## Thesis submitted for the degree of Doctor of Philosophy

# Permutation Invariant Gaussian Matrix Models 

## GEORGE BARNES

Supervisor
Dr Sanjaye Ramgoolam

August 31, 2023
Centre for Theoretical Physics
School of Physical and Chemical Sciences
Queen Mary University of London

## Declaration

I, George Barnes, confirm that the research included within this thesis is my own work or that where it has been carried out in collaboration with, or supported by others, that this is duly acknowledged below and my contribution indicated. Previously published material is also acknowledged below.

I attest that I have exercised reasonable care to ensure that the work is original, and does not to the best of my knowledge break any UK law, infringe any third party's copyright or other Intellectual Property Right, or contain any confidential material.

I accept that the College has the right to use plagiarism detection software to check the electronic version of the thesis.

I confirm that this thesis has not been previously submitted for the award of a degree by this or any other university.

The copyright of this thesis rests with the author and no quotation from it or information derived from it may be published without the prior written consent of the author.

## Signature: George Barnes

Date: 21/07/2023

Details of collaboration and publications:

This thesis is based on the publications [1, 2, 3, 4, The work in [1, 2, 3, 4, was carried out with my supervisor Sanjaye Ramgoolam, along with Adrian Padellaro in [1, 2, 3], and Michael Stephanou in [4. Where other sources have been used, they are cited in the bibliography.

## Abstract

Matrix models are ubiquitous in physics. Commonly arising due to the presence of gauge symmetries in a system, they play an important role in establishing results within the context of the AdS/CFT correspondence. They also capture the statistics of complex systems in a remarkably diverse range of fields within the framework of random matrix theory (RMT), these include nuclear physics, chaos, condensed matter physics, financial correlations and biological networks. Inspired by both of these applications, and motivated by the existence of physical matrix systems possessing discrete symmetries, is the study of Gaussian matrix models invariant under $S_{N}$, the symmetric group of all permutations of $N$ objects. We specialise the most general of these models to the case of symmetric matrices with vanishing diagonal elements. This model is used to study an ensemble of financial correlation matrices and as a tool to detect market states. This problem has a natural permutation symmetry, and the observables of interest are permutation invariant polynomials of the matrix variables (PIMOs). We find that the values of low order PIMOs are generally closely matched by the predictions by the model, and vectors of PIMOs are shown to be efficient indicators of the market state. Turning our attention to the general structure of permutation invariant Gaussian matrix (PIGM) models of general matrices of size $N$, we show that PIMOs of degree $k$ are in one-to-one correspondence with equivalence classes of the diagrammatic partition algebra $P_{k}(N)$. On a subspace of the 13-parameter space of general PIGM models there is an enhanced $O(N)$ symmetry. At a special point within this subspace exists the simplest $O(N)$ invariant action, which we furnish with an inner product on the PIMOs. We prove the large $N$ factorisation of this inner product. Lastly, we study the implications of permutation symmetry for the state space and dynamics of quantum mechanical matrix systems. The general permutation invariant matrix quantum harmonic oscillator Hamiltonian is solved and families of interacting Hamiltonians, which are diagonalised by a representation theoretic basis for the permutation invariant subspace, are described. These include Hamiltonians for which low-energy states are $S_{N}$ invariant and can give rise to large ground state degeneracies related to the dimensions of partition algebras.

## Acknowledgements

I would like to thank my supervisor Sanjaye Ramgoolam for all of his support, guidance and patience throughout my time at Queen Mary. Sanjaye always manages to make enough time to see me, no matter how busy he is, and the enthusiasm he brings to each meeting is infectious. It makes working on challenging problems a lot of fun, and for that among many other things, I'm truly grateful.

I would like to thank Adrian for being a great friend and the best teammate I can imagine. The depth of his insight still manages to surprise me and I'm lucky to have had the privilege of working with him so much throughout my PhD .

I would like to thank Mike for many interesting discussions, illuminating explanations, and for somehow managing to find the time to finish a paper while caring for his newborn daughter.

The theory group at QMUL has been a wonderful environment within which to spend the past three-and-a-half years. I have, at this point, shared an office with many people, and am grateful to each one of them for helping make it a place I look forward to arriving each morning. Thanks to Sam, Bernie, David, Manuel, Stefano, Marcel, Shun-Qing, Linfeng, Rajath, Gergely, Enrico, Nadia, Graham, Josh, Lewis, Kymani, Chinmaya, Tancredi, Alex, Gus, George, and Mahesh.

Finally, thank you to my family, friends, and to Florina. Your love and support has made this possible.

This work was supported by the Science and Technology Facilities Council (STFC) Studentship ST/S505663/1.

## Contents

1 Introduction ..... 10
2 Background ..... 14
2.1 Representation theory of finite groups ..... 14
2.2 Representation theory of the symmetric group ..... 16
2.2.1 Stability of products of symmetric group irreducible representations ..... 18
2.2.2 The natural representation ..... 19
2.3 Partition algebras ..... 20
2.4 5-parameter PIGM models ..... 23
2.5 Counting of observables ..... 24
2.6 13-parameter PIGM models ..... 26
2.6.1 Finding $Q_{i j k l}^{[N-2,2]}$ and $Q_{i j k l}^{[N-1,1,1]}$ ..... 32
2.6.2 Expectation values ..... 34
3 PIGM models for financial correlations ..... 37
3.1 Summary of results on the 4-parameter Gaussian matrix model ..... 39
3.2 4-parameter Gaussian model: detailed construction ..... 44
3.2.1 Symmetric group representation theory and matrix variables ..... 45
3.2.2 Projectors for $V^{\text {phys }}$ ..... 46
3.2.3 The action and physical projectors ..... 50
3.2.4 Observables and correlators ..... 52
3.2.5 Expectation values ..... 54
3.2.6 Embedding within 13-parameter PIGM model ..... 56
3.3 Daily correlation matrices from high-frequency forex data ..... 57
3.3.1 Correlation matrix methodology ..... 59
3.4 Matrix theory and matrix data: near-Gaussianity ..... 62
3.4.1 Theory/experiment deviations normalised by standard deviations of the observables ..... 62
3.4.2 Day capture and balanced accuracy of theoretical typicality predic-tion for days65
3.4.3 Absolute errors relative to standard deviations and standard errors ..... $\square$
of observables ..... 67
3.5 Applications of matrix theory to matrix data: anomaly detection ..... 69
3.5.1 Anomaly detection algorithm ..... 70
3.5.2 Economically significant dates ..... 71
3.5.3 Dimensionality reduction ..... 71
3.5.4 Longest observable vectors and economically significant dates ..... 73
3.6 Discussion ..... 76
4 Hidden symmetries and large N factorisation ..... 78
4.1 Hidden symmetries in permutation invariant Gaussian matrix models ..... 79
4.1.1 Counting matrix observables using partition algebras ..... 79
4.1.2 Enhanced $O(N)$ symmetry in parameter space ..... 84
4.2 Permutation invariant matrix observables (PIMOs) ..... 88
4.2.1 Construction of PIMOs ..... 88
4.2.2 Inner product on PIMOs ..... 90
4.3 Large N factorisation ..... 92
4.3.1 $\quad$ Factorisation for trace form on $P_{k}(N)$ ..... 94
4.3.2 Factorisation for PIMOs ..... 96
4.3.3 Factorisation for multi-matrix observables ..... 98
4.4 Discussion ..... 100
5 Permutation symmetry in large $N$ matrix quantum mechanics ..... 101
5.1 Review: matrix harmonic oscillator ..... 105
5.1.1 Diagram notation ..... 107
5.2 Permutation invariant sectors for quantum matrix systems ..... 109
5.2.1 Partition algebras and invariant tensors ..... 111
5.2.2 Diagram basis ..... 112
5.2.3 Representation basis ..... 114
5.3 Representation basis and algebraic charges ..... 118
5.3.1 Central elements in the partition algebra ..... 119
5.3.2 Multiplicity labels and maximal commuting subalgebras ..... 123
5.3.3 Construction of low degree representation bases ..... 125
5.4 Exactly solvable permutation invariant matrix harmonic oscillator ..... 127
5.4.1 Construction ..... 128
5.4.2 Spectrum ..... 129
5.4.3 Canonical partition function ..... 132
5.4.4 Energy eigenbases ..... 133
5.5 Algebraic Hamiltonians and permutation invariant ground states ..... 134
5.5.1 Partition algebra elements as quantum mechanical operators ..... 135
5.5.2 Decoupling invariant sectors and invariant ground states ..... 139
5.5.3 Resolving the invariant spectrum ..... 143
5.5.4 Precision resolution of the invariant spectrum ..... 144
5.5.5 General invariant Hamiltonians from partition algebras ..... 145
5.5.6 Bosons on a lattice ..... 147
5.6 AdS/CFT inspired extremal correlators in matrix quantum mechanics ..... 149
5.6.1 Two-point correlators ..... 149
5.6.2 Three-point correlators ..... 150
5.7 Discussion ..... 155
6 Discussion and conclusion ..... 157
A Inner product calculations ..... 160
B Matrix units and Fourier inversion from inner product ..... 163
B. 1 Schur-Weyl duality and non-degenerate bilinear forms ..... 163
B. 2 Orthogonality of matrix elements ..... 165
B. 3 Matrix units for $P_{k}(N)$ ..... 165
B. 4 Matrix units for $S P_{k}(N)$ and normalisation constants ..... 166
C Orbit basis ..... 168
D Computing low degree matrix units ..... 175

## List of Figures

3.1 Examples of realised daily correlation estimates over time ..... 62
3.2 Histograms of the standardised values of each of the observables (one value
63
per correlation matrix i.e. per day). ..... 6
3.3 Pearson product-moment correlation of the observables. ..... 64
3.4 Distances of observable vectors and raw correlation vectors from the originusing the standardised Euclidean and Mahalanobis metrics.72
4.1 By identifying the bottom vertices of $d_{2}^{T}$ with the top vertices of $d_{1}$, and the top vertices of $d_{2}^{T}$ with the bottom vertices of $d_{1}$, we have constructed a diagram with all the edges of $d_{1}$ together with all the edges of $d_{2}$.94
5.1 The figure illustrates the type of spectra that can be engineered using the algebraic Hamiltonians discussed in this section. Blue (light) lines correspond to states that are invariant under the adjoint action of $S_{N}$. Black (dark) lines are non-invariant states.135
5.2 Matrix oscillators are naturally associated with a $N$-by- $N$ square lattice. The creation operator $\left(a^{\dagger}\right)_{i}^{j}$ creates a quanta of excitation at row $i$ column $j$ in the lattice.147

## List of Tables

| 3.1 | Currency pair mapping. | 58 |
| :---: | :---: | :---: |
| 3.2 | Descriptive statistics of number of quote updates per 5 minute time interval. |  |
| 3.3 | Summary statistics of regular 5-minute (log) mid-price returns. | 61 |
| 3.4 | For each observable in the first two columns the third column lists the ab- |  |
|  | solute difference between the experimental value and theoretical prediction |  |
|  | normalised by the experimental standard deviation. The fourth column lists |  |
|  | the ratio of the experimental and theoretical standard deviations. The fifth |  |
|  | and sixth columns list the experimental day capture the theoretical day cap- |  |
|  | ture respectively. The seventh column gives the Balanced Accuracy of the |  |
|  | theoretical model's day capture at $\pm 2 \sigma$. The * values were obtained using |  |
|  | an estimate of $\sigma_{\mathrm{T}}$ described at the end of section [3.4.1]] | 6 |
| 3.5 | Definitions of TP, TN, FP and FN in generic binary classification. In our |  |
|  | case positive is a day with an observable value falling within two standard |  |
|  | deviations of the mean and negative is a day with an observable value falling |  |
|  | outside this range. |  |
| 3.6 | The 12 cubic and quartic observables that have the largest normalised dif- |  |
|  | ference from the PIGM predictions. | 73 |
| 3.7 | In-sample anomaly detection results. In the table above, the proportions, |  |
|  | $P_{B}, P_{T}$ and the odds-ratio, OR, are as defined in equation (3.96). The p- |  |
|  | value is obtained using Fisher's exact one-sided test. The * symbol following |  |
|  | a p-value indicates significance at the 0.05 level, ${ }^{* *}$ indicates significance at |  |
|  | the 0.01 level and ${ }^{* * *}$ indicates significance at the 0.001 level. |  |
| 3.8 | Out-of-sample anomaly detection results. In the table above, the propor- |  |
|  | tions, $P_{B}, P_{T}$ and the odds-ratio, OR, are as defined in equation (3.96). The |  |
|  | p -value is obtained using Fisher's exact one-sided test. The * symbol follow- |  |
|  | ing a p-value indicates significance at the 0.05 level, ${ }^{* *}$ indicates significance |  |
|  | at the 0.01 level and ${ }^{* * *}$ indicates significance at the 0.001 level. | 76 |

## Chapter 1

## Introduction

This thesis is about permutation invariant Gaussian matrix (PIGM) models. These models constitute a class of universal statistical mechanics models, and are of potential relevance to any system possessing the appropriate symmetry and Gaussianity properties. The inception of, and continued development of such models is motivated, on the one hand by theoretical physics and the study of holography, and on the other from data science and its application to diverse data sets.

The motivation coming from physics stems from the abundance of systems involving matrix degrees of freedom. In high energy physics these commonly involve matrices transforming in the adjoint or bifundamental of a group, such as $U(N), S U(N), S O(N)$, or $S p(N)$. This group is commonly a gauge symmetry of the system and physical states are defined to be those invariant under the action of $G$. The space of gauge invariants is populated by traces of matrices, and is organised by algebras dual to $G$. For the case of $G=U(N)$ the dual algebras are based on the standard Schur-Weyl duality [5] between $U(N)$ and $S_{k}$ on $V^{\otimes k}$ the $k$-fold tensor product of the fundamental representation of $U(N)$. Applications of SchurWeyl duality to the computation of correlators in matrix models with $U(N)$ symmetry are developed in [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19] and short reviews are [20, 21]. These multi-matrix applications involve dual algebras beyond the symmetric group algebras. For example Brauer algebras, which have a basis of diagrams, are used in [7]. When $U(N)$ is replaced by the permutation group, $S_{N}$, as the invariance of interest, the Schur-Weyl dual algebras are diagrammatic partition algebras $P_{k}(N)$. Partition algebras were first introduced in [22, 23, 24] in application to the statistical mechanics of Potts models. Their relation to the representation theory of symmetric groups is an active area of mathematical research [25, 26, 27, 28].

An important and persistent feature of matrix systems governed by continuous symme-
tries, first pointed out by 't Hooft in 1974 [29], is their simplification in the large $N$ limit. This simplification has played a major role in the development of gauge-string duality in subsequent years including in low-dimensional non-critical string theories dual to zero-dimensional QFTs (matrix models) [30, 31, 32, the string dual of two-dimensional Yang-Mills theories [33], and the generalisation to higher dimensions in the AdS/CFT correspondence [34]. A cornerstone of this simplicity is "large $N$ factorisation". In the context of AdS/CFT, large $N$ factorisation for two-point functions involving gauge invariants built from a complex matrix is an expression of orthogonality for distinct trace structures [35]. This plays an important role in the connection between multi-traces in the CFT constructed from a small number of matrices and perturbative gravitons in the AdS dual [36, 37]. The breakdown of this orthogonality when the number of matrices $k$ becomes comparable to $N$ guides the identification of the CFT duals [35, 6, 38] of giant gravitons [39, 40, 41], the dual $S_{k}$ algebra can be used to neatly classify these operators in terms of features of Young diagrams.

In addition to this holographic motivation, PIGM models were originally developed for their direct application to data structures arising in computational linguistics [42, 43, and more specifically in the field of distributional semantics. The study of distributional semantics was originated by Firth [44] and Harris [45] in the 1950s and '60s and was founded on the precept that a word's meaning can be gleaned from the frequency of its occurrence among specific neighbouring words. In practice, words are represented by meaning vectors recording the frequency of cooccurrence (in the same sentence say) with a set of commonly occurring words that act as a basis for the vector space. As Firth would have it "You shall know a word by the company that it keeps". More recent developments add grammatical structure, represented in these linguistic vector spaces as higher order objects: matrices and tensors. Composing words into phrases or sentences is equivalent to the contraction of indices, the precise form of which is dictated by the grammatical role played by the word in the sentence [46, 47]. Large ensembles of matrices arise naturally in this context as they represent the grammatical role played by any lexical category that modifies a single noun to return a noun phrase, for example adjectives and intransitive verbs. These matrices are not invariant under any continuous symmetries, however, the order of the basis vectors in these matrices is unimportant which gives the permutation invariance.

Since their initial introduction by Wigner 48] and Dyson [49] Gaussian matrix models have demonstrated a remarkable degree of universality, capturing the statistic of a wide variety of complex systems. At first used to explain the statistics of the energy levels of complex nuclei, they have since been applied to chaos, condensed matter physics, biological networks, feature-matrices in bio-statistics, data science, financial correlations and quantum gravity [50, 51, 52, 53, 54]. In the same vein PIGM models were proposed to describe the statistics of matrix ensembles arising in computational linguistics based on the general
symmetries and properties of the matrices rather than their complex, precise relations. The approaches diverge however, since applications of random matrix theory (RMT) typically focus on eigenvalue distributions of the random matrices, whereas PIGM models concentrate on low order expectation values of observables. In this the PIGM approach mirrors that of quantum field theory (QFT) as applied to particle physics: viewing the matrix integrals as zero-dimensional QFTs and employing much of the same technology. Like RMT, PIGM models capture a universal structure present in many systems and we expect they will prove broadly applicable. Indeed, since the definition and solution of the most general 13-parameter PIGM model in [43, they have already been successfully utilised in a variety of computational linguistic applications: the Gaussianity analysed in 55 was based on the construction of matrices by linear regression in [42] while [56] extended the analysis of [55] and also analysed the matrices constructed by neural network methods in [57. Permutation invariant observables are more general functions of matrices than those invariant under continuous symmetries, as such some traditional areas of application of RMT may be enriched by the inclusion of the study of permutation invariant observables.

The structure of this thesis is as follows: in chapter 2 we briefly cover some basic finite group representation theory before considering representation theory of the symmetric group $S_{N}$ in particular. Key results include the correspondence between conjugacy classes and irreducible representations of $S_{N}$, which enables a labelling of the irreducible representations by partitions of $N$ or equivalently, Young diagrams containing $N$ boxes. Many combinatoric results used throughout are most easily expressed in terms of Young diagrams: the Hook length formula for determining the dimension of a irreducible representation (2.16), (2.23) giving the irreducible decomposition of an $S_{N}$ irreducible representation tensored with the Hook representation, which in turn can be applied to find the decomposition of tensors of the natural representation. Detailed reference material is signposted throughout. This representation theory underpins the results of [43] which is the root of the material in the succeeding chapters. Due to its central importance, we recap the definition and solution of the 13 -parameter Gaussian matrix model, originally in [43], albeit with a slightly different presentation, more suitable for our purposes.

Akin to random matrix models, PIGM models are universal structures applicable to any ensemble possessing the requisite symmetry and Gaussianity. Based on [4] chapter 3 moves beyond the initial applications of these models in computation linguistics and applies them to an ensemble of financial correlation data sourced from high frequency foreign exchange (forex) market trades. In contrast to the general $N \times N$ matrices considered in [43] the matrices here are symmetric and have vanishing diagonal elements. We find these models are defined by four coupling parameters and give their solution in terms of the inverse couplings and the size of the matrices. Using the data to fit the model to linear and quadratic order we find a good agreement between the empirical cubic and quartic observable ex-
pectation values and those predicted by the model. Observables that depart from these predictions indicate informative structure beyond that captured by the random model. Indeed, feature vectors constructed from these least Gaussian observables are shown to be useful low dimensional representations of the high dimensional correlation matrices. This usefulness is demonstrated in their application to market state anomaly detection.

In chapter 4 we extend a familiar large $N$ factorisation property of inner products of matrix observables invariant under continuous symmetry to the case of matrix observables invariant under the discrete symmetry of permutations, based on the original work [2]. In order to prove this result we first establish a correspondence between permutation invariant matrix observables and equivalence classes of partition algebra elements. The correspondence is reminiscent of that between $U(N)$ invariants and conjugacy classes of the symmetric group. Indeed, both correspondences are consequences of the Schur-Weyl duality between the two pairs of groups.

Polynomials in matrix variables $M_{i j}$ are closely related to quantum mechanical states constructed from matrix oscillators $\left(a^{\dagger}\right)_{j}^{i}$. This allows us to translate the technology developed for zero-dimensional matrix models in the early chapters to the setting of matrix quantum mechanics. Chapter 5 pursues this theme, based on the work in [3]. We begin by giving a detailed description of the permutation invariant state space. The state space can be organised by the order of the polynomial of matrix oscillators used to create the state. Then at each order $k$, the partition algebra $P_{k}(N)$ proves an important tool in this description and gives an efficient construction of permutation invariant states. The majority of our results up to this point are independent of any particular Hamiltonian. Using Fourier transformation on $P_{k}(N)$ we construct a representation theory basis which forms an energy eigenbasis for the Hamiltonian of the free matrix quantum harmonic oscillator. We introduce an 11-parameter family of exactly solvable Hamiltonians which define the most general matrix harmonic oscillator systems compatible with permutation symmetry. In addition, interacting Hamiltonians are discussed which exhibit a variety of spectral features controlled by sequences of partition algebras. We conclude by calculating two- and three-point correlators, the former are shown to factorise (in an extension of the factorisation result of chapter (4], and the latter are shown to obey selection rules based on Clebsch-Gordan multiplicities of symmetric groups.

## Chapter 2

## Background

This chapter contains much of the background material utilised in the bulk of this thesis. First, we introduce some basic symmetric group representation theory including a treatment of the natural representation $V_{N}$ of the symmetric group and its irreducible decomposition. The natural representation is of particular importance as we will frequently work with matrices $M_{i j}$ transforming like $V_{N} \otimes V_{N}$. The symmetric group acting on $V_{N}$ has a Schur-Weyl dual algebra called the partition algebra. We spend some time introducing this algebra as it is used to prove important results in chapters 4 and 5 . The final sections of this chapter introduce permutation invariant matrix (PIGM) models themselves, following the original papers 58] and 43.

Useful resources, giving a more comprehensive treatment of the representation theory of the symmetric group, include [59, 5, 60]. A very good introduction to partition algebras is given in 61.

### 2.1 Representation theory of finite groups

A representation of a group $G$ on a vector space $V$ is a group homomorphism $D^{V}$ from the group to the space of invertible matrices on the vector space $G L(V)$, that is $D^{V}$ must obey 59]

$$
\begin{equation*}
D^{V}\left(g_{1} g_{2}\right)=D^{V}\left(g_{1}\right) D^{V}\left(g_{2}\right), \quad \forall g_{1}, g_{2} \in G \tag{2.1}
\end{equation*}
$$

Choosing an alternative basis for the vector space $V$ will replace the linear operators $D^{V}(g)$ by their transforms by some matrix $C$. The transformed matrices

$$
\begin{equation*}
D^{V^{\prime}}(g)=C D^{V}(g) C^{-1} \tag{2.2}
\end{equation*}
$$

provide an equivalent representation to the original matrices $D^{V}(g)$, despite the matrices themselves being different.

A subrepresentation of a representation $V$ is a vector subspace $W$ of $V$ which is invariant under the action of $D^{V}(g)$ for all $g \in G$. A representation $V$ is irreducible if there exists no proper nonzero invariant subspace $W$ of $V$, i.e. $V$ admits no subspaces other than the trivial two: $V$ itself and $\{0\}$. Given two representations $V$ and $W$, the direct sum $V \oplus W$ and the tensor product $V \otimes W$ are also representations, the latter via the action

$$
\begin{equation*}
D^{V \otimes W}(g)=D^{V}(g) \otimes D^{W}(g) \tag{2.3}
\end{equation*}
$$

For any representation $V$ of a finite group $G$ there exists a decomposition

$$
\begin{equation*}
V \cong V_{1}^{\oplus \tau_{1}} \oplus V_{2}^{\oplus \tau_{2}} \oplus \cdots \oplus V_{k}^{\oplus \tau_{k}}, \tag{2.4}
\end{equation*}
$$

into irreducible representations. The irreducible representations are labelled $V_{i}$ and each appears with multiplicity $\tau_{i}$. This decomposition is unique.

It is useful to work with quantities that are basis independent. An important example of such a quantity is the character, or trace, of a linear operator

$$
\begin{equation*}
\chi^{V}(g) \equiv \operatorname{Tr}_{V}(D(g))=\sum_{i} D^{V}(g)_{i i} . \tag{2.5}
\end{equation*}
$$

Indeed, under a basis transformation we have

$$
\begin{equation*}
\sum_{i, j, k} C_{i j} D^{V}(g)_{j k} C_{k i}^{-1}=\sum_{j, k} D^{V}(g)_{j k} \delta_{k j}=\sum_{j} D^{V}(g)_{j j} . \tag{2.6}
\end{equation*}
$$

Define the following inner product on characters of a group $G$ 60

$$
\begin{equation*}
\left\langle\chi^{V_{i}}, \chi^{V_{j}}\right\rangle=\frac{1}{|G|} \sum_{g \in G} \chi^{V_{i}}(g) \overline{\chi^{V_{j}}(g)} . \tag{2.7}
\end{equation*}
$$

where the bar stands for complex conjugation and $|G|$ is the order of $G$. Under this inner product irreducible characters are orthogonal

$$
\begin{equation*}
\left\langle\chi^{V_{i}}, \chi^{V_{j}}\right\rangle=\delta\left(V_{i}, V_{j}\right) . \tag{2.8}
\end{equation*}
$$

This can be used to calculate the multiplicities in the irreducible decomposition of a general representation (2.4

$$
\begin{equation*}
\left\langle\chi^{V}, \chi^{V_{i}}\right\rangle=\tau_{i} . \tag{2.9}
\end{equation*}
$$

### 2.2 Representation theory of the symmetric group

A permutation on $N$ objects is an invertible map $\sigma:\{1, \cdots, N\} \rightarrow\{1, \ldots, N\}$. The product of two permutations $\sigma_{1}, \sigma_{2}$, each acting on $N$ objects is defined by composing the maps: $\sigma_{1} \sigma_{2}(i)=\sigma_{2}\left(\sigma_{1}(i)\right)$. As an example, consider the following two permutations acting on the set $\{1,2,3\}$

$$
\begin{align*}
& \sigma_{1}: 1 \mapsto 2,2 \mapsto 3,3 \mapsto 1,  \tag{2.10}\\
& \sigma_{2}: 1 \mapsto 2,2 \mapsto 1,3 \mapsto 3 .
\end{align*}
$$

In this case

$$
\begin{equation*}
\sigma_{1} \sigma_{2}: 1 \mapsto 1,2 \mapsto 3,3 \mapsto 2 . \tag{2.11}
\end{equation*}
$$

Permutations can also be written in cycle notation, in which each permutation map is broken into cycles. Each element in each cycle is mapped to the next element in that cycle under the given permutation, the last element in each cycle is mapped to the first. For example the permutations above are

$$
\begin{equation*}
\sigma_{1}=(123), \quad \sigma_{2}=(12)(3), \quad \sigma_{1} \sigma_{2}=(1)(23) . \tag{2.12}
\end{equation*}
$$

The set of all possible permutations acting on $N$ objects, composed with the above product rule, form a group: the symmetric group on $N$ objects, denoted $S_{N}$. There is one such group for each $N \in \mathbb{Z}^{+}$. Conjugacy classes of $S_{N}$ are composed of all elements that share the same cycle structure. For instance $S_{3}$ has three conjugacy classes

$$
\begin{equation*}
\{(1)(2)(3)\}, \quad\{(1)(23),(12)(3),(13)(2)\}, \quad\{(123),(132)\} . \tag{2.13}
\end{equation*}
$$

The number of irreducible representations of $S_{N}$ is equal to the number of conjugacy classes of the group 5. 5 . In turn this is equal to the number of integer partitions of $N$. An integer partition $\lambda$ of $N$ is a way of writing $N$ as a sum of integers. Partitions that differ only in the order of their elements are the same, so that we choose a canonical ordering for the elements from high to low. For instance, the partitions of four are given by

$$
\begin{equation*}
[4], \quad[3,1], \quad[2,2], \quad[2,1,1], \quad[1,1,1,1] . \tag{2.14}
\end{equation*}
$$

To each integer partition is commonly associated a Young diagram. Young diagrams are composed of boxes such that the $i^{\text {th }}$ element of a partition is equal to the number of boxes in the $i^{\text {th }}$ row of the Young diagram. The five partitions in (2.14) correspond to the following Young diagrams


$[2,2]:$

$[2,1,1]: \square, \quad[1,1,1,1]: \square$.

There are many useful combinatoric results concerning Young diagrams and the representation theory of the symmetric group. The texts cited at the beginning of this chapter are useful references for these. One such result we will make use of is that the dimension of an $S_{N}$ irreducible representation can be calculated from its Young diagram $\lambda$ using the Hook length formula

$$
\begin{equation*}
\operatorname{Dim}\left(V_{\lambda}\right)=\frac{N!}{\prod h_{\lambda}(i, j)}, \tag{2.16}
\end{equation*}
$$

where $h_{\lambda}(i, j)$ is the Hook length of the box indexed by $(i, j)$ in the diagram $\lambda$. The Hook length is the number of boxes directly to the right of or directly below the box in question plus one (as the box itself is also counted once). For example, the boxes in the following Young diagram are labelled by their respective Hook lengths

$$
\begin{equation*}
 \tag{2.17}
\end{equation*}
$$

The diagram 2.16) labels an irreducible representation of $S_{7}$. The dimension of the $S_{7}$ irreducible representation corresponding to $[4,2,1]$ is

$$
\begin{equation*}
\operatorname{Dim}\left(V_{[4,2,1]}\right)=\frac{7!}{6 \cdot 4 \cdot 2 \cdot 3}=35 . \tag{2.18}
\end{equation*}
$$

All representations of the symmetric groups can be expressed in real form. For any group with only real irreducible representations we can choose the matrix representations $D^{V}(g)$ to be real.

For the symmetric group $S_{N}$ the character orthogonality relation (2.8) can be written

$$
\begin{equation*}
\left\langle\chi^{V_{i}}, \chi^{V_{j}}\right\rangle=\frac{1}{N!} \sum_{\sigma \in S_{N}} \chi^{V_{i}}(\sigma) \chi^{V_{j}}(\sigma), \tag{2.19}
\end{equation*}
$$

since we can always choose matrix representations of the symmetric group to be orthogonal, and characters are constant on conjugacy classes, with $\sigma$ and $\sigma^{-1}$ belonging to the same conjugacy class.

### 2.2.1 Stability of products of symmetric group irreducible representations

We begin by illustrating a remarkable stability property of tensors of irreducible symmetric group representations first observed by Murnaghan [62, 63]. For large enough $N$ we see that the irreducible decomposition of the tensor product of $S_{N}$ irreducible representations stabilises in the sense that only the first row of each of the components in the decomposition changes 64]

$$
\begin{align*}
& V_{[1,1]} \otimes V_{[1,1]}=V_{[2]}, \\
& V_{[2,1]} \otimes V_{[2,1]}=V_{[3]}+V_{[2,1]}+V_{[1,1,1]}, \\
& V_{[3,1]} \otimes V_{[3,1]}=V_{[4]}+V_{[3,1]}+V_{[2,1,1]}+V_{[2,2]}, \\
& V_{[4,1]} \otimes V_{[4,1]}=V_{[5]}+V_{[4,1]}+V_{[3,1,1]}+V_{[3,2]}, \\
& V_{[5,1]} \otimes V_{[5,1]}=V_{[6]}+V_{[5,1]}+V_{[4,1,1]}+V_{[4,2]} . \tag{2.20}
\end{align*}
$$

That is, for any $N \geq 4$, the decomposition is given by

$$
\begin{equation*}
V_{[(N-1), 1]} \otimes V_{[(N-1), 1]}=V_{[((N-1)+1)]}+V_{[(N-1), 1]}+V_{[((N-1)-1), 1,1]}+V_{[((N-1)-1), 2]} . \tag{2.21}
\end{equation*}
$$

This suggests a useful bijection between partitions that captures the stable part of the decomposition we are interested in. On one side we have the original partitions $\lambda \vdash N$ with $N$ large and on the other we have the same partition with the first row subtracted $\left[\lambda_{2}, \ldots, \lambda_{l}\right]$. The second type of partitions are equivalent to partitions $\alpha \vdash m$ obeying $m<N$ and $\alpha_{1} \leq \frac{N}{2}$. Then if $\alpha=\left[\alpha_{1}, \alpha_{2}, \ldots, \alpha_{l-1}\right] \vdash m$ the bijection is given by

$$
\begin{equation*}
\bar{\alpha}=\left[N-|\alpha|, \alpha_{1}, \ldots, \alpha_{l-1}\right] \vdash N . \tag{2.22}
\end{equation*}
$$

The Murnaghan rule gives a prescription for constructing the multiplicity of terms in the decomposition of the tensor of two irreducible representations [65]. Specialising this to the case where one of the products in the tensor is the Hook representation we have

$$
\begin{equation*}
V_{\bar{\alpha}} \otimes V_{\overline{[1]}}=c(\alpha)\left(V_{\bar{\alpha}}\right) \oplus \bigoplus_{\beta \in \alpha^{ \pm}} V^{\bar{\beta}}, \tag{2.23}
\end{equation*}
$$

where $c(\alpha)$ is the number of corner boxes (boxes with a Hook length of one) in $\alpha$ and $\alpha^{ \pm}$ is the set of partitions $\beta$ with $\beta_{1} \leq \frac{N}{2}$ constructed by starting with $\alpha$ and either adding a box, removing a box or moving a corner box to a different corner.

It will frequently be useful to find the irreducible decomposition of the $k^{\text {th }}$ tensor product of the $N$ dimensional natural representation of the symmetric group. Decomposing the natural representation according to (2.31)

$$
\begin{equation*}
V_{N}^{\otimes k} \cong\left(V_{0} \oplus V_{\overline{[1]}}\right)^{\otimes k} \tag{2.24}
\end{equation*}
$$

and applying (2.23) iteratively, gives us a way to calculate the irreducible decomposition of tensor powers of the natural representation.

### 2.2.2 The natural representation

We consider the natural representation of the symmetric group, $V_{N}$, as a span of $N$ basis vectors $\left\{e_{1}, e_{2}, \ldots, e_{N}\right\}$ and a set of linear operators $\rho_{V_{N}}(\sigma)$, with $\sigma \in S_{N}$, acting on this basis as

$$
\begin{equation*}
\rho_{V_{N}}(\sigma) e_{i}=e_{\sigma^{-1}(i)} \tag{2.25}
\end{equation*}
$$

and extended by linearity. Following [43], we form the linear combinations

$$
\begin{align*}
E_{0} & =\frac{1}{\sqrt{N}}\left(e_{1}+e_{2}+\cdots+e_{N}\right), \\
E_{1} & =\frac{1}{\sqrt{2}}\left(e_{1}-e_{2}\right), \\
E_{2} & =\frac{1}{\sqrt{6}}\left(e_{1}+e_{2}-2 e_{3}\right), \\
\vdots & \\
E_{a} & =\frac{1}{\sqrt{a(a+1)}}\left(e_{1}+e_{2}+\cdots+e_{a}-a e_{a+1}\right),  \tag{2.26}\\
\vdots & \\
E_{N-1} & =\frac{1}{\sqrt{N(N-1)}}\left(e_{1}+e_{2}+\cdots+e_{N-1}-(N-1) e_{N}\right) .
\end{align*}
$$

$\left\{E_{0}\right\}$ is an invariant under the action of $S_{N}$. It is easily checked that $\left\{E_{1}, E_{2}, \ldots, E_{N-1}\right\}$ form an orthonormal basis of $V_{[N-1,1]}$, the Hook representation, by the orthonormality of the $e_{i}$. We define the overlap of these bases by $C_{a, i}$,

$$
\begin{equation*}
C_{a, i}=\left\langle E_{a} \mid e_{i}\right\rangle=\frac{1}{\sqrt{a(a+1)}}\left(-a \delta_{i, a+1}+\sum_{j=1}^{a} \delta_{i j}\right) . \tag{2.27}
\end{equation*}
$$

The $V_{[N]}$ overlap with the original basis is given by

$$
\begin{equation*}
C_{0, i}=\left\langle E_{0} \mid e_{i}\right\rangle=\frac{1}{\sqrt{N}} . \tag{2.28}
\end{equation*}
$$

From

$$
\begin{equation*}
\sum_{A=0}^{N-1} C_{A, i} C_{A, j}=C_{0, i} C_{0, j}+\sum_{a=1}^{N-1} C_{a, i} C_{a, j}=\delta_{i j} \tag{2.29}
\end{equation*}
$$

we find

$$
\begin{equation*}
\sum_{a=1}^{N-1} C_{a, i} C_{a, j}=\left(\delta_{i j}-\frac{1}{N}\right)=F(i, j) \tag{2.30}
\end{equation*}
$$

$F(i, j)$ is the projector in $V_{N}$ for $V_{[N-1,1]}$.
We have recovered the familiar result, that the natural representation is isomorphic to the direct sum of two irreducible representations: the trivial representation $V_{[N]}$ and the Hook or standard representation $V_{[N-1,1]}$ (associated with partitions $[N]$ and $[N-1,1]$ respectively)

$$
\begin{equation*}
V_{N}=V_{[N]} \oplus V_{[N-1,1]} . \tag{2.31}
\end{equation*}
$$

### 2.3 Partition algebras

We introduce the partition algebras in the diagram basis following the treatment in 61]. This is a convenient starting point because it gives the most straight-forward description of multiplication in $P_{k}(N)$. The partition algebra $P_{k}(N)$ is an algebra of dimension $B(2 k)$ :

$$
\begin{equation*}
\operatorname{Dim}\left(P_{k}(N)\right)=B(2 k) . \tag{2.32}
\end{equation*}
$$

The Bell number $B(2 k)$ is the number of possible partitions of a set with $2 k$ distinct elements. Bell numbers can be computed from the generating function

$$
\begin{equation*}
\sum_{k=0}^{\infty} \frac{B(k)}{k!} x^{k}=e^{e^{x}-1}, \tag{2.33}
\end{equation*}
$$

from which one finds $B(2 k)=2,15,203,4140$ for $k=1,2,3,4$.
A set partition $\pi$ of a set $S$ is a set of disjoint subsets of $S$ such that their union is all of $S$. The diagram basis for $P_{k}(N)$ is labelled by set partitions of the set $\left\{1, \ldots, k, 1^{\prime}, \ldots, k^{\prime}\right\}$. The set of all set partitions of $\left\{1, \ldots, k, 1^{\prime}, \ldots, k^{\prime}\right\}$ is denoted $\Pi_{2 k}$ (see for example [66] for further information on set partitions). For example, the set $\Pi_{4}$ contains the following
$B(4)=15$ set partitions (subsets are separated by a vertical bar)

$$
\begin{align*}
& 1|2| 1^{\prime} \mid 2^{\prime}, \\
& 11^{\prime}|2| 2^{\prime}, \quad 12^{\prime}\left|1^{\prime}\right| 2, \quad 12\left|1^{\prime}\right| 2^{\prime}, \quad 1^{\prime} 2^{\prime}|1| 2, \quad 1^{\prime} 2|1| 2^{\prime}, \quad 22^{\prime}\left|1^{\prime}\right| 1, \\
& 11^{\prime} 2^{\prime}\left|2, \quad 121^{\prime}\right| 2^{\prime}, \quad 122^{\prime}\left|1^{\prime}, \quad 1^{\prime} 2^{\prime} 2\right| 1, \quad 11^{\prime}\left|22^{\prime}, \quad 12^{\prime}\right| 1^{\prime} 2, \quad 12 \mid 1^{\prime} 2^{\prime}, \\
& 121^{\prime} 2^{\prime} \tag{2.34}
\end{align*}
$$

Each $\pi \in \Pi_{2 k}$ labels an element of the diagram basis of $P_{k}(N)$. We write $d_{\pi}$ for the diagram basis element corresponding to $\pi \in \Pi_{2 k}$. As the name suggests, $d_{\pi}$ should be thought of as a diagram. It is a diagram with $2 k$ vertices divided into two rows. The bottom vertices are labelled $1, \ldots, k$ from left to right and the vertices of the top row are labelled $1^{\prime}, \ldots, k^{\prime}$ from left to right. Two vertices are connected by an edge if they belong to the same subset of $\pi$. The diagrams corresponding to the set partitions in (2.34) are


There is a redundancy in the diagram picture, arising from the fact that we are free to choose any set of edges, as long as every vertex in a subset of the set partition can be reached from any other vertex in the same subset, by a path along the edges. For example, the following pairs of diagrams correspond to the same element in $P_{3}(N)$


The partition algebras are so-called diagram algebras because multiplication can be defined through diagram concatenation (in the diagram basis). The product in $P_{k}(N)$ is independent of the choice of representative diagram. Let $d_{\pi}$ and $d_{\pi^{\prime}}$ be two diagrams in $P_{k}(N)$. The composition $d_{\pi^{\prime \prime}}=d_{\pi} d_{\pi^{\prime}}$ is constructed by placing $d_{\pi}$ above $d_{\pi^{\prime}}$ and identifying the bottom vertices of $d_{\pi}$ with the top vertices of $d_{\pi^{\prime}}$. The diagram is simplified by following the edges connecting the bottom vertices of $d_{\pi^{\prime}}$ to the top vertices of $d_{\pi}$. Any connected
components within the middle rows are removed and we multiply by $N^{c}$, where $c$ is the number of these complete blocks removed. For example,

where the factor of $N$ in the first equation comes from removing the middle component at vertex 1 and 2. For linear combinations of diagrams, multiplication is defined by linear extension.

The subset of diagrams with $k$ edges, each connecting a vertex at the top to a vertex at the bottom, where every vertex has exactly one incident edge, span a subalgebra. This subalgebra is isomorphic to the symmetric group algebra $\mathbb{C}\left[S_{k}\right]$. For example, there is a one-to-one correspondence between permutations in $S_{3}$ and the following set of diagrams in $P_{3}(N)$


In the language of set partitions, these diagrams correspond to set partitions with subsets of the form $\left\{i j^{\prime}\right\}$ for $i, j \in 1, \ldots, k$.

The diagram $d_{\pi} \in P_{k}(N)$ corresponds to an element of $\operatorname{End}\left(V_{N}^{\otimes k}\right)$ through the following action

$$
\begin{equation*}
d_{\pi}\left(e_{i_{1}} \otimes \cdots \otimes e_{i_{k}}\right)=\sum_{i_{1^{\prime}}, \ldots, i_{k^{\prime}}}\left(d_{\pi}\right)_{i_{1} \ldots i_{k}}^{i_{1} \ldots i_{k}} e_{i_{1^{\prime}}} \otimes \cdots \otimes e_{i_{k^{\prime}}} \tag{2.39}
\end{equation*}
$$

The matrix elements $\left(d_{\pi}\right)_{i_{1} \ldots i_{k}}^{i_{1} \ldots \ldots i_{k^{\prime}}}$ correspond to the diagram representation by associating a Kronecker delta to every edge connecting a pair of vertices. For example,

$$
\begin{equation*}
\underbrace{1^{\prime}}_{1}=\delta_{i_{1} i_{2}} \delta_{i_{2}}^{i_{2^{\prime}}} \delta^{i_{2^{\prime}} i_{1^{\prime}}} \text { and } \underbrace{1_{2}^{\prime}}_{1} \stackrel{2^{\prime}}{\bullet}=\delta_{i_{1} i_{2}} \delta_{i_{1}}^{i_{1^{\prime}}} . \tag{2.40}
\end{equation*}
$$

We define the following two operations on the partition algebras. Firstly, the tensor product $d_{\pi_{1}} \otimes d_{\pi_{2}}$ is the diagram obtained by horizontal concatenation of $d_{\pi_{1}}$ and $d_{\pi_{2}}$, for example


This can be viewed as an outer product on partition algebra diagrams which maps $P_{k_{1}}(N) \times$ $P_{k_{2}}(N)$ to $P_{k_{1}+k_{2}}(N)$. It is a diagram with $2 k_{1}+2 k_{2}$ vertices. Secondly, the join $d_{\pi_{1}} \vee d_{\pi_{2}}$ of two diagrams, each with $2 k$ vertices, is obtained by adding all the edges of $d_{\pi_{1}}$ to the
edges of $d_{\pi_{2}}$ (or vice versa), for example


The resulting diagram also has $2 k$ vertices. For general elements (linear combinations of diagram basis elements) the two operations are defined by linear extension.

### 2.4 5-parameter PIGM models

Permutation invariant Gaussian matrix (PIGM) models were first developed to model the statistics of large ensembles of matrices appearing in natural language processing [58]. The matrices in these ensembles are $N \times N$ general real matrices $M$. The central objects of study are Gaussian matrix integrals of the form

$$
\begin{equation*}
\int d M e^{L(M)+Q(M)}, \tag{2.43}
\end{equation*}
$$

where $L(M)$ is a linear function of the matrix variables $M_{i j}$ and $Q(M)$ is a quadratic function of the matrix variables, both of which are invariant under the diagonal action of $S_{N}$.

In [58] the authors define a five-parameter model comprised of two linear parameters $J^{0}, J^{s}$ controlling the diagonal and off-diagonal matrix elements respectively and three quadratic parameters: $a$ and $b$ for the square of the off-diagonal matrix elements, $c$ for the square of the diagonal matrix elements. The partition function of this model is

$$
\begin{array}{r}
\mathcal{Z}\left(J^{0}, J^{S}, a, b, c\right)=\int d M e^{J^{0} \sum_{i=1}^{N} M_{i i}+J^{S} \sum_{i<j}\left(M_{i j}+M_{j i}\right)-\frac{c}{2} \sum_{i=1}^{N} M_{i i}^{2}} \\
e^{-\frac{1}{4}(a+b) \sum_{i<j}\left(M_{i j}^{2}+M_{j i}^{2}\right)-\frac{1}{2}(a-b) \sum_{i<j} M_{i j} M_{j i}} \tag{2.44}
\end{array}
$$

The observables of the model are $S_{N}$ invariant polynomials in the matrix variables, that is polynomials $f\left(M_{i j}\right)$ obeying

$$
\begin{equation*}
f\left(M_{i j}\right)=f\left(M_{\sigma(i) \sigma(j)}\right), \quad \forall \sigma \in S_{N} \tag{2.45}
\end{equation*}
$$

Expectation values of these observables are defined in the usual fashion

$$
\begin{equation*}
\langle f(M)\rangle \equiv \frac{1}{\mathcal{Z}} \int d M f(M) e^{-S_{(5)}} \tag{2.46}
\end{equation*}
$$

where we have written the term in the exponent of 2.44 as $-S_{(5)}$.

The expectation values of observables can be calculated by directly performing the integration in (2.46). This is possible because the integral factorises into $N$ single variable integrals for the diagonal elements and $N(N-1) / 2$ two-variable integrals for the off-diagonal elements.

The procedure utilises the following result from multidimensional Gaussian integration 67]

$$
\begin{equation*}
\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} e^{-\frac{1}{2} \mathbf{x}^{T} \Lambda \mathbf{x}+\mathbf{J}^{T} \mathbf{x}} d x_{1} d x_{2} \ldots d x_{n}=\frac{(2 \pi)^{n / 2}}{(\operatorname{det} \Lambda)^{1 / 2}} e^{\frac{1}{2} \mathbf{J}^{T} \Lambda^{-1} \mathbf{J}}, \tag{2.47}
\end{equation*}
$$

with $\mathbf{x}$ and $\mathbf{J}$ n-dimensional vectors, and $\Lambda$ a symmetric, non-singular, $n \times n$ matrix. We have $n$ gaussian integrals, all coupled to each other through the action of $\Lambda$.

### 2.5 Counting of observables

The five-parameter model defined by (2.44) is not the most general permutation invariant Gaussian matrix model. This can be seen by the following counting of degree $k$ permutation invariant polynomials in the matrix variables $M_{i j}$ originally given in [58] and reviewed here. From (2.45) we see that $M_{i j}$ transforms like the tensor product of two copies of the natural representation

$$
\begin{equation*}
M_{i j} \cong V_{N} \otimes V_{N} . \tag{2.48}
\end{equation*}
$$

Since the matrix variables commute, the product of $k$ copies of $M$ transforms like

$$
\begin{equation*}
\left(M_{i j}\right)^{k} \cong \operatorname{Sym}^{k}\left(V_{N} \otimes V_{N}\right) . \tag{2.49}
\end{equation*}
$$

The linear operator for $\sigma \in S_{N}$ on $V_{N}$ we call $D^{V_{N}}(\sigma)$. The linear operator on $V_{N}^{\otimes 2 k}$ is then

$$
\begin{equation*}
D_{N}^{V_{N}^{\otimes 2 k}}(\sigma)=D^{V_{N}^{\otimes 2}}(\sigma) \otimes \cdots \otimes D^{V_{N}^{\otimes 2}}(\sigma), \tag{2.50}
\end{equation*}
$$

where the product is over $k$ factors. The symmetric subspace of $\left(V_{N} \otimes V_{N}\right)^{\otimes k}$ can be projected to by averaging over permutations $\tau \in S_{k}$ which act by permuting each of the $\left(V_{N} \otimes V_{N}\right)$ factors. The dimension of the subspace of invariants within this symmetric
subspace is

$$
\begin{align*}
\operatorname{Dim}(N, k) & =\frac{1}{k!N!} \sum_{\sigma \in S_{N}} \sum_{\tau \in S_{k}} \operatorname{Tr}_{V_{N}^{\otimes 2 k}}\left(\tau D^{V_{N}^{\otimes 2 k}}(\sigma)\right) \\
& =\frac{1}{k!N!} \sum_{\sigma \in S_{N}} \sum_{\tau \in S_{k}} \prod_{i=1}^{k} \operatorname{Tr}_{V_{N}^{\otimes 2}}\left(D^{\left.V_{N}^{\otimes 2}\left(\sigma^{i}\right)\right)^{C_{i}(\tau)}} .\right. \tag{2.51}
\end{align*}
$$

Using

$$
\begin{equation*}
\operatorname{Tr}_{V_{N}}\left(D^{V_{N}}\left(\sigma^{i}\right)\right)=\sum_{l \mid i} l C_{l}(\sigma) \tag{2.52}
\end{equation*}
$$

where the sum is taken over the divisors of $i$, we can rewrite the trace terms

$$
\begin{equation*}
\operatorname{Dim}(N, k)=\frac{1}{k!N!} \sum_{\sigma \in S_{N}} \sum_{\tau \in S_{k}} \prod_{i=1}^{k}\left(\sum_{l \mid i} l C_{l}(\sigma)\right)^{2 C_{i}(\tau)} \tag{2.53}
\end{equation*}
$$

This is a function of the conjugacy classes of $S_{N}$ and $S_{k}$, which can be written in terms of the partitions of $N$ and $k$ respectively. Write these $p=\left\{p_{1}, p_{2}, \ldots, p_{N}\right\}$ and $\{q=$ $\left.q_{1}, q_{2}, \ldots, q_{N}\right\}$ where, for example $p_{i}$ is the number of $i$-cycles in the partition $p$. The number of permutations $\sigma \in S_{N}$ belonging to a conjugacy class $p$ is given by

$$
\begin{equation*}
\frac{N!}{\prod_{i=1}^{N} i^{p_{i}} p_{i}!} . \tag{2.54}
\end{equation*}
$$

Allowing us to write

$$
\begin{equation*}
\operatorname{Dim}(N, k)=\frac{1}{k!N!} \sum_{p \vdash N} \sum_{q \vdash k} \frac{N!}{\prod_{i=1}^{N} i^{p_{i}} p_{i}!} \frac{k!}{\prod_{i=1}^{k} i^{q_{i}} q_{i}!} \prod_{i=1}^{k}\left(\sum_{l \mid i} l p_{l}\right)^{2 q_{i}} . \tag{2.55}
\end{equation*}
$$

For $N \geq 2 k$ the dimension of this space stabilises, in this stable limit we have

$$
\begin{equation*}
\operatorname{Dim}(2 k, k)=\sum_{p \vdash 2 k} \sum_{q \vdash k} \frac{1}{\prod_{i=1}^{N} i^{p_{i}+q_{i}} p_{i}!q_{i}!} \prod_{i=1}^{k}\left(\sum_{l \mid i} l p_{l}\right)^{2 q_{i}} \tag{2.56}
\end{equation*}
$$

For $k=1,2,3,4$ this evaluates to $2,11,52,296$. From this it is clear that the five-parameter model is not the most general Gaussian permutation invariant model it is possible to construct. It includes both possible linear terms, but is missing eight quadratic terms.

As noted in 58 these polynomials can be associated with graphs as follows: each summed index corresponds to a vertex, and each matrix $M_{i j}$ to a directed edge from the vertex corresponding to index $i$ to the vertex corresponding to index $j$. The two linear polynomials
are given by the following graphs

and the 11 quadratic terms correspond to the following graphs


### 2.6 13-parameter PIGM models

In [43] a 13 -parameter family of Gaussian matrix models consistent with permutation invariance was constructed. The expectation values of linear and quadratic permutation invariant polynomials in $M_{i j}$ were given as analytic expressions in $N$, the size of the matrices. Expectation values for a sample of cubic and quartic invariant polynomials were constructed using Wick's theorem.

The schematic form of the PIGM model partition function $Z_{(13)}$ is

$$
\begin{equation*}
Z_{(13)}=\int \mathrm{d} M \exp \left(-S_{(13)}(M)\right)=\int \mathrm{d} M \exp \left(-\sum_{i=1}^{2} \mu_{i} L_{i}(M)-\sum_{i=1}^{11} g_{i} Q_{i}(M)\right) . \tag{2.59}
\end{equation*}
$$

The action $S_{(13)}(M)$ contains two linear terms: $L_{1}, L_{2}$; and eleven quadratic terms $Q_{1}, \ldots, Q_{11}$,
these are listed in 2.57 and 2.58 . The measure is taken to be the measure on $\mathbb{R}^{N^{2}}$,

$$
\begin{equation*}
\mathrm{d} M \equiv \prod_{i=1}^{N} \mathrm{~d} M_{i i} \prod_{i \neq j}^{N} \mathrm{~d} M_{i j} \tag{2.60}
\end{equation*}
$$

It is the most general quadratic action invariant under the diagonal group action of $S_{N}$ (the symmetric group on $N$ objects),

$$
\begin{equation*}
S_{(13)}\left(M_{\sigma(i) \sigma(j)}\right)=S_{(13)}\left(M_{i j}\right), \quad \forall \sigma \in S_{N} \tag{2.61}
\end{equation*}
$$

Again, we note that from this action the vector space spanned by $M_{i j}$ transforms in the same way as $V_{N} \otimes V_{N}$

$$
\begin{equation*}
M_{i j} \cong V_{N} \otimes V_{N} \tag{2.62}
\end{equation*}
$$

This is not an irreducible representation, instead it decomposes into several irreducible components

$$
\begin{equation*}
V_{N} \otimes V_{N} \cong 2 V_{[N]} \oplus 3 V_{[N-1,1]} \oplus V_{[N-2,1,1]} \oplus V_{[N-2,2]} \tag{2.63}
\end{equation*}
$$

Here $V_{[N]}$ corresponds to the trivial one-dimensional representation of $S_{N}$. The representations $V_{[N-1,1]}, V_{[N-2,1,1]}, V_{[N-2,2]}$ are non-trivial irreducible representations, which we label by their corresponding integer partitions of $N$. Their dimensions can be calculated using (2.16), they are

$$
\begin{equation*}
N-1,(N-1)(N-2) / 2, N(N-3) / 2 \tag{2.64}
\end{equation*}
$$

respectively. Detailed descriptions, including explicit constructions of irreducible representations of $S_{N}$ can be found in [60, 68]. Index these irreducible representations by $\Lambda_{1}$ ranging over

$$
\begin{equation*}
\Lambda_{1} \in\{[N],[N-1,1],[N-2,1,1],[N-2,2]\} \tag{2.65}
\end{equation*}
$$

and refer to the corresponding irreducible representations of $S_{N}$ as $V_{\Lambda_{1}}$. The decomposition (2.63) can be deduced using

$$
\begin{equation*}
V_{N} \cong V_{[N]} \oplus V_{[N-1,1]} \tag{2.66}
\end{equation*}
$$

together with the tensor product rule (2.23) (for more detail see section 7.13 of [68]). See also [60] for a dedicated treatment of symmetric group representation theory. The linear transformation which reduces the LHS of (2.63) to a sum of irreducible representations is called the Clebsch-Gordan decomposition. The matrix elements of this transformation are called Clebsch-Gordan coefficients. These coefficients can be used to construct projection
operators for the subspaces of (2.62) corresponding to particular irreducible representations. The application of these projectors effects a change of basis which diagonalises the partition function of the 13-parameter model leaving it amenable to standard techniques of multi-dimensional Gaussian integration.

