, i.e.,

$$
\begin{aligned}
F_U(u;\mu,\sigma) &= \Phi_{sev}\left(\frac{u-\mu}{\sigma}\right), -\infty<u<\infty,\\
f_U(u;\mu,\sigma) &= \frac{1}{\sigma}\phi_{sev}\left(\frac{u-\mu}{\sigma}\right), -\infty<u<\infty,
\end{aligned}
$$

where $\sigma = 1/\beta$, $\mu = \ln(\eta)$, and $\Phi_{sev}(z) = 1-\exp[-\exp(z)]$ and $\phi_{sev}(z) = \exp[z-\exp(z)]$ are the CDF and PDF for the standard SEV distribution.

We consider the experimental planning for a fixed value of N and r when various types of coherent systems are used in the progressively Type-II censored experiments when the component lifetimes follow a Weibull distribution. In Tables 3.4 and 3.5, we present the optimal progressive censoring schemes for n-component coherent systems with Weibull distributed components under settings [**M1**] and [**M2**]. In Table 3.6, we present the optimal progressive censoring schemes for n-component coherent systems with Weibull distributed components ($n = 1, 2$, and 3) by fixing $R_{\max} = 8$ under setting [**M1**] and [**M2**].

Table 3.4: Optimal progressive censoring schemes for n-component systems with Weibull distributed components when $N = 30$, by fixing $n = 1, 2$, and 3 under setting **[M1]** and **[M2]**

System Type	n	Setting	m	r	$E(R)$	$R_{\max}$	R	**[C1]** $\lvert I \rvert$	**[C2]** $V_\mu + V_\sigma$	**[C3]** V_μ	**[C4]** $\overline{TTE}$	**[C5]** $\overline{TTT}$	$(1,1,10)$	$(1,0,10)$	**[C6]** $(1,1,0)$	$(0,1,10)$	$(0,1,5)$	$(0,1,20)$	$(0,1,30)$
Series	1	**[M1]**	30	6	6	6	(0,24,0,0,0,0)	**118.21**	0.2257	0.1749	15.89	44.33	24.15	8.26	21.89	18.15	17.02	20.40	22.66
							(24,0,0,0,0,0)	108.74	**0.2220**	**0.1668**	17.12	44.33	25.34	**8.22**	23.12	19.34	18.23	21.56	23.78
							(0,0,0,0,0,24)	43.19	0.6115	0.4726	**1.62**	**44.33**	**13.73**	12.12	**7.62**	**7.73**	**4.68**	**13.85**	**19.97**
		[M2]	30	6	6	6	(0,24,0,0,0,0)	**118.21**	0.2257	0.1749	15.89	44.33	24.15	8.26	21.89	18.15	17.02	20.40	22.66
							(24,0,0,0,0,0)	108.74	**0.2220**	**0.1668**	17.12	44.33	25.34	**8.22**	23.12	19.34	18.23	21.56	23.78
							(0,0,0,0,0,24)	43.19	0.6115	0.4726	**1.62**	**44.33**	13.73	12.12	**7.62**	7.73	4.68	13.85	19.97
							(12,0,0,0,0,12)	54.69	0.4157	0.3060	2.73	44.33	**12.89**	10.16	8.73	**6.89**	4.81	11.04	15.20
							(6,0,0,0,0,18)	47.67	0.5168	0.3909	2.01	44.33	13.18	11.17	8.01	7.18	**4.59**	12.35	17.51
							(16,0,0,0,0,8)	62.20	0.3464	0.25	3.66	44.33	13.13	9.46	9.66	7.13	5.39	**10.59**	14.05
							(18,0,0,0,0,6)	67.60	0.3121	0.2233	4.46	44.33	13.58	9.12	10.46	7.58	6.02	10.70	**13.82**
	2	**[M1]**	15	6	6	6	(0,9,0,0,0,0)	**87.96**	0.2759	0.2077	8.21	44.33	16.97	8.76	14.21	10.97	9.59	13.73	16.49
							(9,0,0,0,0,0)	84.47	**0.2600**	**0.1890**	8.68	44.33	17.28	**8.60**	14.68	11.28	9.98	13.88	**16.48**
							(0,0,0,0,0,9)	45.28	0.5713	0.4388	**1.81**	44.33	**13.52**	11.71	**7.81**	**7.52**	**4.66**	**13.23**	18.95
		[M2]	15	6	6	6	(0,9,0,0,0,0)	**87.96**	0.2759	0.2077	8.21	44.33	16.97	8.76	14.21	10.97	9.59	13.73	16.49
							(9,0,0,0,0,0)	84.47	**0.2600**	**0.1890**	8.68	44.33	17.28	**8.60**	14.68	11.28	9.98	13.88	16.48
							(0,0,0,0,0,9)	45.28	0.5713	0.4388	**1.81**	44.33	13.52	11.71	**7.81**	7.52	4.66	13.23	18.95
							(4,0,0,0,0,5)	53.45	0.4389	0.3267	2.63	44.33	**13.02**	10.39	8.63	**7.02**	4.83	11.41	15.8
							(1,0,0,0,0,8)	46.88	0.5394	0.4114	1.95	44.33	13.35	11.39	7.95	7.35	**4.65**	12.74	18.13
							(6,0,0,0,0,3)	60.46	0.3687	0.2695	3.51	44.33	13.20	9.69	9.51	7.20	5.36	**10.89**	14.58
							(7,0,0,0,0,2)	65.54	0.3330	0.2414	4.28	44.33	13.61	9.33	10.28	7.61	5.95	10.94	**14.27**
	3	**[M1]**	10	6	6	6	(0,4,0,0,0,0)	**73.65**	0.3251	0.2436	5.65	44.33	14.90	9.25	11.65	8.90	7.28	12.15	15.40
							(4,0,0,0,0,0)	72.31	**0.3091**	**0.2261**	5.87	44.33	14.96	**9.09**	11.87	8.96	7.42	**12.05**	**15.14**
							(0,0,0,0,0,4)	48.04	0.5268	0.4019	**2.08**	44.33	**13.35**	11.27	**8.08**	**7.35**	**4.71**	12.62	17.88
		[M2]	10	6	6	6	(0,4,0,0,0,0)	**73.65**	0.3251	0.2436	5.65	44.33	14.90	9.25	11.65	8.90	7.28	12.15	15.40
							(4,0,0,0,0,0)	72.31	**0.3091**	**0.2261**	5.87	44.33	14.96	**9.09**	11.87	8.96	7.42	12.05	15.14
							(0,0,0,0,0,4)	48.04	0.5268	0.4019	**2.08**	44.33	13.35	11.27	**8.08**	7.35	**4.71**	12.62	17.88
							(2,0,0,0,0,2)	55.62	0.4235	0.3156	2.94	44.33	**13.17**	10.23	8.94	**7.17**	5.06	11.41	15.64
							(3,0,0,0,0,1)	61.80	0.3683	0.2712	3.82	44.33	13.50	9.68	9.82	7.50	5.66	**11.19**	**14.87**
Parallel	2	**[M1]**	15	3	6	18	(12,0,0)	**60.87**	**0.2615**	0.1557	15.95	**97.65**	24.57	**8.62**	21.95	18.57	17.26	21.18	23.80
							(0,12,0)	54.21	0.2891	**0.1552**	12.4	110.03	21.29	8.89	18.4	15.29	13.85	18.18	21.07
							(0,0,12)	33.42	0.4327	0.1940	**4.15**	118.10	**14.48**	10.33	**10.15**	**8.48**	**6.31**	**12.80**	**17.13**
		[M2]	15	3	6	18	(12,0,0)	**60.87**	**0.2615**	0.1557	15.95	**97.65**	24.57	**8.62**	21.95	18.57	17.26	21.18	23.80
							(10,0,2)	47.95	0.2975	**0.1499**	8.45	103.94	17.43	8.98	14.45	11.43	9.94	14.40	17.38
							(0,0,12)	33.42	0.4327	0.1940	**4.15**	118.10	**14.48**	10.33	**10.15**	**8.48**	**6.31**	12.80	17.13
							(4,0,8)	36.36	0.388	0.1744	4.86	113.99	14.74	9.88	10.86	8.74	6.80	**12.62**	16.50
							(7,0,5)	40.11	0.3475	0.16	5.88	109.87	15.36	9.48	11.88	9.36	7.62	12.83	**16.31**
	3	**[M1]**	10	2	6	22	(8,0)	**39.77**	**0.3194**	0.1420	14.60	**153.70**	23.79	**9.19**	20.60	17.79	16.20	20.99	24.18
							(0,8)	26.33	0.4489	**0.1245**	**6.02**	174.83	**16.51**	10.49	**12.02**	**10.51**	**8.26**	**15.00**	**19.49**
		[M2]	10	2	6	22	(8,0)	**39.77**	**0.3194**	0.1420	14.60	**153.70**	23.79	**9.19**	20.60	17.79	16.20	20.99	24.18
							(2,6)	27.34	0.4313	**0.1238**	6.45	172.03	16.76	10.31	12.45	10.76	8.61	15.08	19.39
							(0,8)	26.33	0.4489	0.1245	**6.02**	174.83	**16.51**	10.49	**12.02**	**10.51**	**8.26**	**15.00**	19.49
							(3,5)	28.0495	0.4203	0.124	6.74	170.21	16.94	10.20	12.74	10.94	8.84	15.15	**19.35**
2-out-of-n	3	**[M1]**	10	3	6	13	(7,0,0)	**53.54**	**0.3004**	**0.1786**	8.95	**73.94**	17.95	**9**	14.95	11.95	10.45	14.96	**17.96**
							(0,0,7)	32.7	0.53	0.2971	**3.12**	85.94	**14.42**	11.3	**9.12**	**8.42**	**5.77**	**13.72**	19.02
		[M2]	10	3	6	13	(7,0,0)	**53.54**	**0.3004**	**0.1786**	8.95	**73.94**	17.95	**9.00**	14.95	11.95	10.45	14.96	17.96
							(0,0,7)	32.70	0.5300	0.2971	**3.12**	85.94	14.42	11.30	**9.12**	8.42	**5.77**	13.72	19.02
							(2,0,5)	34.86	0.4804	0.2664	3.53	83.75	**14.33**	10.80	9.53	**8.33**	5.93	13.14	17.94
							(5,0,2)	41.45	0.3835	0.2124	4.93	79.22	14.77	9.84	10.93	8.77	6.85	**12.60**	**16.44**
Series-parallel	3	**[M1]**	10	4	6 2/3	14	(6,0,0,0)	**43.58**	**0.3166**	**0.1964**	8.98	**62.30**	18.81	**9.83**	15.65	12.15	10.56	15.31	18.48
							(0,6,0,0)	43.01	0.3381	0.2152	8.06	64.96	18.11	10.05	14.73	11.44	9.75	14.82	**18.2**
							(0,0,0,6)	24.7	0.5605	0.3377	**2.73**	69.08	**15**	12.27	**9.4**	**8.34**	**5.53**	**13.94**	19.55
		[M2]	10	4	6 2/3	14	(6,0,0,0)	**43.58**	**0.3166**	**0.1964**	8.98	**62.30**	18.81	**9.83**	15.65	12.15	10.56	15.31	18.48
							(0,0,0,6)	24.70	0.5605	0.3377	**2.73**	69.08	15.00	12.27	**9.40**	8.34	**5.53**	13.94	19.55
							(2,0,0,4)	27.67	0.4807	0.282	3.35	67.73	**14.82**	11.47	10.02	**8.16**	5.75	12.96	17.77
							(4,0,0,2)	32.68	0.3967	0.2301	4.6	65.68	15.23	10.63	11.27	8.57	6.58	**12.53**	**16.50**
Parallel-series	3	**[M1]**	10	3	7	23	(7,0,0)	**47.19**	**0.3037**	**0.1824**	12.89	109.98	22.93	**10.04**	19.89	15.93	14.41	18.96	22.00
							(0,0,7)	30.62	0.4589	0.2274	**4.00**	**107.06**	**15.59**	11.59	**11.00**	**8.59**	**6.29**	13.18	17.77
		[M2]	10	3	7	23	(7,0,0)	**47.19**	**0.3037**	0.1824	12.89	109.98	22.93	**10.04**	19.89	15.93	14.41	18.96	22.00
							(6,0,1)	41.50	0.3287	**0.1798**	8.28	**100.83**	18.57	10.29	15.28	11.57	9.92	**14.85**	**18.14**
							(0,0,7)	30.62	0.4589	0.2274	**4.00**	107.06	**15.59**	11.59	**11.00**	**8.59**	**6.29**	13.18	17.77
							(2,0,5)	32.57	0.4217	0.2096	4.56	107.58	15.78	11.22	11.56	8.78	6.67	**12.99**	17.21
							(4,0,3)	35.63	0.3786	0.192	5.59	104.73	16.38	10.79	12.59	9.38	7.48	13.16	**16.95**

For criterion **[C6]**, the objective function is $\psi(\mathbf{R}) = c_1 E(R) + c_2 E(TTE) + c_3 (Var(\hat{\mu}) + Var(\hat{\sigma}))$.

Table 3.5: Optimal progressive censoring schemes for n-component coherent systems with Weibull distributed components when $N = 24$, by fixing $n = 1, 2, 3$, and 4 under setting **[M2]**

								[C1]	**[C2]**	**[C3]**	**[C4]**	**[C5]**			**[C6]**				
System Type	n	Setting	m	r	$E(R)$	$R_{\max}$	R	$\lvert\boldsymbol{I}\rvert$	$V_\mu + V_\sigma$	V_μ	$\overline{TTE}$	$\overline{TTT}$	$(1,1,10)$	$(1,0,10)$	$(1,1,0)$	$(0,1,10)$	$(0,1,5)$	$(0,1,20)$	$(0,1,30)$
Series	1	**[M1]**	24	4	4	4	(20,0,0,0)	**53.81**	**0.3297**	**0.2553**	13.85	29.56	21.15	**7.30**	17.85	17.15	15.5	20.45	23.74
							(0,20,0,0)	52.8	0.373	0.2972	11.71	29.56	19.44	7.73	15.71	15.44	13.58	19.17	**22.9**
							(0,0,20,0)	40.07	0.5025	0.4026	8.35	29.56	17.38	9.02	12.35	13.38	10.87	**18.4**	23.43
							(0,0,0,20)	20.09	0.9841	0.7849	**1.32**	29.56	**15.16**	13.84	**5.32**	**11.16**	**6.24**	21.00	30.84
		[M2]	24	4	4	4	(20,0,0,0)	**53.81**	**0.3297**	**0.2553**	13.85	29.56	21.15	**7.30**	17.85	17.15	15.50	20.45	23.74
							(0,0,0,20)	20.09	0.9841	0.7849	**1.32**	29.56	15.16	13.84	**5.32**	11.16	6.24	21.00	30.84
							(14,0,0,6)	29.77	0.5492	0.4149	3.11	29.56	**12.60**	9.49	7.11	**8.60**	5.85	14.09	19.59
							(10,0,0,10)	25.20	0.6899	0.5312	2.16	29.56	13.06	10.90	6.16	9.06	**5.61**	15.96	22.86
							(16,0,0,4)	33.58	0.4750	0.3559	4.07	29.56	12.82	8.75	8.07	8.82	6.45	**13.57**	18.32
							(17,0,0,3)	36.25	0.4375	0.3271	4.86	29.56	13.24	8.37	8.86	9.24	7.05	13.61	**17.99**
	2	**[M1]**	12	4	4	4	(8,0,0,0)	**41.02**	**0.3984**	**0.3009**	7.08	29.56	15.07	**7.98**	11.08	11.07	9.07	**15.05**	**19.03**
							(0,0,8,0)	32.15	0.5982	0.4738	4.71	29.56	**14.69**	9.98	8.71	**10.69**	7.7	16.67	22.65
							(0,0,0,8)	20.74	0.9377	0.7448	**1.42**	29.56	14.8	13.38	**5.42**	10.8	**6.11**	20.18	29.55
		[M2]	12	4	4	4	(8,0,0,0)	**41.02**	**0.3984**	**0.3009**	7.08	29.56	15.07	**7.98**	11.08	11.07	9.07	15.05	19.03
							(0,0,0,8)	20.74	0.9377	0.7448	**1.42**	29.56	14.80	13.38	**5.42**	10.80	6.11	20.18	29.55
							(6,0,0,2)	29.82	0.5549	0.4208	3.20	29.56	**12.75**	9.55	7.20	**8.75**	5.98	14.30	19.85
							(4,0,0,4)	25.18	0.6966	0.5378	2.19	29.56	13.16	10.97	6.19	9.16	**5.67**	16.12	23.09
							(7,0,0,1)	33.83	0.4787	0.3605	4.31	29.56	13.10	8.79	8.31	9.10	6.70	**13.88**	**18.67**
	3	**[M1]**	8	4	4	4	(4,0,0,0)	**34.79**	**0.4738**	**0.3588**	4.82	29.56	**13.56**	**8.74**	8.82	**9.56**	7.19	**14.30**	**19.04**
							(0,0,0,4)	21.56	0.8869	0.7013	**1.56**	29.56	14.43	12.87	**5.56**	10.43	**6.00**	19.30	28.17
		[M2]	8	4	4	4	(4,0,0,0)	**34.79**	**0.4738**	**0.3588**	4.82	29.56	13.56	**8.74**	8.82	9.56	7.19	**14.30**	**19.04**
							(0,0,0,4)	21.56	0.8869	0.7013	**1.56**	29.56	14.43	12.87	**5.56**	10.43	6.00	19.30	28.17
							(3,0,0,1)	28.60	0.5931	0.4532	2.98	29.56	**12.91**	9.93	6.98	**8.91**	5.94	14.84	20.77
							(2,0,0,2)	25.27	0.7001	0.5418	2.24	29.56	13.24	11.00	6.24	9.24	**5.74**	16.24	23.24
	4	**[M1]**	6	4	4	4	(2,0,0,0)	**30.94**	**0.5481**	**0.4188**	3.69	29.56	**13.18**	**9.48**	7.69	**9.18**	6.43	**14.66**	**20.14**
							(0,0,0,2)	22.61	0.8303	0.6534	**1.75**	29.56	14.06	12.30	**5.75**	10.06	**5.91**	18.36	26.66
		[M2]	6	4	4	4	(2,0,0,0)	**30.94**	**0.5481**	**0.4188**	3.69	29.56	**13.18**	**9.48**	7.69	**9.18**	6.43	**14.66**	**20.14**
							(0,0,0,2)	22.61	0.8303	0.6534	**1.75**	29.56	14.06	12.30	**5.75**	10.06	5.91	18.36	26.66
							(1,0,0,1)	25.47	0.7000	0.5429	2.31	29.56	13.31	11.00	6.31	9.31	**5.81**	16.31	23.31
Parallel	2	**[M1]**	12	2	4	14	(10,0)	**28.18**	**0.3930**	**0.2292**	11.89	**72.51**	19.82	**7.93**	15.89	15.82	13.85	19.75	23.68
							(0,10)	16.14	0.6471	0.3113	**3.60**	83.57	**14.07**	10.47	**7.60**	**10.07**	**6.83**	**16.54**	**23.01**
		[M2]	12	2	4	14	(10,0)	**28.18**	**0.3930**	**0.2292**	11.89	**72.51**	19.82	**7.93**	15.89	15.82	13.85	19.75	23.68
							(0,10)	16.14	0.6471	0.3113	**3.60**	83.57	14.07	10.47	**7.60**	10.07	**6.83**	16.54	23.01
							(2,8)	16.58	0.6223	0.2989	3.84	82.41	**14.06**	10.22	7.84	**10.06**	6.95	16.29	22.51
							(7,3)	19.68	0.5139	0.2538	5.34	78.19	14.48	9.14	9.34	10.48	7.91	**15.62**	20.76
							(8,2)	21.13	0.4810	0.2432	6.15	76.87	14.96	8.81	10.15	10.96	8.56	15.77	**20.58**
	3	**[M1/M2]**	8	2	6	18	(6,0)	**35.86**	**0.3394**	0.1501	14.85	**139.03**	24.24	**9.39**	20.85	18.24	16.54	21.64	25.03
							(0,6)	25.23	0.4445	**0.1246**	**6.77**	155.72	**17.22**	10.44	**12.77**	**11.22**	**8.99**	**15.66**	**20.11**
2-out-of-n	3	**[M1]**	10	2	6	13	(6,0)	**24.78**	**0.4787**	**0.2929**	6.81	**54.21**	15.60	**8.79**	10.81	11.60	9.20	**16.39**	**21.17**
							(0,6)	15.66	0.8093	0.4781	**2.67**	60.81	**14.76**	12.09	**6.67**	**10.76**	**6.71**	18.85	26.95
		[M2]	10	2	6	13	(6,0)	**24.78**	**0.4787**	**0.2929**	6.81	**54.21**	15.60	**8.79**	10.81	11.60	9.20	16.39	**21.17**
							(0,6)	15.66	0.8093	0.4781	**2.67**	60.81	14.76	12.09	**6.67**	10.76	6.71	18.85	26.95
							(4,2)	18.53	0.6432	0.3756	3.72	57.65	**14.16**	10.43	7.72	**10.16**	6.94	16.59	23.02
							(1,5)	16.10	0.7765	0.4565	2.81	60.24	14.58	11.76	6.81	10.58	**6.69**	18.34	26.10
							(5,1)	20.54	0.5724	0.3371	4.55	56.28	14.28	9.72	8.55	10.28	7.42	**16.00**	21.73
Series-parallel	3	**[M1]**	8	2	4 2/3	18	(6,0)	**12.73**	**0.6738**	**0.4688**	5.51	**33.34**	**16.92**	**11.41**	10.18	**12.25**	8.88	**18.99**	**25.73**
							(0,6)	6.93	1.3139	0.9259	**1.62**	36.33	19.42	17.81	**6.28**	14.76	**8.19**	27.89	41.03
		[M2]	8	2	4 2/3	18	(6,0)	**12.73**	**0.6738**	**0.4688**	5.51	**33.34**	16.92	**11.41**	10.18	12.25	8.88	**18.99**	**25.73**
							(0,6)	6.93	1.3139	0.9259	**1.62**	36.33	19.42	17.81	**6.28**	14.76	8.19	27.89	41.03
							(5,1)	10.11	0.8283	0.5637	3.35	34.48	**16.30**	12.95	8.02	**11.63**	7.49	19.92	28.20
							(4,2)	8.78	0.9656	0.6593	2.57	35.13	16.89	14.32	7.24	12.23	**7.40**	21.88	31.54
Parallel-series	3	**[M1]**	8	2	3 1/3	10	(6,0)	**22.11**	**0.4650**	**0.2784**	9.56	**71.24**	17.55	**7.98**	12.90	14.21	11.89	18.86	**23.51**
							(0,6)	14.70	0.6976	0.3678	**3.40**	77.58	**13.71**	10.31	**6.73**	**10.38**	**6.89**	**17.35**	24.33
		[M2]	8	2	3 1/3	10	(6,0)	**22.11**	**0.4650**	**0.2784**	9.56	**71.24**	17.55	**7.98**	12.90	14.21	11.89	18.86	23.51
							(0,6)	14.70	0.6976	0.3678	**3.40**	77.58	13.71	10.31	**6.73**	10.38	**6.89**	17.35	24.33
							(2,4)	15.56	0.6456	0.3402	3.86	76.14	**13.65**	9.79	7.19	**10.32**	7.09	16.77	23.23
							(4,2)	17.21	0.5740	0.3078	4.86	74.15	13.93	9.07	8.19	10.60	7.73	**16.34**	22.08
							(5,1)	18.86	0.5252	0.2908	6.06	72.82	14.64	8.58	9.39	11.31	8.68	16.56	**21.81**

For criterion **[C6]**, the objective function is $\psi(\mathbf{R}) = c_1E(R) + c_2E(TTE) + c_3(Var(\hat{\mu}) + Var(\hat{\sigma}))$.

Table 3.6: Optimal allocations for n-component systems with Weibull distributed components when $N = 18$, by fixing $R_{\max} = 8$ with $n = 1$ and 2 under setting **[M1]** and **[M2]**

								[C1]	[C2]	[C3]	[C4]	[C5]			[C6]				
System Type	n	Setting	m	r	$E(R)$	$R_{\max}$	R	$\lvert\boldsymbol{I}\rvert$	$V_\mu + V_\sigma$	V_μ	$\overline{TTE}$	$\overline{TTT}$	$(1,1,10)$	$(1,0,10)$	$(1,1,0)$	$(0,1,10)$	$(0,1,5)$	$(0,1,20)$	$(0,1,30)$
Series	1	[M1]	18	8	8	8	(0,10,0,0,0,0,0,0,0)	**150.55**	**0.1783**	**0.1251**	18.95	59.11	28.73	**9.78**	26.95	20.73	19.84	22.51	24.3
							(0,0,0,0,0,0,0,0,10)	79.52	0.2664	0.1658	**4.18**	59.11	**14.85**	10.66	**12.18**	**6.85**	**5.52**	**9.51**	**12.17**
		[M2]	18	8	8	8	(0,10,0,0,0,0,0,0,0)	**150.55**	**0.1783**	0.1251	18.95	59.11	28.73	**9.78**	26.95	20.73	19.84	22.51	24.30
							(4,4,1,0,1,0,0,0)	141.87	0.1814	**0.1250**	18.16	59.11	27.97	9.81	26.16	19.97	19.07	21.79	23.60
							(0,0,0,0,0,0,0,0,10)	79.52	0.2664	0.1658	**4.18**	59.11	**14.85**	10.66	**12.18**	**6.85**	**5.52**	**9.51**	**12.17**
	2	[M1]/[M2]	9	8	8	8	(0,1,0,0,0,0,0,0,0)	**110.90**	0.2050	0.1329	9.92	59.11	19.97	10.05	17.92	11.97	10.95	14.02	16.07
							(1,0,0,0,0,0,0,0,0)	110.03	**0.2045**	**0.1318**	9.99	59.11	20.03	**10.04**	17.99	12.03	11.01	14.08	16.12
							(0,0,0,0,0,0,0,0,1)	95.99	0.2261	0.1427	**6.76**	59.11	**17.02**	10.26	**14.76**	**9.02**	**7.89**	**11.28**	**13.54**
2-out-of-n	3	[M1]	6	2	4	8	(4,0)	**21.87**	**0.4807**	**0.2766**	7.01	**49.67**	19.82	**12.81**	15.01	11.82	9.42	**16.63**	**21.43**
							(0,4)	15.14	0.7112	0.3857	**3.25**	54.42	**18.36**	15.11	**11.25**	**10.36**	**6.80**	17.47	24.58
		[M2]	6	2	4	8	(4,0)	**21.87**	**0.4807**	**0.2766**	7.01	**49.67**	19.82	**12.81**	15.01	11.82	9.42	16.63	**21.43**
							(0,4)	15.14	0.7112	0.3857	**3.25**	54.42	18.36	15.11	**11.25**	10.36	**6.80**	17.47	24.58
							(2,2)	16.60	0.6294	0.3407	3.95	52.77	**18.25**	14.29	11.95	**10.25**	7.10	16.54	22.84
							(3,1)	18.24	0.5671	0.3105	4.77	51.56	18.44	13.67	12.77	10.44	7.61	**16.11**	21.78
Series-parallel	3	[M1]	6	2	3 1/3	8	(4,0)	**11.31**	**0.6708**	**0.4379**	5.70	**32.60**	**20.41**	**14.71**	13.70	**12.41**	9.06	**19.12**	**25.83**
							(0,4)	6.94	1.1000	0.7067	**2.13**	35.03	21.13	19.00	**10.13**	13.13	**7.63**	24.13	35.13
		[M2]	6	2	3 1/3	8	(4,0)	**11.31**	**0.6708**	**0.4379**	5.70	**32.60**	20.41	**14.71**	13.70	12.41	9.06	**19.12**	**25.83**
							(0,4)	6.94	1.1000	0.7067	**2.13**	35.03	21.13	19.00	**10.13**	13.13	7.63	24.13	35.13
							(3,1)	8.99	0.8145	0.5141	3.55	33.70	**19.70**	16.15	11.55	**11.70**	7.62	19.84	27.99
							(2,2)	7.91	0.9328	0.5890	2.78	34.31	20.11	17.33	10.78	12.11	**7.44**	21.44	30.76

For criterion **[C6]**, the objective function is $\psi(\mathbf{R}) = c_1 R_{\max} + c_2 E(TTE) + c_3(Var(\hat{\mu}) + Var(\hat{\sigma}))$.

3.6.3. Discussions

Based on the numerical results presented in Table 3.1, for optimality criteria **[C1]** (D-optimality), **[C2]** (A-optimality) and **[C4]** ($\overline{TTE}$), we would choose to perform the life testing experiment using individual components (i.e., $n = 1$) over other types of systems. Specifically, for optimal criterion **[C1]** (D-optimality) that maximizes the determinant of the expected Fisher information matrix, when $N = 30$ and $r = 6$, we would censor all 24 components at the second observed failure. In addition, for optimal criterion **[C2]** (A-optimality) that minimizes the sum of the asymptotic variance of the MLEs, we would choose to perform a progressively Type-II censored experiment by censoring $R_1 = 23$ and $R_6 = 1$ components at the first and last observed failures, respectively. For optimal criterion **[C4]** ($\overline{TTE}$) that minimizes the expected total time on test, the optimal scheme would censor all of the 24 components at the last observed failure, which is the conventional Type-II censoring scheme. This result agrees with our intuition as the conventional Type-II censoring scheme gives the shortest experimental time among all the progressive censoring schemes.

When we consider $E(R) = 6$ and experimental schemes that give $E(R)$ close to 6, the optimal values of the objective functions for **[C1]**, **[C2]**, and **[C4]** when using series, parallel, 2-out-of-n, series-parallel, and parallel-series systems are ($|I|= 194.48$, $V_\mu + V_\sigma = 0.1933$, $\overline{TTE} = 3.13$), ($|I|= 70.43$, $V_\mu + V_\sigma = 0.2465$, $\overline{TTE} = 6.23$), ($|I|= 80.81$, $V_\mu + V_\sigma = 0.2513$, $\overline{TTE} = 4.96$), ($|I|= 79.86$, $V_\mu + V_\sigma = 0.2471$, $\overline{TTE} = 4.48$) and ($|I|= 63.56$, $V_\mu + V_\sigma = 0.2706$, $\overline{TTE} = 6.04$), respectively. If one considers the optimal criterion **[C6]** that minimizes a combined cost function involving the costs due to failures and time and inaccuracy of parameter estimation, based on the choices of c_1, c_2, and c_3, which are the cost per unit if failed components, the cost per unit of time for the duration of the experiment, and the cost of variance of the estimates of the unknown model parameters, respectively, putting individual components in the experiments still produce best performance compared to other settings.

Note that for series-parallel and parallel-series systems, the values of $E(R)$ are slightly larger than those for other systems, which might bring some disadvantages in comparison.

Suppose one considers the optimal criterion **[C3]** (V-optimality) that minimizes the asymptotic variance of the MLE, the optimal values of the objective function (i.e., V_μ) for **[C3]** based on series, parallel, 2-out-of-n, series-parallel, and parallel-series systems are 0.1330, 0.0980, 0.1181, 0.1381, and 0.1144, respectively. From these results, we suggest performing the progressively Type-II censored experiment by putting the 30 components into ten 3-component parallel systems and censoring eight systems at the second failure. However, the trade-off of using 3-component parallel systems for the progressively Type-II censored experiment is that the expected total time on test and the expected total experimental time based on 3-component parallel systems are the longest compared to the other systems considered here.

From Tables 3.4 and 3.5, we observe a similar pattern when the underlying component lifetimes follow Weibull distribution. For example, suppose there are 24 components available for the experiment, and we plan to fail four components. From Table 3.5, when $N = 24$, $E(R) = 4$ and the component lifetimes follow a Weibull distribution, for optimal criterion **[C3]**, the optimal experimental plan based on 3-component and 2-component parallel systems give the optimal values 0.1246 and 0.2292, respectively, while the optimal experimental plan based on individual components gives 0.2553. In other words, using 3-component and 2-component parallel systems for the progressively Type-II censored experiment provides 51.19% and 10.22% improvements, respectively, in terms of the criterion **[C3]** compared to using individual components for the experiment. However, the value of $E(R)$ based on 3-component parallel systems is 6, which is larger than the target value of $E(R) = 4$.

Instead of fixing the value of $E(R)$, in Tables 3.3 and 3.6, we compare the optimal progressive censoring schemes based on different coherent systems by fixing the value of $R_{\max}$. From Tables 3.3 and 3.6, we observe that for a fixed value of $R_{\max}$, although using

2-out-of-3 or series-parallel systems in a progressively Type-II censored experiment do not always give an advantage over using series systems; they always give shorter expected TTE and TTT for both lognormal and Weibull distributed components.

The numerical results show the advantages and disadvantages of using systems in an experiment with progressively Type-II censored samples that do not show a specific pattern. The performance depends on the type of systems, the value of $E(R)$ (or $R_{\max}$), and the underlying component distributions. Therefore, it is crucial to evaluate the performance of the optimal experimental plans under different types of systems and different values of n to determine the optimal experimental planning for a specific situation, and the methodologies proposed here are helpful for this purpose.

3.7. Practical Example

In this section, we use a practical example of insulating fluids testing presented in Viveros and Balakrishnan (1994) and Ng et al. (2004) to illustrate the methodologies developed in this chapter for planning a future progressively Type-II censored experiment. Ng et al. (2004) compared the efficiency of different progressively Type-II censoring plans with $N = 19$ components effective sample size $r = 8$ (i.e., the number of observed failures). For a future test, suppose the total number of insulating fluids available is $N = 19$, and we consider putting the insulating fluids into a n-component series system with $n = 1$ (i.e., component-wise experiment), and $n = 2$ (i.e., nine series systems).

Assuming the lifetimes of the insulating fluids follow Weibull distribution and lognormal distribution and considering the optimality criteria and settings described in Section 3.4, we determine the optimal experimental plans for different optimality criteria under different settings. The optimal progressive censoring schemes for different criteria under the assumption that the component lifetimes follow Weibull and lognormal distributions are presented in Tables 3.7 and 3.8, respectively.

From the results in Table 3.7, the progressively Type-II censored experimental plans based on $n = 1$ are preferred over $n = 2$ for different optimality criteria. For example, for optimality criterion [**C2**], the optimal value of $Var(\hat{\mu}) + Var(\hat{\sigma})$ based on setting [**M2**] for $n = 1$ and $n = 2$ are 0.1770 and 0.2045, respectively. For various combinations of costs c_1, c_2, and c_3, the performance of optimal experimental plans for $n = 1$ still outperforms those for $n = 2$. In practice, engineers might need to do all censoring at one time in certain circumstances. Therefore, the setting [**M1**] (i.e., censoring all of the $m - r$ systems or components at a certain failure) can be considered instead of setting [**M2**] in a practical situation. If setting [**M1**] is considered, based on the results presented in Table 3.7, the optimal experimental plans based on $n = 1$ are still preferred over $n = 2$.

We have similar observations in Table 3.8. If we consider setting [**M2**], the progressively Type-II censored experimental plans with a single component ($n = 1$) consistently outperform the plans with two components ($n = 2$) across multiple criteria. For instance, for optimality criterion [**C3**], the optimal value of $Var(\hat{\mu})$ based on setting [**M2**] for $n = 1$ and $n = 2$ are 0.0958 and 0.0963, respectively. For setting [**M1**], if we use criteria [**C3**] and [**C5**], then using two-component systems in progressively Type-II censored experiments would yield better performance than those of using components.

Table 3.7: Optimal progressive censoring schemes for n-component series systems with Weibull distributed components when $N = 19$, by fixing $r = 8$ with $n = 1$ and 2 under settings **[M1]** and **[M2]**

							[C1]	**[C2]**	**[C3]**	**[C4]**	**[C5]**	**[C6]** with (c_1, c_2, c_3)						
n	Setting	m	r	$E(R)$	$R_{\max}$	R	$\lvert \boldsymbol{I} \rvert$	$Var(\hat{\mu}) + Var(\hat{\sigma})$	$Var(\hat{\mu})$	$\overline{TTE}$	$\overline{TTT}$	$(1,1,10)$	$(1,0,10)$	$(1,1,0)$	$(0,1,10)$	$(0,1,5)$	$(0,1,20)$	$(0,1,30)$
1	**[M1]**	19	8	8	8	(11,0,0,0,0,0,0,0)	144.42	0.1826	0.1272	19.55	59.11	29.37	9.83	27.55	21.37	20.46	23.2	25.02
						(0,11,0,0,0,0,0,0)	**154.11**	**0.1770**	**0.1250**	18.90	59.11	28.67	**9.77**	26.90	20.67	19.79	22.44	24.21
						(0,0,0,0,0,0,0,11)	78.98	0.2756	0.1743	**3.90**	59.11	**14.66**	10.76	**11.9**	**6.66**	**5.28**	**9.41**	**12.17**
	[M2]	19	8	8	8	(0,11,0,0,0,0,0,0)	**154.11**	**0.1770**	0.1250	18.90	59.11	28.67	**9.77**	26.90	20.67	19.79	22.44	24.21
						(8,0,1,1,1,0,0,0)	140.40	0.1820	**0.1250**	18.03	59.11	27.85	9.82	26.03	19.85	18.94	21.67	23.49
						(0,0,0,0,0,0,0,11)	78.98	0.2756	0.1743	**3.90**	59.11	**14.66**	10.76	**11.90**	**6.66**	**5.28**	**9.41**	12.17
						(1,0,0,0,0,0,0,10)	81.24	0.2656	0.1671	4.16	59.11	14.82	10.66	12.16	6.82	5.49	9.47	**12.13**
2	**[M1]** and **[M2]**	9	8	8	8	(0,1,0,0,0,0,0,0)	**110.90**	0.2050	0.1329	9.92	59.11	19.97	10.05	17.92	11.97	10.95	14.02	16.07
						(1,0,0,0,0,0,0,0)	110.03	**0.2045**	**0.1318**	9.99	59.11	20.03	**10.04**	17.99	12.03	11.01	14.08	16.12
						(0,0,0,0,0,0,0,1)	95.99	0.2261	0.1427	**6.76**	59.11	**17.02**	10.26	**14.76**	**9.02**	**7.89**	**11.28**	**13.54**

Table 3.8: Optimal progressive censoring schemes for n-component series systems with lognormal distributed components when $N = 19$, by fixing $r = 8$ with $n = 1$ and 2 under settings [**M1**] and [**M2**]

							[**C1**]	[**C2**]	[**C3**]	[**C4**]	[**C5**]	[**C6**] with (c_1, c_2, c_3)						
n	Setting	m	r	$E(R)$	$R_{\max}$	R	$\lvert \boldsymbol{I} \rvert$	$Var(\hat{\mu}) + Var(\hat{\sigma})$	$Var(\hat{\mu})$	$\overline{TTE}$	$\overline{TTT}$	$(1, 1, 10)$	$(1, 0, 10)$	$(1, 1, 0)$	$(0, 1, 10)$	$(0, 1, 5)$	$(0, 1, 20)$	$(0, 1, 30)$
1	[**M1**]	19	8	8	8	(0,0,11,0,0,0,0,0)	**225.27**	0.1543	0.1020	32.82	105.26	42.36	9.54	40.82	34.36	33.59	35.91	37.45
						(0,11,0,0,0,0,0,0)	221.84	**0.1540**	0.1047	34.70	105.99	44.24	**9.54**	42.70	36.24	35.47	37.78	39.32
						(0,0,0,0,0,0,0,11)	186.53	0.1741	**0.0975**	**5.92**	**93.61**	**15.66**	9.74	**13.92**	**7.66**	**6.79**	**9.40**	**11.14**
	[**M2**]	19	8	8	8	(0,0,11,0,0,0,0,0)	**225.27**	0.1543	0.1020	32.82	105.26	42.36	9.54	40.82	34.36	33.59	35.91	37.45
						(0,10,1,0,0,0,0,0)	222.06	**0.1540**	0.1041	34.28	105.72	43.82	**9.54**	42.28	35.82	35.05	37.36	38.90
						(4,0,0,0,0,0,0,7)	191.72	0.1639	**0.0958**	7.64	**93.02**	17.28	9.64	15.64	9.28	8.46	10.92	12.56
						(0,0,0,0,0,0,0,11)	186.53	0.1741	0.0975	**5.92**	93.61	**15.66**	9.74	**13.92**	**7.66**	**6.79**	**9.40**	**11.14**
2	[**M1**] and [**M2**]	9	8	8	8	(0,1,0,0,0,0,0,0)	**200.68**	0.1590	0.0976	14.76	95.39	24.35	9.59	22.76	16.35	15.56	17.94	19.53
						(1,0,0,0,0,0,0,0)	198.19	**0.1589**	0.0976	14.85	95.11	24.44	**9.59**	22.85	16.44	15.65	18.03	19.62
						(0,0,0,0,0,0,0,1)	192.95	0.1633	**0.0963**	**9.71**	**93.42**	**19.34**	9.63	**17.71**	**11.34**	**10.52**	**12.97**	**14.61**

3.8. Concluding Remarks

In this chapter, we discuss the optimal experimental planning for progressively Type-II censored experiments when the experimental units can be put into coherent systems for the life testing experiments. The component lifetime distributions are assumed to follow a distribution in the log-location-scale family of distributions, a general class of distributions that includes some commonly used lifetime distributions, such as the Weibull and the lognormal distributions. The formulations required to obtain the optimal experimental planning are presented in terms of system signatures for different optimality criteria. Motivated by the insulating fluids testing presented in Balakrishnan and Aggarwala (2000) and Ng et al. (2004), the proposed methodologies are illustrated by the special case that the experimental units are put into series systems for the life testing experiment. The optimal progressive Type-II censoring schemes for different types of systems and different numbers of components available for the experiment are provided. We have shown that considering the use of systems formed by the test units in a progressively Type-II censored experiment provides extra flexibility in planning an experiment and gains efficiency in some situations.

In summary, the performance of utilizing systems/components in progressively Type-II censored experiments depends on the types of systems and criteria. Using systems for the experiment does not always confer advantages over using individual components, but it enhances estimation efficiency and reduces the expected total time spent on the experiment when it does prove superior. Comparing the performance of optimal experimental plans based on different coherent systems with the number of components $n > 3$ would be an intriguing avenue for further research.

CHAPTER 4

Concluding Remarks and Future Research Directions

4.1. Summaries and Concluding Remarks

In this thesis, we investigate the planning of life-testing experiments when the experimental units can be put into coherent systems. Different experimental schemes, including the multi-level constant-stress accelerated life testing scheme and the progressive Type-II censoring scheme, are considered. The advantages and disadvantages of the experimental plans based on components and systems are discussed, and practical recommendations are made based on the results obtained in this thesis. The major contributions of Chapters 2 and 3 are summarized here.

4.1.1. Major contributions of Chapter 2

In Chapter 2, we discuss the optimal experimental planning for multi-level stress experiments with Type-II censoring, where experimental units can be put into coherent systems for the life testing experiment. We assume that the component lifetime distributions belong to a log-location-scale family distribution, encompassing commonly used distributions such as Weibull and lognormal. We present formulations for obtaining optimal experimental planning for different optimality criteria. A reliability test for furniture joints is used to illustrate the proposed methodologies with series systems and a sensitivity analysis for misspecification of the underlying lifetime distribution is also provided. While using systems in multi-level constant-stress experiments with Type-II censoring does not always provide advantages, we demonstrate that it always adds flexibility, improves efficiency, and reduces the expected total time spent on the experiment for all components in certain situations compared to using individual components.

4.1.2. Major contributions of Chapter 3

In Chapter 3, we discuss the optimal experimental planning for progressive Type-II censoring. Similar to the setting in Chapter 2, we assume the component lifetimes follow a log-location-scale family of distribution. We provide formulas for obtaining optimal experimental planning by using different types of systems with $n \leq 4$ components for various optimality criteria. We demonstrate the proposed methodologies with a reliability test with insulating fluids. While the performance of the optimal experimental plans for different types of systems under different optimality criteria are evaluated, we show that the advantages and disadvantages of using systems in progressively Type-II censored experiments vary based on the structure of the system $E(R)$ (or the maximum possible number of component failures $R_{\max}$), and the underlying component lifetime distribution. Based on the results obtained in this thesis, it is recommended to consider optimal progressive censoring schemes with both component-level and system-level experiments and to determine the best experimental planning strategy for a specific situation and optimal criterion.

4.2. Future Research Directions

In this thesis, we study the optimal experimental planning for multi-level and single-level stress constant-stress accelerated life test with Type-II censoring and progressive Type-II censoring in Chapters 2 and 3. The following research directions can be explored for future research on this topic.

4.2.1. Extensions to other experimental schemes

Following the study in this thesis, one potential direction is to study the optimal experimental planning for the step-stress accelerated life test (SSALT) described in Section 1.6.3. The primary interest in the optimal experimental planning for an SSALT is to determine the optimal time points for changing the stress level of the experiment when all the experimental units or the systems formed by those experimental units are placed on the life tests with the same start-up stress level.

4.2.1.1. Step-stress ALT with Type-II censoring

Consider a step-stress ALT with K stress levels, $y_1 < y_2 < \ldots < y_K$, in the test with $m \times n$ experimental units and a pre-fixed stress change points $\tau_1 < \tau_2 < \ldots < \tau_{K-1}$. Suppose the experimental units are put into m n-component coherent systems in which the system signature $(s_1, s_2, \ldots, s_n)$ and the minimal signature $(a_1, a_2, \ldots, a_n)$ defined in Section 1.2.2 are known. Initially, all the m coherent n-component systems are placed on the life testing experiment starting with the first stress level y_1. At time τ_1, the stress level is changed from y_1 to y_2, and we denote the number of systems that failed in period $[0, \tau_1)$ as m_1. Then, the life test continues with the remaining $m - m_1$ systems. At time τ_2, the stress level is changed to y_3. We denote the number of systems failed in $[\tau_1, \tau_2)$ as m_2. The experiment continues in this manner until we reach stress level y_K, and the experiment will be terminated when the r-th system failure is observed, where r is pre-fixed (i.e., Type-II censoring). Here, we denote the total number of failures from the first k stress levels as $r_k = \sum_{q=1}^{k} m_q$. Since there is a positive probability that the r-th system failure (i.e., the termination time of the experiment) occurs before reaching the K-th stress level, we denote the stress level that the experiment is terminated at as k^*-th stress level, i.e.,

$$k^* = \left(\max_{k \in (1,2,\ldots,K-1)} \{k : r_k < r\} \right) + 1.$$

We further denote the system failure times at the k-th stress level as $T_{k1}, T_{k2}, \ldots, T_{km_k}$ for $k = 1, 2, \ldots, K$. Thus, the observed data are

$$
\begin{aligned}
&t_{11}, t_{12}, \ldots, t_{1m_1}, \\
&t_{21}, t_{22} \ldots, t_{2m_2}, \ldots, \ldots, \\
&t_{(k^*-1)1}, t_{(k^*-1)2}, \ldots, t_{(k^*-1)m_{(k^*-1)}}, \\
&t_{k^*1}, t_{k^*2}, \ldots, t_{k^*(r-r_{k^*-1})}.
\end{aligned}
$$

4.2.1.2. Cumulative exposure model

A cumulative exposure model is commonly used to model data obtained from step-stress ALTs. This model connects the distribution of the lifetime on the current stress level with the distribution on the preceding level, assuming no effect from other stress levels. Assume the lifetimes of the experimental units (components) follow a log-location-scale distribution and hence, the log-transformed lifetimes of the experimental units working under stress level k (i.e., y_k) follow a location-scale distribution with location parameter μ_k and scale parameter σ (see, Section 1.4). We also assume the location parameter μ_k of the log-transformed component lifetimes is related to the stress levels with a linear relationship with parameters β_0 and β_1, that is,

$$\mu_k = \beta_0 + \beta_1 y_k,$$

$k = 1, 2, \ldots, K$.

We denote the PDF and SF of the log-transformed component lifetimes on the k-th stress level respectively by $f_{U_k}(\cdot; \mu_k, \sigma)$ and $\bar{F}_{U_k}(\cdot; \mu_k, \sigma)$ with location parameters μ_k and a common scale parameter σ, for $k = 1, 2, \ldots, K$. For the i-th order statistics among the n

log-transformed component lifetimes in a system tested at the k-th stress level, we denote the PDF and SF of the i-th ordered component log-lifetime as $f_{i:n}^{U_k}(\cdot; \mu_k, \sigma)$ and $\bar{F}_{i:n}^{U_k}(\cdot; \mu_k, \sigma)$, respectively. Then, the PDF and SF of the log-transformed system lifetimes at the k-th stress level, denoted as $f_{V_k}(\cdot; \mu_k, \sigma)$ and $\bar{F}_{V_k}(\cdot; \mu_k, \sigma)$, respectively, for $k = 1, 2, \ldots, K$, can be expressed as

$$\begin{aligned} f_{V_k}(v; \mu_k, \sigma) &= \sum_{i=1}^{n} a_i f_{1:i}^{U_k}(v; \mu_k, \sigma) = \sum_{i=1}^{n} a_i i f_{U_k}(v; \mu_k, \sigma)[\bar{F}_{U_k}(v; \mu_k, \sigma)]^{i-1}, \\ \bar{F}_{V_k}(v; \mu_k, \sigma) &= \sum_{i=1}^{n} a_i \bar{F}_{1:i}^{U_k}(v; \mu_k, \sigma) = \sum_{i=1}^{n} a_i [\bar{F}_{U_k}(v; \mu_k, \sigma)]^{i}, \end{aligned}$$

as presented in Section 1.4.

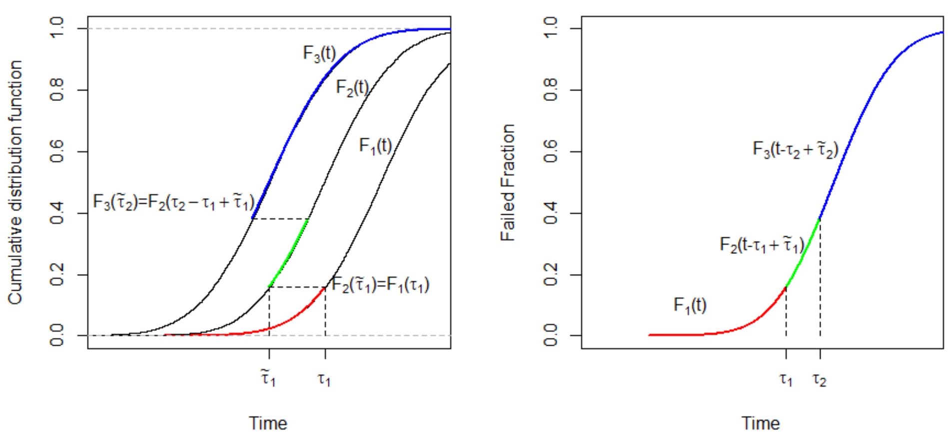

Figure 4.1: Illustration of the CDF of lifetime distribution under step-stress testing with three stress levels based on cumulative exposure model

In the step-stress ALT with Type-II censoring described in Section 4.2.1.1, we start the life test with all m coherent n-component systems at stress level y_1, where the CDF of the log-transformed component lifetime is $F_{U_1}(u; \mu_1, \sigma)$. At time τ_1 (corresponding to log-transformed time $\ln \tau_1$), the stress level changes from y_1 to y_2, at which the surviving systems (and the surviving components in those systems) keep running with the CDF $F_{U_2}(u; \mu_2, \sigma)$. Under the cumulative exposure model, we consider the failed fraction of those remaining components. We assume at time $\tilde{\tau}_1$ $(0 < \tilde{\tau}_1 < \tau_1)$, the component consumes the same fraction of log-lifetime with CDF $F_{U_2}(u;\mu_2,\sigma)$ as that at time $\ln\tau_1$ with CDF $F_{U_1}(u;\mu_1,\sigma)$, that is, $F_{U_1}(\ln\tau_1;\mu_1,\sigma) = F_{U_2}(\ln\tilde{\tau}_1;\mu_2,\sigma)$. Then, for any components failing at stress level y_2, the cumulative fraction of the log-lifetime is $F_{U_2}(\ln(\exp(u) - \tau_1 + \tilde{\tau}_1);\mu_2,\sigma)$, where $F_{U_1}(\ln\tau_1;\mu_1,\sigma) = F_{U_2}(\ln\tilde{\tau}_1;\mu_2,\sigma)$.

For the first two stress levels, under the assumption that the log-lifetime distributions of the components at different stress levels are in the location-scale family of distributions with different location parameters and the same scale parameter, we have

$$\frac{\ln\tau_1 - \mu_1}{\sigma} = \frac{\ln\tilde{\tau}_1 - \mu_2}{\sigma} \Longrightarrow \ln\tilde{\tau}_1 = \ln\tau_1 - \mu_1 + \mu_2 \Longrightarrow \tilde{\tau}_1 = \tau_1\exp(\mu_2 - \mu_1).$$

Similarly, at time τ_2, the stress level changes from y_2 to y_3. If we assume at time $\tilde{\tau}_2(\tau_1 < \tilde{\tau}_2 < \tau_2)$, the component consume the same fraction of log-lifetime with CDF $F_{U_3}(u;\mu_3,\sigma)$ as that at time τ_2 with CDF $F_{U_2}(u;\mu_2,\sigma)$, i.e., $F_{U_2}(\ln(\exp(u) - \tau_2 + \tilde{\tau}_2);\mu_3,\sigma)$, where $F_{U_2}(\ln(\tau_2 - \tau_1 + \tilde{\tau}_1);\mu_2,\sigma) = F_{U_3}(\ln\tilde{\tau}_2;\mu_3,\sigma)$. Then, we have $((\ln(\tau_2 - \tau_1 + \tilde{\tau}_1) - \mu_2)/\sigma = (\ln\tilde{\tau}_2 - \mu_3)/\sigma$, which leads to

$$\ln\tilde{\tau}_2 = (\ln(\tau_2 - \tau_1 + \tilde{\tau}_1) - \mu_2 + \mu_3 \Longrightarrow \tilde{\tau}_2 = (\tau_2 - \tau_1 + \tilde{\tau}_1)\exp(\mu_3 - \mu_2).$$

Following the same argument, in general, we have

$$\tilde{\tau}_k = (\tau_k - \tau_{k-1} + \tilde{\tau}_{k-1})\exp(\mu_{k+1} - \mu_k),$$

for $k = 1, 2, ..., K-1$. We also set $\tau_0 = 0$, $\tilde{\tau}_0 = 0$, and $\tau_K = \infty$. If we use d_k to denote the difference between τ_k and τ_{k-1}, i.e., $\tau_k - \tau_{k-1} \triangleq d_k$, we obtain

$$
\begin{aligned}
\tilde{\tau}_1 &= d_1 \exp(\mu_2 - \mu_1) = \tau_1 \exp(\mu_2 - \mu_1) \\
\tilde{\tau}_2 &= (d_2 + \tilde{\tau}_1) \exp(\mu_3 - \mu_2) = d_2 \exp(\mu_3 - \mu_2) + d_1 \exp(\mu_3 - \mu_1) \\
\tilde{\tau}_3 &= (d_3 + \tilde{\tau}_2) \exp(\mu_4 - \mu_3) = d_3 \exp(\mu_4 - \mu_3) + d_2 \exp(\mu_4 - \mu_2) + d_1 \exp(\mu_4 - \mu_1)
\end{aligned}
$$

Therefore, in general, we have

$$
\tilde{\tau}_k = \sum_{p=1}^{k} d_p \exp(\mu_{k+1} - \mu_p) = \sum_{p=1}^{k} d_p \exp[\beta_1 (y_{k+1} - y_p)],
$$

where $k = 1, 2, ..., K-1$, $\tau_K = \infty$, $d_1 = \tau_1$.

After we obtain the Type-II censored system lifetime data and formulate the cumulative exposure model for step-stress ALTs, a similar concept of examining all the permutations of censored system lifetimes, as described in Chapter 2 and Chapter 3, can be applied. However, considering required permutations for step-stress ALTs with Type-II censoring is more complex than constant-stress ALTs. Special attention and caution are necessary to effectively handle the complexities of analyzing Type-II censoring data for step-stress ALTs.

4.2.2. Consider systems with a larger number of components

In this thesis, our primary objective is to determine the optimal experimental planning for a few components and a limited range of system types. This limitation is primarily due to the limited number of systems that can be used in the experiment when the number of components available for the experiment is small. For example, if we have 20 components available for the experiment, we will only have 10, 5, 4, and 2 systems if we consider 2-, 4-, 5-, and 10-component systems. It would be valuable to expand our study for future research to study experimental planning with a larger number of components, such as hundreds or even thousands, and explore a broader range of system types. By broadening our investigation, we can obtain a more comprehensive understanding of optimal experimental planning across a greater diversity of scenarios.

4.2.3. Consider non-location-scale family of distributions

In this thesis, our focus has been on investigating optimal experimental planning based on location-scale family distributions. This choice is motivated by the flexibility of these distributions in fitting experimental data. However, for future research endeavors, we can consider exploring certain distributions from the scale-shape family, such as the gamma and Pareto distributions. It is important to note that working with scale family distributions poses challenges due to the lack of a standardized form of the distribution. Therefore, to obtain meaningful and reliable results, conducting such investigations requires careful execution due to the increased intricacies involved.

Appendix A

Appendix A: Terms Involved in Second Partial Derivatives of the Log-likelihood Function