Note that the multiplicity of $V_{[N]}$ in 2.63 is precisely why there are two linear terms $L_{1}, L_{2}$ in the action 2.59. Furthermore, the isomorphism in equation (2.63) implies the existence of a set of linear combinations of matrix elements labelled by $\Lambda_{1}$

$$
\begin{equation*}
S_{a}^{\Lambda_{1}, \alpha}=\sum_{i, j} C_{a, i j}^{\Lambda_{1}, \alpha} M_{i j} . \tag{2.67}
\end{equation*}
$$

The index $a$ is a state index for each irreducible representation and $\alpha$ is a multiplicity index

$$
\begin{align*}
& a \in\left\{1, \ldots, \operatorname{Dim} V_{\Lambda_{1}}\right\},  \tag{2.68}\\
& \alpha \in\left\{1, \ldots, \operatorname{Mult}\left(V_{N} \otimes V_{N} \rightarrow V_{\Lambda_{1}}\right)\right\},
\end{align*}
$$

where $\operatorname{Mult}\left(V_{N} \otimes V_{N} \rightarrow V_{\Lambda_{1}}\right)$ is the multiplicity of $V_{\Lambda_{1}}$ in $V_{N} \otimes V_{N}$. The coefficients $C_{a, i j}^{\Lambda_{1}, \alpha}$ are Clebsch-Gordan coefficients and they have the property

$$
\begin{equation*}
\sum_{i, j} C_{a, i j}^{\Lambda_{1}, \alpha} M_{\sigma^{-1}(i) \sigma^{-1}(j)}=\sum_{b} D_{a b}^{\Lambda_{1}}(\sigma) S_{b}^{\Lambda_{1} \alpha}, \tag{2.69}
\end{equation*}
$$

where the matrices $D_{a b}^{\Lambda_{1}}(\sigma)$ are irreducible matrix representations of $S_{N}$ (background on the Clebsch-Gordan coefficients for symmetric groups is available in [68]). Without loss of generality, we can assume that the Clebsch-Gordan coefficients define an orthonormal basis with respect to the inner product

$$
\begin{equation*}
\left(M_{i j}, M_{k l}\right)=\delta_{i k} \delta_{j l} . \tag{2.70}
\end{equation*}
$$

Equivalently, the representation theoretic variables satisfy

$$
\begin{equation*}
\left(S_{a}^{\Lambda_{1}, \alpha}, S_{b}^{\Lambda_{1}^{\prime}, \beta}\right)=\delta_{a b} \delta^{\Lambda_{1} \Lambda_{1}^{\prime}} \delta^{\alpha \beta} . \tag{2.71}
\end{equation*}
$$

Together with the fact that the inner product (2.70) is $S_{N}$ invariant, it follows that

$$
\begin{equation*}
D_{a b}^{\Lambda_{1}}\left(\sigma^{-1}\right)=D_{b a}^{\Lambda_{1}}(\sigma) . \tag{2.72}
\end{equation*}
$$

Using the representation theory basis it immediately follows that the quadratic combination

$$
\begin{equation*}
\sum_{a} S_{a}^{\Lambda_{1}, \alpha} S_{a}^{\Lambda_{1}, \beta}=\sum_{i, j, k, l} M_{i j} Q_{i j k l}^{\Lambda_{1}, \alpha \beta} M_{k l} \tag{2.73}
\end{equation*}
$$

is an invariant polynomial, where

$$
\begin{equation*}
Q_{i j k l}^{\Lambda_{1}, \alpha \beta}=\sum_{a} C_{a, i j}^{\Lambda_{1}, \alpha} C_{a, k l}^{\Lambda_{1}, \beta} \tag{2.74}
\end{equation*}
$$

A useful observation is that, while the Clebsch-Gordan coefficients depend on a choice of basis for every irreducible component on the RHS of (2.63), the tensors $Q_{i j k l}^{\Lambda_{1}, \alpha \beta}$ do not. For all four $\Lambda_{1}$, they can be constructed using only the explicit bases for the subspaces $V_{[N]}$ and $V_{[N-1,1]}$ in 2.63 [43]. Their construction using Clebsch-Gordan coefficients means that they satisfy

$$
\begin{equation*}
Q_{\sigma(i) \sigma(j) \sigma(k) \sigma(l)}^{\Lambda, \alpha \beta}=Q_{i j k l}^{\Lambda, \alpha \beta} \tag{2.75}
\end{equation*}
$$

This follows from the equivariance property 2.69

$$
\begin{align*}
Q_{\sigma(i) \sigma(j) \sigma(k) \sigma(l)}^{\Lambda, \alpha \beta} & =\sum_{a} C_{a, \sigma(i) \sigma(j)}^{\Lambda, \alpha} C_{a, \sigma(k) \sigma(l)}^{\Lambda, \beta}=\sum_{a, b, c} C_{b, i j}^{\Lambda, \alpha} C_{c, k l}^{\Lambda, \beta} D_{a b}^{\Lambda}(\sigma) D_{a c}^{\Lambda}(\sigma) \\
& =\sum_{b, c} C_{b, i j}^{\Lambda, \alpha} C_{c, k l}^{\Lambda, \beta} \delta_{b c}=Q_{i j k l}^{\Lambda, \alpha \beta} \tag{2.76}
\end{align*}
$$

Going to the second line uses $D_{a b}^{\Lambda}(\sigma)=D_{b a}^{\Lambda}\left(\sigma^{-1}\right)$ which follows from the fact that representation matrices for $S_{N}$ can always be chosen to be real and unitary, i.e. orthogonal matrices.

We may associate a unique parameter in the action to each invariant. Since

$$
\begin{equation*}
\sum_{i, j, k, l} M_{i j} Q_{i j k l}^{\Lambda_{1}, \alpha \beta} M_{k l}=\sum_{i, j, k, l} M_{i j} Q_{i j k l}^{\Lambda_{1}, \beta \alpha} M_{k l} \tag{2.77}
\end{equation*}
$$

there is a symmetric matrix of dimension $\operatorname{Mult}\left(V_{N} \otimes V_{N} \rightarrow V_{\Lambda_{1}}^{S_{N}}\right)$ parametrising the quadratic part of the action, for every choice of $\Lambda_{1}$. Using the multiplicities in the decomposition 2.63, we have

$$
\begin{equation*}
11=\frac{2 \cdot 3}{2!}+\frac{3 \cdot 4}{2!}+\frac{1 \cdot 2}{2!}+\frac{1 \cdot 2}{2!} \tag{2.78}
\end{equation*}
$$

independent parameters, which agrees with the counting given by 2.56). The two linear terms in 2.59) are given by

$$
\begin{equation*}
\mu_{1} L_{1}=\mu_{1} S^{[N], 1} \quad \text { and } \quad \mu_{2} L_{2}=\mu_{2} S^{[N], 2} \tag{2.79}
\end{equation*}
$$

The quadratic part of 2.59 is

$$
\begin{equation*}
\sum_{\substack{\Lambda_{1}, a \\ \alpha, \beta}} S_{a}^{\Lambda_{1}, \alpha} g_{\alpha \beta}^{\Lambda_{1}} S_{a}^{\Lambda_{1}, \beta}=\sum_{\substack{i, j, k, l \\ \Lambda_{1}, \alpha, \beta}} g_{\alpha \beta}^{\Lambda_{1}} M_{i j} Q_{i j k l}^{\Lambda_{1}, \alpha \beta} M_{k l} \tag{2.80}
\end{equation*}
$$

where the matrices $g_{\alpha \beta}^{\Lambda_{1}}$ are the parameters of the model in the representation theory basis. In this basis the partition function is

$$
\begin{align*}
\int \mathrm{d} M \exp \left(-S_{(13)}(M)\right) & =\int \mathrm{d} S \exp \left(-\sum_{\alpha=1}^{2} \mu_{\alpha} \sum_{i, j=1}^{N} C_{i j}^{[N], \alpha} M_{i j}-\sum_{\Lambda_{1}, \alpha, \beta} g_{\alpha \beta}^{\Lambda_{1}} \sum_{i, j, k, l=1}^{N} M_{i j} Q_{i j k l}^{\Lambda_{1}, \alpha \beta} M_{k l}\right) \\
& =\int \mathrm{d} S \exp \left(-\mu_{1} S^{[N], 1}-\mu_{2} S^{[N], 2}-\sum_{\substack{\Lambda_{1}, a \\
\alpha, \beta}} S_{a}^{\Lambda_{1}, \alpha} g_{\alpha \beta}^{\Lambda_{1}} S_{a}^{\Lambda_{1}, \beta}\right) \tag{2.81}
\end{align*}
$$

the change of measure has unit Jacobian since the $S^{\Lambda_{1} ; \alpha}$ variables are given by an orthogonal change of basis, with

$$
\begin{align*}
\mathrm{d} S & =\prod_{\Lambda_{1}, \alpha, a} \mathrm{~d} S_{a}^{\Lambda_{1}, \alpha} \\
& =\mathrm{d} S^{V_{[N]} ; 1} \mathrm{~d} S^{V_{[N]} ; 1} \prod_{a=1}^{N-1} \mathrm{~d} S_{a}^{V_{[N-1,1]} ; 1} \mathrm{~d} S_{a}^{V_{[N-1,1]} ; 2} \mathrm{~d} S_{a}^{V_{[N-1,1]} ; 3} \prod_{a=1}^{\operatorname{Dim}\left(V_{2}\right)} \mathrm{d} S_{a}^{V_{[N-2,2]}} \prod_{a=1}^{\operatorname{Dim}\left(V_{3}\right)} \mathrm{d} S_{a}^{V_{[N-2,1,1]}} \tag{2.82}
\end{align*}
$$

The matrices $g_{\alpha \beta}^{\Lambda_{1}}$ must have non-negative eigenvalues to define a convergent integral.
The action $S_{(13)}$ written in terms of the representation theory variables, i.e. the RHS of (2.81), is written in terms of the projectors and the $[N]$ Clebschs coefficients as

$$
\begin{equation*}
S_{(13)}=\sum_{\alpha=1}^{2} \mu_{\alpha} \sum_{i, j=1}^{N} C_{i j}^{[N], \alpha} M_{i j}+\sum_{\Lambda_{1}, \alpha, \beta} g_{\alpha \beta}^{\Lambda_{1}} \sum_{i, j, k, l=1}^{N} M_{i j} Q_{i j k l}^{\Lambda_{1}, \alpha \beta} M_{k l} \tag{2.83}
\end{equation*}
$$

The linear terms can be constructed with the $V_{N} \otimes V_{N} \rightarrow V_{[N]}$ Clebschs

$$
\begin{align*}
C_{i j}^{[N], 1} & =C_{0, i} C_{0, j}=\frac{1}{N}  \tag{2.84}\\
C_{i j}^{[N], 2} & =\sum_{a, b} C_{a, i} C_{b, j} C_{a, b}^{[N-1,1],[N-1,1] \rightarrow[N]} \\
& =\sum_{a, b} C_{a, i} C_{b, j} \frac{\delta_{a b}}{\sqrt{N-1}} \\
& =\frac{1}{\sqrt{N-1}} F(i, j) \tag{2.85}
\end{align*}
$$

where $C_{a, b}^{[N-1,1],[N-1,1] \rightarrow[N]}$ is the Clebsch coefficient from $V_{[N-1,1]} \otimes V_{[N-1,1]}$ to the trivial representation, and $F(i, j)$ is the projector for $V_{[N-1,1]}$ in $V_{N}$ defined in 2.30. In addition to the two Clebschs above, the quadratic terms require the three following $V_{N} \otimes V_{N} \rightarrow$

## $V_{[N-1,1]}$ Clebschs

$$
\begin{align*}
C_{a, i j}^{[N-1,1], 1} & =C_{0, i} C_{a, j}=\frac{1}{\sqrt{N}} C_{a, j},  \tag{2.86}\\
C_{a, i j}^{[N-1,1], 2} & =C_{a, i} C_{0, j}=\frac{1}{\sqrt{N}} C_{a, i},  \tag{2.87}\\
C_{a, i j}^{[N-1,1], 3} & =\sum_{b, c=1}^{N-1} C_{b, i} C_{c, j} \sqrt{\frac{N}{N-2}} \sum_{p=1}^{N} C_{b, p} C_{c, p} C_{a, p} \\
& =\sqrt{\frac{N}{N-2}} \sum_{p=1}^{N} F(i, p) F(j, p) C_{a, p} . \tag{2.88}
\end{align*}
$$

Detailed calculations of these Clebschs, particularly $C_{i j}^{[N], 2}$ and $C_{a, i j}^{[N-1,1], 3}$, can be found in 43]. Using these Clebsch coefficients and (2.74) we can write down expressions for all but the $\Lambda_{1}=[N-2,2],[N-2,1,1] Q \mathrm{~s}$ appearing in equation (2.83):

$$
\begin{align*}
& Q_{i j k l}^{[N], 11}=C_{i j}^{[N], 1} C_{k l}^{[N], 1}=\frac{1}{N^{2}},  \tag{2.89}\\
& Q_{i j k l}^{[N], 12}=C_{i j}^{[N], 1} C_{k l}^{[N], 2}=\frac{1}{N \sqrt{N-1}} F(k, l),  \tag{2.90}\\
& Q_{i j k l}^{[N], 21}=C_{i j}^{[N], 2} C_{k l}^{[N], 1}=\frac{1}{N \sqrt{N-1}} F(i, j),  \tag{2.91}\\
& Q_{i j k l}^{[N], 22}=C_{i j}^{[N], 2} C_{k l}^{[N], 2}=\frac{1}{N-1} F(i, j) F(k, l),  \tag{2.92}\\
& Q_{i j k l}^{[N-1,1], 11}=\sum_{a=1}^{N-1} C_{a, i j}^{[N-1,1], 1} C_{a, k l}^{[N-1,1], 1}=\frac{1}{N} F(j, l),  \tag{2.93}\\
& Q_{i j k l}^{[N-1,1], 12}=\sum_{a=1}^{N-1} C_{a, i j}^{[N-1,1], 1} C_{a, k l}^{[N-1,1], 2}=\frac{1}{N} F(j, k),  \tag{2.94}\\
& Q_{i j k l}^{[N-1,1], 21}=\sum_{a=1}^{N-1} C_{a, i j}^{[N-1,1], 2} C_{a, k l}^{[N-1,1], 1}=\frac{1}{N} F(i, l),  \tag{2.95}\\
& Q_{i j k l}^{[N-1,1], 22}=\sum_{a=1}^{N-1} C_{a, i j}^{[N-1,1], 2} C_{a, k l}^{[N-1,1], 2}=\frac{1}{N} F(i, k),  \tag{2.96}\\
& Q_{i j k l}^{[N-1,1], 13}=\sum_{a=1}^{N-1} C_{a, i j}^{[N-1,1], 1} C_{a, k l}^{[N-1,1], 3}=\frac{1}{\sqrt{N-2}} \sum_{p=1}^{N} F(j, p) F(k, p) F(l, p),  \tag{2.97}\\
& Q_{i j k l}^{[N-1,1], 31}=\sum_{a=1}^{N-1} C_{a, i j}^{[N-1,1], 3} C_{a, k l}^{[N-1,1], 1}=\frac{1}{\sqrt{N-2}} \sum_{p=1}^{N} F(i, p) F(j, p) F(l, p),  \tag{2.98}\\
& Q_{i j k l}^{[N-1,1], 23}=\sum_{a=1}^{N-1} C_{a, i j}^{[N-1,1], 2} C_{a, k l}^{[N-1,1], 3}=\frac{1}{\sqrt{N-2}} \sum_{p=1}^{N} F(i, p) F(k, p) F(l, p), \tag{2.99}
\end{align*}
$$

$$
\begin{align*}
& Q_{i j k l}^{[N-1,1], 32}=\sum_{a=1}^{N-1} C_{a, i j}^{[N-1,1], 3} C_{a, k l}^{[N-1,1], 2}=\frac{1}{\sqrt{N-2}} \sum_{p=1}^{N} F(i, p) F(j, p) F(k, p),  \tag{2.100}\\
& Q_{i j k l}^{[N-1,1], 33}=\sum_{a=1}^{N-1} C_{a, i j}^{[N-1,1], 3} C_{a, k l}^{[N-1,1], 3}=\frac{N}{N-2} \sum_{p, q=1}^{N} F(i, p) F(j, p) F(k, q) F(l, q) F(p, q) . \tag{2.101}
\end{align*}
$$

At first glance (2.74), along with the $Q \mathrm{~s}$ appearing in the partition function (2.81), appear to also demand the $V_{N} \otimes V_{N} \rightarrow V_{[N-2,2]}$ and $V_{N} \otimes V_{N} \rightarrow V_{[N-2,1,1]}$ Clebsch coefficients

$$
\begin{align*}
C_{a, i j}^{[N-2,2]} & =\sum_{b, c=1}^{N-1} C_{b, i} C_{c, j} C_{b, c ; a}^{[N-1,1],[N-1,1] \rightarrow[N-2,2]},  \tag{2.102}\\
C_{a, i j}^{[N-2,1,1]} & =\sum_{b, c=1}^{N-1} C_{b, i} C_{c, j} C_{b, c ; a}^{[N-1,1],[N-1,1] \rightarrow[N-2,1,1]} . \tag{2.103}
\end{align*}
$$

Despite this, it is possible to find expressions for $Q_{i j k l}^{[N-2,2]}$ and $Q_{i j k l}^{[N-2,1,1]}$ directly, in terms of simple projectors and others that we have already calculated.

### 2.6.1 Finding $Q_{i j k l}^{[N-2,2]}$ and $Q_{i j k l}^{[N-1,1,1]}$

We begin by noting the orthogonal decomposition

$$
\begin{align*}
V_{[N-1,1]} \otimes V_{[N-1,1]} & \cong \operatorname{Sym}^{2}\left(V_{[N-1,1]}\right) \oplus \Lambda^{2}\left(V_{[N-1,1]}\right) \\
& \cong V_{[N]} \oplus V_{[N-1,1]} \oplus V_{[N-2,2]} \oplus V_{[N-2,1,1]} . \tag{2.104}
\end{align*}
$$

The symmetric and anti-symmetric subspaces decompose as

$$
\begin{align*}
\operatorname{Sym}^{2}\left(V_{[N-1,1]}\right) & \cong V_{[N]} \oplus V_{[N-1,1]} \oplus V_{[N-2,2]}  \tag{2.105}\\
\Lambda^{2}\left(V_{[N-1,1]}\right) & \cong V_{[N-2,1,1]} . \tag{2.106}
\end{align*}
$$

Denoting general symmetric group projectors from some representation $\Lambda_{1}$ to another representation $\Lambda_{2}$ as $P^{\Lambda_{1} \rightarrow \Lambda_{2}}$, and reserving $Q^{\Lambda_{2}}$ for projectors from $V_{N} \otimes V_{N}$ to $\Lambda_{2}$. Equation 2.106) gives a straightforward construction of $Q_{i j k l}^{[N-2,1,1]}$ :
$Q_{i j k l}^{[N-2,1,1]}=\sum_{e=1}^{(N-1)(N-2) / 2} C_{e, i j}^{[N-2,1,1]} C_{e, k l}^{[N-2,1,1]}$

$$
\begin{align*}
& =\sum_{a, b, c, d=1}^{N-1} \sum_{e} C_{e, a b}^{[N-1,1] \otimes[N-1,1] \rightarrow[N-2,1,1]} C_{e, c d}^{[N-1,1] \otimes[N-1,1] \rightarrow[N-2,1,1]} C_{a, i} C_{b, j} C_{c, k} C_{d, l} \\
& =\sum_{a, b, c, d=1}^{N-1} P_{a b, c d}^{[N-1,1] \otimes[N-1,1] \rightarrow[N-2,1,1]} C_{a, i} C_{b, j} C_{c, k} C_{d, l} \\
& =\sum_{a, b, c, d=1}^{N-1} P_{a b, c d}^{[N-1,1] \otimes[N-1,1] \rightarrow \Lambda^{2}([N-1,1])} C_{a, i} C_{b, j} C_{c, k} C_{d, l} \\
& =\sum_{a, b, c, d=1}^{N-1} \frac{\left(\delta_{a c} \delta_{b d}-\delta_{a d} \delta_{b c}\right)}{2} C_{a, i} C_{b, j} C_{c, k} C_{d, l} \\
& =\frac{1}{2}(F(i, k) F(j, l)-F(i, l) F(j, k)) . \tag{2.107}
\end{align*}
$$

Similarly, the decomposition 2.104 can be used to calculate the $V_{[N-2,2]}$ projector

$$
\begin{align*}
Q_{i j k l}^{[N-2,2]}= & \sum_{e=1}^{N(N-3) / 2} C_{e, i j}^{[N-2,2]} C_{e, k l}^{[N-2,2]} \\
= & \sum_{a, b, c, d=1}^{N-1} \sum_{e} C_{e, a b}^{[N-1,1] \otimes[N-1,1] \rightarrow[N-2,2]} C_{e, c d}^{[N-1,1] \otimes[N-1,1] \rightarrow[N-2,2]} C_{a, i} C_{b, j} C_{c, k} C_{d, l} \\
= & \sum_{a, b, c, d=1}^{N-1} P_{a b, c d}^{[N-1,1] \otimes[N-1,1] \rightarrow[N-2,2]} C_{a, i} C_{b, j} C_{c, k} C_{d, l} \\
= & \sum_{a, b, c, d=1}^{N-1}\left(\left(1-P^{\operatorname{Sym}^{2}([N-1,1]) \rightarrow[N]}-P^{\mathrm{Sym}^{2}([N-1,1]) \rightarrow[N-1,1]}\right)\right. \\
= & \left.\sum_{a}^{[N-1,1] \otimes[N-1,1] \rightarrow \operatorname{Sym}^{2}([N-1,1])}\right)_{a b, c d} C_{a, i} C_{b, j} C_{c, k} C_{d, l} \\
= & \frac{1}{2}\left(Q_{i j k l}^{\operatorname{Sym}^{2}([N-1,1])}-Q_{i j k l}^{[N], 22}-Q_{i j k l}^{[N-1,1], 33}\right) \\
& \left.-\frac{N}{N-2} \sum_{p, q=1}^{N} F(j, l)+F(i, l) F(j, k)\right)-\frac{1}{N-1} F(i, j) F(k, l)
\end{align*}
$$

where we have used the expressions given in 2.92 and 2.101 in going to the final line.

### 2.6.2 Expectation values

The expectation values of permutation invariant polynomials $f(M)$ are defined by

$$
\begin{equation*}
\langle f(M)\rangle=\frac{1}{\mathcal{Z}_{(13)}} \int \mathrm{d} M f(M) e^{-S_{(13)}} \tag{2.109}
\end{equation*}
$$

The non-vanishing expectation values of linear observables are those that transform like $V_{[N]}$ under the diagonal $S_{N}$. These can be calculated with 2.47) in the usual way: introduce linear generating terms for each of the $S$ variables, take derivatives with respect to the appropriate linear couplings, and then set to zero all but the $S^{[N] ; \alpha}$ linear couplings. This procedure yields

$$
\begin{equation*}
\left\langle S^{[N] ; \alpha}\right\rangle=\sum_{\beta=1}^{2}\left(g_{[N]}^{-1}\right)_{\alpha \beta} \mu_{\beta} \equiv \widetilde{\mu}_{\alpha} \tag{2.110}
\end{equation*}
$$

The expectation values of quadratic observables are given by

$$
\begin{align*}
\left\langle S_{a}^{\Lambda_{1} ; \alpha} S_{b}^{\Lambda_{2} ; \beta}\right\rangle & =\left\langle S_{a}^{\Lambda_{1} ; \alpha} S_{b}^{\Lambda_{2} ; \beta}\right\rangle_{\mathrm{conn}}+\left\langle S_{a}^{\Lambda_{1} ; \alpha}\right\rangle\left\langle S_{b}^{\Lambda_{2} ; \beta}\right\rangle \\
& =\delta\left(\Lambda_{1}, \Lambda_{2}\right) \delta_{a b}\left(g_{\Lambda_{1}}^{-1}\right)_{\alpha \beta}+\left\langle S_{a}^{\Lambda_{1} ; \alpha}\right\rangle\left\langle S_{b}^{\Lambda_{2} ; \beta}\right\rangle \tag{2.111}
\end{align*}
$$

where we have defined the connected two point function

$$
\begin{equation*}
\left\langle S_{a}^{\Lambda_{1} ; \alpha} S_{b}^{\Lambda_{2} ; \beta}\right\rangle_{\mathrm{conn}}=\left\langle S_{a}^{\Lambda_{1} ; \alpha} S_{b}^{\Lambda_{2} ; \beta}\right\rangle-\left\langle S_{a}^{\Lambda_{1} ; \alpha}\right\rangle\left\langle S_{b}^{\Lambda_{2} ; \beta}\right\rangle \tag{2.112}
\end{equation*}
$$

The expectation values of the observables in the original $M_{i j}$ basis can be recovered from these. The linear expectation values are

$$
\begin{align*}
\left\langle M_{i j}\right\rangle & =\sum_{\Lambda_{1}} \sum_{\alpha} \sum_{a=1}^{\operatorname{dim} \Lambda_{1}} C_{a, i j}^{\Lambda_{1}, \alpha}\left\langle S_{a}^{\Lambda_{1}, \alpha}\right\rangle \\
& =C_{i j}^{[N], 1}\left\langle S^{[N], 1}\right\rangle+C_{i j}^{[N], 2}\left\langle S^{[N], 2}\right\rangle \\
& =\frac{\widetilde{\mu}_{1}}{N}+\frac{\widetilde{\mu}_{2}}{\sqrt{N-1}} F(i, j) \tag{2.113}
\end{align*}
$$

Similarly, performing a change of basis for each of the $M$ s appearing in quadratic expec-
tation values gives

$$
\begin{align*}
\left\langle M_{i j} M_{k l}\right\rangle_{\text {conn }} & =\sum_{\Lambda_{1}, \Lambda_{2}} \sum_{\alpha, \beta} \sum_{a=1}^{\operatorname{dim} \Lambda_{1}} \sum_{b=1}^{\operatorname{dim} \Lambda_{2}} C_{a, i j}^{\Lambda_{1}, \alpha} C_{b, k l}^{\Lambda_{2}, \beta}\left\langle S_{a}^{\Lambda_{1}, \alpha} S_{b}^{\Lambda_{2}, \beta}\right\rangle_{\mathrm{conn}} \\
& =\sum_{\Lambda_{1}, \Lambda_{2}} \sum_{\alpha, \beta} \sum_{a=1}^{\operatorname{dim} \Lambda_{1}} \sum_{b=1}^{\operatorname{dim} \Lambda_{2}} C_{a, j}^{\Lambda_{1}, \alpha} C_{b, k l}^{\Lambda_{2}, \beta} \delta\left(\Lambda_{1}, \Lambda_{2}\right) \delta_{a b}\left(g_{\Lambda_{1}}^{-1}\right)_{\alpha \beta} \\
& =\sum_{\Lambda_{1}} \sum_{\alpha, \beta} \sum_{a=1}^{\operatorname{dim} \Lambda_{1}} C_{a, i j}^{\Lambda_{1}, \alpha} C_{a, k l}^{\Lambda_{1}, \beta}\left(g_{\Lambda_{1}}^{-1}\right)_{\alpha \beta} \\
& =\sum_{\Lambda_{1}} \sum_{\alpha, \beta} Q_{i j k l}^{\Lambda_{1}, \alpha \beta}\left(g_{\Lambda_{1}}^{-1}\right)_{\alpha \beta} . \tag{2.114}
\end{align*}
$$

Using the expressions for the $Q \mathrm{~s}$ we have already given, and remembering that they are symmetric in the multiplicity indices (see 2.77) , we can write the two-point function of the $S$ variables as

$$
\begin{align*}
& \left\langle M_{i j} M_{k l}\right\rangle_{\mathrm{conn}}=\frac{1}{N^{2}}\left(g_{[N]}^{-1}\right)_{11}+\frac{\left(g_{[N]}^{-1}\right)_{22}}{(N-1)} F(i, j) F(k, l)+\frac{\left(g_{[N]}^{-1}\right)_{12}}{N \sqrt{N-1}}(F(k, l)+F(i, j)) \\
& +\frac{\left(g_{[N-1,1]}^{-1}\right)_{11}}{N} F(j, l)+\frac{\left(g_{[N-1,1]}^{-1}\right)_{22}}{N} F(i, k) \\
& +\frac{N\left(g_{[N-1,1]}^{-1}\right)_{33}}{(N-2)} \sum_{p, q=1}^{N} F(i, p) F(j, p) F(k, q) F(l, q) F(p, q) \\
& +\frac{\left(g_{[N-1,1]}^{-1}\right)_{12}}{N}(F(j, k)+F(i, l))+\frac{\left(g_{[N-1,1]}^{-1}\right)_{13}}{\sqrt{N-2}}\left(\sum_{p=1}^{N} F(j, p) F(k, p) F(l, p)\right. \\
& +F(i, p) F(j, p) F(l, p))+\frac{\left(g_{[N-1,1]}^{-1}\right)_{23}}{\sqrt{N-2}}\left(\sum_{p=1}^{N} F(i, p) F(k, p) F(l, p)+F(i, p) F(j, p) F(k, p)\right) \\
& +\left(g_{[N-2,2]}^{-1}\right)\left(\frac{1}{2} F(i, k) F(j, l)+\frac{1}{2} F(i, l) F(j, k)\right. \\
& \left.-\frac{N}{N-2} \sum_{p, q=1}^{N} F(i, p) F(j, p) F(k, q) F(l, q) F(p, q)-\frac{1}{(N-1)} F(i, j) F(k, l)\right) \\
& +\frac{\left(g_{[N-2,1,1]}^{-1}\right)}{2}(F(i, k) F(j, l)-F(i, l) F(j, k)) . \tag{2.115}
\end{align*}
$$

This can be used directly to calculate expectation values of the original matrix basis PIMOs as analytic expressions in $N$, the dimension of the matrices. For example a few quadratic
observables are

$$
\begin{align*}
\sum_{i, j}\left\langle M_{i i} M_{j j}\right\rangle & =\widetilde{\mu}_{1}^{2}+2 \widetilde{\mu}_{1} \widetilde{\mu}_{2} \sqrt{N-1}+\widetilde{\mu}_{2}^{2}(N-1)  \tag{2.116}\\
\sum_{i, j, k}\left\langle M_{i j} M_{i k}\right\rangle & =N\left(g_{[N]}^{-1}\right)_{11}+N(N-1)\left(g_{[N-1,1]}^{-1}\right)_{22}+\widetilde{\mu}_{1}^{2} N  \tag{2.117}\\
\sum_{i, j, k, l}\left\langle M_{i j} M_{k l}\right\rangle & =N^{2}\left(g_{[N]}^{-1}\right)_{11}+\widetilde{\mu}_{1}^{2} N^{2} \tag{2.118}
\end{align*}
$$

For a full list of quadratic observables along with some cubic and quartic, accompanied by detailed calculations, see sections 3,4 and 5 of [43].

## Chapter 3

## PIGM models for financial correlations

The PIGM model described in section 2.6 uses an integration over $N \times N$ general matrices and has 13 parameters. An important subspace of these general matrices is that of symmetric matrices with vanishing diagonal. These restricted matrices are relevant to many physical systems, for example correlation matrices are of this type (by subtracting the identity). In this chapter, we study the most general PIGM model containing symmetric matrices with vanishing diagonal. We find there is a reduction of the 13 -parameter model to a four-parameter model, which we explicitly construct, and solve to find analytic formulae in $N$ for the expectation values of permutation invariant polynomials of the matrix variables, which form the observables of the theory. As an initial application of this model, we construct and analyse financial correlation matrices for a sequence of days obtained by calculating correlations between price movements in high frequency foreign exchange (forex) market data. The correlation matrices are symmetric and have vanishing matrix elements along the diagonal (by subtracting the identity). The PIGM model we define is used to demonstrate approximate permutation invariant Gaussianity in this ensemble of forex correlation matrices.

This application of PIGM models is in part motivated by a rich history of applying random matrix theory (RMT) to the study of financial correlation matrices. In particular, the eigenvalue distributions of these matrices have been studied, demonstrating close agreement between the majority of eigenvalues and the eigenvalue distribution as given by the so-called Marchenko-Pastur (M-P) law [69] applied to random correlation matrices [70, 71, 72]. Evidence has also been presented that the largest eigenvalues - those that deviate most strongly from the M-P distribution - are associated with non-random overall market, sector
and stock correlation structure (see [72] for example). Practical applications of these findings have been developed such as cleaning/de-noising correlation matrices, amongst others [73, 74, 75, 76].

The PIGM model provides a new approach to studying and describing financial correlation matrices that is distinct from existing approaches based on RMT. It focuses on low degree permutation invariant polynomial functions of matrices (observables) instead of eigenvalue distributions, which are the focus of traditional RMT. This perspective is based on the postulate that near-Gaussian permutation invariant sectors of real world matrix data contain useful information. The PIGM model furnishes a parsimonious specification of the probability density function of these matrices using only four free parameters for the symmetric, vanishing diagonal model. This is close to the one or two parameters of the simplest RMT and far smaller than a multi-variate Gaussian distribution for an $N \times N$ matrix, which has order $N^{2}$ parameters. It provides an analytic solution to the expectation values of permutation invariant products of matrix elements. The empirical higher order observables (cubic, quartic etc.) that agree closely with the model - which is only fit to linear and quadratic observables - reveal consistency with random matrices implied by the fitted model. The empirical higher order observables that depart from theoretical expectations indicate informative structure beyond that encoded in the random model. A vector of observables therefore provides a signature for a particular correlation matrix, which may provide a useful, low dimensional, permutation invariant representation of correlation matrices. The effectiveness of this representation in anomaly detection is explored as an initial example.

Motivation for investigating the broad theme of Gaussianity within a financial setting comes from the known statistical properties of correlation matrices. Existing results establish the asymptotic $N^{2}$-variate Gaussianity of the sampling distribution of $N \times N$ dimensional correlation matrices under fairly general conditions for example (related to finite fourth moments of the underlying observations from which the correlation matrices are constructed, see [77], Theorem 3.4.4 and subsequent comments). This differs from our non-asymptotic i.e. finite sample setting, but does imply that the distribution of ensembles of correlation matrices approaches a multivariate Gaussian in the large observation sample limit.

In section 3.1 we summarise the theoretical results concerning PIGM models contained within the rest of this chapter. We define general permutation invariant Gaussian matrix models and consider the restriction of these models appropriate to the financial data described in section 3.3, namely that the matrices must be symmetric and have vanishing diagonal elements. We also define the permutation invariant observables of the model and explain a useful bijection between these observables and loopless graphs, examples of which
are given.

Section 3.2 contains the bulk of the theory, the primary goal is to solve the most general PIGM model of symmetric matrices with vanishing diagonal. This is achieved with the help of representation theory of the symmetric group and builds on the results of [43] and [1]. We find that these models are characterised by one linear and three quadratic couplings. Linear and quadratic expectation values of observables can be expressed simply in terms of these coupling parameters, 3.75 and 3.74 respectively. Higher order expectation values are simply constructed from these with the application of Wick's theorem.

Section 3.3 gives details of the high-frequency forex data used to construct the matrix ensemble studied in the remainder of the chapter, as well as the method by which the members of this ensemble are constructed.

Section 3.4 contains a description of the empirical statistical properties of the observables. This includes practical measures of their Gaussianity and comparison of their properties with those predicted by the model presented in section 3.2 .

In section 3.5 we construct vectors of observables for each correlation matrix. These observable vectors are low dimensional representations of the correlation matrices. There are 31 cubic and quartic observables for general matrix size $N$ (as long as $N \geq 8$, a condition which is generally satisfied in large $N$ applications such as the one here). In general, we find that the observable vectors provide a good representation of the original correlation matrices, performing well in anomaly detection applications. The best performances are achieved by selecting subsets of the cubic and quartic observables, based on the ranking of their small non-Gaussianities, and on the postulate that the more non-Gaussian observables are most informative of economic factors driving atypicality of the days. The performance of observable vectors in these applications compares favourably with standard dimensionality reduction techniques, namely, Principal Component Analysis (PCA). We conclude in section 3.6 .

### 3.1 Summary of results on the 4-parameter Gaussian matrix model

Here we summarise the main technical results of this chapter and outline the key ideas behind the construction of the general PIGM model for an ensemble of symmetric matrices which have vanishing matrix elements along the diagonal. This section is intended to provide an understanding of the key theoretical points of the chapter, without getting into the details of the construction of section 3.2 . We will review the description of probability
distributions using a Euclidean action which is Gaussian or near-Gaussian, using the simple case of a one-variable statistics and motivate the measure of non-Gaussianity we use later in the case of permutation invariant matrix distributions. We explain the structure of the 4 -parameter PIGM models and the connection between the permutation invariant polynomial functions of the matrices with loopless graphs.

It is useful to recall that a one-variable Gaussian distribution, with mean $\mu$ and standard deviation $\sigma$, for a random variable $x$, is described by a probability density function

$$
\begin{equation*}
f(x)=\frac{1}{\sigma \sqrt{2 \pi}} e^{-\frac{(x-\mu)^{2}}{2 \sigma^{2}}} . \tag{3.1}
\end{equation*}
$$

The moments of the distribution are expectation values $\left\langle x^{k}\right\rangle$, defined as

$$
\begin{equation*}
\left\langle x^{k}\right\rangle=\int_{-\infty}^{\infty} d x f(x) x^{k} \tag{3.2}
\end{equation*}
$$

It is also useful to define, by analogy with statistical physics and quantum mechanical path integrals, the action $S=\frac{(x-\mu)^{2}}{2 \sigma^{2}}$. The partition function is

$$
\begin{equation*}
Z=\int d x e^{-S} \tag{3.3}
\end{equation*}
$$

while the moments are

$$
\begin{equation*}
\left\langle x^{k}\right\rangle=\frac{1}{Z} \int d x e^{-S} x^{k} \tag{3.4}
\end{equation*}
$$

The action $S$ is a quadratic function of $x$.
It is often the case that the action of a theory is approximately Gaussian - deviating from Gaussianity by some small higher order terms. The full action of the system $S^{\prime}$ can then be written as a Gaussian piece $S$, plus an additional non-Gaussian piece $\delta S$

$$
\begin{equation*}
S^{\prime}=S+\lambda \delta S \tag{3.5}
\end{equation*}
$$

The smallness of the higher order terms is governed by the interaction strength $\lambda$, whose smallness is required to ensure

$$
\begin{equation*}
\left\langle\mathcal{O}_{\alpha}\right\rangle_{S}-\left\langle\mathcal{O}_{\alpha}\right\rangle_{S^{\prime}}<\sigma_{\left\langle\mathcal{O}_{\alpha}\right\rangle_{S^{\prime}}}, \tag{3.6}
\end{equation*}
$$

i.e. the expectation value of some observable $\mathcal{O}_{\alpha}$, is largely insensitive to the non-Gaussian contribution to the true action governing the theory, $\delta S$.

More concretely, take $\mu=0$ in the simple, pure Gaussian, one parameter toy model defined
above. The partition function is

$$
\begin{equation*}
Z=\int \mathrm{d} x e^{-S}=\int \mathrm{d} x e^{-\frac{x^{2}}{2 \sigma^{2}}} \tag{3.7}
\end{equation*}
$$

View this as an approximation to some true physical partition function, which includes some small non-Gaussian perturbation. We explain by way of a simple example which captures the mechanism, that the absolute differences between expectation values of low order polynomials in the random variables (observables) in the Gaussian model and the perturbed model are small compared to the standard deviation of the observable. The simple example consists of a Gaussian action perturbed by a small quartic correction, so that the perturbed partition function $Z^{\prime}$ reads

$$
\begin{equation*}
Z^{\prime}=\int \mathrm{d} x e^{-S^{\prime}}=\int \mathrm{d} x e^{-\frac{x^{2}}{2 \sigma^{2}}-\frac{\lambda}{4!} x^{4}}=Z\left(1-\frac{\lambda}{4!}\left\langle x^{4}\right\rangle+\ldots\right) . \tag{3.8}
\end{equation*}
$$

Using (3.4) we calculate the absolute difference between the fourth moment of each of the theories. In the purely Gaussian case we have

$$
\begin{equation*}
\left\langle x^{4}\right\rangle=3 \sigma^{4}, \tag{3.9}
\end{equation*}
$$

and in the perturbed theory

$$
\begin{align*}
\left\langle x^{4}\right\rangle_{S^{\prime}}=\frac{Z}{Z^{\prime}}\left(\left\langle x^{4}\right\rangle-\frac{\lambda}{4!}\left\langle x^{8}\right\rangle+\ldots\right) & \approx\left\langle x^{4}\right\rangle+\frac{\lambda}{4!}\left(\left\langle x^{4}\right\rangle^{2}-\left\langle x^{8}\right\rangle\right), \\
& =3 \sigma^{4}+\frac{\lambda}{4!}\left(9 \sigma^{8}-105 \sigma^{8}\right) \\
& =3 \sigma^{4}-\frac{96}{4!} \lambda \sigma^{8} . \tag{3.10}
\end{align*}
$$

Taking the difference of these values

$$
\begin{equation*}
\left|\left\langle x^{4}\right\rangle-\left\langle x^{4}\right\rangle_{S^{\prime}}\right| \approx \frac{96}{4!} \lambda \sigma^{8} . \tag{3.11}
\end{equation*}
$$

The standard deviation of the fourth moment in the perturbed theory requires

$$
\begin{align*}
\left\langle x^{8}\right\rangle_{S^{\prime}}=\frac{Z}{Z^{\prime}}\left(\left\langle x^{8}\right\rangle-\frac{\lambda}{4!}\left\langle x^{12}\right\rangle+\ldots\right) & \approx\left\langle x^{8}\right\rangle+\frac{\lambda}{4!}\left(\left\langle x^{8}\right\rangle\left\langle x^{4}\right\rangle-\left\langle x^{12}\right\rangle\right) \\
& =105 \sigma^{8}+\frac{\lambda}{4!}\left(315 \sigma^{12}-11!!\sigma^{12}\right) \\
& =105 \sigma^{8}-\frac{(11!!-315)}{4!} \lambda \sigma^{12} \tag{3.12}
\end{align*}
$$

Which gives

$$
\begin{align*}
\sigma_{\left\langle x^{4}\right\rangle_{S^{\prime}}} & =\sqrt{\left\langle x^{8}\right\rangle_{S^{\prime}}-\left\langle x^{4}\right\rangle_{S^{\prime}}^{2}}+\mathcal{O}(\sqrt{\lambda}) \\
& =\sqrt{105 \sigma^{8}-\left(3 \sigma^{4}\right)^{2}} \\
& \approx \sqrt{96} \sigma^{4} \tag{3.13}
\end{align*}
$$

Finally, we see that the absolute difference between the fourth moments normalised by the standard deviation is

$$
\begin{equation*}
\frac{\left|\left\langle x^{4}\right\rangle-\left\langle x^{4}\right\rangle_{S^{\prime}}\right|}{\sigma_{\left\langle x^{4}\right\rangle_{S^{\prime}}}} \sim \lambda \sigma^{4} \tag{3.14}
\end{equation*}
$$

Therefore, as long as the physical theory is approximately Gaussian, i.e. $\lambda$ is small, its normalised fourth moment is well approximated by that of the purely Gaussian theory.

We postulate that real market effects governing the interactions between currency rates included in this study are modelled analogously by a Gaussian action plus some small nonGaussian perturbation. The smallness of the non-Gaussian terms allows us to approximate expectation values using a purely Gaussian theory. Evidence for this near-Gaussianity is provided primarily by the smallness of the measured observable deviations from those predicted by a purely Gaussian theory. These are listed in table 3.4 of section 3.4 .

We show in section 3.2 that a general PIGM model for symmetric matrices is a 9-parameter model, and for symmetric matrices with diagonally vanishing matrix elements the permutation invariant Gaussian model is a 4-parameter model. The 9-dimensional parameter space for the symmetric matrices and the 4-dimensional parameter space for the symmetric diagonally vanishing matrices are subspaces of the 13-dimensional parameter space for generic matrices. The embedding of the 4-dimensional parameter space in the 13-parameter space is described in section 3.2 ,

An important ingredient in understanding permutation invariant random matrix models is the structure of the permutation invariant polynomial functions of matrices, which are closely related to graphs. For symmetric matrices with vanishing diagonal elements there is one linear invariant function and three quadratic invariant functions, these are





Graphs are associated with these polynomials, however in contrast to the case of general matrices 2.58), the edges are no longer directed - a consequence of the symmetry of the
matrices - and they no longer contain loops - a consequence of their vanishing diagonal elements.

For a fixed degree, the permutation invariant polynomial functions form a vector space. As long as the matrix dimension is larger than twice the degree of the polynomial the graphs are in one-to-one correspondence with basis elements of this vector space. In our present financial application this degree condition is always satisfied. A detailed discussion of this condition and the effects of going beyond it are given in [1].

The action of the reduced PIGM model is given by the most general combination of permutation invariant linear and quadratic terms

$$
\begin{equation*}
\mathcal{S}^{\mathrm{FX}}=\mu \sum_{i, j=1}^{N} M_{i j}+g_{1} \sum_{i, j=1}^{N} M_{i j} M_{i j}+g_{2} \sum_{i, j, k=1}^{N} M_{i j} M_{j k}+g_{3} \sum_{i, j, k, l=1}^{N} M_{i j} M_{k l} \tag{3.16}
\end{equation*}
$$

where $\mu$ is the linear coupling strength and $g_{1}, g_{2}$ and $g_{3}$ are the quadratic couplings. Label observables of this theory $\mathcal{O}_{\alpha}\left(M_{i j}\right)$, where $\alpha$ indexes the particular observable, they are permutation invariant polynomials of the random matrix variables of symmetric matrices with vanishing diagonal. Expectation values of these variables are defined as

$$
\begin{equation*}
\left\langle\mathcal{O}_{\alpha}\left(M_{i j}\right)\right\rangle=\frac{\int \mathrm{d} M \mathcal{O}_{\alpha} e^{-\mathcal{S}^{\mathrm{FX}}}}{\int \mathrm{~d} M e^{-\mathcal{S}^{\mathrm{FX}}}} \tag{3.17}
\end{equation*}
$$

In order to solve (3.17) for any choice of $\mathcal{O}_{\alpha}$ we must find a change of basis that factorises the RHS. This is possible with the application of appropriate projectors $Q^{\text {phys }}$. Given these the action for the 4-parameter model can be written

$$
\begin{align*}
\mathcal{S}^{\mathrm{FX}}= & \sum_{i, j, k, l=1}^{N} \frac{1}{2}\left(g_{[N]} M_{i j} Q_{i j k l}^{\mathrm{phys} ;[N]} M_{k l}+g_{[N-1,1]} M_{i j} Q_{i j k l}^{\mathrm{phys} ;[N-1,1]} M_{k l}+g_{[N-2,2]} M_{i j} Q_{i j k l}^{\mathrm{phys} ;[N-2,2]} M_{k l}\right) \\
& -\sum_{i, j=1}^{N} \mu_{[N]} C_{i j}^{\mathrm{phys} ;[N]} M_{i j} \tag{3.18}
\end{align*}
$$

Where $\mu_{[N]}, g_{[N]}, g_{[N-1,1]}$ and $g_{[N-2,2]}$ are the couplings in the transformed basis. Performing the projections in 3.18 allows for the solution of the linear and quadratic expectation values via standard techniques of Gaussian integration. Cubic, quartic and higher expectation values can be calculated with the application of Wick's theorem, which allows them to be expressed as sums of products of linear and quadratic expectation values.

The equations relating the 13 -parameters appearing in the action 2.81 to the 4 -parameters of the physical action (3.18) are given in (3.81) and (3.82). As expected both models give consistent results for the expectation values of physical observables.

To determine how well this PIGM model predicts the statistics of the forex correlation data described in section 3.3 we first use the correlation matrices to define the Gaussian model i.e. to fix the linear and quadratic couplings of the model. We then calculate the theoretical expectation values $\left\langle\mathcal{O}_{\alpha}(M)\right\rangle_{\mathrm{T}}$ defined by (3.17) using this action. These are then compared to the experimental expectation values $\left\langle\mathcal{O}_{\alpha}(M)\right\rangle_{\mathrm{E}}$ calculated from the financial correlation matrices themselves using the following similarity measure

$$
\begin{equation*}
\Delta_{\alpha}=\frac{\left|\left\langle\mathcal{O}_{\alpha}(M)\right\rangle_{\mathrm{T}}-\left\langle\mathcal{O}_{\alpha}(M)\right\rangle_{\mathrm{E}}\right|}{\sigma_{E, \alpha}(M)}, \tag{3.19}
\end{equation*}
$$

where $\sigma_{E, \alpha}$ is the standard deviation of the expectation value with respect to the ensemble of correlation matrices. This measure of similarity is used to identify the observables which deviate most significantly from Gaussianity. In section 3.5 these least Gaussian observables are shown to be the optimal candidates for data reduction in a variety of anomaly detection tests.

### 3.2 4-parameter Gaussian model: detailed construction

In this section we give a detailed account of the construction of the 4 -parameter PIGM model which was outlined in section 3.1. The aim is to model the statistics of the ensemble of correlation matrices introduced in section 3.3. In addition this model should be applicable to any matrix ensemble in the same universality class. This class is composed of symmetric square matrices with zeros along the diagonal, for which physical quantities are invariant under simultaneous permutations of the rows and columns. Given the universality of the model we label our matrices $M$ and index them with lowercase indices $1 \leq i, j \leq N$, to distinguish them from the financial correlation matrices specifically, which we label $\hat{\rho}$. We will refer to matrices with zeros on the diagonal as "diagonally vanishing".

We begin by defining the action of permutations on the matrix variables and establishing their irreducible decomposition under this group action. We then define the most general action of a PIGM model, containing diagonally vanishing symmetric matrices. It is parameterised by one linear and three quadratic couplings. Finding projectors that project from the original matrix basis $M_{i j}$ to a basis transforming according to this irreducible decomposition allows us to rewrite this action in a diagonalised form. In turn, this diagonalisation permits the use of standard multi-dimensional Gaussian integration techniques which, along with the application of Wick's theorem, produce analytic formulae for the expectation values of observables as a function of $N$ - the dimension of the matrices.

### 3.2.1 Symmetric group representation theory and matrix variables

Recall the diagonal action of $S_{N}$, which simultaneously permutes the rows and columns of a matrix

$$
\begin{equation*}
M_{i j} \rightarrow M_{\sigma(i) \sigma(j)}, \quad \forall \sigma \in S_{N} \tag{3.20}
\end{equation*}
$$

We refer to the space of symmetric matrices with vanishing diagonal as the physical subspace $V^{\text {phys }}$ of general $N \times N$ matrices and label matrices in the space with a superscript i.e. $M^{\text {phys }} \in V^{\text {phys }}$. They obey the conditions

$$
\begin{equation*}
M_{i j}^{\text {phys }}=M_{j i}^{\text {phys }}, \quad M_{i i}^{\text {phys }}=0, \quad 1 \leq i, j \leq N \tag{3.21}
\end{equation*}
$$

Since these conditions are $S_{N}$ equivariant with respect to the action defined in (3.20) the physical subspace is invariant under $S_{N}$

$$
\begin{equation*}
M_{i j} \in V^{\text {phys }} \Rightarrow M_{\sigma(i) \sigma(j)} \in V^{\text {phys }}, \quad \forall \sigma \in S_{N} \tag{3.22}
\end{equation*}
$$

By physical we mean only to restrict to the non-trivial data of interest. In the correlation matrices described in section 3.3 all the data is contained within symmetric matrices with vanishing diagonal.

The space of diagonally vanishing symmetric matrices form a subspace of the representations in the decomposition 2.63 . Firstly, symmetric matrices transform as $\operatorname{Sym}^{2}\left(V_{N}\right)$. This is a reducible representation with the following decomposition

$$
\begin{equation*}
\operatorname{Sym}^{2}\left(V_{N}\right) \cong 2 V_{[N]} \oplus 2 V_{[N-1,1]} \oplus V_{[N-2,2]} \tag{3.23}
\end{equation*}
$$

The matrix elements along the diagonal $\left\{M_{i i} \mid 1 \leq i \leq N\right\}$ transform like the natural representation $V_{N}$. Removing a copy of $V_{N} \cong V_{[N]} \oplus V_{[N-1,1]}$ from the symmetric product of $V_{N}$ in (3.23) gives the following decomposition of the physical subspace

$$
\begin{equation*}
V^{\text {phys }} \cong \operatorname{Sym}^{2}\left(V_{N}\right) / V_{N} \cong V_{[N]} \oplus V_{[N-1,1]} \oplus V_{[N-2,2]} \tag{3.24}
\end{equation*}
$$

The decomposition (3.24) tells us that the enforcement of permutation invariance on the action of a Gaussian theory containing symmetric $N$ dimensional matrices without diagonal permits a single independent linear term (i.e. the number of trivial representations appearing on the RHS). Quadratic products of physical matrices transform as

$$
\begin{equation*}
V^{\text {phys }} \otimes V^{\text {phys }} \cong\left(V_{[N]} \oplus V_{[N-1,1]} \oplus V_{[N-2,2]}\right) \otimes\left(V_{[N]} \oplus V_{[N-1,1]} \oplus V_{[N-2,2]}\right) \tag{3.25}
\end{equation*}
$$

Using the orthogonality property of characters, as well as the reality property of irreducible representations of the symmetric group, it can be shown that the tensor product of two irreducible representations contains the trivial representation if and only if the irreducible representations are identical - and in this case the decomposition contains exactly one copy of the trivial representation. This enables us to count the number of independent quadratic terms. We find three independent quadratic contributions to the action corresponding to the following three terms in 3.25

$$
\begin{align*}
V_{[N]} \otimes V_{[N]} & \cong V_{[N]}+\ldots,  \tag{3.26}\\
V_{[N-1,1]} \otimes V_{[N-1,1]} & \cong V_{[N]}+\ldots,  \tag{3.27}\\
V_{[N-2,2]} \otimes V_{[N-2,2]} & \cong V_{[N]}+\ldots \tag{3.28}
\end{align*}
$$

Much of our task in solving the physical model for diagonally vanishing symmetric matrices amounts to finding a change of basis for $V_{N} \otimes V_{N}$ from the original $e_{i} \otimes e_{j}$ to one which transforms in the same manner as the irreducible decomposition of $V^{\text {phys }}$, i.e. from the LHS of (3.24) to the RHS. Once found, this diagonalises the physical action and consequently permits the calculation of expectation values of observables. The coefficients that define this change of basis are called Clebsch-Gordon coefficients. Define the following Clebsch-Gordon coefficients $C_{i j}^{\text {phys; }[N]}, C_{i j, a}^{\text {phys; }[N-1,1]}, C_{i j, a}^{\text {phys; }[N-2,2]}$, one for each irreducible representation on the RHS of $(3.24)$ respectively, where $a$ is a state index running over the dimension of the irreducible representation.

Note, that if we only imposed the condition of symmetry $M_{i j}=M_{j i}$ on the matrices, we would have two linear couplings, corresponding to the two copies of $V_{[N]}$ in 3.23). We would have three parameters of a $2 \times 2$ symmetric matrix of couplings for quadratic terms arising from the two copies of $V_{[N]}$ in 3.23 , three parameters of a $2 \times 2$ symmetric matrix of couplings for quadratic terms arising from the two copies of $V_{[N-1,1]}$ in (3.23), and finally one parameter for $V_{[N-2,2]}$. For symmetric matrices, therefore, there is a nine parameter family of PIGM models. We will focus, in the following, on the four parameter models which incorporate the symmetry condition $M_{i j}=M_{j i}$ as well as the condition of vanishing diagonal.

### 3.2.2 Projectors for $V^{\text {phys }}$

The projectors to the trivial representations appearing in the quadratic products of $M_{i j}$ are given by squaring the relevant Clebsch coefficients and summing over intermediate states,
as in equation 2.74 , i.e.

$$
\begin{align*}
Q_{i j k l}^{\mathrm{phys} ;[N]} & =C_{i j}^{\mathrm{phys} ;[N]} C_{k l}^{\mathrm{phys} ;[N]},  \tag{3.29}\\
Q_{i j k l}^{\mathrm{phys} ;[N-1,1]} & =\sum_{a=1}^{N-1} C_{i j, a}^{\mathrm{phys} ;[N-1,1]} C_{k l, a}^{\mathrm{phys} ;[N-1,1]},  \tag{3.30}\\
Q_{i j k l}^{\mathrm{phys} ;[N-2,2]} & =\sum_{a=1}^{N(N-3) / 2} C_{i j, a}^{\mathrm{phys} ;[N-2,2]} C_{k l, a}^{\mathrm{phys} ;[N-2,2]} \tag{3.31}
\end{align*}
$$

In this section we find explicit formulae for these projectors. The $V_{[N]}$ and $V_{[N-1,1]}$ projectors are constructed by finding the Clebschs on the RHS of 3.29 and 3.30 explicitly. In the case of the $V_{[N-2,2]}$ projector things are not so simple, as the Clebsch is not so easily to calculate, none-the-less we are able to construct the projector using general properties of Clebsch coefficients and other known projectors, bypassing the need for knowledge of the $V_{[N-2,2]}$ Clebsch. Thankfully we have seen this before: the $V_{[N-2,2]}$ projector is given by 2.108

To find $C_{i j}^{\text {phys; }[N]}$ and $C_{i j, a}^{\text {phys; }[N-1,1]}$ we first write down a representation theory basis for $V_{N} \otimes V_{N}$ given by the variables (2.67), in terms of the change of basis coefficients given in (2.84), 2.85 and (2.86) - 2.88) as was done in 43] i.e. a basis that transforms like the RHS of (2.63),

$$
\begin{align*}
S^{V_{[N]} ; 1} & \equiv \sum_{i, j=1}^{N} C_{0, i} C_{0, j} M_{i j}=\frac{1}{N} \sum_{i, j=1}^{N} e_{i} \otimes e_{j}  \tag{3.32}\\
S^{V_{[N]} ; 2} & \equiv \frac{1}{\sqrt{N-1}} \sum_{a=1}^{N-1} \sum_{i, j=1}^{N} C_{a, i} C_{a, j} M_{i j}=\frac{1}{\sqrt{N-1}} \sum_{a=1}^{N-1} E_{a} \otimes E_{a}  \tag{3.33}\\
S_{a}^{V_{[N-1,1] ; 1}} \equiv & \sum_{i, j=1}^{N} C_{0, i} C_{a, j} M_{i j}=\frac{1}{\sqrt{N}} \sum_{i=1}^{N} e_{i} \otimes E_{a}  \tag{3.34}\\
S_{a}^{V_{[N-1,1]} ; 2} \equiv & \sum_{i, j=1}^{N} C_{a, i} C_{0, j} M_{i j}=\frac{1}{\sqrt{N}} \sum_{i=1}^{N} E_{a} \otimes e_{i}  \tag{3.35}\\
S_{a}^{V_{[N-1,1]} ; 3} \equiv & \sqrt{\frac{N}{N-2}} \sum_{b, c=1}^{N-1} \sum_{i, j, k=1}^{N} C_{a, k} C_{b, k} C_{c, k} C_{b, i} C_{c, j} M_{i j} \\
& =\sqrt{\frac{N}{N-2}} \sum_{b, c=1}^{N-1} \sum_{i=1}^{N} C_{a, i} C_{b, i} C_{c, i} E_{b} \otimes E_{c} \tag{3.36}
\end{align*}
$$

We also note the orthogonal decomposition

$$
\begin{equation*}
M_{i j} \cong V_{N} \otimes V_{N} \cong \operatorname{Sym}^{2}\left(V_{N}\right) \oplus \Lambda^{2}\left(V_{N}\right) \cong V^{\text {phys }} \oplus V^{\text {diag }} \oplus \Lambda^{2}\left(V_{N}\right) \tag{3.37}
\end{equation*}
$$

in which $V^{\text {diag }}$ is the subspace of diagonal matrix elements and $\Lambda^{2}\left(V_{N}\right)$ is the antisymmetric subspace of $V_{N} \otimes V_{N}$.