$$
\begin{aligned}
& \frac{\partial^2 \bar{F}_{T_k}(t_{k\ell};\boldsymbol{\theta})}{\partial \beta_0^2} \\
= & \left(-\frac{1}{\sigma}\frac{\partial \phi_{sev}(z_{\ell:m_k})}{\partial \beta_0}\right) \\
& \times \sum_{i=1}^{n_k} s_{ki} \sum_{j}^{i-1} \binom{n_k}{j} \left[j \left[F^*(z_{\ell:m_k})\right]^{j-1} \left[\bar{F}^*(z_{\ell:m_k})\right]^{n_k-j} - \left[F^*(z_{\ell:m_k})\right]^{j} (n_k-j)\left[\bar{F}^*(z_{\ell:m_k})\right]^{n_k-j-1} \right] \\
& + \left(-\frac{1}{\sigma} f^*(z_{\ell:m_k})\right) \sum_{i=1}^{n_k} s_{ki} \sum_{j}^{i-1} \binom{n_k}{j} \\
& \Bigg\{ j \left(\frac{\partial F^*(z_{\ell:m_k})}{\partial \beta_0}\right) \Bigg[(j-1) \left[F^*(z_{\ell:m_k})\right]^{j-2} \left[\bar{F}^*(z_{\ell:m_k})\right]^{n_k-j} \\
& - \left[F^*(z_{\ell:m_k})\right]^{j-1} (n_k-j) \left[\bar{F}^*(z_{\ell:m_k})\right]^{n_k-j-1} \Bigg] \\
& -(n_k-j) \left(\frac{\partial F^*(z_{\ell:m_k})}{\partial \beta_0}\right) \Bigg[j \left[F^*(z_{\ell:m_k})\right]^{j-1} \left[\bar{F}^*(z_{\ell:m_k})\right]^{n_k-j-1} \\
& - \left[F^*(z_{\ell:m_k})\right]^{j} (n_k-j-1) \left[\bar{F}^*(z_{\ell:m_k})\right]^{n_k-j-2} \Bigg] \Bigg\} \\
= & \left(-\frac{1}{\sigma}\frac{\partial f^*(z_{\ell:m_k})}{\partial \beta_0}\right) \sum_{i=1}^{n_k} s_{ki} \sum_{j}^{i-1} \binom{n_k}{j} \Bigg[j \left[F^*(z_{\ell:m_k})\right]^{j-1} \left[\bar{F}^*(z_{\ell:m_k})\right]^{n_k-j} \\
& - \left[F^*(z_{\ell:m_k})\right]^{j} (n_k-j)\left[\bar{F}^*(z_{\ell:m_k})\right]^{n_k-j-1} \Bigg] \\
& + \left(\frac{1}{\sigma^2} \left[f^*(z_{\ell:m_k})\right]^2\right) \sum_{i=1}^{n_k} s_{ki} \sum_{j}^{i-1} \binom{n_k}{j} \\
& \Bigg\{ j \left[(j-1) \left[F^*(z_{\ell:m_k})\right]^{j-2} \left[\bar{F}^*(z_{\ell:m_k})\right]^{n_k-j} - \left[F^*(z_{\ell:m_k})\right]^{j-1} (n_k-j) \left[\bar{F}^*(z_{\ell:m_k})\right]^{n_k-j-1} \right] \\
& -(n_k-j) \left[j \left[F^*(z_{\ell:m_k})\right]^{j-1} \left[\bar{F}^*(z_{\ell:m_k})\right]^{n_k-j-1} - \left[F^*(z_{\ell:m_k})\right]^{j} (n_k-j-1) \left[\bar{F}^*(z_{\ell:m_k})\right]^{n_k-j-2} \right] \Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial^2 \bar{F}_{T_k}(t_{k\ell};\boldsymbol{\theta})}{\partial\beta_0\partial\beta_1} \\
= \quad & \left(-\frac{1}{\sigma}\frac{\partial f^*(z_{\ell:m_k})}{\partial\beta_1}\right)\sum_{i=1}^{n_k} s_{ki}\sum_{j}^{i-1}\binom{n_k}{j} \\
& \left[j\,[F^*(z_{\ell:m_k})]^{j-1}\,[\bar{F}^*(z_{\ell:m_k})]^{n_k-j} - [F^*(z_{\ell:m_k})]^{j}\,(n_k-j)[\bar{F}^*(z_{\ell:m_k})]^{n_k-j-1}\right] \\
& + \left(\frac{y_k}{\sigma^2}\,[f^*(z_{\ell:m_k})]^2\right)\sum_{i=1}^{n_k} s_{ki}\sum_{j}^{i-1}\binom{n_k}{j} \\
& \Bigg\{j\left[(j-1)\,[F^*(z_{\ell:m_k})]^{j-2}\left[\bar{F}^*(z_{\ell:m_k})\right]^{n_k-j} - [F^*(z_{\ell:m_k})]^{j-1}\,(n_k-j)\left[\bar{F}^*(z_{\ell:m_k})\right]^{n_k-j-1}\right] \\
& -(n_k-j)\left[j\,[F^*(z_{\ell:m_k})]^{j-1}\left[\bar{F}^*(z_{\ell:m_k})\right]^{n_k-j-1} - [F^*(z_{\ell:m_k})]^{j}\,(n_k-j-1)\left[\bar{F}^*(z_{\ell:m_k})\right]^{n_k-j-2}\right]\Bigg\} \\
&\frac{\partial^2 \bar{F}_{T_k}(t_{k\ell};\boldsymbol{\theta})}{\partial\beta_0\partial\sigma} \\
= \quad & \left(-\frac{1}{\sigma}\frac{\partial f^*(z_{\ell:m_k})}{\partial\sigma} + \frac{1}{\sigma^2}f^*(z_{\ell:m_k})\right) \\
& \times\sum_{i=1}^{n_k} s_{ki}\sum_{j}^{i-1}\binom{n_k}{j}\left[j\,[F^*(z_{\ell:m_k})]^{j-1}\,[\bar{F}^*(z_{\ell:m_k})]^{n_k-j} - [F^*(z_{\ell:m_k})]^{j}\,(n_k-j)[\bar{F}^*(z_{\ell:m_k})]^{n_k-j-1}\right] \\
& + \left(\frac{z_{\ell:m_k}}{\sigma^2}\,[f^*(z_{\ell:m_k})]^2\right)\sum_{i=1}^{n_k} s_{ki}\sum_{j}^{i-1}\binom{n_k}{j} \\
& \Bigg\{j\left[(j-1)\,[F^*(z_{\ell:m_k})]^{j-2}\left[\bar{F}^*(z_{\ell:m_k})\right]^{n_k-j} - [F^*(z_{\ell:m_k})]^{j-1}\,(n_k-j)\left[\bar{F}^*(z_{\ell:m_k})\right]^{n_k-j-1}\right] \\
& -(n_k-j)\left[j\,[F^*(z_{\ell:m_k})]^{j-1}\left[\bar{F}^*(z_{\ell:m_k})\right]^{n_k-j-1} - [F^*(z_{\ell:m_k})]^{j}\,(n_k-j-1)\left[\bar{F}^*(z_{\ell:m_k})\right]^{n_k-j-2}\right]\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial^2 \bar{F}_{T_k}(t_{k\ell};\boldsymbol{\theta})}{\partial \beta_1^2} \\
= \quad & \left(-\frac{y_k}{\sigma}\frac{\partial f^*(z_{\ell:m_k})}{\partial \beta_1}\right) \\
& \times \sum_{i=1}^{n_k} s_{ki} \sum_{j}^{i-1} \binom{n_k}{j} \left[j \left[F^*(z_{\ell:m_k})\right]^{j-1} \left[\bar{F}^*(z_{\ell:m_k})\right]^{n_k-j} - \left[F^*(z_{\ell:m_k})\right]^{j} (n_k-j) [\bar{F}^*(z_{\ell:m_k})]^{n_k-j-1} \right] \\
& + \left(\frac{y_k^2}{\sigma^2}\left[f^*(z_{\ell:m_k})\right]^2\right) \sum_{i=1}^{n_k} s_{ki} \sum_{j}^{i-1} \binom{n_k}{j} \\
& \Bigg\{ j \left[(j-1) \left[F^*(z_{\ell:m_k})\right]^{j-2} \left[\bar{F}^*(z_{\ell:m_k})\right]^{n_k-j} - \left[F^*(z_{\ell:m_k})\right]^{j-1} (n_k-j) \left[\bar{F}^*(z_{\ell:m_k})\right]^{n_k-j-1} \right] \\
& -(n_k-j) \left[j \left[F^*(z_{\ell:m_k})\right]^{j-1} \left[\bar{F}^*(z_{\ell:m_k})\right]^{n_k-j-1} - \left[F^*(z_{\ell:m_k})\right]^{j} (n_k-j-1) \left[\bar{F}^*(z_{\ell:m_k})\right]^{n_k-j-2} \right] \Bigg\} \\
&\frac{\partial^2 \bar{F}_{T_k}(t_{k\ell};\boldsymbol{\theta})}{\partial \beta_1 \partial \sigma} \\
= \quad & \left(-\frac{y_k}{\sigma}\frac{\partial f^*(z_{\ell:m_k})}{\partial \sigma} + \frac{y_k}{\sigma^2} f^*(z_{\ell:m_k})\right) \\
& \times \sum_{i=1}^{n_k} s_{ki} \sum_{j}^{i-1} \binom{n_k}{j} \left[j \left[F^*(z_{\ell:m_k})\right]^{j-1} \left[\bar{F}^*(z_{\ell:m_k})\right]^{n_k-j} - \left[F^*(z_{\ell:m_k})\right]^{j} (n_k-j) [\bar{F}^*(z_{\ell:m_k})]^{n_k-j-1} \right] \\
& + \left(\frac{y_k z_{\ell:m_k}}{\sigma^2}\left[f^*(z_{\ell:m_k})\right]^2\right) \sum_{i=1}^{n_k} s_{ki} \sum_{j}^{i-1} \binom{n_k}{j} \\
& \Bigg\{ j \left[(j-1) \left[F^*(z_{\ell:m_k})\right]^{j-2} \left[\bar{F}^*(z_{\ell:m_k})\right]^{n_k-j} - \left[F^*(z_{\ell:m_k})\right]^{j-1} (n_k-j) \left[\bar{F}^*(z_{\ell:m_k})\right]^{n_k-j-1} \right] \\
& -(n_k-j) \left[j \left[F^*(z_{\ell:m_k})\right]^{j-1} \left[\bar{F}^*(z_{\ell:m_k})\right]^{n_k-j-1} - \left[F^*(z_{\ell:m_k})\right]^{j} (n_k-j-1) \left[\bar{F}^*(z_{\ell:m_k})\right]^{n_k-j-2} \right] \Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial^2 \bar{F}_{T_k}(t_{k\ell};\boldsymbol{\theta})}{\partial\sigma^2}\\
= \quad &\left(-\frac{z_{\ell:m_k}}{\sigma}\frac{\partial f^*(z_{\ell:m_k})}{\partial\sigma}+\frac{2z_{\ell:m_k}}{\sigma^2}f^*(z_{\ell:m_k})\right)\\
&\times\sum_{i=1}^{n_k}s_{ki}\sum_{j}^{i-1}\binom{n_k}{j}\left[j\left[F^*(z_{\ell:m_k})\right]^{j-1}\left[\bar{F}^*(z_{\ell:m_k})\right]^{n_k-j}-\left[F^*(z_{\ell:m_k})\right]^{j}(n_k-j)[\bar{F}^*(z_{\ell:m_k})]^{n_k-j-1}\right]\\
&+\left(\frac{z_{\ell:m_k}^2}{\sigma^2}\left[f^*(z_{\ell:m_k})\right]^2\right)\sum_{i=1}^{n_k}s_{ki}\sum_{j}^{i-1}\binom{n_k}{j}\\
&\Bigg\{j\Bigg[(j-1)\left[F^*(z_{\ell:m_k})\right]^{j-2}\left[\bar{F}^*(z_{\ell:m_k})\right]^{n_k-j}\\
&-\left[F^*(z_{\ell:m_k})\right]^{j-1}(n_k-j)\left[\bar{F}^*(z_{\ell:m_k})\right]^{n_k-j-1}\Bigg]\\
&-(n_k-j)\Bigg[j\left[F^*(z_{\ell:m_k})\right]^{j-1}\left[\bar{F}^*(z_{\ell:m_k})\right]^{n_k-j-1}\\
&-\left[F^*(z_{\ell:m_k})\right]^{j}(n_k-j-1)\left[\bar{F}^*(z_{\ell:m_k})\right]^{n_k-j-2}\Bigg]\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
& \frac{\partial^2 f_{T_k}(t_{k\ell};\boldsymbol{\theta})}{\partial \beta_0^2} \\
= \quad & \frac{1}{\sigma}\Bigg\{ \frac{\partial^2 f^*(z_{\ell:m_k})}{\partial \beta_0^2} \sum_{i=1}^{n_k} s_{ki} \binom{n_k}{i} i [F^*(z_{\ell:m_k})]^{i-1} [\bar{F}^*(z_{\ell:m_k})]^{n_k-i} \\
& + \frac{\partial f^*(z_{\ell:m_k})}{\partial \beta_0} \sum_{i=1}^{n_k} s_{ki} \binom{n_k}{i} i \left(\frac{\partial F^*(z_{\ell:m_k})}{\partial \beta_0} \right) \\
& \times \big[(i-1)[F^*(z_{\ell:m_k})]^{i-2} [\bar{F}^*(z_{\ell:m_k})]^{n_k-i} - [F^*(z_{\ell:m_k})]^{i-1} (n_k - i) [\bar{F}^*(z_{\ell:m_k})]^{n_k-i-1} \big] \Bigg\} \\
& - \frac{1}{\sigma^2}\Bigg\{ 2 f^*(z_{\ell:m_k}) \frac{\partial f^*(z_{\ell:m_k})}{\partial \beta_0} \sum_{i=1}^{n_k} s_{ki} \binom{n_k}{i} i \Bigg[(i-1)[F^*(z_{\ell:m_k})]^{i-2} [\bar{F}^*(z_{\ell:m_k})]^{n_k-i} \\
& - [F^*(z_{\ell:m_k})]^{i-1} (n_k - i) [\bar{F}^*(z_{\ell:m_k})]^{n_k-i-1} \Bigg] \\
& + (f^*(z_{\ell:m_k}))^2 \left(\frac{\partial F^*(z_{\ell:m_k})}{\partial \beta_0} \right) \sum_{i=1}^{n_k} s_{ki} \binom{n_k}{i} i \\
& \Bigg[[(i-1)(i-2)[F^*(z_{\ell:m_k})]^{i-3} [\bar{F}^*(z_{\ell:m_k})]^{n_k-i} \\
& - (i-1)[F^*(z_{\ell:m_k})]^{i-2} (n_k - i) [\bar{F}^*(z_{\ell:m_k})]^{n_k-i-1}] \\
& - [(i-1)[F^*(z_{\ell:m_k})]^{i-2} (n_k - i) [\bar{F}^*(z_{\ell:m_k})]^{n_k-i-1} \\
& - [F^*(z_{\ell:m_k})]^{i-1} (n_k - i)(n_k - i - 1) [\bar{F}^*(z_{\ell:m_k})]^{n_k-i-2}] \Bigg] \Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial^2 f_{T_k}(t_{k\ell};\boldsymbol{\theta})}{\partial\beta_0\partial\beta_1}\\
= \quad & \frac{1}{\sigma}\Bigg\{\frac{\partial^2 f^*(z_{\ell:m_k})}{\partial\beta_0\partial\beta_1}\sum_{i=1}^{n_k} s_{ki}\binom{n_k}{i} i[F^*(z_{\ell:m_k})]^{i-1}[\bar{F}^*(z_{\ell:m_k})]^{n_k-i}\\
&+\frac{\partial f^*(z_{\ell:m_k})}{\partial\beta_0}\sum_{i=1}^{n_k} s_{ki}\binom{n_k}{i} i\left(\frac{\partial F^*(z_{\ell:m_k})}{\partial\beta_1}\right)\\
&\times\left[(i-1)[F^*(z_{\ell:m_k})]^{i-2}[\bar{F}^*(z_{\ell:m_k})]^{n_k-i}-[F^*(z_{\ell:m_k})]^{i-1}(n_k-i)[\bar{F}^*(z_{\ell:m_k})]^{n_k-i-1}\right]\Bigg\}\\
&-\frac{1}{\sigma^2}\Bigg\{2f^*(z_{\ell:m_k})\frac{\partial f^*(z_{\ell:m_k})}{\partial\beta_1}\sum_{i=1}^{n_k} s_{ki}\binom{n_k}{i} i\Bigg[(i-1)[F^*(z_{\ell:m_k})]^{i-2}[\bar{F}^*(z_{\ell:m_k})]^{n_k-i}\\
&-[F^*(z_{\ell:m_k})]^{i-1}(n_k-i)[\bar{F}^*(z_{\ell:m_k})]^{n_k-i-1}\Bigg]\\
&+(f^*(z_{\ell:m_k}))^2\left(\frac{\partial F^*(z_{\ell:m_k})}{\partial\beta_1}\right)\sum_{i=1}^{n_k} s_{ki}\binom{n_k}{i} i\\
&\Bigg[[(i-1)(i-2)[F^*(z_{\ell:m_k})]^{i-3}[\bar{F}^*(z_{\ell:m_k})]^{n_k-i}\\
&-(i-1)[F^*(z_{\ell:m_k})]^{i-2}(n_k-i)[\bar{F}^*(z_{\ell:m_k})]^{n_k-i-1}]\\
&-[(i-1)[F^*(z_{\ell:m_k})]^{i-2}(n_k-i)[\bar{F}^*(z_{\ell:m_k})]^{n_k-i-1}\\
&-[F^*(z_{\ell:m_k})]^{i-1}(n_k-i)(n_k-i-1)[\bar{F}^*(z_{\ell:m_k})]^{n_k-i-2}]\Bigg]\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
& \frac{\partial^2 f_{T_k}(t_{k\ell};\boldsymbol{\theta})}{\partial\beta_0\partial\sigma} \\
= \; & \frac{1}{\sigma^2}\Bigg\{ \left[\frac{\partial f^*(z_{\ell:m_k})}{\partial\beta_0} + \sigma\frac{\partial^2 f^*(z_{\ell:m_k})}{\partial\beta_0\partial\sigma}\right] \sum_{i=1}^{n_k} s_{ki}\binom{n_k}{i} i[F^*(z_{\ell:m_k})]^{i-1}[\bar{F}^*(z_{\ell:m_k})]^{n_k-i} \\
& +\sigma\frac{\partial f^*(z_{\ell:m_k})}{\partial\beta_0}\sum_{i=1}^{n_k} s_{ki}\binom{n_k}{i} i\left(\frac{\partial F^*(z_{\ell:m_k})}{\partial\sigma}\right) \\
& \times\left[(i-1)[F^*(z_{\ell:m_k})]^{i-2}[\bar{F}^*(z_{\ell:m_k})]^{n_k-i} - [F^*(z_{\ell:m_k})]^{i-1}(n_k-i)[\bar{F}^*(z_{\ell:m_k})]^{n_k-i-1}\right]\Bigg\} \\
& -\frac{1}{\sigma^2}\Bigg\{2f^*(z_{\ell:m_k})\frac{\partial f^*(z_{\ell:m_k})}{\partial\sigma}\sum_{i=1}^{n_k} s_{ki}\binom{n_k}{i} i\Bigg[(i-1)[F^*(z_{\ell:m_k})]^{i-2}[\bar{F}^*(z_{\ell:m_k})]^{n_k-i} \\
& -[F^*(z_{\ell:m_k})]^{i-1}(n_k-i)[\bar{F}^*(z_{\ell:m_k})]^{n_k-i-1}\Bigg] \\
& +(f^*(z_{\ell:m_k}))^2\left(\frac{\partial F^*(z_{\ell:m_k})}{\partial\sigma}\right)\sum_{i=1}^{n_k} s_{ki}\binom{n_k}{i} i \\
& \Bigg[[(i-1)(i-2)[F^*(z_{\ell:m_k})]^{i-3}[\bar{F}^*(z_{\ell:m_k})]^{n_k-i} \\
& -(i-1)[F^*(z_{\ell:m_k})]^{i-2}(n_k-i)[\bar{F}^*(z_{\ell:m_k})]^{n_k-i-1}] \\
& -[(i-1)[F^*(z_{\ell:m_k})]^{i-2}(n_k-i)[\bar{F}^*(z_{\ell:m_k})]^{n_k-i-1} \\
& -[F^*(z_{\ell:m_k})]^{i-1}(n_k-i)(n_k-i-1)[\bar{F}^*(z_{\ell:m_k})]^{n_k-i-2}]\Bigg]\Bigg\} \\
& +\left(-\frac{2}{\sigma}\right)\frac{\partial f_{T_k}(t_{k\ell};\boldsymbol{\theta})}{\partial\beta_0}
\end{aligned}
$$

$$
\begin{aligned}
& \frac{\partial^2 f_{T_k}(t_{k\ell};\boldsymbol{\theta})}{\partial \beta_1^2} \\
= \quad & \frac{1}{\sigma}\Bigg\{ \frac{\partial^2 f^*(z_{\ell:m_k})}{\partial \beta_1^2} \sum_{i=1}^{n_k} s_{ki} \binom{n_k}{i} i [F^*(z_{\ell:m_k})]^{i-1} [\bar{F}^*(z_{\ell:m_k})]^{n_k-i} \\
& + \frac{\partial f^*(z_{\ell:m_k})}{\partial \beta_1} \sum_{i=1}^{n_k} s_{ki} \binom{n_k}{i} i \left(\frac{\partial F^*(z_{\ell:m_k})}{\partial \beta_1} \right) \\
& \times \left[(i-1)[F^*(z_{\ell:m_k})]^{i-2} [\bar{F}^*(z_{\ell:m_k})]^{n_k-i} - [F^*(z_{\ell:m_k})]^{i-1} (n_k - i) [\bar{F}^*(z_{\ell:m_k})]^{n_k-i-1} \right] \Bigg\} \\
& - \frac{y_k}{\sigma^2} \Bigg\{ 2 f^*(z_{\ell:m_k}) \frac{\partial f^*(z_{\ell:m_k})}{\partial \beta_0} \sum_{i=1}^{n_k} s_{ki} \binom{n_k}{i} i \Bigg[(i-1)[F^*(z_{\ell:m_k})]^{i-2} [\bar{F}^*(z_{\ell:m_k})]^{n_k-i} \\
& - [F^*(z_{\ell:m_k})]^{i-1} (n_k - i) [\bar{F}^*(z_{\ell:m_k})]^{n_k-i-1} \Bigg] \\
& + (f^*(z_{\ell:m_k}))^2 \left(\frac{\partial F^*(z_{\ell:m_k})}{\partial \beta_1} \right) \sum_{i=1}^{n_k} s_{ki} \binom{n_k}{i} i \\
& \Bigg[[(i-1)(i-2)[F^*(z_{\ell:m_k})]^{i-3} [\bar{F}^*(z_{\ell:m_k})]^{n_k-i} \\
& - (i-1)[F^*(z_{\ell:m_k})]^{i-2} (n_k - i) [\bar{F}^*(z_{\ell:m_k})]^{n_k-i-1}] \\
& - [(i-1)[F^*(z_{\ell:m_k})]^{i-2} (n_k - i) [\bar{F}^*(z_{\ell:m_k})]^{n_k-i-1} \\
& - [F^*(z_{\ell:m_k})]^{i-1} (n_k - i)(n_k - i - 1) [\bar{F}^*(z_{\ell:m_k})]^{n_k-i-2}] \Bigg] \Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
& \frac{\partial^2 f_{T_k}(t_{k\ell};\boldsymbol{\theta})}{\partial\beta_1\partial\sigma} \\
= \; & \frac{1}{\sigma^2}\Bigg\{\left[\frac{\partial f^*(z_{\ell:m_k})}{\partial\beta_1}+\sigma\frac{\partial^2 f^*(z_{\ell:m_k})}{\partial\beta_1\partial\sigma}\right]\sum_{i=1}^{n_k} s_{ki}\binom{n_k}{i} i[F^*(z_{\ell:m_k})]^{i-1}[\bar{F}^*(z_{\ell:m_k})]^{n_k-i} \\
& +\sigma\frac{\partial f^*(z_{\ell:m_k})}{\partial\beta_1}\sum_{i=1}^{n_k} s_{ki}\binom{n_k}{i} i\left(\frac{\partial F^*(z_{\ell:m_k})}{\partial\sigma}\right) \\
& \times\big[(i-1)[F^*(z_{\ell:m_k})]^{i-2}[\bar{F}^*(z_{\ell:m_k})]^{n_k-i}-[F^*(z_{\ell:m_k})]^{i-1}(n_k-i)[\bar{F}^*(z_{\ell:m_k})]^{n_k-i-1}\big]\Bigg\} \\
& -\frac{y_k}{\sigma^2}\Bigg\{2f^*(z_{\ell:m_k})\frac{\partial f^*(z_{\ell:m_k})}{\partial\sigma}\sum_{i=1}^{n_k} s_{ki}\binom{n_k}{i} i\Big[(i-1)[F^*(z_{\ell:m_k})]^{i-2}[\bar{F}^*(z_{\ell:m_k})]^{n_k-i} \\
& -[F^*(z_{\ell:m_k})]^{i-1}(n_k-i)[\bar{F}^*(z_{\ell:m_k})]^{n_k-i-1}\Big] \\
& +(f^*(z_{\ell:m_k}))^2\left(\frac{\partial F^*(z_{\ell:m_k})}{\partial\sigma}\right)\sum_{i=1}^{n_k} s_{ki}\binom{n_k}{i} i \\
& \Big[[(i-1)(i-2)[F^*(z_{\ell:m_k})]^{i-3}[\bar{F}^*(z_{\ell:m_k})]^{n_k-i} \\
& -(i-1)[F^*(z_{\ell:m_k})]^{i-2}(n_k-i)[\bar{F}^*(z_{\ell:m_k})]^{n_k-i-1}] \\
& -[(i-1)[F^*(z_{\ell:m_k})]^{i-2}(n_k-i)[\bar{F}^*(z_{\ell:m_k})]^{n_k-i-1} \\
& -[F^*(z_{\ell:m_k})]^{i-1}(n_k-i)(n_k-i-1)[\bar{F}^*(z_{\ell:m_k})]^{n_k-i-2}]\Big]\Bigg\} \\
& +\left(-\frac{2}{\sigma}\right)\frac{\partial f_{T_k}(t_{k\ell};\boldsymbol{\theta})}{\partial\beta_1}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 f_{V_k}(v_{\ell:m_k};\boldsymbol{\theta})}{\partial\sigma^2} &= \frac{1}{\sigma^2}\Bigg\{\left[\sigma\frac{\partial^2 f^*(z_{\ell:m_k})}{\partial\sigma^2}\right]\sum_{i=1}^{n_k}s_{ki}\binom{n_k}{i}i[F^*(z_{\ell:m_k})]^{i-1}[\bar{F}^*(z_{\ell:m_k})]^{n_k-i} \\
&+\left[\sigma\frac{\partial f^*(z_{\ell:m_k})}{\partial\sigma}-f^*(z_{\ell:m_k})\right]\sum_{i=1}^{n_k}s_{ki}\binom{n_k}{i}i\left(\frac{\partial F^*(z_{\ell:m_k})}{\partial\sigma}\right) \\
&\times\left[(i-1)[F^*(z_{\ell:m_k})]^{i-2}[\bar{F}^*(z_{\ell:m_k})]^{n_k-i}-[F^*(z_{\ell:m_k})]^{i-1}(n_k-i)[\bar{F}^*(z_{\ell:m_k})]^{n_k-i-1}\right]\Bigg\} \\
&-\frac{1}{\sigma^2}\Bigg\{\left[-\frac{z_{\ell:m_k}}{\sigma}(f^*(z_{\ell:m_k}))^2+z_{\ell:m_k}2f^*(z_{\ell:m_k})\frac{\partial f^*(z_{\ell:m_k})}{\partial\sigma}\right] \\
&\times\sum_{i=1}^{n_k}s_{ki}\binom{n_k}{i}i\bigg[(i-1)[F^*(z_{\ell:m_k})]^{i-2}[\bar{F}^*(z_{\ell:m_k})]^{n_k-i} \\
&-[F^*(z_{\ell:m_k})]^{i-1}(n_k-i)[\bar{F}^*(z_{\ell:m_k})]^{n_k-i-1}\bigg] \\
&+z_{\ell:m_k}(f^*(z_{\ell:m_k}))^2\left(\frac{\partial F^*(z_{\ell:m_k})}{\partial\sigma}\right)\sum_{i=1}^{n_k}s_{ki}\binom{n_k}{i}i \\
&\bigg[[(i-1)(i-2)[F^*(z_{\ell:m_k})]^{i-3}[\bar{F}^*(z_{\ell:m_k})]^{n_k-i} \\
&-(i-1)[F^*(z_{\ell:m_k})]^{i-2}(n_k-i)[\bar{F}^*(z_{\ell:m_k})]^{n_k-i-1}] \\
&-[(i-1)[F^*(z_{\ell:m_k})]^{i-2}(n_k-i)[\bar{F}^*(z_{\ell:m_k})]^{n_k-i-1} \\
&-[F^*(z_{\ell:m_k})]^{i-1}(n_k-i)(n_k-i-1)[\bar{F}^*(z_{\ell:m_k})]^{n_k-i-2}]\bigg]\Bigg\} \\
&+\left(-\frac{2}{\sigma}\right)\frac{\partial f_{T_k}(t_{k\ell};\boldsymbol{\theta})}{\partial\sigma}
\end{aligned}
$$

Appendix B

Appendix B: First and second derivatives of the PDF and CDF of standard normal distribution

$$C_{10}(z) = \frac{\partial \Phi_{nor}(z)}{\partial \beta_0} = -\frac{1}{\sigma}\phi_{nor}(z) \qquad P_{10}(z) = \frac{\partial \phi_{nor}(z)}{\partial \beta_0} = \frac{z}{\sigma}\phi_{nor}(z)$$

$$C_{20}(z) = \frac{\partial \Phi_{nor}(z)}{\partial \beta_1} = -\frac{y}{\sigma}\phi_{nor}(z) \qquad P_{20}(z) = \frac{\partial \phi_{nor}(z)}{\partial \beta_1} = \frac{zy}{\sigma}\phi_{nor}(z)$$

$$C_{30}(z) = \frac{\partial \Phi_{nor}(z)}{\partial \sigma} = -\frac{z}{\sigma}\phi_{nor}(z) \qquad P_{30}(z) = \frac{\partial \phi_{nor}(z)}{\partial \sigma} = \frac{z^2}{\sigma}\phi_{nor}(z)$$

$$C_{11}(z) = \frac{\partial^2 \Phi_{nor}(z)}{\partial \beta_0^2} = -\frac{z}{\sigma^2}\phi_{nor}(z) \qquad P_{11}(z) = \frac{\partial^2 \phi_{nor}(z)}{\partial \beta_0^2} = \frac{z^2-1}{\sigma^2}\phi_{nor}(z)$$

$$C_{12}(z) = \frac{\partial^2 \Phi_{nor}(z)}{\partial \beta_0 \partial \beta_1} = -\frac{zy}{\sigma^2}\phi_{nor}(z) \qquad P_{12}(z) = \frac{\partial^2 \phi_{nor}(z)}{\partial \beta_0 \partial \beta_1} = \frac{(z^2-1)y}{\sigma^2}\phi_{nor}(z)$$

$$C_{13}(z) = \frac{\partial^2 \Phi_{nor}(z)}{\partial \beta_0 \partial \sigma} = -\frac{z^2-1}{\sigma^2}\phi_{nor}(z) \qquad P_{13}(z) = \frac{\partial^2 \phi_{nor}(z)}{\partial \beta_0 \partial \sigma} = \frac{z^3-2z}{\sigma^2}\phi_{nor}(z)$$

$$C_{22}(z) = \frac{\partial^2 \Phi_{nor}(z)}{\partial \beta_1^2} = -\frac{zy^2}{\sigma^2}\phi_{nor}(z) \qquad P_{22}(z) = \frac{\partial^2 \phi_{nor}(z)}{\partial \beta_1^2} = \frac{(z^2-1)y^2}{\sigma^2}\phi_{nor}(z)$$

$$C_{23}(z) = \frac{\partial^2 \Phi_{nor}(z)}{\partial \beta_1 \partial \sigma} = -\frac{z^2 y - y}{\sigma^2}\phi_{nor}(z) \qquad P_{23}(z) = \frac{\partial^2 \phi_{nor}(z)}{\partial \beta_1 \partial \sigma} = \frac{zy(z^2-2)}{\sigma^2}\phi_{nor}(z)$$

$$C_{33}(z) = \frac{\partial^2 \Phi_{nor}(z)}{\partial \sigma^2} = -\frac{z^3-2z}{\sigma^2}\phi_{nor}(z) \qquad P_{33}(z) = \frac{\partial^2 \phi_{nor}(z)}{\partial \sigma^2} = \frac{z^4-3z^2}{\sigma^2}\phi_{nor}(z)$$

Appendix C

Appendix C: First and second derivatives of the PDF and CDF of standard SEV distribution

$$C_{10}(z) = \frac{\partial \Phi_{sev}(z)}{\partial \beta_0} = -\frac{1}{\sigma}\phi_{sev}(z)$$
$$C_{20}(z) = \frac{\partial \Phi_{sev}(z)}{\partial \beta_1} = -\frac{y}{\sigma}\phi_{sev}(z)$$
$$C_{30}(z) = \frac{\partial \Phi_{sev}(z)}{\partial \sigma} = -\frac{z}{\sigma}\phi_{sev}(z)$$
$$C_{11}(z) = \frac{\partial^2 \Phi_{sev}(z)}{\partial \beta_0^2} = -\frac{(e^z - 1)}{\sigma^2}\phi_{sev}(z)$$
$$C_{12}(z) = \frac{\partial^2 \Phi_{sev}(z)}{\partial \beta_0 \partial \beta_1} = -\frac{(e^z - 1)y}{\sigma^2}\phi_{sev}(z)$$
$$C_{13}(z) = \frac{\partial^2 \Phi_{sev}(z)}{\partial \beta_0 \partial \sigma} = -\frac{ze^z - z - 1}{\sigma^2}\phi_{sev}(z)$$
$$C_{22}(z) = \frac{\partial^2 \Phi_{sev}(z)}{\partial \beta_1^2} = -\frac{(e^z - 1)y^2}{\sigma^2}\phi_{sev}(z)$$
$$C_{23}(z) = \frac{\partial^2 \Phi_{sev}(z)}{\partial \beta_1 \partial \sigma} = -\frac{(ze^z - z - 1)y}{\sigma^2}\phi_{sev}(z)$$
$$C_{33}(z) = \frac{\partial^2 \Phi_{sev}(z)}{\partial \sigma^2} = -\frac{z(ze^z - z - 2)}{\sigma^2}\phi_{sev}(z)$$
$$P_{10}(z) = \frac{\partial \phi_{sev}(z)}{\partial \beta_0} = \frac{(e^z - 1)}{\sigma}\phi_{sev}(z)$$
$$P_{20}(z) = \frac{\partial \phi_{sev}(z)}{\partial \beta_1} = \frac{(e^z - 1)y}{\sigma}\phi_{sev}(z)$$
$$P_{30}(z) = \frac{\partial \phi_{sev}(z)}{\partial \sigma} = \frac{(e^z - 1)z}{\sigma}\phi_{sev}(z)$$
$$P_{11}(z) = \frac{\partial^2 \phi_{sev}(z)}{\partial \beta_0^2} = \frac{(e^{2z} - 3e^z + 1)}{\sigma^2}\phi_{sev}(z)$$
$$P_{12}(z) = \frac{\partial^2 \phi_{sev}(z)}{\partial \beta_0 \partial \beta_1} = \frac{((e^{2z} - 3e^z + 1)y}{\sigma^2}\phi_{sev}(z)$$
$$P_{13}(z) = \frac{\partial^2 \phi_{sev}(z)}{\partial \beta_0 \partial \sigma} = \frac{(ze^{2z} - 3ze^z - 2e^z + z + 1)}{\sigma^2}\phi_{sev}(z)$$
$$P_{22}(z) = \frac{\partial^2 \phi_{sev}(z)}{\partial \beta_1^2} = \frac{(e^{2z} - 3e^z + 1)y^2}{\sigma^2}\phi_{sev}(z)$$
$$P_{23}(z) = \frac{\partial^2 \phi_{sev}(z)}{\partial \beta_1 \partial \sigma} = \frac{y(ze^{2z} - 3ze^z - 2e^z + z + 1)}{\sigma^2}\phi_{sev}(z)$$
$$P_{33}(z) = \frac{\partial^2 \phi_{sev}(z)}{\partial \sigma^2} = \frac{z(ze^{2z} - 3ze^z - 2e^z + z + 2)}{\sigma^2}\phi_{sev}(z)$$