Define further representation variables $S^{\text {diag; } V_{[N]}}$ and $S_{a}^{\text {diag; } V_{[N-1,1]}}$ composed of the diagonal elements of $M_{i j}$, that transform according to the first and second terms on the RHS of the $V^{\text {diag }}$ decomposition respectively

$$
\begin{align*}
S^{\mathrm{diag} ; V_{[N]}} & \equiv \frac{1}{\sqrt{N}} \sum_{i=1}^{N} e_{i} \otimes e_{i}  \tag{3.38}\\
S_{a}^{\mathrm{diag} ; V_{[N-1,1]}} & \equiv E_{a} \otimes E_{a} \tag{3.39}
\end{align*}
$$

Using the inner product on $V_{N} \otimes V_{N}$ in 2.70 we can express these in terms of the original representation variables (3.32) - 3.36

$$
\begin{align*}
S^{\operatorname{diag} ; V_{[N]}} & =\left(S^{\mathrm{diag} ; V_{[N]}}, S^{V_{[N]} ; 1}\right) S^{V_{[N]} ; 1}+\left(S^{\mathrm{diag} ; V_{[N]}}, S^{V_{[N]} ; 2}\right) S^{V_{[N]} ; 2} \\
& =\frac{1}{\sqrt{N}} S^{V_{[N] ; 1}}+\sqrt{\frac{N-1}{N}} S^{V_{[N]} ; 2} \tag{3.40}
\end{align*}
$$

and

$$
\begin{align*}
S_{a}^{\operatorname{diag} ; V_{[N-1,1]}}= & \frac{1}{2} \sum_{b=1}^{N-1}\left(\left(S_{a}^{\mathrm{diag} ; V_{[N-1,1]}}, S_{b}^{V_{[N-1,1]} ; 1}\right) S_{b}^{V_{[N-1,1]} ; 1}+\left(S_{a}^{\operatorname{diag} ; V_{[N-1,1]}}, S_{b}^{V_{[N-1,1]} ; 2}\right) S_{b}^{V_{[N-1,1]} ; 2}\right) \\
& +\sum_{b=1}^{N-1}\left(S_{a}^{\operatorname{diag} ; V_{[N-1,1]}}, S_{b}^{V_{[N-1,1]} ; 3}\right) S_{b}^{V_{[N-1,1]} ; 3} \\
= & \frac{1}{2} \sum_{b=1}^{N-1}\left(\sqrt{\frac{2}{N}} \delta_{a b} S_{b}^{V_{[N-1,1]} ; 1}+\sqrt{\frac{2}{N}} \delta_{a b} S_{b}^{V_{[N-1,1]} ; 2}\right)+\sum_{b=1}^{N-1} \sqrt{\frac{N-2}{N}} \delta_{a b} S_{b}^{V_{[N-1,1]} ; 3} \\
= & \frac{1}{\sqrt{2 N}}\left(S_{a}^{V_{[N-1,1]} ; 1}+S_{a}^{V_{[N-1,1]} ; 2}\right)+\sqrt{\frac{N-2}{N}} S_{a}^{V_{[N-1,1]} ; 3} \tag{3.41}
\end{align*}
$$

Detailed calculations of the inner products appearing in these expressions can be found in appendix A. The physical variables $S^{\text {phys } ; V_{[N]}}$ and $S_{a}^{\text {phys } ; V_{[N-1,1]}}$ span the orthogonal complement of the diagonal variables in the $V_{[N]}$ and $V_{[N-1,1]}$ subspaces (as given in 3.38)
of $\operatorname{Sym}^{2}\left(V_{N}\right)$

$$
\begin{align*}
S^{\mathrm{phys} ; V_{[N]}} & =\sqrt{\frac{N-1}{N}} S^{V_{[N]} ; 1}-\frac{1}{\sqrt{N}} S^{V_{[N] ; 2}} \\
& =\sqrt{\frac{N-1}{N}} \sum_{i, j=1}^{N} C_{0, i} C_{0, j} M_{i j}-\frac{1}{\sqrt{N(N-1)}} \sum_{a=1}^{N-1} \sum_{i, j=1}^{N} C_{a, i} C_{a, j} M_{i j}  \tag{3.42}\\
S_{a}^{\text {phys } ; V_{[N-1,1]}} & =\sqrt{\frac{N-2}{2 N}}\left(S_{a}^{V_{[N-1,1]} ; 1}+S_{a}^{V_{[N-1,1]} ; 2}\right)-\sqrt{\frac{2}{N}} S_{a}^{V_{[N-1,1] ; 3}} \\
& =\sqrt{\frac{N-2}{2 N^{2}}} \sum_{i, j=1}^{N}\left(C_{a, i}+C_{a, j}\right) M_{i j}-\sqrt{\frac{2}{N}} \sum_{i, j=1}^{N} \sum_{b, c=1}^{N-1} C_{b, i} C_{c, j} C_{b, c}^{[N-1,1][N-1,1] \rightarrow[N-1,1]} M_{i j} . \tag{3.43}
\end{align*}
$$

From (3.42) and 3.43 we can read off the Clebsch coefficients needed in the construction of the projectors 3.29 and 3.30 :

$$
\begin{equation*}
S^{\mathrm{phys} ; V_{[N]}}=\sum_{i, j} C_{i j}^{\mathrm{phys} ;[N]} M_{i j} \Rightarrow C_{i j}^{\mathrm{phys} ;[N]}=\sqrt{\frac{N-1}{N}} C_{0, i} C_{0, j}-\frac{1}{\sqrt{N(N-1)}} \sum_{a=1}^{N-1} C_{a, i} C_{a, j}, \tag{3.44}
\end{equation*}
$$

$$
\begin{align*}
S_{a}^{\mathrm{phys} ; V_{[N-1,1]}}=\sum_{i, j} C_{i j, a}^{\mathrm{phys} ;[N-1,1]} M_{i j} \Rightarrow C_{i j, a}^{\mathrm{phys} ;[N-1,1]} & =\sqrt{\frac{(N-2)}{2 N^{2}}}\left(C_{a, i}+C_{a, j}\right) \\
& -\sqrt{\frac{2}{(N-2)}} \sum_{b, c=1}^{N-1} \sum_{k=1}^{N} C_{b, i} C_{c, j} C_{a, k} C_{b, k} C_{c, k} \tag{3.45}
\end{align*}
$$

The action of the physical $V_{[N]}, V_{[N-1,1]}$ projectors, given by the square of the Clebsch coefficients in (3.44) and 3.45, along with the $V_{[N-2,2]}$ projector given in 2.108 can be found by acting on a generic state $e_{i} \otimes e_{j}$ in $V_{N} \otimes V_{N}$. Doing so leaves us with the following delta expressions

$$
\begin{align*}
Q_{i j k l}^{\mathrm{phys} ;[N]} & =\frac{1}{N(N-1)}\left(\delta_{i j}-1\right)\left(\delta_{k l}-1\right),  \tag{3.46}\\
Q_{i j k l}^{\mathrm{phys} ;[N-1,1]} & =\frac{1}{2(N-1)}\left(1-\delta_{i j}\right)\left(1-\delta_{k l}\right)\left(\delta_{i k}+\delta_{i l}+\delta_{j k}+\delta_{j l}-\frac{4}{N}\right),  \tag{3.47}\\
Q_{i j k l}^{\mathrm{phys} ;[N-2,2]} & =\frac{1}{N-2}\left(-N \delta_{i j} \delta_{j k} \delta_{k l}+\delta_{i j} \delta_{i k}+\delta_{i j} \delta_{i l}+\delta_{i k} \delta_{i l}+\delta_{j k} \delta_{j l}\right. \\
& \left.+\frac{1}{N-1}\left(\delta_{i j}-1\right)\left(\delta_{k l}-1\right)-\frac{1}{2}\left(\delta_{i k}+\delta_{i l}+\delta_{j k}+\delta_{j l}\right)\right) . \tag{3.48}
\end{align*}
$$

As a simple check of these expressions we write down the projector from $V_{N} \otimes V_{N}$ to the entire physical subspace. This projects general $N \times N$ matrices onto the space of symmetric matrices with vanishing diagonal. The action of this projector on a general state $e_{k} \otimes e_{l}$ can be written using the inner product 2.70,

$$
\begin{align*}
Q^{\mathrm{phys}} e_{k} \otimes e_{l} & =\frac{1}{2} \sum_{i<j}^{N} e_{i} \otimes e_{j}\left(e_{i} \otimes e_{j}, e_{k} \otimes e_{l}\right) \\
& =\frac{1}{2}\left(e_{k} \otimes e_{l}+e_{l} \otimes e_{k}\right)-\delta_{k l}\left(e_{k} \otimes e_{l}\right) . \tag{3.49}
\end{align*}
$$

It can be written as a delta expression as

$$
\begin{equation*}
Q_{i j k l}^{\text {phys }}=\frac{1}{2}\left(\delta_{i k} \delta_{j l}+\delta_{i l} \delta_{j k}\right)-\delta_{i j} \delta_{j k} \delta_{k l} . \tag{3.50}
\end{equation*}
$$

As expected, given the orthogonality of the physical projectors (3.46) - (3.48)

$$
\begin{equation*}
Q^{\text {phys }}=Q^{\text {phys } ;[N]}+Q^{\text {phys } ;[N-1,1]}+Q^{\text {phys } ;[N-2,2]} . \tag{3.51}
\end{equation*}
$$

### 3.2.3 The action and physical projectors

Define the partition function of the most general permutation invariant Gaussian matrix model of symmetric matrices with vanishing diagonal as

$$
\begin{equation*}
\mathcal{Z}^{\mathrm{FX}} \equiv \int \mathrm{~d} M^{\mathrm{phys}} e^{-\mathcal{S}^{\mathrm{FX}}} \tag{3.52}
\end{equation*}
$$

where the action is given by (3.16) and the measure is

$$
\begin{equation*}
\mathrm{d} M^{\mathrm{phys}} \equiv \prod_{i<j} \mathrm{~d} M_{i j} \tag{3.53}
\end{equation*}
$$

We are interested in calculating permutation invariant expectation values of operators composed of the $M_{i j}$ variables. These are defined as

$$
\begin{equation*}
\left\langle\mathcal{O}_{\alpha}(M)\right\rangle \equiv \frac{1}{\mathcal{Z}} \int \mathrm{~d} M^{\mathrm{phys}} \mathcal{O}_{\alpha}(M) e^{-\mathcal{S}^{\mathrm{FX}}} \tag{3.54}
\end{equation*}
$$

In order to evaluate (3.54) we must factorise the integral. The action, as written in (3.16), contains $\frac{N(N-1)}{2}$ matrix variables mixed in a non-trivial way. The solution to this problem exploits the decomposition (3.24), we write the action in terms of a basis for the RHS of this decomposition using the $Q^{\text {phys }, \Lambda_{1}}$ projectors, where $\Lambda_{1}$ runs over the set $\{[N],[N-$
$1,1],[N-2,2]\}$,

$$
\begin{align*}
\mathcal{S}^{\mathrm{FX}}= & \sum_{\Lambda_{1}} g_{\Lambda_{1}} \sum_{i, j, k, l=1}^{N} M_{i j} Q_{i j k l}^{\mathrm{phys} ; \Lambda_{1}} M_{k l}-\sum_{i, j=1}^{N} \mu_{[N]} C_{i j}^{\mathrm{phys} ;[N]} M_{i j} \\
= & \sum_{i, j, k, l} \frac{1}{2}\left(g_{[N]} M_{i j} Q_{i j k l}^{\mathrm{phys} ;[N]} M_{k l}+g_{[N-1,1]} M_{i j} Q_{i j k l}^{\mathrm{phys} ;[N-1,1]} M_{k l}+g_{[N-2,2]} M_{i j} Q_{i j k l}^{\mathrm{phys} ;[N-2,2]} M_{k l}\right) \\
& -\sum_{i, j} \mu_{[N]} C_{i j}^{\mathrm{phys} ;[N]} M_{i j} \tag{3.55}
\end{align*}
$$

Applying these projectors amounts to a change of basis, from the matrix variables $M_{i j}$ to the representation theory variables $S^{\text {phys } ; V_{[N]}}, S_{a}^{\text {phys } ; V_{[N-1,1]}}, S_{a}^{\text {phys } ; V_{[N-2,2]}}$ :

$$
\begin{gather*}
\mathcal{S}^{\mathrm{FX}}=-\mu_{[N]} S^{\mathrm{phys} ; V_{[N]}}+\frac{g_{[N]}}{2} S^{\mathrm{phys} ; V_{[N]}} S^{\mathrm{phys} ; V_{[N]}}+\frac{g_{[N-1,1]}}{2} \sum_{a=1}^{N-1} S_{a}^{\mathrm{phys} ; V_{[N-1,1]}} S_{a}^{\mathrm{phys} ; V_{[N-1,1]}} \\
+\frac{g_{[N-2,2]}}{2} \sum_{a=1}^{N(N-3) / 2} S_{a}^{\mathrm{phys} ; V_{[N-2,2]}} S_{a}^{\mathrm{phys} ; V_{[N-2,2]}} \tag{3.56}
\end{gather*}
$$

With this the partition function reads

$$
\begin{equation*}
\mathcal{Z}^{\mathrm{FX}} \equiv \int \mathrm{~d} S^{\mathrm{phys}} e^{-\mathcal{S}^{\mathrm{FX}}} \tag{3.57}
\end{equation*}
$$

with

$$
\begin{equation*}
\mathrm{d} S^{\mathrm{phys}}=\mathrm{d} S^{\mathrm{phys} ; V_{[N]}} \prod_{a_{1}=1}^{N-1} \mathrm{~d} S_{a_{1}}^{\mathrm{phys} ; V_{[N-1,1]}} \prod_{a_{2}=1}^{N(N-3) / 2} \mathrm{~d} S_{a_{2}}^{\mathrm{phys} ; V_{[N-2,2]}} \tag{3.58}
\end{equation*}
$$

The factorised expression permits the application of standard techniques of Gaussian integration. Substituting the expressions for the projectors into 3.55 and performing the summations we can also write the action in terms of the full $V_{N} \otimes V_{N}$ representation variables (3.32-3.36

$$
\begin{align*}
\mathcal{S}^{\mathrm{FX}}= & -\mu_{[N]}\left(\sqrt{\frac{N-1}{N}} S^{V_{[N]} ; 1}-\frac{1}{\sqrt{N}} S^{V_{[N]} ; 2}\right)+\frac{g_{[N]}}{2}\left(\sqrt{\frac{N-1}{N}} S^{V_{[N]} ; 1}-\frac{1}{\sqrt{N}} S^{V_{[N]} ; 2}\right)^{2} \\
& +\frac{g_{[N-1,1]}}{2} \sum_{a=1}^{N-1}\left(\sqrt{\frac{N-2}{2 N}}\left(S_{a}^{V_{[N-1,1]} ; 1}+S_{a}^{V_{[N-1,1]} ; 2}\right)-\sqrt{\frac{2}{N}} S_{a}^{V_{[N-1,1]} ; 3}\right)^{2} \\
& +\frac{g_{[N-2,2]}}{2} \sum_{a=1}^{N(N-3) / 2} S_{a}^{V_{[N-2,2]}} S_{a}^{V_{[N-2,2]}} \tag{3.59}
\end{align*}
$$

### 3.2.4 Observables and correlators

The observables of our theory $\mathcal{O}_{\alpha}$ are permutation invariant functions of the matrix variables $M_{i j}$, they obey

$$
\begin{equation*}
\mathcal{O}_{\alpha}\left(M_{i j}\right)=\mathcal{O}_{\alpha}\left(M_{\sigma(i) \sigma(j)}\right), \quad \forall \sigma \in S_{N} \tag{3.60}
\end{equation*}
$$

The physical permutation invariant observables of order $k$ are in one-to-one correspondence with undirected, loopless multigraphs with $k$ edges. Each matrix describes an edge connecting vertices labelled by the row and column indices of the matrix. Requiring the matrices to be symmetric is equivalent to considering undirected edges. The further requirement that the matrices have vanishing diagonal entries is equivalent to restricting to loopless multigraphs.

Below we list the complete set of quadratic, cubic and quartic graphs of general matrices that survive the projection to the physical subspace

$$
\begin{equation*}
M_{i j}^{\mathrm{phys}}=Q_{i j k l}^{\mathrm{phys}} M_{k l} \tag{3.61}
\end{equation*}
$$

The counting of these graphs organised by number of edges is given by the OEIS sequence A050535 [78]. The three quadratic observables are

$$
\begin{equation*}
\sum_{i, j} M_{i j}^{2}, \quad \stackrel{\sum_{i, j, k} M_{i j} M_{j k},}{\sum_{i, j, k, l} M_{i j} M_{k l}} \tag{3.62}
\end{equation*}
$$

The eight cubic observables are


The 23 quartic observables are


### 3.2.5 Expectation values

We can write the partition function in the following form

$$
\begin{align*}
\mathcal{Z}^{\mathrm{FX}} & =\int \mathrm{d} S^{\mathrm{phys}} \exp \left(\sum_{\Lambda_{1}, a} \mu_{\Lambda_{1}, a} S_{a}^{\text {phys } ; \Lambda_{1}}-\frac{1}{2} \sum_{\Lambda_{1}, a} S_{a}^{\text {phys } ; \Lambda_{1}} g_{\Lambda_{1}} S_{a}^{\text {phys } ; \Lambda_{1}}\right) \\
& =\frac{(2 \pi)^{\frac{N(N-1)}{2}}}{(\operatorname{det} g)^{\frac{1}{2}}} \exp \left(\frac{1}{2} \sum_{\Lambda_{1}} \sum_{a} \mu_{\Lambda_{1}, a} g_{\Lambda_{1}}^{-1} \mu_{\Lambda_{1}, a}\right), \tag{3.65}
\end{align*}
$$

in which we have included linear couplings for all $S^{\text {phys }}$ variables. This is the usual trick employed to generate expectation values from the partition function. All linear couplings are included to source expectation values of any observable by taking derivatives with respect to the relevant linear coupling. In order to recover the permutation invariant model all but the $V_{[N]}$ linear coupling are set to zero. The integral in (3.65) is performed using the result (2.47).

Expectation values of the physical variables $\mathcal{O}(S)$ are defined by

$$
\begin{equation*}
\langle\mathcal{O}(S)\rangle \equiv \frac{\int \mathrm{d} S^{\mathrm{phys}} \mathcal{O}(S) e^{-\mathcal{S}^{\mathrm{FX}}}}{\int \mathrm{~d} S^{\mathrm{phys}} e^{-\mathcal{S}^{\mathrm{FX}}}} \tag{3.66}
\end{equation*}
$$

Calculating these by taking derivatives of the RHS of (3.65), with respect to $\mu_{\Lambda_{1}, a}$ we find for example, the linear expectation values are given by

$$
\begin{align*}
\left\langle S_{a}^{\mathrm{phys} ; \Lambda_{1}}\right\rangle & =\frac{1}{\mathcal{Z}^{\mathrm{FX}}} \int \mathrm{~d} S^{\mathrm{phys}} S_{a}^{\mathrm{phys} ; \Lambda_{1}} e^{-\mathcal{S}^{\mathrm{FX}}}=\left.\frac{1}{\mathcal{Z}^{\mathrm{FX}}} \frac{\partial \mathcal{Z}^{\mathrm{FX}}}{\partial \mu_{\Lambda_{1}, a}}\right|_{\mu_{\Lambda_{1}, a} \neq 0 \text { iff } \Lambda_{1}=V_{[N]}} \\
& =g_{\Lambda_{1}}^{-1} \mu_{\Lambda_{1}} \delta\left(\Lambda_{1}, V_{[N]}\right) . \tag{3.67}
\end{align*}
$$

That is, the only non-zero linear expectation value is

$$
\begin{equation*}
\left\langle S^{\text {phys } \left.; V_{[N]}\right\rangle}\right\rangle=g_{[N]}^{-1} \mu_{[N]} . \tag{3.68}
\end{equation*}
$$

For later convenience we define

$$
\begin{equation*}
\widetilde{\mu}_{[N]} \equiv g_{[N]}^{-1} \mu_{[N]} . \tag{3.69}
\end{equation*}
$$

Similarly, we can calculate the two-point function by taking two derivatives of 3.65

$$
\begin{align*}
\left\langle S_{a}^{\text {phys } ; \Lambda_{1}} S_{b}^{\text {phys } ; \Lambda_{2}}\right\rangle & =\frac{1}{\mathcal{Z}^{\mathrm{FX}}} \int \mathrm{~d} S^{\text {phys }} S_{a}^{\text {phys } ; \Lambda_{1}} S_{b}^{\text {phy } ; \Lambda_{2}} e^{-\mathcal{S}^{\mathrm{FX}}}=\left.\frac{1}{\mathcal{Z}^{\mathrm{FX}}} \frac{\partial}{\partial \mu_{\Lambda_{2}, b}} \frac{\partial \mathcal{Z}^{\mathrm{FX}}}{\partial \mu_{\Lambda_{1}, a}}\right|_{\mu_{\Lambda, a} \neq 0 \text { iff } \Lambda=V_{[N]}} \\
& =\left.\frac{1}{\mathcal{Z}^{\mathrm{FX}}} \frac{\partial}{\partial \mu_{\Lambda_{2}, b}}\left(g_{\Lambda_{1}}\right)_{c d}^{-1} \mu_{\Lambda_{1}, c} \delta_{a c} \mathcal{Z}^{\mathrm{FX}}\right|_{\mu_{\Lambda, a} \neq 0 \text { iff } \Lambda=V_{[N]}} \\
& =g_{\Lambda_{1}}^{-1} \delta_{a b} \delta\left(\Lambda_{1}, \Lambda_{2}\right)+\left\langle S_{a}^{\text {phys } ; \Lambda_{1}}\right\rangle\left\langle S_{b}^{\text {phy } ; \Lambda_{2}}\right\rangle \\
& =g_{\Lambda_{1}}^{-1} \delta_{a b} \delta\left(\Lambda_{1}, \Lambda_{2}\right)+\widetilde{\mu}_{[N]}^{2} \delta\left(\Lambda_{1}, V_{[N]}\right) \delta\left(\Lambda_{2}, V_{[N]}\right) . \tag{3.70}
\end{align*}
$$

Again, we define the connected piece of the two point function

$$
\begin{equation*}
\left\langle S_{a}^{\mathrm{phys} ; \Lambda_{1}} S_{b}^{\mathrm{phys} ; \Lambda_{2}}\right\rangle_{\mathrm{conn}}=g_{\Lambda_{1}}^{-1} \delta_{a b} \delta\left(\Lambda_{1}, \Lambda_{2}\right) \tag{3.71}
\end{equation*}
$$

We can find the one-point function of the $M_{i j}^{\text {phys }}$ by writing it in terms of the physical $S$ variables and applying 3.68

$$
\begin{align*}
\left\langle M_{i j}^{\mathrm{phys}}\right\rangle & =C_{i j}^{\mathrm{phys} ;[N]}\left\langle S^{\left.\mathrm{phys} ; V_{[N]}\right\rangle}\right. \\
& =\left(\sqrt{\frac{N-1}{N^{3}}}-\frac{1}{\sqrt{N(N-1)}} F(i, j)\right) \widetilde{\mu}_{[N]} \tag{3.72}
\end{align*}
$$

Similarly, we can find the two-point function of the $M_{i j}^{\text {phys }}$ variables by writing each $M^{\text {phys }}$ in terms of the physical $S$ variables and then using 3.68 and 3.70 to evaluate each of the resulting expectation values

$$
\begin{align*}
& \left\langle M_{i j}^{\text {phys }} M_{k l}^{\text {phys }}\right\rangle_{\text {conn }}=\sum_{\Lambda_{1}, \Lambda_{2}} \sum_{a=1}^{\operatorname{dim} \Lambda_{1}} \sum_{b=1}^{\operatorname{dim} \Lambda_{2}} C_{i j, a}^{\text {phys } ; \Lambda_{1}} C_{k l, b}^{\text {phys } ; \Lambda_{2}}\left\langle S_{a}^{\text {phys } ; \Lambda_{1}} S_{b}^{\text {phys } ; \Lambda_{2}}\right\rangle_{\text {conn }} \\
& \quad=\sum_{\Lambda_{1}, \Lambda_{2}} \sum_{a=1}^{\operatorname{dim} \Lambda_{1}} \sum_{b=1}^{\operatorname{dim} \Lambda_{2}} C_{i j, a}^{\text {phys } ; \Lambda_{1}} C_{k l, b}^{\text {phys } ; \Lambda_{2}} g_{\Lambda_{1}}^{-1} \delta_{a b} \delta\left(\Lambda_{1}, \Lambda_{2}\right) \\
& \quad=\sum_{\Lambda_{1}} \sum_{a=1}^{\operatorname{dim} \Lambda_{1}} C_{i j, a}^{\text {phys }, \Lambda_{1}} C_{k l, a}^{\text {phys } ; \Lambda_{1}} g_{\Lambda_{1}}^{-1} \\
& \quad=\sum_{\Lambda_{1}} Q_{i j k l}^{\text {phys } ; \Lambda_{1}} g_{\Lambda_{1}}^{-1} . \tag{3.73}
\end{align*}
$$

Plugging in the expressions for the physical $Q_{\mathrm{s}}$ in equations 3.46 - 3.48) and rewriting
in terms of $F$ s gives the entire two-point function as

$$
\begin{align*}
& \left\langle M_{i j}^{\text {phys }} M_{k l}^{\text {phys }}\right\rangle=\left(\sqrt{\frac{N-1}{N^{3}}}-\frac{1}{\sqrt{N(N-1)}} F(i, j)\right)\left(\sqrt{\frac{N-1}{N^{3}}}-\frac{1}{\sqrt{N(N-1)}} F(k, l)\right) \widetilde{\mu}_{[N]}^{2} \\
& \quad+\frac{1}{N}\left(\frac{1}{N-1} F(i, j) F(k, l)-\frac{1}{N}(F(i, j)+F(k, l))+\frac{N-1}{N^{2}}\right) g_{[N]}^{-1} \\
& \quad+\frac{1}{2(N-2)}\left(1-\delta_{i j}\right)\left(1-\delta_{k l}\right)(F(i, k)+F(j, k)+F(i, l)+F(j, l)) g_{[N-1,1]}^{-1} \\
& \quad+\left(\frac{1}{2} F(i, k) F(j, l)+\frac{1}{2} F(i, l) F(j, k)-\frac{N}{N-2} \sum_{p, q=1}^{N} F(i, p) F(j, p) F(k, q) F(l, q) F(p, q)\right. \\
& \left.\quad-\frac{1}{N-1} F(i, j) F(k, l)\right) g_{[N-2,2]}^{-1} . \tag{3.74}
\end{align*}
$$

Evaluating this expressions for the linear and quadratic PIMOs gives

$$
\begin{equation*}
\sum_{i, j}\left\langle M_{i j}^{\text {phys }}\right\rangle=\sqrt{N(N-1)} \widetilde{\mu}_{[N]} \tag{3.75}
\end{equation*}
$$

and

$$
\begin{align*}
& \sum_{i, j}\left\langle M_{i j}^{\text {phys }} M_{i j}^{\text {phys }}\right\rangle=\widetilde{\mu}_{[N]}^{2}+g_{[N]}^{-1}+(N-1) g_{[N-1,1]}^{-1}+\frac{N(N-3)}{2} g_{[N-2,2]}^{-1},  \tag{3.76}\\
& \sum_{i, j, k}\left\langle M_{i j}^{\text {phys }} M_{j k}^{\text {phys }}\right\rangle_{\mathrm{conn}}=(N-1) \widetilde{\mu}_{[N]}^{2}+(N-1) g_{[N]}^{-1}+\frac{(N-1)(N-2)}{2} g_{[N-1,1]}^{-1},  \tag{3.77}\\
& \sum_{i, j, k, l}\left\langle M_{i j}^{\text {phys }} M_{k l}^{\text {phys }}\right\rangle_{\text {conn }}=N(N-1) \widetilde{\mu}_{[N]}^{2}+N(N-1) g_{[N]}^{-1} . \tag{3.78}
\end{align*}
$$

Since our theory is Gaussian, Wick's theorem allows us to calculate higher point expectation values from the linear and quadratic expectation values.

### 3.2.6 Embedding within 13-parameter PIGM model

Previous work has solved the most general Gaussian matrix models for general $N \times N$ matrices [43]. These models are defined by 2 linear and 11 quadratic coupling parameters. We expect the $1+3$ parameter model considered in this chapter to be embedded in the larger $2+11$ parameter model.

Indeed, we now compare the action of the of the physical model 3.59) written in terms of $S$ variables (3.32) - (3.36) to that of the 13-parameter model. For convenience we reprint
the action of the physical model

$$
\begin{align*}
\mathcal{S}^{\mathrm{FX}}= & -\mu_{[N]}\left(\sqrt{\frac{N-1}{N}} S^{V_{[N]} ; 1}-\frac{1}{\sqrt{N}} S^{V_{[N]} ; 2}\right)+\frac{g_{[N]}}{2}\left(\sqrt{\frac{N-1}{N}} S^{V_{[N]} ; 1}-\frac{1}{\sqrt{N}} S^{V_{[N]} ; 2}\right)^{2} \\
& +\frac{g_{[N-1,1]}}{2} \sum_{a=1}^{N-1}\left(\sqrt{\frac{N-2}{2 N}}\left(S_{a}^{V_{[N-1,1] ; 1}}+S_{a}^{V_{[N-1,1] ; 2}}\right)-\sqrt{\frac{2}{N}} S_{a}^{V_{[N-1,1] ; 3}}\right)^{2} \\
& +\frac{g_{[N-2,2]}}{2} \sum_{a=1}^{N(N-3) / 2} S_{a}^{V_{[N-2,2]}} S_{a}^{V_{[N-2,2]}} . \tag{3.79}
\end{align*}
$$

We then express the action of the 13-parameter model in terms of the same variables,

$$
\begin{align*}
\mathcal{S} & =-\sum_{\alpha=1}^{2} \mu_{[N], \alpha}^{(13)} S^{V_{[N]} ; \alpha}+\frac{1}{2} \sum_{\alpha, \beta=1}^{2} S_{[N] ; \alpha}^{V_{[N]}}\left(g_{[N]}^{(13)}\right)_{\alpha \beta} S_{[N] ; \beta}^{V_{[N]}}+\frac{1}{2} \sum_{a=1}^{N-1} \sum_{\alpha, \beta=1}^{3} S_{a}^{[N-1,1] ; \alpha}\left(g_{[N-1,1]}^{(13)}\right)_{\alpha \beta} S_{a}^{[N-1,1] ; \beta} \\
& +\frac{1}{2} g_{[N-2,2]}^{(13)} \sum_{a=1}^{N(N-3) / 2} S_{a}^{V_{[N-2,2]}} S_{a}^{V_{[N-2,2]}}+\frac{1}{2} g_{V_{[N-2,1,1]}}^{(13)} \sum_{a=1}^{(N-1)(N-2) / 2} S_{a}^{V_{[N-2,1,1]}} S_{a}^{V_{[N-2,1,1]}}, \tag{3.80}
\end{align*}
$$

labelling the couplings of this model with a superscript "(13)" here to distinguish them. This expression 3.80 is obtained by plugging the $Q$ s given in section 2.6 into the equation for the 13-parameter action 2.83 . We see the point at which the PIGM model of general matrices reduces to that of the symmetric matrices with vanishing diagonal is given by

$$
\begin{equation*}
\mu_{[N]}^{(13)}=\mu_{[N]}\left[\sqrt{\frac{N-1}{N}} \quad-\frac{1}{\sqrt{N}}\right] \tag{3.81}
\end{equation*}
$$

and

$$
\begin{array}{ll}
g_{[N]}^{(13)}=g_{[N]}\left[\begin{array}{cc}
\frac{N-1}{N} & -\frac{\sqrt{N-1}}{N} \\
-\frac{\sqrt{N-1}}{N} & \frac{1}{N}
\end{array}\right], & g_{[N-1,1]}^{(13)}=g_{[N-1,1]}\left[\begin{array}{ccc}
\frac{N-2}{2 N} & \frac{N-2}{2 N} & -2 \frac{\sqrt{N-2}}{N} \\
\frac{N-2}{2 N} & \frac{N-2}{2 N} & -2 \frac{\sqrt{N-2}}{N} \\
-2 \frac{\sqrt{N-2}}{N} & -2 \frac{\sqrt{N-2}}{N} & \frac{2}{N}
\end{array}\right], \\
g_{[N-2,2]}^{(13)}=g_{[N-2,2]}, & g_{[N-2,1,1]}^{(13)}=0 . \tag{3.82}
\end{array}
$$

### 3.3 Daily correlation matrices from high-frequency forex data

The high-frequency forex data that we analyse pertain to 19 of the most liquidly traded currency pairs and cover the date range from 1 April 2020 to 31 January 2022. The data is sourced from TrueFX [79] and is comprised of all updates of the best price quotes at which any market participant is willing to buy (top-of-book bid quotes) or sell (top-of-book offer quotes). Market participants providing such price quotes include banks, brokers and asset
managers on the Integral OCX platform 1 . The data precision is in milliseconds for time stamps and fractions of a pir $2^{2}$ for prices. We exclude United States currency settlement holidays (days where no settlements of prior transactions are made) due to the central importance of the US Dollar to forex trading. We also exclude the 24th, 25th, 26th, 31st of December and the 1st, 2nd of January due to reduced liquidity. In total, around one billion pricing updates were analysed. For each currency pair, the mid-price series, $p_{j}^{(I)}$, is calculated from the bid and offer quotes as

$$
\begin{equation*}
p_{j}^{(I)}=\left(b_{j}^{(I)}+a_{j}^{(I)}\right) / 2 \quad I \in\{1, \ldots, 19\}, j \in\left\{1, \ldots, n^{I}\right\} \tag{3.83}
\end{equation*}
$$

where $b^{(I)}$ and $a^{(I)}$ are contemporaneous bid and offer quotes respectively, $I \in\{1, \ldots, 19\}$, $j$ indexes the quotes and $n^{I}$ corresponds to the number of quotes for the currency pair $I$ per day. Table 3.1 gives the mapping of these indices to actual currency pair names.

| Index (I) | Currency Pair |
| :---: | :---: |
| 1 | AUD/JPY |
| 2 | AUD/NZD |
| 3 | AUD/USD |
| 4 | CAD/JPY |
| 5 | CHF/JPY |
| 6 | EUR/CHF |
| 7 | EUR/GBP |
| 8 | EUR/JPY |
| 9 | EUR/PLN |
| 10 | EUR/USD |
| 11 | GBP/JPY |
| 12 | GBP/USD |
| 13 | NZD/USD |
| 14 | USD/CAD |
| 15 | USD/CHF |
| 16 | USD/JPY |
| 17 | USD/MXN |
| 18 | USD/TRY |
| 19 | USD/ZAR |

Table 3.1: Currency pair mapping.

These mid-prices are then sampled on a regular time grid using the last-tick methodology, where the most recent quotes in each currency pair are used to calculate the mid-price for

[^0]that time interval. The regularly sampled mid-prices are then
\[

$$
\begin{equation*}
p_{t_{(1)}}^{(I)}, \ldots, p_{t_{(n)}}^{(I)}, \quad t_{(i+1)}-t_{(i)}=5 \text { minutes, } \quad i \in\{1, \ldots, n\} \tag{3.84}
\end{equation*}
$$

\]

where $t_{(i)}, i \in\{1, \ldots, n\}$ are the time stamps on a regularly sampled grid and $n$ is the number of 5 minute intervals per day. If we denote the time stamp of each quote as $\tau_{j}, j \in\left\{1, \ldots, n^{I}\right\}$, then the quote used for each 5 minute interval can be described as,

$$
\begin{equation*}
p_{t_{(i)}}^{(I)}=p_{\max \left\{1 \leq j \leq n^{I} \mid \tau_{j} \leq t_{(i)}\right\}}^{(I)} . \tag{3.85}
\end{equation*}
$$

We note that the choice of 5 minutes as a time interval is common in high frequency financial correlation analyses. We obtain the (log) mid-price returns via

$$
\begin{equation*}
r_{(i)}^{(I)}=\log \frac{p_{t_{(i+1)}}^{(I)}}{p_{t_{(i)}}^{(I)}}, \quad i \in\{1, \ldots, n-1\} . \tag{3.86}
\end{equation*}
$$

Note that the first time interval of each day, for all currency pairs, begins at 00:00:00.000 UTC/GMT and ends at 00:04:59:59.999 UTC/GMT. The last time interval begins at 23:55:00.000 UTC/GMT and ends at 23:59:59.999 UTC/GMT. The advantage of determining calendar date based on UTC/GMT is that the major forex trading sessions are all captured on the same calendar date, namely Asia, then Europe, then North America. There are $n=288$ five minute intervals per day. The time intervals are not only regular, but also aligned across all the currency pairs. See table 3.2 for the statistics on the number of quotes per time interval for each currency pair, aggregated across all days. See table 3.3 for the descriptive statistics of the regularly sampled (log) returns per currency pair, again aggregated across all days in the data set. It is readily apparent from the descriptive statistics in table 3.3 that the means of the $(\log )$ return distributions are very close to zero and that the standard deviations vary between currency pairs. In addition, the returns have high kurtosis consistent with the expected behaviour of price movements of financial instruments with a calendar time clock. The only currency pair that has a markedly asymmetric distribution is USD/TRY as evidenced by a large negative skewness (i.e. a left-skewed distribution). The large volatility, kurtosis and negative skewness of the USD/TRY distribution can be related to the sharp depreciation of the Turkish Lira during the Turkish currency and debt crisis which occurred during the period of analysis.

### 3.3.1 Correlation matrix methodology

In statistics, various measures of association between two random variables have been defined. In our context, we apply certain measures of correlation to ascertain the degree to

| Currency Pair | Mean | Std Dev. | Q1 | Med. | Q3 |
| :--- | ---: | ---: | ---: | ---: | ---: |
| AUD/JPY | 419 | 374.5 | 187 | 319 | 531 |
| AUD/NZD | 304 | 279.3 | 135 | 231 | 383 |
| AUD/USD | 404 | 396.9 | 165 | 296 | 511 |
| CAD/JPY | 272 | 257.0 | 113 | 197 | 348 |
| CHF/JPY | 287 | 271.7 | 118 | 212 | 368 |
| EUR/CHF | 269 | 295.5 | 90 | 173 | 338 |
| EUR/GBP | 308 | 316.7 | 103 | 206 | 407 |
| EUR/JPY | 506 | 435.8 | 202 | 387 | 685 |
| EUR/PLN | 277 | 487.4 | 47 | 125 | 296 |
| EUR/USD | 512 | 506.4 | 181 | 376 | 683 |
| GBP/JPY | 553 | 471.5 | 240 | 433 | 724 |
| GBP/USD | 492 | 472.6 | 174 | 361 | 668 |
| NZD/USD | 269 | 269.3 | 114 | 199 | 333 |
| USD/CAD | 373 | 371.7 | 152 | 271 | 470 |
| USD/CHF | 237 | 255.4 | 82 | 161 | 305 |
| USD/JPY | 322 | 319.9 | 135 | 234 | 399 |
| USD/MXN | 447 | 434.1 | 143 | 324 | 610 |
| USD/TRY | 205 | 580.4 | 9 | 36 | 138 |
| USD/ZAR | 445 | 498.5 | 139 | 325 | 588 |

Table 3.2: Descriptive statistics of number of quote updates per 5 minute time interval.

| Currency Pair | Mean $\left(\mathrm{x} 10^{-4}\right)$ | Std Dev. $\left(\mathrm{x} 10^{-4}\right)$ | Skewness | Kurtosis |
| :--- | ---: | ---: | ---: | ---: |
| AUD/JPY | 0.0 | 3.8 | 0.0 | 24.4 |
| AUD/NZD | 0.0 | 2.1 | 0.2 | 36.6 |
| AUD/USD | 0.0 | 3.8 | 0.0 | 15.3 |
| CAD/JPY | 0.0 | 3.1 | 0.1 | 16.2 |
| CHF/JPY | 0.0 | 2.4 | 0.0 | 12.5 |
| EUR/CHF | 0.0 | 1.8 | -0.6 | 80.5 |
| EUR/GBP | 0.0 | 2.6 | -0.2 | 37.5 |
| EUR/JPY | 0.0 | 2.4 | 0.1 | 16.8 |
| EUR/PLN | 0.0 | 2.7 | 0.0 | 31.1 |
| EUR/USD | 0.0 | 2.4 | -0.2 | 32.9 |
| GBP/JPY | 0.0 | 3.1 | 0.0 | 20.0 |
| GBP/USD | 0.0 | 3.1 | 0.0 | 14.9 |
| NZD/USD | 0.0 | 3.8 | 0.1 | 19.8 |
| USD/CAD | 0.0 | 2.7 | 0.0 | 18.8 |
| USD/CHF | 0.0 | 2.5 | -0.2 | 19.8 |
| USD/JPY | 0.0 | 2.1 | 0.2 | 17.3 |
| USD/MXN | 0.0 | 5.5 | 0.0 | 18.8 |
| USD/TRY | 0.1 | 11.7 | -7.3 | 79.7 |
| USD/ZAR | 0.0 | 6.4 | -0.1 | 17.2 |

Table 3.3: Summary statistics of regular 5-minute (log) mid-price returns.
which currency ( $\log$ ) returns are concordant or discordant. Intuitively, this should capture an important aspect of the relationship of one currency pair with another. Calculating correlations on high frequency financial data is complicated by two main effects. The first is the fact that observations occur irregularly in time and moreover, asynchronously across instruments. The second is the presence of microstructure noise due to various factors such as bid-ask bounce (relevant mainly for transaction based data), minimum tick intervals, latency effects etc. See [80] and the references therein for more detail on these two complicating issues and various approaches to address them. In this article we utilise the correlation estimator (3.87) on (log) mid-price returns. This estimator is referred to as the realised correlation estimator in the finance literature and is defined as,

$$
\begin{equation*}
\hat{\rho}^{I J}=\frac{\sum_{i=1}^{n-1} r_{(i)}^{(I)} r_{(i)}^{(J)}}{\sqrt{\left(\sum_{i=1}^{n-1} r_{(i)}^{(I)}\right)^{2}\left(\sum_{i=1}^{n-1} r_{(i)}^{(J)}\right)^{2}}}, \quad I, J \in\{1, \ldots, 19\} \tag{3.87}
\end{equation*}
$$

where $I, J$ are currency pair indices. This estimator captures the normalised, aggregated co-movement (i.e. covariance) of two series of returns over a given time period (one day in our case). It is well established that the realised correlation estimator is, in general, sensitive to the issues described above. However, it is widely acknowledged in the literature that the impact of these issues can be mitigated by sampling regularly at a lower frequency i.e. 5-15 minute intervals. We utilise 5 minute time intervals in particular, as is common in analysing high frequency financial data. We have also verified empirically that the correlation results are not very sensitive to the choice of the time interval length (beyond a certain length). We do acknowledge however that the procedure we have applied is not likely to be the most efficient and discards some information (see [80] or [81, 82] for approaches that are likely to be more efficient for example). However, the simplicity of the realised correlation estimator is appealing and it allows us to make contact with asymptotic Gaussian sampling properties as discussed in the introduction. The main focus of the present chapter is to explore the phenomenological modelling of ensembles of correlation matrices and not particular correlation estimators. The impact of using more sophisticated and potentially more efficient estimators in our context can be explored in future research. Note that the resultant $\hat{\rho}^{I J}$ correlation matrix is a symmetric, real matrix with 19 (19$1) / 2=171$ independent entries. This figure accounts for the fact that the diagonal elements are fixed and equal and do not contribute to the degrees of freedom (we subtract the identity to get a correlation matrix with vanishing diagonal elements). As mentioned previously, we are concerned with the ensemble statistics of such matrices. There are several ways to construct such an ensemble. We choose to calculate the correlation matrix for each trading day (aligned with UTC/GMT boundaries), and study the sampling distribution of


Figure 3.1: Examples of realised daily correlation estimates over time.
the matrices. In particular, we study, $\hat{\rho}_{A}^{I J}$, where $A \in\left\{1, \ldots, N_{D}\right\}$ indexes trading dates. In our data, there are 446 unique trading days, i.e. $N_{D}=446$. We plot examples of the evolution of two elements of these correlation matrices over time in figure 3.1.

### 3.4 Matrix theory and matrix data: near-Gaussianity

In this section we apply the Gaussian model of section 3.2 to the ensemble of correlation matrices defined in section 3.3. We show that the vast majority of cubic and quartic observables closely match the predictions of the Gaussian theory.

We form a compact representation of the original correlation matrix data by defining a vector of observables for each correlation matrix. The observable vectors are shown to perform well in anomaly detection tasks in section 3.5. Optimal performance is achieved in these tasks by constructing observable vectors from the least Gaussian observables.

### 3.4.1 Theory/experiment deviations normalised by standard deviations of the observables

We begin by elucidating some empirical statistical properties of the observables - the permutation invariant polynomials of the correlation matrix elements. These are listed in table 3.4 and the distributions of their standardised values over the 446 days are plotted in the histograms in figure 3.2. Many appear roughly Gaussian, while others exhibit a right/positive skew along with heavier tails than the normal distribution. The estimated


Figure 3.2: Histograms of the standardised values of each of the observables (one value per correlation matrix i.e. per day).

Pearson product-moment correlation of the observable elements is plotted in figure 3.3. It is noteworthy that all correlations are positive and that most correlations are very strongly positive. The strength of the correlations is particularly relevant in our choice of statistical distance measure in the anomaly detection analysis presented in section 3.5. Indeed, this motivated utilising the Mahalanobis distance which typically performs well even in the presence of such correlations.

Equipped with the Gaussian model and its solution, given in section 3.2 , and the financial data described in section 3.3 we now perform a variety of tests to assess how well this model describes the statistics of the forex correlation data. Firstly, we calculate the normalised absolute error

$$
\begin{equation*}
\Delta_{\alpha}=\frac{\left|\left\langle\mathcal{O}_{\alpha}\right\rangle_{\mathrm{T}}-\left\langle\mathcal{O}_{\alpha}\right\rangle_{\mathrm{E}}\right|}{\sigma_{\mathrm{E}, \alpha}} \tag{3.88}
\end{equation*}
$$

between the experimental cubic and quartic observable average values

$$
\begin{equation*}
\left\langle\mathcal{O}_{\alpha}\right\rangle_{\mathrm{E}}=\frac{1}{n_{A}} \sum_{A=1}^{N_{D}} \mathcal{O}_{\alpha}\left(\hat{\rho}_{A}^{I J}\right) \tag{3.89}
\end{equation*}
$$

and the Gaussian model's prediction of those expectation values $\left\langle\mathcal{O}_{\alpha}\right\rangle_{\mathrm{T}}$ defined in (3.54).


Figure 3.3: Pearson product-moment correlation of the observables.

In both equations $\alpha$ indexes the observables and in we have normalised by the standard deviation of the experimental observable values.

As argued in section 3.1 these normalised errors are expected to be small where the underlying data is approximately Gaussian. The normalised absolute error for each observable is listed in the third column of table 3.4. In general these are in very good agreement: only four observables differ from the theoretical prediction by more than one standard deviation, and the average normalised absolute error of the cubic and quartic observables is 0.42 standard deviations. This can be regarded as strong evidence for Gaussianity in the permutation invariant sector of the FX-rate correlation matrix data.

The model defined in section 3.2 can be used to predict the standard deviation of cubic and quartic observables. We call this the theoretical standard deviation, and define it for each observable $\mathcal{O}_{\alpha}$ as

$$
\begin{equation*}
\sigma_{\mathrm{T}, \alpha} \equiv \sqrt{\left|\left\langle\left(\mathcal{O}_{\alpha}\right)^{2}\right\rangle_{\mathrm{T}}-\left\langle\mathcal{O}_{\alpha}\right\rangle_{\mathrm{T}}^{2}\right|} . \tag{3.90}
\end{equation*}
$$

In itself this is an interesting quantity to compare to the experimental observable standard deviations $\sigma_{\mathrm{E}, \alpha}$. The ratio of the two standard deviations is shown for each observable in the fourth column of table 3.4 . The values of $\sigma_{\mathrm{T}, 3}$ and $\sigma_{\mathrm{T}, 22}$ provided by the model are much smaller than the observed values. This is consistent with the finding in column three of table 3.4, in which we see the expectation values of these observables deviating the most from the model. It is these large deviations from Gaussianity that lend these observables their power in the construction of lower-dimensional representations of the correlation matrices (see section 3.5.3).

We briefly note an alternative approach to estimating the theoretical standard deviation, also employed in 55 to give good theoretical predictions of the experimental standard deviations of observables. This estimate is obtained by calculating the absolute difference between $\left\langle\mathcal{O}_{\alpha}\right\rangle_{\mathrm{T}}$ and $\left\langle\mathcal{O}_{\alpha}\right\rangle_{\mathrm{T}}$, where $\left\langle\mathcal{O}_{\alpha}\right\rangle_{\mathrm{T}}$, is the expectation value evaluated with the quadratic couplings that parametrise the model shifted by one standard deviation. Taking the average of this difference over all eight possible permutations of sign for the shifts of the three parameters gives us our estimate of the standard deviation. This method was used to estimate the standard deviations of $\mathcal{O}_{12}$ and $\mathcal{O}_{19}$ due to the prohibitive computational demands of calculating the octic expectation values $\left\langle\mathcal{O}_{12}^{2}\right\rangle$ and $\left\langle\mathcal{O}_{19}^{2}\right\rangle$.

### 3.4.2 Day capture and balanced accuracy of theoretical typicality prediction for days

The normalised errors presented in the previous section are encouragingly small, but rather abstract. In order to get a more intuitive sense of the agreement between the data and the Gaussian model, and with an eye toward developing useful applications, we consider a more practical measure. We call this second measure as day capture and define it as the proportion of days for which the value of an observable lies within two standard deviations of the mean value. If the observables were exactly Gaussian distributed we would expect the proportion of days captured to be close to $95.4 \%$, in line with the expectation of a onevariable Gaussian. This practical measure seems a sensible one given the approximately Gaussian distributions in figure 3.2. First, we list the experimetal day capture of each of the observables. These values are presented in the fifth column of table 3.4, all of which are very close to the expected $95.4 \%$.

Secondly, as a test of the Gaussian model we introduce the theoretical day capture and define it as the proportion of days falling within two theoretical standard deviations of the theoretical expectation value of an observable. To test the model's predicted day capture rates we calculate the balanced accuracy of the theoretical day capture for each observable. The process for which we now describe.

| Label | Observable | $\Delta_{\alpha}$ | $\sigma_{\mathrm{E}} / \sigma_{\mathrm{T}}$ | $\mu_{\mathrm{E}} \pm 2 \sigma_{\mathrm{E}}$ | $\mu_{\mathrm{T}} \pm 2 \sigma_{\mathrm{T}}$ | Balanced Accuracy |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathcal{O}_{1}$ | $\sum_{i, j} \hat{\rho}_{i j}^{3}$ | 0.02 | 0.90 | 95.74 | 97.31 | 0.82 |
| $\mathcal{O}_{2}$ | $\sum_{i, j, k} \hat{\rho}_{i j}^{2} \hat{\rho}_{j k}$ | 0.33 | 1.49 | 95.29 | 85.87 | 0.95 |
| $\mathcal{O}_{3}$ | $\sum_{i, j, k} \hat{\rho}_{i j} \hat{\rho}_{j k} \hat{\rho}_{k i}$ | 2.04 | 9.39 | 95.07 | 0.22 | 0.50 |
| $\mathcal{O}_{4}$ | $\sum_{i, j, k, l} \hat{\rho}_{i j}^{2} \hat{\rho}_{k l}$ | 0.01 | 1.06 | 95.74 | 95.07 | 1.00 |
| $\mathcal{O}_{5}$ | $\sum_{i, j, k, l} \hat{\rho}_{i j} \hat{\rho}_{j k} \hat{\rho}_{k l}$ | 0.97 | 3.36 | 95.52 | 41.70 | 0.72 |
| $\mathcal{O}_{6}$ | $\sum_{i, j, k, l} \hat{\rho}_{i j} \hat{\rho}_{i k} \hat{\rho}_{i l}$ | 0.33 | 0.73 | 95.74 | 98.43 | 0.68 |
| $\mathcal{O}_{7}$ | $\sum_{i, j, k, l, m} \hat{\rho}_{i j} \hat{\rho}_{j k} \hat{\rho}_{l m}$ | 0.12 | 1.05 | 95.96 | 94.62 | 0.99 |
| $\mathcal{O}_{8}$ | $\sum_{i, j, k, l, m, n} \hat{\rho}_{i j} \hat{\rho}_{k l} \hat{\rho}_{m n}$ | 0.01 | 0.65 | 95.74 | 98.43 | 0.68 |
| $\mathcal{O}_{9}$ | $\sum_{i, j} \hat{\rho}_{i j}^{4}$ | 0.54 | 1.10 | 94.84 | 92.83 | 0.81 |
| $\mathcal{O}_{10}$ | $\sum_{i, j, k} \hat{\rho}_{i j}^{2} \hat{\rho}_{j k}^{2}$ | 0.42 | 2.42 | 94.39 | 71.08 | 0.88 |
| $\mathcal{O}_{11}$ | $\sum_{i, j, k} \hat{\rho}_{i j} \hat{\rho}_{j k}^{3}$ | 0.05 | 1.30 | 95.29 | 93.50 | 0.99 |
| $\mathcal{O}_{12}$ | $\sum_{i, j, k} \hat{\rho}_{i j} \hat{\rho}_{i k} \hat{\rho}_{j k}^{2}$ | 0.88 | 3.27 * | 95.07 | 45.74 | 0.74* |
| $\mathcal{O}_{13}$ | $\sum_{i, j, k, l} \hat{\rho}_{i j} \hat{\rho}_{k j} \hat{\rho}_{l j}^{2}$ | 0.19 | 1.67 | 95.29 | 88.57 | 0.96 |
| $\mathcal{O}_{14}$ | $\sum_{i, j, k, l} \hat{\rho}_{i j} \hat{\rho}_{k l}^{3}$ | 0.04 | 0.64 | 95.96 | 97.98 | 0.75 |
| $\mathcal{O}_{15}$ | $\sum_{i, j, k, l} \hat{\rho}_{i j} \hat{\rho}_{j k}^{2} \hat{\rho}_{k l}$ | 0.21 | 1.07 | 95.74 | 94.39 | 0.99 |
| $\mathcal{O}_{16}$ | $\sum_{i, j, k, l} \hat{\rho}_{i j} \hat{\rho}_{j k} \hat{\rho}_{k l}^{2}$ | 0.49 | 2.45 | 95.07 | 73.32 | 0.89 |
| $\mathcal{O}_{17}$ | $\sum_{i, j, k, l} \hat{\rho}_{i j} \hat{\rho}_{j k} \hat{\rho}_{i k} \hat{\rho}_{k l}$ | 1.00 | 7.44 | 95.29 | 20.85 | 0.61 |
| $\mathcal{O}_{18}$ | $\sum_{i, j, k, l} \hat{\rho}_{i j}^{2} \hat{\rho}_{k l}^{2}$ | 0.07 | 2.06 | 94.84 | 76.68 | 0.90 |
| $\mathcal{O}_{19}$ | $\sum_{i, j, k, l} \hat{\rho}_{i j} \hat{\rho}_{j k} \hat{\rho}_{k l} \hat{\rho}_{l i}$ | 1.24 | 8.57* | 95.07 | 8.07 | 0.54* |
| $\mathcal{O}_{20}$ | $\sum_{i, j, k, l, m} \hat{\rho}_{i k} \hat{\rho}_{j k} \hat{\rho}_{l k} \hat{\rho}_{m k}$ | 0.36 | 0.79 | 94.84 | 97.31 | 0.76 |
| $\mathcal{O}_{21}$ | $\sum_{i, j, k, l, m} \hat{\rho}_{i l} \hat{\rho}_{j k} \hat{\rho}_{l k} \hat{\rho}_{m k}$ | 0.39 | 1.67 | 96.19 | 86.10 | 0.95 |
| $\mathcal{O}_{22}$ | $\sum_{i, j, k, l, m} \hat{\rho}_{i j} \hat{\rho}_{k l} \hat{\rho}_{l m} \hat{\rho}_{m k}$ | 1.31 | 10.1 | 95.96 | 1.79 | 0.51 |
| $\mathcal{O}_{23}$ | $\sum_{i, j, k, l, m} \hat{\rho}_{i j}^{2} \hat{\rho}_{k l} \hat{\rho}_{l m}$ | 0.05 | 1.63 | 95.52 | 91.03 | 0.98 |
| $\mathcal{O}_{24}$ | $\sum_{i, j, k, l, m} \hat{\rho}_{i j} \hat{\rho}_{k l} \hat{\rho}_{l m}^{2}$ | 0.24 | 0.81 | 96.41 | 96.41 | 1.00 |
| $\mathcal{O}_{25}$ | $\sum_{i, j, k, l, m} \hat{\rho}_{i j} \hat{\rho}_{j k} \hat{\rho}_{k l} \hat{\rho}_{l m}$ | 0.79 | 5.98 | 95.52 | 38.57 | 0.70 |
| $\mathcal{O}_{26}$ | $\sum_{i, j, k, l, m, n} \hat{\rho}_{i j} \hat{\rho}_{k l} \hat{\rho}_{k m} \hat{\rho}_{k n}$ | 0.12 | 0.54 | 95.74 | 99.10 | 0.61 |
| $\mathcal{O}_{27}$ | $\sum_{i, j, k, l, m, n} \hat{\rho}_{i j} \hat{\rho}_{j k} \hat{\rho}_{l m} \hat{\rho}_{l n}$ | 0.10 | 1.47 | 95.52 | 92.38 | 0.98 |
| $\mathcal{O}_{28}$ | $\sum_{i, j, k, l, m, n} \hat{\rho}_{i j}^{2} \hat{\rho}_{k l} \hat{\rho}_{m n}$ | 0.02 | 0.58 | 95.96 | 98.65 | 0.67 |
| $\mathcal{O}_{29}$ | $\sum_{i, j, k, l, m, n} \hat{\rho}_{i j} \hat{\rho}_{k l} \hat{\rho}_{l m} \hat{\rho}_{m n}$ | 0.65 | 2.10 | 95.96 | 76.01 | 0.90 |
| $\mathcal{O}_{30}$ | $\sum_{i, j, k, l, m, n, o} \hat{\rho}_{i j} \hat{\rho}_{j k} \hat{\rho}_{l m} \hat{\rho}_{n o}$ | 0.13 | 0.67 | 95.07 | 97.53 | 0.75 |
| $\mathcal{O}_{31}$ | $\sum_{i, j, k, l, m, n, o, p} \hat{\rho}_{i j} \hat{\rho}_{k l} \hat{\rho}_{m n} \hat{\rho}_{o p}$ | 0.03 | 0.38 | 95.74 | 99.33 | 0.58 |
| Average |  | 0.42 | 2.50 |  |  | 0.80 |

Table 3.4: For each observable in the first two columns the third column lists the absolute difference between the experimental value and theoretical prediction normalised by the experimental standard deviation. The fourth column lists the ratio of the experimental and theoretical standard deviations. The fifth and sixth columns list the experimental day capture the theoretical day capture respectively. The seventh column gives the Balanced Accuracy of the theoretical model's day capture at $\pm 2 \sigma$. The * values were obtained using an estimate of $\sigma_{\mathrm{T}}$ described at the end of section 3.4.1.

Consider the following binary classification problem: classify a day as typical (positive) if it falls within two standard deviations of the observable mean as calculated from the data, and a day as atypical (negative) if it falls outside this range. Then define the True Positive

| Actual condition Predicted condition | Positive (typical) | Negative (atypical) |
| :---: | :---: | :---: |
| Positive (typical) | True Positive (TP) | False Negative (FN) |
| Negative (atypical) | False Positive (FP) | True Negative (TN) |

Table 3.5: Definitions of TP, TN, FP and FN in generic binary classification. In our case positive is a day with an observable value falling within two standard deviations of the mean and negative is a day with an observable value falling outside this range.

Rate (TPR) and True Negative Rate (TNR) as follows with reference to the quantities defined in table 3.5

$$
\begin{equation*}
\mathrm{TPR}=\frac{\mathrm{TP}}{\mathrm{TP}+\mathrm{FN}}, \quad \mathrm{TNR}=\frac{\mathrm{TN}}{\mathrm{TN}+\mathrm{FP}} . \tag{3.91}
\end{equation*}
$$

Translating back into the terminology of day capture we have

$$
\begin{align*}
& \mathrm{TPR}=\frac{\text { Correctly predicted typical days }}{\text { Total number of typical days }}, \\
& \mathrm{TNR}=\frac{\text { Correctly predicted atypical days }}{\text { Total number of atypical days }} . \tag{3.92}
\end{align*}
$$

From the average of these quantities we define the Balanced Accuracy of the model

$$
\begin{equation*}
\text { Balanced Accuracy }=\frac{\mathrm{TPR}+\mathrm{TNR}}{2} . \tag{3.93}
\end{equation*}
$$

The Balanced Accuracy of the day capture of each observable is listed in the seventh column of table 3.4 Many observables have a Balanced Accuracy of, or very close to 1. The average Balanced Accuracy of the theoretical day capture over all observables is 0.8 , which is generally considered to be a good score in data sciences. This agreement between theory and experiment for day capture is robust to changes in the size of the sample used for the analysis.

### 3.4.3 Absolute errors relative to standard deviations and standard errors of observables

Thus far, we have analysed the differences between theoretical observable mean values and experimental observable mean values, normalised by the experimental standard deviation of the observable values. We have found that 27 out of 31 observables deviate from the model's prediction by less than one experimental standard deviation ${ }^{3}$. These normalised differences have a physical interpretation. Small normalised differences in this case are suggestive of

[^1]small coupling constants for higher order corrections in the action (see section 3.1) which is evidence for near-Gaussianity of the experimental data generating process. Furthermore, we have found that "physical" tests of the theoretical model versus experiment such as calculating the proportion of days captured and balanced accuracy of a typicality classifier provide additional evidence for the pure Gaussian model being a good approximation.

Another possible choice for normalising the differences between the theoretical observable mean values and the experimental observable mean values is the experimental standard error. The standard error in this case is the standard deviation of the experimental observable mean values. We denote the standard error, $\sigma_{\bar{x}}$. Given the definition of the mean estimator, i.e. $\bar{x}=1 / n \sum_{i=1}^{n} x_{i}$, the standard error is equal to the standard deviation of the permutation invariant polynomial values for each matrix, $\sigma_{\mathrm{E}}$, divided by $\sqrt{n}$ i.e. $\sigma_{\bar{x}}=\sigma_{\mathrm{E}} / \sqrt{n}$. The standard error is useful in determining whether the differences between the theoretical observable mean values and experimental observable mean values are plausibly due to sampling variation. The larger the sample size, the smaller the departures between theory and experiment that can be distinguished from sampling variation i.e. "experimental error". In particular, genuine departures correspond to large standard errors e.g. larger than three standard errors. Such departures can be interpreted as highly statistically significant differences.

In our data set, which has a fairly large sample size of 446, we have observed that 13 out of 31 observables have a difference between the theoretical observable mean and experimental observable mean value of fewer than three standard errors (i.e. 18 out of 31 observables exhibit a departure of more than three standard errors). To further explore the statistical significance of differences in the theoretical versus experimental observable mean values we have also calculated the percentile bootstrap confidence intervals of the experimental observable mean values. The bootstrap procedure involved re-sampling from the original set of observable values, uniformly with replacement, to construct 1000 bootstrap samples of the same size as the original sample (i.e. 446). The mean of each such bootstrap sample was then calculated. Given the asymptotic normality of the mean estimator, it is expected that 99.7th percentile bootstrap confidence intervals will approximately correspond to three standard errors on either side of the original experimental mean estimate. This is reflected in our results, which reveal 12 of 31 observables with theoretical observable mean values within this confidence interval and 19 of 31 observables with theoretical mean values outside the interval. This is in close agreement with the aforementioned basic standard error results where we had 13 of 31 observables with theoretical observable mean values lying within three standard errors of the experimental observable mean values.

It is also worth noting that statistically significant differences may nevertheless be small
in terms of relative error, which we recall is defined as

$$
\begin{equation*}
\frac{\left|\left\langle\mathcal{O}_{\alpha}\right\rangle_{\mathrm{T}}-\left\langle\mathcal{O}_{\alpha}\right\rangle_{\mathrm{E}}\right|}{\left\langle\mathcal{O}_{\alpha}\right\rangle_{\mathrm{E}}} . \tag{3.94}
\end{equation*}
$$

Indeed we have observed that 20 out of 31 observables have a relative error of less than $30 \%$ when comparing theoretical versus experimental mean observable values.

The key point from this section is that when we consider the absolute error of observables in comparison to the standard deviation, a measure motivated by consideration of perturbative corrections to the toy Gaussian model, we have 27 of 31 observables which are within one standard deviation (all are within three standard deviations). On the other hand six of 31 are within one standard error ( 13 are within three standard errors). This suggests that developing computations of expectation values in theoretical models which contain small cubic and quartic terms, as guided by the data, is likely to give statistically significant improvement, given our current sample sizes, of the agreement between theoretical and experimental expectation values of observables. This is technically more intricate than computing in the Gaussian model and is left for future investigation.

### 3.5 Applications of matrix theory to matrix data: anomaly detection

Observable vectors, formed using lists of permutation invariant polynomial functions labelled by graphs, provide the key bridge between permutation invariant Gaussian matrix theory and the matrix data. The observable vectors associated with the correlation matrices can be regarded as lower-dimensional representations of the correlation matrices. The observable vectors themselves are random vectors, for which the statistics entailed by the PIGM model are a good approximation in general. A natural question to ask is whether the observable vectors provide a more compact representation of correlation matrices which accentuate statistical "signal" in the data as opposed to noise. Such a representation would be closely linked to an accurate characterisation of the market state and applications could include classification/regression models, clustering analysis, anomaly/outlier detection etc. In this section, we consider the task of anomaly detection.

We demonstrate that the observable vectors do indeed constitute a promising representation for anomaly/outlier detection. The task of anomaly/outlier detection pertains to identifying observations that differ significantly from the majority of the data set. In our context, we seek to identify unusual and noteworthy observable vectors, each of which is associated with the correlation matrix of a particular date. To verify that a meaningful
result has been obtained, we need a notion of unusual and noteworthy dates in the forex market as a reference. The natural approach we take is to consider special dates in the forex trading calendar corresponding to the highest impact economic news announcements. These announcements often lead to a flurry of trading activity along with associated price movements, market volatility and changes in the relationships between currency pairs.

### 3.5.1 Anomaly detection algorithm

A common approach to detecting anomalous/outlier observations, is to utilise a statistical distance measure to determine the distance of each random vector from the mean vector (a natural multivariate measure of centrality), see [83] for example. One can equivalently think of this as determining the length of the random vectors as measured from the origin for centred data. We will utilise the Mahalanobis distance measure to assess these distances. The Mahalanobis distance is similar to the Euclidean distance, when the Euclidean distance is applied to vectors where each element has been scaled by the respective standard deviation, but it better handles the fact that different elements of the vector are correlated in general. Geometrically, surfaces of constant Euclidean distance are spheres, whereas surfaces of constant Mahalanobis distance are ellipsoids in general. The Mahalanobis distance better treats the case of highly correlated elements in the random vectors, as is the case for the observable vectors (as noted in section 3.4.1, see figure 3.3 in particular). Concretely, given a multivariate probability distribution $F$ on $\mathbb{R}^{N}$ (i.e. generating random vectors $\vec{y} \in \mathbb{R}^{N}$ ), with mean vector $\vec{\mu}$ and covariance matrix $\Sigma$, the Mahalanobis distance of a point $\vec{x} \in \mathbb{R}^{N}$ from the mean $\vec{\mu}$ is defined as,

$$
\begin{equation*}
d(\vec{x}, \vec{\mu})=\sqrt{(\vec{x}-\vec{\mu}) \Sigma^{-1}(\vec{x}-\vec{\mu})}, \tag{3.95}
\end{equation*}
$$

where $\mu_{i}=E\left(y_{i}\right)$ and $\Sigma_{i j}=E\left[\left(y_{i}-\mu_{i}\right)\left(y_{j}-\mu_{j}\right)\right]$. The utility of the Mahalanobis distance in anomaly detection is thus in identifying points that are far from the mean, accounting for the covariance structure implied by the distribution $F$. If the distribution $F$ is a multivariate normal distribution for example, there is a particularly direct link between the Mahalanobis distance at a point $\vec{x}$ and the probability density at $\vec{x}$. Distant points have exponentially lower probability density in this case. The Mahalanobis distance can be fruitfully applied even when the distribution $F$ is not known to be a multivariate normal distribution however and we will not need to make this assumption.

### 3.5.2 Economically significant dates

It is well known amongst forex trading practitioners that there are certain currencies and certain types of economic announcements that typically have the highest impact on the forex markets (see 84 for example). These currencies are associated with the countries or blocs with the largest economies, namely the US Dollar, Chinese Renminbi, Japanese Yen, European Union Euro (Germany is the largest economy in the EU at time of writing) and Great British Pound.

Four of the most important classes of economic announcements are the following,

- Central bank meetings and announcements relating to interest rate decisions etc. These include the FOMC (Federal Open Market Committee), ECB (European Central Bank), BoE (Bank of England), PBoC (People's Bank of China) and BoJ (Bank of Japan) meetings associated with the United States, European Union, Great Britain, China and Japan respectively.
- Unemployment data releases. One of the most important examples of this is the US Non-Farm Payrolls release.
- Consumer price index releases. The most important release in this category is the US consumer price index release.
- Unplanned forex news including special central bank meetings and speeches, political speeches etc.

In our subsequent analyses we utilise the economic calendar sourced from [85] of high impact events and filter for only those events pertaining to the aforementioned currencies (and associated economies) and economic announcements specifically. The exact strings used for filtering the events based on event name are: "ECB Press Conference", "BoE MPC", "FOMC Press Conference", "BoJ Press Conference", "PBoC Interest Rate Decision", "Nonfarm Payrolls", "Consumer Price Index ex Food \& Energy", "European Council Meeting", "EU Leaders Special Summit" and "ECB Special Strategy Meeting". During the period 2020-04-01 to 2022-01-31 one or more of these high impact events occurred on approximately $27 \%$ of business days.

### 3.5.3 Dimensionality reduction

The construction of lower-dimensional representations of the correlation matrices - namely the observable vectors - is effectively a dimensionality reduction procedure. As is common


Figure 3.4: Distances of observable vectors and raw correlation vectors from the origin using the standardised Euclidean and Mahalanobis metrics.
with such procedures (e.g. Principal Component Analysis (PCA) ), there is a trade-off between reducing dimensionality and preserving information content. Balancing these trade-offs through a good choice of the number of components often leads to better results in various applications. In PCA, the cumulative variance of the first principal components is typically used as an organising quantity to select how many such components to include. In our case, we take the normalised magnitude of the differences between the empirical expectation values of the cubic and quartic observables and the theoretical predictions of the PIGM model (3.19), as an organising quantity for determining which observables to retain. The thesis is that the empirical higher order observables that depart from theoretical expectations indicate additional information beyond the linear and quadratic structure encoded in the PIGM model. We have empirically determined that the 12 "least Gaussian" observables yield optimal anomaly detection results (i.e. statistical significance and odds-ratios). Notably, the results broadly improve as more observables are added starting from a small number of observables, reach a peak and then decline somewhat as more observables are added. These "least Gaussian" observables are listed in table 3.6. We have also observed that useful information remains in the other observables

| Obervable Label | Observable Def. | Observable Order |
| :--- | :---: | :---: |
| $\mathcal{O}_{3}$ | $\sum_{i, j, k} \hat{\rho}_{i j} \hat{\rho}_{j k} \hat{\rho}_{k i}$ | Cubic |
| $\mathcal{O}_{5}$ | $\sum_{i, j, k, l} \hat{\rho}_{i j} \hat{\rho}_{j k} \hat{\rho}_{k l}$ | Cubic |
| $\mathcal{O}_{9}$ | $\sum_{i, j} \hat{\rho}_{i j}^{4}$ | Quartic |
| $\mathcal{O}_{10}$ | $\sum_{i, j, k} \hat{\rho}_{i j}^{2} \hat{\rho}_{j k}^{2}$ | Quartic |
| $\mathcal{O}_{12}$ | $\sum_{i, j, k} \hat{\rho}_{i j} \hat{\rho}_{i k} \hat{\rho}_{j k}^{2}$ | Quartic |
| $\mathcal{O}_{16}$ | $\sum_{i, j, k, l} \hat{\rho}_{i j} \hat{\rho}_{j k} \hat{\rho}_{k l l}^{2}$ | Quartic |
| $\mathcal{O}_{17}$ | $\sum_{i, j, k, l} \hat{\rho}_{i j} \hat{\rho}_{j k} \hat{\rho}_{i k} \hat{\rho}_{k l}$ | Quartic |
| $\mathcal{O}_{19}$ | $\sum_{i, j, k, l} \hat{\rho}_{i j} \hat{\rho}_{j k} \hat{\rho}_{k l} \hat{\rho}_{l i}$ | Quartic |
| $\mathcal{O}_{21}$ | $\sum_{i, j, k, l, m} \hat{\rho}_{i l} \hat{\rho}_{j k} \hat{\rho}_{l k} \hat{\rho}_{m k}$ | Quartic |
| $\mathcal{O}_{22}$ | $\sum_{i, j, k, l, m} \hat{\rho}_{i j} \hat{\rho}_{k l} \hat{\rho}_{l m} \hat{\rho}_{m k}$ | Quartic |
| $\mathcal{O}_{25}$ | $\sum_{i, j, k, l, m} \hat{\rho}_{i j} \hat{\rho}_{j k} \hat{\rho}_{k l} \hat{\rho}_{l m}$ | Quartic |
| $\mathcal{O}_{29}$ | $\sum_{i, j, k, l, m, n} \hat{\rho}_{i j} \hat{\rho}_{k l} \hat{\rho}_{l m} \hat{\rho}_{m n}$ | Quartic |

Table 3.6: The 12 cubic and quartic observables that have the largest normalised difference from the PIGM predictions.
however. The cubic and quartic observables that are best predicted by the PIGM model still have reasonable effectiveness in anomaly detection for example, as do random subsets of observables and the complete set of observables. This aligns with our overarching findings that the PIGM model is a good fit overall and captures meaningful statistical structure. There does appear to be additional information captured in the least well fit observables however as supported by the results below.

### 3.5.4 Longest observable vectors and economically significant dates

We assess how strongly the lengths of the observable vectors constructed from the observables in table 3.6 are associated with the presence or absence of economically significant events. To investigate the utility of the observable representation, we also compare to the results obtained for the original correlation matrices, applying both standardised Euclidean and Mahalanobis distance measures. Finally, we compare the observable representation to a representation obtained by applying PCA to the original correlation matrices. In particular, we select the smallest number of principal components that captures at least $70 \%$ of the variance, corresponding to the first 10 principal components in this case (this value also matches the number of components to retain as determined by the elbow method [86]). Only the standardised Euclidean metric is applied to the PCA vector since the principal components are uncorrelated and thus the Mahalanobis distance yields equivalent results. The methodology is as follows.

1. Calculate the standardised Euclidean and Mahalanobis vector lengths for the observable vector associated with each date in the dataset (representing distance from the mean observable vector or equivalently the origin in this case). The maximal dimension of the vector space considered here is $D=31$ while the optimal number of least Gaussian observables, as stated earlier, is 12 .
2. Calculate the standardised Euclidean and Mahalanobis vector lengths for the correlation feature vector associated with each date in the dataset. The correlation feature vector for each date is comprised of the 171 pairwise correlations calculated between all 19 currency pairs. We term these features raw correlations.
3. Calculate the standardised Euclidean vector lengths for the PCA feature vector associated with each date in the dataset.
4. Rank the dates in the dataset by Euclidean and Mahalanobis vector length, in descending order for the observable vectors, raw correlation and PCA feature vectors.
5. Assess whether the top 25,50 and 100 dates have a statistically significantly higher number of economically significant events than the bottom 25, 50 and 100 dates ordered by distance in a descending manner. In addition, we calculate the ratio between the odds of observing an economically significant news event in the top/most anomalous $25,50,100$ dates and the odds of such an event occurring in the bottom/most typical $25,50,100$ dates. This odds-ratio (OR) is defined as,

$$
\begin{equation*}
\mathrm{OR}=\frac{P_{T} /\left(1-P_{T}\right)}{P_{B} /\left(1-P_{B}\right)}, \tag{3.96}
\end{equation*}
$$

where $P_{B}$ corresponds to the proportion of the $25,50,100$ closest dates to the origin that are associated with economically significant events. Similarly, $P_{T}$ corresponds to the proportion of the $25,50,100$ furthest dates from the origin that are associated with economically significant events.

The distances for the respective metrics and features are presented in figure 3.4 Notably, the observable feature vectors appear to have more distinct outlier days and less noise. In addition, we note that the PCA vector lengths yield a fairly similar pattern to the observable vector lengths with the Mahalanobis distance. The results of comparing the top $25,50,100$ dates by distance from the origin with the bottom $25,50,100$ dates respectively are collected in table 3.7. The best contrast of the number of economic events appearing in the furthest days from the origin compared to the closest days from the origin respectively is given by the Mahalanobis distance evaluated on observable vectors. Indeed, for the Mahalanobis distance on observable vectors, there is a higher degree of statistical significance and higher odds-ratios than the other combinations of metric and features in
almost all cases. The high odds-ratios imply that the odds of a economically significant event occurring in the anomalous groups (most distant) are much higher than the odds in the typical groups (least distant). This provides evidence that the observable vectors are a good, low-dimensional characterisation of the market state and accentuate meaningful financial "signal".

An additional note relates to the correlation matrices and associated observable vectors for February 2022. During this period, there were several extremely anomalous dates (with extreme vector lengths), coinciding with the beginning of the war in Ukraine. These had the effect of masking the anomalous nature of earlier events and reducing the sensitivity of the detection algorithm. This is a well known consequence of applying the Mahalanobis distance to anomaly detection, termed the masking effect [83]. We therefore excluded February 2022 from all our analyses. The analysis conducted thus far can be regarded as pertaining to in-sample anomaly detection. We also conducted an out-of-sample analysis using the same observables and number of principal components for PCA as the in-sample analysis, now for the date range 2022-03-01 to 2023-03-31. The results are collected in table 3.8 and reveal that the in-sample anomaly detection results generalise well, and the Mahalanobis distance calculated on observable vectors continues to out-perform the alternatives in the majority of cases. Two other robustness checks that were conducted include rerunning the analysis with correlation matrices constructed with 10 and 15 minute sampling intervals for the log returns (as opposed to 5 minutes) as well as testing subsets of the most important economic announcements. The results of these analyses were qualitatively similar to those already presented.

| Metric | Features | Subset Size | $P_{T}$ | $P_{B}$ | p-value | Odds-Ratio |
| :--- | :--- | ---: | ---: | ---: | ---: | ---: |
| Euclidean | Observables | 25 | 0.40 | 0.20 | $10.8 \times 10^{-2}$ | 2.67 |
| Euclidean | Observables | 50 | 0.38 | 0.24 | $9.7 \times 10^{-2}$ | 1.94 |
| Euclidean | Observables | 100 | 0.38 | 0.28 | $8.8 \times 10^{-2}$ | 1.58 |
| Mahalanobis | Observables | 25 | 0.44 | 0.16 | $3.1 \times 10^{-2 *}$ | 4.12 |
| Mahalanobis | Observables | 50 | 0.38 | 0.16 | $1.2 \times 10^{-2 *}$ | 3.22 |
| Mahalanobis | Observables | 100 | 0.39 | 0.13 | $0.0 \times 10^{-2 * * *}$ | 4.28 |
| Euclidean | PCA Correlations | 25 | 0.60 | 0.32 | $4.4 \times 10^{-2 *}$ | 3.19 |
| Euclidean | PCA Correlations | 50 | 0.46 | 0.24 | $1.8 \times 10^{-2 *}$ | 2.70 |
| Euclidean | PCA Correlations | 100 | 0.41 | 0.23 | $0.5 \times 10^{-2 * *}$ | 2.33 |
| Euclidean | Raw Correlations | 25 | 0.52 | 0.32 | $12.6 \times 10^{-2}$ | 2.30 |
| Euclidean | Raw Correlations | 50 | 0.44 | 0.28 | $7.2 \times 10^{-2}$ | 2.02 |
| Euclidean | Raw Correlations | 100 | 0.39 | 0.22 | $0.7 \times 10^{-2 * *}$ | 2.27 |
| Mahalanobis | Raw Correlations | 25 | 0.24 | 0.16 | $36.3 \times 10^{-2}$ | 1.66 |
| Mahalanobis | Raw Correlations | 50 | 0.32 | 0.14 | $2.8 \times 10^{-2 *}$ | 2.89 |
| Mahalanobis | Raw Correlations | 100 | 0.28 | 0.20 | $12.3 \times 10^{-2}$ | 1.56 |

Table 3.7: In-sample anomaly detection results. In the table above, the proportions, $P_{B}, P_{T}$ and the odds-ratio, OR, are as defined in equation (3.96). The p -value is obtained using Fisher's exact one-sided test. The * symbol following a p-value indicates significance at the 0.05 level, ${ }^{* *}$ indicates significance at the 0.01 level and ${ }^{* * *}$ indicates significance at the 0.001 level.

| Metric | Features | Subset Size | $P_{T}$ | $P_{B}$ | p-value | Odds-Ratio |
| :--- | :--- | ---: | ---: | ---: | ---: | ---: |
| Euclidean | Observables | 25 | 0.44 | 0.20 | $6.4 \times 10^{-2}$ | 3.14 |
| Euclidean | Observables | 50 | 0.44 | 0.24 | $2.8 \times 10^{-2 *}$ | 2.49 |
| Euclidean | Observables | 100 | 0.29 | 0.23 | $21.0 \times 10^{-2}$ | 1.37 |
| Mahalanobis | Observables | 25 | 0.56 | 0.16 | $0.4 \times 10^{-2 * *}$ | 6.68 |
| Mahalanobis | Observables | 50 | 0.50 | 0.10 | $0.0 \times 10^{-2 * * *}$ | 9.00 |
| Mahalanobis | Observables | 100 | 0.37 | 0.12 | $0.0 \times 10^{-2 * * *}$ | 4.31 |
| Euclidean | PCA Correlations | 25 | 0.48 | 0.20 | $3.6 \times 10^{-2 *}$ | 3.69 |
| Euclidean | PCA Correlations | 50 | 0.48 | 0.14 | $0.0 \times 10^{-2 * * *}$ | 5.67 |
| Euclidean | PCA Correlations | 100 | 0.42 | 0.13 | $0.0 \times 10^{-2 * * *}$ | 4.85 |
| Euclidean | Raw Correlations | 25 | 0.48 | 0.20 | $3.6 \times 10^{-2 *}$ | 3.69 |
| Euclidean | Raw Correlations | 50 | 0.48 | 0.16 | $0.1 \times 10^{-2 * * *}$ | 4.85 |
| Euclidean | Raw Correlations | 100 | 0.42 | 0.14 | $0.0 \times 10^{-2 * * *}$ | 4.45 |
| Mahalanobis | Raw Correlations | 25 | 0.28 | 0.16 | $24.8 \times 10^{-2}$ | 2.04 |
| Mahalanobis | Raw Correlations | 50 | 0.36 | 0.20 | $5.9 \times 10^{-2}$ | 2.25 |
| Mahalanobis | Raw Correlations | 100 | 0.31 | 0.25 | $21.6 \times 10^{-2}$ | 1.35 |

Table 3.8: Out-of-sample anomaly detection results. In the table above, the proportions, $P_{B}, P_{T}$ and the odds-ratio, OR, are as defined in equation (3.96). The p-value is obtained using Fisher's exact one-sided test. The * symbol following a p-value indicates significance at the 0.05 level, ${ }^{* *}$ indicates significance at the 0.01 level and ${ }^{* * *}$ indicates significance at the 0.001 level.

### 3.6 Discussion

We have developed the most general four-parameter PIGM models appropriate for ensembles of matrices which are symmetric and diagonally vanishing. We have used the models to find evidence for near-Gaussianity in ensembles of matrices, one matrix for every day over a period, constructed from high-frequency foreign exchange price quotes. The nearGaussianity was found to be robust against changes in how the ensemble was constructed: we varied the time intervals between the quote updates used to construct the daily averages, as well as the number of days used in our ensemble.

The near-Gaussianity is used to motivate a data-reduction technique based on the use of low degree permutation invariant functions of matrices (observables) as characteristics
of the entities represented by the matrices in the ensemble, in this case the days in the period under consideration. The small non-Gaussianities of each observable were used to rank the observables in order of decreasing non-Gaussianity and to find an optimal number of least Gaussian observables for data analysis. The degree of non-Gaussianity is thus being used as an analog of the magnitude of singular values in principal component analysis (PCA). The sets of observables considered, either the full set of observables up to quartic degree or the subsets with optimal number of least Gaussian observables, are much smaller than the number of matrix elements in the matrices. We found successful results in anomaly detection based on the observables to find the most atypical and the most typical days in the ensemble. We demonstrated statistically significant matching between these typicality/atypicality results extracted from the data of financial correlation matrices and corresponding results based on human economic judgement of significant events affecting foreign exchange markets. We propose that the success of the use of a set of least Gaussian observables in anomaly detection should be interpreted as indicating that these ensembles of daily foreign exchange matrices capture an economic reality best described by the Gaussian model perturbed by specific small cubic and quartic couplings in the action. The non-Gaussianities capture system-specific non-universalities while the overall approximate Gaussianity is a universal characteristic which holds across diverse systems, as indeed already evidenced in ensembles of words [55, 56].

## Chapter 4

## Hidden symmetries and large $\mathbf{N}$ factorisation

In this chapter, we develop the theme of large $N$ factorisation for PIGM models. The formulation of large $N$ factorisation we use is similar to the one in [35. We will use the simplest inner product on the space of PIMOs. It comes from a special point on the moduli space of PIGM models of general $N \times N$ matrices where the action has an enhanced $O(N)$ symmetry. This is the first sense in which hidden symmetries appear in this chapter. We note that this large $N$ factorisation result carries over to the quantum mechanical regime. In fact, in section 5.6 of the next chapter we generalise this result to show the large $N$ factorisation of quantum mechanical permutation invariant states.

The second kind of hidden symmetry appearing in this chapter is based on Schur-Weyl duality. Observables invariant under the action of a symmetry group $G$ are organised by algebras dual to $G$. For the case of $U(N)$ symmetry the dual algebras are based on the standard Schur-Weyl duality [5] between $U(N)$ and $S_{k}$ in the $k$-fold tensor product $V^{\otimes k}$ of the fundamental representation $V$ of $U(N)$. Applications of Schur-Weyl duality to the computation of correlators in matrix models with $U(N)$ symmetry are developed in [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19] and short reviews are [20, 21]. The $U(N)$ case serves as a powerful source of analogies throughout the chapter. When $U(N)$ is replaced by $S_{N}$ as the invariance of interest, the Schur-Weyl dual algebras are diagrammatic partition algebras $P_{k}(N)$. An introduction to these algebras suitable for our purposes can be found in section 2.3, more detailed information can be found in the references therein.

The chapter is organised as follows. In section 4.1 we review the counting of PIMOs described in section [2.5. We give a new description of the counting, which emphasises the underlying hidden partition algebra symmetry arising as a consequence of Schur-Weyl
duality. We then give a derivation of the $O(N)$ symmetric point in the moduli space of $S_{N}$ invariant one-matrix models.

Section 4.2 is dedicated to the construction of PIMOs by means of partition algebras. The analogous construction of $U(N)$ invariants using symmetric group algebras is reviewed as a warm-up exercise. This is generalised to give a map from partition algebra elements to PIMOs (equation (4.51)), leading to a correspondence between PIMOs and equivalence classes of partition algebra elements. These equivalence classes are defined in equation (4.54). The simplest $O(N)$ invariant action is used to define an inner product on the space of PIMOs, which can be written as a trace of partition algebra elements (equation 4.56)).

Section 4.3 proves the large $N$ factorisation of the inner product on PIMOs thus defined. That is, we show the inner product obeys

$$
\begin{equation*}
\left\langle\hat{\mathcal{O}}_{i} \hat{\mathcal{O}}_{j}\right\rangle=\delta_{i j}+O(1 / \sqrt{N}), \tag{4.1}
\end{equation*}
$$

where $\hat{\mathcal{O}}_{i}, \hat{\mathcal{O}}_{j}$ are normalised PIMOs labelled by indices $i, j$ running over equivalence classes of partition algebra elements. The proof of large $N$ factorisation relies on the existence of a partial ordering on the diagram basis for the partition algebra. The partial ordering is related to an inclusion of diagrams, and can itself be described by another diagram of diagrams called a Hasse diagram [87. Equation (4.1) generalises a familiar large $N$ factorisation property of inner products of matrix traces invariant under continuous symmetries. We also extend the proof to show the large $N$ factorisation of multi-matrix observables.

### 4.1 Hidden symmetries in permutation invariant Gaussian matrix models

This section begins with a counting of PIMOs which emphasises the role played by the dual algebra to $S_{N}$ acting on $V_{N}^{\otimes k}$, this dual algebra is called the partition algebra. After this we derive the parameter limit of the 13 parameter space of general PIGM models in which there is an enhanced $O(N)$ symmetry.

### 4.1.1 Counting matrix observables using partition algebras

Permutation invariant matrix polynomials are defined to obey

$$
\begin{equation*}
\mathcal{O}\left(M_{\sigma(i) \sigma(j)}\right)=\mathcal{O}\left(M_{i j}\right), \quad \forall \sigma \in S_{N} . \tag{4.2}
\end{equation*}
$$

These PIMOs can be organised by their degree. At degree $k$, the matrix monomials

$$
\begin{equation*}
M_{i_{1} i_{1}}, M_{i_{2} i_{2^{\prime}}} \ldots M_{i_{k} i_{k^{\prime}}} \tag{4.3}
\end{equation*}
$$

form a basis for a vector space isomorphic to $\operatorname{Sym}^{k}\left(V_{N} \otimes V_{N}\right)$. The symmetric group $S_{k}$ acts on $\left(V_{N} \otimes V_{N}\right)^{\otimes k}$ by permuting the $k$ tensor factors. The subspace $\operatorname{Sym}^{k}\left(V_{N} \otimes V_{N}\right)$ is the subspace of $S_{k}$ invariants in $\left(V_{N} \otimes V_{N}\right)^{\otimes k}$. This $S_{k}$ invariance is imposed by the bosonic symmetry of the matrix variables $M_{i j}$. The PIMOs form the $S_{N} \times S_{k}$ invariant subspace of $\left(V_{N} \otimes V_{N}\right)^{\otimes k}$ :

$$
\begin{align*}
& \text { Matrix polynomials of degree } k \text { invariant under } S_{N} \\
& =\text { Invariants } S_{N} \times S_{k}\left(\left(V_{N} \otimes V_{N}\right)^{\otimes k}\right) \equiv\left[\left(V_{N} \otimes V_{N}\right)^{\otimes k}\right]_{N} S_{N} \\
& =\left\{v \in\left(V_{N} \otimes V_{N}\right)^{\otimes k}: \sigma v=v, \tau v=v \mid \forall \sigma \in S_{N}, \tau \in S_{k}\right\} . \tag{4.4}
\end{align*}
$$

Note that the action of $\tau \in S_{k}$ on $\left(V_{N} \otimes V_{N}\right)^{\otimes k}$ commutes with the action of $\sigma \in S_{N}$. This follows since the same $\sigma$ is applied to all tensor factors.

In 2.5 we saw the dimension of the space of independent PIMOs for matrices of size $N$ and polynomial degree $k$ was given by

$$
\begin{equation*}
\mathcal{N}(N, k)=\frac{1}{N!k!} \sum_{p \vdash N} \sum_{q \vdash k} \frac{N!}{\prod_{i=1}^{N} i^{p_{i}} p_{i}!} \frac{k!}{\prod_{i=1}^{k} i^{q_{i}} q_{i}!} \prod_{i=1}^{k}\left(\sum_{l \mid i} l p_{l}\right)^{2 q_{i}} . \tag{4.5}
\end{equation*}
$$

The initial sums run over integer partitions (Young diagrams) $p$ of $N$, and integer partitions $q$ of $k$ while the final sum is over the integer divisors $l$ of $i$. The equation (4.5) computes the multiplicity of the trivial representation of $S_{N} \times S_{k}$ in the decomposition of $\left(V_{N} \otimes V_{N}\right)^{\otimes k}$, which is the dimension of $\left[\left(V_{N} \otimes V_{N}\right)^{\otimes k}\right]^{S_{N} \times S_{k}}$. There exists an isomorphism

$$
\begin{equation*}
\left(V_{N} \otimes V_{N}\right)^{\otimes k} \cong \bigoplus_{\Lambda_{1}, \Lambda_{2}} V_{\Lambda_{1}}^{S_{N}} \otimes V_{\Lambda_{2}}^{S_{k}} \otimes V_{\Lambda_{1} \Lambda_{2}} \tag{4.6}
\end{equation*}
$$

organising the space into irreducible representations of $S_{N} \times S_{k}$, with multiplicities $V_{\Lambda_{1} \Lambda_{2}}$. Let $V_{[N]} \otimes V_{[k]}$ denote the trivial representation of $S_{N} \times S_{k}$ with multiplicity space $V_{[N][k]}$, then the dimension of $S_{N} \times S_{k}$ invariants is given by

$$
\begin{equation*}
\mathcal{N}(N, k)=\operatorname{Dim} V_{[N][k]} . \tag{4.7}
\end{equation*}
$$

The generalisation to multi-matrix observables and a proof of their correspondence with colored directed graphs was developed in 11. The approach we use in this chapter is based on a new way of counting PIMOs, utilising the connection between dual algebras and matrix invariants.

We begin by reviewing this connection in the case of $U(N)$ invariants. Tensor products of the defining representation $V$ of $U(N)$ have a multiplicity free decomposition into irreducible representations of $U(N) \times S_{k}$ labelled by Young diagrams

$$
\begin{equation*}
V^{\otimes k} \cong \bigoplus_{\substack{\Lambda \vdash k \\ l(\Lambda) \leq N}} V_{\Lambda}^{U(N)} \otimes V_{\Lambda}^{\mathbb{C} S_{k}} \tag{4.8}
\end{equation*}
$$

The sum runs over Young diagrams $\Lambda$ with $k$ boxes, and for $k>N$ is restricted such that the number of rows $l(\Lambda)$ in the Young diagram $\Lambda$ is not greater than $N$. In the remainder of this chapter we will assume $N \geq k$ for discussions of the unitary group. This result is known as Schur-Weyl duality (see chapter 6 in [5]). On the left-hand side of this equation we have a basis $e_{i_{1}} \otimes e_{i_{2}} \otimes \cdots \otimes e_{i_{k}}$ with each index $i$ running from 1 to $N$. On the right-hand side we have a basis $E_{M m}^{\Lambda}$ with

$$
\begin{align*}
& m \in\left\{1, \ldots, \operatorname{Dim} V_{\Lambda}^{\mathbb{C} S_{k}}\right\}, \\
& M \in\left\{1, \ldots, \operatorname{Dim} V_{\Lambda}^{U(N)}\right\} . \tag{4.9}
\end{align*}
$$

For a fixed Young diagram $\Lambda$ and a fixed state $M$ in $V_{\Lambda}^{U(N)}$, there is a multiplicity of $\operatorname{Dim} V_{\Lambda}^{\mathbb{C} S_{k}}$. That is, we have

$$
\begin{equation*}
\operatorname{Mult}\left(V^{\otimes k} \rightarrow V_{\Lambda}^{U(N)}\right)=\operatorname{Dim} V_{\Lambda}^{\mathbb{C} S_{k}} . \tag{4.10}
\end{equation*}
$$

It is well-known that $U(N)$ invariant matrix observables have a basis of multi-traces. These traces can be parameterised by conjugacy classes of permutations. A description of the connection between gauge invariant observables and equivalence classes of permutations for single matrix as well as multi-matrix problems, with applications to AdS/CFT is given in [21. We review the connection here with an emphasis on Schur-Weyl duality from the outset. This framework, as explained in [21], can be used to give a description of finite $N$ effects on the counting and construction of gauge invariant observables, but we focus here, as previously mentioned, on the case $N \geq k$. For the unitary group the matrix elements $M_{i j}$ are isomorphic to $V \otimes V^{*}$, where $V^{*}$ is the complex conjugate representation of $V$. In other words, $U \in U(N)$ acts on $M$ by conjugation,

$$
\begin{equation*}
M \mapsto U M U^{\dagger} . \tag{4.11}
\end{equation*}
$$

Since $\left(V \otimes V^{*}\right)^{\otimes k} \cong V^{\otimes k} \otimes\left(V^{*}\right)^{\otimes k}$, we have

$$
\begin{equation*}
\left(V \otimes V^{*}\right)^{\otimes k} \cong\left(\bigoplus_{\Lambda \vdash k} V_{\Lambda}^{U(N)} \otimes V_{\Lambda}^{\mathbb{C} S_{k}}\right) \otimes\left(\bigoplus_{\Lambda^{\prime} \vdash k}\left(V^{*}\right)_{\Lambda^{\prime}}^{U(N)} \otimes V_{\Lambda^{\prime}}^{\mathbb{C} S_{k}}\right) . \tag{4.12}
\end{equation*}
$$

$U(N)$ invariants appear in a tensor product $V_{\Lambda}^{U(N)} \otimes\left(V^{*}\right)_{\Lambda^{\prime}}^{U(N)}$ (with multiplicity 1) if and only if $\Lambda=\Lambda^{\prime}$ :

$$
\begin{equation*}
\operatorname{Dim}\left[V_{\Lambda}^{U(N)} \otimes\left(V^{*}\right)_{\Lambda^{\prime}}^{U(N)}\right]^{U(N)}=\delta_{\Lambda \Lambda^{\prime}} . \tag{4.13}
\end{equation*}
$$

We are using $[W]^{U(N)}$ to refer to the $U(N)$ invariant subspace of the representation $W$. We have

$$
\begin{align*}
{\left[\left(V \otimes V^{*}\right)^{\otimes k}\right]^{U(N)} } & \cong \bigoplus_{\Lambda, \Lambda^{\prime} \vdash k}\left[V_{\Lambda}^{U(N)} \otimes\left(V^{*}\right)_{\Lambda^{\prime}}^{U(N)}\right]^{U(N)} \otimes V_{\Lambda}^{\mathbb{C} S_{k}} \otimes V_{\Lambda^{\prime}}^{\mathbb{C} S_{k}} \\
& \cong \bigoplus_{\Lambda, \Lambda^{\prime} \vdash k} \delta_{\Lambda \Lambda^{\prime}} V_{\Lambda}^{\mathbb{C} S_{k}} \otimes V_{\Lambda^{\prime}}^{\mathbb{C} S_{k}}  \tag{4.14}\\
& \cong \bigoplus_{\Lambda \vdash k} V_{\Lambda}^{\mathbb{C} S_{k}} \otimes V_{\Lambda}^{\mathbb{C} S_{k}}
\end{align*}
$$

where the second line follows from Schur's Lemma which implies equation (4.13). Since we are looking for $U(N)$ invariant polynomials of degree $k$ in $M_{i j}$, the counting is given by the $U(N)$ invariant subspace of $\operatorname{Sym}^{k}\left(V \otimes V^{*}\right)$. Equivalently this is the space $\left[\left(V \otimes V^{*}\right)^{\otimes k}\right]^{U(N) \times S_{k}}$. There is a one-dimensional space of $S_{k}$ invariants in $V_{\Lambda}^{\mathbb{C} S_{k}} \otimes V_{\Lambda}^{\mathbb{C} S_{k}}$ for each $\Lambda$. Hence the counting is given by

Dimension of the space of $U(N)$ invariant polynomials of degree $k$ in $M_{i j}$

$$
=\sum_{\Lambda \vdash k} 1
$$

$=$ Number of integer partitions of $k$
$=$ Number of multi-trace structures with $k$ copies of $M$.
Thus the counting of $U(N)$ invariants is controlled by the symmetric group algebra, which appeared through Schur-Weyl duality.

Similarly, in the case of $S_{N}$ invariant observables there is a dual algebra at play. The dual algebra for the natural representation of $S_{N}$ is called the partition algebra, denoted $P_{k}(N)$ [22, 24]. The representations of the partition algebra determine the multiplicities of $S_{N}$ irreducible representations through the decomposition (see section 2.5 in [28])

$$
\begin{equation*}
V_{N}^{\otimes k} \cong \bigoplus_{l=0}^{k} \bigoplus_{\Lambda_{1}^{\#} \vdash l} V_{\left[N-l, \Lambda_{1}^{\#}\right]}^{S_{N}} \otimes V_{\left[N-l, \Lambda_{1}^{\#}\right]}^{P_{P}(N)} . \tag{4.16}
\end{equation*}
$$

The Young diagram $\Lambda_{1}=\left[N-l, \Lambda_{1}^{\#}\right]$, which is an integer partition of $N$, is constructed by placing the diagram $\Lambda_{1}^{\#}$ below a first row of $N-l$ boxes. Of course $\Lambda_{1}$ must be a valid Young diagram, this imposes some constraints on $\Lambda_{1}^{\#}$ which are not manifest in 4.16. This occurs for $N<2 k$ as we explain, while it does not occur for $N \geq 2 k$. The latter is called the stable limit. To understand this, we denote the first row length of $\Lambda_{1}^{\#}$ by
$r_{1}\left(\Lambda_{1}^{\#}\right)$. For $N \geq 2 k$, all values of $l$ and all choices of $\Lambda_{1}^{\#}$ give valid Young diagrams $\Lambda_{1}$, since $N-l \geq r_{1}\left(\Lambda_{1}^{\#}\right)$. Indeed writing $N=2 k+a$ for some $a \geq 0$, we have

$$
\begin{equation*}
N-l=2 k+a-l \geq k+a \tag{4.17}
\end{equation*}
$$

The inequality follows since $l \leq k$ in equation 4.16). We also have

$$
\begin{equation*}
k+a \geq r_{1}\left(\Lambda_{1}^{\#}\right) \tag{4.18}
\end{equation*}
$$

which follows because $\Lambda_{1}^{\#}$ has no more than $k$ boxes. For $N<2 k$, the condition $N-l \geq$ $r_{1}\left(\Lambda_{1}^{\#}\right)$ imposes a non-trivial $N$-dependent restriction on $\Lambda_{1}^{\#}$. Indeed let $N=2 k-a$ for $a>0$, then the condition $N-l \geq r_{1}\left(\Lambda_{1}^{\#}\right)$ becomes

$$
\begin{equation*}
k-a \geq r_{1}\left(\Lambda_{1}^{\#}\right) \tag{4.19}
\end{equation*}
$$

This is non-trivial condition since $\Lambda_{1}^{\#}$ can have up to $k$ boxes.
Note that the symmetric group algebra $\mathbb{C} S_{k}$ is a subalgebra of $P_{k}(N)$ (permutations of the tensor factors commute with the action of $S_{N}$ on $V_{N}^{\otimes k}$ ), see section 2.3 a detailed description of this. We can restrict any representation $V_{\Lambda_{1}}^{P_{k}(N)}$ to $\mathbb{C} S_{k}$ to give a decomposition of the form

$$
\begin{equation*}
V_{\Lambda_{1}}^{P_{k}(N)} \cong \bigoplus_{\Lambda_{2} \vdash k} V_{\Lambda_{2}}^{S_{k}} \otimes V_{\Lambda_{1} \Lambda_{2}}^{P_{k}(N) \rightarrow \mathbb{C} S_{k}} \tag{4.20}
\end{equation*}
$$

The dimension of $V_{\Lambda_{1} \Lambda_{2}}^{P_{k}(N) \rightarrow \mathbb{C} S_{k}}$ is the branching multiplicity

$$
\begin{equation*}
\operatorname{Dim}\left(V_{\Lambda_{1} \Lambda_{2}}^{P_{k}(N) \rightarrow \mathbb{C} S_{k}}\right)=\operatorname{Mult}\left(V_{\Lambda_{1}}^{P_{k}(N)} \rightarrow V_{\Lambda_{2}}^{\mathbb{C}\left(S_{k}\right)}\right) \tag{4.21}
\end{equation*}
$$

Since $\left(V_{N} \otimes V_{N}\right)^{\otimes k} \cong V_{N}^{\otimes k} \otimes V_{N}^{\otimes k}$ we have

$$
\begin{equation*}
\left(V_{N} \otimes V_{N}\right)^{\otimes k} \cong\left(\underset{\substack{\Lambda_{1} \vdash N \\ \Lambda_{2} \vdash k}}{ } V_{\Lambda_{1}}^{S_{N}} \otimes V_{\Lambda_{2}}^{S_{k}} \otimes V_{\Lambda_{1} \Lambda_{2}}^{P_{k}(N) \rightarrow \mathbb{C} S_{k}}\right) \otimes\left(\underset{\substack{\Lambda_{1}^{\prime} \vdash N \\ \Lambda_{2}^{\prime} \vdash k}}{ } V_{\Lambda_{1}^{\prime}}^{S_{N}} \otimes V_{\Lambda_{2}^{\prime}}^{S_{k}} \otimes V_{\Lambda_{1}^{\prime} \Lambda_{2}^{\prime}}^{P_{k}(N) \rightarrow \mathbb{C} S_{k}}\right) \tag{4.22}
\end{equation*}
$$

There is a single $S_{N}$ invariant state in every tensor product $V_{\Lambda_{1}} \otimes V_{\Lambda_{1}^{\prime}}$ if and only if $V_{\Lambda_{1}} \cong V_{\Lambda_{1}^{\prime}}$, and similarly for $S_{k}$. Therefore

$$
\begin{equation*}
\left[\left(V_{N} \otimes V_{N}\right)^{\otimes k}\right]_{N}^{S_{N} \times S_{k}} \cong \bigoplus_{\substack{\Lambda_{1} \vdash N \\ \Lambda_{2} \vdash k}} V_{\Lambda_{1} \Lambda_{2}}^{P_{k}(N) \rightarrow \mathbb{C} S_{k}} \otimes V_{\Lambda_{1} \Lambda_{2}}^{P_{k}(N) \rightarrow \mathbb{C} S_{k}} \tag{4.23}
\end{equation*}
$$

and considering the dimension of this subspace of $S_{N} \times S_{k}$ invariants, $V_{[N][k]}$, we find

$$
\begin{equation*}
\operatorname{Dim} V_{[N][k]}=\mathcal{N}(N, k)=\sum_{\Lambda_{1} \vdash N} \sum_{\Lambda_{1} \vdash k} \operatorname{Mult}\left(V_{\Lambda_{1}}^{P_{k}(N)} \rightarrow V_{\Lambda_{2}}^{\mathbb{C}\left(S_{k}\right)}\right)^{2} . \tag{4.24}
\end{equation*}
$$

The sum of squares is indicative of a matrix (Artin-Wedderburn) decomposition [88, 89] of a hidden algebra parametrising PIMOs (we found the exposition of the Artin-Wedderburn decomposition in [90 to be useful). We will turn to an explicit construction of PIMOs using partition algebra elements in line with the counting (4.24) in section 4.2. This sum of squares form of counting invariants, and their connection to the Artin-Wedderburn structure of algebras, has been used in a number of multi-matrix and tensor model applications, e.g. 91, 92, 93, 94, 95).

### 4.1.2 Enhanced $O(N)$ symmetry in parameter space

The quadratic GOE (Gaussian Orthogonal Ensemble) is determined by the probability density function

$$
\begin{equation*}
\exp \left(-S_{\mathrm{GOE}}(M)\right)=\exp \left(-\operatorname{Tr}\left(M M^{T}\right)\right) \tag{4.25}
\end{equation*}
$$

on the space of real symmetric matrices (see definition 2.3.1. in [96]). The matrix elements $M_{i j}$ for $i \leq j$ in this ensemble of matrices are statistically independent. There are no mixing terms. Here we consider the underlying space to be the space of real matrices, with no symmetry constraint. There is a four-parameter family of $O(N)$ invariant quadratic action: ${ }^{11}$

$$
\begin{equation*}
S_{O(N)}(M)=N \epsilon \operatorname{Tr}(M)-\left(N \alpha \operatorname{Tr}\left(M M^{T}\right)+N \beta \operatorname{Tr}(M M)+\gamma(\operatorname{Tr} M)^{2}\right) . \tag{4.26}
\end{equation*}
$$

In this model, the matrix elements are not statistically independent, but the linear and quadratic moments are readily solvable, as we now show. Higher moments can be obtained using Wick's theorem.

This four parameter family is a special case of the general PIGM models considered in 2.6. We now solve for the second moments of matrix variables for the model defined by 4.26) and compare with the second moments of the 13 parameter PIGM model. This gives the limit in which the 13 parameter PIGM model reduces to the $O(N)$ invariant matrix model.

[^2]We begin by rewriting the action:

$$
\begin{align*}
S_{O(N)}(M)=N \epsilon & \sum_{i} M_{i i} \\
& -N(\alpha+\beta) \sum_{i} M_{i i}^{2}-N \alpha \sum_{i \neq j} M_{i j}^{2}-N \beta \sum_{i \neq j} M_{i j} M_{j i}-\gamma \sum_{i, j} M_{i i} M_{j j} . \tag{4.27}
\end{align*}
$$

Let

$$
\begin{equation*}
z=\left(M_{11}, M_{22}, \ldots, M_{N N}, M_{12}, M_{21}, M_{13}, M_{31}, \ldots, M_{N-1 N}, M_{N N-1}\right), \tag{4.28}
\end{equation*}
$$

then the action can be expressed as

$$
\begin{equation*}
S_{O(N)}(z)=z \mu-z G z^{T} \tag{4.29}
\end{equation*}
$$

The vector $\mu$ is

$$
\mu=\left(\begin{array}{c}
N \epsilon  \tag{4.30}\\
\vdots \\
N \epsilon \\
0 \\
\vdots \\
0
\end{array}\right)
$$

with the first $N$ terms equal to $\epsilon$ and the rest 0 and

$$
\begin{align*}
& G=\left(\begin{array}{cccc}
G_{1} & & & \\
& G_{2} & & \\
& & \ddots & \\
& & & G_{2}
\end{array}\right), \quad G_{1}=N\left(\begin{array}{ccc}
\alpha+\beta & & \\
& \ddots & \\
& & \alpha+\beta
\end{array}\right)+\left(\begin{array}{ccc}
\gamma & \cdots & \gamma \\
\vdots & \vdots & \vdots \\
\gamma & \cdots & \gamma
\end{array}\right), \\
& G_{2}=N\left(\begin{array}{cc}
\alpha & \beta \\
\beta & \alpha
\end{array}\right) . \tag{4.31}
\end{align*}
$$

The inverse of $G_{2}$ is

$$
\left(G_{2}\right)^{-1}=\frac{1}{N\left(\alpha^{2}-\beta^{2}\right)}\left(\begin{array}{cc}
\alpha & -\beta  \tag{4.32}\\
-\beta & \alpha
\end{array}\right)
$$

while the inverse of $G_{1}$ is given by

$$
\left(G_{1}\right)_{i j}^{-1}= \begin{cases}\frac{1}{N^{2}}\left(\frac{N-1}{\alpha+\beta}+\frac{1}{\alpha+\beta+\gamma}\right), & \text { if } i=j,  \tag{4.33}\\ -\frac{1}{N^{2}}\left(\frac{\gamma}{(\alpha+\beta)(\alpha+\beta+\gamma)}\right), & \text { if } i \neq j .\end{cases}
$$

From the form of these inverse matrices we can write down the connected two-point function

$$
\begin{align*}
\left\langle M_{i j} M_{k l}\right\rangle_{o(N)} & =\delta_{i j} \delta_{k l} \delta_{i l} \frac{1}{N^{2}}\left(\frac{N-1}{\alpha+\beta}+\frac{1}{\alpha+\beta+\gamma}\right)-\left(\delta_{i j} \delta_{k l}-\delta_{i j} \delta_{k l} \delta_{i l}\right) \frac{1}{N^{2}} \frac{\gamma}{(\alpha+\beta)(\alpha+\beta+\gamma)} \\
& +\left(\delta_{i k} \delta_{j l}-\delta_{i k} \delta_{j l} \delta_{i j}\right) \frac{1}{N} \frac{\alpha}{\alpha^{2}-\beta^{2}}-\left(\delta_{i l} \delta_{k j}-\delta_{i l} \delta_{k j} \delta_{i j}\right) \frac{1}{N} \frac{\beta}{\alpha^{2}-\beta^{2}} . \tag{4.34}
\end{align*}
$$

Defining

$$
\begin{array}{ll}
a=\frac{1}{N^{2}}\left(\frac{N-1}{\alpha+\beta}+\frac{1}{\alpha+\beta+\gamma}\right), & b=\frac{1}{N^{2}} \frac{\gamma}{(\alpha+\beta)(\alpha+\beta+\gamma)} \\
c=\frac{1}{N} \frac{\alpha}{\alpha^{2}-\beta^{2}}, & d=\frac{1}{N} \frac{\beta}{\alpha^{2}-\beta^{2}} \tag{4.35}
\end{array}
$$

and collecting like terms we are left with the following expression for the two-point function

$$
\begin{equation*}
\left\langle M_{i j} M_{k l}\right\rangle_{o(N)}=\delta_{i j} \delta_{k l} \delta_{i l}(a+b-c+d)-\delta_{i j} \delta_{k l} b+\delta_{i k} \delta_{j l} c-\delta_{i l} \delta_{k j} d . \tag{4.36}
\end{equation*}
$$

The parameters $a, b, c, d$ satisfy $a+b+d=c$, therefore the two-point function can be simplified to

$$
\begin{equation*}
\left\langle M_{i j} M_{k l}\right\rangle_{o(N)}=-\delta_{i j} \delta_{k l} b+\delta_{i k} \delta_{j l}(a+b+d)-\delta_{i l} \delta_{k j} d \tag{4.37}
\end{equation*}
$$

Comparing this to the two-point function of the 13 parameter PIGM model (2.115) we find
that it is reproduced in the following parameter limit

$$
\begin{align*}
& \left(g_{[N]}^{-1}\right)_{11}=a, \\
& \left(g_{[N]}^{-1}\right)_{22}=a-(N-2) b, \\
& \left(g_{[N]}^{-1}\right)_{12}=-\sqrt{N-1} b, \\
& \left(g_{[N-1,1]}^{-1}\right)_{11}=a+b+d, \\
& \left(g_{[N-1,1]}^{-1}\right)_{22}=a+b+d, \\
& \left(g_{[N-1,1]}^{-1}\right)_{33}=a+b, \\
& \left(g_{[N-1,1]}^{-1}\right)_{12}=-d, \\
& \left(g_{[N-1,1]}^{-1}\right)_{13}=0, \\
& \left(g_{[N-1,1]}^{-1}\right)_{23}=0, \\
& \left(g_{[N-2,2]}^{-1}\right)=a+b, \\
& \left(g_{[N-2,1,1]}^{-1}\right)=a+b+2 d . \tag{4.38}
\end{align*}
$$

There is a special point in this limit that recovers the two-point function

$$
\begin{equation*}
\left\langle M_{i j} M_{k l}\right\rangle_{\mathrm{GOE}}=\delta_{i k} \delta_{j l} \tag{4.39}
\end{equation*}
$$

for the simple $O(N)$ model with action

$$
\begin{equation*}
S_{\mathrm{GOE}}(M)=\operatorname{Tr}\left(M M^{T}\right) . \tag{4.40}
\end{equation*}
$$

Setting $\epsilon=\beta=\gamma=0$ in equation (4.26) reproduces this action. Therefore, the relevant limit of the permutation invariant Gaussian model is found by taking $a=1$ and $b=d=0$ in (4.38) leaving us with
$\left(g_{[N]}^{-1}\right)_{11}=\left(g_{[N]}^{-1}\right)_{22}=\left(g_{[N-1,1]}^{-1}\right)_{11}=\left(g_{[N-1,1]}^{-1}\right)_{22}=\left(g_{[N-1,1]}^{-1}\right)_{33}=\left(g_{[N-2,2]}^{-1}\right)=\left(g_{[N-2,1,1]}^{-1}\right)=1$
as the only non-zero parameters.
A quick check on the above computation is the following. Using Clebsch-Gordan coefficients
we have

$$
\begin{align*}
\operatorname{Tr}\left(M M^{T}\right)=\sum_{i, j} M_{i j} M_{i j} & =\sum_{i, j} \sum_{a, b, \Lambda_{1} \Lambda_{1}^{\prime}, \alpha, \beta} C_{a, i j}^{\Lambda_{1}, \alpha} C_{b, i j}^{\Lambda_{1}^{\prime}, \beta} S_{a}^{\Lambda_{1}, \alpha} S_{b}^{\Lambda_{1}^{\prime}, \beta} \\
& =\sum_{a, b, \Lambda_{1} \Lambda_{1}^{\prime}, \alpha, \beta} \delta_{a b} \delta^{\Lambda_{1} \Lambda_{1}^{\prime}} \delta^{\alpha \beta} S_{a}^{\Lambda_{1}, \alpha} S_{b}^{\Lambda_{1}^{\prime}, \beta}=\sum_{a, \Lambda_{1}, \alpha} S_{a}^{\Lambda_{1}, \alpha} S_{a}^{\Lambda_{1}, \alpha}, \tag{4.42}
\end{align*}
$$

where the second line uses orthogonality of the Clebsch-Gordan coefficients. Comparing with equation (2.81) recovers the parameter limit (4.41). ${ }^{2}$

### 4.2 Permutation invariant matrix observables (PIMOs)

We now describe how PIMOs of order $k$ can be constructed from the $S_{k}$ invariant subalgebra of $P_{k}(N)$. Properties of the partition algebra [23, 22, 24, 61] will allow us to prove large $N$ factorisation of PIMOs in the $O(N)$ symmetric matrix model.