Appendix D

Appendix D: Fisher Information Matrix for Series Systems

$$\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \beta_0^2} = n\bigg\{ \sum_{k=1}^{K}\sum_{\ell=1}^{r} \Big[- A^{-2}(z_{\ell:m_k})A_1^2(z_{\ell:m_k}) + A^{-1}(z_{\ell:m_k})A_{11}(z_{\ell:m_k})\Big]$$

$$- r\sum_{k=1}^{K} \Big[- A^{-2}(z_{\ell:m_k})A_1^2(z_{\ell:m_k}) + A^{-1}(z_{\ell:m_k})A_{11}(z_{\ell:m_k})\Big]\bigg\}$$

$$+ \bigg\{ \sum_{k=1}^{K}\sum_{\ell=1}^{r} \Big[- B^{-2}(z_{\ell:m_k})B_1^2(z_{\ell:m_k}) + B^{-1}(z_{\ell:m_k})B_{11}(z_{\ell:m_k})$$

$$+ A^{-2}(z_{\ell:m_k})A_1^2(z_{\ell:m_k}) - A^{-1}(z_{\ell:m_k})A_{11}(z_{\ell:m_k})\Big]$$

$$+ \frac{N}{2}\sum_{k=1}^{K} \Big[- A^{-2}(z_{\ell:m_k})A_1^2(z_{\ell:m_k}) + A^{-1}(z_{\ell:m_k})A_{11}(z_{\ell:m_k})\Big]\bigg\}$$

$$\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \beta_1^2} = n\bigg\{ \sum_{k=1}^{K}\sum_{\ell=1}^{r} \Big[- A^{-2}(z_{\ell:m_k})A_2^2(z_{\ell:m_k}) + A^{-1}(z_{\ell:m_k})A_{22}(z_{\ell:m_k})\Big]$$

$$- r\sum_{k=1}^{K} \Big[- A^{-2}(z_{\ell:m_k})A_2^2(z_{\ell:m_k}) + A^{-1}(z_{\ell:m_k})A_{22}(z_{\ell:m_k})\Big]\bigg\}$$

$$+ \bigg\{ \sum_{k=1}^{K}\sum_{\ell=1}^{r} \Big[- B^{-2}(z_{\ell:m_k})B_2^2(z_{\ell:m_k}) + B^{-1}(z_{\ell:m_k})B_{22}(z_{\ell:m_k})$$

$$+ A^{-2}(z_{\ell:m_k})A_2^2(z_{\ell:m_k}) - A^{-1}(z_{\ell:m_k})A_{22}(z_{\ell:m_k})\Big]$$

$$+ \frac{N}{2}\sum_{k=1}^{K} \Big[- A^{-2}(z_{\ell:m_k})A_2^2(z_{\ell:m_k}) + A^{-1}(z_{\ell:m_k})A_{22}(z_{\ell:m_k})\Big]\bigg\}$$

$$
\begin{aligned}
\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \sigma^2} &= n\Bigg\{ \sum_{k=1}^{K}\sum_{\ell=1}^{r} \Big[-A^{-2}(z_{\ell:m_k})A_3^2(z_{\ell:m_k}) + A^{-1}(z_{\ell:m_k})A_{33}(z_{\ell:m_k})\Big] \\
&- r\sum_{k=1}^{K} \Big[-A^{-2}(z_{\ell:m_k})A_3^2(z_{\ell:m_k}) + A^{-1}(z_{\ell:m_k})A_{33}(z_{\ell:m_k})\Big]\Bigg\} \\
&+ \Bigg\{ \frac{1}{\sigma^2} + \sum_{k=1}^{K}\sum_{\ell=1}^{r} \Big[-B^{-2}(z_{\ell:m_k})B_3^2(z_{\ell:m_k}) + B^{-1}(z_{\ell:m_k})B_{33}(z_{\ell:m_k}) \\
&+ A^{-2}(z_{\ell:m_k})A_3^2(z_{\ell:m_k}) - A^{-1}(z_{\ell:m_k})A_{33}(z_{\ell:m_k})\Big] \\
&+ \frac{N}{2}\sum_{k=1}^{K} \Big[-A^{-2}(z_{\ell:m_k})A_3^2(z_{\ell:m_k}) + A^{-1}(z_{\ell:m_k})A_{33}(z_{\ell:m_k})\Big]\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \beta_0 \partial \beta_1} &= n\Bigg\{ \sum_{k=1}^{K}\sum_{\ell=1}^{r} \Big[-A^{-2}(z_{\ell:m_k})A_1(z_{\ell:m_k})A_2(z_{\ell:m_k}) + A^{-1}(z_{\ell:m_k})A_{12}(z_{\ell:m_k})\Big] \\
&- r\sum_{k=1}^{K} \Big[-A^{-2}(z_{\ell:m_k})A_1(z_{\ell:m_k})A_2(z_{\ell:m_k}) + A^{-1}(z_{\ell:m_k})A_{12}(z_{\ell:m_k})\Big]\Bigg\} \\
&+ \Bigg\{ \sum_{k=1}^{K}\sum_{\ell=1}^{r} \Big[-B^{-2}(z_{\ell:m_k})B_1(z_{\ell:m_k})B_2(z_{\ell:m_k}) + B^{-1}(z_{\ell:m_k})B_{12}(z_{\ell:m_k}) \\
&+ -A^{-2}(z_{\ell:m_k})A_1(z_{\ell:m_k})A_2(z_{\ell:m_k}) + A^{-1}(z_{\ell:m_k})A_{12}(z_{\ell:m_k})\Big] \\
&+ \frac{N}{2}\sum_{k=1}^{K} \Big[-A^{-2}(z_{\ell:m_k})A_1(z_{\ell:m_k})A_2(z_{\ell:m_k}) + A^{-1}(z_{\ell:m_k})A_{12}(z_{\ell:m_k})\Big]\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \beta_0 \partial \sigma} &= n\Bigg\{ \sum_{k=1}^{K}\sum_{\ell=1}^{r} \Big[-A^{-2}(z_{\ell:m_k})A_1(z_{\ell:m_k})A_3(z_{\ell:m_k}) + A^{-1}(z_{\ell:m_k})A_{13}(z_{\ell:m_k})\Big] \\
&- r\sum_{k=1}^{K} \Big[-A^{-2}(z_{\ell:m_k})A_1(z_{\ell:m_k})A_3(z_{\ell:m_k}) + A^{-1}(z_{\ell:m_k})A_{13}(z_{\ell:m_k})\Big]\Bigg\} \\
&+ \Bigg\{ \sum_{k=1}^{K}\sum_{\ell=1}^{r} \Big[-B^{-2}(z_{\ell:m_k})B_1(z_{\ell:m_k})B_3(z_{\ell:m_k}) + B^{-1}(z_{\ell:m_k})B_{13}(z_{\ell:m_k}) \\
&+ -A^{-2}(z_{\ell:m_k})A_1(z_{\ell:m_k})A_3(z_{\ell:m_k}) + A^{-1}(z_{\ell:m_k})A_{13}(z_{\ell:m_k})\Big] \\
&+ \frac{N}{2}\sum_{k=1}^{K} \Big[-A^{-2}(z_{\ell:m_k})A_1(z_{\ell:m_k})A_3(z_{\ell:m_k}) + A^{-1}(z_{\ell:m_k})A_{13}(z_{\ell:m_k})\Big]\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 l(\boldsymbol{\theta})}{\partial\beta_1\partial\sigma} &= n\bigg\{ \sum_{k=1}^{K}\sum_{\ell=1}^{r}\Big[-A^{-2}(z_{\ell:m_k})A_2(z_{\ell:m_k})A_3(z_{\ell:m_k}) + A^{-1}(z_{\ell:m_k})A_{23}(z_{\ell:m_k})\Big] \\
&\quad - r\sum_{k=1}^{K}\Big[-A^{-2}(z_{\ell:m_k})A_2(z_{\ell:m_k})A_3(z_{\ell:m_k}) + A^{-1}(z_{\ell:m_k})A_{23}(z_{\ell:m_k})\Big]\bigg\} \\
&\quad + \bigg\{ \sum_{k=1}^{K}\sum_{\ell=1}^{r}\Big[-B^{-2}(z_{\ell:m_k})B_2(z_{\ell:m_k})B_3(z_{\ell:m_k}) + B^{-1}(z_{\ell:m_k})B_{23}(z_{\ell:m_k}) \\
&\quad + -A^{-2}(z_{\ell:m_k})A_2(z_{\ell:m_k})A_3(z_{\ell:m_k}) + A^{-1}(z_{\ell:m_k})A_{23}(z_{\ell:m_k})\Big] \\
&\quad + \frac{N}{2}\sum_{k=1}^{K}\Big[-A^{-2}(z_{\ell:m_k})A_2(z_{\ell:m_k})A_3(z_{\ell:m_k}) + A^{-1}(z_{\ell:m_k})A_{23}(z_{\ell:m_k})\Big]\bigg\}
\end{aligned}
$$

Appendix E

Appendix E: First and second derivatives of the PDF and CDF of standard Normal distribution

$$CC_{10}(z) = \frac{\partial \Phi_{nor}(z)}{\partial \mu} = -\frac{1}{\sigma}\phi_{nor}(z) \qquad PP_{10}(z) = \frac{\partial \phi_{nor}(z)}{\partial \mu} = \frac{z}{\sigma}\phi_{nor}(z)$$

$$CC_{20}(z) = \frac{\partial \Phi_{nor}(z)}{\partial \sigma} = -\frac{z}{\sigma}\phi_{nor}(z) \qquad PP_{20}(z) = \frac{\partial \phi_{nor}(z)}{\partial \sigma} = \frac{z^2}{\sigma}\phi_{nor}(z)$$

$$CC_{11}(z) = \frac{\partial^2 \Phi_{nor}(z)}{\partial \mu^2} = -\frac{z}{\sigma^2}\phi_{nor}(z) \qquad PP_{11}(z) = \frac{\partial^2 \phi_{nor}(z)}{\partial \mu^2} = \frac{z^2-1}{\sigma^2}\phi_{nor}(z)$$

$$CC_{12}(z) = \frac{\partial^2 \Phi_{nor}(z)}{\partial \mu \partial \sigma} = -\frac{z^2-1}{\sigma^2}\phi_{nor}(z) \qquad PP_{12}(z) = \frac{\partial^2 \phi_{nor}(z)}{\partial \mu \partial \sigma} = \frac{z^3-2z}{\sigma^2}\phi_{nor}(z)$$

$$CC_{22}(z) = \frac{\partial^2 \Phi_{nor}(z)}{\partial \sigma^2} = -\frac{z^3-2z}{\sigma^2}\phi_{nor}(z) \qquad PP_{22}(z) = \frac{\partial^2 \phi_{nor}(z)}{\partial \sigma^2} = \frac{z^4-3z^2}{\sigma^2}\phi_{nor}(z)$$

Appendix F

Appendix F: First and second derivatives of the PDF and CDF of standard SEV distribution

$$
\begin{aligned}
CC_{10}(z) &= \frac{\partial \Phi_{sev}(z)}{\partial \mu} = -\frac{1}{\sigma}\phi_{sev}(z) \\
CC_{20}(z) &= \frac{\partial \Phi_{sev}(z)}{\partial \sigma} = -\frac{z}{\sigma}\phi_{sev}(z) \\
CC_{11}(z) &= \frac{\partial^2 \Phi_{sev}(z)}{\partial \mu^2} = -\frac{(e^z - 1)}{\sigma^2}\phi_{sev}(z) \\
CC_{12}(z) &= \frac{\partial^2 \Phi_{sev}(z)}{\partial \mu \partial \sigma} = -\frac{ze^z - z - 1}{\sigma^2}\phi_{sev}(z) \\
CC_{22}(z) &= \frac{\partial^2 \Phi_{sev}(z)}{\partial \sigma^2} = -\frac{z(ze^z - z - 2)}{\sigma^2}\phi_{sev}(z) \\
PP_{10}(z) &= \frac{\partial \phi_{sev}(z)}{\partial \mu} = \frac{(e^z - 1)}{\sigma}\phi_{sev}(z) \\
PP_{20}(z) &= \frac{\partial \phi_{sev}(z)}{\partial \sigma} = \frac{(e^z - 1)z}{\sigma}\phi_{sev}(z) \\
PP_{11}(z) &= \frac{\partial^2 \phi_{sev}(z)}{\partial \mu^2} = \frac{(e^{2z} - 3e^z + 1)}{\sigma^2}\phi_{sev}(z) \\
PP_{12}(z) &= \frac{\partial^2 \phi_{sev}(z)}{\partial \mu \partial \sigma} = \frac{(ze^{2z} - 3ze^z - 2e^z + z + 1)}{\sigma^2}\phi_{sev}(z) \\
PP_{22}(z) &= \frac{\partial^2 \phi_{sev}(z)}{\partial \sigma^2} = \frac{z(ze^{2z} - 3ze^z - 2e^z + z + 2)}{\sigma^2}\phi_{sev}(z)
\end{aligned}
$$

Appendix G

Appendix G: Detailed Formula for LogNormal Distribution with Various Types of Systems

G.1. $l = 1$

$$\begin{aligned}\frac{\partial l(\boldsymbol{\theta})}{\partial \mu} &= \sum_{\ell=1}^{r}\left[\frac{z_{\ell:r:m}}{\sigma} + \frac{R_\ell n + n - 1}{\sigma}\frac{\phi_{nor}(z_{\ell:r:m})}{\bar{\Phi}_{nor}(z_{\ell:r:m})}\right]\\ \frac{\partial l(\boldsymbol{\theta})}{\partial \sigma} &= \sum_{\ell=1}^{r}\left[-\frac{1}{\sigma} + \frac{z_{\ell:r:m}^2}{\sigma} + \frac{z_{\ell:r:m}(R_\ell n + n - 1)}{\sigma}\frac{\phi_{nor}(z_{\ell:r:m})}{\bar{\Phi}_{nor}(z_{\ell:r:m})}\right].\end{aligned}$$

So the likelihood equations for μ and σ are, respectively,

$$\sum_{\ell=1}^{r} z_{\ell:r:m} + \sum_{\ell=1}^{r}(R_\ell n + n - 1)\frac{\phi_{nor}(z_{\ell:r:m})}{\bar{\Phi}_{nor}(z_{\ell:r:m})} = 0;$$

$$-r + \sum_{\ell=1}^{r} z_{\ell:r:m}^2 + \sum_{\ell=1}^{r}(R_\ell n + n - 1)z_{\ell:r:m}\frac{\phi_{nor}(z_{\ell:r:m})}{\bar{\Phi}_{nor}(z_{\ell:r:m})} = 0.$$

From the log-likelihood function, the second derivatives forming the observed Fisher Information are

$$\begin{aligned}\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \mu^2} &= -\frac{r}{\sigma^2} + \frac{1}{\sigma^2}\sum_{\ell=1}^{r}(R_\ell n + n - 1)\frac{z_{\ell:r:m}\phi_{nor}(z_{\ell:r:m})\bar{\Phi}_{nor}(z_{\ell:r:m}) - \phi_{nor}^2(z_{\ell:r:m})}{\bar{\Phi}_{nor}^2(z_{\ell:r:m})}\\ \frac{\partial^2 l(\boldsymbol{\theta})}{\partial \mu \partial \sigma} &= -\frac{2}{\sigma^2}\sum_{\ell=1}^{r} z_{\ell:r:m} - \frac{1}{\sigma^2}\sum_{\ell=1}^{r}(R_\ell n + n - 1)\frac{\phi_{nor}(z_{\ell:r:m})}{\bar{\Phi}_{nor}(z_{\ell:r:m})}\\ &\quad + \frac{1}{\sigma^2}\sum_{\ell=1}^{r}(R_\ell n + n - 1)z_{\ell:r:m}\frac{z_{\ell:r:m}\phi_{nor}(z_{\ell:r:m})\bar{\Phi}_{nor}(z_{\ell:r:m}) - \phi_{nor}^2(z_{\ell:r:m})}{\bar{\Phi}_{nor}^2(z_{\ell:r:m})}\\ \frac{\partial^2 l(\boldsymbol{\theta})}{\partial \sigma^2} &= \frac{r}{\sigma^2} - \frac{3}{\sigma^2}\sum_{\ell=1}^{r} z_{\ell:r:m}^2 - \frac{2}{\sigma^2}\sum_{\ell=1}^{r}(R_\ell n + n - 1)z_{\ell:r:m}\frac{\phi_{nor}(z_{\ell:r:m})}{\bar{\Phi}_{nor}(z_{\ell:r:m})}\\ &\quad + \frac{1}{\sigma^2}\sum_{\ell=1}^{r}(R_\ell n + n - 1)z_{\ell:r:m}^2\frac{z_{\ell:r:m}\phi_{nor}(z_{\ell:r:m})\bar{\Phi}_{nor}(z_{\ell:r:m}) - \phi_{nor}^2(z_{\ell:r:m})}{\bar{\Phi}_{nor}^2(z_{\ell:r:m})}.\end{aligned}$$

We can obtain optimal planning with two methods. The first method uses approximation with Taylor series expansion. The approximated first and second partial derivatives of the log-likelihood function involved in the MLEs and Fisher information matrix are described and presented in Appendix F. The second method uses the Monte Carlo integration, similar to what we do in Chapter 2.

G.2. $l = n$

The likelihood equations for μ and σ are, respectively,

$$\sum_{\ell=1}^{r}\left[z_{\ell:r:m}+\frac{(R_\ell n+n-1)\left[\Phi_{nor}(z_{\ell:r:m})\right]^n-(n-1)}{1-\left[\Phi_{nor}(z_{\ell:r:m})\right]^n}\frac{\phi_{nor}(z_{\ell:r:m})}{\Phi_{nor}(z_{\ell:r:m})}\right]=0;$$

$$\sum_{\ell=1}^{r}\left[-1+z_{\ell:r:m}^2+\frac{z_{\ell:r:m}\left[(R_\ell n+n-1)\left[\Phi_{nor}(z_{\ell:r:m})\right]^n-(n-1)\right]}{1-\left[\Phi_{nor}(z_{\ell:r:m})\right]^n}\frac{\phi_{nor}(z_{\ell:r:m})}{\Phi_{nor}(z_{\ell:r:m})}\right]=0.$$

Let

$$\begin{aligned} p_1 &\triangleq R_\ell n+n-1 \\ p_2 &\triangleq n-1, \end{aligned}$$

then we have

$$\begin{aligned} \frac{\partial l(\boldsymbol{\theta})}{\partial \mu} &= \frac{1}{\sigma}\sum_{\ell=1}^{r}\left[z_{\ell:r:m}+\frac{p_1\left[\Phi_{nor}(z_{\ell:r:m})\right]^n-p_2}{1-\left[\Phi_{nor}(z_{\ell:r:m})\right]^n}\frac{\phi_{nor}(z_{\ell:r:m})}{\Phi_{nor}(z_{\ell:r:m})}\right] \\ \frac{\partial l(\boldsymbol{\theta})}{\partial \sigma} &= \frac{1}{\sigma}\sum_{\ell=1}^{r}\left[-1+z_{\ell:r:m}^2+z_{\ell:r:m}\frac{p_1\left[\Phi_{nor}(z_{\ell:r:m})\right]^n-p_2}{1-\left[\Phi_{nor}(z_{\ell:r:m})\right]^n}\frac{\phi_{nor}(z_{\ell:r:m})}{\Phi_{nor}(z_{\ell:r:m})}\right]. \end{aligned}$$

Suppose

$$g(z)=\frac{p_1\left[\Phi_{nor}(z)\right]^n-p_2}{1-\left[\Phi_{nor}(z)\right]^n}\frac{\phi_{nor}(z)}{\Phi_{nor}(z)}=\frac{p_1\phi_{nor}(z)\left[\Phi_{nor}(z)\right]^n-p_2\phi_{nor}(z)}{\Phi_{nor}(z)-\left[\Phi_{nor}(z)\right]^{n+1}},$$

then we have

$$\frac{\partial g(z)}{\partial \mu} = \frac{1}{\sigma}\left[zg(z) + [\phi_{nor}(z)]^2 \frac{p_2(2-Rn)\,[\Phi_{nor}(z)]^n - p_1\,[\Phi_{nor}(z)]^{2n} - p_2}{\left[\Phi_{nor}(z) - [\Phi_{nor}(z)]^{n+1}\right]^2}\right]$$
$$\frac{\partial g(z)}{\partial \sigma} = \frac{z}{\sigma}\left[zg(z) + [\phi_{nor}(z)]^2 \frac{p_2(2-Rn)\,[\Phi_{nor}(z)]^n - p_1\,[\Phi_{nor}(z)]^{2n} - p_2}{\left[\Phi_{nor}(z) - [\Phi_{nor}(z)]^{n+1}\right]^2}\right].$$

We can also obtain

$$\begin{aligned}
\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \mu^2} &= \frac{1}{\sigma}\sum_{\ell=1}^{r}\left[-\frac{1}{\sigma} + \frac{\partial g(z_{\ell:r:m})}{\partial \mu}\right] \\
&= \frac{1}{\sigma^2}\sum_{\ell=1}^{r}\left[z_{\ell:r:m}g(z_{\ell:r:m}) + [\phi_{nor}(z_{\ell:r:m})]^2 \frac{p_2(2-Rn)\,[\Phi_{nor}(z_{\ell:r:m})]^n - p_1\,[\Phi_{nor}(z_{\ell:r:m})]^{2n} - p_2}{\left[\Phi_{nor}(z_{\ell:r:m}) - [\Phi_{nor}(z_{\ell:r:m})]^{n+1}\right]^2} - 1\right] \\
\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \mu \partial \sigma} &= \frac{1}{\sigma}\sum_{\ell=1}^{r}\left[-\frac{2z_{\ell:r:m}}{\sigma} + z_{\ell:r:m}\frac{\partial g(z_{\ell:r:m})}{\partial \mu} - \frac{g(z_{\ell:r:m})}{\sigma}\right] \\
&= \frac{1}{\sigma^2}\sum_{\ell=1}^{r}\left[(z_{\ell:r:m}^2 - 1)g(z_{\ell:r:m}) + z_{\ell:r:m}\,[\phi_{nor}(z_{\ell:r:m})]^2 \frac{p_2(2-Rn)\,[\Phi_{nor}(z_{\ell:r:m})]^n - p_1\,[\Phi_{nor}(z_{\ell:r:m})]^{2n} - p_2}{\left[\Phi_{nor}(z_{\ell:r:m}) - [\Phi_{nor}(z_{\ell:r:m})]^{n+1}\right]^2}\right. \\
&\quad \left. -2z_{\ell:r:m}\right] \\
\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \sigma^2} &= \frac{1}{\sigma}\sum_{\ell=1}^{r}\left[-2z_{\ell:r:m}\frac{z_{\ell:r:m}}{\sigma} + z_{\ell:r:m}\frac{\partial g(z_{\ell:r:m})}{\partial \sigma} + (-\frac{z_{\ell:r:m}}{\sigma})g(z_{\ell:r:m})\right] \\
&\quad -\frac{1}{\sigma^2}\sum_{\ell=1}^{r}\left[-1 + z_{\ell:r:m}^2 + z_{\ell:r:m}g(z_{\ell:r:m})\right] \\
&= \frac{1}{\sigma^2}\sum_{\ell=1}^{r}\left[z_{\ell:r:m}^3 g(z_{\ell:r:m}) + z_{\ell:r:m}^2\,[\phi_{nor}(z_{\ell:r:m})]^2 \frac{p_2(2-Rn)\,[\Phi_{nor}(z_{\ell:r:m})]^n - p_1\,[\Phi_{nor}(z_{\ell:r:m})]^{2n} - p_2}{\left[\Phi_{nor}(z_{\ell:r:m}) - [\Phi_{nor}(z_{\ell:r:m})]^{n+1}\right]^2}\right. \\
&\quad \left. -2z_{\ell:r:m}g(z_{\ell:r:m}) - 3z_{\ell:r:m}^2 + 1\right].
\end{aligned}$$

G.3. 2-out-of-n

With lognormal distributed components, if we use the following

$$\begin{aligned}
g_1(z_{\ell:r:m}) &= \Phi_{nor}(z_{\ell:r:m})\,[1 - \Phi_{nor}(z_{\ell:r:m})]^{n-2} \\
\text{and } g_2(z_{\ell:r:m}) &= [1 - \Phi_{nor}(z_{\ell:r:m})]^n + n\,[\Phi_{nor}(z_{\ell:r:m})]\,[1 - \Phi_{nor}(z_{\ell:r:m})]^{n-1},
\end{aligned}$$

we have

$$
\begin{aligned}
\frac{\partial l(\boldsymbol{\theta})}{\partial \mu} &= \sum_{\ell=1}^{r}\left[\frac{z_{\ell:r:m}}{\sigma} + \frac{\frac{\partial g_1(z_{\ell:r:m})}{\partial \mu}}{g_1(z_{\ell:r:m})} + R_\ell \frac{\frac{\partial g_2(z_{\ell:r:m})}{\partial \mu}}{g_2(z_{\ell:r:m})}\right] \\
\frac{\partial l(\boldsymbol{\theta})}{\partial \sigma} &= \sum_{\ell=1}^{r}\left[\frac{z_{\ell:r:m}^2 - 1}{\sigma} + \frac{\frac{\partial g_1(z_{\ell:r:m})}{\partial \sigma}}{g_1(z_{\ell:r:m})} + R_\ell \frac{\frac{\partial g_2(z_{\ell:r:m})}{\partial \sigma}}{g_2(z_{\ell:r:m})}\right],
\end{aligned}
$$

where

$$
\begin{aligned}
\frac{\partial g_1(z_{\ell:r:m})}{\partial \mu} &= \frac{(n-2)\phi_{nor}(z_{\ell:r:m})\Phi_{nor}(z_{\ell:r:m})\left[1-\Phi_{nor}(z_{\ell:r:m})\right]^{n-3}}{\sigma} \\
&\quad - \frac{\phi_{nor}(z_{\ell:r:m})\left[1-\Phi_{nor}(z_{\ell:r:m})\right]^{n-2}}{\sigma} \\
\frac{\partial g_1(z_{\ell:r:m})}{\partial \sigma} &= \frac{(n-2)z_{\ell:r:m}\phi_{nor}(z_{\ell:r:m})\Phi_{nor}(z_{\ell:r:m})\left[1-\Phi_{nor}(z_{\ell:r:m})\right]^{n-3}}{\sigma} \\
&\quad - \frac{z_{\ell:r:m}\phi_{nor}(z_{\ell:r:m})\left[1-\Phi_{nor}(z_{\ell:r:m})\right]^{n-2}}{\sigma}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial g_2(z_{\ell:r:m})}{\partial \mu} &= \frac{n(n-1)\phi_{nor}(z_{\ell:r:m})\Phi_{nor}(z_{\ell:r:m})\left[1-\Phi_{nor}(z_{\ell:r:m})\right]^{n-2}}{\sigma} \\
\frac{\partial g_2(z_{\ell:r:m})}{\partial \sigma} &= \frac{n(n-1)z_{\ell:r:m}\phi_{nor}(z_{\ell:r:m})\Phi_{nor}(z_{\ell:r:m})\left[1-\Phi_{nor}(z_{\ell:r:m})\right]^{n-2}}{\sigma}.
\end{aligned}
$$

Then, we have

$$\begin{aligned}
\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \mu^2} &= \sum_{\ell=1}^{r}\Bigg[-\frac{1}{\sigma^2}+\frac{\frac{\partial^2 g_1(z_{\ell:r:m})}{\partial \mu^2}g_1(z_{\ell:r:m})-\left[\frac{\partial g_1(z_{\ell:r:m})}{\partial \mu}\right]^2}{g_1(z_{\ell:r:m})^2} \\
&\quad +R_\ell\frac{\frac{\partial^2 g_2(z_{\ell:r:m})}{\partial \mu^2}g_2(z_{\ell:r:m})-\left[\frac{\partial g_2(z_{\ell:r:m})}{\partial \mu}\right]^2}{g_2(z_{\ell:r:m})}\Bigg] \\
\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \mu \partial \sigma} &= \sum_{\ell=1}^{r}\Bigg[-\frac{2z_{\ell:r:m}}{\sigma^2}+\frac{\frac{\partial^2 g_1(z_{\ell:r:m})}{\partial \sigma \partial \mu}g_1(z_{\ell:r:m})-\frac{\partial g_1(z_{\ell:r:m})}{\partial \sigma}\frac{\partial g_1(z_{\ell:r:m})}{\partial \mu}}{g_1(z_{\ell:r:m})^2} \\
&\quad +R_\ell\frac{\frac{\partial^2 g_2(z_{\ell:r:m})}{\partial \sigma \partial \mu}g_2(z_{\ell:r:m})-\frac{\partial g_2(z_{\ell:r:m})}{\partial \sigma}\frac{\partial g_2(z_{\ell:r:m})}{\partial \mu}}{g_2(z_{\ell:r:m})}\Bigg] \\
\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \sigma^2} &= \frac{1}{\sigma}\sum_{\ell=1}^{r}\Bigg[\frac{1-3z_{\ell:r:m}^2}{\sigma^2}+\frac{\frac{\partial^2 g_1(z_{\ell:r:m})}{\partial \sigma^2}g_1(z_{\ell:r:m})-\left[\frac{\partial g_1(z_{\ell:r:m})}{\partial \sigma}\right]^2}{g_1(z_{\ell:r:m})^2} \\
&\quad +R_\ell\frac{\frac{\partial^2 g_2(z_{\ell:r:m})}{\partial \sigma^2}g_2(z_{\ell:r:m})-\left[\frac{\partial g_2(z_{\ell:r:m})}{\partial \sigma}\right]^2}{g_2(z_{\ell:r:m})}\Bigg].
\end{aligned}$$