### 4.2.1 Construction of PIMOs

Before constructing degree $k$ PIMOs from elements $d \in P_{k}(N)$, as a warm-up, we recap the construction of $U(N)$ invariants using elements in $\mathbb{C} S_{k}$. See [21] for a review of the background literature.

For this construction it will be useful to rewrite $M_{i j}$ as $M_{j}^{i}$ and think of these as the matrix elements of a linear operator acting on $V$, the defining representation of $U(N)$. Define $M$ to be the linear operator $M: V \rightarrow V$ with matrix elements

$$
\begin{equation*}
M e_{i}=\sum_{j} M_{i}^{j} e_{j}, \tag{4.43}
\end{equation*}
$$

in a basis $e_{i}$ for $V$. In diagram notation the linear operator $M$ is represented by a box labelled $M$, with one incoming and one outgoing edge (read from top to bottom),

$$
M_{i}^{j}=\begin{gather*}
\dot{j}  \tag{4.44}\\
\frac{\dot{\bullet}}{\dot{\bullet}} \\
\hline i
\end{gather*}
$$

[^3]The operator $M^{\otimes k}$ acts on $V^{\otimes k}$ as

$$
\begin{equation*}
M^{\otimes k} e_{i_{1}} \otimes \cdots \otimes e_{i_{k}}=M e_{i_{1}} \otimes \cdots \otimes M e_{i_{k}} \tag{4.45}
\end{equation*}
$$

Diagrammatically, tensor products of operators are represented by horizontally composing the diagrams,

When viewed as a matrix polynomial, the trace

$$
\mathcal{O}_{\tau}=\operatorname{Tr}_{V \otimes k}\left(M^{\otimes k} \tau\right)=\sum_{\substack{i_{1}, \ldots, i_{k}  \tag{4.47}\\
i_{1}, \ldots, i_{k}}}(\tau)_{i_{1} \ldots i_{k}}^{i_{1}, \ldots i_{k^{\prime}}} M_{i_{1_{1}}}^{i_{1}} \ldots M_{i_{k^{\prime}}}^{i_{k}}=\begin{align*}
& \square \\
& \hline \\
& \hline \ldots \ldots \square \\
& \hline
\end{align*},
$$

is a unitary invariant of degree $k$. The horizontal lines in equation (4.47) are used to indicate that the incoming and outgoing edges are identified, as expected from a trace. The matrix elements of the permutation $\tau$ as a linear operator on $V^{\otimes k}$ are

$$
\begin{equation*}
(\tau)_{i_{1} \ldots i_{k}}^{i_{1} \ldots i_{k^{\prime}}}=\delta_{i_{\tau(1)}}^{i_{1}^{\prime}} \ldots \delta_{i_{\tau(k)}}^{i_{k^{\prime}}} . \tag{4.48}
\end{equation*}
$$

The diagram representing $\tau$ is obtained by associating an edge with every Kronecker delta. For example, for $\tau=(12)$ we have the diagram


Invariance under the action of $U(N)$ follows because $\tau \in S_{k}$ commutes with any $U(N)$ acting on $V^{\otimes k}$. The correspondence between gauge invariant operators and permutations has a redundancy given by,

$$
\begin{equation*}
\mathcal{O}_{\gamma \tau \gamma^{-1}}=\mathcal{O}_{\tau}, \quad \text { for all } \gamma \in S_{k} \tag{4.50}
\end{equation*}
$$

This follows because $\gamma^{-1} M^{\otimes k} \gamma=M^{\otimes k}$. Therefore, a basis of multi-trace observables is in one-to-one correspondence with conjugacy classes of $S_{k}$, as expected from the counting in equation 4.15).

The construction of degree $k$ PIMOs is analogous, only now the dual algebra in the partition algebra $P_{k}(N)$. For any $d \in P_{k}(N)$, the matrix polynomial

$$
\mathcal{O}_{d}=\operatorname{Tr}_{V_{N}^{\otimes k}}\left(M^{\otimes k} d\right)=\sum_{\substack{i_{1}, \ldots, i_{k}  \tag{4.51}\\
i_{1}, \ldots, i_{k^{\prime}}}}(d)_{i_{1} \ldots . i_{k}}^{i_{1} \ldots . i_{k^{\prime}}} M_{i_{1}}^{i_{1}} \ldots M_{i_{k^{\prime}}}^{i_{k}}=\begin{array}{|}
\square \\
\hline \\
\hline \ldots \ldots \square \\
\hline
\end{array},
$$

is a PIMO, because $d$ commutes with the action of $S_{N}$ on $V_{N}^{\otimes k}$. The matrix elements $(d)_{i_{1} \ldots i_{k}}^{i_{1}, \ldots i_{k}}$ also correspond to the diagram representation by associating every Kronecker delta to an edge connecting a pair of vertices. For example,

As before, for any $\gamma \in S_{k}$ we have

$$
\begin{equation*}
\mathcal{O}_{\gamma d \gamma^{-1}}=\mathcal{O}_{d} \tag{4.53}
\end{equation*}
$$

Degree $k$ PIMOs are in one-to-one correspondence with the $S_{k}$ invariant subalgebra of $P_{k}(N)$. A basis is given by the set of distinct equivalence classes

$$
\begin{equation*}
[d]=\left\{\gamma d \gamma^{-1} \mid \forall \gamma \in S_{k}\right\} . \tag{4.54}
\end{equation*}
$$

### 4.2.2 Inner product on PIMOs

The simplest $O(N)$ invariant matrix model has the quadratic expectation value

$$
\begin{equation*}
\left\langle M_{j}^{i} M_{l}^{k}\right\rangle=\delta^{i k} \delta_{j l} . \tag{4.55}
\end{equation*}
$$

Let $d_{1}, d_{2} \in P_{k}(N)$, and define the two-point function of PIMOs $\mathcal{O}_{d_{1}}, \mathcal{O}_{d_{2}}$ using Wick's theorem and equation (4.55), keeping only Wick contractions between the two observables i.e. we are treating them as "normal-ordered". There is a trace expression for this two-point function

$$
\begin{equation*}
\left\langle\mathcal{O}_{d_{1}} \mathcal{O}_{d_{2}}\right\rangle=\sum_{\gamma \in S_{k}} \operatorname{Tr}_{V_{N}^{\otimes k}}\left(d_{1} \gamma d_{2}^{T} \gamma^{-1}\right), \tag{4.56}
\end{equation*}
$$

where $d^{T}$ is the transpose of the diagram $d$, obtained from $d$ by flipping the top and bottom vertices. The permutations $\gamma$ parameterise the Wick contractions. The proof of (4.56) goes as follows. Note that the quadratic expectation value 4.55) diagrammatically corresponds
to the replacement
where the Kronecker deltas appearing on the RHS of 4.55 have been replaced by edges. The two-point function in equation 4.56 can be represented by the diagram in the first line below


The second line is the sum over Wick contractions parameterised by $\gamma \in S_{k}$. The last equality comes from straightening the diagram. Following the lines and recording the operators encountered on the way, we recognise the last diagram as the representation of $\operatorname{Tr}_{V_{N}^{\otimes k}}\left(d_{1} \gamma d_{2}^{T} \gamma^{-1}\right)$.

The symmetry of the two-point function is proved by observing that

$$
\begin{equation*}
\sum_{\gamma \in S_{k}} \operatorname{Tr}_{V_{N}^{\otimes k}}\left(d_{1} \gamma d_{2}^{T} \gamma^{-1}\right)=\sum_{\gamma \in S_{k}} \operatorname{Tr}_{V_{N}^{\otimes k}}\left(\gamma d_{2} \gamma^{-1} d_{1}^{T}\right)=\sum_{\gamma \in S_{k}} \operatorname{Tr}_{V_{N}^{\otimes k}}\left(d_{2} \gamma d_{1}^{T} \gamma^{-1}\right) \tag{4.59}
\end{equation*}
$$

We have used the invariance of the trace under transposition, cyclicity of the trace and a relabelling of $\gamma \rightarrow \gamma^{-1}$. The non-degeneracy of the two-point function at large $N$ follows from the factorisation property in the next section. The non-degeneracy at all orders in $1 / \sqrt{N}$ is proved in [3] by exhibiting an orthogonal basis constructed using representation theory data. This shows that the two-point function defines an inner product.

### 4.3 Large N factorisation

In this section, we show that normalised PIMOs

$$
\begin{equation*}
\hat{\mathcal{O}}_{d}=\frac{\mathcal{O}_{d}}{\sqrt{\left\langle\mathcal{O}_{d} \mathcal{O}_{d}\right\rangle}} \tag{4.60}
\end{equation*}
$$

factorise at large $N$

$$
\left\langle\hat{\mathcal{O}}_{d_{1}} \hat{\mathcal{O}}_{d_{2}}\right\rangle= \begin{cases}1+O(1 / \sqrt{N}) & \text { if }\left[d_{1}\right]=\left[d_{2}\right]  \tag{4.61}\\ 0+O(1 / \sqrt{N}) & \text { if }\left[d_{1}\right] \neq\left[d_{2}\right]\end{cases}
$$

To prove large $N$ factorisation we will study the powers of $N$ appearing in the two-point function, written as on the RHS of equation (4.56)

$$
\begin{equation*}
\operatorname{Tr}_{V_{N}^{\otimes k}}\left(d_{1} \gamma d_{2}^{T} \gamma^{-1}\right) . \tag{4.62}
\end{equation*}
$$

It is instructive to first consider the simpler case

$$
\begin{equation*}
\operatorname{Tr}_{V_{N}^{\otimes k}}^{\otimes k}\left(d_{1} d_{2}^{T}\right) . \tag{4.63}
\end{equation*}
$$

This trace can be computed in terms of the number of connected components in the diagram $d_{1} \vee d_{2}$, given by a diagram with all the edges of $d_{1}$ and $d_{2}$. In the mathematics literature, this operation is called the join on the partition lattice (see [87]). It is given by

$$
\begin{equation*}
\operatorname{Tr}_{V_{N}^{\otimes k}}\left(d_{1} d_{2}^{T}\right)=N^{c\left(d_{1} \vee d_{2}\right)} . \tag{4.64}
\end{equation*}
$$

where $c(d)$ is the number of connected components in the diagram $d$. Examples of the join operation are

$$
\begin{equation*}
\stackrel{\bullet}{\bullet} \cdot \stackrel{\bullet}{\bullet} \text {, and } \stackrel{\bullet}{\bullet} \cdot \stackrel{\bullet}{\bullet}=\stackrel{\bullet}{\bullet} \tag{4.65}
\end{equation*}
$$

Examples of $c(d)$ are

$$
c(!\quad \bullet)=2, \quad c\left(\begin{array}{ll}
\bullet & \bullet \tag{4.66}
\end{array}\right)=3 .
$$

To illustrate equation (4.64) consider the following pair of diagrams

$$
\begin{equation*}
(!\cdot)_{i_{1} i_{2}}^{i_{1} i_{2^{\prime}}}=\delta_{i_{1}}^{i_{1^{\prime}}}, \quad(\bullet \cdot \bullet)_{i_{1} i_{2}}^{i_{1} i_{i^{\prime}}}=\delta_{i_{2}}^{i_{2^{\prime}}} . \tag{4.67}
\end{equation*}
$$

The join is given by

$$
\begin{equation*}
\left(!\cdot{ }^{\bullet} \cdot \bullet \cdot!\right)_{i_{1} i_{2}}^{i_{1}^{\prime} i_{2^{\prime}}}=(\bullet \cdot \bullet)_{i_{1} i_{2}}^{i_{1} i_{2}^{\prime}}=\delta_{i_{1}}^{i_{1}^{\prime}} \delta_{i_{2}}^{i_{2^{\prime}}} \tag{4.68}
\end{equation*}
$$

Diagram multiplication in the partition algebra gives

$$
\operatorname{Tr}_{V_{N}^{\otimes 2}}\left(\bullet \cdot(\cdot) \cdot()^{T}\right)=\operatorname{Tr}_{V_{N}^{\otimes 2}}\left(\begin{array}{ll}
\bullet & \bullet  \tag{4.69}\\
\bullet & \bullet \\
\bullet & \bullet
\end{array}\right)=\operatorname{Tr}_{V_{N}^{\otimes 2}}\left(\begin{array}{ll}
\bullet & \bullet \\
\bullet & \bullet
\end{array}\right)=N^{2}
$$

while the corresponding expression using the join gives

$$
\begin{equation*}
\left.\operatorname{Tr}_{V_{N}^{\otimes 2}}\left(\bullet \cdot(\bullet \cdot)^{T}\right)=N^{c(\bullet} \cdot \bullet \vee \cdot l\right)=N^{c}(\emptyset \cdot \emptyset)=N^{2} \tag{4.70}
\end{equation*}
$$

To prove this equivalence in general, recall that every edge in a diagram corresponds to a Kronecker delta when acting on $V_{N}^{\otimes k}$ (see examples in 4.52). Consequently

$$
\begin{equation*}
\left(d_{1} \vee d_{2}\right)_{i_{1} \ldots i_{k}}^{i_{1} \prime \ldots i_{k^{\prime}}}=\left(d_{1}\right)_{i_{1} \ldots i_{k}}^{i_{1} \prime \ldots i_{k^{\prime}}}\left(d_{2}\right)_{i_{1} \ldots i_{k}}^{i_{1^{\prime}} \ldots i_{k^{\prime}}} \tag{4.71}
\end{equation*}
$$

It follows that

$$
\begin{align*}
\operatorname{Tr}_{V_{N} k k}^{\otimes k}\left(d_{1} d_{2}^{T}\right) & =\sum_{\substack{i_{1}, \ldots, i_{k} \\
i_{1^{\prime}}, \ldots i_{k^{\prime}}}}\left(d_{1}\right)_{i_{1} \ldots i_{k}}^{i_{1} \ldots i_{k^{\prime}}}\left(d_{2}^{T}\right)_{i_{1^{\prime} \ldots i_{k^{\prime}}}^{i_{1} \ldots i_{k}}}=\sum_{\substack{i_{1}, \ldots, i_{k} \\
i_{1^{\prime}}, \ldots i_{k^{\prime}}}}\left(d_{1}\right)_{i_{1} \ldots i_{k}}^{i_{1^{\prime} \prime \ldots i_{k^{\prime}}}}\left(d_{2}\right)_{i_{1} \ldots i_{k}}^{i_{1^{\prime} \ldots i_{k^{\prime}}}} \\
& =\sum_{\substack{i_{1}, \ldots, i_{k} \\
i_{1^{\prime}}, \ldots i_{k}}}\left(d_{1} \vee d_{2}\right)_{i_{1} \ldots i_{k}}^{i_{1^{\prime}} \ldots i_{k^{\prime}}} . \tag{4.72}
\end{align*}
$$

Equivalently, the diagrammatic representation of a trace identifies the bottom vertices with the top vertices,

$$
\operatorname{Tr}_{V_{N}^{\otimes k}}^{\otimes}\left(d_{1} d_{2}^{T}\right)=\begin{align*}
& \square d_{1}  \tag{4.73}\\
& \square d_{2}^{T} \\
& \square
\end{align*}
$$

Taken literally, this means that we identify the bottom vertices of $d_{2}^{T}$ with the top vertices of $d_{1}$, and the top vertices of $d_{2}^{T}$ with the bottom vertices of $d_{1}$. The diagram constructed in this manner has all the edges of $d_{1}$ together with all the edges of $d_{2}$, which is precisely equal to $d_{1} \vee d_{2}$. See figure 4.1 for an illustration.

To complete the proof we show that

$$
\begin{equation*}
\operatorname{Tr}_{V_{N}^{\otimes k}}\left(d_{1} d_{2}^{T}\right)=\frac{\square d_{1}}{\square d_{2}^{T}}=\sum_{\substack{i_{1}, \ldots, i_{k} \\ i_{1^{\prime}, \ldots, i_{k}}}}\left(d_{1} \vee d_{2}\right)_{i_{1} \ldots i_{k}}^{i_{1^{\prime}} \ldots i_{k^{\prime}}}=N^{c\left(d_{1} \vee d_{2}\right)} \tag{4.74}
\end{equation*}
$$



Figure 4.1: By identifying the bottom vertices of $d_{2}^{T}$ with the top vertices of $d_{1}$, and the top vertices of $d_{2}^{T}$ with the bottom vertices of $d_{1}$, we have constructed a diagram with all the edges of $d_{1}$ together with all the edges of $d_{2}$.

Let $b_{1}, \ldots, b_{l}$ be sets containing the vertices of connected components of $d_{1} \vee d_{2}$. Then,

$$
\begin{equation*}
\sum_{\substack{i_{1}, \ldots, i_{k} \\ i_{1}, \ldots, i_{k^{\prime}}}}\left(d_{1} \vee d_{2}\right)_{i_{1} \ldots i_{k}}^{i_{1} \ldots i_{k^{\prime}}}=\left(\sum_{b_{1}} 1\right)\left(\sum_{b_{2}} 1\right) \ldots\left(\sum_{b_{l}} 1\right)=N^{c\left(d_{1} \vee d_{2}\right)}, \tag{4.75}
\end{equation*}
$$

where the sums over connected components correspond to sums where the indices in each component are set equal. For example, if $b_{1}=\left\{1,3,5^{\prime}, 8\right\}$ then

$$
\begin{equation*}
\sum_{b_{1}} 1 \equiv \sum_{i_{1}, i_{3}, i_{5^{\prime}}, i_{8}} \delta_{i_{1} i_{3}} \delta_{i_{3} i_{5^{\prime}}} \delta_{i_{5^{\prime}} i_{8}}=\sum_{i_{1}=i_{3}=i_{5^{\prime}}=i_{8}} 1=N . \tag{4.76}
\end{equation*}
$$

### 4.3.1 Factorisation for trace form on $P_{k}(N)$

The proof of the following version of factorisation

$$
\frac{\operatorname{Tr}_{V_{N}^{\otimes k}}\left(d_{1} d_{2}^{T}\right)}{\sqrt{\operatorname{Tr}_{V_{N}^{\otimes k}}\left(d_{1} d_{1}^{T}\right) \operatorname{Tr}_{V_{N}^{\otimes k}}\left(d_{2} d_{2}^{T}\right)}}= \begin{cases}1+O(1 / \sqrt{N}) & \text { if } d_{1}=d_{2}  \tag{4.77}\\ 0+O(1 / \sqrt{N}) & \text { if } d_{1} \neq d_{2},\end{cases}
$$

contains most of the essential ingredients necessary for the one-matrix case. This is a useful warm-up exercise and, as we will see in section 4.3.2, a special case of factorisation in multi-matrix models. This equation (4.77) is related to the properties of the distance function defined in proposition 3.1 of $97{ }^{3}$

The factorisation in equation (4.77) is a consequence of the following

$$
\begin{array}{lll}
2 c\left(d_{1} \vee d_{2}\right)=c\left(d_{1} \vee d_{1}\right)+c\left(d_{2} \vee d_{2}\right)=c\left(d_{1}\right)+c\left(d_{2}\right) & \text { if } & d_{1}=d_{2},  \tag{4.78}\\
2 c\left(d_{1} \vee d_{2}\right)<c\left(d_{1} \vee d_{1}\right)+c\left(d_{2} \vee d_{2}\right)=c\left(d_{1}\right)+c\left(d_{2}\right) & \text { if } & d_{1} \neq d_{2},
\end{array}
$$

where we have used $c\left(d_{1} \vee d_{1}\right)+c\left(d_{2} \vee d_{2}\right)=c\left(d_{1}\right)+c\left(d_{2}\right)$ since $d \vee d=d$. We will prove (4.78) by separating the general pairs $d_{1}, d_{2}$ into three distinct cases:

1. If $d_{1}$ only contains edges that are also contained in $d_{2}$, but $d_{1} \neq d_{2}$, we write $d_{1}<d_{2}$. For example,

$$
\begin{equation*}
\stackrel{\bullet}{0}, \text { and } \stackrel{\bullet}{\bullet}< \tag{4.74}
\end{equation*}
$$

In this case, $d_{1} \vee d_{2}=d_{2}$ and it follows that,

$$
\begin{equation*}
c\left(d_{1} \vee d_{2}\right)=c\left(d_{2}\right) \tag{4.80}
\end{equation*}
$$

Note that $d_{1}<d_{2}$ implies $c\left(d_{1}\right)>c\left(d_{2}\right)$. Therefore,

$$
\begin{equation*}
2 c\left(d_{1} \vee d_{2}\right)=c\left(d_{2}\right)+c\left(d_{2}\right)<c\left(d_{1}\right)+c\left(d_{2}\right) . \tag{4.81}
\end{equation*}
$$

Since the LHS and RHS are symmetric under exchanging $d_{1} \leftrightarrow d_{2}$, the inequality $2 c\left(d_{1} \vee d_{2}\right)<c\left(d_{1}\right)+c\left(d_{2}\right)$ holds for $d_{2}<d_{1}$ as well.
2. Suppose $d_{1} \neq d_{2}$ and that there is no set of edges that can be added to $d_{1}$ to turn it into $d_{2}$, nor is there a set of edges that can be added to $d_{2}$ to obtain $d_{1}$. Then, we say that $d_{1}$ and $d_{2}$ are incomparable. We denote this by $d_{1} \nRightarrow d_{2}$. The following diagrams are examples of incomparable diagrams


In this incomparable case, we have

$$
\begin{equation*}
c\left(d_{1} \vee d_{2}\right)<c\left(d_{1}\right) \text { and } c\left(d_{1} \vee d_{2}\right)<c\left(d_{2}\right) \tag{4.83}
\end{equation*}
$$

since forming the join involves adding to $d_{1}$, additional edges creating connections which did not exist in $d_{1}$, or alternatively adding to $d_{2}$ additional edges that did not

[^4]exist in $d_{2}$. Consequently we have the inequality
\[

$$
\begin{equation*}
2 c\left(d_{1} \vee d_{2}\right)<c\left(d_{1}\right)+c\left(d_{2}\right) . \tag{4.84}
\end{equation*}
$$

\]

3. If $d_{1}=d_{2}$ we have

$$
\begin{equation*}
c\left(d_{1} \vee d_{2}\right)=c\left(d_{1} \vee d_{1}\right)=c\left(d_{1}\right)=c\left(d_{2}\right), \tag{4.85}
\end{equation*}
$$

and therefore,

$$
\begin{equation*}
2 c\left(d_{1} \vee d_{2}\right)=c\left(d_{1}\right)+c\left(d_{2}\right) . \tag{4.86}
\end{equation*}
$$

To summarise, $2 c\left(d_{1} \vee d_{2}\right) \leq c\left(d_{1}\right)+c\left(d_{2}\right)$ with equality if and only if $d_{1}=d_{2}$.
As a corollary of the above discussion, which will be useful in the next sub-section, note that if we consider a fixed diagram $d_{1}$ and a family of diagrams $d_{3}$ with fixed $c\left(d_{3}\right)$ such that $c\left(d_{1}\right)>c\left(d_{3}\right)$, then we have for each $d_{3}$ in the family one of the following

$$
\begin{array}{ll}
c\left(d_{1} \vee d_{3}\right)<c\left(d_{3}\right) & \text { if } d_{1} \not \equiv d_{3} \\
c\left(d_{1} \vee d_{3}\right)=c\left(d_{3}\right) & \text { if } d_{1}<d_{3} \tag{4.87}
\end{array}
$$

This follows from 4.80) and 4.83).

### 4.3.2 Factorisation for PIMOs

The one-matrix connected two-point function 4.56) includes a sum over $\gamma \in S_{k}$,

$$
\begin{equation*}
\left\langle\hat{\mathcal{O}}_{d_{1}} \hat{\mathcal{O}}_{d_{2}}\right\rangle=\frac{\sum_{\gamma_{1} \in S_{k}} N^{c\left(d_{1} \vee \gamma_{1} d_{2} \gamma_{1}^{-1}\right)}}{\sqrt{\sum_{\gamma_{2} \in S_{k}} N^{c\left(d_{1} \vee \gamma_{2} d_{1} \gamma_{2}^{-1}\right)} \sum_{\gamma_{3} \in S_{k}} N^{c\left(d_{2} \vee \gamma_{3} d_{2} \gamma_{3}^{-1}\right)}}} . \tag{4.88}
\end{equation*}
$$

Large $N$ factorisation of PIMOs follows from the inequalities

$$
\begin{array}{lll}
2 \max _{\gamma_{1}} c\left(d_{1} \vee \gamma_{1} d_{2} \gamma_{1}^{-1}\right)=\max _{\gamma_{2}} c\left(d_{1} \vee \gamma_{2} d_{1} \gamma_{2}^{-1}\right)+\max _{\gamma_{3}} c\left(d_{2} \vee \gamma_{3} d_{2} \gamma_{3}^{-1}\right) & \text { if } & {\left[d_{1}\right]=\left[d_{2}\right],} \\
2 \max _{\gamma_{1}} c\left(d_{1} \vee \gamma_{1} d_{2} \gamma_{1}^{-1}\right)<\max _{\gamma_{2}} c\left(d_{1} \vee \gamma_{2} d_{1} \gamma_{2}^{-1}\right)+\max _{\gamma_{3}} c\left(d_{2} \vee \gamma_{3} d_{2} \gamma_{3}^{-1}\right) & \text { if } & {\left[d_{1}\right] \neq\left[d_{2}\right] .} \tag{4.89}
\end{array}
$$

The first step in proving equation (4.89) is to simplify the terms on the RHS. The inequalities in equation 4.78) imply that $c\left(d \vee \gamma d \gamma^{-1}\right)$ is maximised when $d=\gamma d \gamma^{-1}$. Of course, the identity permutation always satisfies this equality. Therefore,

$$
\begin{equation*}
\max _{\gamma} c\left(d \vee \gamma d \gamma^{-1}\right)=c(d) . \tag{4.90}
\end{equation*}
$$

We are left with proving

$$
\begin{array}{lll}
2 \max _{\gamma} c\left(d_{1} \vee \gamma d_{2} \gamma^{-1}\right)=c\left(d_{1}\right)+c\left(d_{2}\right) & \text { if } & {\left[d_{1}\right]=\left[d_{2}\right]}  \tag{4.91}\\
2 \max _{\gamma} c\left(d_{1} \vee \gamma d_{2} \gamma^{-1}\right)<c\left(d_{1}\right)+c\left(d_{2}\right) & \text { if } & {\left[d_{1}\right] \neq\left[d_{2}\right]}
\end{array}
$$

We employ the same strategy as before, and consider three distinct cases:

1. Suppose $c\left(d_{1}\right)>c\left(d_{2}\right)$, and consider the diagrams $\gamma d_{2} \gamma^{-1}$ for $\gamma \in S_{k}$. We have $c\left(d_{1}\right)>c\left(\gamma d_{2} \gamma^{-1}\right)=c\left(d_{2}\right)$. Assume $d_{1}, d_{2}$ are such there exists some $\gamma^{*}$ such that $d_{1}<\gamma^{*} d_{2}\left(\gamma^{*}\right)^{-1}$. For any such $\gamma^{*}$, the equality in 4.87 implies that

$$
\begin{equation*}
2 c\left(d_{1} \vee \gamma^{*} d_{2}\left(\gamma^{*}\right)^{-1}\right)=2 c\left(d_{2}\right)<c\left(d_{1}\right)+c\left(d_{2}\right) \tag{4.92}
\end{equation*}
$$

Any $\gamma$ not satisfying this condition leads to $d_{1} \not \equiv \gamma d_{2} \gamma^{-1}$, and the inequality in 4.87) implies that

$$
\begin{equation*}
2 c\left(d_{1} \vee \gamma d_{2} \gamma^{-1}\right)<2 c\left(d_{2}\right) \tag{4.93}
\end{equation*}
$$

This implies that

$$
\begin{equation*}
2 \max _{\gamma} c\left(d_{1} \vee \gamma d_{2} \gamma^{-1}\right)=2 c\left(d_{1} \vee \gamma^{*} d_{2}\left(\gamma^{*}\right)^{-1}\right)=2 c\left(d_{2}\right)<c\left(d_{1}\right)+c\left(d_{2}\right) \tag{4.94}
\end{equation*}
$$

The pair

$$
\begin{equation*}
d_{1}=\bullet \bullet, \quad d_{2}=\bullet \bullet \tag{4.95}
\end{equation*}
$$

exemplify this case since

$$
\begin{equation*}
\cdot \bullet<\square=+: \tag{4.96}
\end{equation*}
$$

The argument is identical for the case where $c\left(d_{1}\right)<c\left(d_{2}\right)$, and there exists some $\gamma^{*} \in S_{k}$ such that $d_{2}<\gamma^{*} d_{1}\left(\gamma^{*}\right)^{-1}$. Here, by renaming $d_{1} \leftrightarrow d_{2}$ in 4.94), we have

$$
\begin{equation*}
2 \max _{\gamma} c\left(d_{2} \vee \gamma d_{1} \gamma^{-1}\right)=2 c\left(d_{2} \vee \gamma^{*} d_{1}\left(\gamma^{*}\right)^{-1}\right)=2 c\left(d_{1}\right)<c\left(d_{1}\right)+c\left(d_{2}\right) \tag{4.97}
\end{equation*}
$$

Using the symmetry of the inner product 4.59 it follows

$$
\begin{equation*}
2 \max _{\gamma} c\left(d_{1} \vee \gamma d_{2} \gamma^{-1}\right)<c\left(d_{1}\right)+c\left(d_{2}\right) \tag{4.98}
\end{equation*}
$$

2. Secondly, consider the case of incomparability,

$$
\begin{equation*}
d_{1} \not \equiv \gamma d_{2} \gamma^{-1} \quad \forall \gamma \in S_{k} . \tag{4.99}
\end{equation*}
$$

Recall that for incomparable diagrams (4.84,

$$
\begin{equation*}
2 c\left(d_{1} \vee \gamma d_{2} \gamma^{-1}\right)<c\left(d_{1}\right)+c\left(\gamma d_{2} \gamma^{-1}\right)=c\left(d_{1}\right)+c\left(d_{2}\right), \tag{4.100}
\end{equation*}
$$

where the last equality follows because conjugation by a permutation does not change the number of connected components. Therefore

$$
\begin{equation*}
2 \max _{\gamma} c\left(d_{1} \vee \gamma d_{2} \gamma^{-1}\right)<c\left(d_{1}\right)+c\left(d_{2}\right), \tag{4.101}
\end{equation*}
$$

in this case as well.
3. When $d_{1}=\gamma d_{2} \gamma^{-1}$ for some $\gamma \in S_{k}$, the bound is saturated and

$$
\begin{equation*}
2 \max _{\gamma} c\left(d_{1} \vee \gamma d_{2} \gamma^{-1}\right)=2 c\left(d_{1}\right) . \tag{4.102}
\end{equation*}
$$

The condition $d_{1}=\gamma d_{2} \gamma^{-1}$ implies $\left[d_{1}\right]=\left[d_{2}\right]$. We have proven the inequalities in equation (4.89) and consequently have proven permutation invariant matrix observables factorise at large $N$.

### 4.3.3 Factorisation for multi-matrix observables

The above argument generalises to multi-matrix models. Let $M^{(f)}$ be $n$ matrices with flavour label $f=1, \ldots, n$ and second moment

$$
\begin{equation*}
\left\langle\left(M^{(f)}\right)^{i}{ }_{j}\left(M^{\left(f^{\prime}\right)}\right)^{k}{ }_{l}\right\rangle=\delta^{f f^{\prime}} \delta^{i k} \delta_{j l} . \tag{4.103}
\end{equation*}
$$

Permutation invariant multi-matrix observables of degree $k=k_{1}+k_{2}+\cdots+k_{n}$, where $k_{f}$ is the degree of matrix $M^{(f)}$, are constructed using partition algebra elements. Multi-matrix observables are labelled by $\vec{k}=\left(k_{1}, \ldots, k_{n}\right)$ and $d \in P_{k}(N)$

$$
\begin{equation*}
\mathcal{O}_{\vec{k}, d}=\operatorname{Tr}_{V_{N}^{\otimes k}}\left(\left(M^{(1)}\right)^{\otimes k_{1}} \ldots\left(M^{(n)}\right)^{\otimes k_{n}} d\right) . \tag{4.104}
\end{equation*}
$$

As before, we have bosonic symmetry. For any $\gamma \in S_{\vec{k}} \equiv S_{k_{1}} \times \cdots \times S_{k_{n}}$ observables are invariant

$$
\begin{align*}
\mathcal{O}_{\vec{k}, \gamma d \gamma^{-1}} & =\operatorname{Tr}_{V_{N}^{\otimes k}}\left(\left(M^{(1)}\right)^{k_{1}} \otimes \cdots \otimes\left(M^{(n)}\right)^{k_{n}} \gamma d \gamma^{-1}\right) \\
& =\operatorname{Tr}_{V_{N}^{\otimes k}}\left(\gamma^{-1}\left(M^{(1)}\right)^{k_{1}} \otimes \cdots \otimes\left(M^{(n)}\right)^{k_{n}} \gamma d\right) \\
& =\operatorname{Tr}_{V_{N}^{\otimes k}}^{\otimes k}\left(\left(M^{(1)}\right)^{k_{1}} \otimes \cdots \otimes\left(M^{(n)}\right)^{k_{n}} d\right) \\
& =\mathcal{O}_{\vec{k}, d} . \tag{4.105}
\end{align*}
$$

Multi-matrix observables are in one-to-one correspondence with partition algebra equivalence classes

$$
\begin{equation*}
[d]=\left\{\gamma d \gamma^{-1} \mid \gamma \in S_{\vec{k}}\right\} . \tag{4.106}
\end{equation*}
$$

Wick contractions vanish unless the flavour indices match, and the sum over $\gamma \in S_{k}$ reduces to a sum over $\gamma \in S_{\vec{k}}$

$$
\begin{equation*}
\left\langle\mathcal{O}_{\vec{k}, d_{1}} \mathcal{O}_{\overrightarrow{k^{\prime}}, d_{2}}\right\rangle=\delta_{\vec{k} \vec{k}^{\prime}} \sum_{\gamma \in S_{\vec{k}}} \operatorname{Tr}_{V_{N}^{\otimes k}}\left(d_{1} \gamma d_{2} \gamma^{-1}\right)=\delta_{\vec{k} \vec{k}^{\prime}} \sum_{\gamma \in S_{\vec{k}}} N^{c\left(d_{1} \vee \gamma d_{2} \gamma^{-1}\right)} . \tag{4.107}
\end{equation*}
$$

The same argument holds for the inequality

$$
\begin{equation*}
2 \max _{\gamma} c\left(d_{1} \vee \gamma d_{2} \gamma^{-1}\right) \leq \max _{\gamma} c\left(d_{1} \vee \gamma d_{2} \gamma^{-1}\right)+\max _{\gamma} c\left(d_{1} \vee \gamma d_{2} \gamma^{-1}\right)=c\left(d_{1}\right)+c\left(d_{2}\right) . \tag{4.108}
\end{equation*}
$$

It is saturated if and only if there exists a $\gamma \in S_{\vec{k}}$ such that $d_{1}=\gamma d_{2} \gamma^{-1}$. That is, if and only if $\left[d_{1}\right]=\left[d_{2}\right]$ or

$$
\begin{equation*}
\mathcal{O}_{\vec{k}, d_{1}}=\mathcal{O}_{\vec{k}, d_{2}} \tag{4.109}
\end{equation*}
$$

To summarise we have

$$
\left\langle\hat{\mathcal{O}}_{\vec{k}, d_{1}} \hat{\mathcal{O}}_{\vec{k}^{\prime}, d_{2}}\right\rangle=\delta_{\vec{k} \vec{k}^{\prime}} \times \begin{cases}1+O(1 / \sqrt{N}) & \text { if }\left[d_{1}\right]=\left[d_{2}\right],  \tag{4.110}\\ 0+O(1 / \sqrt{N}) & \text { if }\left[d_{1}\right] \neq\left[d_{2}\right],\end{cases}
$$

for permutation invariant multi-matrix observables in the above Gaussian $O(N)$ model.
Note that in the case $n=k, k_{f}=1$ (all matrices distinct), we have

$$
\begin{equation*}
S_{\vec{k}}=\underbrace{S_{1} \times \cdots \times S_{1}}_{n} . \tag{4.111}
\end{equation*}
$$

Therefore, the sum over Wick contractions reduces to a single element (the identity element). The corresponding two-point function is the first case we considered (equation (4.77).

Finally, we observe that the proof of factorisation presented here for general observables labelled by partition algebra diagrams specialises to a new way of thinking about factorisation in the case of matrix invariants with continuous symmetry, where the partition algebra diagrams specialise to permutations. The previously known proof based on permutation products can be understood, in the one-matrix case, from the equation

$$
\begin{equation*}
\left\langle\mathcal{O}_{\sigma_{1}}(Z) \mathcal{O}_{\sigma_{2}}\left(Z^{\dagger}\right)\right\rangle=\frac{k!}{\left|T_{1}\right|\left|T_{2}\right|} \sum_{\sigma_{1}^{\prime} \in T_{1}} \sum_{\sigma_{2}^{\prime} \in T_{2}} \sum_{\sigma_{3} \in S_{k}} \delta\left(\sigma_{1}^{\prime} \sigma_{2}^{\prime} \sigma_{3}\right) N^{C_{\sigma_{3}}} \tag{4.112}
\end{equation*}
$$

This equation is derived and explained as equation (2.12) in 21] (multi-matrix generalisations are discussed in references therein). Gauge invariant operators are labelled by permutations $\sigma_{1}, \sigma_{2}$ in conjugacy classes $T_{1}, T_{2}$, while $\left|T_{1}\right|,\left|T_{2}\right|$ are the sizes of these conjugacy classes. Large $N$ factorisation follows from the fact that the largest power of $N$ comes from the case where $\sigma_{3}$ is the identity and this only occurs when $T_{1}=T_{2}$. In the present way of looking at permutations as special cases of partition algebra diagrams, permutations belonging to distinct conjugacy classes are always incomparable in the partial order on set partitions associated to the diagrams. This corresponds to case 2 in of the proofs in sections 4.3.1 and 4.3.2.

### 4.4 Discussion

In this chapter we considered $S_{N}$ invariant matrix models, viewed as generalisations of their more familiar cousins invariant under continuous symmetries. We have shown that there exists a four-dimensional subspace of the most general 13-dimensional parameter space of PIGM models in which the $S_{N}$ symmetry is enhanced to $O(N)$. The parameter limit in which this enhancement takes place is given by equation (4.38). The special case of the simplest $O(N)$ invariant Gaussian 4.40) arises at the parameters given in 4.41).

The factorisation property of multi-trace matrix observables invariant under continuous symmetries such as $U(N)$ in the large $N$ limit is a well known result. We have shown that this continues to hold for $S_{N}$ invariant observables. In the $U(N)$ case this can be seen using properties of the symmetric group by exploiting the Schur-Weyl duality of $U(N)$ and $S_{k}$ in order to establish a correspondence between observables and conjugacy classes of $S_{k}$. Analogously, we gave a description of the permutation invariant matrix polynomial functions in terms of a diagram basis for partition algebras. We used the inner product on the permutation invariant polynomials arising from the simplest $O(N)$ invariant action, and proved large $N$ factorisation.

## Chapter 5

## Permutation symmetry in large $N$ matrix quantum mechanics

Regarding PIGM models as zero-dimensional quantum field theories, in this chapter we take the natural next step of considering one-dimensional QFTs, i.e. matrix quantum mechanical systems with permutation symmetry. We pay particular attention to methods which are applicable for general $N$ and allow large $N$ expansions. We give a general description of the permutation invariant subspace in matrix quantum mechanical systems, drawing on relevant results from the mathematical literature on partition algebras. This is followed by a discussion of interesting Hamiltonians for many-body quantum physics. This is motivated by the vibrant interplay between holography and many-body quantum mechanical systems which manifests itself, for example, in the connection between free fermions and large $N$ two-dimensional Yang Mills theory [98]; free fermions and the halfBPS sector of $\mathcal{N}=4$ SYM [6, 99]; free fermions and supersymmetric indices [100], bosons in a 3D harmonic oscillator and eighth BPS states in $\mathcal{N}=4$ SYM [101, 102, 103]; quantum mechanical spin matrix theory which is used as a simplified set-up to study the emergence mechanisms of AdS/CFT [104, 105. This interplay is also visible in the prominent role of coherent states, a technique widely used in many body quantum physics, in the study of large $N$ systems. This theme appears in early work on large $N$ (e.g. 106, 107) as well as more recent developments (e.g. [108, 109, 110]).

Many aspects of large $N$ simplifications in matrix systems are consequences of Schur-Weyl duality. The standard instance of Schur-Weyl duality [5 concerns the tensor product $V^{\otimes k}$ of the fundamental representation $V$ of $U(N)$. The symmetric group $S_{k}$ of all permutations of $k$ objects acts on $V^{\otimes k}$ by permuting the factors of the tensor product. Schur-Weyl duality states that the algebra of operators commuting with the standard $U(N)$ action on
the tensor product $V^{\otimes k}$ is the group algebra $\mathbb{C} S_{k}$. This has important implications for the classification of $U(N)$ gauge invariant polynomial functions of matrix variables, where a matrix $X$ transforms as $X \rightarrow U X U^{\dagger}$ for $U \in U(N)$. Schur-Weyl duality relates this problem to the rich combinatorics and representation theory of symmetric groups (see e.g. [68]). For example, the gauge invariant polynomial functions of degree $k$ for one matrix of size $N$, taking $N>k$ for simplicity, are labelled by conjugacy classes of $S_{k}$. Finite $N$ effects are captured with the use of Young diagrams. Schur-Weyl duality has been used as a powerful tool in the construction of gauge invariant observables in one-matrix and multi-matrix systems in connection with the AdS/CFT correspondence. This played an important role in identifying the CFT duals [35, 6, 99] of giant gravitons [39, 40, 41] in the AdS/CFT correspondence. The Schur-Weyl duality framework has been further applied to the computation of 1-matrix and multi-matrix correlators [6, 7, 8, 9, 10, 12, 11, 13, 15, [14, 16, 111, 18, 17. A short review is [21]. These multi-matrix applications involve dual algebras beyond the symmetric group algebras. For example Brauer algebras, which have a basis of diagrams, are used in [7]. The symmetric group algebra $\mathbb{C} S_{k}$ can also be viewed as a diagram algebra with multiplication given by the composition of diagrams. For example the following six diagrams give a basis of $\mathbb{C} S_{3}$, the corresponding permutations are given in cycle notation

$(12)(3)=\underbrace{1}_{1} \underbrace{2}_{3}$,
$(13)(2)=\underbrace{2}_{1}$
$(1)(23)=\underbrace{1}_{0} \underbrace{2}_{2}$,
$(132)=\underbrace{1}_{1} \underbrace{2}_{2}$
$(123)=\underbrace{1}_{1}$

The same general philosophy can be applied to the case where we are considering polynomial functions of a matrix $X$ invariant under the transformation $X \rightarrow M_{\sigma} X M_{\sigma}^{T}$, where $M_{\sigma}$ is a matrix representing the permutation $\sigma \in S_{N}$ in the $N$-dimensional natural representation of $S_{N}$, satisfying $M_{\sigma}^{T}=M_{\sigma}^{-1}$. Matrix systems with $S_{N}$ symmetry together with partition algebras allow us to study large $N$ simplifications in the case of discrete (finite) groups.

Polynomials in matrix variables $M_{j}^{i}$ are closely related to quantum mechanical states constructed from matrix oscillators $\left(a^{\dagger}\right)_{j}^{i}$. This allows us to translate the technology developed for zero-dimensional matrix models in the preceding chapters to the setting of matrix quantum mechanics. We will give a detailed description of the space of $S_{N}$ invariant states constructed from matrix oscillators. Polynomials in matrix oscillators can be organised by the degree of the polynomials. At degree $k$, the state space is isomorphic to an $S_{k}$
symmetric subspace $\mathcal{H}^{(k)}$ of $\operatorname{End}\left(V_{N}^{\otimes k}\right)$ :

$$
\begin{equation*}
\mathcal{H}^{(k)} \rightarrow \operatorname{End}\left(V_{N}^{\otimes k}\right) . \tag{5.2}
\end{equation*}
$$

There is a one-to-one correspondence between tensors

$$
\begin{equation*}
\left\langle e^{i_{1}} \cdots e^{i_{k}}\right| T\left|e_{j_{1}} \cdots e_{j_{k}}\right\rangle=T_{j_{1} \cdots j_{k}}^{i_{1} \cdots i_{k}} \tag{5.3}
\end{equation*}
$$

and elements in $\operatorname{End}\left(V_{N}^{\otimes k}\right)$. The bosonic symmetry of the oscillators imposes an invariance under simultaneous re-ordering of the upper and lower indices. Commuting with the $S_{k}$ action is the $S_{N}$ action on $V_{N}^{\otimes k}$ which we denote $\mathcal{L}(\sigma)$, and is made explicit here in order to avoid confusion with other permutation group actions. The $S_{N}$ permutation invariance translates to an invariance of $T$ under an adjoint action

$$
\begin{equation*}
\operatorname{Ad}(\sigma)[T]=\mathcal{L}(\sigma) T \mathcal{L}\left(\sigma^{-1}\right) . \tag{5.4}
\end{equation*}
$$

Many of our results on the $S_{N}$ invariant state space of matrix oscillators, particularly in section 5.2 are independent of the Hamiltonian. They can be viewed as a detailed account of the $S_{N}$ invariant subspace in matrix quantum mechanics using partition algebras and representation theory. The use of the partition algebra $P_{k}(N)$ to study operators and quantum states in $\mathcal{H}^{(k)}$ allows us to take advantage of simplifications in the limit where $k$ is kept fixed as $N \rightarrow \infty$.

The representation theoretic approach allows the construction of solvable algebraic Hamiltonians where the $S_{N}$ invariant states are resolved according to representation theoretic characteristics. Sections 5.4 and 5.5 discuss different classes of solvable $S_{N}$ invariant Hamiltonians obeying

$$
\begin{equation*}
\operatorname{Ad}(\sigma) H=H \operatorname{Ad}(\sigma) . \tag{5.5}
\end{equation*}
$$

The chapter is organised as follows. For concreteness, section 5.1 contains a review of the simplest quantum mechanical model with matrix degrees of freedom. This is the free matrix quantum harmonic oscillator. It is a model containing $N^{2}$ decoupled harmonic oscillators $X_{i j}, i, j=1, \ldots, N$ with a global $U\left(N^{2}\right)$ symmetry. The Hilbert space of this model is a Fock space $\mathcal{H}$ of states constructed using matrix oscillators $\left(a^{\dagger}\right)_{j}^{i}$. This model also serves as a good place to introduce the diagram notation that we will use in the rest of the chapter.

In section 5.2 we consider the $S_{N}$ invariant subspace $\mathcal{H}_{\text {inv }}$ of the total Hilbert space $\mathcal{H}$ of a general quantum mechanics matrix system. This is the subspace of states invariant
under $a^{\dagger} \rightarrow M_{\sigma} a^{\dagger} M_{\sigma}^{T}$, where $M_{\sigma}$ is a permutation matrix of size $N$. We explain the correspondence between permutation invariant matrix states of degree $k$ and partition algebras $P_{k}(N)$. The partition algebras have three natural bases, and each one gives rise to a different basis for $\mathcal{H}_{\text {inv }}$. The diagram basis is natural when discussing inner and outer products. The factorisation property proven in the previous chapter (and originally in [2]) translates to orthogonality of the diagram basis at large $N$. The second basis is the socalled orbit basis which gives rise to an orthogonal basis for all $N$. We call the third basis the representation basis. In the mathematical literature, the representation basis is called a complete set of matrix units. The product in the matrix unit basis is a generalisation of the product for elementary matrices for matrix algebras. The representation basis can be constructed using Fourier transformation on $P_{k}(N)$ and is a direct analogue of the Schur basis for $U(N)$ invariants. Appendix B contains some necessary results for Fourier transforms on semi-simple algebras, closely following 90 but with some modifications that are important for our application, the proofs of many of these results can be found in appendix A of [3]. Physically, the representation basis can be understood as a basis that diagonalises a set of algebraic commuting charges.

Section 5.3 is devoted to the construction and diagonalisation of these charges, which can be used to give the explicit transformation from the diagram basis to the representation basis at large $N$. We illustrate the method for small $k$ and large $N$. These are tabulated in appendix D . The representation basis forms an energy eigenbasis for the Hamiltonian of the free matrix quantum harmonic oscillator presented in section 5.1.

In section 5.4 we introduce an 11 parameter family of exactly solvable quantum matrix systems. The potential in these systems is the most general permutation invariant quadratic function of the matrix variables. These quantum systems can therefore be viewed as general matrix harmonic oscillator systems compatible with permutation symmetry. We find the spectrum for general choices of the parameters by adapting the representation theoretic techniques which have been used to compute correlators in PIGM models throughout the previous chapters and introduced in [43]. Further, we write the canonical partition function in a simple closed form. The representation basis states from section 5.2 do not form an eigenbasis for the general Hamiltonians considered here. The action of the Hamiltonians on the representation basis states leads to a mixing which is constrained by Clebsch-Gordan multiplicities for the symmetric groups. We briefly discuss this mixing.

In section 5.5 we discuss interacting Hamiltonians, parametrised by a positive integer $K$, constructed using partition algebra elements, with the property that the ground states are all permutation invariant states and have degeneracies controlled by a sequence of partition algebras $P_{k}(N)$ for $k \in\{0,1, \ldots, K\}$. The energy gap between the ground states and the lowest excited state is also determined by $K$. By deforming these Hamiltonians with other
partition algebra elements, we design Hamiltonians where the degeneracy of the ground states is broken by small amounts - these two scenarios are illustrated in figure 5.1. We also include a general description of permutation invariant Hamiltonians, finding an interesting relation to the counting of two-matrix permutation invariants of the kind considered in [1]. We conclude this section with an interpretation of the oscillators $\left(a^{\dagger}\right)_{i}^{j}$ as creation operators on a square lattice with sites labelled $(i, j)$.

We compute a set of two- and three-point correlators of invariant operators in section 5.6. The two-point correlators have a large $N$ factorisation property described in the context of matrix models in the previous chapter. The three-point functions are similar to extremal correlators which are relevant to quantum mechanical models considered in AdS/CFT. The extremal correlators are shown to obey selection rules based on ClebschGordan multiplicities (Kronecker coefficients) of symmetric groups.

### 5.1 Review: matrix harmonic oscillator

This section is a review of the simplest matrix quantum harmonic oscillator. The Lagrangian (5.6) describes $N^{2}$ free harmonic oscillators. The corresponding Hamiltonian has a global $U\left(N^{2}\right)$ symmetry. This has a $U(N) \times U(N)$ subgroup of unitary matrices acting by left and right multiplication. There is also a smaller $S_{N} \times S_{N}$ subgroup of the $U(N) \times U(N)$ which plays an important role in subsequent sections. The simplest, noninteracting $U\left(N^{2}\right)$ invariant model will serve as an ideal set-up to introduce the notation used in the rest of the chapter. In particular, we describe how to construct states and operators in $\mathcal{H}$, the Hilbert space of the theory, by considering the oscillators $a_{i j}, a_{i j}^{\dagger}$ as endomorphisms on $V_{N}$. We frequently have this view in mind when manipulating states and operators and it is often practical to employ diagrammatic notation in order to do so. The basics of this diagrammatic notation is introduced at the end of this section.

The simplest matrix harmonic oscillator is described by the Lagrangian

$$
\begin{equation*}
L_{0}=\frac{1}{2}\left(\sum_{i, j=1}^{N} \partial_{t} X_{i j} \partial_{t} X_{i j}-X_{i j} X_{i j}\right) \tag{5.6}
\end{equation*}
$$

It describes $N^{2}$ decoupled oscillators. The conjugate momenta are

$$
\begin{equation*}
\Pi_{i j}=\frac{\partial L_{0}}{\partial\left(\partial_{t} X_{i j}\right)}=\frac{\partial}{\partial t} X_{i j} \tag{5.7}
\end{equation*}
$$

The Hamiltonian corresponding to $L_{0}$ is

$$
\begin{equation*}
H_{0}=\frac{1}{2}\left(\sum_{i, j=1}^{N} \Pi_{i j} \Pi_{i j}+X_{i j} X_{i j}\right) . \tag{5.8}
\end{equation*}
$$

The canonical commutation relations are

$$
\begin{equation*}
\left[X_{i j}, \Pi_{k l}\right]=i \delta_{i k} \delta_{j l} . \tag{5.9}
\end{equation*}
$$

The Hamiltonian given in (5.8) is diagonalised in the usual way - introducing oscillators $a_{i j}^{\dagger}, a_{i j}$ defined by

$$
\begin{align*}
X_{i j} & =\sqrt{\frac{1}{2}}\left(a_{i j}^{\dagger}+a_{i j}\right),  \tag{5.10}\\
\Pi_{i j} & =i \sqrt{\frac{1}{2}}\left(a_{i j}^{\dagger}-a_{i j}\right),
\end{align*}
$$

with commutation relations

$$
\begin{equation*}
\left[a_{i j}, a_{k l}^{\dagger}\right]=\delta_{i k} \delta_{j l} . \tag{5.11}
\end{equation*}
$$

Normal ordering $H_{0}$ gives

$$
\begin{equation*}
H_{0}=\sum_{i, j=1}^{N} a_{i j}^{\dagger} a_{i j}, \tag{5.12}
\end{equation*}
$$

the number operator.
$H_{0}$ is invariant under a $U\left(N^{2}\right)$ symmetry that acts on oscillators as

$$
\begin{align*}
& a_{i j} \rightarrow \sum_{k, l=1}^{N} U_{i j ; k l} a_{k l},  \tag{5.13}\\
& a_{i j}^{\dagger} \rightarrow \sum_{k, l=1}^{N} U_{k l ; i j}^{\dagger} a_{k l}^{\dagger}, \tag{5.14}
\end{align*}
$$

with $U_{i j ; k l}$ an $N^{2} \times N^{2}$ unitary matrix satisfying

$$
\begin{equation*}
\sum_{k, l=1}^{N} U_{i j ; k l} U_{k l ; m n}^{\dagger}=\delta_{i m} \delta_{j n} \tag{5.15}
\end{equation*}
$$

Under the $U\left(N^{2}\right)$ transformation $H_{0}$ is invariant,

$$
\begin{align*}
H_{0} & \rightarrow \sum_{i, j, k, l, m, n} U_{k l ; i j}^{\dagger} U_{i j ; m n} a_{k l}^{\dagger} a_{m n} \\
& =\sum_{k, l, m, n} \delta_{k m} \delta_{l n} a_{k l}^{\dagger} a_{m n} \\
& =\sum_{k, l} a_{k l}^{\dagger} a_{k l} \tag{5.16}
\end{align*}
$$

The oscillator states

$$
\begin{equation*}
\prod_{i, j} \frac{\left(a_{i j}^{\dagger}\right)^{k_{i j}}}{\sqrt{k_{i j}!}}|0\rangle \tag{5.17}
\end{equation*}
$$

labelled by non-negative integers $k_{i j}$ with $i, j=1, \ldots, N$ are energy eigenstates of $H_{0}$. The total Hilbert (Fock) space $\mathcal{H}$ decomposes into subspaces $\mathcal{H}^{(k)}$ with fixed number of oscillators (degree) $k$,

$$
\begin{equation*}
\mathcal{H} \cong \bigoplus_{k=0}^{\infty} \mathcal{H}^{(k)} \tag{5.18}
\end{equation*}
$$

The subset of states with $k=\sum_{i, j} k_{i j}$ form an eigenbasis for the subspace $\mathcal{H}^{(k)}$ and have energy $k$. In general the spectrum is highly degenerate. The number of states with energy $k$ is

$$
\begin{equation*}
\operatorname{Dim} \mathcal{H}^{(k)}=\binom{N^{2}+k-1}{k}=\frac{N^{2}\left(N^{2}+1\right) \ldots\left(N^{2}+k-1\right)}{k!} \tag{5.19}
\end{equation*}
$$

This is the number of ways to choose $k$ elements from a set of $N^{2}$ with replacement. It is also the dimension of the symmetric part of a $k$-fold tensor product of a vector space with dimension $N^{2}$. Equivalently, it is the dimension of the vector space of states composed of $k$ bosonic oscillators $a_{i j}^{\dagger}$. For fixed $k$ and $N \gg 2 k$ the dimension grows as $N^{2 k}$.

### 5.1.1 Diagram notation

Throughout this chapter we use diagrammatic notation to describe states and operators in $\mathcal{H}^{(k)}$. For this purpose, it is useful to introduce the following matrices of oscillators $\left(a^{\dagger}\right)_{j}^{i}=a_{j i}^{\dagger}$ and $a_{j}^{i}=a_{i j}$ which satisfy

$$
\begin{equation*}
\left[a_{j}^{i},\left(a^{\dagger}\right)_{k}^{l}\right]=\delta_{k}^{i} \delta_{j}^{l} \tag{5.20}
\end{equation*}
$$

Let $V_{N}$ be an $N$-dimensional vector space with basis $\left\{e_{1}, \ldots, e_{N}\right\}$. The matrices of oscillators can be viewed as (operator-valued) elements in $\operatorname{End}\left(V_{N}\right)$, where $\operatorname{End}\left(V_{N}\right)$ is the set
of all linear maps $V_{N} \rightarrow V_{N}$. In this language, the above oscillators are matrix elements,

$$
\begin{equation*}
a^{\dagger}\left(e_{i}\right)=\sum_{j=1}^{N}\left(a^{\dagger}\right)_{i}^{j} e_{j} \quad \text { and } \quad a\left(e_{i}\right)=\sum_{j=1}^{N} a_{i}^{j} e_{j} . \tag{5.21}
\end{equation*}
$$

Consequently, a general degree one state in $\mathcal{H}$ can be written as

$$
\begin{equation*}
\operatorname{Tr}_{V_{N}}\left(T a^{\dagger}\right)|0\rangle=\sum_{i, j=1}^{N} T_{j}^{i}\left(a^{\dagger}\right)_{i}^{j}|0\rangle \equiv|T\rangle, \tag{5.22}
\end{equation*}
$$

where $T \in \operatorname{End}\left(V_{N}\right)$ (i.e. an $N$-by- $N$ matrix) and the last equality is a definition of the state $|T\rangle$.

The degree $k$ subspace is given by

$$
\begin{equation*}
\mathcal{H}^{(k)} \cong \operatorname{span}_{\mathbb{C}}\left\{\left(a^{\dagger}\right)_{j_{1}}^{i_{1}} \ldots\left(a^{\dagger}\right)_{j_{k}}^{i_{k}}|0\rangle\right\} \tag{5.23}
\end{equation*}
$$

and general states are parametrised by tensors $T_{i_{1} \ldots i_{k}}^{j_{1} \ldots j_{k}}$. It is convenient to view these tensors as elements of $\operatorname{End}\left(V_{N}^{\otimes k}\right)$. That is, in the usual basis for tensor product spaces

$$
\begin{equation*}
T\left(e_{i_{1}} \otimes e_{i_{2}} \otimes \ldots \otimes e_{i_{k}}\right)=\sum_{j_{1}, j_{2}, \ldots, j_{k}=1}^{N} T_{i_{1} \ldots i_{k}}^{j_{1} \ldots j_{k}} e_{j_{1}} \otimes e_{j_{2}} \otimes \ldots \otimes e_{j_{k}} . \tag{5.24}
\end{equation*}
$$

Generalising the degree one case, a general state $|T\rangle \in \mathcal{H}^{(k)}$ can be written as a trace

$$
\begin{equation*}
|T\rangle=\operatorname{Tr}_{V_{N}^{\otimes k}}\left(T\left(a^{\dagger}\right)^{\otimes k}\right)|0\rangle=\sum_{\substack{i_{1}, \ldots, i_{k} \\ j_{1}, \ldots, j_{k}}} T_{i_{1} \ldots i_{k}}^{j_{1} \ldots j_{k}}\left(a^{\dagger}\right)_{j_{1}}^{i_{1}} \ldots\left(a^{\dagger}\right)_{j_{k}}^{i_{k}}|0\rangle, \tag{5.25}
\end{equation*}
$$

for $T \in \operatorname{End}\left(V_{N}^{\otimes k}\right)$ and $\left(a^{\dagger}\right)^{\otimes k}=a^{\dagger} \otimes \cdots \otimes a^{\dagger}$ with matrix elements

$$
\begin{equation*}
\left(a^{\dagger}\right)^{\otimes k}\left(e_{j_{1}} \otimes \cdots \otimes e_{j_{k}}\right)=\sum_{i_{1}, \ldots, i_{k}}\left(a^{\dagger}\right)_{j_{1}}^{i_{1}} \ldots\left(a^{\dagger}\right)_{j_{k}}^{i_{k}} e_{i_{1}} \otimes \cdots \otimes e_{i_{k}} . \tag{5.26}
\end{equation*}
$$

It should be emphasised that, due to the bosonic symmetry of the oscillators, $T_{i_{1} \ldots i_{k}}^{j_{1} \ldots j_{k}}$ is a symmetric tensor (under simultaneous permutations of upper and lower indices), for example $T_{i_{1} i_{2} \ldots i_{k}}^{j_{1} j_{2} \ldots j_{k}}=T_{i_{2} i_{1} \ldots i_{k}}^{j_{2} j_{1} \ldots j_{k}}$.

It is useful to formulate this restriction in terms of $S_{k}$ invariance. An element $\tau \in S_{k}$, viewed as a bijective map $\tau:\{1, \ldots, k\} \rightarrow\{1, \ldots, k\}$ defines a linear operator $\mathcal{L}_{\tau^{-1}}$ which acts on $V_{N}^{\otimes k}$ as

$$
\begin{equation*}
\mathcal{L}_{\tau^{-1}}\left(e_{i_{1}} \otimes \cdots \otimes e_{i_{k}}\right)=e_{i_{\tau(1)}} \otimes \cdots \otimes e_{i_{\tau(k)}} . \tag{5.27}
\end{equation*}
$$

The symmetry of $T$ is equivalent to the statement

$$
\begin{equation*}
\mathcal{L}_{\tau} T \mathcal{L}_{\tau^{-1}}=T, \quad \forall \tau \in S_{k} \tag{5.28}
\end{equation*}
$$

or with indices

$$
\begin{equation*}
T_{i_{\tau(1)} \ldots i_{\tau(k)}}^{j_{\tau(1)} \ldots j_{\tau(k)}}=T_{i_{1} \ldots i_{k}}^{j_{1} \ldots j_{k}}, \quad \forall \tau \in S_{k} . \tag{5.29}
\end{equation*}
$$

Therefore, states in $\mathcal{H}^{(k)}$ are in one-to-one correspondence with elements $T \in \operatorname{End}_{S_{k}}\left(V_{N}^{\otimes k}\right)$, the subspace of linear maps that commute with the action of $S_{k}$.

We introduce diagrammatic notation to simplify manipulations involving tensor equations with many indices. A map $T \in \operatorname{End}\left(V_{N}^{\otimes k}\right)$ is represented by a box

$$
\begin{equation*}
T_{i_{1} \ldots i_{k}}^{j_{1} \ldots j_{k}}=\frac{j_{1} \ldots j_{k}}{\substack{T \\ i_{1} \ldots i_{k}}} \tag{5.30}
\end{equation*}
$$

where the incoming and outgoing edges correspond to states in $V_{N}^{\otimes k}$. Internal lines in a diagram correspond to contracted indices. For example, the state $|T\rangle \in \mathcal{H}^{(k)}$ can be represented diagrammatically as


Again, the horizontal lines identify the top edge with the bottom edge to give a trace, and the line between the $\left(a^{\dagger}\right)^{\otimes k}$ and $T$ boxes signifies that the corresponding indices are identified and summed over. This diagram should be compared to 5.25 .

### 5.2 Permutation invariant sectors for quantum matrix systems

In this section we consider the action of $S_{N}$ on the subspace $\mathcal{H}^{(k)}$, spanned by degree $k$ polynomials in matrix oscillators $\left(a^{\dagger}\right)_{j}^{i}$ acting on the vacuum. The adjoint action of
permutations $\sigma \in S_{N}$ on the quantum mechanical matrix variables

$$
\begin{equation*}
\sigma: X_{j}^{i} \rightarrow\left(M_{\sigma} X M_{\sigma^{-1}}\right)_{j}^{i}=X_{\sigma(j)}^{\sigma(i)} \tag{5.32}
\end{equation*}
$$

translates into action on oscillators

$$
\begin{equation*}
\sigma:\left(a^{\dagger}\right)_{j}^{i} \rightarrow\left(a^{\dagger}\right)_{\sigma(j)}^{\sigma(i)} \tag{5.33}
\end{equation*}
$$

We turn our attention to the subspace $\mathcal{H}_{\text {inv }}^{(k)} \subset \mathcal{H}^{(k)}$ of $S_{N}$ invariant states constructed from polynomials in these oscillators. We will construct bases for $\mathcal{H}_{\text {inv }}^{(k)}$, for general $k$, taking inspiration from the construction of representation theory bases for PIGM models in chapter 4 (originally [2]). There, a basis for the space of $S_{N}$ invariant polynomials in matrix indeterminates $X_{j}^{i}$ of degree $k$ was given in terms of elements of the diagrammatic partition algebra $P_{k}(N)$ 61. With the identification

$$
\begin{equation*}
X_{j}^{i} \leftrightarrow\left(a^{\dagger}\right)_{j}^{i}, \tag{5.34}
\end{equation*}
$$

we can employ these techniques to construct $S_{N}$ invariant states in matrix quantum mechanics.

The algebra $\operatorname{End}_{S_{N}}\left(V_{N}^{\otimes k}\right)$, of linear operators on $V_{N}^{\otimes k}$ that commute with $S_{N}$, is of central importance in understanding the implications of permutation invariance in quantum mechanical matrix systems. For $N \geq 2 k$ this algebra is isomorphic to the partition algebra $P_{k}(N) 61$

$$
\begin{equation*}
\operatorname{End}_{S_{N}}\left(V_{N}^{\otimes k}\right) \cong P_{k}(N) . \tag{5.35}
\end{equation*}
$$

The Hilbert space $\mathcal{H}_{\text {inv }}^{(k)}$ spanned by degree $k$ polynomials in the oscillators is isomorphic to an $S_{k}$ invariant subalgebra of $P_{k}(N)$ :

$$
\begin{equation*}
\mathcal{H}_{\mathrm{inv}}^{(k)} \cong \operatorname{End}_{S_{N} \times S_{k}}\left(V_{N}^{\otimes k}\right) \subseteq \operatorname{End}_{S_{N}}\left(V_{N}^{\otimes k}\right), \tag{5.36}
\end{equation*}
$$

The partition algebras are finite-dimensional associative algebras with dimension $B(2 k)$, the Bell numbers, 2.32 . Notably, $B(2 k)$ does not depend on $N$. Consequently, $\operatorname{Dim} \mathcal{H}_{\text {inv }}^{(k)}$ does not grow with $N$ for $N \geq 2 k$. This is in contrast to $\operatorname{Dim} \mathcal{H}^{(k)}$, which grows like $N^{2 k}$ for $N \gg 2 k$.

We have chosen to construct states using the oscillators $\left(a^{\dagger}\right)_{j}^{i}$. This produces a basis for $\mathcal{H}_{\text {inv }}$ that is simultaneously an energy eigenbasis of $H_{0}$, given by 5.8. However, it is worth emphasising that the resulting description of the state space $\mathcal{H}_{\text {inv }}$ is applicable to any quantum matrix system, not only the system with Hamiltonian $H_{0}$. For example, the
description of $\mathcal{H}_{\text {inv }}$ in terms of partition algebras holds equally well if the Hamiltonian is a perturbation of $H_{0}$ by a polynomial in the matrix creation and annihilation operators.

We begin this section in 5.2 .1 by reviewing the connection between partition algebras and states in $\mathcal{H}_{\text {inv }}$. In section 5.2 .2 we explore this connection in the diagram basis of partition algebras, and rephrase important results from chapter 4 in the language of matrix quantum mechanics. In section 5.2 .3 we introduce the representation basis for the partition algebras, so called because it is labelled by a set of representation theoretic data. This basis uses Fourier transforms [90] on $P_{k}(N)$ to construct an all-orders orthogonal basis for $N \geq 2 k$, which diagonalises a set of algebraic charges. These charges are discussed in detail in section 5.3, and used in section 5.5 to construct algebraic Hamiltonians with interesting spectra.

### 5.2.1 Partition algebras and invariant tensors

For any $\sigma \in S_{N}$ we have a linear operator $\mathcal{L}(\sigma) \in \operatorname{End}\left(V_{N}^{\otimes k}\right)$ defined by

$$
\begin{equation*}
\mathcal{L}\left(\sigma^{-1}\right)\left(e_{i_{1}} \otimes e_{i_{2}} \otimes \ldots \otimes e_{i_{k}}\right)=e_{\sigma\left(i_{1}\right)} \otimes e_{\sigma\left(i_{2}\right)} \otimes \ldots \otimes e_{\sigma\left(i_{k}\right)} \tag{5.37}
\end{equation*}
$$

Here $\sigma \in S_{N}$ is a bijective $\operatorname{map}\{1, \ldots, N\} \rightarrow\{1, \ldots, N\}$. This is used to define the adjoint action $\operatorname{Ad}(\sigma)$ of $\sigma \in S_{N}$ on states $|T\rangle \in \mathcal{H}^{(k)}$

$$
\begin{align*}
\operatorname{Ad}(\sigma)|T\rangle & =\operatorname{Tr}_{V_{N}^{\otimes k}}\left[\mathcal{L}(\sigma) T \mathcal{L}\left(\sigma^{-1}\right)\left(a^{\dagger}\right)^{\otimes k}\right]|0\rangle \\
& =\sum_{\substack{i_{1}, \ldots, i_{k} \\
j_{1}, \ldots, j_{k}}} T_{i_{1} \ldots i_{k}}^{j_{1} \ldots j_{k}}\left(a^{\dagger}\right)_{\sigma^{-1}\left(j_{1}\right)}^{\sigma_{1}^{-1}\left(i_{1}\right)} \ldots\left(a^{\dagger}\right)_{\sigma^{-1}\left(j_{k}\right)}^{\sigma^{-1}\left(i_{k}\right)}|0\rangle \\
& =\sum_{\substack{i_{1}, \ldots, i_{k} \\
j_{1}, \ldots, j_{k}}} T_{\sigma\left(i_{1}\right) \ldots \sigma\left(i_{k}\right)}^{\sigma\left(j_{1}\right) \ldots \sigma\left(j_{k}\right)}\left(a^{\dagger}\right)_{j_{1}}^{i_{1}} \ldots\left(a^{\dagger}\right)_{j_{k}}^{i_{k}}|0\rangle \tag{5.38}
\end{align*}
$$

This adjoint action on the tensor coefficients of the oscillators corresponds to the adjoint action on the oscillators which follows from (5.33). States $|T\rangle \in \mathcal{H}_{\mathrm{inv}}^{(k)}$ are $S_{N}$ invariant if they satisfy

$$
\begin{equation*}
\operatorname{Ad}(\sigma)|T\rangle=|T\rangle \tag{5.39}
\end{equation*}
$$

That is, all states in $\mathcal{H}_{\mathrm{inv}}^{(k)}$ can be constructed from tensors satisfying

$$
\begin{equation*}
T_{\sigma\left(i_{1}\right) \ldots \sigma\left(i_{k}\right)}^{\sigma\left(j_{1}\right) \ldots \sigma\left(j_{k}\right)}=T_{i_{1} \ldots i_{k}}^{j_{1} \ldots j_{k}}, \quad \forall \sigma \in S_{N} \tag{5.40}
\end{equation*}
$$

or equivalently

$$
\begin{equation*}
\mathcal{L}(\sigma) T \mathcal{L}\left(\sigma^{-1}\right)=T \tag{5.41}
\end{equation*}
$$

For $N \geq 2 k, \operatorname{End}_{S_{N}}\left(V_{N}^{\otimes k}\right)$ is isomorphic to the partition algebra $P_{k}(N)$

$$
\begin{equation*}
\operatorname{End}_{S_{N}}\left(V_{N}^{\otimes k}\right)=\operatorname{Span}_{\mathbb{C}}\left\{T \in \operatorname{End}\left(V_{N}^{\otimes k}\right): \mathcal{L}(\sigma) T \mathcal{L}\left(\sigma^{-1}\right)=T, \forall \sigma \in S_{N}\right\} \cong P_{k}(N) \tag{5.42}
\end{equation*}
$$

For tensors labelling states we have a further $S_{k}$ invariance. The vector space of $S_{N} \times S_{k}$ invariant linear maps is denoted

$$
\begin{align*}
& \operatorname{End}_{S_{N} \times S_{k}}\left(V_{N}^{\otimes k}\right)= \\
& \operatorname{Span}_{\mathbb{C}}\left\{T \in \operatorname{End}\left(V_{N}^{\otimes k}\right): \mathcal{L}(\sigma) T \mathcal{L}\left(\sigma^{-1}\right)=\mathcal{L}_{\tau} T \mathcal{L}_{\tau^{-1}}=T, \forall \sigma \in S_{N}, \tau \in S_{k}\right\} . \tag{5.43}
\end{align*}
$$

and we have the correspondence

$$
\begin{equation*}
\mathcal{H}_{\mathrm{inv}}^{(k)} \cong \operatorname{End}_{S_{N} \times S_{k}}\left(V_{N}^{\otimes k}\right) \tag{5.44}
\end{equation*}
$$

The partition algebra $P_{k}(N)$ contains a subalgebra $S P_{k}(N)$, spanned by elements that commute with $\mathbb{C} S_{k} \subset P_{k}(N)$, called the symmetrised partition algebra. For $N \geq 2 k$, $S P_{k}(N)$ is isomorphic to $\operatorname{End}_{S_{N} \times S_{k}}\left(V_{N}^{\otimes k}\right)$, and by extension $\mathcal{H}_{\text {inv }}^{(k)}$

$$
\begin{equation*}
\mathcal{H}_{\mathrm{inv}}^{(k)} \cong \operatorname{End}_{S_{N} \times S_{k}}\left(V_{N}^{\otimes k}\right) \cong S P_{k}(N) \tag{5.45}
\end{equation*}
$$

This motivates a description of the state space in terms of $S P_{k}(N)$, the symmetrised subalgebra of $P_{k}(N)$, which we turn to in the next subsection.

To summarise the above steps in words, we are investigating the adjoint action of permutations in $S_{N}$ on $N \times N$ quantum mechanical matrix variables $X_{j}^{i}$. The corresponding oscillators inherit the adjoint $S_{N}$ action. Oscillator states with $k$ oscillators correspond to tensors $T$ with $k$ upper and lower indices, subject to an $S_{k}$ symmetry permuting the $k$ upper-lower index pairs of $T$. This $S_{k}$ symmetry has its origin in the bosonic nature of the oscillators. The $S_{N}$ invariant $k$-oscillator states correspond to tensors having $k$ upper and $k$ lower indices, subject to an $S_{N} \times S_{k}$ invariance. This subspace of tensors can be described as a symmetrised sub-algebra $S P_{k}(N)$ of the partition algebra $P_{k}(N)$.

### 5.2.2 Diagram basis

Every state in $\mathcal{H}_{\mathrm{inv}}^{(k)}$ corresponds to an element in the $S_{k}$ invariant sub-algebra of $P_{k}(N)$, which we call the symmetrised partition algebra and denote $S P_{k}(N)$. Consider the action of $S_{k}$ on the diagrams given by

$$
\begin{equation*}
\tau: d_{\pi} \rightarrow \tau d_{\pi} \tau^{-1} \tag{5.46}
\end{equation*}
$$

for any $\tau \in S_{k}, d_{\pi} \in P_{k}(N)$. A basis for $S P_{k}(N)$ is labeled by distinct orbits under this action. We denote by $\left[d_{\pi}\right] \in S P_{k}(N)$ the invariant element obtained by averaging over the $S_{k}$ orbit of $d_{\pi}$

$$
\begin{equation*}
\left[d_{\pi}\right]=\frac{1}{k!} \sum_{\tau \in S_{k}} \tau d_{\pi} \tau^{-1}=\frac{1}{\left|\left[d_{\pi}\right]\right|} \sum_{d_{\pi^{\prime}} \in\left[d_{\pi}\right]} d_{\pi^{\prime}} \tag{5.47}
\end{equation*}
$$

where $\left|\left[d_{\pi}\right]\right|$ is the size of the orbit. The equality follows because $\left|\left[d_{\pi}\right]\right|$ is equal to $k$ ! divided by the number of permutations $\tau$ leaving $d_{\pi}$ fixed (orbit stabiliser theorem). It follows that a basis for $\mathcal{H}_{\mathrm{inv}}^{(k)}$ is labeled by $\left[d_{\pi}\right] \in S P_{k}(N)$ through the correspondence

$$
\begin{equation*}
\left|\left[d_{\pi}\right]\right\rangle=\operatorname{Tr}_{V_{N}^{\otimes k}}^{\otimes k}\left(\left[d_{\pi}\right]\left(a^{\dagger}\right)^{\otimes k}\right)|0\rangle=\sum_{\substack{i_{1}, \ldots, i_{k} \\ i_{1^{\prime}}, \ldots, i_{k^{\prime}}}}\left(\left[d_{\pi}\right]\right)_{i_{1} \ldots i_{k}}^{i_{1^{\prime} \ldots i_{k^{\prime}}}}\left(a^{\dagger}\right)_{i_{1^{\prime}}}^{i_{1}} \ldots\left(a^{\dagger}\right)_{i_{k^{\prime}}}^{i_{k}}|0\rangle \tag{5.48}
\end{equation*}
$$

Note that

$$
\begin{equation*}
\left|d_{\pi}\right\rangle=\operatorname{Tr}_{V_{N}^{\otimes k}}\left(d_{\pi}\left(a^{\dagger}\right)^{\otimes k}\right)|0\rangle=\left|\left[d_{\pi}\right]\right\rangle \tag{5.49}
\end{equation*}
$$

and so, for the sake of notational efficiency, we will often label states with $d_{\pi}$ instead of $\left[d_{\pi}\right]$. For example,

$$
\begin{equation*}
|[\bowtie]\rangle\rangle=|\curvearrowleft .\rangle=\sum_{i}\left(a^{\dagger}\right)_{i}^{i}\left(a^{\dagger}\right)_{i}^{i}|0\rangle \tag{5.50}
\end{equation*}
$$

and

$$
\left|\left[\begin{array}{l}
\bullet  \tag{5.51}\\
\bullet
\end{array}\right]\right\rangle=\left|\frac{1}{2}(\mathfrak{\bullet}+\bullet \bullet)\right\rangle=|\mathfrak{\llcorner}\rangle=\sum_{i, j}\left(a^{\dagger}\right)_{i}^{i}\left(a^{\dagger}\right)_{j}^{i}|0\rangle
$$

States obtained by acting with the annihilation operators $a_{j}^{i}$ on the dual vacuum $\langle 0|$ can similarly be labelled by partition algebra diagrams

$$
\begin{align*}
\left\langle d_{\pi}\right| & =\langle 0| \operatorname{Tr}_{V_{N}^{\otimes k}}^{\otimes k}\left(d_{\pi}^{T} a^{\otimes k}\right) \\
& =\langle 0| \operatorname{Tr}_{V_{N}^{\otimes k}}^{\otimes k}\left(\left[d_{\pi}^{T}\right] a^{\otimes k}\right) \\
& =\langle 0| \sum_{\substack{i_{1}, \ldots, i_{k} \\
i_{1}^{\prime}, \ldots, i_{k^{\prime}}}}\left(\left[d_{\pi}\right]\right)_{i_{1} \ldots i_{k}}^{i_{1^{\prime}} \ldots i_{k^{\prime}}} a_{i_{1}}^{i_{1^{\prime}}} \ldots a_{i_{k}}^{i_{k^{\prime}}} \\
& =\langle 0| \sum_{\substack{i_{1}, \ldots, i_{k} \\
i_{1^{\prime}}, \ldots, i_{k^{\prime}}}}\left(\left[d_{\pi}^{T}\right]\right)_{i_{1^{\prime} \ldots i_{k^{\prime}}}^{i_{1} \ldots i_{k}}} a_{i_{1}}^{i_{1_{1}^{\prime}}} \ldots a_{i_{k}}^{i_{k^{\prime}}}, \tag{5.52}
\end{align*}
$$

where $d_{\pi}^{T}$ is the transpose of $d_{\pi}$. As a diagram, $d_{\pi}^{T}$ is the reflection of $d_{\pi}$ across a horizontal line, for example

$$
\begin{equation*}
(\cdot \cdot \cdot)^{T}=\text { • } \tag{5.53}
\end{equation*}
$$

The use of the transpose in this definition is motivated by the orthonormality property below 5.56. As shown in section 4.2.2 the inner product can be written as a trace of products of elements in $S P_{k}(N)$,

$$
\begin{equation*}
\left\langle d_{\pi} \mid d_{\pi^{\prime}}\right\rangle=\sum_{\tau \in S_{k}}\left(d_{\pi}^{T} \tau d_{\pi^{\prime}} \tau^{-1}\right)_{i_{1} \ldots i_{k}}^{i_{1} \ldots i_{k}}=\sum_{\tau \in S_{k}} \operatorname{Tr}_{V_{N}^{\otimes k}}\left(d_{\pi}^{T} \tau d_{\pi^{\prime}} \tau^{-1}\right) . \tag{5.54}
\end{equation*}
$$

The large $N$ factorisation result 4.61) derived in chapter 4 (and originally in [2]) implies that the normalised states

$$
\begin{equation*}
\left|\hat{d}_{\pi}\right\rangle=\frac{1}{\sqrt{\left\langle d_{\pi} \mid d_{\pi}\right\rangle}}\left|d_{\pi}\right\rangle, \tag{5.55}
\end{equation*}
$$

are orthonormal at large $N$ (to leading order in $1 / \sqrt{N}$ )

$$
\left\langle\hat{d}_{\pi} \mid \hat{d}_{\pi^{\prime}}\right\rangle= \begin{cases}1+O(1 / \sqrt{N}) & \text { if }\left[d_{\pi}\right]=\left[d_{\pi^{\prime}}\right]  \tag{5.56}\\ 0+O(1 / \sqrt{N}) & \text { otherwise }\end{cases}
$$

### 5.2.3 Representation basis

The connection between $S_{N}$ invariant states and partition algebras gives rise to a natural basis, labelled by representation theoretic data. The representation basis diagonalises a set of commuting algebraic charges that we introduce in section 5.3. This observation gives a concrete construction algorithm for the change of basis matrix (from diagram basis to representation basis).

Recall the Schur-Weyl decomposition of $V_{N}^{\otimes k}$ 4.16) given in the previous chapter. In the limit $N \geq 2 k$ we can write this in a simplified form

$$
\begin{equation*}
V_{N}^{\otimes k}=\bigoplus_{\Lambda_{1} \in \mathcal{Y}_{\mathcal{S}}(k)} V_{\Lambda_{1}}^{S_{N}} \otimes V_{\Lambda_{1}}^{P_{k}(N)} \tag{5.57}
\end{equation*}
$$

in which the sum can be labelled by the set of all Young diagrams $\Lambda_{1}^{\#}$ having up to $k$ boxes: these are inserted below a first row to form Young diagrams with $N$ boxes. This stable set of Young diagrams having $N$ boxes is denoted $\mathcal{Y}_{\mathcal{S}}(k)$. We work within this limit throughout.

Equation (5.38) implies that we can identify

$$
\begin{equation*}
\operatorname{End}\left(V_{N}^{\otimes k}\right) \cong V_{N}^{\otimes k} \otimes V_{N}^{\otimes k} \tag{5.58}
\end{equation*}
$$

as a representation of $S_{N}$. We use Schur-Weyl duality (5.57) to decompose each factor on
the RHS as

$$
\begin{equation*}
V_{N}^{\otimes k} \otimes V_{N}^{\otimes k}=\left(\underset{\Lambda_{1} \in \mathcal{Y}_{\mathcal{S}}(k)}{ } V_{\Lambda_{1}}^{S_{N}} \otimes V_{\Lambda_{1}}^{P_{k}(N)}\right) \otimes\left(\underset{\Lambda_{1}^{\prime} \in \mathcal{Y}_{\mathcal{S}}(k)}{ } V_{\Lambda_{1}^{\prime}}^{S_{N}} \otimes V_{\Lambda_{1}^{\prime}}^{P_{k}(N)}\right) \tag{5.59}
\end{equation*}
$$

where we are assuming the stable limit. Projecting to $S_{N}$ invariants on both sides gives

$$
\begin{equation*}
P_{k}(N) \cong \operatorname{End}_{S_{N}}\left(V_{N}^{\otimes k}\right) \cong \bigoplus_{\Lambda_{1} \in \mathcal{Y}_{\mathcal{S}}(k)} V_{\Lambda_{1}}^{P_{k}(N)} \otimes V_{\Lambda_{1}}^{P_{k}(N)} \tag{5.60}
\end{equation*}
$$

This follows because the decomposition of $V_{\Lambda_{1}}^{S_{N}} \otimes V_{\Lambda_{1}^{\prime}}^{S_{N}}$ contains an invariant if and only if $\Lambda_{1}=\Lambda_{1}^{\prime}$.

As noted underneath 4.24 in chapter 4 , the RHS of 5.60 reflects a decomposition of $P_{k}(N)$ into a direct sum of matrix algebras. Such a decomposition always exists for a semi-simple algebra by the Artin-Wedderburn theorem. This implies that there exists a basis of generalised elementary matrices (also called a complete set of matrix units) for $P_{k}(N)$. A complete set of matrix units is a basis

$$
\begin{equation*}
Q_{\alpha \beta}^{\Lambda_{1}}, \quad \Lambda_{1} \in \mathcal{Y}_{\mathcal{S}}(k), \quad \alpha, \beta \in\left\{1, \ldots, \operatorname{Dim}\left(V_{\Lambda_{1}}^{P_{k}(N)}\right)\right\} \tag{5.61}
\end{equation*}
$$

with the property

$$
\begin{equation*}
Q_{\alpha \beta}^{\Lambda_{1}} Q_{\alpha^{\prime} \beta^{\prime}}^{\Lambda_{1}^{\prime}}=\delta^{\Lambda_{1} \Lambda_{1}^{\prime}} \delta_{\beta \alpha^{\prime}} Q_{\alpha \beta^{\prime}}^{\Lambda_{1}} \tag{5.62}
\end{equation*}
$$

In other words, $P_{k}(N)$ can be realised as block-diagonal matrices, with each block labelled by an irreducible representation $\Lambda_{1}$ of $P_{k}(N)$. The Artin-Wedderburn decomposition implies

$$
\begin{equation*}
\operatorname{Dim}\left(P_{k}(N)\right)=B(2 k)=\sum_{\Lambda_{1} \in \mathcal{Y}_{\mathcal{S}}(k)}\left(\operatorname{Dim} V_{\Lambda_{1}}^{P_{k}(N)}\right)^{2} \tag{5.63}
\end{equation*}
$$

which is analogous to the expression

$$
\begin{equation*}
|G|=\sum_{R \in \operatorname{Rep}(G)}\left(\operatorname{Dim} V_{R}^{G}\right)^{2} \tag{5.64}
\end{equation*}
$$

for the order of a finite group $G$ in terms of its irreducible representations $R$.

As is proven in Appendix A of [3], the following set of linear combinations of elements in $P_{k}(N)$ form a complete set of matrix units for $P_{k}(N)$,

$$
\begin{equation*}
Q_{\alpha \beta}^{\Lambda_{1}}=\sum_{i=1}^{B(2 k)} \operatorname{Dim}\left(V_{\Lambda_{1}}^{S_{N}}\right) D_{\beta \alpha}^{\Lambda_{1}}\left(\left(b_{i}^{*}\right)^{T}\right) b_{i} . \tag{5.65}
\end{equation*}
$$

The coefficients $D_{\beta \alpha}^{\Lambda_{1}}(d)$ are matrix elements of the representation of $P_{k}(N)$, labelled by $\Lambda_{1} \vdash N$. The sum is over a basis $b_{i}, i \in\{1, \ldots, B(2 k)\}$ for $P_{k}(N)$ (for example the diagram basis). The element $b_{i}^{*}$ is called the dual of $b_{i}$. It has an explicit construction in terms of the inverse of the Gram matrix defined by

$$
\begin{equation*}
g_{i j}=\operatorname{Tr}_{V_{N}^{\otimes k}}\left(b_{i} b_{j}^{T}\right) \tag{5.66}
\end{equation*}
$$

The dual of $b_{i}$ is

$$
\begin{equation*}
b_{i}^{*}=\sum_{j=1}^{B(2 k)} g_{i j}^{-1} b_{j} \tag{5.67}
\end{equation*}
$$

and the inverse of the Gram matrix in the diagram basis can be written as a series expansion in $N$.

To construct a representation basis for $\mathcal{H}_{\mathrm{inv}}^{(k)}$, we need to construct matrix units for $S P_{k}(N)$. They can be constructed from matrix units for $P_{k}(N)$ as follows. The partition algebra $P_{k}(N)$ contains a subalgebra $\mathbb{C} S_{k}$. Consequently, we can restrict an irreducible representation $V_{\Lambda_{1}}^{P_{k}(N)}$ to a representation of $\mathbb{C} S_{k}$, which in general is reducible. Letting $V_{\Lambda_{2}}^{\mathbb{C} S_{k}}$ be an irreducible representation of $\mathbb{C} S_{k}$ labelled by a Young diagram $\Lambda_{2}$ with $k$ boxes, we have

$$
\begin{equation*}
V_{\Lambda_{1}}^{P_{k}(N)} \cong \bigoplus_{\Lambda_{2} \vdash k} V_{\Lambda_{2}}^{\mathbb{C} S_{k}} \otimes V_{\Lambda_{1} \Lambda_{2}}^{P_{k}(N) \rightarrow \mathbb{C} S_{k}} \tag{5.68}
\end{equation*}
$$

The dimension of $V_{\Lambda_{1} \Lambda_{2}}^{P_{k}(N) \rightarrow \mathbb{C} S_{k}}$ is the branching multiplicity

$$
\begin{equation*}
\operatorname{Dim}\left(V_{\Lambda_{1} \Lambda_{2}}^{P_{k}(N) \rightarrow \mathbb{C} S_{k}}\right)=\operatorname{Mult}\left(V_{\Lambda_{1}}^{P_{k}(N)} \rightarrow V_{\Lambda_{2}}^{\mathbb{C} S_{k}}\right) \tag{5.69}
\end{equation*}
$$

In the rest of the chapter we will use $\Lambda_{1}$ to label irreducible representations of $S_{N}$ and $P_{k}(N)$. Irreducible representations of $S_{k}$ are denoted by $\Lambda_{2}$. Inserting the decomposition (5.68) into equation (5.60) and projecting to $S_{k}$ invariants gives

$$
\begin{equation*}
\mathcal{H}_{\mathrm{inv}}^{(k)} \cong \operatorname{End}_{S_{N} \times S_{k}}\left(V_{N}^{\otimes k}\right) \cong \bigoplus_{\substack{\Lambda_{1} \in \mathcal{Y}_{\mathcal{S}}(k) \\ \Lambda_{2} \vdash k}} V_{\Lambda_{1} \Lambda_{2}}^{P_{k}(N) \rightarrow \mathbb{C} S_{k}} \otimes V_{\Lambda_{1} \Lambda_{2}}^{P_{k}(N) \rightarrow \mathbb{C} S_{k}} \tag{5.70}
\end{equation*}
$$

This should be understood as an Artin-Wedderburn decomposition of $S P_{k}(N)$.
Equation (5.68) points us towards a construction of matrix units for $S P_{k}(N)$ from matrix units of $P_{k}(N)$. On the LHS we have a basis

$$
\begin{equation*}
E_{\alpha}^{\Lambda_{1}}, \quad \alpha \in\left\{1, \ldots \operatorname{Dim}\left(V_{\Lambda_{1}}^{P_{k}(N)}\right)\right\} \tag{5.71}
\end{equation*}
$$

where the representation of $d \in P_{k}(N)$ is irreducible,

$$
\begin{equation*}
d\left(E_{\alpha}^{\Lambda_{1}}\right)=\sum_{\beta} D_{\beta \alpha}^{\Lambda_{1}}(d) E_{\beta}^{\Lambda_{1}} \tag{5.72}
\end{equation*}
$$

The RHS has a basis

$$
\begin{align*}
E_{\Lambda_{2}, p}^{\Lambda_{1}, \mu}, & p \in\left\{1, \ldots, \operatorname{Dim}\left(V_{\Lambda_{1}}^{\mathbb{C} S_{k}}\right)\right\}  \tag{5.73}\\
& \mu \in\left\{1, \ldots, \operatorname{Dim}\left(V_{\Lambda_{1} \Lambda_{2}}^{P_{k}(N) \rightarrow \mathbb{C} S_{k}}\right)\right\},
\end{align*}
$$

where $\mu$ is a multiplicity label for $V_{\Lambda_{2}}^{\mathbb{C} S_{k}}$ in the decomposition. We demand that the representation of $\tau \in \mathbb{C} S_{k}$ is irreducible in this basis,

$$
\begin{equation*}
\tau\left(E_{\Lambda_{2}, p}^{\Lambda_{1}, \mu}\right)=\sum_{q} D_{q p}^{\Lambda_{2}}(\tau) E_{\Lambda_{2}, q}^{\Lambda_{1}, \mu} \tag{5.74}
\end{equation*}
$$

where $D_{q p}^{\Lambda_{2}}(\tau)$ is an irreducible representation of $\tau \in \mathbb{C} S_{k}$. The change of basis coefficients are called Branching coefficients

$$
\begin{equation*}
E_{\Lambda_{2}, p}^{\Lambda_{1}, \mu}=\sum_{\alpha} B_{\Lambda_{1}, \alpha \rightarrow \Lambda_{2}, p ; \mu}^{P_{k}(N) \rightarrow \mathbb{C} S_{k}} E_{\alpha}^{\Lambda_{1}} \tag{5.75}
\end{equation*}
$$

or in braket notation

$$
\begin{equation*}
B_{\Lambda_{1}, \alpha \rightarrow \Lambda_{2}, p ; \mu}^{P_{k}(N) \rightarrow \mathbb{C} S_{k}}=\left\langle E_{\alpha}^{\Lambda_{1}} \mid E_{\Lambda_{2}, p}^{\Lambda_{1}, \mu}\right\rangle \tag{5.76}
\end{equation*}
$$

The elements

$$
\begin{equation*}
Q_{\Lambda_{2}, \mu \nu}^{\Lambda_{1}}=\sum_{\alpha, \beta, p} Q_{\alpha \beta}^{\Lambda_{1}} B_{\Lambda_{1}, \alpha \rightarrow \Lambda_{2}, p ; \mu}^{P_{k}(N) \rightarrow \mathbb{C} S_{k}} B_{\Lambda_{1}, \beta \rightarrow \Lambda_{2}, p ; \nu}^{P_{k}(N) \rightarrow \mathbb{C} S_{k}} \tag{5.77}
\end{equation*}
$$

form a complete set of matrix units for $S P_{k}(N)$. The sum over $p$ implements the projection to $S_{k}$ invariants. The above elements satisfy (see appendix A of [3] for a proof of this)

$$
\begin{equation*}
Q_{\Lambda_{2}, \mu \nu}^{\Lambda_{1}} Q_{\Lambda_{2}^{\prime}, \mu^{\prime} \nu^{\prime}}^{\Lambda_{1}^{\prime}}=\delta^{\Lambda_{1} \Lambda_{1}^{\prime}} \delta_{\Lambda_{2} \Lambda_{2}^{\prime}} \delta_{\nu \mu^{\prime}} Q_{\Lambda_{2} \mu \nu^{\prime}}^{\Lambda_{1}} \tag{5.78}
\end{equation*}
$$

and orthogonality of states

$$
\begin{equation*}
\left|Q_{\Lambda_{2}, \mu \nu}^{\Lambda_{1}}\right\rangle=\operatorname{Tr}_{V_{N}^{\otimes k}}\left(Q_{\Lambda_{2}, \mu \nu}^{\Lambda_{1}}\left(a^{\dagger}\right)^{\otimes k}\right) \tag{5.79}
\end{equation*}
$$

follows from the form of the inner product 5.54 . The proof goes as follows

$$
\begin{align*}
\left\langle Q_{\Lambda_{2}, \mu \nu}^{\Lambda_{1}} \mid Q_{\Lambda_{2}^{\prime}, \mu^{\prime} \nu^{\prime}}^{\Lambda_{1}^{\prime}}\right\rangle & =\sum_{\tau \in S_{k}} \operatorname{Tr}_{V_{N}^{\otimes k}}\left(Q_{\Lambda_{2}, \mu \nu}^{\Lambda_{1}} \tau\left(Q_{\Lambda_{2}^{\prime}, \mu^{\prime} \nu^{\prime}}^{\Lambda_{1}^{\prime}}\right)^{T} \tau^{-1}\right) \\
& =\sum_{\tau \in S_{k}} \operatorname{Tr}_{V_{N}^{\otimes k}}\left(Q_{\Lambda_{2}, \mu \nu}^{\Lambda_{1}} \tau Q_{\Lambda_{2}^{\prime}, \nu^{\prime} \mu^{\prime}}^{\Lambda_{1}^{\prime}} \tau^{-1}\right)  \tag{5.80}\\
& =k!\operatorname{Tr}_{V_{N}^{\otimes k}}\left(Q_{\Lambda_{2}, \mu \nu}^{\Lambda_{1}} Q_{\Lambda_{2}^{\prime}, \nu^{\prime} \mu^{\prime}}^{\Lambda_{1}^{\prime}}\right) \\
& =k!\delta^{\Lambda_{1} \Lambda_{1}^{\prime}} \delta_{\Lambda_{2} \Lambda_{2}^{\prime}} \delta_{\nu \nu^{\prime}} \operatorname{Tr}_{V_{N}^{\otimes k} k}\left(Q_{\Lambda_{2}, \mu \mu^{\prime}}^{\Lambda_{1}}\right) .
\end{align*}
$$

In the second equality we used $\left(Q_{\Lambda_{2}^{\prime}, \mu^{\prime} \nu^{\prime}}^{\Lambda_{1}^{\prime}}\right)^{T}=Q_{\Lambda_{2}^{\prime}, \nu^{\prime} \mu^{\prime}}^{\Lambda_{1}^{\prime}}$ which follows from equation B.13). Note that

$$
\begin{align*}
\operatorname{Tr}_{V_{N}^{\otimes k}}\left(Q_{\Lambda_{2}, \mu \mu^{\prime}}^{\Lambda_{1}}\right) & =\operatorname{Tr}_{V_{N}^{\otimes k}}\left(Q_{\Lambda_{2}, \mu 1}^{\Lambda_{1}} Q_{\Lambda_{2}, 1 \mu^{\prime}}^{\Lambda_{1}}\right) \\
& =\operatorname{Tr}_{V_{N}^{\otimes k}}\left(Q_{\Lambda_{2}, 1 \mu^{\prime}}^{\Lambda_{1}} Q_{\Lambda_{2}, \mu 1}^{\Lambda_{1}}\right)  \tag{5.81}\\
& =\delta_{\mu \mu^{\prime}} \operatorname{Tr}_{V_{N}^{\otimes k}}\left(Q_{\Lambda_{2}, 11}^{\Lambda_{1}}\right) \\
& =\delta_{\mu \mu^{\prime}} \mathcal{N}_{\Lambda_{1} \Lambda_{2}}
\end{align*}
$$

such that the normalisation (see equation $\overline{B .27}$ )

$$
\begin{equation*}
\mathcal{N}_{\Lambda_{1} \Lambda_{2}}=\operatorname{Dim} V_{\Lambda_{1}}^{S_{N}} \operatorname{Dim} V_{\Lambda_{2}}^{S_{k}} \tag{5.82}
\end{equation*}
$$

only depends on irreducible representations $\Lambda_{1}, \Lambda_{2}$, which proves orthogonality.
To summarise, we have shown that there exists an orthogonal basis for $\mathcal{H}_{\mathrm{inv}}^{(k)}$ labelled by representation theoretic data, for arbitrary $N \geq 2 k$, using Fourier transforms on semisimple algebras. The detailed proofs of these results can be found in the appendices of [3]. In the next section we will provide explicit formulas for the change of basis from the diagram basis to the basis of matrix units. We leave the elucidation of finite $N$ effects (the case $N<2 k$ which lies beyond the stable limit) in the representation basis for future work.

### 5.3 Representation basis and algebraic charges

In this section we discuss the construction of the representation basis elements $Q_{\Lambda_{2}, \mu \nu}^{\Lambda_{1}}$ as linear combinations of diagrams in $P_{k}(N)$. These can, in principle, be computed using equation (5.77) by first computing the branching coefficients. The computation of these requires explicit choices of basis in the representations $V_{\Lambda_{1}}^{P_{k}(N)}$ and $V_{\Lambda_{2}}^{\mathbb{C}\left[S_{k}\right]}$. Such choices can be bypassed. The basic idea is to find the $Q_{\Lambda_{2}, \mu \nu}^{\Lambda_{1}}$ as eigenvectors of appropriate elements of $P_{k}(N)$ which can be viewed as operators on $P_{k}(N)$ acting by the algebra multiplication. The subspaces labelled by $\Lambda_{1}, \Lambda_{2}$, associated with irreducible representations of $S_{N}$ and $S_{k}$
respectively, are identified using central elements (Casimirs) in the group algebras $\mathbb{C}\left[S_{N}\right]$ and $\mathbb{C}\left[S_{k}\right]$. These Casimirs can be expressed as elements of $P_{k}(N)$ using Schur-Weyl duality. This is particularly useful in the large $N$ limit where $k$ is kept fixed and $N \gg k$, since the dimension of $P_{k}(N)$ does not grow with $N$. The more refined determination of subspaces labelled by $\mu$ and $\nu$ is achieved by picking non-central elements of $P_{k}(N)$ which nevertheless generate a maximally commuting subalgebra.

We explicitly construct the change of basis for the special cases of degree $k=1,2$. Tables of these basis elements are found in appendix $D$. The expansion coefficients are given as functions of $N$ and are therefore valid for all $N \geq 2 k$.

Analogous constructions in multi-matrix systems with continuous gauge symmetry, relevant to AdS/CFT, are given in [12, 112]. They also played a role, using developments in tensor models with $U(N)$ symmetries, in [95] in giving a combinatorial interpretation of Kronecker coefficients.

### 5.3.1 Central elements in the partition algebra

For a fixed pair $\Lambda_{1}, \Lambda_{2}$, the linear span of $Q_{\Lambda_{2}, \mu \nu}^{\Lambda_{1}}$ for $\mu, \nu=1, \ldots, \operatorname{Dim} V_{\Lambda_{1} \Lambda_{2}}^{P_{k}(N) \rightarrow \mathbb{C}\left[S_{k}\right]}$ forms a subspace of $S P_{k}(N)$. We now describe how this subspace can be identified with simultaneous eigenspaces of Casimirs associated with $\mathbb{C}\left[S_{N}\right]$ and $\mathbb{C}\left[S_{k}\right]$.

First, we define Casimirs of $\mathbb{C}\left[S_{N}\right]$, and explain their relation to $P_{k}(N)$. The center $\mathcal{Z}\left(\mathbb{C}\left[S_{N}\right]\right)$ of $\mathbb{C}\left[S_{N}\right]$ consists of elements

$$
\begin{equation*}
\mathcal{Z}\left(\mathbb{C}\left[S_{N}\right]\right)=\left\{z \in \mathbb{C}\left[S_{N}\right]: z \sigma=\sigma z, \quad \forall \sigma \in \mathbb{C}\left[S_{N}\right]\right\} . \tag{5.83}
\end{equation*}
$$

Elements in the center are called central elements. For a central element $z$, the homomorphism property of representations implies

$$
\begin{equation*}
\mathcal{L}(z) \mathcal{L}(\sigma)=\mathcal{L}(\sigma) \mathcal{L}(z), \quad \forall \sigma \in S_{N} \tag{5.84}
\end{equation*}
$$

and it follows that $\mathcal{L}(z)$ is an element of the algebra of operators acting on $V_{N}^{\otimes k}$ which commutes with $S_{N}$, i.e. $\mathcal{L}(z) \in \operatorname{End}_{S_{N}}\left(V_{N}^{\otimes k}\right)$.

As we reviewed in the previous section, $P_{k}(N) \cong \operatorname{End}_{S_{N}}\left(V_{N}^{\otimes k}\right)$ for $N \geq 2 k$. This establishes a connection between $\mathcal{Z}\left(\mathbb{C}\left[S_{N}\right]\right)$ and $P_{k}(N)$ as linear operators acting on $V_{N}^{\otimes k}$. In particular, for every $z \in \mathcal{Z}\left(\mathbb{C}\left[S_{N}\right]\right)$, there exists an element $\bar{z} \in P_{k}(N)$ defined by

$$
\begin{equation*}
\bar{z}\left(e_{i_{1}} \otimes \cdots \otimes e_{i_{k}}\right)=\mathcal{L}(z)\left(e_{i_{1}} \otimes \cdots \otimes e_{i_{k}}\right) . \tag{5.85}
\end{equation*}
$$

Note that the definition of $\bar{z}$ depends on $k$. Further, observe that

$$
\begin{equation*}
\mathcal{L}(z) d\left(e_{i_{1}} \otimes \cdots \otimes e_{i_{k}}\right)=d \mathcal{L}(z)\left(e_{i_{1}} \otimes \cdots \otimes e_{i_{k}}\right) \tag{5.86}
\end{equation*}
$$

for all $d \in P_{k}(N)$ because $P_{k}(N)$ and $\mathbb{C}\left[S_{N}\right]$ are mutual commutants in $\operatorname{End}\left(V_{N}^{\otimes k}\right)$. This implies that $\bar{z}$ is automatically in the center of $P_{k}(N)$, which we denote $\mathcal{Z}\left(P_{k}(N)\right)$. In other words, equation (5.85) defines a homomorphism from $\mathcal{Z}\left(\mathbb{C}\left[S_{N}\right]\right)$ to $\mathcal{Z}\left(P_{k}(N)\right)$. As a particular case of being central in $P_{k}(N), \bar{z}$ commutes with $\mathbb{C}\left[S_{k}\right] \subset P_{k}(N)$.

Central elements play a special role in representation theory. Schur's lemma implies that an irreducible matrix representation of a central element is proportional to the identity matrix. The proportionality constant is a normalised character. In particular we have

$$
\begin{equation*}
D_{a b}^{\Lambda_{1}}(z)=\hat{\chi}^{\Lambda_{1}}(z) \delta_{a b} \tag{5.87}
\end{equation*}
$$

where we have introduced the short-hand

$$
\begin{equation*}
\hat{\chi}^{\Lambda_{1}}(z)=\frac{\chi^{\Lambda_{1}}(z)}{\operatorname{Dim} V_{\Lambda_{1}}^{S_{N}}} \tag{5.88}
\end{equation*}
$$

for normalised characters. In this sense central elements are Casimirs, they act by constants on irreducible subspaces, and the constants can be used to determine the particular representation.

The element of $\mathbb{C}\left[S_{N}\right]$ formed by summing over all elements in a distinct conjugacy class of $S_{N}$ is central. For example, we define the element $T_{2} \in \mathcal{Z}\left(\mathbb{C}\left[S_{N}\right]\right)$ as

$$
\begin{equation*}
T_{2}=\sum_{1 \leq i<j \leq N}(i j) \tag{5.89}
\end{equation*}
$$

where the sum is over all transpositions. By the argument in the previous paragraph, there exists an element $\bar{T}_{2}^{(k)} \in \mathcal{Z}\left(P_{k}(N)\right)$ such that

$$
\begin{align*}
\bar{T}_{2}^{(k)}\left(e_{i_{1}} \otimes \cdots \otimes e_{i_{k}}\right)= & \mathcal{L}\left(T_{2}\right)\left(e_{i_{1}} \otimes \cdots \otimes e_{i_{k}}\right) \\
\sum_{i_{1^{\prime} \ldots i_{k^{\prime}}}}\left(\bar{T}_{2}^{(k)}\right)_{i_{1} \ldots i_{k}}^{i_{1^{\prime} \ldots i_{k^{\prime}}}} e_{i_{1^{\prime}}} \otimes \cdots \otimes e_{i_{k^{\prime}}} & =\sum_{\substack{\sigma=(i j) \\
1 \leq i<j \leq N}} e_{\sigma^{-1}\left(i_{1}\right)} \otimes \cdots \otimes e_{\sigma^{-1}\left(i_{k}\right)} .
\end{align*}
$$

As we will explain, the eigenvalues of the central element $\bar{T}_{2}^{(k)}$ can be used to distinguish the label $\Lambda_{1}$ on matrix units. Since $\bar{T}_{2}^{(k)}$ is an element of $S P_{k}(N)$, it has an expansion in
terms of diagrams (see [61, Equation 3.32, Theorem 3.35])

$$
\begin{equation*}
\bar{T}_{2}^{(k)}=\sum_{\pi \in \Pi_{2 k}}\left(\bar{T}_{2}^{(k)}\right)^{\pi} d_{\pi} \tag{5.91}
\end{equation*}
$$

The equality in (5.90) implies a radical simplification for large $N$. The element $T_{2}$ contains order $N^{2}$ transpositions, while $\bar{T}_{2}^{(k)}$ contains at most $B(2 k)$ diagrams. The dependence on $N$ is incorporated in the coefficients $\left(\bar{T}_{2}^{(k)}\right)^{\pi}$, which are polynomial functions of $N$. Explicit examples are 5.118 and 5.123 .