When $n=3$, the explicit formula are given by

$$\begin{aligned}
\frac{\partial g_1(z_{\ell:r:m})}{\partial \mu} &= \frac{2\phi_{nor}(z_{\ell:r:m})\Phi_{nor}(z_{\ell:r:m})-\phi_{nor}(z_{\ell:r:m})}{\sigma} \\
\frac{\partial g_1(z_{\ell:r:m})}{\partial \sigma} &= \frac{z_{\ell:r:m}\left[2\phi_{nor}(z_{\ell:r:m})\Phi_{nor}(z_{\ell:r:m})-\phi_{nor}(z_{\ell:r:m})\right]}{\sigma},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial g_2(z_{\ell:r:m})}{\partial \mu} &= \frac{6\phi_{nor}(z_{\ell:r:m})\Phi_{nor}(z_{\ell:r:m})\left[1-\Phi_{nor}(z_{\ell:r:m})\right]}{\sigma} \\
\frac{\partial g_2(z_{\ell:r:m})}{\partial \sigma} &= \frac{z_{\ell:r:m}6\phi_{nor}(z_{\ell:r:m})\Phi_{nor}(z_{\ell:r:m})\left[1-\Phi_{nor}(z_{\ell:r:m})\right]}{\sigma},
\end{aligned}$$

$$\frac{\partial^2 g_1(z_{\ell:r:m})}{\partial\mu^2} = \frac{\phi_{nor}(z_{\ell:r:m})[2\Phi_{nor}(z_{\ell:r:m})-1]z_{\ell:r:m} - 2\phi^2_{nor}(z_{\ell:r:m})}{\sigma^2}$$

$$\frac{\partial^2 g_1(z_{\ell:r:m})}{\partial\mu\partial\sigma} = \frac{\phi_{nor}(z_{\ell:r:m})[2\Phi_{nor}(z_{\ell:r:m})-1]z^2_{\ell:r:m} - 2\phi^2_{nor}(z_{\ell:r:m})z_{\ell:r:m}}{\sigma^2} - \frac{\phi_{nor}(z_{\ell:r:m})[2\Phi_{nor}(z_{\ell:r:m})-1]}{\sigma^2}$$

$$\frac{\partial^2 g_1(z_{\ell:r:m})}{\partial\sigma^2} = \frac{\phi_{nor}(z_{\ell:r:m})[2\Phi_{nor}(z_{\ell:r:m})-1]z^3_{\ell:r:m} - 2\phi^2_{nor}(z_{\ell:r:m})z^2_{\ell:r:m}}{\sigma^2} - \frac{2\phi_{nor}(z_{\ell:r:m})[2\Phi_{nor}(z_{\ell:r:m})-1]z_{\ell:r:m}}{\sigma^2}$$

and

$$\frac{\partial^2 g_2(z_{\ell:r:m})}{\partial\mu^2} = \frac{6\phi_{nor}(z_{\ell:r:m})\left(-\left[\Phi_{nor}(z_{\ell:r:m})\right]^2 + \left[\Phi_{nor}(z_{\ell:r:m})\right]\right)z_{\ell:r:m}}{\sigma^2} + \frac{6\phi^2_{nor}(z_{\ell:r:m})\left(2\Phi_{nor}(z_{\ell:r:m})-1\right)}{\sigma^2}$$

$$\frac{\partial^2 g_2(z_{\ell:r:m})}{\partial\mu\partial\sigma} = \frac{6\phi_{nor}(z_{\ell:r:m})\left(-\left[\Phi_{nor}(z_{\ell:r:m})\right]^2 + \left[\Phi_{nor}(z_{\ell:r:m})\right]\right)z^2_{\ell:r:m}}{\sigma^2} + \frac{6\phi^2_{nor}(z_{\ell:r:m})\left(2\Phi_{nor}(z_{\ell:r:m})-1\right)z_{\ell:r:m}}{\sigma^2} - \frac{6\phi_{nor}(z_{\ell:r:m})\left(-\left[\Phi_{nor}(z_{\ell:r:m})\right]^2 + \left[\Phi_{nor}(z_{\ell:r:m})\right]\right)}{\sigma^2}$$

$$\frac{\partial^2 g_2(z_{\ell:r:m})}{\partial\sigma^2} = \frac{6\phi_{nor}(z_{\ell:r:m})\left(-\left[\Phi_{nor}(z_{\ell:r:m})\right]^2 + \left[\Phi_{nor}(z_{\ell:r:m})\right]\right)z^3_{\ell:r:m}}{\sigma^2} + \frac{6\phi^2_{nor}(z_{\ell:r:m})\left(2\Phi_{nor}(z_{\ell:r:m})-1\right)z^2_{\ell:r:m}}{\sigma^2} - \frac{12\phi_{nor}(z_{\ell:r:m})\left(-\left[\Phi_{nor}(z_{\ell:r:m})\right]^2 + \left[\Phi_{nor}(z_{\ell:r:m})\right]\right)z_{\ell:r:m}}{\sigma^2}$$

When $n = 4$, the explicit formula are given by

$$\begin{aligned}
\frac{\partial g_1(z_{\ell:r:m})}{\partial \mu} &= \frac{2\phi_{nor}(z_{\ell:r:m})\Phi_{nor}(z_{\ell:r:m})\left[1-\Phi_{nor}(z_{\ell:r:m})\right]}{\sigma} \\
&\quad -\frac{\phi_{nor}(z_{\ell:r:m})\left[1-\Phi_{nor}(z_{\ell:r:m})\right]^2}{\sigma} \\
\frac{\partial g_1(z_{\ell:r:m})}{\partial \sigma} &= \frac{2\phi_{nor}(z_{\ell:r:m})\Phi_{nor}(z_{\ell:r:m})\left[1-\Phi_{nor}(z_{\ell:r:m})\right]}{\sigma} \\
&\quad -\frac{\phi_{nor}(z_{\ell:r:m})\left[1-\Phi_{nor}(z_{\ell:r:m})\right]^2}{\sigma}(z_{\ell:r:m}),
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial g_2(z_{\ell:r:m})}{\partial \mu} &= \frac{12\phi_{nor}(z_{\ell:r:m})\Phi_{nor}(z_{\ell:r:m})\left[1-\Phi_{nor}(z_{\ell:r:m})\right]^2}{\sigma} \\
\frac{\partial g_2(z_{\ell:r:m})}{\partial \sigma} &= \frac{12\phi_{nor}(z_{\ell:r:m})\Phi_{nor}(z_{\ell:r:m})\left[1-\Phi_{nor}(z_{\ell:r:m})\right]^2}{\sigma}(z_{\ell:r:m}).
\end{aligned}$$

Then, we have

$$\begin{aligned}
\frac{\partial^2 g_1(z_{\ell:r:m})}{\partial \mu^2} &= \frac{\phi_{nor}(z_{\ell:r:m})\left[1-\Phi_{nor}(z_{\ell:r:m})\right]\left[3\Phi_{nor}(z_{\ell:r:m})-1\right]z_{\ell:r:m}}{\sigma^2} \\
&\quad -\frac{\phi_{nor}^2(z_{\ell:r:m})\left[6\Phi_{nor}(z_{\ell:r:m})-4\right]}{\sigma^2} \\
\frac{\partial^2 g_1(z_{\ell:r:m})}{\partial \mu \partial \sigma} &= \frac{\phi_{nor}(z_{\ell:r:m})\left[1-\Phi_{nor}(z_{\ell:r:m})\right]\left[3\Phi_{nor}(z_{\ell:r:m})-1\right]z_{\ell:r:m}^2}{\sigma^2} \\
&\quad -\frac{\phi_{nor}^2(z_{\ell:r:m})\left[6\Phi_{nor}(z_{\ell:r:m})-4\right]z_{\ell:r:m}}{\sigma^2} \\
&\quad -\frac{\phi_{nor}(z_{\ell:r:m})\left[1-\Phi_{nor}(z_{\ell:r:m})\right]\left[3\Phi_{nor}(z_{\ell:r:m})-1\right]}{\sigma^2} \\
\frac{\partial^2 g_1(z_{\ell:r:m})}{\partial \sigma^2} &= \frac{\phi_{nor}(z_{\ell:r:m})\left[1-\Phi_{nor}(z_{\ell:r:m})\right]\left[3\Phi_{nor}(z_{\ell:r:m})-1\right]z_{\ell:r:m}^3}{\sigma^2} \\
&\quad -\frac{\phi_{nor}^2(z_{\ell:r:m})\left[6\Phi_{nor}(z_{\ell:r:m})-4\right]z_{\ell:r:m}^2}{\sigma^2} \\
&\quad -\frac{2\phi_{nor}(z_{\ell:r:m})\left[1-\Phi_{nor}(z_{\ell:r:m})\right]\left[3\Phi_{nor}(z_{\ell:r:m})-1\right]z_{\ell:r:m}}{\sigma^2}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial^2 g_2(z_{\ell:r:m})}{\partial \mu^2} &= \frac{12\phi_{nor}(z_{\ell:r:m})\Phi_{nor}(z_{\ell:r:m})\left[1-\Phi_{nor}(z_{\ell:r:m})\right]^2 z_{\ell:r:m}}{\sigma^2} \\
&\quad + \frac{12\phi_{nor}^2(z_{\ell:r:m})\left(4\Phi_{nor}^2(z_{\ell:r:m})-3\Phi_{nor}(z_{\ell:r:m})-1\right)}{\sigma^2} \\
\frac{\partial^2 g_2(z_{\ell:r:m})}{\partial \mu \partial \sigma} &= \frac{12\phi_{nor}(z_{\ell:r:m})\Phi_{nor}(z_{\ell:r:m})\left[1-\Phi_{nor}(z_{\ell:r:m})\right]^2 z_{\ell:r:m}^2}{\sigma^2} \\
&\quad + \frac{12\phi_{nor}^2(z_{\ell:r:m})\left(4\Phi_{nor}^2(z_{\ell:r:m})-3\Phi_{nor}(z_{\ell:r:m})-1\right) z_{\ell:r:m}}{\sigma^2} \\
&\quad - \frac{12\phi_{nor}(z_{\ell:r:m})\Phi_{nor}(z_{\ell:r:m})\left[1-\Phi_{nor}(z_{\ell:r:m})\right]^2}{\sigma^2} \\
\frac{\partial^2 g_2(z_{\ell:r:m})}{\partial \sigma^2} &= \frac{12\phi_{nor}(z_{\ell:r:m})\Phi_{nor}(z_{\ell:r:m})\left[1-\Phi_{nor}(z_{\ell:r:m})\right]^2 z_{\ell:r:m}^3}{\sigma^2} \\
&\quad + \frac{12\phi_{nor}^2(z_{\ell:r:m})\left(4\Phi_{nor}^2(z_{\ell:r:m})-3\Phi_{nor}(z_{\ell:r:m})-1\right) z_{\ell:r:m}^2}{\sigma^2} \\
&\quad - \frac{24\phi_{nor}(z_{\ell:r:m})\Phi_{nor}(z_{\ell:r:m})\left[1-\Phi_{nor}(z_{\ell:r:m})\right]^2 z_{\ell:r:m}}{\sigma^2}.
\end{aligned}$$

G.4. 3-out-of-n for $n = 4$

If we use the following

$$\begin{aligned}
g_1(z_{\ell:r:m}) &= \Phi_{nor}^2(z_{\ell:r:m})\left[1-\Phi_{nor}(z_{\ell:r:m})\right]^{n-3} \\
&= \Phi_{nor}^2(z_{\ell:r:m})\left[1-\Phi_{nor}(z_{\ell:r:m})\right] \\
\text{and } g_2(z_{\ell:r:m}) &= \left[1-\Phi_{nor}(z_{\ell:r:m})\right]^n + n\Phi_{nor}(z_{\ell:r:m})\left[1-\Phi_{nor}(z_{\ell:r:m})\right]^{n-1} \\
&\quad + \frac{n(n-1)}{2}\Phi_{nor}^2(z_{\ell:r:m})\left[1-\Phi_{nor}(z_{\ell:r:m})\right]^{n-2} \\
&= \left[3\Phi_{nor}^2(z_{\ell:r:m})+2\Phi_{nor}(z_{\ell:r:m})+1\right]\left[1-\Phi_{nor}(z_{\ell:r:m})\right]^2 \\
&= 3\Phi_{nor}^4(z_{\ell:r:m})-4\Phi_{nor}^3(z_{\ell:r:m})+1,
\end{aligned}$$

we have

$$\begin{aligned}
\frac{\partial l(\boldsymbol{\theta})}{\partial \mu} &= \sum_{\ell=1}^{r}\left[\frac{z_{\ell:r:m}}{\sigma}+\frac{\frac{\partial g_1(z_{\ell:r:m})}{\partial \mu}}{g_1(z_{\ell:r:m})}+R_\ell\frac{\frac{\partial g_2(z_{\ell:r:m})}{\partial \mu}}{g_2(z_{\ell:r:m})}\right]\\
\frac{\partial l(\boldsymbol{\theta})}{\partial \sigma} &= \sum_{\ell=1}^{r}\left[\frac{z_{\ell:r:m}^2-1}{\sigma}+\frac{\frac{\partial g_1(z_{\ell:r:m})}{\partial \sigma}}{g_1(z_{\ell:r:m})}+R_\ell\frac{\frac{\partial g_2(z_{\ell:r:m})}{\partial \sigma}}{g_2(z_{\ell:r:m})}\right],
\end{aligned}$$

where

$$\begin{aligned}
\frac{\partial g_1(z_{\ell:r:m})}{\partial \mu} &= \frac{3\phi_{nor}(z_{\ell:r:m})\Phi_{nor}^2(z_{\ell:r:m})-2\phi_{nor}(z_{\ell:r:m})\Phi_{nor}(z_{\ell:r:m})}{\sigma}\\
\frac{\partial g_1(z_{\ell:r:m})}{\partial \sigma} &= \frac{3\phi_{nor}(z_{\ell:r:m})\Phi_{nor}^2(z_{\ell:r:m})-2\phi_{nor}(z_{\ell:r:m})\Phi_{nor}(z_{\ell:r:m})}{\sigma}(z_{\ell:r:m})
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial g_2(z_{\ell:r:m})}{\partial \mu} &= \frac{12\phi_{nor}(z_{\ell:r:m})\left(\Phi_{nor}^2(z_{\ell:r:m})-\Phi_{nor}^3(z_{\ell:r:m})\right)}{\sigma}\\
\frac{\partial g_2(z_{\ell:r:m})}{\partial \sigma} &= \frac{12\phi_{nor}(z_{\ell:r:m})\left(\Phi_{nor}^2(z_{\ell:r:m})-\Phi_{nor}^3(z_{\ell:r:m})\right)}{\sigma}(z_{\ell:r:m}).
\end{aligned}$$

Then,

$$\begin{aligned}
\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \mu^2} &= \sum_{\ell=1}^{r}\Bigg[-\frac{1}{\sigma^2}+\frac{\frac{\partial^2 g_1(z_{\ell:r:m})}{\partial \mu^2}g_1(z_{\ell:r:m})-\left[\frac{\partial g_1(z_{\ell:r:m})}{\partial \mu}\right]^2}{g_1(z_{\ell:r:m})^2}\\
&\quad +R_\ell\frac{\frac{\partial^2 g_2(z_{\ell:r:m})}{\partial \mu^2}g_2(z_{\ell:r:m})-\left[\frac{\partial g_2(z_{\ell:r:m})}{\partial \mu}\right]^2}{g_2(z_{\ell:r:m})}\Bigg]\\
\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \mu \partial \sigma} &= \sum_{\ell=1}^{r}\Bigg[-\frac{2z_{\ell:r:m}}{\sigma^2}+\frac{\frac{\partial^2 g_1(z_{\ell:r:m})}{\partial \sigma \partial \mu}g_1(z_{\ell:r:m})-\frac{\partial g_1(z_{\ell:r:m})}{\partial \sigma}\frac{\partial g_1(z_{\ell:r:m})}{\partial \mu}}{g_1(z_{\ell:r:m})^2}\\
&\quad +R_\ell\frac{\frac{\partial^2 g_2(z_{\ell:r:m})}{\partial \sigma \partial \mu}g_2(z_{\ell:r:m})-\frac{\partial g_2(z_{\ell:r:m})}{\partial \sigma}\frac{\partial g_2(z_{\ell:r:m})}{\partial \mu}}{g_2(z_{\ell:r:m})}\Bigg]\\
\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \sigma^2} &= \frac{1}{\sigma}\sum_{\ell=1}^{r}\Bigg[\frac{1-3z_{\ell:r:m}^2}{\sigma^2}+\frac{\frac{\partial^2 g_1(z_{\ell:r:m})}{\partial \sigma^2}g_1(z_{\ell:r:m})-\left[\frac{\partial g_1(z_{\ell:r:m})}{\partial \sigma}\right]^2}{g_1(z_{\ell:r:m})^2}\\
&\quad +R_\ell\frac{\frac{\partial^2 g_2(z_{\ell:r:m})}{\partial \sigma^2}g_2(z_{\ell:r:m})-\left[\frac{\partial g_2(z_{\ell:r:m})}{\partial \sigma}\right]^2}{g_2(z_{\ell:r:m})}\Bigg],
\end{aligned}$$

where

$$\begin{aligned}
\frac{\partial^2 g_1(z_{\ell:r:m})}{\partial \mu^2} &= \frac{\phi_{nor}(z_{\ell:r:m})[3\Phi_{nor}^2(z_{\ell:r:m})}{\sigma^2}\\
&\quad -\frac{2\Phi_{nor}(z_{\ell:r:m})]z_{\ell:r:m}-2\phi_{nor}^2(z_{\ell:r:m})[3\Phi_{nor}(z_{\ell:r:m})-1]}{\sigma^2}\\
\frac{\partial^2 g_1(z_{\ell:r:m})}{\partial \mu \partial \sigma} &= \frac{\phi_{nor}(z_{\ell:r:m})[3\Phi_{nor}^2(z_{\ell:r:m})-2\Phi_{nor}(z_{\ell:r:m})]z_{\ell:r:m}^2}{\sigma^2}\\
&\quad -\frac{2\phi_{nor}^2(z_{\ell:r:m})[3\Phi_{nor}(z_{\ell:r:m})-1]z_{\ell:r:m}}{\sigma^2}\\
&\quad -\frac{\phi_{nor}(z_{\ell:r:m})[3\Phi_{nor}^2(z_{\ell:r:m})-2\Phi_{nor}(z_{\ell:r:m})]}{\sigma^2}\\
\frac{\partial^2 g_1(z_{\ell:r:m})}{\partial \sigma^2} &= \frac{\phi_{nor}(z_{\ell:r:m})[3\Phi_{nor}^2(z_{\ell:r:m})}{\sigma^2}\\
&\quad -\frac{2\Phi_{nor}(z_{\ell:r:m})]z_{\ell:r:m}^3-2\phi_{nor}^2(z_{\ell:r:m})[3\Phi_{nor}(z_{\ell:r:m})-1]z_{\ell:r:m}^2}{\sigma^2}\\
&\quad -\frac{2\phi_{nor}(z_{\ell:r:m})[3\Phi_{nor}^2(z_{\ell:r:m})-2\Phi_{nor}(z_{\ell:r:m})]z_{\ell:r:m}}{\sigma^2}
\end{aligned}$$

and

$$
\begin{aligned}
\frac{\partial^2 g_2(z_{\ell:r:m})}{\partial \mu^2} &= \frac{12\phi_{nor}(z_{\ell:r:m})\left[\Phi_{nor}^2(z_{\ell:r:m}) - \Phi_{nor}^3(z_{\ell:r:m})\right] z_{\ell:r:m}}{\sigma^2} \\
&\quad + \frac{12\phi_{nor}^2(z_{\ell:r:m})\left[3\Phi_{nor}^2(z_{\ell:r:m}) - 2\Phi_{nor}(z_{\ell:r:m})\right]}{\sigma^2} \\
\frac{\partial^2 g_2(z_{\ell:r:m})}{\partial \mu \partial \sigma} &= \frac{12\phi_{nor}(z_{\ell:r:m})\left[\Phi_{nor}^2(z_{\ell:r:m}) - \Phi_{nor}^3(z_{\ell:r:m})\right] z_{\ell:r:m}^2}{\sigma^2} \\
&\quad + \frac{12\phi_{nor}^2(z_{\ell:r:m})\left[3\Phi_{nor}^2(z_{\ell:r:m}) - 2\Phi_{nor}(z_{\ell:r:m}) z_{\ell:r:m}\right]}{\sigma^2} \\
&\quad - \frac{12\phi_{nor}(z_{\ell:r:m})\left[\Phi_{nor}^2(z_{\ell:r:m}) - \Phi_{nor}^3(z_{\ell:r:m})\right]}{\sigma^2} \\
\frac{\partial^2 g_2(z_{\ell:r:m})}{\partial \sigma^2} &= \frac{12\phi_{nor}(z_{\ell:r:m})\left[\Phi_{nor}^2(z_{\ell:r:m}) - \Phi_{nor}^3(z_{\ell:r:m})\right] z_{\ell:r:m}^3}{\sigma^2} \\
&\quad + \frac{12\phi_{nor}^2(z_{\ell:r:m})\left[3\Phi_{nor}^2(z_{\ell:r:m}) - 2\Phi_{nor}(z_{\ell:r:m}) z_{\ell:r:m}^2\right]}{\sigma^2} \\
&\quad - \frac{24\phi_{nor}(z_{\ell:r:m})\left[\Phi_{nor}^2(z_{\ell:r:m}) - \Phi_{nor}^3(z_{\ell:r:m})\right] z_{\ell:r:m}}{\sigma^2}.
\end{aligned}
$$

G.5. Series-parallel systems

With lognormal distributed components, we have

$$
\begin{aligned}
\frac{\partial l(\boldsymbol{\theta})}{\partial \mu} &= \frac{1}{\sigma}\sum_{\ell=1}^{r}\Bigg[z_{\ell:r:m} + \frac{2\phi_{nor}(z_{\ell:r:m})\Phi_{nor}(z_{\ell:r:m})}{1 - \left[\Phi_{nor}(z_{\ell:r:m})\right]^2} \\
&\quad + R_\ell \frac{\phi_{nor}(z_{\ell:r:m})\left(-3\left[\Phi_{nor}(z_{\ell:r:m})\right]^2 + 2\left[\Phi_{nor}(z_{\ell:r:m})\right] + 1\right)}{\left[\Phi_{nor}(z_{\ell:r:m})\right]^3 - \left[\Phi_{nor}(z_{\ell:r:m})\right]^2 - \Phi_{nor}(z_{\ell:r:m}) + 1}\Bigg] \\
\frac{\partial l(\boldsymbol{\theta})}{\partial \sigma} &= \frac{1}{\sigma}\sum_{\ell=1}^{r}\Bigg[-1 + z_{\ell:r:m}^2 + \frac{2\phi_{nor}(z_{\ell:r:m})\Phi_{nor}(z_{\ell:r:m})}{1 - \left[\Phi_{nor}(z_{\ell:r:m})\right]^2} z_{\ell:r:m} \\
&\quad + R_\ell \frac{\phi_{nor}(z_{\ell:r:m})\left(-3\left[\Phi_{nor}(z_{\ell:r:m})\right]^2 + 2\left[\Phi_{nor}(z_{\ell:r:m})\right] + 1\right)}{\left[\Phi_{nor}(z_{\ell:r:m})\right]^3 - \left[\Phi_{nor}(z_{\ell:r:m})\right]^2 - \Phi_{nor}(z_{\ell:r:m}) + 1} z_{\ell:r:m}\Bigg].
\end{aligned}
$$

To find the maximum likelihood estimates of the parameters, we could set the above equations to 0 and solve them simultaneously.

Similarly, if we use the following

$$\begin{aligned} g_1(z_{\ell:r:m}) &= 1 - [\Phi_{nor}(z_{\ell:r:m})]^2 \\ \text{and } g_2(z_{\ell:r:m}) &= [\Phi_{nor}(z_{\ell:r:m})]^3 - [\Phi_{nor}(z_{\ell:r:m})]^2 - \Phi_{nor}(z_{\ell:r:m}) + 1, \end{aligned}$$

we can obtain

$$\begin{aligned} \frac{\partial l(\boldsymbol{\theta})}{\partial \mu} &= \sum_{\ell=1}^{r} \left[\frac{z_{\ell:r:m}}{\sigma} + \frac{\frac{\partial g_1(z_{\ell:r:m})}{\partial \mu}}{g_1(z_{\ell:r:m})} + R_\ell \frac{\frac{\partial g_2(z_{\ell:r:m})}{\partial \mu}}{g_2(z_{\ell:r:m})} \right] \\ \frac{\partial l(\boldsymbol{\theta})}{\partial \sigma} &= \sum_{\ell=1}^{r} \left[\frac{z_{\ell:r:m}^2 - 1}{\sigma} + \frac{\frac{\partial g_1(z_{\ell:r:m})}{\partial \sigma}}{g_1(z_{\ell:r:m})} + R_\ell \frac{\frac{\partial g_2(z_{\ell:r:m})}{\partial \sigma}}{g_2(z_{\ell:r:m})} \right], \end{aligned}$$

where

$$\begin{aligned} \frac{\partial g_1(z_{\ell:r:m})}{\partial \mu} &= \frac{2\phi_{nor}(z_{\ell:r:m})\Phi_{nor}(z_{\ell:r:m})}{\sigma} \\ \frac{\partial g_1(z_{\ell:r:m})}{\partial \sigma} &= \frac{2\phi_{nor}(z_{\ell:r:m})\Phi_{nor}(z_{\ell:r:m})}{\sigma}(z_{\ell:r:m}) \end{aligned}$$

and

$$\begin{aligned} \frac{\partial g_2(z_{\ell:r:m})}{\partial \mu} &= \frac{\phi_{nor}(z_{\ell:r:m}) \left(-3 \left[\Phi_{nor}(z_{\ell:r:m})\right]^2 + 2 \left[\Phi_{nor}(z_{\ell:r:m})\right] + 1 \right)}{\sigma} \\ \frac{\partial g_2(z_{\ell:r:m})}{\partial \sigma} &= \frac{\phi_{nor}(z_{\ell:r:m}) \left(-3 \left[\Phi_{nor}(z_{\ell:r:m})\right]^2 + 2 \left[\Phi_{nor}(z_{\ell:r:m})\right] + 1 \right)}{\sigma}(z_{\ell:r:m}). \end{aligned}$$

Then,

$$\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \mu^2} = \sum_{\ell=1}^{r}\Bigg[-\frac{1}{\sigma^2} + \frac{\frac{\partial^2 g_1(z_{\ell:r:m})}{\partial \mu^2} g_1(z_{\ell:r:m}) - \left[\frac{\partial g_1(z_{\ell:r:m})}{\partial \mu}\right]^2}{g_1(z_{\ell:r:m})^2} + R_\ell \frac{\frac{\partial^2 g_2(z_{\ell:r:m})}{\partial \mu^2} g_2(z_{\ell:r:m}) - \left[\frac{\partial g_2(z_{\ell:r:m})}{\partial \mu}\right]^2}{g_2(z_{\ell:r:m})}\Bigg]$$

$$\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \mu \partial \sigma} = \sum_{\ell=1}^{r}\Bigg[-\frac{2z_{\ell:r:m}}{\sigma^2} + \frac{\frac{\partial^2 g_1(z_{\ell:r:m})}{\partial \sigma \partial \mu} g_1(z_{\ell:r:m}) - \frac{\partial g_1(z_{\ell:r:m})}{\partial \sigma}\frac{\partial g_1(z_{\ell:r:m})}{\partial \mu}}{g_1(z_{\ell:r:m})^2} + R_\ell \frac{\frac{\partial^2 g_2(z_{\ell:r:m})}{\partial \sigma \partial \mu} g_2(z_{\ell:r:m}) - \frac{\partial g_2(z_{\ell:r:m})}{\partial \sigma}\frac{\partial g_2(z_{\ell:r:m})}{\partial \mu}}{g_2(z_{\ell:r:m})}\Bigg]$$

$$\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \sigma^2} = \frac{1}{\sigma}\sum_{\ell=1}^{r}\Bigg[\frac{1-3z_{\ell:r:m}^2}{\sigma^2} + \frac{\frac{\partial^2 g_1(z_{\ell:r:m})}{\partial \sigma^2} g_1(z_{\ell:r:m}) - \left[\frac{\partial g_1(z_{\ell:r:m})}{\partial \sigma}\right]^2}{g_1(z_{\ell:r:m})^2} + R_\ell \frac{\frac{\partial^2 g_2(z_{\ell:r:m})}{\partial \sigma^2} g_2(z_{\ell:r:m}) - \left[\frac{\partial g_2(z_{\ell:r:m})}{\partial \sigma}\right]^2}{g_2(z_{\ell:r:m})}\Bigg],$$

where

$$\frac{\partial^2 g_1(z_{\ell:r:m})}{\partial \mu^2} = \frac{2\phi_{nor}(z_{\ell:r:m})\Phi_{nor}(z_{\ell:r:m})z_{\ell:r:m} - 2\phi_{nor}^2(z_{\ell:r:m})}{\sigma^2}$$

$$\frac{\partial^2 g_1(z_{\ell:r:m})}{\partial \mu \partial \sigma} = \frac{2\phi_{nor}(z_{\ell:r:m})\Phi_{nor}(z_{\ell:r:m})z_{\ell:r:m}^2 - 2\phi_{nor}^2(z_{\ell:r:m})z_{\ell:r:m}}{\sigma^2} - \frac{2\phi_{nor}(z_{\ell:r:m})\Phi_{nor}(z_{\ell:r:m})}{\sigma^2}$$

$$\frac{\partial^2 g_1(z_{\ell:r:m})}{\partial \sigma^2} = \frac{2\phi_{nor}(z_{\ell:r:m})\Phi_{nor}(z_{\ell:r:m})z_{\ell:r:m}^3 - 2\phi_{nor}^2(z_{\ell:r:m})z_{\ell:r:m}^2}{\sigma^2} - \frac{2\phi_{nor}(z_{\ell:r:m})\Phi_{nor}(z_{\ell:r:m})z_{\ell:r:m}}{\sigma^2}$$

and

$$\begin{aligned}
\frac{\partial^2 g_2(z_{\ell:r:m})}{\partial \mu^2} &= \frac{\phi_{nor}(z_{\ell:r:m})\left(-3\left[\Phi_{nor}(z_{\ell:r:m})\right]^2 + 2\left[\Phi_{nor}(z_{\ell:r:m})\right] + 1\right) z_{\ell:r:m}}{\sigma^2} \\
&\quad + \frac{\phi_{nor}^2(z_{\ell:r:m})\left(6\Phi_{nor}(z_{\ell:r:m}) - 2\right)}{\sigma^2} \\
\frac{\partial^2 g_2(z_{\ell:r:m})}{\partial \mu \partial \sigma} &= \frac{\phi_{nor}(z_{\ell:r:m})\left(-3\left[\Phi_{nor}(z_{\ell:r:m})\right]^2 + 2\left[\Phi_{nor}(z_{\ell:r:m})\right] + 1\right) z_{\ell:r:m}^2}{\sigma^2} \\
&\quad + \frac{\phi_{nor}^2(z_{\ell:r:m})\left(6\Phi_{nor}(z_{\ell:r:m}) - 2\right) z_{\ell:r:m}}{\sigma^2} \\
&\quad - \frac{\phi_{nor}(z_{\ell:r:m})\left(-3\left[\Phi_{nor}(z_{\ell:r:m})\right]^2 + 2\left[\Phi_{nor}(z_{\ell:r:m})\right] + 1\right)}{\sigma^2} \\
\frac{\partial^2 g_2(z_{\ell:r:m})}{\partial \sigma^2} &= \frac{\phi_{nor}(z_{\ell:r:m})\left(-3\left[\Phi_{nor}(z_{\ell:r:m})\right]^2 + 2\left[\Phi_{nor}(z_{\ell:r:m})\right] + 1\right) z_{\ell:r:m}^3}{\sigma^2} \\
&\quad + \frac{\phi_{nor}^2(z_{\ell:r:m})\left(6\Phi_{nor}(z_{\ell:r:m}) - 2\right) z_{\ell:r:m}^2}{\sigma^2} \\
&\quad - \frac{2\phi_{nor}(z_{\ell:r:m})\left(-3\left[\Phi_{nor}(z_{\ell:r:m})\right]^2 + 2\left[\Phi_{nor}(z_{\ell:r:m})\right] + 1\right) z_{\ell:r:m}}{\sigma^2}.
\end{aligned}$$

G.6. Parallel-series systems

If we use the following

$$\begin{aligned}
g_1(z_{\ell:r:m}) &= 2\Phi_{nor}(z_{\ell:r:m}) - \frac{5}{3}\left[\Phi_{nor}(z_{\ell:r:m})\right]^2 \\
\text{and } g_2(z_{\ell:r:m}) &= \left[\Phi_{nor}(z_{\ell:r:m})\right]^3 - 2\left[\Phi_{nor}(z_{\ell:r:m})\right]^2 + 1,
\end{aligned}$$

we have

$$\begin{aligned}
\frac{\partial l(\boldsymbol{\theta})}{\partial \mu} &= \sum_{\ell=1}^{r}\left[\frac{z_{\ell:r:m}}{\sigma} + \frac{\frac{\partial g_1(z_{\ell:r:m})}{\partial \mu}}{g_1(z_{\ell:r:m})} + R_\ell \frac{\frac{\partial g_2(z_{\ell:r:m})}{\partial \mu}}{g_2(z_{\ell:r:m})}\right] \\
\frac{\partial l(\boldsymbol{\theta})}{\partial \sigma} &= \sum_{\ell=1}^{r}\left[\frac{z_{\ell:r:m}^2 - 1}{\sigma} + \frac{\frac{\partial g_1(z_{\ell:r:m})}{\partial \sigma}}{g_1(z_{\ell:r:m})} + R_\ell \frac{\frac{\partial g_2(z_{\ell:r:m})}{\partial \sigma}}{g_2(z_{\ell:r:m})}\right],
\end{aligned}$$

where

$$\frac{\partial g_1(z_{\ell:r:m})}{\partial \mu} = \frac{10\phi_{nor}(z_{\ell:r:m})\Phi_{nor}(z_{\ell:r:m}) - 6\phi_{nor}(z_{\ell:r:m})}{3\sigma}$$

$$\frac{\partial g_1(z_{\ell:r:m})}{\partial \sigma} = \frac{10\phi_{nor}(z_{\ell:r:m})\Phi_{nor}(z_{\ell:r:m}) - 6\phi_{nor}(z_{\ell:r:m})}{3\sigma}(z_{\ell:r:m})$$

and

$$\frac{\partial g_2(z_{\ell:r:m})}{\partial \mu} = \frac{\phi_{nor}(z_{\ell:r:m})\left(-3\left[\Phi_{nor}(z_{\ell:r:m})\right]^2 + 4\left[\Phi_{nor}(z_{\ell:r:m})\right]\right)}{\sigma}$$

$$\frac{\partial g_2(z_{\ell:r:m})}{\partial \sigma} = \frac{\phi_{nor}(z_{\ell:r:m})\left(-3\left[\Phi_{nor}(z_{\ell:r:m})\right]^2 + 4\left[\Phi_{nor}(z_{\ell:r:m})\right]\right)}{\sigma}(z_{\ell:r:m}).$$

Then,

$$\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \mu^2} = \sum_{\ell=1}^{r}\Bigg[-\frac{1}{\sigma^2} + \frac{\frac{\partial^2 g_1(z_{\ell:r:m})}{\partial \mu^2} g_1(z_{\ell:r:m}) - \left[\frac{\partial g_1(z_{\ell:r:m})}{\partial \mu}\right]^2}{g_1(z_{\ell:r:m})^2} + R_\ell \frac{\frac{\partial^2 g_2(z_{\ell:r:m})}{\partial \mu^2} g_2(z_{\ell:r:m}) - \left[\frac{\partial g_2(z_{\ell:r:m})}{\partial \mu}\right]^2}{g_2(z_{\ell:r:m})}\Bigg]$$

$$\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \mu \partial \sigma} = \sum_{\ell=1}^{r}\Bigg[-\frac{2z_{\ell:r:m}}{\sigma^2} + \frac{\frac{\partial^2 g_1(z_{\ell:r:m})}{\partial \sigma \partial \mu} g_1(z_{\ell:r:m}) - \frac{\partial g_1(z_{\ell:r:m})}{\partial \sigma}\frac{\partial g_1(z_{\ell:r:m})}{\partial \mu}}{g_1(z_{\ell:r:m})^2} + R_\ell \frac{\frac{\partial^2 g_2(z_{\ell:r:m})}{\partial \sigma \partial \mu} g_2(z_{\ell:r:m}) - \frac{\partial g_2(z_{\ell:r:m})}{\partial \sigma}\frac{\partial g_2(z_{\ell:r:m})}{\partial \mu}}{g_2(z_{\ell:r:m})}\Bigg]$$

$$\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \sigma^2} = \frac{1}{\sigma}\sum_{\ell=1}^{r}\Bigg[\frac{1 - 3z_{\ell:r:m}^2}{\sigma^2} + \frac{\frac{\partial^2 g_1(z_{\ell:r:m})}{\partial \sigma^2} g_1(z_{\ell:r:m}) - \left[\frac{\partial g_1(z_{\ell:r:m})}{\partial \sigma}\right]^2}{g_1(z_{\ell:r:m})^2} + R_\ell \frac{\frac{\partial^2 g_2(z_{\ell:r:m})}{\partial \sigma^2} g_2(z_{\ell:r:m}) - \left[\frac{\partial g_2(z_{\ell:r:m})}{\partial \sigma}\right]^2}{g_2(z_{\ell:r:m})}\Bigg],$$

where

$$\begin{aligned}
\frac{\partial^2 g_1(z_{\ell:r:m})}{\partial \mu^2} &= \frac{\phi_{nor}(z_{\ell:r:m})[10\Phi_{nor}(z_{\ell:r:m})-6]z_{\ell:r:m}-10\phi_{nor}^2(z_{\ell:r:m})}{3\sigma^2}\\
\frac{\partial^2 g_1(z_{\ell:r:m})}{\partial \mu \partial \sigma} &= \frac{\phi_{nor}(z_{\ell:r:m})[10\Phi_{nor}(z_{\ell:r:m})-6]z_{\ell:r:m}^2-10\phi_{nor}^2(z_{\ell:r:m})z_{\ell:r:m}}{3\sigma^2}\\
&\quad -\frac{\phi_{nor}(z_{\ell:r:m})[10\Phi_{nor}(z_{\ell:r:m})-6]}{3\sigma^2}\\
\frac{\partial^2 g_1(z_{\ell:r:m})}{\partial \sigma^2} &= \frac{\phi_{nor}(z_{\ell:r:m})[10\Phi_{nor}(z_{\ell:r:m})-6]z_{\ell:r:m}^3-10\phi_{nor}^2(z_{\ell:r:m})z_{\ell:r:m}^2}{3\sigma^2}\\
&\quad -\frac{2\phi_{nor}(z_{\ell:r:m})[10\Phi_{nor}(z_{\ell:r:m})-6]z_{\ell:r:m}}{3\sigma^2}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial^2 g_2(z_{\ell:r:m})}{\partial \mu^2} &= \frac{\phi_{nor}(z_{\ell:r:m})\left(-3\left[\Phi_{nor}(z_{\ell:r:m})\right]^2+4\left[\Phi_{nor}(z_{\ell:r:m})\right]\right)z_{\ell:r:m}}{\sigma^2}\\
&\quad +\frac{\phi_{nor}^2(z_{\ell:r:m})\left(6\Phi_{nor}(z_{\ell:r:m})-4\right)}{\sigma^2}\\
\frac{\partial^2 g_2(z_{\ell:r:m})}{\partial \mu \partial \sigma} &= \frac{\phi_{nor}(z_{\ell:r:m})\left(-3\left[\Phi_{nor}(z_{\ell:r:m})\right]^2+4\left[\Phi_{nor}(z_{\ell:r:m})\right]\right)z_{\ell:r:m}^2}{\sigma^2}\\
&\quad +\frac{\phi_{nor}^2(z_{\ell:r:m})\left(6\Phi_{nor}(z_{\ell:r:m})-4\right)z_{\ell:r:m}}{\sigma^2}\\
&\quad -\frac{\phi_{nor}(z_{\ell:r:m})\left(-3\left[\Phi_{nor}(z_{\ell:r:m})\right]^2+4\left[\Phi_{nor}(z_{\ell:r:m})\right]\right)}{\sigma^2}\\
\frac{\partial^2 g_2(z_{\ell:r:m})}{\partial \sigma^2} &= \frac{\phi_{nor}(z_{\ell:r:m})\left(-3\left[\Phi_{nor}(z_{\ell:r:m})\right]^2+4\left[\Phi_{nor}(z_{\ell:r:m})\right]\right)z_{\ell:r:m}^3}{\sigma^2}\\
&\quad +\frac{\phi_{nor}^2(z_{\ell:r:m})\left(6\Phi_{nor}(z_{\ell:r:m})-4\right)z_{\ell:r:m}^2}{\sigma^2}\\
&\quad -\frac{2\phi_{nor}(z_{\ell:r:m})\left(-3\left[\Phi_{nor}(z_{\ell:r:m})\right]^2+4\left[\Phi_{nor}(z_{\ell:r:m})\right]\right)z_{\ell:r:m}}{\sigma^2}.
\end{aligned}$$

Appendix H

Appendix H: Detailed Formula for Weibull Distribution with Various Types of Systems

H.1. $l = 1$

With $l = 1$, we are using series systems. Thus, the SF and PDF of the log-transformed system failure times can be simplified as

$$\begin{aligned} \bar{F}_V(v;\boldsymbol{\theta}) &= \left[\bar{F}^*(z)\right]^n, \\ \text{and } f_V(v;\boldsymbol{\theta}) &= \frac{n}{\sigma} f^*(z)\left[\bar{F}^*(z)\right]^{n-1}, \quad n = 1, 2, 3. \end{aligned}$$

Thus, we have

$$\begin{aligned} \ln f_V(v_{\ell:r:m};\boldsymbol{\theta}) &= \ln\left\{\frac{n}{\sigma} f^*(z_{\ell:r:m})\left[\bar{F}^*(z_{\ell:r:m})\right]^{n-1}\right\} \\ &= \ln n - \ln\sigma + \ln f^*(z_{\ell:r:m}) + (n-1)\ln\bar{F}^*(z_{\ell:r:m}) \\ \text{and } \ln\bar{F}_V(v_{\ell:r:m};\boldsymbol{\theta}) &= \ln\left\{\left[\bar{F}^*(z_{\ell:r:m})\right]^n\right\} = n\ln\bar{F}^*(z_{\ell:r:m}). \end{aligned}$$

Hence, we have the first derivatives as

$$\begin{aligned} \frac{\partial \ln f_V(v_{\ell:r:m};\boldsymbol{\theta})}{\partial\mu} &= \frac{\partial \ln f^*(z_{\ell:r:m})}{\partial\mu} + (n-1)\frac{\partial \ln\bar{F}^*(z_{\ell:r:m})}{\partial\mu} \\ \frac{\partial \ln f_V(v_{\ell:r:m};\boldsymbol{\theta})}{\partial\sigma} &= -\frac{1}{\sigma} + \frac{\partial \ln f^*(z_{\ell:r:m})}{\partial\sigma} + (n-1)\frac{\partial \ln\bar{F}^*(z_{\ell:r:m})}{\partial\sigma} \end{aligned}$$

and

$$\begin{aligned}\frac{\partial \ln \bar{F}_V(v_{\ell:r:m};\boldsymbol{\theta})}{\partial \mu} &= n\frac{\partial \ln \bar{F}^*(z_{\ell:r:m})}{\partial \mu}\\ \frac{\partial \ln \bar{F}_V(v_{\ell:r:m};\boldsymbol{\theta})}{\partial \sigma} &= n\frac{\partial \ln \bar{F}^*(z_{\ell:r:m})}{\partial \sigma}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\frac{\partial l(\boldsymbol{\theta})}{\partial \mu} &= \sum_{\ell=1}^{r}\left[\frac{\partial \ln f^*(z_{\ell:r:m})}{\partial \mu} + (R_\ell n + n - 1)\frac{\partial \ln \bar{F}^*(z_{\ell:r:m})}{\partial \mu}\right]\\ &= \sum_{\ell=1}^{r}\left[\frac{1}{f^*(z_{\ell:r:m})}\frac{\partial f^*(z_{\ell:r:m})}{\partial \mu} + (R_\ell n + n - 1)\frac{1}{\bar{F}^*(z_{\ell:r:m})}\frac{\partial \bar{F}^*(z_{\ell:r:m})}{\partial \mu}\right]\\ \frac{\partial l(\boldsymbol{\theta})}{\partial \sigma} &= \sum_{\ell=1}^{r}\left[-\frac{1}{\sigma} + \frac{\partial \ln f^*(z_{\ell:r:m})}{\partial \sigma} + (R_\ell n + n - 1)\frac{\partial \ln \bar{F}^*(z_{\ell:r:m})}{\partial \sigma}\right]\\ &= \sum_{\ell=1}^{r}\left[-\frac{1}{\sigma} + \frac{1}{f^*(z_{\ell:r:m})}\frac{\partial f^*(z_{\ell:r:m})}{\partial \sigma} + (R_\ell n + n - 1)\frac{1}{\bar{F}^*(z_{\ell:r:m})}\frac{\partial \bar{F}^*(z_{\ell:r:m})}{\partial \sigma}\right].\end{aligned}$$

With Weibull distributed components, we have

$$\begin{aligned}\frac{\partial l(\boldsymbol{\theta})}{\partial \mu} &= \sum_{\ell=1}^{r}\left[\frac{e^{z_{\ell:r:m}} - 1}{\sigma} + \frac{R_\ell n + n - 1}{\sigma}\frac{\phi_{sev}(z_{\ell:r:m})}{\bar{\Phi}_{sev}(z_{\ell:r:m})}\right]\\ &= \sum_{\ell=1}^{r}\left[\frac{e^{z_{\ell:r:m}} - 1}{\sigma} + \frac{R_\ell n + n - 1}{\sigma}e^{z_{\ell:r:m}}\right]\\ &= \sum_{\ell=1}^{r}\left[-\frac{1}{\sigma} + \frac{R_\ell n + n}{\sigma}e^{z_{\ell:r:m}}\right]\\ \frac{\partial l(\boldsymbol{\theta})}{\partial \sigma} &= \sum_{\ell=1}^{r}\left[-\frac{1}{\sigma} + \frac{(e^{z_{\ell:r:m}} - 1)z_{\ell:r:m}}{\sigma} + \frac{z_{\ell:r:m}(R_\ell n + n - 1)}{\sigma}\frac{\phi_{sev}(z_{\ell:r:m})}{\bar{\Phi}_{sev}(z_{\ell:r:m})}\right]\\ &= \sum_{\ell=1}^{r}\left[-\frac{1}{\sigma} + \frac{(e^{z_{\ell:r:m}} - 1)z_{\ell:r:m}}{\sigma} + \frac{z_{\ell:r:m}(R_\ell n + n - 1)}{\sigma}e^{z_{\ell:r:m}}\right]\\ &= \sum_{\ell=1}^{r}\left[-\frac{1}{\sigma} - \frac{z_{\ell:r:m}}{\sigma} + \frac{z_{\ell:r:m}(R_\ell n + n)}{\sigma}e^{z_{\ell:r:m}}\right].\end{aligned}$$

Hence, the likelihood equations for μ and σ are, respectively,

$$-r + \sum_{\ell=1}^{r}(R_\ell n + n)e^{z_{\ell:r:m}} = 0;$$
$$-r - \sum_{\ell=1}^{r} z_{\ell:r:m} + \sum_{\ell=1}^{r}(R_\ell n + n)z_{\ell:r:m}e^{z_{\ell:r:m}} = 0.$$

From the log-likelihood function, the second derivatives forming the observed Fisher Information are

$$\begin{aligned}
\frac{\partial^2 l(\boldsymbol{\theta})}{\partial\mu^2} &= -\frac{1}{\sigma^2}\sum_{\ell=1}^{r}(R_\ell n + n)e^{z_{\ell:r:m}} \\
\frac{\partial^2 l(\boldsymbol{\theta})}{\partial\mu\partial\sigma} &= \frac{r}{\sigma^2} - \frac{1}{\sigma^2}\sum_{\ell=1}^{r}(R_\ell n + n)\left(z_{\ell:r:m}e^{z_{\ell:r:m}} + e^{z_{\ell:r:m}}\right) \\
\frac{\partial^2 l(\boldsymbol{\theta})}{\partial\sigma^2} &= \frac{r}{\sigma^2} + \frac{2}{\sigma^2}\sum_{\ell=1}^{r} z_{\ell:r:m} - \frac{1}{\sigma^2}\sum_{\ell=1}^{r}(R_\ell n + n)\left(z_{\ell:r:m}^2 e^{z_{\ell:r:m}} + 2z_{\ell:r:m}e^{z_{\ell:r:m}}\right).
\end{aligned}$$

In order to compute the expected Fisher information matrix, we need to compute the expectations $E(Z_{\ell:r:m})$ and $E(Z_{\ell:r:m}^d \exp(Z_{\ell:r:m}))$, $d = 0, 1, 2$. Consider the PDF of the ℓ-th order statistic of a sample of size m, $\ell = 1, \ldots, r$ of the minimum among n random variables from the standard SEV distribution is given by

$$\begin{aligned}
f_{\ell:r:m}(z_\ell) &= c' \sum_{i=0}^{\ell-1} c_{i,\ell-1}(R_1+1, \ldots, R_{\ell-1}+1) f_v(z_\ell)\left[\bar{F}_v(z_\ell)\right]^{R_i''-1}, \\
&= nc' \sum_{i=0}^{\ell-1} c_{i,\ell-1}(R_1+1, \ldots, R_{\ell-1}+1) f^*(z_\ell)\left[\bar{F}^*(z_\ell)\right]^{nR_i''-1} \\
&= nc' \sum_{i=0}^{\ell-1} c_{i,\ell-1}(R_1+1, \ldots, R_{\ell-1}+1) e^{z_\ell - nR_i'' e^{z_\ell}}, \quad -\infty < z_\ell < \infty,
\end{aligned}$$

where

$$
\begin{aligned}
R_i'' &= R_\ell^* + \sum_{j=\ell-i}^{\ell-1}(R_j+1) \\
R_\ell^* &= m-\ell-R_1-\cdots-R_{\ell-1}+1 \\
c' &= m(m-R_1-1)\ldots(m-\ell-R_1-\cdots-R_{\ell-1}+1),
\end{aligned}
$$

and

$$
c_{i,q}(a_1,\ldots,a_r) = \frac{(-1)^i}{\left\{\prod_{j=1}^{i}\left(\sum_{k=q-i+1}^{q-i+j} a_k\right)\right\}\left\{\prod_{j=1}^{q-i}\left(\sum_{k=j}^{q-i} a_k\right)\right\}},
$$

where the conventions are that $\prod_{j=1}^{0} a_j = 1$. Then, the expectations $E(\exp(Z_{\ell:m_k}))$ and $E(Z_{\ell:m_k}^d \exp(Z_{\ell:m_k}))$ can be expressed as

$$
E(Z_{\ell:m_k}) = nc' \sum_{i=0}^{\ell-1} c_{i,\ell-1}(R_1+1,\ldots,R_{\ell-1}+1)\int_{-\infty}^{\infty} z e^{z-nR_i''e^z} dz
$$

and

$$
\begin{aligned}
&E[Z_{\ell:m_k}^d \exp(Z_{\ell:m_k})] \\
&= nc' \sum_{i=0}^{\ell-1} c_{i,\ell-1}(R_1+1,\ldots,R_{\ell-1}+1)\int_{-\infty}^{\infty} z^d e^{2z-nR_i''e^z} dz,
\end{aligned}
$$

respectively, which can be computed by evaluating the integrals

$$
h(c) = \int_{-\infty}^{\infty} z e^{z-ce^z} dz \text{ and } g_d(c) = \int_{-\infty}^{\infty} z^d e^{2z-ce^z} dz,
$$

for $d = 0, 1, 2$ as

$$
h(c) = -\frac{\gamma + \log c}{c}
$$

and

$$\begin{aligned}
g_0(c) &= \frac{1}{c^2}, \\
g_1(c) &= \frac{1-\gamma-\log c}{c^2}, \\
g_2(c) &= \frac{\log^2 c - 2(1-\gamma)\log c + \gamma^2 - 2\gamma + \pi^2/6}{c^2},
\end{aligned}$$

where $\gamma = -\Gamma'(1) = 0.57721566...$ is Euler's constant.

Hence, we find the expected values as

$$\begin{aligned}
E(\exp(Z_{\ell:m_k})) &= nc' \sum_{i=0}^{\ell-1} c_{i,\ell-1}(R_1+1,\dots,R_{\ell-1}+1)\frac{-\gamma-\log(nR''_i)}{nR''_i} \\
E(\exp(Z_{\ell:m_k})) &= nc' \sum_{i=0}^{\ell-1} c_{i,\ell-1}(R_1+1,\dots,R_{\ell-1}+1)\frac{1}{(nR''_i)^2} \\
E(Z_{\ell:m_k}\exp(Z_{\ell:m_k})) &= nc' \sum_{i=0}^{\ell-1} c_{i,\ell-1}(R_1+1,\dots,R_{\ell-1}+1)\frac{1-\gamma-\log(nR''_i)}{(nR''_i)^2} \\
E(Z^2_{\ell:m_k}\exp(Z_{\ell:m_k})) &= nc' \sum_{i=0}^{\ell-1} c_{i,\ell-1}(R_1+1,\dots,R_{\ell-1}+1) \\
&\quad \times \frac{(\log nR''_i)^2 - 2(1-\gamma)\log(nR''_i) + \gamma^2 - 2\gamma + \pi^2/6}{(nR''_i)^2}.
\end{aligned}$$

H.2. $l = n$

With $l = n$, we have parallel systems with system structure being $\boldsymbol{s} = (0, 0, \dots, 0, 1)$. The SF and PDF of the log-transformed system failure times are

$$\begin{aligned}
\bar{F}_V(v;\boldsymbol{\theta}) &= \sum_{j=0}^{n-1} \binom{n}{j} \left[\bar{F}^*(z)\right]^j \left[\bar{F}^*(z)\right]^{n-j} \\
&= 1 - \left[F^*(z)\right]^n, \\
\text{and } f_V(v;\boldsymbol{\theta}) &= \frac{n}{\sigma} f^*(z)\left[F^*(z)\right]^{n-1}.
\end{aligned}$$

Thus, we have

$$
\begin{aligned}
\ln f_V(v_{\ell:r:m};\boldsymbol{\theta}) &= \ln\left\{\frac{n}{\sigma}f^*(z)\,[F^*(z)]^{n-1}\right\} \\
&= \ln n - \ln\sigma + \ln f^*(z_{\ell:r:m}) + (n-1)\ln F^*(z_{\ell:r:m}) \\
\text{and }\ \ln \bar{F}_V(v_{\ell:r:m};\boldsymbol{\theta}) &= \ln\left\{1-[F^*(z_{\ell:r:m})]^n\right\} \\
&= \ln\left\{1-[F^*(z_{\ell:r:m})]^n\right\}.
\end{aligned}
$$

Hence, we have the first derivatives as

$$
\begin{aligned}
\frac{\partial \ln f_V(v_{\ell:r:m};\boldsymbol{\theta})}{\partial\mu} &= \frac{\partial \ln f^*(z_{\ell:r:m})}{\partial\mu} + (n-1)\frac{\partial \ln F^*(z_{\ell:r:m})}{\partial\mu} \\
\frac{\partial \ln f_V(v_{\ell:r:m};\boldsymbol{\theta})}{\partial\sigma} &= -\frac{1}{\sigma} + \frac{\partial \ln f^*(z_{\ell:r:m})}{\partial\sigma} + (n-1)\frac{\partial \ln F^*(z_{\ell:r:m})}{\partial\sigma}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial \ln \bar{F}_V(v_{\ell:r:m};\boldsymbol{\theta})}{\partial\mu} &= \frac{\partial \ln\left\{1-[F^*(z_{\ell:r:m})]^n\right\}}{\partial\mu} \\
\frac{\partial \ln \bar{F}_V(v_{\ell:r:m};\boldsymbol{\theta})}{\partial\sigma} &= \frac{\partial \ln\left\{1-[F^*(z_{\ell:r:m})]^n\right\}}{\partial\sigma}.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
\frac{\partial l(\boldsymbol{\theta})}{\partial\mu} &= \sum_{\ell=1}^{r}\Bigg[\frac{1}{f^*(z_{\ell:r:m})}\frac{\partial f^*(z_{\ell:r:m})}{\partial\mu} \\
&\qquad + \frac{(n-1)-(R_\ell n+n-1)\,[F^*(z_{\ell:r:m})]^n}{F^*(z_{\ell:r:m})\,(1-[F^*(z_{\ell:r:m})]^n)}\frac{\partial F^*(z_{\ell:r:m})}{\partial\mu}\Bigg] \\
\frac{\partial l(\boldsymbol{\theta})}{\partial\sigma} &= \sum_{\ell=1}^{r}\Bigg[-\frac{1}{\sigma} + \frac{1}{f^*(z_{\ell:r:m})}\frac{\partial f^*(z_{\ell:r:m})}{\partial\sigma} \\
&\qquad + \frac{(n-1)-(R_\ell n+n-1)\,[F^*(z_{\ell:r:m})]^n}{F^*(z_{\ell:r:m})\,(1-[F^*(z_{\ell:r:m})]^n)}\frac{\partial F^*(z_{\ell:r:m})}{\partial\sigma}\Bigg].
\end{aligned}
$$

With Weibull distributed components, we have

$$\begin{aligned}
\frac{\partial l(\boldsymbol{\theta})}{\partial \mu} &= \frac{1}{\sigma}\sum_{\ell=1}^{r}\left[(e^{z_{\ell:r:m}}-1)+\frac{(R_\ell n+n-1)\left[\Phi_{sev}(z_{\ell:r:m})\right]^n-(n-1)}{1-\left[\Phi_{sev}(z_{\ell:r:m})\right]^n}\frac{\phi_{sev}(z_{\ell:r:m})}{\Phi_{sev}(z_{\ell:r:m})}\right]\\
\frac{\partial l(\boldsymbol{\theta})}{\partial \sigma} &= \frac{1}{\sigma}\sum_{\ell=1}^{r}\Bigg[-1+(e^{z_{\ell:r:m}}-1)z_{\ell:r:m}\\
&\quad+\frac{z_{\ell:r:m}\left[(R_\ell n+n-1)\left[\Phi_{sev}(z_{\ell:r:m})\right]^n-(n-1)\right]}{1-\left[\Phi_{sev}(z_{\ell:r:m})\right]^n}\frac{\phi_{sev}(z_{\ell:r:m})}{\Phi_{sev}(z_{\ell:r:m})}\Bigg].
\end{aligned}$$

Hence, the likelihood equations for μ and σ are, respectively,

$$\sum_{\ell=1}^{r}\left[(e^{z_{\ell:r:m}}-1)+\frac{(R_\ell n+n-1)\left[\Phi_{sev}(z_{\ell:r:m})\right]^n-(n-1)}{1-\left[\Phi_{sev}(z_{\ell:r:m})\right]^n}\frac{\phi_{sev}(z_{\ell:r:m})}{\Phi_{sev}(z_{\ell:r:m})}\right]=0;$$

$$\sum_{\ell=1}^{r}\left[-1+(e^{z_{\ell:r:m}}-1)z_{\ell:r:m}+\frac{z_{\ell:r:m}\left[(R_\ell n+n-1)\left[\Phi_{sev}(z_{\ell:r:m})\right]^n-(n-1)\right]}{1-\left[\Phi_{sev}(z_{\ell:r:m})\right]^n}\frac{\phi_{sev}(z_{\ell:r:m})}{\Phi_{sev}(z_{\ell:r:m})}\right]=0.$$

If we use the following

$$\begin{aligned}
g_1(z_{\ell:r:m}) &= \Phi_{sev}(z_{\ell:r:m})\\
\text{and } g_2(z_{\ell:r:m}) &= 1-\left[\Phi_{sev}(z_{\ell:r:m})\right]^n,
\end{aligned}$$

we have

$$\begin{aligned}
\frac{\partial l(\boldsymbol{\theta})}{\partial \mu} &= \sum_{\ell=1}^{r}\left[\frac{(e^{z_{\ell:r:m}}-1)}{\sigma}+(n-1)\frac{\frac{\partial g_1(z_{\ell:r:m})}{\partial\mu}}{g_1(z_{\ell:r:m})}+R_\ell\frac{\frac{\partial g_2(z_{\ell:r:m})}{\partial\mu}}{g_2(z_{\ell:r:m})}\right]\\
\frac{\partial l(\boldsymbol{\theta})}{\partial \sigma} &= \sum_{\ell=1}^{r}\left[-\frac{1}{\sigma}+\frac{(e^{z_{\ell:r:m}}-1)z_{\ell:r:m}}{\sigma}+(n-1)\frac{\frac{\partial g_1(z_{\ell:r:m})}{\partial\sigma}}{g_1(z_{\ell:r:m})}+R_\ell\frac{\frac{\partial g_2(z_{\ell:r:m})}{\partial\sigma}}{g_2(z_{\ell:r:m})}\right],
\end{aligned}$$

where

$$\begin{aligned}
\frac{\partial g_1(z_{\ell:r:m})}{\partial \mu} &= -\frac{\phi_{sev}(z_{\ell:r:m})}{\sigma} \\
\frac{\partial g_1(z_{\ell:r:m})}{\partial \sigma} &= -\frac{\phi_{sev}(z_{\ell:r:m})}{\sigma} z_{\ell:r:m}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial g_2(z_{\ell:r:m})}{\partial \mu} &= n\frac{\phi_{sev}(z_{\ell:r:m})\left[\Phi_{sev}(z_{\ell:r:m})\right]^{n-1}}{\sigma} \\
\frac{\partial g_2(z_{\ell:r:m})}{\partial \sigma} &= n\frac{\phi_{sev}(z_{\ell:r:m})\left[\Phi_{sev}(z_{\ell:r:m})\right]^{n-1}}{\sigma} z_{\ell:r:m}.
\end{aligned}$$