There exist similar elements $t_{2}^{(k)} \in \mathcal{Z}\left(\mathbb{C}\left[S_{k}\right]\right) \subset \mathcal{Z}\left(P_{k}(N)\right)$ defined by summing over transposition diagrams. For example,


The eigenvalues of $t_{2}^{(k)}$ will be used to distinguish the label $\Lambda_{2}$.
Equation (5.90) together with equation (5.87 gives

$$
\begin{equation*}
D_{\alpha \beta}^{\Lambda_{1}}\left(\bar{T}_{2}^{(k)}\right)=\frac{\chi^{\Lambda_{1}}\left(\bar{T}_{2}^{(k)}\right)}{\operatorname{Dim} V_{\Lambda_{1}}^{P_{k}(N)}} \delta_{\alpha \beta}=\hat{\chi}^{\Lambda_{1}}\left(T_{2}\right) \delta_{\alpha \beta} \tag{5.93}
\end{equation*}
$$

where the distinction between the two characters is

$$
\begin{equation*}
\chi^{\Lambda_{1}}\left(\bar{T}_{2}^{(k)}\right)=\sum_{\alpha=1}^{\operatorname{Dim} V_{\Lambda_{1}}^{P_{k}(N)}} D_{\alpha \alpha}^{\Lambda_{1}}\left(\bar{T}_{2}^{(k)}\right), \quad \text { and } \quad \chi^{\Lambda_{1}}\left(T_{2}\right)=\sum_{a=1}^{\operatorname{Dim} V_{\Lambda_{1}}^{S_{N}}} D_{a a}^{\Lambda_{1}}\left(T_{2}\right) \tag{5.94}
\end{equation*}
$$

That is, the first character is a character of $P_{k}(N)$, the second is a character of $\mathbb{C}\left[S_{N}\right]$. Similarly,

$$
\begin{equation*}
D_{p q}^{\Lambda_{2}}\left(t_{2}^{(k)}\right)=\hat{\chi}^{\Lambda_{2}}\left(t_{2}^{(k)}\right) \delta_{p q} \tag{5.95}
\end{equation*}
$$

where

$$
\begin{equation*}
\hat{\chi}^{\Lambda_{2}}\left(t_{2}^{(k)}\right)=\frac{\chi^{\Lambda_{2}}\left(t_{2}^{(k)}\right)}{\operatorname{Dim} V_{\Lambda_{2}}^{S_{k}}} \tag{5.96}
\end{equation*}
$$

Normalised characters of $T_{2}$ and $t_{2}^{(k)}$ can be expressed in terms of combinatorial quantities (known as the contents) of boxes of Young diagrams (see example 7 in section I. 7 of [113]). Let $Y_{\Lambda_{1}}, Y_{\Lambda_{2}}$ be the Young diagrams corresponding to integer partitions $\Lambda_{1} \in \mathcal{Y}_{\mathcal{S}}(k), \Lambda_{2} \vdash k$. Then

$$
\begin{equation*}
\hat{\chi}^{\Lambda_{1}}\left(T_{2}\right)=\sum_{(i, j) \in Y_{\Lambda_{1}}}(j-i), \quad \hat{\chi}^{\Lambda_{2}}\left(t_{2}^{(k)}\right)=\sum_{(i, j) \in Y_{\Lambda_{2}}}(j-i), \tag{5.97}
\end{equation*}
$$

where $(i, j)$ corresponds to the cell in the $i$ th row and $j$ th column of the Young diagram
(the top left box has coordinate $(1,1)$ ).
With the above facts at hand, we can understand how the $\Lambda_{1}, \Lambda_{2}$ labels correspond to eigenvalues of $\bar{T}_{2}^{(k)}, t_{2}^{(k)}$. $P_{k}(N)$ matrix units have the following property (proven in appendix A of (3)

$$
\begin{equation*}
d Q_{\alpha \beta}^{\Lambda_{1}}=\sum_{\gamma} D_{\gamma \alpha}^{\Lambda_{1}}(d) Q_{\gamma \beta}^{\Lambda_{1}}, \quad \text { for } d \in P_{k}(N), \tag{5.98}
\end{equation*}
$$

and therefore

$$
\begin{equation*}
Q_{\alpha \beta}^{\Lambda_{1}} \bar{T}_{2}^{(k)}=\bar{T}_{2}^{(k)} Q_{\alpha \beta}^{\Lambda_{1}}=\sum_{\gamma} D_{\gamma \alpha}^{\Lambda_{1}}\left(\bar{T}_{2}^{(k)}\right) Q_{\gamma \beta}^{\Lambda_{1}}=\hat{\chi}^{\Lambda_{1}}\left(T_{2}\right) Q_{\alpha \beta}^{\Lambda_{1}} . \tag{5.99}
\end{equation*}
$$

We derive a similar equation for $t_{2}^{(k)}$ acting on $Q_{\Lambda_{2}, \mu \nu}^{\Lambda_{1}}$ using the definition in 5.77. From the definition we have

$$
\begin{align*}
t_{2}^{(k)} Q_{\Lambda_{2}, \mu \nu}^{\Lambda_{1}} & =\sum_{\alpha, \beta, p} t_{2}^{(k)} Q_{\alpha \beta}^{\Lambda_{1}} B_{\Lambda_{1}, \alpha \rightarrow \Lambda_{2}, p ; \mu}^{P_{k}(N) \rightarrow \mathbb{C}\left[S_{k}\right]} B_{\Lambda_{1}, \beta \rightarrow \Lambda_{2}, p ; \nu}^{P_{k}(N) \rightarrow \mathbb{C}\left[S_{k}\right]}  \tag{5.100}\\
& =\sum_{\alpha, \beta, \gamma, \gamma^{\prime}, p} D_{\gamma \alpha}^{\Lambda_{1}}\left(t_{2}^{(k)}\right) \delta_{\gamma \gamma^{\prime}} Q_{\gamma^{\prime} \beta}^{\Lambda_{1}} B_{\Lambda_{1}, \alpha \rightarrow \Lambda_{2}, p, \mu}^{P_{k}(N) \rightarrow \mathbb{C}\left[S_{k}\right]} B_{\Lambda_{1}, \beta \rightarrow \Lambda_{2}, p ; \nu}^{P_{k}(N) \rightarrow \mathbb{C}\left[S_{k}\right]} .
\end{align*}
$$

We re-write the Kronecker delta using the completeness relation

$$
\begin{equation*}
\sum_{\Lambda_{2}^{\prime}, p^{\prime}, \mu^{\prime}} B_{\Lambda_{1}, \gamma \rightarrow \Lambda_{2}, p^{\prime} ; \mu^{\prime}}^{P_{k}(N) \rightarrow \mathbb{C}\left[S_{k}\right]}{ }_{\Lambda_{1}, \gamma^{\prime} \rightarrow \Lambda_{2}, p^{\prime} ; \mu^{\prime}}^{P_{k}(N) \rightarrow \mathbb{C}\left[S_{k}\right]}=\delta_{\gamma \gamma^{\prime}} . \tag{5.101}
\end{equation*}
$$

Inserting this into (5.100) gives

$$
\begin{align*}
& \sum_{\alpha, \beta, \gamma, \gamma^{\prime}, p} D_{\gamma \alpha}^{\Lambda_{1}}\left(t_{2}^{(k)}\right) \delta_{\gamma \gamma^{\prime}} Q_{\gamma^{\prime} \beta}^{\Lambda_{1}} B_{\Lambda_{1}, \alpha \rightarrow \Lambda_{2}, p ; \mu}^{P_{k}(N) \rightarrow \mathbb{C}\left[S_{k}\right]} B_{\Lambda_{1}, \beta \rightarrow \Lambda_{2}, p ; \nu}^{P_{k}(N) \rightarrow \mathbb{C}\left[S_{k}\right]}= \\
& \sum_{\alpha, \beta, \gamma, \gamma^{\prime}, p, p \Lambda_{2}^{\prime}, p^{\prime}, \mu^{\prime}} D_{\gamma \alpha} D_{\gamma_{1}}^{\Lambda_{1}}\left(t_{2}^{(k)}\right) B_{\Lambda_{1}, \gamma \rightarrow \Lambda_{2}, p^{\prime} ; \mu^{\prime}}^{P_{k}(N) \rightarrow \mathbb{C}\left[S_{k}\right]} B_{\Lambda_{1}, \gamma^{\prime} \rightarrow \Lambda_{2}^{\prime}, p^{\prime} ; \mu^{\prime}}^{P_{k}(N) \rightarrow \mathbb{C}\left[S_{k}\right]} Q_{\gamma^{\prime} \beta}^{\Lambda_{1}} B_{\Lambda_{1}, \alpha \rightarrow \Lambda_{2}, p ; \mu}^{P_{k}(N) \rightarrow \mathbb{C}\left[S_{k}\right]} B_{\Lambda_{1}, \beta \rightarrow \Lambda_{2}, p ; \nu}^{P_{k}(N) \rightarrow \mathbb{C}\left[S_{k}\right]} . \tag{5.102}
\end{align*}
$$

Now note that

$$
\begin{equation*}
\sum_{\gamma, \alpha} D_{\gamma \alpha}^{\Lambda_{1}}\left(t_{2}^{(k)}\right) B_{\Lambda_{1}, \gamma \rightarrow \Lambda_{2}^{2}, p^{\prime} ; \mu^{\prime}}^{P_{k}(N) \rightarrow \mathbb{C}\left[S_{k}\right]} B_{\Lambda_{1}, \alpha \rightarrow \Lambda_{2}, p ; \mu}^{P_{k}(N) \rightarrow \mathbb{C}\left[S_{k}\right]}=\delta_{\Lambda_{2} \Lambda_{2}^{\prime}} \delta_{\mu^{\prime} \mu} D_{p^{\prime} p}^{\Lambda_{2}}\left(t_{2}^{(k)}\right)=\delta_{\Lambda_{2} \Lambda_{2}^{\prime}} \delta_{\mu^{\prime} \mu} \delta_{p^{\prime} p} \hat{\chi}^{\Lambda_{2}}\left(t_{2}^{(k)}\right) . \tag{5.103}
\end{equation*}
$$

We substitute this into (5.102) and find

$$
\begin{align*}
& \sum_{\alpha, \beta, \gamma, \gamma^{\prime}, p, p \Lambda_{2}^{\prime}, p^{\prime}, \mu^{\prime}} D_{\gamma \alpha}^{\Lambda_{1}}\left(t_{2}^{(k)}\right) B_{\Lambda_{1}, \gamma \rightarrow \Lambda_{2}^{\prime}, p^{\prime} ; \mu^{\prime}}^{P_{k}(N) \rightarrow \mathbb{C}\left[S_{k}\right]} B_{\Lambda_{1}, \gamma^{\prime} \rightarrow \Lambda_{2}^{\prime}, p^{\prime} ; \mu^{\prime}}^{P_{k}(N) \rightarrow \mathbb{C}\left[S_{k}\right]} Q_{\gamma^{\prime} \beta}^{\Lambda_{1}} B_{\Lambda_{1}, \alpha \rightarrow \Lambda_{2}, p ; \mu}^{P_{k}(N) \rightarrow \mathbb{C}\left[S_{k}\right]} B_{\Lambda_{1}, \beta \rightarrow \Lambda_{2}, p ; \nu}^{P_{k}(N) \rightarrow \mathbb{C}\left[S_{k}\right]}=  \tag{5.104}\\
& \sum_{\beta, \gamma^{\prime}, p, p} \sum_{\Lambda_{2}^{\prime}, p^{\prime}, \mu^{\prime}} \delta_{\Lambda_{2} \Lambda_{2}^{\prime}} \delta_{\mu^{\prime} \mu} \delta_{p^{\prime} p} \hat{\chi}^{\Lambda_{2}}\left(t_{2}^{(k)}\right) B_{\Lambda_{1}, \gamma^{\prime} \rightarrow \Lambda_{2}, p^{\prime} ; \mu^{\prime}}^{P_{k}(N) \rightarrow \mathbb{C}\left[S_{k}\right]} Q_{\gamma^{\prime} \beta}^{\Lambda_{1}} B_{\Lambda_{1}, \beta \rightarrow \Lambda_{2}, p ; \nu}^{P_{k}(N) \rightarrow \mathbb{C}\left[S_{k}\right]}=  \tag{5.105}\\
& \sum_{\beta, \gamma^{\prime}, p} \hat{\chi}^{\Lambda_{2}}\left(t_{2}^{(k)}\right) Q_{\gamma^{\prime} \beta}^{\Lambda_{1}} B_{\Lambda_{1}, \gamma^{\prime} \rightarrow \Lambda_{2}, p ; \mu}^{P_{k}(N) \rightarrow \mathbb{C}\left[S_{k}\right]} B_{\Lambda_{1}, \beta \rightarrow \Lambda_{2}, p ; \nu}^{P_{k}(N) \rightarrow \mathbb{C}\left[S_{k}\right]}=\hat{\chi}^{\Lambda_{2}}\left(t_{2}^{(k)}\right) Q_{\Lambda_{2}, \mu \nu}^{\Lambda_{1}}, \tag{5.106}
\end{align*}
$$

which proves the analogue of (5.99) in the case of $t_{2}^{(k)}$.
We define linear operators on $S P_{k}(N)$ using multiplication by $\bar{T}_{2}^{(k)}, t_{2}^{(k)}$

$$
\begin{equation*}
\bar{T}_{2}^{(k)}\left(Q_{\Lambda_{2}, \mu \nu}^{\Lambda_{1}}\right)=\bar{T}_{2}^{(k)} Q_{\Lambda_{2}, \mu \nu}^{\Lambda_{1}}=\hat{\chi}^{\Lambda_{1}}\left(T_{2}\right) Q_{\Lambda_{2}, \mu \nu}^{\Lambda_{1}}, \tag{5.107}
\end{equation*}
$$

and

$$
\begin{equation*}
t_{2}^{(k)}\left(Q_{\Lambda_{2}, \mu \nu}^{\Lambda_{1}}\right)=t_{2}^{(k)} Q_{\Lambda_{2}, \mu \nu}^{\Lambda_{1}}=\hat{\chi}^{\Lambda_{2}}\left(t_{2}^{(k)}\right) Q_{\Lambda_{2}, \mu \nu}^{\Lambda_{1}} . \tag{5.108}
\end{equation*}
$$

That is, the matrix units for $S P_{k}(N)$ are eigenvectors of the linear operators associated with $\bar{T}_{2}^{(k)}$ and $t_{2}^{(k)}$. The eigenvalues are sufficient to determine the subspaces labelled by irreducible representations $\Lambda_{1}, \Lambda_{2}$ for $k=1,2$ and general $N$. As discussed in detail in [112], a larger set of central elements is needed to distinguish different pairs $\Lambda_{1}, \Lambda_{2}$ for general $k$ and $N$.

### 5.3.2 Multiplicity labels and maximal commuting subalgebras

In the previous subsection we described how the subspace spanned by $Q_{\Lambda_{2}, \mu \nu}^{\Lambda_{1}}$ for fixed $\Lambda_{1}, \Lambda_{2}$ is a simultaneous eigenspace of central elements $\bar{T}_{2}^{(k)}, t_{2}^{(k)}$. The subspaces labeled by fixed $\mu, \nu$ are not eigenspaces of any central elements of $S P_{k}(N)$. Nevertheless, they are eigenspaces of elements that (multiplicatively) generate a maximal commutative subalgebra of $S P_{k}(N)$.

We illustrate this in the simple case of a single matrix algebra. This is directly relevant, because the matrix units $Q_{\Lambda_{2}, \mu \nu}^{\Lambda_{1}}$ form (are isomorphic to) a matrix algebra $M_{n}$ with $n=$ $\operatorname{Dim} V_{\Lambda_{1} \Lambda_{2}}^{P_{k}(N) \rightarrow \mathbb{C}\left[S_{k}\right]}$, for fixed $\Lambda_{1}, \Lambda_{2}$. The matrix algebra $M_{n}$ has a basis of matrix units $E_{r s}$ for $r, s=1, \ldots, n$. These are just the elementary matrices with zeroes everywhere except in row $r$, column $s$ where there is a one. In this explicitly realised algebra, it is straight-forward to verify that

$$
\begin{equation*}
E_{r s} E_{r^{\prime} s^{\prime}}=\delta_{s r^{\prime}} E_{r s^{\prime}} . \tag{5.109}
\end{equation*}
$$

It follows from equation (5.109) that

$$
E_{t t} E_{r s}=\delta_{t r} E_{t s}= \begin{cases}E_{r s} \quad \text { if } r=t  \tag{5.110}\\ 0 \quad \text { otherwise }\end{cases}
$$

This fact will be useful in what follows.

We now define a pair of linear operators acting on $M_{n}$ whose eigenvalues uniquely determine the indices $r, s$ on $E_{r s}$. Let

$$
\begin{equation*}
T=1 E_{11}+2 E_{22}+\cdots+n E_{n n} \tag{5.111}
\end{equation*}
$$

and $T^{L}, T^{R}$ be the linear operators on $M_{n}$ defined by left and right action of $T$ respectively,

$$
\begin{equation*}
T^{L}\left(E_{r s}\right)=T E_{r s}, \quad T^{R}\left(E_{r s}\right)=E_{r s} T \tag{5.112}
\end{equation*}
$$

The $n^{2} \times n^{2}$ matrix $\left(T^{L}\right)_{r s}^{t u}$ associated with the linear operator $T^{L}$ has eigenvalues $\{1,2, \ldots, n\}$ (each one is $n$-fold degenerate) with eigenvectors $E_{r s}$,

$$
\begin{equation*}
\sum_{t, u}\left(T^{L}\right)_{r s}^{t u} E_{t u}=T^{L}\left(E_{r s}\right)=r E_{r s} \tag{5.113}
\end{equation*}
$$

Similarly for the matrix $\left(T^{R}\right)_{r s}^{t u}$ associated with the linear operator $T^{R}$,

$$
\begin{equation*}
\sum_{t, u}\left(T^{R}\right)_{r s}^{t u} E_{t u}=T^{R}\left(E_{r s}\right)=s E_{r s} \tag{5.114}
\end{equation*}
$$

The operators $T^{L}$ and $T^{R}$ commute, and their simultaneous eigenvectors $E_{r s}$ have eigenvalues $r$ and $s$, respectively.

The algebra spanned by $\left\{E_{11}, E_{22}, \ldots, E_{n n}\right\}$ is a maximal commuting subalgebra of $M_{n}$. It is multiplicatively generated by $T$. In particular (see [112, Lemma 3.3.1] or [114, Lemma 2.1])

$$
\begin{equation*}
E_{r r}=\prod_{s \neq r} \frac{(T-s)}{(r-s)} \tag{5.115}
\end{equation*}
$$

These ideas generalise to $Q_{\Lambda_{2}, \mu \nu}^{\Lambda_{1}}$, and in the next section we will give the appropriate operators corresponding to $T^{L}, T^{R}$ for $S P_{2}(N)$.

### 5.3.3 Construction of low degree representation bases

We now use the tools presented in this section to explicitly construct the representation basis elements as sums of diagrams, for $k=1,2$ and large $N$. Tables of the representation basis elements expanded in terms of diagrams are found in appendix $D$.

## Degree one basis

For $k=1$ it is enough to use $\bar{T}_{2}^{(1)}$ to distinguish the irreducible representations. We expect to find matrix units

$$
\begin{equation*}
Q_{[1]}^{[N]}, Q_{[1]}^{[N-1,1]} \tag{5.116}
\end{equation*}
$$

since $S_{1}$ only has the trivial representation and the decomposition in 5.57) only contains irreducible representations $[N]$ and $[N-1,1]$ of $P_{1}(N)$.

The map

$$
\begin{equation*}
T_{2} \mapsto \bar{T}_{2}^{(1)} \tag{5.117}
\end{equation*}
$$

is given by (see the section called Murphy elements for $\mathbb{C} A_{k}(N)$ in 61)

$$
\begin{equation*}
\bar{T}_{2}^{(1)}=\frac{N(N-3)}{2}!+\stackrel{\bullet}{\bullet} \tag{5.118}
\end{equation*}
$$

It is straight-forward to diagonalise $\bar{T}_{2}^{(1)}$ acting on $P_{1}(N)$ from the left. Define

$$
\begin{equation*}
Q_{[1]}^{[N]}=\frac{1}{N} \bullet, \quad Q_{[1]}^{[N-1,1]}=\mathfrak{\varrho}-\frac{1}{N} \bullet \tag{5.119}
\end{equation*}
$$

they satisfy

$$
\begin{equation*}
Q_{[1]}^{[N]} Q_{[1]}^{[N-1,1]}=0, \quad Q_{[1]}^{[N]} Q_{[1]}^{[N]}=Q_{[1]}^{[N]}, \quad Q_{[1]}^{[N-1,1]} Q_{[1]}^{[N-1,1]}=Q_{[1]}^{[N-1,1]} \tag{5.120}
\end{equation*}
$$

and have eigenvalues

$$
\begin{align*}
\bar{T}_{2}^{(1)} Q_{[1]}^{[N]} & =\frac{N(N-1)}{2} Q_{[1]}^{[N]}, \\
\bar{T}_{2}^{(1)} Q_{[1]}^{[N-1,1]} & =\frac{N(N-3)}{2} Q_{[1]}^{[N-1,1]}, \tag{5.121}
\end{align*}
$$

which are exactly equal to the normalised characters. Note that $S_{1}$ has no non-trivial representations, and $t_{2}^{(1)}=0$, which is consistent with the normalised character $\frac{k(k-1)}{2}=0$ of the trivial representation.

The orthogonal basis elements for $\mathcal{H}_{\text {inv }}^{(1)}$, corresponding to these matrix units, are

$$
\begin{equation*}
\left|Q_{[1]}^{[N]}\right\rangle=\frac{1}{N} \sum_{i_{1}, i_{1}{ }^{\prime}}\left(a^{\dagger}\right)_{i_{1}}^{i_{1}^{\prime}}|0\rangle, \quad \text { and } \quad\left|Q_{[1]}^{[N-1,1]}\right\rangle=\sum_{i_{1}}\left(a^{\dagger}\right)_{i_{1}}^{i_{1}}-\frac{1}{N} \sum_{i_{1}, i_{1^{\prime}}}\left(a^{\dagger}\right)_{i_{1}}^{i_{1_{1}^{\prime}}}|0\rangle . \tag{5.122}
\end{equation*}
$$

## Degree two basis

The procedure was particularly easy at degree one because $S_{1}$ is trivial, and there were no multiplicities appearing. For $k=2$ we have the sign representation $[1,1]$ and the trivial representation [2] of $S_{2}$, and pairs of irreducible representations $\Lambda_{1}, \Lambda_{2}$ appear with multiplicity larger than one. To distinguish multiplicities we will have to introduce noncentral elements, as discussed in subsection 5.3.2.

At degree two, the partition algebra element we use to distinguish $\Lambda_{1}$ is 61]

$$
\begin{equation*}
\bar{T}_{2}^{(2)}=\frac{(N-2)(N-3)-4}{2}!!+!\cdot+\bullet \cdot+\cdots+\mathbb{C}+N!! \tag{5.123}
\end{equation*}
$$

As a linear map (acting on the left or right) on $P_{2}(N)$, it has eigenvalues

$$
\begin{align*}
\bar{T}_{2}^{(2)}\left(Q_{\Lambda_{2}, \mu \nu}^{[N]}\right) & =\frac{N(N-1)}{2} Q_{\Lambda_{2}, \mu \nu}^{[N]}, \\
\bar{T}_{2}^{(2)}\left(Q_{\Lambda_{2}, \mu \nu}^{[N-1,1]}\right) & =\frac{N(N-3)}{2} Q_{\Lambda_{2}, \mu \nu}^{[N-1,1]}, \\
\bar{T}_{2}^{(2)}\left(Q_{\Lambda_{2}, \mu \nu}^{[N-2,2]}\right) & =\frac{(N-1)(N-4)}{2} Q_{\Lambda_{2}, \mu \nu}^{[N-2,2]}, \\
\bar{T}_{2}^{(2)}\left(Q_{\Lambda_{2}, \mu \nu}^{[N-2,1,1]}\right) & =\frac{N(N-5)}{2} Q_{\Lambda_{2}, \mu \nu}^{[N-2,1,1]} . \tag{5.124}
\end{align*}
$$

The element we use to distinguish $\Lambda_{2}$ is

$$
\begin{equation*}
t_{2}^{(2)}=\boldsymbol{X} . \tag{5.125}
\end{equation*}
$$

The eigenvalues of the corresponding linear map are 1 for $[2]$ and -1 for $[1,1]$.
The non-central element we will use to distinguish multiplicities is

$$
\begin{equation*}
\bar{T}_{2,1}^{(2)}=!\cdot{ }^{+}+\boldsymbol{\bullet} \tag{5.126}
\end{equation*}
$$

It is closely related to $\bar{T}_{2}^{(1)} \in \mathcal{Z}\left(P_{1}\right)$ in equation (5.118) because

$$
\begin{equation*}
\bar{T}_{2}^{(1)} \otimes 1+1 \otimes \bar{T}_{2}^{(1)}=\stackrel{\bullet}{\bullet}+\bullet!+N(N-3)!! \tag{5.127}
\end{equation*}
$$

Roughly speaking, $\bar{T}_{2,1}^{(2)}$ comes from the embedding of $\bar{T}_{2}^{(1)}$ into $S P_{2}(N)$ by adding strands. Symmetrisation has been used to ensure that we have an element in $S P_{2}(N)$.

To determine the multiplicity labels we need to act from the left as well as the right using $\bar{T}_{2,1}^{(2)}$. We define $\bar{T}_{2,1}^{(2), L}$ and $\bar{T}_{2,1}^{(2), R}$ acting on $d \in P_{2}(N)$ by

$$
\begin{equation*}
\bar{T}_{2,1}^{(2), L} d=\bar{T}_{2,1}^{(2)} d, \quad \bar{T}_{2,1}^{(2), R} d=d \bar{T}_{2,1}^{(2)} \tag{5.128}
\end{equation*}
$$

Appendix D gives a representation theoretic argument for why these operators fully distinguish all labels on matrix units, together with a complete table of all $k=2$ matrix units. As an example, we find a matrix unit (see D.30)

$$
\begin{equation*}
\left(Q_{[1,1]}^{[N-2,1,1]}\right)_{22}=\frac{1}{N} \cdot \bullet-\frac{1}{N} \cdot \bullet-\frac{1}{N} \cdot \bullet+\sum+\frac{1}{N} \cdot \bullet-\mathfrak{C} \tag{5.129}
\end{equation*}
$$

which corresponds to the (unnormalised) $S_{N}$ invariant state

$$
\begin{equation*}
\left|\left(Q_{[1,1]}^{[N-2,1,1]}\right)_{22}\right\rangle=\frac{2}{N}\left(\sum_{i, j, k=1}^{N}\left[\left(a^{\dagger}\right)_{i}^{i}\left(a^{\dagger}\right)_{k}^{j}-\left(a^{\dagger}\right)_{i}^{j}\left(a^{\dagger}\right)_{k}^{i}\right]+\sum_{i, j=1}^{N}\left[\left(a^{\dagger}\right)_{j}^{i}\left(a^{\dagger}\right)_{i}^{j}-\left(a^{\dagger}\right)_{i}^{i}\left(a^{\dagger}\right)_{j}^{j}\right]\right)|0\rangle \tag{5.130}
\end{equation*}
$$

### 5.4 Exactly solvable permutation invariant matrix harmonic oscillator

The simplest quantum mechanical matrix Hamiltonian (5.8), considered in section 5.1, is invariant under the symmetric group action

$$
\begin{equation*}
\sigma: X_{i j} \rightarrow X_{\sigma(i) \sigma(j)}, \quad \forall \sigma \in S_{N} \tag{5.131}
\end{equation*}
$$

It is also invariant under the much larger symmetry of continuous transformations by $U\left(N^{2}\right)$. In this section we generalise the quadratic potential to the most general quadratic function $V(X)$ invariant under the above permutation symmetry. We will thus present a quantum mechanical model of $N^{2}$ matrix variables $X_{i j}$ in a permutation invariant quadratic potential $V(X)$. The most general permutation invariant quadratic action in a zero-dimensional matrix model was constructed in [43] using representation theory. Borrowing these techniques, we explicitly construct an 11 parameter family of permutation invariant quadratic potentials. The corresponding Hamiltonian is exactly diagonalisable. We describe the spectrum of the full Hamiltonian and discuss the degeneracy when the quanta of energy are generic, and when they satisfy integrality properties. In the former
case we are able to give a lower bound on the order of the degeneracy, this is given in equation 5.155). In the latter case, the degeneracy is given in terms of an integer partition problem. The integer partition problem has a solution in terms of a canonical partition function (generating function) given by equation 5.158. We end this section in 5.4.4 with a brief discussion of the role that the representation basis could play in simplifying the diagonalisation of $H$, given in equation (5.147), on $\mathcal{H}_{\text {inv }}$.

### 5.4.1 Construction

A matrix harmonic oscillator in a potential $V(X)$ is described by the Lagrangian

$$
\begin{equation*}
L=\frac{1}{2} \sum_{i, j=1}^{N} \partial_{t} X_{i j} \partial_{t} X_{i j}-\frac{1}{2} V(X) . \tag{5.132}
\end{equation*}
$$

We take the potential to be a general quadratic $S_{N}$ invariant potential

$$
\begin{equation*}
V\left(X_{i j}\right)=V\left(X_{\sigma(i) \sigma(j)}\right), \quad \forall \sigma \in S_{N} \tag{5.133}
\end{equation*}
$$

The action of $S_{N}$ on $X_{i j}$ defined in (5.133) corresponds to the diagonal action on the tensor product $V_{N} \otimes V_{N}$. This is given in (5.37) for general $k$, for the $k=2$ case at hand we have

$$
\begin{equation*}
\mathcal{L}\left(\sigma^{-1}\right)\left(e_{i} \otimes e_{j}\right)=e_{\sigma(i)} \otimes e_{\sigma(j)} . \tag{5.134}
\end{equation*}
$$

The vector space $V_{N} \otimes V_{N}$ is reducible with respect to the diagonal action, and the decomposition is given by 2.63). Again, we take the RHS of this isomorphism to be a vector space with orthonormal basis $X_{a}^{\Lambda, \alpha}$ labelled by

$$
\begin{align*}
& \Lambda \in\{[N],[N-1,1],[N-2,2],[N-2,1,1]\} \\
& a \in\left\{1, \ldots, \operatorname{Dim} V_{\Lambda}^{S_{N}}\right\}  \tag{5.135}\\
& \alpha \in\left\{1, \ldots, \operatorname{Mult}\left(V_{N} \otimes V_{N} \rightarrow V_{\Lambda}^{S_{N}}\right)\right\}
\end{align*}
$$

In the representation basis the potential has a simple form,

$$
\begin{equation*}
V(X)=\sum_{\Lambda, \alpha, \beta, a} X_{a}^{\Lambda, \alpha} g_{\alpha \beta}^{\Lambda} X_{a}^{\Lambda, \beta} \tag{5.136}
\end{equation*}
$$

which we can again write in terms of the original matrix variables $X_{i j}$ and projectors $Q_{i j k l}^{\Lambda, \alpha \beta}$
defined in section 2.6

$$
\begin{equation*}
V(X)=\sum_{\Lambda, \alpha, \beta} \sum_{i, j, k, l} Q_{i j k l}^{\Lambda, \alpha \beta} g_{\alpha \beta}^{\Lambda} X_{i j} X_{k l} \tag{5.137}
\end{equation*}
$$

### 5.4.2 Spectrum

The full Hamiltonian with quadratic potential given in 5.137) can be diagonalised using oscillators. We will see that diagonalising the Hamiltonian only requires the diagonalisation of a set of small parameter matrices (one $3 \times 3$ and another $2 \times 2$ ), despite having a potentially large number of harmonic oscillators $\left(N^{2}\right)$.

The full Lagrangian in the representation basis is

$$
\begin{equation*}
L=\sum_{\Lambda, \alpha, \beta, a} \delta_{\alpha \beta} \partial_{t} X_{a}^{\Lambda, \alpha} \partial_{t} X_{a}^{\Lambda, \beta}-X_{a}^{\Lambda, \alpha} g_{\alpha \beta}^{\Lambda} X_{a}^{\Lambda, \beta} \tag{5.138}
\end{equation*}
$$

It describes a set of coupled harmonic oscillators. We write the Lagrangian in decoupled form in the usual way. Let $\Omega_{\alpha \beta}^{\Lambda}=\left(\omega_{\alpha}^{\Lambda}\right)^{2} \delta_{\alpha \beta}$ be the diagonal matrix ${ }^{1}$ such that

$$
\begin{equation*}
g_{\alpha \beta}^{\Lambda}=\sum_{\gamma, \delta} U_{\alpha \gamma}^{\Lambda} \Omega_{\gamma \delta}^{\Lambda} U_{\beta \delta}^{\Lambda} \tag{5.139}
\end{equation*}
$$

where $U^{\Lambda}$ are orthogonal change of basis matrices. In the decoupled basis

$$
\begin{equation*}
S_{a}^{\Lambda, \alpha}=\sum_{\beta} X_{a}^{\Lambda, \beta} U_{\beta \alpha}^{\Lambda} \tag{5.140}
\end{equation*}
$$

we have

$$
\begin{equation*}
L=\sum_{\Lambda, \alpha, a} \frac{1}{2} \partial_{t} S_{a}^{\Lambda, \alpha} \partial_{t} S_{a}^{\Lambda, \alpha}-\frac{1}{2}\left(\omega_{\alpha}^{\Lambda}\right)^{2} S_{a}^{\Lambda, \alpha} S_{a}^{\Lambda, \alpha} \tag{5.141}
\end{equation*}
$$

The canonical momenta are given by

$$
\begin{equation*}
\Sigma_{a}^{\Lambda, \alpha}=\partial_{t} S_{a}^{\Lambda, \alpha} \tag{5.142}
\end{equation*}
$$

The new canonical coordinates satisfy

$$
\begin{equation*}
\left[\Sigma_{a}^{\Lambda, \alpha}, S_{b}^{\Lambda^{\prime}, \beta}\right]=i \delta^{\Lambda \Lambda^{\prime}} \delta^{\alpha \beta} \delta_{a b} \tag{5.143}
\end{equation*}
$$

since $U^{\Lambda}$ are orthogonal matrices.

[^5]The corresponding Hamiltonian,

$$
\begin{equation*}
H=\frac{1}{2} \sum_{\Lambda, \alpha, a} \Sigma_{a}^{\Lambda, \alpha} \Sigma_{a}^{\Lambda, \alpha}+\left(\omega_{\alpha}^{\Lambda}\right)^{2} S_{a}^{\Lambda, \alpha} S_{a}^{\Lambda, \alpha}, \tag{5.144}
\end{equation*}
$$

is diagonalised by introducing oscillators

$$
\begin{align*}
S_{a}^{\Lambda, \alpha} & =\sqrt{\frac{1}{2 \omega_{\alpha}^{\Lambda}}}\left(\left(A^{\dagger}\right)_{a}^{\Lambda, \alpha}+A_{a}^{\Lambda, \alpha}\right),  \tag{5.145}\\
\Sigma_{a}^{\Lambda, \alpha} & =i \sqrt{\frac{\omega_{\alpha}^{\Lambda}}{2}}\left(\left(A^{\dagger}\right)_{a}^{\Lambda, \alpha}-A_{a}^{\Lambda, \alpha}\right),
\end{align*}
$$

which satisfy

$$
\begin{equation*}
\left[A_{a}^{\Lambda, \alpha},\left(A^{\dagger}\right)_{a^{\prime}}^{\Lambda^{\prime}, \alpha^{\prime}}\right]=\delta^{\Lambda \Lambda^{\prime}} \delta^{\alpha \alpha^{\prime}} \delta_{a a^{\prime}} . \tag{5.146}
\end{equation*}
$$

In the oscillator basis, the normal ordered Hamiltonian has the form

$$
\begin{equation*}
H=\sum_{\Lambda, \alpha, a} \omega_{\alpha}^{\Lambda}\left(A^{\dagger}\right)_{a}^{\Lambda, \alpha} A_{a}^{\Lambda, \alpha} . \tag{5.147}
\end{equation*}
$$

Defining number operators $\widehat{N}_{a}^{\Lambda, \alpha}$ and $\widehat{N}^{\Lambda, \alpha}$ as

$$
\begin{gather*}
\widehat{N}_{a}^{\Lambda, \alpha}=\left(A^{\dagger}\right)_{a}^{\Lambda, \alpha} A_{a}^{\Lambda, \alpha},  \tag{5.148}\\
\widehat{N}^{\Lambda, \alpha}=\sum_{a} \widehat{N}_{a}^{\Lambda, \alpha}, \tag{5.149}
\end{gather*}
$$

we may write

$$
\begin{equation*}
H=\sum_{\Lambda, \alpha, a} \widehat{N}_{a}^{\Lambda, \alpha}=\sum_{\Lambda, \alpha} \widehat{N}^{\Lambda, \alpha} . \tag{5.150}
\end{equation*}
$$

The energy quanta $\omega_{\alpha}^{\Lambda}$ do not depend on the oscillator state index $a$. This is a manifestation of the $S_{N}$ invariance of the Hamiltonian $H$.

The Hilbert space $\mathcal{H}^{(k)}$ has a basis of energy eigenstates

$$
\begin{equation*}
\prod_{\substack{\Lambda \in\{[N],[N-1,1],[N-2,2],[N-2,1,1]\} \\ \alpha \in\left\{1, \ldots \operatorname{Mult}\left(V_{N} \otimes V_{N} \rightarrow V_{\Lambda}^{S_{N}^{N}}\right)\right\} \\ a \in\left\{1, \ldots, \operatorname{Dim} V_{\Lambda}^{S_{N}^{N}}\right\}}} \frac{\left[\left(A^{\dagger}\right)_{a}^{\Lambda, \alpha}\right]^{N_{a}^{\Lambda, \alpha}}}{\sqrt{N_{a}^{\Lambda, \alpha}!}}|0\rangle \tag{5.151}
\end{equation*}
$$

where $k=\sum_{\Lambda, \alpha, a} N_{a}^{\Lambda, \alpha}$ is the eigenvalue of the (total) number operator

$$
\begin{equation*}
\widehat{N}=\sum_{\Lambda, \alpha, a} \widehat{N}_{a}^{\Lambda, \alpha} \tag{5.152}
\end{equation*}
$$

and $N_{a}^{\Lambda, \alpha}$ is the eigenvalue of $\widehat{N}_{a}^{\Lambda, \alpha}$.
Since the Hamiltonian (5.147) is a linear combination of number operators $\widehat{N}^{\Lambda, \alpha}$, it is natural to organise $\mathcal{H}^{(k)}$ into eigenspaces of $\widehat{N}^{\Lambda, \alpha}$ with eigenvalues $N^{\Lambda, \alpha}=\sum_{a} N_{a}^{\Lambda, \alpha}$ satisfying $k=\sum_{\Lambda, \alpha} N^{\Lambda, \alpha}$. Diagonalising the number operators $\widehat{N}^{\Lambda, \alpha}$ organises $\mathcal{H}^{(k)}$ into subspaces

$$
\begin{equation*}
\mathcal{H}^{(k)} \cong \bigoplus_{\Sigma N^{\Lambda, \alpha}=k} \bigotimes_{\Lambda, \alpha} \mathcal{H}^{\left[N^{\Lambda, \alpha}\right]} \tag{5.153}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathcal{H}^{\left[N^{\Lambda, \alpha}\right]} \cong \operatorname{Sym}^{N^{\Lambda, \alpha}}\left(V_{\Lambda}^{S_{N}}\right) \tag{5.154}
\end{equation*}
$$

Each summand in 5.153 is a vector space of dimension

$$
\begin{align*}
& \operatorname{Dim}\left(\bigotimes_{\Lambda, \alpha} \mathcal{H}^{\left[N^{\Lambda, \alpha]}\right.}\right)=\prod_{\Lambda, \alpha}\binom{\operatorname{Dim} V_{\Lambda}^{S_{N}}+N^{\Lambda, \alpha}-1}{N^{\Lambda, \alpha}}= \\
& \left(\begin{array}{c}
\left.1+\begin{array}{c}
{[N], 1} \\
N^{[N], 1}
\end{array}\right)\binom{1+N^{[N], 2}-1}{N^{[N], 2}} \times . ~(N \text {. }
\end{array}\right) \\
& \binom{N-1+N^{[N-1,1], 1}-1}{N^{[N-1,1], 1}}\binom{N-1+N^{[N-1,1], 2}-1}{N^{[N-1,1], 2}}\binom{N-1+N^{[N-1,1], 3}-1}{N^{[N-1,1], 3}} \times \\
& \binom{(N-1)(N-2) / 2+N^{[N-2,2]}-1}{N^{[N-2,2]}}\binom{N(N-3) / 2+N^{[N-2,1,1]}-1}{N^{[N-2,2]}}= \\
& \binom{N-2+N^{[N-1,1], 1}}{N^{[N-1,1], 1}}\binom{N-2+N^{[N-1,1], 2}}{N^{[N-1,1], 2}}\binom{N-2+N^{[N-1,1], 3}}{N^{[N-1,1], 3}} \times \\
& \binom{N(N-3) / 2+N^{[N-2,2]}}{N^{[N-2,2]}}\binom{N(N-3) / 2-1+N^{[N-2,1,1]}}{N^{[N-2,2]}} . \tag{5.155}
\end{align*}
$$

Vectors in $\bigotimes_{\Lambda, \alpha} \mathcal{H}^{\left[N^{\Lambda, \alpha}\right]}$ have energy

$$
\begin{equation*}
E\left(\left\{N^{\Lambda, \alpha}\right\}\right)=\sum_{\Lambda, \alpha} N^{\Lambda, \alpha} \omega_{\alpha}^{\Lambda} \tag{5.156}
\end{equation*}
$$

Equation 5.155 thus gives the degeneracy of energy eigenstates for the specified integers $\left\{N^{\Lambda, \alpha}\right\}$, associated with $\Lambda, \alpha$ as given in 5.135. This puts a lower bound on the degeneracy of energy eigenstates. Further degeneracy may occur for particular choices of the constants $\omega_{\alpha}^{\Lambda}$, which can lead to the same numerical value of $E\left(\left\{N^{\Lambda, \alpha}\right\}\right)$ for different choices of $\left\{N^{\Lambda, \alpha}\right\}$.

### 5.4.3 Canonical partition function

The canonical partition function is defined as

$$
\begin{equation*}
Z(\beta)=\operatorname{Tr}_{\mathcal{H}} e^{-\beta H}=\sum_{\mathcal{E}} N(\mathcal{E}) e^{-\beta \mathcal{E}} \tag{5.157}
\end{equation*}
$$

where $N(\mathcal{E})$ is the degeneracy of eigenstates at energy $\mathcal{E}$ and $\beta$ is the inverse temperature.

The binomial factors in 5.155 arise in the expansion of simple rational functions. Defining $x=e^{-\beta}$ for convenience, we can therefore write

$$
\begin{align*}
Z(\beta)= & \frac{1}{\left(1-x^{\omega_{1}^{[N]}}\right)\left(1-x^{\omega_{2}^{[N]}}\right)} \frac{1}{\left(1-x^{\omega_{1}^{[N-1,1]}}\right)^{N-1}\left(1-x^{\omega_{2}^{[N-1,1]}}\right)^{N-1}\left(1-x^{\omega_{3}^{[N-1,1]}}\right)^{N-1}} \times \\
& \frac{1}{\left(1-x^{\omega^{[N-2,2]}}\right)^{(N-1)(N-2) / 2}\left(1-x^{\omega^{[N-2,1,1]}}\right)^{N(N-3) / 2}} \tag{5.158}
\end{align*}
$$

When the quanta of energy $\left(\omega_{\alpha}^{\Lambda}\right)$ in 5.147) are integers, the possible state energies $\mathcal{E}$ are integers and $N(\mathcal{E})$ is related to what we refer to as an integer partition problem. The integer partition problem is the following: pick any integer $\mathcal{E}$, enumerate the set of solutions (choices of $N_{a}^{\Lambda, \alpha}$ ) to

$$
\begin{equation*}
\mathcal{E}=\sum_{\Lambda, \alpha, a} N_{a}^{\Lambda, \alpha} \omega_{\alpha}^{\Lambda} \tag{5.159}
\end{equation*}
$$

The number of solutions is equal to $N(\mathcal{E})$ and a single solution is denoted $N_{a}^{\Lambda, \alpha}(\mathcal{E})$. This problem depends on $N$ because the state label $a$ ranges over $\left\{1, \ldots, \operatorname{Dim} V_{\Lambda}^{S_{N}}\right\}$. Fortunately, the $N$-dependence can be factorised due to the $S_{N}$ symmetry, which greatly simplifies the problem.

To see this, consider the $N$-independent integer partition problem

$$
\begin{equation*}
\mathcal{E}=\sum_{\Lambda, \alpha} N^{\Lambda, \alpha} \omega_{\alpha}^{\Lambda} \tag{5.160}
\end{equation*}
$$

where a solution is given by a list of seven integers $N^{\Lambda, \alpha}(\mathcal{E})$. For every solution $N^{\Lambda, \alpha}(\mathcal{E})$ to 5.160 the number of solutions to the integer partition problem in 5.159 is given by

$$
\begin{equation*}
\operatorname{Dim}\left(\bigotimes_{\Lambda, \alpha} \mathcal{H}^{\left[N^{\Lambda, \alpha}(\mathcal{E})\right]}\right) \tag{5.161}
\end{equation*}
$$

In this sense, the $N$-dependence in the problem has factorised: we only need to find solutions to the $N$-independent equation (5.160) and multiply each solution by a known
$N$-dependent factor. The total number of solutions to 5.159 is given by

$$
\begin{equation*}
\sum_{N^{\Lambda, \alpha}(\mathcal{E})} \operatorname{Dim}\left(\bigotimes_{\Lambda, \alpha} \mathcal{H}^{\left[N^{\Lambda, \alpha}(\mathcal{E})\right]}\right) \tag{5.162}
\end{equation*}
$$

where the sum is over the set of solutions to 5.160 .

### 5.4.4 Energy eigenbases

We have observed that the oscillator states constructed using partition algebra diagram operators in tensor space, contracted with oscillators $\left(a^{\dagger}\right)_{i}^{j}$ obeying (5.11), are eigenstates of the simplest matrix hamiltonian $H_{0}$, given by (5.12). Contracting representation basis elements in the partition algebra with oscillators produces quantum states,

$$
\begin{equation*}
\left|Q_{\Lambda_{2}, \mu \nu}^{\Lambda_{1}}\right\rangle=\operatorname{Tr}_{V_{N}^{\otimes k}}\left(Q_{\Lambda_{2}, \mu \nu}^{\Lambda_{1}}\left(a^{\dagger}\right)^{\otimes k}\right)|0\rangle, \tag{5.163}
\end{equation*}
$$

which are eigenstates of $H_{0}$ and which also diagonalise algebraic conserved charges.
The representation basis states are not eigenstates of the general permutation invariant harmonic oscillator Hamiltonians $H$ in 5.144. There is mixing of the representation basis labels $\left(\Lambda_{1}, \Lambda_{2}, \mu, \nu\right)$ caused by the different weights for the representations $\Lambda$ appearing in the expansion of the $S_{N}$ invariant harmonic oscillator Hamiltonian defined in equation 5.147). We expect this mixing of the labels in the $\left(\Lambda_{1}, \Lambda_{2}, \mu, \nu\right)$ basis to be constrained, for example by the $S_{N}$ Clebsch-Gordan decompositions of $\Lambda \otimes \Lambda_{1}$. Such constrained mixing of representation theory bases for matrix systems arises in Hamiltonians of interest in AdS/CFT. A number of representation theory bases for $U(N)$ invariant multi-matrix systems have been described which capture information about finite $N$ effects and are eigenstates of the Hamiltonian (in radial quantisation) in the free Yang-Mills limit [7, 8, 9, 10, 11, 12]. However, the one-loop dilatation operator defines a non-trivial Hamiltonian which is, in general, not diagonalised by these representation theoretic bases (although there are some interesting exceptions to this statement, see [115). Representation theoretic constraints on the mixing caused by the one-loop dilatation operator are described in [116, 117, 118, 115, 119], following earlier work on one-loop mixings related to strings attached to giant gravitons, e.g. [120, 121].

### 5.5 Algebraic Hamiltonians and permutation invariant ground states

So far our discussion of $S_{N}$ invariant subspaces in quantum mechanical matrix systems has largely (with the exception of the previous section) been independent of any choice of Hamiltonian acting on the Hilbert space. It can be viewed as a general description of the kinematics of $S_{N}$ invariance, independent of the dynamics determined by the Hamiltonian. In this section we present Hamiltonians which realise the eigenspectrum scenarios depicted in figure 5.1, this includes Hamiltonians for which the low energy eigenstates are permutation invariant states.

The Hamiltonians we consider preserve the $S_{N}$ invariant subspace $\mathcal{H}_{\text {inv }}$ defined as

$$
\begin{equation*}
\mathcal{H}_{\mathrm{inv}}=\left\{|T\rangle \in \mathcal{H}: \operatorname{Ad}(\sigma)|T\rangle=|T\rangle, \forall \sigma \in S_{N}\right\} \tag{5.164}
\end{equation*}
$$

The adjoint action of permutations $\sigma \in S_{N}$ on the tensors $T$ labelling the states simultaneously transforms the upper and lower indices of $T$ according to 5.38). For any state $|T\rangle \in \mathcal{H}_{\text {inv }}$ the Hamiltonians $H$ obey the condition

$$
\begin{equation*}
H|T\rangle \in \mathcal{H}_{\mathrm{inv}} \tag{5.165}
\end{equation*}
$$

A sufficient condition for $H$ to satisfy 5.165 is for $H$ itself to be $S_{N}$ invariant i.e. $[\operatorname{Ad}(\sigma), H]=0$ for all $\sigma \in S_{N}$.

We show how to construct Hamiltonians $H_{K}$ of this type, depending on an integer parameter $K$, with finite-dimensional space of $S_{N}$ invariant ground states. Both the energy gap between the ground states and the lowest non-zero energy level, and the ground state degeneracy depend on $K$ in a way that is determined by the algebraic construction. As sketched in the left-hand figure of 5.1a, $H_{K}$ has an energy gap of order $K$. The construction of $H_{K}$ can be viewed as including, in the Hamiltonian, central elements in $\mathbb{C} S_{N}$ acting on $\mathcal{H}^{(k)}$ using $\operatorname{Ad}(\sigma)$ for $k \leq K$. This can be related to the action of elements of $P_{2 k}(N)$ acting on $\mathcal{H}^{(k)}$ for $k \leq K$. We will briefly mention some analogies between the present construction and the phenomenon of topological degeneracy which is widely studied in condensed matter physics.

The ground state degeneracy of $H_{K}$ can be resolved by adding a term $H_{\text {res }}$, made from the central algebraic charges discussed in section 5.3. This breaks the degeneracy of the invariant ground states as illustrated in the spectrum on the right of figure 5.1a. The representation basis $\left|Q_{\Lambda_{2}, \mu \nu}^{\Lambda_{1}}\right\rangle$ presented in section 5.2 .3 diagonalises these Hamiltonians in the invariant subspace, and the state energies depend on labels $\Lambda_{1}, \Lambda_{2}$.

Multiplicity labels $\mu, \nu$ are not distinguished by the central algebraic charges. Distinguishing multiplicity labels requires more general elements of $P_{k}(N)$. Generalising the construction of $H_{\mathrm{res}}$ naturally leads to a large class of $S_{N}$ invariant Hamiltonians related to the left action of elements of $P_{k}(N)$, which can be used to break the degeneracy associated with multiplicity labels. Hamiltonians of this type can have non-trivial spectra, in which invariant states are distributed across the energy spectrum, with no discernible pattern of difference compared to non-invariant states, as illustrated in figure 5.1b,

The 11-parameter Hamiltonians in section 5.4 typically have such non-trivial spectra. Given the non-trivial index contractions in 5.137),

$$
\begin{equation*}
\sum_{i, j, k, l} Q_{i j k l}^{\Lambda, \alpha \beta} X_{i j} X_{k l} \longrightarrow\left(a^{\dagger}\right)_{j}^{i} a_{l}^{k} Q_{i j k l}^{\Lambda, \alpha \beta}, \tag{5.166}
\end{equation*}
$$

these Hamiltonians are not of the kind involving only the left action of $P_{k}(N)$. Similarly, $H_{K}$ is not of this kind. This implies that a more general construction of $S_{N}$ invariant Hamiltonians exists. We give a description of this more general construction, which involves elements of $P_{2 k}(N)$. We end the section with a lattice interpretation of the matrix oscillators.


Figure 5.1: The figure illustrates the type of spectra that can be engineered using the algebraic Hamiltonians discussed in this section. Blue (light) lines correspond to states that are invariant under the adjoint action of $S_{N}$. Black (dark) lines are non-invariant states.

### 5.5.1 Partition algebra elements as quantum mechanical operators

We now translate much of the discussion in section 5.3 into the language of quantum mechanical operators on $\mathcal{H}$. Finding representation bases corresponds to the diagonalisation of commuting operators on $\mathcal{H}$. Notably, elements of $S P_{k}(N)$ naturally correspond to operators for fixed $k$, or maps $\mathcal{H}^{(k)} \rightarrow \mathcal{H}^{(k)}$. However, it will be useful to have expressions for these fixed $k$ operators in terms of oscillators, which act on the entire Hilbert space $\mathcal{H}$.

These two kinds of operators are related by projectors $\mathcal{P}_{k}: \mathcal{H} \rightarrow \mathcal{H}^{(k)}$ to fixed $k$ subspaces. We use this in the construction of Hamiltonians in the remainder of section 5.5.

For a general state $|T\rangle \in \mathcal{H}^{(k)}$ (see 5.25$)$ and element $[d] \in S P_{k}(N)$ there is a corresponding operator defined as

$$
\begin{equation*}
[d]^{L}|T\rangle=|[d] T\rangle=|d T\rangle \tag{5.167}
\end{equation*}
$$

where the superscript $L$ stands for left action, and

$$
\begin{equation*}
(d T)_{i_{1^{\prime} \ldots i_{k^{\prime}}}^{i_{1} \ldots i_{k}}}^{i_{1}}=\sum_{j_{1}, \ldots, j_{k}} d_{j_{1} \ldots j_{k}}^{i_{1} \ldots i_{k}} T_{i_{1^{\prime} \ldots i_{k^{\prime}}}^{j_{1} \ldots j_{k}}} \tag{5.168}
\end{equation*}
$$

The second equality in 5.167 follows since

$$
\begin{align*}
|[d] T\rangle=\operatorname{Tr}_{V_{N}^{\otimes k}}\left([d] T\left(a^{\dagger}\right)^{\otimes k}\right)|0\rangle & =\frac{1}{k!} \sum_{\gamma \in S_{k}} \operatorname{Tr}_{V_{N}^{\otimes k}}\left(\mathcal{L}_{\gamma} d \mathcal{L}_{\gamma^{-1}} T\left(a^{\dagger}\right)^{\otimes k}\right)|0\rangle  \tag{5.169}\\
& =\operatorname{Tr}_{V_{N}^{\otimes k}}\left(d T\left(a^{\dagger}\right)^{\otimes k}\right)|0\rangle=|d T\rangle,
\end{align*}
$$

where $\mathcal{L}(\sigma)$ is defined in equation (5.37). We have used $\mathcal{L}_{\gamma} T=T \mathcal{L}_{\gamma}$ together with $\mathcal{L}_{\gamma}\left(a^{\dagger}\right)^{\otimes k}=\left(a^{\dagger}\right)^{\otimes k} \mathcal{L}_{\gamma}$ to go to the second line. We may also define operators corresponding to right action,

$$
\begin{equation*}
[d]^{R}|T\rangle=|T d\rangle \tag{5.170}
\end{equation*}
$$

We extend $[d]^{L}$ to an operator on $\mathcal{H}$, expressible in terms of oscillators and projectors $\mathcal{P}_{k}: \mathcal{H} \rightarrow \mathcal{H}^{(k)}$ as

$$
\begin{equation*}
[d]^{L}=\frac{1}{k!} \mathcal{P}_{k} \operatorname{Tr}_{V_{N}^{\otimes k}}\left(\left(a^{\dagger}\right)^{\otimes k} d a^{\otimes k}\right) \mathcal{P}_{k} \tag{5.171}
\end{equation*}
$$

Similarly, we can extend $[d]^{R}$ to an operator on $\mathcal{H}$,

$$
\begin{equation*}
[d]^{R}=\frac{1}{k!} \mathcal{P}_{k} \operatorname{Tr}_{V_{N}^{\otimes k}}\left(d\left(a^{\dagger}\right)^{\otimes k} a^{\otimes k}\right) \mathcal{P}_{k} \tag{5.172}
\end{equation*}
$$

In what follows we will prove results explicitly for the left action. For the sake of brevity we omit the analogous proofs for the right action.

The definition of $\mathcal{P}_{k}$ in the oscillator basis is

$$
\begin{equation*}
\mathcal{P}_{k^{\prime}}\left(a^{\dagger}\right)_{j_{1}}^{i_{1}} \ldots\left(a^{\dagger}\right)_{j_{k}}^{i_{k}}|0\rangle=\delta_{k k^{\prime}}\left(a^{\dagger}\right)_{j_{1}}^{i_{1}} \ldots\left(a^{\dagger}\right)_{j_{k}}^{i_{k}}|0\rangle \tag{5.173}
\end{equation*}
$$

We now prove

$$
\begin{equation*}
\frac{1}{k!} \mathcal{P}_{k} \operatorname{Tr}_{V_{N}^{\otimes k}}\left(\left(a^{\dagger}\right)^{\otimes k} d a^{\otimes k}\right) \mathcal{P}_{k}|T\rangle=\left|d T^{(k)}\right\rangle \tag{5.174}
\end{equation*}
$$

where $|T\rangle=\sum_{k=0}^{\infty}\left|T^{(k)}\right\rangle$ and $\left|T^{(k)}\right\rangle \in \mathcal{H}^{(k)}$. The projector immediately gives $\mathcal{P}_{k}|T\rangle=$
$\left|T^{(k)}\right\rangle$. It remains to prove

$$
\begin{equation*}
\frac{1}{k!} \mathcal{P}_{k} \operatorname{Tr}_{V_{N}^{\otimes k}}\left(\left(a^{\dagger}\right)^{\otimes k} d a^{\otimes k}\right)\left|T^{(k)}\right\rangle=\left|d T^{(k)}\right\rangle \tag{5.175}
\end{equation*}
$$

We prove this diagrammatically, using the state definition in terms of diagrams 5.31


In the second equality we have moved all annihilation operators past the creation operators, giving a sum over contractions. The sum over $\gamma \in S_{k}$ encodes the contractions and in the second line we have straightened the diagram. The last identification follows since $\mathcal{L}_{\gamma^{-1}} T^{(k)} \mathcal{L}_{\gamma}=T^{(k)}$. Because $\left|d T^{(k)}\right\rangle \in \mathcal{H}^{(k)}$ we have $\mathcal{P}_{k}\left|d T^{(k)}\right\rangle=\left|d T^{(k)}\right\rangle$, which establishes the equality in (5.174).