Then,

$$\begin{aligned}
\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \mu^2} &= \sum_{\ell=1}^{r}\Bigg[-\frac{e^{z_{\ell:r:m}}}{\sigma^2} + (n-1)\frac{\frac{\partial^2 g_1(z_{\ell:r:m})}{\partial \mu^2} g_1(z_{\ell:r:m}) - \left[\frac{\partial g_1(z_{\ell:r:m})}{\partial \mu}\right]^2}{g_1(z_{\ell:r:m})^2} \\
&+ R_\ell \frac{\frac{\partial^2 g_2(z_{\ell:r:m})}{\partial \mu^2} g_2(z_{\ell:r:m}) - \left[\frac{\partial g_2(z_{\ell:r:m})}{\partial \mu}\right]^2}{g_2(z_{\ell:r:m})}\Bigg] \\
\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \mu \partial \sigma} &= \sum_{\ell=1}^{r}\Bigg[-\frac{z_{\ell:r:m}e^{z_{\ell:r:m}} + e^{z_{\ell:r:m}} - 1}{\sigma^2} \\
&\quad +(n-1)\frac{\frac{\partial^2 g_1(z_{\ell:r:m})}{\partial \sigma \partial \mu} g_1(z_{\ell:r:m}) - \frac{\partial g_1(z_{\ell:r:m})}{\partial \sigma}\frac{\partial g_1(z_{\ell:r:m})}{\partial \mu}}{g_1(z_{\ell:r:m})^2} \\
&+ R_\ell \frac{\frac{\partial^2 g_2(z_{\ell:r:m})}{\partial \sigma \partial \mu} g_2(z_{\ell:r:m}) - \frac{\partial g_2(z_{\ell:r:m})}{\partial \sigma}\frac{\partial g_2(z_{\ell:r:m})}{\partial \mu}}{g_2(z_{\ell:r:m})}\Bigg] \\
\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \sigma^2} &= \frac{1}{\sigma}\sum_{\ell=1}^{r}\Bigg[\frac{1 - z_{\ell:r:m}^2 e^{z_{\ell:r:m}} - 2z_{\ell:r:m}e^{z_{\ell:r:m}} + 2z_{\ell:r:m}}{\sigma^2} \\
&\quad +(n-1)\frac{\frac{\partial^2 g_1(z_{\ell:r:m})}{\partial \sigma^2} g_1(z_{\ell:r:m}) - \left[\frac{\partial g_1(z_{\ell:r:m})}{\partial \sigma}\right]^2}{g_1(z_{\ell:r:m})^2} \\
&+ R_\ell \frac{\frac{\partial^2 g_2(z_{\ell:r:m})}{\partial \sigma^2} g_2(z_{\ell:r:m}) - \left[\frac{\partial g_2(z_{\ell:r:m})}{\partial \sigma}\right]^2}{g_2(z_{\ell:r:m})}\Bigg],
\end{aligned}$$

where

$$\frac{\partial^2 g_1(z_{\ell:r:m})}{\partial \mu^2} = -\frac{(e^{z_{\ell:r:m}}-1))\phi_{sev}(z_{\ell:r:m})}{\sigma^2}$$

$$\frac{\partial^2 g_1(z_{\ell:r:m})}{\partial \mu \partial \sigma} = \frac{-(e^{z_{\ell:r:m}}-1))\phi_{sev}(z_{\ell:r:m})z_{\ell:r:m}+\phi_{sev}(z_{\ell:r:m})}{\sigma^2}$$

$$\frac{\partial^2 g_1(z_{\ell:r:m})}{\partial \sigma^2} = \frac{-(e^{z_{\ell:r:m}}-1))\phi_{sev}(z_{\ell:r:m})z_{\ell:r:m}^2+2\phi_{sev}(z_{\ell:r:m})z_{\ell:r:m}}{\sigma^2}$$

and

$$\begin{aligned}
\frac{\partial^2 g_2(z_{\ell:r:m})}{\partial \mu^2} &= n\left\{\frac{(e^{z_{\ell:r:m}}-1)\phi_{sev}(z_{\ell:r:m})\left[\Phi_{sev}(z_{\ell:r:m})\right]^{n-1}}{\sigma^2}\right.\\
&\quad\left.-\frac{(n-1)\phi_{sev}^2(z_{\ell:r:m})\left[\Phi_{sev}(z_{\ell:r:m})\right]^{n-2}}{\sigma^2}\right\}\\
\frac{\partial^2 g_2(z_{\ell:r:m})}{\partial \mu \partial \sigma} &= n\left\{\frac{\left[(e^{z_{\ell:r:m}}-1)\phi_{sev}(z_{\ell:r:m})\left[\Phi_{sev}(z_{\ell:r:m})\right]^{n-1}\right.}{\sigma^2}\right.\\
&\quad-\frac{\left.(n-1)\phi_{sev}^2(z_{\ell:r:m})\left[\Phi_{sev}(z_{\ell:r:m})\right]^{n-2}\right]z_{\ell:r:m}}{\sigma^2}\\
&\quad\left.-\quad\frac{\phi_{sev}(z_{\ell:r:m})\left[\Phi_{sev}(z_{\ell:r:m})\right]^{n-1}}{\sigma^2}\right\}\\
\frac{\partial^2 g_2(z_{\ell:r:m})}{\partial \sigma^2} &= n\left\{\frac{\left[(e^{z_{\ell:r:m}}-1)\phi_{sev}(z_{\ell:r:m})\left[\Phi_{sev}(z_{\ell:r:m})\right]^{n-1}\right.}{\sigma^2}\right.\\
&\quad-\frac{\left.(n-1)\phi_{sev}^2(z_{\ell:r:m})\left[\Phi_{sev}(z_{\ell:r:m})\right]^{n-2}\right]z_{\ell:r:m}^2}{\sigma^2}\\
&\quad\left.-\quad\frac{2\phi_{sev}(z_{\ell:r:m})\left[\Phi_{sev}(z_{\ell:r:m})\right]^{n-1}z_{\ell:r:m}}{\sigma^2}\right\}.
\end{aligned}$$

H.3. 2-out-of-n

With Weibull distributed components, when $n = 3$, we have If we use the following

$$\begin{aligned} g_1(z_{\ell:r:m}) &= \Phi_{sev}(z_{\ell:r:m}) - \Phi^2_{sev}(z_{\ell:r:m}) \\ \text{and } g_2(z_{\ell:r:m}) &= 2\Phi^3_{sev}(z_{\ell:r:m}) - 3\Phi^2_{sev}(z_{\ell:r:m}) + 1, \end{aligned}$$

we have

$$\begin{aligned} \frac{\partial l(\boldsymbol{\theta})}{\partial \mu} &= \sum_{\ell=1}^{r} \left[\frac{(e^{z_{\ell:r:m}} - 1)}{\sigma} + \frac{\frac{\partial g_1(z_{\ell:r:m})}{\partial \mu}}{g_1(z_{\ell:r:m})} + R_\ell \frac{\frac{\partial g_2(z_{\ell:r:m})}{\partial \mu}}{g_2(z_{\ell:r:m})} \right] \\ \frac{\partial l(\boldsymbol{\theta})}{\partial \sigma} &= \sum_{\ell=1}^{r} \left[-\frac{1}{\sigma} + \frac{(e^{z_{\ell:r:m}} - 1)z_{\ell:r:m}}{\sigma} + \frac{\frac{\partial g_1(z_{\ell:r:m})}{\partial \sigma}}{g_1(z_{\ell:r:m})} + R_\ell \frac{\frac{\partial g_2(z_{\ell:r:m})}{\partial \sigma}}{g_2(z_{\ell:r:m})} \right], \end{aligned}$$

where

$$\begin{aligned} \frac{\partial g_1(z_{\ell:r:m})}{\partial \mu} &= \frac{2\phi_{sev}(z_{\ell:r:m})\Phi_{sev}(z_{\ell:r:m}) - \phi_{sev}(z_{\ell:r:m})}{\sigma} \\ \frac{\partial g_1(z_{\ell:r:m})}{\partial \sigma} &= \frac{2\phi_{sev}(z_{\ell:r:m})\Phi_{sev}(z_{\ell:r:m}) - \phi_{sev}(z_{\ell:r:m})}{\sigma}(z_{\ell:r:m}) \end{aligned}$$

and

$$\begin{aligned} \frac{\partial g_2(z_{\ell:r:m})}{\partial \mu} &= \frac{6\phi_{sev}(z_{\ell:r:m})\left(-\Phi^2_{sev}(z_{\ell:r:m}) + \Phi_{sev}(z_{\ell:r:m})\right)}{\sigma} \\ \frac{\partial g_2(z_{\ell:r:m})}{\partial \sigma} &= \frac{6\phi_{sev}(z_{\ell:r:m})\left(-\Phi^2_{sev}(z_{\ell:r:m}) + \Phi_{sev}(z_{\ell:r:m})\right)}{\sigma}(z_{\ell:r:m}). \end{aligned}$$

Then,

$$\frac{\partial^2 l(\boldsymbol{\theta})}{\partial\mu^2} = \sum_{\ell=1}^{r}\Bigg[-\frac{e^{z_{\ell:r:m}}}{\sigma^2} + \frac{\frac{\partial^2 g_1(z_{\ell:r:m})}{\partial\mu^2} g_1(z_{\ell:r:m}) - \left[\frac{\partial g_1(z_{\ell:r:m})}{\partial\mu}\right]^2}{g_1(z_{\ell:r:m})^2} + R_\ell \frac{\frac{\partial^2 g_2(z_{\ell:r:m})}{\partial\mu^2} g_2(z_{\ell:r:m}) - \left[\frac{\partial g_2(z_{\ell:r:m})}{\partial\mu}\right]^2}{g_2(z_{\ell:r:m})}\Bigg]$$

$$\frac{\partial^2 l(\boldsymbol{\theta})}{\partial\mu\partial\sigma} = \sum_{\ell=1}^{r}\Bigg[-\frac{z_{\ell:r:m}e^{z_{\ell:r:m}} + e^{z_{\ell:r:m}} - 1}{\sigma^2} + \frac{\frac{\partial^2 g_1(z_{\ell:r:m})}{\partial\sigma\partial\mu} g_1(z_{\ell:r:m}) - \frac{\partial g_1(z_{\ell:r:m})}{\partial\sigma}\frac{\partial g_1(z_{\ell:r:m})}{\partial\mu}}{g_1(z_{\ell:r:m})^2} + R_\ell \frac{\frac{\partial^2 g_2(z_{\ell:r:m})}{\partial\sigma\partial\mu} g_2(z_{\ell:r:m}) - \frac{\partial g_2(z_{\ell:r:m})}{\partial\sigma}\frac{\partial g_2(z_{\ell:r:m})}{\partial\mu}}{g_2(z_{\ell:r:m})}\Bigg]$$

$$\frac{\partial^2 l(\boldsymbol{\theta})}{\partial\sigma^2} = \frac{1}{\sigma}\sum_{\ell=1}^{r}\Bigg[\frac{1 - 3z_{\ell:r:m}^2}{\sigma^2} + \frac{\frac{\partial^2 g_1(z_{\ell:r:m})}{\partial\sigma^2} g_1(z_{\ell:r:m}) - \left[\frac{\partial g_1(z_{\ell:r:m})}{\partial\sigma}\right]^2}{g_1(z_{\ell:r:m})^2} + R_\ell \frac{\frac{\partial^2 g_2(z_{\ell:r:m})}{\partial\sigma^2} g_2(z_{\ell:r:m}) - \left[\frac{\partial g_2(z_{\ell:r:m})}{\partial\sigma}\right]^2}{g_2(z_{\ell:r:m})}\Bigg],$$

where

$$\frac{\partial^2 g_1(z_{\ell:r:m})}{\partial\mu^2} = \frac{\phi_{sev}(z_{\ell:r:m})[2\Phi_{sev}(z_{\ell:r:m}) - 1][e^{z_{\ell:r:m}} - 1] - 2\phi_{sev}^2(z_{\ell:r:m})}{\sigma^2}$$

$$\frac{\partial^2 g_1(z_{\ell:r:m})}{\partial\mu\partial\sigma} = \frac{\phi_{sev}(z_{\ell:r:m})[2\Phi_{sev}(z_{\ell:r:m}) - 1][e^{z_{\ell:r:m}} - 1]z_{\ell:r:m}}{\sigma^2} - \frac{2\phi_{sev}^2(z_{\ell:r:m})z_{\ell:r:m} - \phi_{sev}(z_{\ell:r:m})[2\Phi_{sev}(z_{\ell:r:m}) - 1]}{\sigma^2}$$

$$\frac{\partial^2 g_1(z_{\ell:r:m})}{\partial\sigma^2} = \frac{\phi_{sev}(z_{\ell:r:m})[2\Phi_{sev}(z_{\ell:r:m}) - 1][e^{z_{\ell:r:m}} - 1]z_{\ell:r:m}^2}{\sigma^2} - \frac{2\phi_{sev}^2(z_{\ell:r:m})z_{\ell:r:m}^2 - 2\phi_{sev}(z_{\ell:r:m})[2\Phi_{sev}(z_{\ell:r:m}) - 1]z_{\ell:r:m}}{\sigma^2}$$

and

$$
\begin{aligned}
\frac{\partial^2 g_2(z_{\ell:r:m})}{\partial\mu^2} &= \frac{6\phi_{sev}(z_{\ell:r:m})\left(-\left[\Phi_{sev}(z_{\ell:r:m})\right]^2+\left[\Phi_{sev}(z_{\ell:r:m})\right]\right)\left[e^{z_{\ell:r:m}}-1\right]}{\sigma^2} \\
&\quad + \frac{6\phi^2_{sev}(z_{\ell:r:m})\left(2\Phi_{sev}(z_{\ell:r:m})-1\right)}{\sigma^2} \\
\frac{\partial^2 g_2(z_{\ell:r:m})}{\partial\mu\partial\sigma} &= \frac{6\phi_{sev}(z_{\ell:r:m})\left(-\left[\Phi_{sev}(z_{\ell:r:m})\right]^2+\left[\Phi_{sev}(z_{\ell:r:m})\right]\right)\left[e^{z_{\ell:r:m}}-1\right]z_{\ell:r:m}}{\sigma^2} \\
&\quad + \frac{6\phi^2_{sev}(z_{\ell:r:m})\left(2\Phi_{sev}(z_{\ell:r:m})-1\right)z_{\ell:r:m}}{\sigma^2} \\
&\quad - \frac{6\phi_{sev}(z_{\ell:r:m})\left(-\left[\Phi_{sev}(z_{\ell:r:m})\right]^2+\left[\Phi_{sev}(z_{\ell:r:m})\right]\right)}{\sigma^2} \\
\frac{\partial^2 g_2(z_{\ell:r:m})}{\partial\sigma^2} &= \frac{6\phi_{sev}(z_{\ell:r:m})\left(-\left[\Phi_{sev}(z_{\ell:r:m})\right]^2+\left[\Phi_{sev}(z_{\ell:r:m})\right]\right)\left[e^{z_{\ell:r:m}}-1\right]z^2_{\ell:r:m}}{\sigma^2} \\
&\quad + \frac{6\phi^2_{sev}(z_{\ell:r:m})\left(2\Phi_{sev}(z_{\ell:r:m})-1\right)z^2_{\ell:r:m}}{\sigma^2} \\
&\quad - \frac{12\phi_{sev}(z_{\ell:r:m})\left(-\left[\Phi_{sev}(z_{\ell:r:m})\right]^2+\left[\Phi_{sev}(z_{\ell:r:m})\right]\right)z_{\ell:r:m}}{\sigma^2}.
\end{aligned}
$$

When $n = 4$, the explicit formulas are given by

$$
\begin{aligned}
\frac{\partial g_1(z_{\ell:r:m})}{\partial\mu} &= \frac{2\phi_{sev}(z_{\ell:r:m})\Phi_{sev}(z_{\ell:r:m})\left[1-\Phi_{sev}(z_{\ell:r:m})\right]-\phi_{sev}(z_{\ell:r:m})\left[1-\Phi_{sev}(z_{\ell:r:m})\right]^2}{\sigma} \\
\frac{\partial g_1(z_{\ell:r:m})}{\partial\sigma} &= \frac{2z_{\ell:r:m}\phi_{sev}(z_{\ell:r:m})\Phi_{sev}(z_{\ell:r:m})\left[1-\Phi_{sev}(z_{\ell:r:m})\right]-z_{\ell:r:m}\phi_{sev}(z_{\ell:r:m})\left[1-\Phi_{sev}(z_{\ell:r:m})\right]^2}{\sigma},
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial g_2(z_{\ell:r:m})}{\partial\mu} &= \frac{12\phi_{sev}(z_{\ell:r:m})\Phi_{sev}(z_{\ell:r:m})\left[1-\Phi_{sev}(z_{\ell:r:m})\right]^2}{\sigma} \\
\frac{\partial g_2(z_{\ell:r:m})}{\partial\sigma} &= \frac{12z_{\ell:r:m}\phi_{sev}(z_{\ell:r:m})\Phi_{sev}(z_{\ell:r:m})\left[1-\Phi_{sev}(z_{\ell:r:m})\right]^2}{\sigma}.
\end{aligned}
$$

Then, we have

$$\frac{\partial^2 g_1(z_{\ell:r:m})}{\partial \mu^2} = \frac{\phi_{sev}(z_{\ell:r:m})\left[1-\Phi_{sev}(z_{\ell:r:m})\right]\left[3\Phi_{sev}(z_{\ell:r:m})-1\right]\left[e^{z_{\ell:r:m}}-1\right]}{\sigma^2} - \frac{\phi_{sev}^2(z_{\ell:r:m})\left[6\Phi_{sev}(z_{\ell:r:m})-4\right]}{\sigma^2}$$

$$\frac{\partial^2 g_1(z_{\ell:r:m})}{\partial \mu \partial \sigma} = \frac{\phi_{sev}(z_{\ell:r:m})\left[1-\Phi_{sev}(z_{\ell:r:m})\right]\left[3\Phi_{sev}(z_{\ell:r:m})-1\right]\left[e^{z_{\ell:r:m}}-1\right]z_{\ell:r:m}}{\sigma^2} - \frac{\phi_{sev}^2(z_{\ell:r:m})\left[6\Phi_{sev}(z_{\ell:r:m})-4\right]z_{\ell:r:m}}{\sigma^2} - \frac{\phi_{sev}(z_{\ell:r:m})\left[1-\Phi_{sev}(z_{\ell:r:m})\right]\left[3\Phi_{sev}(z_{\ell:r:m})-1\right]}{\sigma^2}$$

$$\frac{\partial^2 g_1(z_{\ell:r:m})}{\partial \sigma^2} = \frac{\phi_{sev}(z_{\ell:r:m})\left[1-\Phi_{sev}(z_{\ell:r:m})\right]\left[3\Phi_{sev}(z_{\ell:r:m})-1\right]\left[e^{z_{\ell:r:m}}-1\right]z_{\ell:r:m}^2}{\sigma^2} - \frac{\phi_{sev}^2(z_{\ell:r:m})\left[6\Phi_{sev}(z_{\ell:r:m})-4\right]z_{\ell:r:m}^2}{\sigma^2} - \frac{2\phi_{sev}(z_{\ell:r:m})\left[1-\Phi_{sev}(z_{\ell:r:m})\right]\left[3\Phi_{sev}(z_{\ell:r:m})-1\right]z_{\ell:r:m}}{\sigma^2}$$

and

$$\frac{\partial^2 g_2(z_{\ell:r:m})}{\partial \mu^2} = \frac{12\phi_{sev}(z_{\ell:r:m})\Phi_{sev}(z_{\ell:r:m})\left[1-\Phi_{sev}(z_{\ell:r:m})\right]^2\left[e^{z_{\ell:r:m}}-1\right]}{\sigma^2} + \frac{12\phi_{sev}^2(z_{\ell:r:m})\left(4\Phi_{sev}^2(z_{\ell:r:m})-3\Phi_{sev}(z_{\ell:r:m})-1\right)}{\sigma^2}$$

$$\frac{\partial^2 g_2(z_{\ell:r:m})}{\partial \mu \partial \sigma} = \frac{12\phi_{sev}(z_{\ell:r:m})\Phi_{sev}(z_{\ell:r:m})\left[1-\Phi_{sev}(z_{\ell:r:m})\right]^2\left[e^{z_{\ell:r:m}}-1\right]z_{\ell:r:m}}{\sigma^2} + \frac{12\phi_{sev}^2(z_{\ell:r:m})\left(4\Phi_{sev}^2(z_{\ell:r:m})-3\Phi_{sev}(z_{\ell:r:m})-1\right)z_{\ell:r:m}}{\sigma^2} - \frac{12\phi_{sev}(z_{\ell:r:m})\Phi_{sev}(z_{\ell:r:m})\left[1-\Phi_{sev}(z_{\ell:r:m})\right]^2}{\sigma^2}$$

$$\frac{\partial^2 g_2(z_{\ell:r:m})}{\partial \sigma^2} = \frac{12\phi_{sev}(z_{\ell:r:m})\Phi_{sev}(z_{\ell:r:m})\left[1-\Phi_{sev}(z_{\ell:r:m})\right]^2\left[e^{z_{\ell:r:m}}-1\right]z_{\ell:r:m}^2}{\sigma^2} + \frac{12\phi_{sev}^2(z_{\ell:r:m})\left(4\Phi_{sev}^2(z_{\ell:r:m})-3\Phi_{sev}(z_{\ell:r:m})-1\right)z_{\ell:r:m}^2}{\sigma^2} - \frac{24\phi_{sev}(z_{\ell:r:m})\Phi_{sev}(z_{\ell:r:m})\left[1-\Phi_{sev}(z_{\ell:r:m})\right]^2 z_{\ell:r:m}}{\sigma^2}.$$

H.4. 3-out-of-n for $n = 4$

For Weibull distributed components, from the following equations

$$\begin{aligned} g_1(z_{\ell:r:m}) &= \Phi^2_{sev}(z_{\ell:r:m})\left[1 - \Phi_{sev}(z_{\ell:r:m})\right] \\ \text{and } g_2(z_{\ell:r:m}) &= 3\Phi^4_{sev}(z_{\ell:r:m}) - 4\Phi^3_{sev}(z_{\ell:r:m}) + 1 \end{aligned}$$

we have

$$\begin{aligned} \frac{\partial l(\boldsymbol{\theta})}{\partial \mu} &= \sum_{\ell=1}^{r}\left[\frac{z_{\ell:r:m}}{\sigma} + \frac{\frac{\partial g_1(z_{\ell:r:m})}{\partial \mu}}{g_1(z_{\ell:r:m})} + R_\ell \frac{\frac{\partial g_2(z_{\ell:r:m})}{\partial \mu}}{g_2(z_{\ell:r:m})}\right] \\ \frac{\partial l(\boldsymbol{\theta})}{\partial \sigma} &= \sum_{\ell=1}^{r}\left[\frac{z^2_{\ell:r:m} - 1}{\sigma} + \frac{\frac{\partial g_1(z_{\ell:r:m})}{\partial \sigma}}{g_1(z_{\ell:r:m})} + R_\ell \frac{\frac{\partial g_2(z_{\ell:r:m})}{\partial \sigma}}{g_2(z_{\ell:r:m})}\right], \end{aligned}$$

where

$$\begin{aligned} \frac{\partial g_1(z_{\ell:r:m})}{\partial \mu} &= \frac{3\phi_{sev}(z_{\ell:r:m})\Phi^2_{sev}(z_{\ell:r:m}) - 2\phi_{sev}(z_{\ell:r:m})\Phi_{sev}(z_{\ell:r:m})}{\sigma} \\ \frac{\partial g_1(z_{\ell:r:m})}{\partial \sigma} &= \frac{3\phi_{sev}(z_{\ell:r:m})\Phi^2_{sev}(z_{\ell:r:m}) - 2\phi_{sev}(z_{\ell:r:m})\Phi_{sev}(z_{\ell:r:m})}{\sigma}(z_{\ell:r:m}) \end{aligned}$$

and

$$\begin{aligned} \frac{\partial g_2(z_{\ell:r:m})}{\partial \mu} &= \frac{12\phi_{sev}(z_{\ell:r:m})\left(\Phi^2_{sev}(z_{\ell:r:m}) - \Phi^3_{sev}(z_{\ell:r:m})\right)}{\sigma} \\ \frac{\partial g_2(z_{\ell:r:m})}{\partial \sigma} &= \frac{12\phi_{sev}(z_{\ell:r:m})\left(\Phi^2_{sev}(z_{\ell:r:m}) - \Phi^3_{sev}(z_{\ell:r:m})\right)}{\sigma}(z_{\ell:r:m}). \end{aligned}$$

Then,

$$\begin{aligned}
\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \mu^2} &= \sum_{\ell=1}^{r}\Bigg[-\frac{1}{\sigma^2}+\frac{\frac{\partial^2 g_1(z_{\ell:r:m})}{\partial \mu^2}g_1(z_{\ell:r:m})-\left[\frac{\partial g_1(z_{\ell:r:m})}{\partial \mu}\right]^2}{g_1(z_{\ell:r:m})^2}\\
&\quad +R_\ell\frac{\frac{\partial^2 g_2(z_{\ell:r:m})}{\partial \mu^2}g_2(z_{\ell:r:m})-\left[\frac{\partial g_2(z_{\ell:r:m})}{\partial \mu}\right]^2}{g_2(z_{\ell:r:m})}\Bigg]\\
\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \mu\partial\sigma} &= \sum_{\ell=1}^{r}\Bigg[-\frac{2z_{\ell:r:m}}{\sigma^2}+\frac{\frac{\partial^2 g_1(z_{\ell:r:m})}{\partial \sigma\partial\mu}g_1(z_{\ell:r:m})-\frac{\partial g_1(z_{\ell:r:m})}{\partial \sigma}\frac{\partial g_1(z_{\ell:r:m})}{\partial \mu}}{g_1(z_{\ell:r:m})^2}\\
&\quad +R_\ell\frac{\frac{\partial^2 g_2(z_{\ell:r:m})}{\partial \sigma\partial\mu}g_2(z_{\ell:r:m})-\frac{\partial g_2(z_{\ell:r:m})}{\partial \sigma}\frac{\partial g_2(z_{\ell:r:m})}{\partial \mu}}{g_2(z_{\ell:r:m})}\Bigg]\\
\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \sigma^2} &= \frac{1}{\sigma}\sum_{\ell=1}^{r}\Bigg[\frac{1-3z_{\ell:r:m}^2}{\sigma^2}+\frac{\frac{\partial^2 g_1(z_{\ell:r:m})}{\partial \sigma^2}g_1(z_{\ell:r:m})-\left[\frac{\partial g_1(z_{\ell:r:m})}{\partial \sigma}\right]^2}{g_1(z_{\ell:r:m})^2}\\
&\quad +R_\ell\frac{\frac{\partial^2 g_2(z_{\ell:r:m})}{\partial \sigma^2}g_2(z_{\ell:r:m})-\left[\frac{\partial g_2(z_{\ell:r:m})}{\partial \sigma}\right]^2}{g_2(z_{\ell:r:m})}\Bigg],
\end{aligned}$$

where

$$\begin{aligned}
\frac{\partial^2 g_1(z_{\ell:r:m})}{\partial \mu^2} &= \frac{\phi_{sev}(z_{\ell:r:m})[3\Phi_{sev}^2(z_{\ell:r:m})-2\Phi_{sev}(z_{\ell:r:m})](e^{z_{\ell:r:m}}-1)}{\sigma^2}\\
&\quad -\frac{2\phi_{sev}^2(z_{\ell:r:m})[3\Phi_{sev}(z_{\ell:r:m})-1]}{\sigma^2}\\
\frac{\partial^2 g_1(z_{\ell:r:m})}{\partial \mu\partial\sigma} &= \frac{\phi_{sev}(z_{\ell:r:m})[3\Phi_{sev}^2(z_{\ell:r:m})-2\Phi_{sev}(z_{\ell:r:m})](e^{z_{\ell:r:m}}-1)z_{\ell:r:m}}{\sigma^2}\\
&\quad -\frac{2\phi_{sev}^2(z_{\ell:r:m})[3\Phi_{sev}(z_{\ell:r:m})-1]z_{\ell:r:m}}{\sigma^2}\\
&\quad -\frac{\phi_{sev}(z_{\ell:r:m})[3\Phi_{sev}^2(z_{\ell:r:m})-2\Phi_{sev}(z_{\ell:r:m})]}{\sigma^2}\\
\frac{\partial^2 g_1(z_{\ell:r:m})}{\partial \sigma^2} &= \frac{\phi_{sev}(z_{\ell:r:m})[3\Phi_{sev}^2(z_{\ell:r:m})-2\Phi_{sev}(z_{\ell:r:m})](e^{z_{\ell:r:m}}-1)z_{\ell:r:m}^2}{\sigma^2}\\
&\quad -\frac{2\phi_{sev}^2(z_{\ell:r:m})[3\Phi_{sev}(z_{\ell:r:m})-1]z_{\ell:r:m}^2}{\sigma^2}\\
&\quad -\frac{2\phi_{sev}(z_{\ell:r:m})[3\Phi_{sev}^2(z_{\ell:r:m})-2\Phi_{sev}(z_{\ell:r:m})]z_{\ell:r:m}}{\sigma^2}
\end{aligned}$$

and

$$
\begin{aligned}
\frac{\partial^2 g_2(z_{\ell:r:m})}{\partial \mu^2} &= \frac{12\phi_{sev}(z_{\ell:r:m})\left[\Phi^2_{sev}(z_{\ell:r:m}) - \Phi^3_{sev}(z_{\ell:r:m})\right](e^{z_{\ell:r:m}} - 1)}{\sigma^2} \\
&\quad + \frac{12\phi^2_{sev}(z_{\ell:r:m})\left[3\Phi^2_{sev}(z_{\ell:r:m}) - 2\Phi_{sev}(z_{\ell:r:m})\right]}{\sigma^2} \\
\frac{\partial^2 g_2(z_{\ell:r:m})}{\partial \mu \partial \sigma} &= \frac{12\phi_{sev}(z_{\ell:r:m})\left[\Phi^2_{sev}(z_{\ell:r:m}) - \Phi^3_{sev}(z_{\ell:r:m})\right](e^{z_{\ell:r:m}} - 1)z_{\ell:r:m}}{\sigma^2} \\
&\quad + \frac{12\phi^2_{sev}(z_{\ell:r:m})\left[3\Phi^2_{sev}(z_{\ell:r:m}) - 2\Phi_{sev}(z_{\ell:r:m})z_{\ell:r:m}\right]}{\sigma^2} \\
&\quad - \frac{12\phi_{sev}(z_{\ell:r:m})\left[\Phi^2_{sev}(z_{\ell:r:m}) - \Phi^3_{sev}(z_{\ell:r:m})\right]}{\sigma^2} \\
\frac{\partial^2 g_2(z_{\ell:r:m})}{\partial \sigma^2} &= \frac{12\phi_{sev}(z_{\ell:r:m})\left[\Phi^2_{sev}(z_{\ell:r:m}) - \Phi^3_{sev}(z_{\ell:r:m})\right](e^{z_{\ell:r:m}} - 1)z^2_{\ell:r:m}}{\sigma^2} \\
&\quad + \frac{12\phi^2_{sev}(z_{\ell:r:m})\left[3\Phi^2_{sev}(z_{\ell:r:m}) - 2\Phi_{sev}(z_{\ell:r:m})z^2_{\ell:r:m}\right]}{\sigma^2} \\
&\quad - \frac{24\phi_{sev}(z_{\ell:r:m})\left[\Phi^2_{sev}(z_{\ell:r:m}) - \Phi^3_{sev}(z_{\ell:r:m})\right]z_{\ell:r:m}}{\sigma^2}.
\end{aligned}
$$

H.5. Series-Parallel Systems

For $n = 3$, the system structure for series-parallel systems is $\boldsymbol{s} = (1/3, 2/3, 0)$, and now the SF and PDF of the log-transformed system failure times are

$$
\begin{aligned}
\bar{F}_V(v;\boldsymbol{\theta}) &= \left[\bar{F}^*(z)\right]^3 + 2[F^*(z)]\left[\bar{F}^*(z)\right]^2 \\
&= [F^*(z)]^3 - [F^*(z)]^2 - F^*(z) + 1, \\
\text{and } f_V(v;\boldsymbol{\theta}) &= \frac{1}{\sigma} f^*(z)\left\{\left[\bar{F}^*(z)\right]^2 + 2[F^*(z)]\left[\bar{F}^*(z)\right]\right\} \\
&= \frac{1}{\sigma} f^*(z)\left\{1 - [F^*(z)]^2\right\}.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
\ln f_V(v_{\ell:r:m};\boldsymbol{\theta}) &= -\ln\sigma + \ln f^*(z_{\ell:r:m}) + \ln\left\{1 - [F^*(z_{\ell:r:m})]^2\right\} \\
\text{and } \ln \bar{F}_V(v_{\ell:r:m};\boldsymbol{\theta}) &= \ln\left\{[F^*(z)]^3 - [F^*(z)]^2 - F^*(z) + 1\right\}.
\end{aligned}
$$

Hence, we have the first derivatives as

$$\begin{aligned}
\frac{\partial \ln f_V(v_{\ell:r:m};\boldsymbol{\theta})}{\partial\mu} &= \frac{\partial \ln f^*(z_{\ell:r:m})}{\partial\mu} + \frac{\partial \ln\left\{1-[F^*(z_{\ell:r:m})]^2\right\}}{\partial\mu}\\
\frac{\partial \ln f_V(v_{\ell:r:m};\boldsymbol{\theta})}{\partial\sigma} &= -\frac{1}{\sigma} + \frac{\partial \ln f^*(z_{\ell:r:m})}{\partial\sigma} + \frac{\partial \ln\left\{1-[F^*(z_{\ell:r:m})]^2\right\}}{\partial\sigma}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial \ln \bar{F}_V(v_{\ell:r:m};\boldsymbol{\theta})}{\partial\mu} &= \frac{\partial \ln\left\{[F^*(z)]^3-[F^*(z)]^2-F^*(z)+1\right\}}{\partial\mu}\\
\frac{\partial \ln \bar{F}_V(v_{\ell:r:m};\boldsymbol{\theta})}{\partial\sigma} &= \frac{\partial \ln\left\{[F^*(z)]^3-[F^*(z)]^2-F^*(z)+1\right\}}{\partial\sigma}.
\end{aligned}$$

With Weibull distributed components, we have

$$\begin{aligned}
\frac{\partial l(\boldsymbol{\theta})}{\partial\mu} &= \frac{1}{\sigma}\sum_{\ell=1}^{r}\left[(e^{z_{\ell:r:m}}-1) + \frac{2\phi_{sev}(z_{\ell:r:m})\Phi_{sev}(z_{\ell:r:m})}{1-[\Phi_{sev}(z_{\ell:r:m})]^2}\right.\\
&+ \left. R_\ell\frac{\phi_{sev}(z_{\ell:r:m})\left(-3[\Phi_{sev}(z_{\ell:r:m})]^2+2[\Phi_{sev}(z_{\ell:r:m})]+1\right)}{[\Phi_{sev}(z_{\ell:r:m})]^3-[\Phi_{sev}(z_{\ell:r:m})]^2-\Phi_{sev}(z_{\ell:r:m})+1}\right]\\
\frac{\partial l(\boldsymbol{\theta})}{\partial\sigma} &= \frac{1}{\sigma}\sum_{\ell=1}^{r}\left[-1+(e^{z_{\ell:r:m}}-1)z_{\ell:r:m} + \frac{2\phi_{sev}(z_{\ell:r:m})\Phi_{sev}(z_{\ell:r:m})}{1-[\Phi_{sev}(z_{\ell:r:m})]^2}z_{\ell:r:m}\right.\\
&+ \left. R_\ell\frac{\phi_{sev}(z_{\ell:r:m})\left(-3[\Phi_{sev}(z_{\ell:r:m})]^2+2[\Phi_{sev}(z_{\ell:r:m})]+1\right)}{[\Phi_{sev}(z_{\ell:r:m})]^3-[\Phi_{sev}(z_{\ell:r:m})]^2-\Phi_{sev}(z_{\ell:r:m})+1}z_{\ell:r:m}\right].
\end{aligned}$$

To find the maximum likelihood estimates of the parameters, we could set the above equations to 0 and solve them simultaneously.

Similarly, from the following equations

$$\begin{aligned}
g_1(z_{\ell:r:m}) &= 1-[\Phi_{sev}(z_{\ell:r:m})]^2\\
\text{and } g_2(z_{\ell:r:m}) &= [\Phi_{sev}(z_{\ell:r:m})]^3-[\Phi_{sev}(z_{\ell:r:m})]^2-\Phi_{sev}(z_{\ell:r:m})+1,
\end{aligned}$$

we have

$$\frac{\partial l(\boldsymbol{\theta})}{\partial \mu} = \sum_{\ell=1}^{r}\left[\frac{(e^{z_{\ell:r:m}}-1)}{\sigma}+\frac{\frac{\partial g_1(z_{\ell:r:m})}{\partial \mu}}{g_1(z_{\ell:r:m})}+R_\ell\frac{\frac{\partial g_2(z_{\ell:r:m})}{\partial \mu}}{g_2(z_{\ell:r:m})}\right]$$

$$\frac{\partial l(\boldsymbol{\theta})}{\partial \sigma} = \sum_{\ell=1}^{r}\left[-\frac{1}{\sigma}+\frac{(e^{z_{\ell:r:m}}-1)z_{\ell:r:m}}{\sigma}+\frac{\frac{\partial g_1(z_{\ell:r:m})}{\partial \sigma}}{g_1(z_{\ell:r:m})}+R_\ell\frac{\frac{\partial g_2(z_{\ell:r:m})}{\partial \sigma}}{g_2(z_{\ell:r:m})}\right],$$

where

$$\frac{\partial g_1(z_{\ell:r:m})}{\partial \mu} = \frac{2\phi_{sev}(z_{\ell:r:m})\Phi_{sev}(z_{\ell:r:m})}{\sigma}$$

$$\frac{\partial g_1(z_{\ell:r:m})}{\partial \sigma} = \frac{2z_{\ell:r:m}\phi_{sev}(z_{\ell:r:m})\Phi_{sev}(z_{\ell:r:m})}{\sigma}$$

$$\frac{\partial g_2(z_{\ell:r:m})}{\partial \mu} = \frac{\phi_{sev}(z_{\ell:r:m})\left(-3\left[\Phi_{sev}(z_{\ell:r:m})\right]^2+2\left[\Phi_{sev}(z_{\ell:r:m})\right]+1\right)}{\sigma}$$

$$\frac{\partial g_2(z_{\ell:r:m})}{\partial \sigma} = \frac{z_{\ell:r:m}\phi_{sev}(z_{\ell:r:m})\left(-3\left[\Phi_{sev}(z_{\ell:r:m})\right]^2+2\left[\Phi_{sev}(z_{\ell:r:m})\right]+1\right)}{\sigma}$$

and

$$\frac{\partial^2 g_1(z_{\ell:r:m})}{\partial \mu^2} = \frac{2\phi_{sev}(z_{\ell:r:m})\Phi_{sev}(z_{\ell:r:m})(e^{z_{\ell:r:m}}-1)-2\phi_{sev}^2(z_{\ell:r:m})}{\sigma^2}$$

$$\frac{\partial^2 g_1(z_{\ell:r:m})}{\partial \mu \partial \sigma} = \frac{2\phi_{sev}(z_{\ell:r:m})\Phi_{sev}(z_{\ell:r:m})(e^{z_{\ell:r:m}}-1)z_{\ell:r:m}-2\phi_{sev}^2(z_{\ell:r:m})z_{\ell:r:m}}{\sigma^2} - \frac{2\phi_{sev}(z_{\ell:r:m})\Phi_{nor}(z_{\ell:r:m})}{\sigma^2}$$

$$\frac{\partial^2 g_1(z_{\ell:r:m})}{\partial \sigma^2} = \frac{2\phi_{nor}(z_{\ell:r:m})\Phi_{nor}(z_{\ell:r:m})(e^{z_{\ell:r:m}}-1)z_{\ell:r:m}^2-2\phi_{nor}^2(z_{\ell:r:m})z_{\ell:r:m}^2}{\sigma^2} - \frac{2\phi_{nor}(z_{\ell:r:m})\Phi_{nor}(z_{\ell:r:m})z_{\ell:r:m}}{\sigma^2}$$

and

$$
\begin{aligned}
\frac{\partial^2 g_2(z_{\ell:r:m})}{\partial \mu^2} &= \frac{\phi_{nor}(z_{\ell:r:m})\left(-3\left[\Phi_{nor}(z_{\ell:r:m})\right]^2 + 2\left[\Phi_{nor}(z_{\ell:r:m})\right] + 1\right)(e^{z_{\ell:r:m}} - 1)}{\sigma^2} \\
&\quad + \frac{\phi_{nor}^2(z_{\ell:r:m})\left(6\Phi_{nor}(z_{\ell:r:m}) - 2\right)}{\sigma^2} \\
\frac{\partial^2 g_2(z_{\ell:r:m})}{\partial \mu \partial \sigma} &= \frac{\phi_{nor}(z_{\ell:r:m})\left(-3\left[\Phi_{nor}(z_{\ell:r:m})\right]^2 + 2\left[\Phi_{nor}(z_{\ell:r:m})\right] + 1\right)(e^{z_{\ell:r:m}} - 1)z_{\ell:r:m}}{\sigma^2} \\
&\quad + \frac{\phi_{nor}^2(z_{\ell:r:m})\left(6\Phi_{nor}(z_{\ell:r:m}) - 2\right)z_{\ell:r:m}}{\sigma^2} \\
&\quad - \frac{\phi_{nor}(z_{\ell:r:m})\left(-3\left[\Phi_{nor}(z_{\ell:r:m})\right]^2 + 2\left[\Phi_{nor}(z_{\ell:r:m})\right] + 1\right)}{\sigma^2} \\
\frac{\partial^2 g_2(z_{\ell:r:m})}{\partial \sigma^2} &= \frac{\phi_{nor}(z_{\ell:r:m})\left(-3\left[\Phi_{nor}(z_{\ell:r:m})\right]^2 + 2\left[\Phi_{nor}(z_{\ell:r:m})\right] + 1\right)(e^{z_{\ell:r:m}} - 1)z_{\ell:r:m}^2}{\sigma^2} \\
&\quad + \frac{\phi_{nor}^2(z_{\ell:r:m})\left(6\Phi_{nor}(z_{\ell:r:m}) - 2\right)z_{\ell:r:m}^2}{\sigma^2} \\
&\quad - \frac{2\phi_{nor}(z_{\ell:r:m})\left(-3\left[\Phi_{nor}(z_{\ell:r:m})\right]^2 + 2\left[\Phi_{nor}(z_{\ell:r:m})\right] + 1\right)z_{\ell:r:m}}{\sigma^2}.
\end{aligned}
$$

H.6. Parallel-Series Systems

For $n = 3$, the system structure for parallel-series systems is $\boldsymbol{s} = (0, 2/3, 1/3)$. Thus, the SF and PDF of the log-transformed system failure times are, respectively,

$$\begin{aligned}
\bar{F}_V(v;\boldsymbol{\theta}) &= \left[\bar{F}^*(z)\right]^3 + 3[F^*(z)]\left[\bar{F}^*(z)\right]^2 + [F^*(z)]^2\left[\bar{F}^*(z)\right], \\
&= [F^*(z)]^3 - 2[F^*(z)]^2 + 1, \\
\text{and } f_V(v;\boldsymbol{\theta}) &= \frac{1}{\sigma} f^*(z)\left\{2[F^*(z)]\left[\bar{F}^*(z)\right] + \frac{1}{3}[F^*(z)]^2\right\} \\
&= \frac{1}{\sigma} f^*(z)\left\{2[F^*(z)] - \frac{5}{3}[F^*(z)]^2\right\}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\ln f_V(v_{\ell:r:m};\boldsymbol{\theta}) &= -\ln\sigma + \ln f^*(z_{\ell:r:m}) + \ln\left\{2[F^*(z)] - \frac{5}{3}\left[F^*(z)\right]^2\right\} \\
\text{and } \ln\bar{F}_V(v_{\ell:r:m};\boldsymbol{\theta}) &= \ln\left\{\left[F^*(z)\right]^3 - 2\left[F^*(z)\right]^2 + 1\right\}.
\end{aligned}$$

If we use the following

$$\begin{aligned}
g_1(z_{\ell:r:m}) &= 2\Phi_{nor}(z_{\ell:r:m}) - \frac{5}{3}\left[\Phi_{nor}(z_{\ell:r:m})\right]^2 \\
\text{and } g_2(z_{\ell:r:m}) &= \left[\Phi_{nor}(z_{\ell:r:m})\right]^3 - 2\left[\Phi_{nor}(z_{\ell:r:m})\right]^2 + 1,
\end{aligned}$$

we have

$$\begin{aligned}
\frac{\partial l(\boldsymbol{\theta})}{\partial\mu} &= \sum_{\ell=1}^{r}\left[\frac{e^{z_{\ell:r:m}} - 1}{\sigma} + \frac{\frac{\partial g_1(z_{\ell:r:m})}{\partial\mu}}{g_1(z_{\ell:r:m})} + R_\ell\frac{\frac{\partial g_2(z_{\ell:r:m})}{\partial\mu}}{g_2(z_{\ell:r:m})}\right] \\
\frac{\partial l(\boldsymbol{\theta})}{\partial\sigma} &= \sum_{\ell=1}^{r}\left[-\frac{1}{\sigma} + \frac{(e^{z_{\ell:r:m}} - 1)z_{\ell:r:m}}{\sigma} + \frac{\frac{\partial g_1(z_{\ell:r:m})}{\partial\sigma}}{g_1(z_{\ell:r:m})} + R_\ell\frac{\frac{\partial g_2(z_{\ell:r:m})}{\partial\sigma}}{g_2(z_{\ell:r:m})}\right],
\end{aligned}$$

where

$$\frac{\partial g_1(z_{\ell:r:m})}{\partial \mu} = \frac{10\phi_{nor}(z_{\ell:r:m})\Phi_{nor}(z_{\ell:r:m}) - 6\phi_{nor}(z_{\ell:r:m})}{3\sigma}$$

$$\frac{\partial g_1(z_{\ell:r:m})}{\partial \sigma} = \frac{10\phi_{nor}(z_{\ell:r:m})\Phi_{nor}(z_{\ell:r:m}) - 6\phi_{nor}(z_{\ell:r:m})}{3\sigma}(z_{\ell:r:m})$$

and

$$\frac{\partial g_2(z_{\ell:r:m})}{\partial \mu} = \frac{\phi_{nor}(z_{\ell:r:m})\left(-3\left[\Phi_{nor}(z_{\ell:r:m})\right]^2 + 4\left[\Phi_{nor}(z_{\ell:r:m})\right]\right)}{\sigma}$$

$$\frac{\partial g_2(z_{\ell:r:m})}{\partial \sigma} = \frac{\phi_{nor}(z_{\ell:r:m})\left(-3\left[\Phi_{nor}(z_{\ell:r:m})\right]^2 + 4\left[\Phi_{nor}(z_{\ell:r:m})\right]\right)}{\sigma}(z_{\ell:r:m})$$

The second derivatives of g_1 and g_2 can be obtained as

$$\frac{\partial^2 g_1(z_{\ell:r:m})}{\partial \mu^2} = \frac{\phi_{nor}(z_{\ell:r:m})[10\Phi_{nor}(z_{\ell:r:m}) - 6](e^{z_{\ell:r:m}} - 1) - 10\phi^2_{nor}(z_{\ell:r:m})}{3\sigma^2}$$

$$\frac{\partial^2 g_1(z_{\ell:r:m})}{\partial \mu \partial \sigma} = \frac{\phi_{nor}(z_{\ell:r:m})[10\Phi_{nor}(z_{\ell:r:m}) - 6](e^{z_{\ell:r:m}} - 1)z_{\ell:r:m}}{3\sigma^2} - \frac{10\phi^2_{nor}(z_{\ell:r:m})z_{\ell:r:m} - \phi_{nor}(z_{\ell:r:m})[10\Phi_{nor}(z_{\ell:r:m}) - 6]}{3\sigma^2}$$

$$\frac{\partial^2 g_1(z_{\ell:r:m})}{\partial \sigma^2} = \frac{\phi_{nor}(z_{\ell:r:m})[10\Phi_{nor}(z_{\ell:r:m}) - 6](e^{z_{\ell:r:m}} - 1)z^2_{\ell:r:m}}{3\sigma^2} - \frac{10\phi^2_{nor}(z_{\ell:r:m})z^2_{\ell:r:m} - 2\phi_{nor}(z_{\ell:r:m})[10\Phi_{nor}(z_{\ell:r:m}) - 6]z_{\ell:r:m}}{3\sigma^2}$$

and

$$
\begin{aligned}
\frac{\partial^2 g_2(z_{\ell:r:m})}{\partial \mu^2} &= \frac{\phi_{nor}(z_{\ell:r:m})\left(-3\left[\Phi_{nor}(z_{\ell:r:m})\right]^2+4\left[\Phi_{nor}(z_{\ell:r:m})\right]\right)(e^{z_{\ell:r:m}}-1)}{\sigma^2} \\
&\quad+\frac{\phi_{nor}^2(z_{\ell:r:m})\left(6\Phi_{nor}(z_{\ell:r:m})-4\right)}{\sigma^2} \\
\frac{\partial^2 g_2(z_{\ell:r:m})}{\partial \mu \partial \sigma} &= \frac{\phi_{nor}(z_{\ell:r:m})\left(-3\left[\Phi_{nor}(z_{\ell:r:m})\right]^2+4\left[\Phi_{nor}(z_{\ell:r:m})\right]\right)(e^{z_{\ell:r:m}}-1)z_{\ell:r:m}}{\sigma^2} \\
&\quad+\frac{\phi_{nor}^2(z_{\ell:r:m})\left(6\Phi_{nor}(z_{\ell:r:m})-4\right)z_{\ell:r:m}}{\sigma^2} \\
&- \frac{\phi_{nor}(z_{\ell:r:m})\left(-3\left[\Phi_{nor}(z_{\ell:r:m})\right]^2+4\left[\Phi_{nor}(z_{\ell:r:m})\right]\right)}{\sigma^2} \\
\frac{\partial^2 g_2(z_{\ell:r:m})}{\partial \sigma^2} &= \frac{\phi_{nor}(z_{\ell:r:m})\left(-3\left[\Phi_{nor}(z_{\ell:r:m})\right]^2+4\left[\Phi_{nor}(z_{\ell:r:m})\right]\right)(e^{z_{\ell:r:m}}-1)z_{\ell:r:m}^2}{\sigma^2} \\
&\quad+\frac{\phi_{nor}^2(z_{\ell:r:m})\left(6\Phi_{nor}(z_{\ell:r:m})-4\right)z_{\ell:r:m}^2}{\sigma^2} \\
&- \frac{2\phi_{nor}(z_{\ell:r:m})\left(-3\left[\Phi_{nor}(z_{\ell:r:m})\right]^2+4\left[\Phi_{nor}(z_{\ell:r:m})\right]\right)z_{\ell:r:m}}{\sigma^2}.
\end{aligned}
$$

References

Alam, F. M. A. (2017). *Likelihood inference for multiple step-stress models from a generalized Birnbaum-Saunders distribution under time constraint*. PhD thesis, McMaster University. 15

Alhadeed, A. A. and Yang, S.-S. (2005). Optimal simple step-stress plan for cumulative exposure model using log-normal distribution. *IEEE Transactions on Reliability*, 54(1):64–68. 16

Alkhalfan, L. A. (2012). *Inference for a gamma step-stress model under censoring*. PhD thesis, McMaster University. 15

Aslett, L. J. M., Coolen, F. P. A., and Wilson, S. P. (2015). Bayesian inference for reliability of systems and networks using the survival signature. *Risk Analysis*, 35(9):1640–1651. 12

Bagdonavicius, V. and Nikulin, M. (2002). *Accelerated Life Models: Modeling and Statistical Analysis*. Chapman and Hall/CRC Press, Boca Raton, FL. 17

Bai, D. S., Kim, M. S., and Lee, S. H. (1989). Optimum simple step-stress accelerated life tests with censoring. *IEEE Transactions on Reliability*, 38(5):528–532. 16

Balakrishnan, N. (2007). Progressive censoring methodology: an appraisal. *TEST*, 16:211–259. 6

Balakrishnan, N. (2008). A synthesis of exact inferential results for exponential step-stress models and associated optimal accelerated life-tests. *Metrika*, 69(2):351. 15

Balakrishnan, N. and Aggarwala, R. (2000). *Progressive Censoring: Theory, Methods, and Applications*. Springer, New York. 6, 68, 107

Balakrishnan, N. and Cramer, E. (2014). *The Art of Progressive censoring*. Birkhäuser, New York, NY. 6

Balakrishnan, N. and Han, D. (2007). Optimal progressive Type-II censoring schemes for nonparametric confidence intervals of quantiles. *Communications in Statistics - Simulation and Computation*, 46:1247 – 1262. 69

Balakrishnan, N. and Han, D. (2008). Exact inference for a simple step-stress model with competing risks for failure from exponential distribution under Type-II censoring. *Journal of Statistical Planning and Inference*, 138(12):4172 – 4186. 15

Balakrishnan, N. and Han, D. (2009). Optimal step-stress testing for progressively Type-I censored data from exponential distribution. *Journal of Statistical Planning and Inference*, 139(5):1782–1798. 16

Balakrishnan, N., Kundu, D., Ng, H. K. T., and Kannan, N. (2007). Point and interval estimation for a simple step-stress model with Type-II censoring. *Journal of Quality Technology*, 39(1):35–47. 15

Balakrishnan, N., Ng, H. K. T., and Navarro, J. (2011a). Exact nonparametric inference for component lifetime distribution based on lifetime data from systems with known signatures. *Journal of Nonparametric Statistics*, 23(3):741–752. 12

Balakrishnan, N., Ng, H. K. T., and Navarro, J. (2011b). Linear inference for Type-II censored lifetime data of reliability systems with known signatures. *IEEE Transactions on Reliability*, 60(2):426–440. 11

Balakrishnan, N. and Xie, Q. (2007a). Exact inference for a simple step-stress model with Type-I hybrid censored data from the exponential distribution. *Journal of Statistical Planning and Inference*, 137(11):3268–3290. 15

Balakrishnan, N. and Xie, Q. (2007b). Exact inference for a simple step-stress model with Type-II hybrid censored data from the exponential distribution. *Journal of Statistical Planning and Inference*, 137(8):2543–2563. 15

Bhattacharya, R. and Balakrishnan, N. (2023). A MCMC-type simple probabilistic approach for determining optimal progressive censoring schemes. *Communications in Statistics - Simulation and Computation*, to appear. 69

Bobotas, P. and Kateri, M. (2019). Optimal designs for step-stress models under interval censoring. *Journal of Statistical Theory and Practice*, 13(4):54. 16

Burkschat, M., Cramer, E., and Kamps, U. (2006). On optimal schemes in progressive censoring. *Statistics & Probability Letters*, 76:1032 – 1036. 69

Burkschat, M., Cramer, E., and Kamps, U. (2007). Optimality criteria and optimal schemes in progressive censoring. *Communications in Statistics - Theory and Methods*, 36:1419 – 1431. 69

Chan, P. S., Balakrishnan, N., So, H. Y., and Ng, H. K. T. (2016). Optimal sample size allocation for multi-level stress testing with exponential regression under Type-I censoring. *Communications in Statistics - Theory and Methods*, 45(6):1831–1852. 14

Chan, P. S., Ng, H. K. T., Balakrishnan, N., and Zhou, Q. (2008). Point and interval estimation for extreme-value regression model under Type-II censoring. *Computational Statistics & Data Analysis*, 52(8):4040–4058. 20

Chan, P. S., So, H. Y., Ng, H. K. T., and Gao, W. (2020). Optimal allocation for extreme value regression under time censoring. In Bekker, A., Chen, D. G., and Ferreira, J., editors, *Computational and Methodological Statistics and Biostatistics – Essays in Contemporary Advancement*, chapter 12, pages 269–307. Springer Nature, Switzerland. 14

Coolen, F. P. A. and Al-nefaiee, A. H. (2012). Nonparametric predictive inference for failure times of systems with exchangeable components. *Proceedings of the Institution of Mechanical Engineers, Part O: Journal of Risk and Reliability*, 226(3):262–273. 12

Dahmen, K., Burkschat, M., and Cramer, E. (2012). A- and D-optimal progressive Type-II censoring designs based on fisher information. *Journal of Statistical Computation and Simulation*, 82:879 – 905. 69

El-Raheem, A. M. A. (2019). Inference and optimal design of multiple constant-stress testing for generalized half-normal distribution under Type-II progressive censoring. *Journal of Statistical Computation and Simulation*, 89(16):3075–3104. 14

Elperin, T. and Gertsbakh, I. (1987). Maximum likelihood estimation in a Weibull regression model with Type-I censoring: A Monte Carlo study. *Communications in Statistics - Simulation and Computation*, 16:349–371. 20

Fallah, A., Asgharzadeh, A., and Ng, H. K. T. (2020). Statistical inference for component lifetime distribution from coherent system lifetimes under a proportional reversed hazard model. *Communications in Statistics - Theory and Methods*, 0(0):1–25. 12

Fan, T. and Wang, W. (2011). Accelerated life tests for Weibull series systems with masked data. *IEEE Transactions on Reliability*, 60(3):557–569. 11

Gouno, E., Sen, A., and Balakrishnan, N. (2004). Optimal step-stress test under progressive type-I censoring. *IEEE Transactions on Reliability*, 53(3):388–393. 16

Guan, Q. and Tang, Y. (2012). Optimal step-stress test under Type-I censoring for multivariate exponential distribution. *Journal of Statistical Planning and Inference*, 142(7):1908–1923. 16

Hakamipour, N. (2018). Optimal plan for step-stress accelerated life test with two stress variables for lognormal data. *Iranian Journal of Science and Technology, Transactions A: Science*, 42(4):2259–2271. 16

Hall, P., Jin, Y., and Samaniego, F. J. (2015). Nonparametric estimation of component reliability based on lifetime data from systems of varying design. *Statistica Sinica*, 25(4):1313–1335. 12

Han, D. and Ng, H. K. T. (2013). Comparison between constant-stress and step-stress accelerated life tests under time constraint. *Naval Research Logistics (NRL)*, 60(7):541–556. 17

Han, D. and Ng, H. K. T. (2014). Asymptotic comparison between constant-stress testing and step-stress testing for Type-I censored data from exponential distribution. *Communications in Statistics - Theory and Methods*, 43(10-12):2384–2394. 17

Hermanns, M. and Cramer, E. (2017). Likelihood inference for the component lifetime distribution based on progressively censored parallel systems data. *Journal of Statistical Computation and Simulation*, 87(3):607–630. 11, 68

Hermanns, M. and Cramer, E. (2018). Inference with progressively censored k-out-of-n system lifetime data. *TEST*, 27:787–810. 11, 68

Hermanns, M., Cramer, E., and Ng, H. K. T. (2020). EM algorithms for ordered and censored system lifetime data under a proportional hazard rate model. *Journal of Statistical Computation and Simulation*, 0(0):1–37. 12

Huang, S.-R. and Wu, S.-J. (2017). Optimal sample size allocation for accelerated life test under progressive Type-II censoring with competing risks. *Journal of Statistical Computation and Simulation*, 87(1):1–16. 14

Jin, Y., Hall, P. G., Jiang, J., and Samaniego, F. J. (2017). Estimating component reliability based on failure time data from a system of unknown design. *Statistica Sinica*, 27(2):479–499. 12

Ka, C. Y., Chan, P. S., Ng, H. K. T., and Balakrishnan, N. (2011). Optimal sample size allocation for multi-level stress testing with Weibull regression under Type-II censoring. *Statistics*, 45(3):257–279. 14

Khamis, I. and Higgins, J. (1998). A new model for step-stress testing. *IEEE Transactions on Reliability*, 47(2):131–134. 16

Kłos, R., Fabisiak, B., and Ng, H. K. T. (2018). Comparative reliability analysis of selected joints for case furniture. *BioResources*, 13(3):5111–5123. 2, 31, 58

Li, C. and Fard, N. (2007). Optimum bivariate step-stress accelerated life test for censored data. *IEEE Transactions on Reliability*, 56(1):77–84. 17

Lin, C.-T., Chou, C.-C., and Balakrishnan, N. (2013). Planning step-stress test plans under Type-I censoring for the log-location-scale case. *Journal of Statistical Computation and Simulation*, 83(10):1852–1867. 16

Lin, C.-T., Chou, C.-C., and Balakrishnan, N. (2020). Planning step-stress test plans under Type-I hybrid censoring for the log-location-scale distribution. *Statistical Methods & Applications*, 29(2):265–288. 16

Miller, R. and Nelson, W. (1983). Optimum simple step-stress plans for accelerated life testing. *IEEE Transactions on Reliability*, R-32(1):59–65. 16

Monroe, E. M. and Pan, R. (2008). Experimental design considerations for accelerated life tests with nonlinear constraints and censoring. *Journal of Quality Technology*, 40(4):355–367. 14

Navarro, J., Ruiz, J., and Sandoval, C. (2007). Properties of coherent systems with dependent components. *Communications in Statistics - Theory and Methods*, 36(1):175–191. 5

Nelson, W. (1980). Accelerated life testing – step-stress models and data analyses. *IEEE Transactions on Reliability*, R-29(2):103–108. 15

Nelson, W. B. (1990). *Accelerated Testing: Statistical Models, Test Plans, and Data Analysis*. John Wiley & Sons, Inc., New York, NY. 17

Nelson, W. B. (2005). A bibliography of accelerated test plans. *IEEE Transactions on Reliability*, 54(2):194–197. 17

Nelson, W. B. (2015). An updated bibliography of accelerated test plans. In *2015 Annual Reliability and Maintainability Symposium (RAMS)*, pages 1–6. 17

Ng, H. K. T. (2010). Censoring methodology. In Lovric, M., editor, *International Encyclopedia of Statistical Science*, page 221–224. Springer. 5

Ng, H. K. T., Balakrishnan, N., and Chan, P. S. (2007). Optimal sample size allocation for tests with multiple levels of stress with extreme value regression. *Naval Research Logistics (NRL)*, 54(3):237–249. 14

Ng, H. K. T., Chan, P. S., and Balakrishnan, N. (2004). Optimal progressive censoring plans for the weibull distribution. *Technometrics*, 46:470 – 481. 69, 103, 107

Ng, H. K. T., Kınacı, İ., Kuş, C., and Chan, P. S. (2017). Optimal experimental plan for multi-level stress testing with Weibull regression under progressive Type-II extremal censoring. *Communications in Statistics - Simulation and Computation*, 46(4):2611–2637. 14, 78

Pan, R., Yang, T., and Seo, K. (2015). Planning constant-stress accelerated life tests for acceleration model selection. *IEEE Transactions on Reliability*, 64(4):1356–1366. 14

Paul, S. R. and Thiagarajah, K. (1996). Approximate variance covariance of maximum likelihood 341 estimators for the parameters of extreme value regression models for censored data. *Sankhya, Series B*, 58:28–37. 20

Proschan, F. and Barlow, R. E. (1965). *Mathematical Theory of Reliability*. John Wiley & Sons, New York, NY. 4

Rausand, M. and Hoyland, A. (2004). *System Reliability Theory – Models, Statistical Methods, and Applications*. Wiley, Hoboken, New Jersey. 4

Salemi, U. H., Rezaei, S., and Nadarajah, S. (2019). A-optimal and D-optimal censoring plans in progressively Type-II right censored order statistics. *Journal of Statistical Computation and Simulation*, 60:1349 – 1367. 69

Samaniego, F. J. (1985). On closure of the IFR class under formation of coherent systems. *IEEE Transactions on Reliability*, R-34(1):69–72. 4, 5

Samaniego, F. J. (2007). *System Signatures and their Applications in Engineering Reliability*. Springer, New York. 4

Sediakin, N. M. (1966). About the one physical principle of theory of reliability. *Proceedings of Academy of Sciences of USSR, Technical Cybernetics*, 3:80–87. 15

Tavangar, M. and Asadi, M. (2020). Component reliability estimation based on system failure-time data. *Journal of Statistical Computation and Simulation*, 90(17):3232–3249. 12

Viveros, R. and Balakrishnan, N. (1994). Interval estimation of parameters of life from progressively censored data. *Technometrics*, 36(1):84–91. 103

Wu, S.-J. and Kuş, C. (2009). On estimation based on progressive first-failure-censored sampling. *Computational Statistics & Data Analysis*, 53(10):3659 – 3670. 11, 31, 68

Yang, Y., Ng, H. K. T., and Balakrishnan, N. (2016). A stochastic expectation-maximization algorithm for the analysis of system lifetime data with known signature. *Computational Statistics*, 31(2):609–641. 12

Yang, Y., Ng, H. K. T., and Balakrishnan, N. (2019). Expectation–maximization algorithm for system-based lifetime data with unknown system structure. *AStA Advances in Statistical Analysis*, 103(1):69–98. 12

Yue, H. B. and Shi, Y. M. (2013). Optimal sample size allocation for multi-level stress testing under progressive hybrid interval censoring. https://www.scientific.net/AMM.423-426.2423. 14

Yurkowsky, W., Schafer, R. E., and Finkelstein, J. M. (1967). Accelerated testing technology. *Rome Air Development Center Technical Report 61GL223. Report No. RADC-TR-67-420*, page 89. 1

Zhang, J., Ng, H. K. T., and Balakrishnan, N. (2015). Statistical inference of component lifetimes with location-scale distributions from censored system failure data with known signature. *IEEE Transactions on Reliability*, 64(2):613–626. 12