As we now show, the Hermitian conjugate of the operator $\left[d_{\pi}\right]^{L}$ is $\left[d_{\pi}^{T}\right]^{L}$, where $d_{\pi}^{T}$ is the element obtained by flipping the diagram $d_{\pi}$ horizontally. This follows from the inner product

$$
\begin{equation*}
\left\langle T^{\prime} \mid T\right\rangle=\sum_{\gamma \in S_{k}} \operatorname{Tr}_{V_{N}^{\otimes k}}\left(\left(T^{\prime}\right)^{T} \gamma T \gamma^{-1}\right) \tag{5.177}
\end{equation*}
$$

defined in (5.54) and

$$
\begin{align*}
\left\langle T^{\prime} \mid d_{\pi} T\right\rangle & =\sum_{\gamma \in S_{k}} \operatorname{Tr}_{V_{N}^{\otimes k}}\left(\left(T^{\prime}\right)^{T} \gamma d_{\pi} T \gamma^{-1}\right)=k!\operatorname{Tr}_{V_{N}^{\otimes k}}\left(\left(T^{\prime}\right)^{T} d_{\pi} T\right) \\
& =\sum_{\gamma \in S_{k}} \operatorname{Tr}_{V_{N}^{\otimes k}}\left(\left(d_{\pi}^{T} T^{\prime}\right)^{T} \gamma^{-1} T \gamma\right)=\left\langle d_{\pi}^{T} T^{\prime} \mid T\right\rangle \tag{5.178}
\end{align*}
$$

As operators on $\mathcal{H}$ the central elements introduced in section 5.3.1, $T_{2} \in \mathcal{Z}\left(\mathbb{C} S_{N}\right), \bar{T}_{2} \in$ $\mathcal{Z}\left(P_{k}(N)\right)$ and $t_{2} \in \mathcal{Z}\left(\mathbb{C} S_{k}\right)$ can be written as oscillators. From the definition of the action
of $T_{2}$ in 5.90 we have

$$
\begin{align*}
T_{2}^{(k), L} & \equiv \frac{1}{k!} \mathcal{P}_{k} \operatorname{Tr}_{V_{N}^{\otimes k}}\left[\left(a^{\dagger}\right)^{\otimes k} \mathcal{L}\left(T_{2}\right) a^{\otimes k}\right] \mathcal{P}_{k} \\
& =\frac{1}{k!} \mathcal{P}_{k} \sum_{\substack{\sigma=(i j) \\
1 \leq i<j \leq N}} \operatorname{Tr}_{V_{N} \otimes k}\left[\left(a^{\dagger}\right)^{\otimes k} \mathcal{L}(\sigma) a^{\otimes k}\right]_{k} \\
& =\frac{1}{k!} \mathcal{P}_{k} \sum_{\substack{\sigma=(i j) \\
1 \leq i<j \leq N}} \sum_{i_{1} \ldots i_{k}}^{i_{1^{\prime} \ldots i_{k^{\prime}}}}\left(a^{\dagger}\right)_{\sigma^{-1}\left(i_{1}\right)}^{i_{i^{\prime}}} \ldots\left(a^{\dagger}\right)_{\sigma^{-1}\left(i_{k}\right)}^{i_{k^{\prime}}} a_{i_{1^{\prime}}}^{i_{1}} \ldots a_{i_{k^{\prime}}}^{i_{k}} \mathcal{P}_{k} \tag{5.179}
\end{align*}
$$

Similarly, the fixed $k$ operators corresponding to $\bar{T}_{2}$ are

$$
\begin{align*}
\bar{T}_{2}^{(k), L} & =\frac{1}{k!} \mathcal{P}_{k} \operatorname{Tr}_{V_{N}^{\otimes k}}^{\otimes}\left[\left(a^{\dagger}\right)^{\otimes k} \bar{T}_{2} a^{\otimes k}\right] \mathcal{P}_{k} \\
& =\frac{1}{k!} \mathcal{P}_{k} \sum_{\substack{i_{1} \ldots i_{k} \\
j_{1} \ldots j_{k} \\
j_{1^{\prime} \ldots j_{k^{\prime}}}}}\left(a^{\dagger}\right)_{i_{1^{\prime}}}^{i_{1}} \ldots\left(a^{\dagger}\right)_{i_{k^{\prime}}}^{i_{k}}\left(\bar{T}_{2}\right)_{j_{1} \ldots j_{k}}^{i_{1^{\prime}} \ldots i_{k^{\prime}}} a_{i_{1}}^{j_{1}} \ldots a_{i_{k}}^{j_{k}} \mathcal{P}_{k} \tag{5.180}
\end{align*}
$$

where $\bar{T}_{2}$ can be expanded in in the diagram basis as in 5.91. Finally, the fixed $k$ operators corresponding to $t_{2}$ are

$$
\begin{align*}
t_{2}^{(k), L} & =\frac{1}{k!} \mathcal{P}_{k} \sum_{\substack{\tau=(i j) \\
1 \leq i<j \leq k}} \operatorname{Tr}_{V_{N}^{\otimes k}}^{\otimes}\left[\left(a^{\dagger}\right)^{\otimes k} \mathcal{L}_{\tau^{-1}} a^{\otimes k}\right] \mathcal{P}_{k} \\
& =\frac{1}{k!} \mathcal{P}_{k} \sum_{\substack{\tau=(i j) \\
1 \leq i<j \leq k}} \sum_{\substack{i_{1} \ldots i_{k} \\
i_{1^{\prime}} \ldots i_{k^{\prime}}}}\left(a^{\dagger}\right)_{i_{\tau(1)}}^{i_{i^{\prime}}} \ldots\left(a^{\dagger}\right)_{i_{\tau(k)}}^{i_{k^{\prime}}} a_{i_{1^{\prime}}}^{i_{1}} \ldots a_{i_{k^{\prime}}}^{i_{k}} \mathcal{P}_{k} . \tag{5.181}
\end{align*}
$$

All three of these operators are Hermitian, because $\left(T_{2}\right)^{T}=T_{2}$ and $\left(t_{2}\right)^{T}=t_{2}$, and consequently their eigenvectors with distinct eigenvalues are orthogonal. They are difficult to diagonalise over the entirety of $\mathcal{H}^{(k)}$, since the dimension grows as $N^{2 k}$ for $N \gg k$. But the diagonalisation over $\mathcal{H}_{\mathrm{inv}}^{(k)}$ is feasible since the dimension is bounded by $B(2 k)$, which does not scale with $N$. Further simplification arises when acting on states $|d\rangle \in \mathcal{H}_{\text {inv }}^{(k)}$, since the action can be formulated as multiplication in $S P_{k}(N)$, thus bypassing the computation of large index contractions. That is, for $|d\rangle \in \mathcal{H}_{\text {inv }}^{(k)}$

$$
\begin{equation*}
\bar{T}_{2}{ }^{(k), L}|d\rangle=\left|\bar{T}_{2} d\right\rangle \tag{5.182}
\end{equation*}
$$

where the product $\bar{T}_{2}{ }^{(k)} d$ can be taken in $P_{k}(N)$. It follows that,

$$
\begin{equation*}
\bar{T}_{2}^{(k), L}\left|Q_{\Lambda_{2}, \mu \nu}^{\Lambda_{1}}\right\rangle=\left|\bar{T}_{2} Q_{\Lambda_{2}, \mu \nu}^{\Lambda_{1}}\right\rangle=\hat{\chi}^{\Lambda_{1}}\left(T_{2}\right)\left|Q_{\Lambda_{2}, \mu \nu}^{\Lambda_{1}}\right\rangle \tag{5.183}
\end{equation*}
$$

and similarly for $t_{2}^{(k), L}$.
The free Hamiltonian $H_{0}$ in equation $(5.8)$ is just the number operator. The above operators conserve the number of particles. Consequently,

$$
\begin{equation*}
\left[H_{0}, T_{2}^{(k), L}\right]=\left[H_{0}, \bar{T}_{2}^{(k), L}\right]=\left[H_{0}, t_{2}^{(k), L}\right]=0 \tag{5.184}
\end{equation*}
$$

and the corresponding charges are conserved.

### 5.5.2 Decoupling invariant sectors and invariant ground states

We now present a Hermitian operator with algebraic origin that can be used to control the energies of states invariant under the adjoint action of $S_{N}$ on $\mathcal{H}^{(k)}$. We use the operator to construct a Hamiltonian with a large number of invariant ground states.

The adjoint action of $\sigma \in S_{N}$ on $\mathcal{H}^{(k)}$ is defined in equation (5.38) as

We may write $\operatorname{Ad}(\sigma)$ in terms of oscillators and projectors $\mathcal{P}_{k}: \mathcal{H} \rightarrow \mathcal{H}^{(k)}$ defined in equation (5.173). For $|T\rangle \in \mathcal{H}^{(k)}$,

$$
\begin{equation*}
\operatorname{Ad}(\sigma)|T\rangle=\frac{1}{k!} \mathcal{P}_{k} \operatorname{Tr}_{V_{N}^{\otimes k}}\left(\mathcal{L}\left(\sigma^{-1}\right)\left(a^{\dagger}\right)^{\otimes k} \mathcal{L}(\sigma) a^{\otimes k}\right) \mathcal{P}_{k}|T\rangle \tag{5.186}
\end{equation*}
$$

We note that the the ordering of $a^{\dagger}$ relative to $a$ is understood to be as shown in the above equation. To understand the equality in (5.186), we evaluate

$$
\begin{equation*}
\operatorname{Tr}_{V_{N}^{\otimes k}}\left(\mathcal{L}\left(\sigma^{-1}\right)\left(a^{\dagger}\right)^{\otimes k} \mathcal{L}(\sigma) a^{\otimes k}\right)|T\rangle \tag{5.187}
\end{equation*}
$$

where we take $|T\rangle \in \mathcal{H}^{(k)}$ (there is no loss of generality since $\mathcal{P}_{k}$ projects to $\mathcal{H}^{(k)}$ ). Dia-
grammatically we have


The first equality follows by encoding the contraction of annihilation/creation operators in a sum over $\gamma \in S_{k}$, and the last equality follows by $\mathcal{L}_{\gamma} T=T \mathcal{L}_{\gamma}$. This establishes the equality (5.186).

We are now in a position to define the Hermitian operator of interest. Let $C_{3}^{(k)}$ be the operator defined to act on $|T\rangle \in \mathcal{H}^{(k)}$ as

$$
\begin{equation*}
C_{3}^{(k)}|T\rangle=\frac{1}{3} \sum_{\substack{\sigma=(i j k) \\ 1 \leq i \neq j \neq k \leq N}} \operatorname{Ad}(\sigma)|T\rangle=\frac{1}{3} \sum_{\substack{\sigma=(i j k) \\ 1 \leq i \neq j \neq k \leq N}} \frac{\square}{\frac{T^{\prime}(\sigma)}{\square}}|0\rangle, \tag{5.189}
\end{equation*}
$$

where the sum is over all 3 -cycles. It commutes with the adjoint action of $S_{N}$,

$$
\begin{equation*}
\operatorname{Ad}(\gamma) C_{3}^{(k)}=C_{3}^{(k)} \operatorname{Ad}(\gamma), \quad \forall \gamma \in S_{N}, \tag{5.190}
\end{equation*}
$$

because $C_{3}^{(k)}$ is a sum over an entire conjugacy class. We now use a sequence of diagrammatic manipulations to show that the action of $C_{3}^{(k)}$ can equivalently be expressed using an element $\bar{T}_{3}^{(2 k)} \in P_{2 k}(N)$. A useful way to rewrite the diagram in (5.189) is
where we have gone from a trace in $V_{N}^{\otimes k}$ to a trace in $V_{N}^{\otimes 2 k}$. By arguments analogous to those in section 5.3.1, the action of

$$
\begin{equation*}
\frac{1}{3} \sum_{\substack{\sigma=(i j k) \\ 1 \leq i \neq j \neq k \leq N}} \mathcal{L}(\sigma) \tag{5.192}
\end{equation*}
$$

on $V_{N}^{\otimes 2 k}$ is related to an element in $P_{2 k}(N)$, that we call $\bar{T}_{3}^{(2 k)}$. Diagrammatically, this is understood from the following sequence of identifications,


That is, we have

$$
\begin{equation*}
\left.C_{3}^{(k)}|T\rangle=\operatorname{Tr}_{V_{N}^{\otimes 2 k}}\left(c(T \otimes 1) \bar{T}_{3}^{(2 k)}\left(\left(a^{\dagger}\right)^{\otimes k} \otimes 1\right)\right)\right)|0\rangle \tag{5.194}
\end{equation*}
$$

where $c \in P_{2 k}(N)$ is the bottom box in the diagram on the RHS of 5.193 and

$$
\begin{equation*}
(c)_{j_{1} \ldots j_{2 k}}^{i_{1} \ldots i_{2 k}}=\delta^{i_{1} i_{k+1}} \ldots \delta^{i_{k} i_{2 k}} \delta_{j_{1} j_{k+1}} \ldots \delta_{j_{k} j_{2 k}} \tag{5.195}
\end{equation*}
$$

The explicit formula for $\bar{T}_{3}^{(2 k)}$ could be derived using steps similar to the derivation of the relation between $\bar{T}_{2}^{(k)}$ and $T_{2}^{(k)}$ in section 5.3.1. Relating $C_{3}^{(k)}$ to an element $\bar{T}_{3}^{(2 k)}$ using $P_{2 k}(N)$ allows for two kinds of large $N$ simplification. Firstly, in place of $N!/(N-3)!3$ ! terms in $C_{3}^{(k)}$ we have no more than $B(2 k)$ terms in $\bar{T}_{3}^{(2 k)}$, where $B(2 k)$ are the Bell numbers. Additionally, index contractions ranging over $N$ can be replaced by multiplication in the partition algebra $P_{2 k}(N)$ when $|T\rangle \in \mathcal{H}_{\text {inv }}$, the complexity of this multiplication scales with $k$.

We now move on to discuss the spectrum of $C_{3}^{(k)}$. Since $\mathcal{H}^{(k)}$ is reducible with respect to the adjoint action of $S_{N}$, it decomposes into irreducible representations of $S_{N}$, labeled by Young diagrams $Y$ with $N$ boxes. By Schur's lemma the action of $C_{3}^{(k)}$ on each irreducible subspace of this decomposition is proportional to the identity. The constant of proportionality is the normalised character of $C_{3}^{(k)}$ in the irreducible representation $Y$,

$$
\begin{equation*}
\hat{\chi}_{Y}\left(C_{3}^{(k)}\right)=\frac{\chi_{Y}\left(C_{3}^{(k)}\right)}{\operatorname{Dim} V_{Y}^{S_{N}}} \tag{5.196}
\end{equation*}
$$

Normalised characters of $C_{3}^{(k)}$ are known [122, Theorem 4] to equal

$$
\begin{equation*}
\hat{\chi}_{Y}\left(C_{3}^{(k)}\right)=\sum_{(p, q) \in Y}(q-p)^{2}-\frac{N(N-1)}{2}, \tag{5.197}
\end{equation*}
$$

where the sum is over all cells in the Young diagram $Y$, using coordinates $(p, q)$ for rows and columns respectively. For example, the largest eigenvalue of $C_{3}^{(k)}$ corresponds to the trivial representation (Young diagram with all $N$ boxes in the first row) where

$$
\begin{equation*}
\sum_{(p, q) \in Y}(q-p)^{2}=0^{2}+1^{2}+2^{2}+\cdots+(N-1)^{2}=\frac{N(N-1)(2 N-1)}{6} \tag{5.198}
\end{equation*}
$$

which gives the eigenvalue $\frac{N(N-1)(N-2)}{3}$ in 5.197 . In what follows it will be useful to shift the eigenvalue of the trivial representation to zero by considering the operator

$$
\begin{equation*}
\hat{C}_{3}^{(k)}=\frac{N(N-1)(N-2)}{3}-C_{3}^{(k)} . \tag{5.199}
\end{equation*}
$$

In terms of oscillators and projectors, $\hat{C}_{3}^{(k)}$ is written as

$$
\begin{equation*}
\hat{C}_{3}^{(k)}=\frac{1}{k!} \mathcal{P}_{k}\left[\frac{N(N-1)(N-2)}{3}-\sum_{\substack{\sigma=(i j k) \\ 1 \leq i \neq j \neq k \leq N}} \operatorname{Tr}_{V_{N}^{\otimes k}( }\left(\mathcal{L}\left(\sigma^{-1}\right)\left(a^{\dagger}\right)^{\otimes k} \mathcal{L}(\sigma) a^{\otimes k}\right)\right] \mathcal{P}_{k} . \tag{5.200}
\end{equation*}
$$

We can use $\hat{C}_{3}^{(k)}$ to construct Hamiltonians with interesting spectra. Consider the family of Hamiltonians (depending on $K$ )

$$
\begin{equation*}
H_{K}=\sum_{k=0}^{K} \hat{C}_{3}^{(k)} H_{0}+\sum_{k=K+1}^{\infty} \mathcal{P}_{k} H_{0} \tag{5.201}
\end{equation*}
$$

where $H_{0}$ is the free Hamiltonian (number operator) defined in 5.8. In this model, all invariant states of degree $k \leq K$ have zero energy, while non-invariant states have energies that scale with $N$. For example, degree $k \leq K$ states in the representation [ $N-1,1$ ] of $S_{N}$ have energies $k N(N-2)$. More generally, degree $k \leq K$ states in the representation $[N-a, a]$ for $1 \leq a<\lfloor N / 2\rfloor$ have energy $k(N-a+1)(N-2) a$. States of degree $k>K$ have energy $k$. The spectrum of $H_{K}$ is illustrated on the left hand side of figure 5.1a. Taking $N \gg K$, there is a $K$-dependent degeneracy of invariant ground states and a gap of order $K$. In this scenario, the subspace of ground states has dimension

$$
\begin{equation*}
\sum_{k=0}^{K} \operatorname{Dim} \mathcal{H}_{\mathrm{inv}}^{(k)}=1+\sum_{k=1}^{K} \operatorname{Dim} S P_{k}(N) \tag{5.202}
\end{equation*}
$$

where $\mathcal{H}_{\text {inv }}^{(k)}$ is the degree $k$ subspace of $\mathcal{H}_{\text {inv }}$ (see equation B. 11 in 58 for explicit formulas computing $\operatorname{Dim} \mathcal{H}_{\text {inv }}^{(k)}$ ). By taking $N \gg K \gg 1$, we can have a large degeneracy of ground states alongside the interesting correlations between the degeneracy of ground states and the energy gap. A large ground state degeneracy associated with elements of a diagrammatic algebra, in this case the partition algebras $S P_{k}(N)$ for $k \leq K$, is reminiscent of topological degeneracy and its links to anyons [123, 124]. We leave a more detailed investigation of the analogies between the present algebraic constructions and topological degeneracy for the future.

### 5.5.3 Resolving the invariant spectrum

In the previous section we discussed a Hamiltonian (5.201) with degenerate ground state. We will now use the commuting algebraic charges $\bar{T}_{2}^{(k)}, t_{2}^{(k)} \in P_{k}(N)$, constructed in section 5.3. to resolve this degeneracy. Note that the charges commute with $\operatorname{Ad}(\sigma)$ and in particular they commute with $\hat{C}_{3}^{(k)}$. We prove this in the next subsection, where we consider more general operators coming from elements of $P_{k}(N)$. Note that because $\bar{T}_{2}^{(k)}$ and $t_{2}^{(k)}$ are central elements of $P_{k}(N)$, and the representation basis states $\left|Q_{\Lambda_{2}, \mu \nu}^{\Lambda_{1}}\right\rangle$ correspond to elements in $P_{k}(N)$, the left and right actions of the charges are equivalent on these basis states.

The algebraic charges can be written in terms of oscillators and projectors as in 5.180 and 5.181. Importantly, the representation basis states $\left|Q_{\Lambda_{2}, \mu \nu}^{\Lambda_{1}}\right\rangle$ are eigenstates of $\bar{T}_{2}^{(k), L}, t_{2}^{(k), L}$. The eigenvalues are normalised characters of the representations $\Lambda_{1}$ of $S_{N}$ and $\Lambda_{2}$ of $S_{k}$ respectively (see 5.183 ). That is

$$
\begin{align*}
& \bar{T}_{2}^{(k), L}\left|Q_{\Lambda_{2}, \mu \nu}^{\Lambda_{1}}\right\rangle=\left|\bar{T}_{2}^{(k)} Q_{\Lambda_{2}, \mu \nu}^{\Lambda_{1}}\right\rangle=\bar{T}_{2}^{(k), R}\left|Q_{\Lambda_{2}, \mu \nu}^{\Lambda_{1}}\right\rangle=\hat{\chi}^{\Lambda_{1}}\left(T_{2}\right)\left|Q_{\Lambda_{2}, \mu \nu}^{\Lambda_{1}}\right\rangle  \tag{5.203}\\
& t_{2}^{(k), L}\left|Q_{\Lambda_{2}, \mu \nu}^{\Lambda_{1}}\right\rangle=\left|t_{2}^{(k)} Q_{\Lambda_{2}, \mu \nu}^{\Lambda_{1}}\right\rangle=t_{2}^{(k), R}\left|Q_{\Lambda_{2}, \mu \nu}^{\Lambda_{1}}\right\rangle=\hat{\chi}^{\Lambda_{2}}\left(t_{2}\right)\left|Q_{\Lambda_{2}, \mu \nu}^{\Lambda_{1}}\right\rangle \tag{5.204}
\end{align*}
$$

where the normalised characters $\hat{\chi}$ are defined in (5.97). Note that the eigenvalues of the operator $t_{2}^{(k), L}$ range between $\pm \frac{k(k-1)}{2}$, and those of $\bar{T}_{2}^{(k), L}$ between $\pm \frac{N(N-1)}{2}$, including an infinite number of such operators in a Hamiltonian may result in a spectrum that is not bounded from below. By adding these algebraic charges to the Hamiltonian (5.201 the energy of the states $\left|Q_{\Lambda_{2}, \mu \nu}^{\Lambda_{1}}\right\rangle$ labeled by distinct pairs $\Lambda_{1}, \Lambda_{2}$ will split. The multiplicity labels $\mu, \nu$ are not distinguished by these central algebraic charges. Hamiltonians that resolve more detailed information such as multiplicity labels are discussed in the next subsection.

For concreteness consider the spectrum of the Hamiltonian

$$
\begin{align*}
H_{K}^{\prime} & =H_{K}+H_{\mathrm{res}} \\
& =H_{K}-\frac{2}{N(N-1)} \sum_{k=1}^{K} \bar{T}_{2}^{(k), L} \\
& =\sum_{k=0}^{K} \hat{C}_{3}^{(k)} H_{0}+\sum_{k=K+1}^{\infty} \mathcal{P}_{k} H_{0}-\frac{2}{N(N-1)} \sum_{k=1}^{K} \bar{T}_{2}^{(k), L} . \tag{5.205}
\end{align*}
$$

The ground state degeneracy is reduced compared to $H_{K}$. The lowest energy states are degree $k \leq K$ states $\left|Q_{\Lambda_{2}, \mu \nu}^{[N]}\right\rangle$ with energy -1 . The highest energy state with degree $k \leq K$ is $\left|Q_{\left[1^{K}\right]}^{\left[N-K, 1^{K}\right]}\right\rangle$, it has degree $K$ and energy $-\frac{(N-2 K-1)}{(N-1)}$. The gap of order $K$ remains, as illustrated on the right of figure 5.1a. The label $\Lambda_{2}$ can be resolved by including $t_{2}^{(k), L}$ in the Hamiltonian.

To fully resolve the labels $\Lambda_{1}, \Lambda_{2}$ for general $k$ and $N$, new charges are necessary. Detailed discussions of the problem of using such charges in the centre of the symmetric group algebra $\mathbb{C}\left[S_{n}\right]$, with motivations coming from a model for information loss in AdS/CFT [125], are given in [112, 126]. It can be proven that $\left\{T_{2}, T_{3}, \cdots, T_{n}\right\}$ provide an adequate set of charges and these also provide a multiplicative generating set for the centre of the group algebra. Typically, a smaller set $\left\{T_{2}, T_{3}, \cdots, T_{k_{*}(n)}\right\}$ suffices. For example $k_{*}(5)=2, k_{*}(14)=3, k_{*}(80)=6$. In the present discussion these results can be applied by choosing $n=k$ and $n=N$ respectively.

### 5.5.4 Precision resolution of the invariant spectrum

In the previous section we presented Hamiltonians involving commuting algebraic charges, constructed from central elements in $P_{k}(N)$, that resolve the representation labels $\Lambda_{1}, \Lambda_{2}$ of representation basis elements $\left|Q_{\Lambda_{2}, \mu \nu}^{\Lambda_{1}}\right\rangle$. As discussed in section 5.3.2, and illustrated in an explicit example in section 5.3.3, more general elements of $S P_{k}(N)$ are necessary to resolve the multiplicity labels $\mu, \nu$. We will use this observation to construct $S_{N}$ invariant Hamiltonians, involving operators $[d]^{L}$ and $[d]^{R}$ constructed from non-central elements $[d] \in S P_{k}(N)$, with non-degenerate eigenvalues.

Since we want to construct Hamiltonians $H$ satisfying $[\operatorname{Ad}(\sigma), H]=0$, built from operators $[d]^{L},[d]^{R}$, we will now prove that $\left[\operatorname{Ad}(\sigma),[d]^{L}\right]=\left[\operatorname{Ad}(\sigma),[d]^{R}\right]=0$. To show that
$[d]^{L} \operatorname{Ad}(\sigma)=\operatorname{Ad}(\sigma)[d]^{L}$ we combine equation (5.167) with equation 5.185)

$$
\begin{align*}
\operatorname{Ad}(\sigma)[d]^{L}|T\rangle & =\operatorname{Tr}_{V_{N}^{\otimes k}}\left(\mathcal{L}(\sigma) d T \mathcal{L}\left(\sigma^{-1}\right)\left(a^{\dagger}\right)^{\otimes k}\right)|0\rangle \\
& =\operatorname{Tr}_{V_{N}^{\otimes k}}\left(d \mathcal{L}(\sigma) T \mathcal{L}\left(\sigma^{-1}\right)\left(a^{\dagger}\right)^{\otimes k}\right)|0\rangle  \tag{5.206}\\
& =[d]^{L} \operatorname{Ad}(\sigma)|T\rangle
\end{align*}
$$

where the second line follows since $\mathcal{L}(\sigma) d=d \mathcal{L}(\sigma)$ as elements of $\operatorname{End}\left(V_{N}^{\otimes k}\right)$ (linear maps $\left.V_{N}^{\otimes k} \rightarrow V_{N}^{\otimes k}\right)$. The argument is identical for $[d]^{R} \operatorname{Ad}(\sigma)=\operatorname{Ad}(\sigma)[d]^{R}$.

To construct Hamiltonians $H$, using the above operators, we need to ensure that any operator we include in $H$ is Hermitian. The operators $[d]^{L},[d]^{R}$ are not Hermitian in general, unless $\left[d^{T}\right]=[d]$. Taking this into account, we can parametrise a large family of $S_{N}$ invariant Hamiltonians using the diagram basis for $P_{k}(N)$. We write

$$
\begin{equation*}
H=\frac{1}{2} \sum_{k=1}^{\infty} \sum_{\left[d_{\pi}\right]}\left(L_{k, \pi}\left[d_{\pi}\right]^{L}+L_{k, \pi}^{*}\left[d_{\pi}^{T}\right]^{L}+R_{k, \pi}\left[d_{\pi}\right]^{R}+R_{k, \pi}^{*}\left[d_{\pi}^{T}\right]^{R}\right) \tag{5.207}
\end{equation*}
$$

where the sum over $\left[d_{\pi}\right]$ runs over a basis for $S P_{k}(N)$ and $L_{k, \pi}, R_{k, \pi}$ are complex parameters with the constraint $L_{k, \pi}^{*}=L_{k, \pi^{\prime}}$ and $R_{k, \pi}^{*}=R_{k, \pi^{\prime}}$ if $d_{\pi}^{T}=d_{\pi^{\prime}}$. The equivalent expression for $H$ in terms of oscillators and projectors is

$$
\begin{align*}
H= & \frac{1}{2} \sum_{k=1}^{\infty} \sum_{\left[d_{\pi}\right]} \mathcal{P}_{k} \operatorname{Tr}_{V_{N}^{\otimes k}}\left(\left(a^{\dagger}\right)^{\otimes k} \frac{L_{k, \pi} d_{\pi}+L_{k, \pi}^{*} d_{\pi}^{T}}{k!} a^{\otimes k}\right) \mathcal{P}_{k} \\
& +\frac{1}{2} \sum_{k=1}^{\infty} \sum_{\left[d_{\pi}\right]} \mathcal{P}_{k} \operatorname{Tr}_{V_{N}^{\otimes k}}\left(\frac{R_{k, \pi} d_{\pi}+R_{k, \pi}^{*} d_{\pi}^{T}}{k!}\left(a^{\dagger}\right)^{\otimes k} a^{\otimes k}\right) \mathcal{P}_{k} \tag{5.208}
\end{align*}
$$

Progressively turning on parameters in equation 5.207 will tend to break degeneracy in the spectrum. Eventually, the spectrum may take the form in figure 5.1 b where invariant and non-invariant states are mixed, and most of the degeneracy is broken.

### 5.5.5 General invariant Hamiltonians from partition algebras

The Hamiltonian $H$ in (5.207) is not the most general Hamiltonian satisfying $[H, \operatorname{Ad}(\sigma)]=$ 0. For example, it does not include the Hamiltonian (5.147) constructed in section 5.4 nor $H_{K}$ in 5.201. As we noticed in 5.193, $C_{3}^{(k)}$ is related to an element in $P_{2 k}(N)$. We now generalise this observation to give a construction of general $S_{N}$ invariant operators from elements in $P_{2 k}(N)$.

General degree preserving operators that commute with $\operatorname{Ad}(\sigma)$ can be constructed from
elements $d \in P_{2 k}(N)$ as

$$
\begin{equation*}
\frac{1}{k!} \mathcal{P}_{k} \operatorname{Tr}_{V_{N}^{\otimes 2 k}}\left(d\left(a^{\dagger}\right)^{\otimes k} \otimes a^{\otimes k}\right) \mathcal{P}_{k} \leftrightarrow \frac{1}{k!} \mathcal{P}_{k} \frac{\square}{\left(a^{\dagger}\right)^{\otimes k} \sqrt{a^{\otimes k}}} \mathcal{P}_{k} \tag{5.209}
\end{equation*}
$$

The action of these operators on $|T\rangle \in \mathcal{H}^{(k)}$ is


Commutativity with $\operatorname{Ad}(\sigma)$ follows from the following diagrammatic manipulations

where the first equality uses equation 5.210 . The second line introduces an identity operator of the form $\mathcal{L}\left(\sigma^{-1}\right) \mathcal{L}(\sigma)$ acting on the left-hand vector space $V_{N}^{\otimes k}$. The third equality follows from $\mathcal{L}\left(\sigma^{-1}\right) d=d \mathcal{L}\left(\sigma^{-1}\right)$ and the cyclicity of the trace. The last line removes the identity operator $\mathcal{L}(\sigma) \mathcal{L}\left(\sigma^{-1}\right)$ acting on the right-hand vector space $V_{N}^{\otimes k}$.

The last diagram is equal to

$$
\begin{equation*}
\operatorname{Ad}(\sigma) \frac{1}{k!} \operatorname{Tr}_{V_{N}^{\otimes 2 k}}\left(d\left(a^{\dagger}\right)^{\otimes k} \otimes a^{\otimes k}\right)|T\rangle \tag{5.212}
\end{equation*}
$$

which proves that they commute.
The construction readily generalises to operators that do not preserve the degree of states. Consider

$$
\begin{equation*}
\frac{1}{k_{1}!} \mathcal{P}_{k_{2}} \operatorname{Tr}_{V_{N}^{\otimes 2 k}}\left(d\left(a^{\dagger}\right)^{\otimes k_{2}} \otimes a^{\otimes k_{1}}\right) \mathcal{P}_{k_{1}} \tag{5.213}
\end{equation*}
$$

this gives a map $d: \mathcal{H}^{\left(k_{1}\right)} \rightarrow \mathcal{H}^{\left(k_{2}\right)}$ labeled by elements $d \in P_{k_{1}+k_{2}}(N)$. Note that these operators have an $S_{k_{2}} \times S_{k_{1}}$ symmetry, which permutes the creation operators and annihilation operators separately. Therefore, the dimension of the space of these operators is related to the counting of two-matrix permutation invariants, which was studied in section 2 of [1.

### 5.5.6 Bosons on a lattice

The Fock space of matrix oscillators can be interpreted as the Fock space of bosons on a two-dimensional lattice of size $N^{2}$. The lattice is parameterised by ordered pairs $(i, j)$ for $i, j=1, \ldots, N$ which label the site in the $i^{\text {th }}$ row, $j^{\text {th }}$ column as in figure 5.2. The


Figure 5.2: Matrix oscillators are naturally associated with a $N$-by- $N$ square lattice. The creation operator $\left(a^{\dagger}\right)_{i}^{j}$ creates a quanta of excitation at row $i$ column $j$ in the lattice.
creation operator $\left(a^{\dagger}\right)_{i}^{j}$ creates a quantum of excitation at the site $(i, j)$. In our conventions, $a_{j}^{i}$ annihilates a quantum at site $(i, j)$. Permutation invariant states naturally contain excitations spread throughout the entire lattice. For example, the state

$$
\begin{equation*}
|\emptyset\rangle=\sum_{i=1}^{N}\left(a^{\dagger}\right)_{i}^{i}|0\rangle \tag{5.214}
\end{equation*}
$$

contains an excitation of every site on the diagonal, and the state

$$
\begin{equation*}
|\bullet .\rangle-|\bullet .\rangle=\sum_{i \neq j}\left(a^{\dagger}\right)_{j}^{i}|0\rangle, \tag{5.215}
\end{equation*}
$$

contains an excitation on every off-diagonal site.
Most choices of $S_{N}$ invariant Hamiltonians constructed in equation (5.207) contain nonlocal interactions, connecting sites at opposite sides of the lattice. Note that the left acting terms in the Hamiltonian (5.207) leave the columns fixed while the right acting terms fix the rows. An example of the non-locality is seen by considering

$$
\begin{equation*}
H=P_{1} \operatorname{Tr}_{V_{N}}\left(a^{\dagger} \bullet a\right) P_{1}=P_{1} \sum_{i, j, k=1}^{N}\left(a^{\dagger}\right)_{j}^{i}(a)_{i}^{k} P_{1} \tag{5.216}
\end{equation*}
$$

This interaction moves a single excitation at site $(i, j)$ to every row in column $j$. In particular,

$$
\begin{equation*}
H\left(a^{\dagger}\right)_{1}^{1}|0\rangle=\sum_{i=1}^{N}\left(a^{\dagger}\right)_{i}^{1}|0\rangle \tag{5.217}
\end{equation*}
$$

contains the state $\left(a^{\dagger}\right)_{N}^{1}$.
We can enumerate a set of diagrams that give local $S_{N}$ invariant terms, through left and right action, as follows. First note that the identity element in $P_{k}(N)$ gives a local term. For example, in $k=2$

$$
\begin{equation*}
\operatorname{Tr}_{V_{N}^{\otimes 2}}\left(\left(a^{\dagger}\right)^{\otimes 2} \cdot!a^{\otimes 2}\right)=\sum_{i_{1}, i_{2}, j_{1}, j_{2}=1}^{N}\left(a^{\dagger}\right)_{j_{1}}^{i_{1}}\left(a^{\dagger}\right)_{j_{2}}^{i_{2}}(a)_{i_{1}}^{j_{1}}(a)_{i_{2}}^{j_{2}} . \tag{5.218}
\end{equation*}
$$

It follows that any diagram that can be constructed from the identity element by adding additional edges is local. For example

$$
\begin{equation*}
\operatorname{Tr}_{V_{N}^{\otimes 2}}\left(\left(a^{\dagger}\right)^{\otimes 2} \sqsupseteq a^{\otimes 2}\right)=\sum_{i_{1}, i_{2}, j=1}^{N}\left(a^{\dagger}\right)_{j}^{i_{1}}\left(a^{\dagger}\right)_{j}^{i_{2}}(a)_{i_{1}}^{j}(a)_{i_{2}}^{j}, \tag{5.219}
\end{equation*}
$$

which is still local.

### 5.6 AdS/CFT inspired extremal correlators in matrix quantum mechanics

Extremal correlators in $\mathcal{N}=4$ SYM form interesting sectors having non-renormalisation properties [127. They are closely connected to representation theoretic quantities such as Littlewood-Richardsson coefficients, and form a crucial set of examples for checking the AdS/CFT correspondence. In the quantum mechanical model presented in this chapter, vacuum expectation values similar to extremal correlators can be computed exactly. In this section we make use the factorisation result concerning the two-point function of permutation invariant matrix observables (4.61) proven in the previous chapter - this is used to demonstrate that a similar factorisation property holds for quantum mechanical permutation invariant states. We then compute an expression for extremal three-point correlators associated with $S_{N}$ invariant states, which are simple in the diagram basis, and obey representation theoretic selection rules.

### 5.6.1 Two-point correlators

The equation (5.48) can be interpreted as a quantum mechanical operator-state correspondence for $S_{N}$ invariant states labelled by $\left[d_{\pi}\right] \in S P_{k}(N)$,

$$
\begin{equation*}
\left|d_{\pi}\right\rangle \longleftrightarrow \mathcal{O}_{\pi}=\operatorname{Tr}_{V_{N}^{\otimes k}}\left(\left[d_{\pi}\right]\left(a^{\dagger}\right)^{\otimes k}\right) \tag{5.220}
\end{equation*}
$$

From equation (5.52) we have

$$
\begin{equation*}
\mathcal{O}_{\pi}^{\dagger}=\operatorname{Tr}_{V_{N}^{\otimes k}}\left(\left[d_{\pi}^{T}\right] a^{\otimes k}\right) \tag{5.221}
\end{equation*}
$$

where the transpose $d_{\pi}^{T}$ is the diagram obtained by reflecting $d_{\pi}$ across a horizontal line, as illustrated in (5.53). The time-dependent operators are given by

$$
\begin{equation*}
\mathcal{O}_{\pi}(t)=\mathrm{e}^{-i H_{0} t} \mathcal{O}_{\pi} \mathrm{e}^{i H_{0} t}=\mathrm{e}^{-i k t} \mathcal{O}_{\pi} \tag{5.222}
\end{equation*}
$$

where $H_{0}$ is the free Hamiltonian, defined in equation 5.12).
In section 4.3 the two-point function of permutation invariant matrix observables was shown to factorise in the large $N$ limit. Here, this result is used to show an equivalent factorisation property for the two-point function of permutation invariant quantum mechanical states. Let $\left[d_{\pi_{1}}\right] \in S P_{k_{1}}(N),\left[d_{\pi_{2}}\right] \in S P_{k_{2}}(N)$, and define the two-point correlator to be the
vacuum expectation value

$$
\begin{align*}
& \langle 0| \mathcal{O}_{\pi_{1}}^{\dagger}\left(t_{1}\right) \mathcal{O}_{\pi_{2}}\left(t_{2}\right)|0\rangle=  \tag{5.223}\\
& \quad \mathrm{e}^{i k_{1} t_{1}-i k_{2} t_{2}}\langle 0| \operatorname{Tr}_{V_{N}^{\otimes k_{1}}}\left(\left[d_{\pi_{1}}^{T}\right] a^{\otimes k_{1}}\right) \operatorname{Tr}_{V_{N}^{\otimes k_{2}}}\left(\left[d_{\pi_{2}}\right]\left(a^{\dagger}\right)^{\otimes k_{2}}\right)|0\rangle .
\end{align*}
$$

Ignoring the trivial time dependence and taking normalised operators $\left[\hat{d}_{\pi_{1}}\right],\left[\hat{d}_{\pi_{2}}\right]$, as defined in (5.55), in the large $N$ limit we have

$$
\begin{align*}
\langle 0| \operatorname{Tr}_{V_{N}^{\otimes k_{1}}}\left(\left[\hat{d}_{\pi_{1}}^{T}\right] a^{\otimes k_{1}}\right) \operatorname{Tr}_{V_{N}^{\otimes k_{2}}}\left(\left[\hat{d}_{\pi_{2}}\right]\left(a^{\dagger}\right)^{\otimes k_{2}}\right)|0\rangle & =\delta_{k_{1} k_{2}} \sum_{\gamma \in S_{k_{1}}} \operatorname{Tr}_{V_{N}^{\otimes k_{2}}}\left(\gamma^{-1} \hat{d}_{\pi_{1}}^{T} \gamma \hat{d}_{\pi_{2}}\right) \\
& = \begin{cases}1+O(1 / \sqrt{N}) & \text { if }\left[d_{\pi_{1}}\right]=\left[d_{\pi_{2}}\right], \\
0+O(1 / \sqrt{N}) & \text { otherwise } .\end{cases} \tag{5.224}
\end{align*}
$$

In the first line we have absorbed the $S_{k_{1}}$ averaging into the sum over $\gamma \in S_{k_{1}}$ arising from the Wick contractions of $a$ and $a^{\dagger}$. In the second line we have used the factorisation result (4.61).

### 5.6.2 Three-point correlators

Let $\left[d_{\pi_{1}}\right] \in S P_{k_{1}}(N),\left[d_{\pi_{2}}\right] \in S P_{k_{2}}(N),\left[d_{\pi}\right] \in S P_{k}(N)$, and define the extremal three-point correlator to be the vacuum expectation value

$$
\begin{align*}
& \langle 0| \mathcal{O}_{\pi_{1}}^{\dagger}\left(t_{1}\right) \mathcal{O}_{\pi_{2}}^{\dagger}\left(t_{2}\right) \mathcal{O}_{\pi}(t)|0\rangle= \\
& \mathrm{e}^{i k_{1} t_{1}+i k_{2} t_{2}-i k t}\langle 0| \operatorname{Tr}_{V_{N}^{\otimes k_{1}}}\left(\left[d_{\pi_{1}}^{T}\right] a^{\otimes k_{1}}\right) \operatorname{Tr}_{V_{N}^{\otimes k_{2}}}\left(\left[d_{\pi_{2}}^{T}\right] a^{\otimes k_{2}}\right) \operatorname{Tr}_{V_{N}^{\otimes k}}\left(\left[d_{\pi}\right]\left(a^{\dagger}\right)^{\otimes k}\right)|0\rangle, \tag{5.225}
\end{align*}
$$

with the constraint that $k=k_{2}+k_{1}$. As we now show, extremal correlators are simple when expressed in the diagram basis. We compute 5.225) by Wick contractions, which are encoded in a sum over $\gamma \in S_{k}$. Once again, ignoring the trivial time-dependence we have

$$
\begin{align*}
& \langle 0| \operatorname{Tr}_{V_{N}^{\otimes k_{1}}}\left[\left[d_{\pi_{1}}^{T}\right] a^{\otimes k_{1}}\right) \operatorname{Tr}_{V_{N}^{\otimes k_{2}}}\left(\left[d_{\pi_{2}}^{T}\right] a^{\otimes k_{2}}\right) \operatorname{Tr}_{V_{N}^{\otimes k}}\left(\left[d_{\pi}\right]\left(a^{\dagger}\right)^{\otimes k}\right)|0\rangle \\
& =\sum_{\gamma \in S_{k}} \operatorname{Tr}_{V_{N}^{\otimes k}}\left(\gamma^{-1}\left(d_{\pi_{1}}^{T} \otimes d_{\pi_{2}}^{T}\right) \gamma d_{\pi}\right) \\
& =\sum_{\gamma \in S_{k}} N^{c\left(\gamma^{-1}\left(d_{\pi_{1}} \otimes d_{\pi_{2}}\right) \gamma \vee d_{\pi}\right)} . \tag{5.226}
\end{align*}
$$

We will now derive a set of representation theoretic selection rules for the extremal corre-
lators. To state the result we are going to prove, we define the operators

$$
\begin{equation*}
\mathcal{O}_{\Lambda_{2}, \mu \nu}^{\Lambda_{1}}=\operatorname{Tr}_{V_{N}^{\otimes k}}\left(Q_{\Lambda_{2}, \mu \nu}^{\Lambda_{1}}\left(a^{\dagger}\right)^{\otimes k}\right), \tag{5.227}
\end{equation*}
$$

associated with representation basis elements $Q_{\Lambda_{2}, \mu \nu}^{\Lambda_{1}} \in S P_{k}(N)$. Consider the extremal correlator (the time-independent part of it)

$$
\begin{equation*}
\langle 0|\left(\mathcal{O}_{\Lambda_{2}, \mu \nu}^{\Lambda_{1}}\right)^{\dagger}\left(\mathcal{O}_{\Lambda_{2}^{\Lambda_{1}, \mu^{\prime} \nu^{\prime}}}^{\Lambda_{1}^{\prime}}\right)^{\dagger} \mathcal{O}_{\Lambda_{2}^{\prime \prime}, \mu^{\prime \prime} \nu^{\prime \prime}}^{\Lambda_{1}^{\prime \prime}}|0\rangle=k!\operatorname{Tr}_{V_{N}^{\otimes k}}^{\otimes}\left(\left(Q_{\Lambda_{2}, \nu \mu}^{\Lambda_{1}} \otimes Q_{\Lambda_{2}^{\prime}, \nu^{\prime} \mu^{\prime}}^{\Lambda_{1}^{\prime}}\right) Q_{\Lambda_{2}^{\prime \prime}, \mu^{\prime \prime} \nu^{\prime \prime}}^{\Lambda_{1}^{\prime \prime}}\right), \tag{5.228}
\end{equation*}
$$

for $Q_{\Lambda_{2}, \mu \nu}^{\Lambda_{1}} \in S P_{k_{1}}(N), Q_{\Lambda_{2}^{\prime}, \mu^{\prime} \nu^{\prime}}^{\Lambda_{1}^{\prime}} \in S P_{k_{2}}(N), Q_{\Lambda_{2}^{\prime \prime}, \mu^{\prime \prime} \nu^{\prime \prime}}^{\Lambda^{\prime \prime}} \in S P_{k}(N)$. The factor of $k$ ! follows since the matrix units for $S P_{k}(N)$ are invariant under conjugation by $S_{k}$. Note that the multiplicity labels are exchanged under diagram transposition, which follows from (B.13). The resulting selection rule tells us that the trace in (5.228) vanishes if $C\left(\Lambda_{1}, \Lambda_{1}^{\prime}, \Lambda_{1}^{\prime \prime}\right)=0$, where $C\left(\Lambda_{1}, \Lambda_{1}^{\prime}, \Lambda_{1}^{\prime \prime}\right)$ is the Kronecker coefficient for tensor products of irreducible representations of $S_{N}$.

We start with the simpler, but analogous expression for matrix units of $P_{k}(N)$,


Using (see e.g. equation 5.98)

$$
\begin{equation*}
\left(Q_{\beta \alpha}^{\Lambda_{1}} \otimes Q_{\beta^{\prime} \alpha^{\prime}}^{\Lambda_{1}^{\prime}}\right) Q_{\alpha^{\prime \prime} \beta^{\prime \prime}}^{\Lambda_{1 \prime \prime}^{\prime \prime}}=\sum_{\gamma^{\prime \prime}} D_{\gamma^{\prime \prime} \alpha^{\prime \prime}}^{\Lambda_{1}^{\prime \prime}}\left(Q_{\beta \alpha}^{\Lambda_{1}} \otimes Q_{\beta^{\prime} \alpha^{\prime}}^{\Lambda_{1}^{\prime}}\right) Q_{\gamma^{\prime \prime} \beta^{\prime \prime}}^{\Lambda_{1}^{\prime \prime}}, \tag{5.230}
\end{equation*}
$$

we have

$$
\begin{align*}
& \operatorname{Tr}_{V_{N}^{\otimes k}}\left(\left(Q_{\beta \alpha}^{\Lambda_{1}} \otimes Q_{\beta^{\prime} \alpha^{\prime}}^{\Lambda_{1}^{\prime}}\right) Q_{\alpha^{\prime \prime} \beta^{\prime \prime}}^{\Lambda_{1}^{\prime \prime}}\right)=\sum_{\gamma^{\prime \prime}} D_{\gamma^{\prime \prime} \alpha^{\prime \prime}}^{\Lambda_{1}^{\prime \prime}}\left(Q_{\beta \alpha}^{\Lambda_{1}} \otimes Q_{\beta^{\prime} \alpha^{\prime}}^{\Lambda_{1}^{\prime}}\right) \operatorname{Tr}_{V_{N}^{\otimes k}}\left(Q_{\gamma^{\prime \prime} \beta^{\prime \prime}}^{\Lambda_{1}^{\prime \prime}}\right) \\
&=D_{\beta^{\prime \prime} \alpha^{\prime \prime}}^{\Lambda_{1 \prime}^{\prime \prime}}\left(Q_{\beta \alpha}^{\Lambda_{1}} \otimes Q_{\beta^{\prime} \alpha^{\prime}}^{\Lambda_{1}^{\prime}}\right) \operatorname{Dim} V_{\Lambda_{1}^{\prime \prime}}^{S_{N}} \\
& \alpha^{\prime \prime}  \tag{5.231}\\
&=\operatorname{Dim} V_{\Lambda_{1}^{\prime \prime}}^{S_{N}^{\prime \prime}} \begin{array}{|c|c|}
Q_{\alpha \beta}^{\prime \prime} \\
\Lambda_{1}
\end{array} Q_{\alpha^{\prime} \beta^{\prime}}^{\Lambda_{1}^{\prime}} \\
& \Lambda_{1}^{\prime \prime}
\end{align*}
$$

The second equality uses B.26).
To further simplify, we want to turn the RHS into a product of matrix elements. This is achieved by inserting a resolution of the identity using representations of $P_{k_{1}}(N) \otimes P_{k_{2}}(N)$. This resolves to a set of branching coefficients for $P_{k}(N) \rightarrow P_{k_{1}}(N) \otimes P_{k_{2}}(N)$. We denote these by

$$
\begin{equation*}
B_{\gamma_{\gamma^{\prime \prime} \rightarrow \gamma \gamma^{\prime}}^{\Lambda_{1}^{\prime \prime}}+\tilde{\Lambda}_{1} \otimes \tilde{\Lambda}_{1}^{\prime}, \xi} \tag{5.232}
\end{equation*}
$$

where it is implicit that $k=k_{1}+k_{2}$. The ranges of the labels are

$$
\begin{align*}
& \gamma \in\left[1, \ldots, \operatorname{Dim}\left(V_{\widetilde{\Lambda}_{1}}^{P_{k_{1}}(N)}\right)\right], \\
& \gamma^{\prime} \in\left[1, \ldots, \operatorname{Dim}\left(V_{\widetilde{\Lambda}_{1}^{\prime}}^{P_{k_{2}}(N)}\right)\right], \\
& \gamma^{\prime \prime} \in\left[1, \ldots, \operatorname{Dim}\left(V_{\Lambda_{1}^{\prime}}^{P_{k}(N)}\right)\right], \\
& \xi \in\left[1, \ldots, \operatorname{Mult}\left(V_{\Lambda_{1}^{\prime \prime}}^{P_{k}(N)} \rightarrow V_{\widetilde{\Lambda}_{1}}^{P_{k_{1}}(N)} \otimes V_{\widetilde{\Lambda}_{1}^{\prime}}^{P_{k_{2}}(N)}\right)\right], \tag{5.233}
\end{align*}
$$

the final label, $\xi$, gives the multiplicity of $\Lambda_{1}^{\prime \prime}$ in the decomposition. Branching coefficients are represented by the following diagrams

$$
\begin{equation*}
B_{\gamma^{\prime \prime} \rightarrow \gamma \gamma^{\prime}}^{\Lambda_{1}^{\prime \prime} \rightarrow \widetilde{\Lambda}_{1} \otimes \widetilde{\Lambda_{1}^{\prime}, \xi}}={\widetilde{\Lambda_{1}}}_{\underbrace{\overbrace{1}^{\prime}}_{\gamma^{\prime}}}^{\gamma_{1}^{\prime \prime}} \widetilde{\Lambda_{1}^{\prime}} \tag{5.234}
\end{equation*}
$$

It is worth noting that by Schur-Weyl duality the branching multiplicities for partition algebras are related to the multiplicities $C\left(\widetilde{\Lambda}_{1}, \widetilde{\Lambda}_{1}^{\prime}, \Lambda_{1}^{\prime \prime}\right)$, known as Kronecker coefficients, of irreducible representations $\Lambda_{1}^{\prime \prime}$ in tensor products of $S_{N}$ representations $\widetilde{\Lambda}_{1} \otimes \widetilde{\Lambda}_{1}^{\prime}$ (see eq.
(3.1.3) of [128])

$$
\begin{equation*}
\operatorname{Mult}\left(V_{\Lambda_{1}^{\prime \prime}}^{P_{k}(N)} \rightarrow V_{\widetilde{\Lambda}_{1}}^{P_{k_{1}}(N)} \otimes V_{\widetilde{\Lambda}_{1}^{\prime}}^{P_{k_{2}}(N)}\right)=C\left(\widetilde{\Lambda}_{1}, \widetilde{\Lambda}_{1}^{\prime}, \Lambda_{1}^{\prime \prime}\right) \tag{5.235}
\end{equation*}
$$

For simiplicity we are assuming $N \geq\left(2 k_{1}+2 k_{2}\right)$. For comparison, in Schur-Weyl duality between $U(N)$ and $\mathbb{C} S_{k}$, Littlewood-Richardson coefficients are branching multiplicities for $S_{k_{1}+k_{2}} \rightarrow S_{k_{1}} \times S_{k_{2}}$ but correspond to decomposition of tensor products of $U(N)$ representations.

Branching coefficients are equivariant:

$$
\begin{equation*}
D_{\gamma^{\prime \prime} \delta^{\prime \prime}}^{\Lambda_{1}^{\prime \prime}}\left(d_{\pi_{1}} \otimes d_{\pi_{2}}\right)=\sum_{\widetilde{\Lambda}_{1}, \widetilde{\Lambda_{1}^{\prime}}, \gamma, \delta, \gamma^{\prime}, \delta^{\prime}, \xi} B_{\gamma^{\prime \prime} \rightarrow \gamma \gamma^{\prime}}^{\Lambda_{1}^{\prime \prime} \rightarrow \widetilde{\Lambda}_{1} \otimes \widetilde{\Lambda}_{1}^{\prime}, \xi} D_{\gamma \delta}^{\widetilde{\Lambda}_{1}}\left(d_{\pi_{1}}\right) D_{\gamma^{\prime} \delta \delta^{\prime}}^{\widetilde{\Lambda}_{1}^{\prime}}\left(d_{\pi_{2}}\right) B_{\delta^{\prime \prime} \rightarrow \delta \delta^{\prime}}^{\Lambda_{1}^{\prime \prime} \rightarrow \widetilde{\Lambda}_{1} \otimes \widetilde{\Lambda}_{1}^{\prime}, \xi}, \tag{5.236}
\end{equation*}
$$

for $d_{\pi_{1}} \in P_{k_{1}}(N), d_{\pi_{2}} \in P_{k_{2}}(N)$. Setting $d_{\pi_{1}}=Q_{\alpha \beta}^{\Lambda_{1}}, d_{\pi_{2}}=Q_{\alpha^{\prime} \beta^{\prime}}^{\Lambda^{\prime}}$, equation (5.236) corresponds to the diagram identity


Inserting this into equation (5.231) gives

$$
\begin{align*}
& \operatorname{Tr}_{V_{N}^{\otimes k}}^{\otimes k}\left(\left(Q_{\beta \alpha}^{\Lambda_{1}} \otimes Q_{\beta^{\prime} \alpha^{\prime}}^{\Lambda_{1}^{\prime}}\right) Q_{\alpha^{\prime \prime} \beta^{\prime \prime}}^{\Lambda_{1}^{\prime \prime}}\right)= \\
& \quad \operatorname{Dim} V_{\Lambda_{1}^{\prime \prime}}^{S_{N}} \sum_{\widetilde{\Lambda}_{1}, \widetilde{\Lambda}_{1}^{\prime}, \gamma, \eta, \gamma^{\prime}, \eta^{\prime}, \xi} B_{\gamma_{1}^{\prime \prime} \rightarrow \gamma \gamma^{\prime}}^{\Lambda_{1}^{\prime \prime} \rightarrow \widetilde{\Lambda}_{1} \otimes \widetilde{\Lambda}_{1}^{\prime}, \xi} D_{\gamma \eta}^{\widetilde{\Lambda}_{1}}\left(Q_{\alpha \beta}^{\Lambda_{1}}\right) D_{\gamma^{\prime} \eta^{\prime}}^{\widetilde{\Lambda}_{1}^{\prime}}\left(Q_{\alpha^{\prime} \beta^{\prime}}^{\Lambda_{1}^{\prime}}\right) B_{\eta^{\prime \prime} \rightarrow \eta \eta^{\prime}}^{\Lambda_{1}^{\prime \prime} \rightarrow \widetilde{\Lambda}_{1} \otimes \widetilde{\Lambda}_{1}^{\prime}, \xi} \tag{5.238}
\end{align*}
$$

Matrix elements of irreducible representations are orthogonal (see equation (B.12). This implies

$$
\begin{equation*}
D_{\eta^{\prime \prime} \gamma^{\prime \prime}}^{\widetilde{\Lambda}_{1 \prime}^{\prime \prime}}\left(Q_{\alpha^{\prime \prime} \beta^{\prime \prime}}^{\Lambda_{1}^{\prime \prime}}\right)=\delta^{\widetilde{\Lambda}_{1}^{\prime \prime} \Lambda_{1}^{\prime \prime}} \delta_{\eta^{\prime \prime} \beta^{\prime \prime}} \delta_{\gamma^{\prime \prime} \alpha^{\prime \prime}} \tag{5.239}
\end{equation*}
$$

or the equivalent diagrammatic expression


Substituting this identity into (5.238) reduces it to


This gives the final result for matrix units of $P_{k}(N)$.
The full expression for (5.228) - extremal three-point correlators in the representation basis - is given by 5.241) together with branching coefficients from the partition algebras to symmetric group algebras (see (5.75)),

$$
\begin{align*}
& \operatorname{Tr}_{V_{N}^{\otimes k}}\left(\left(Q_{\Lambda_{2}, \nu \mu}^{\Lambda_{1}} \otimes Q_{\Lambda_{2}^{\prime}, \nu^{\prime} \mu^{\prime}}^{\Lambda_{1}^{\prime}}\right) Q_{\Lambda_{2}^{\prime \prime}, \mu^{\prime \prime} \nu^{\prime \prime}}^{\Lambda_{1}^{\prime \prime}}\right)=\operatorname{Dim} V_{\Lambda_{1}^{\prime \prime}}^{S_{N}^{\prime}} \sum_{\substack{, \beta, \alpha^{\prime}, \beta^{\prime}, \alpha^{\prime \prime}, \beta^{\prime \prime}, p, p^{\prime}, p^{\prime \prime}, \xi}} B_{\alpha^{\prime \prime} \rightarrow \alpha \alpha^{\prime}}^{\Lambda_{1}^{\prime \prime} \rightarrow \Lambda_{1} \otimes \Lambda_{1}^{\prime}, \xi} B_{\beta^{\prime \prime} \rightarrow \beta \beta^{\prime}}^{\Lambda_{1}^{\prime \prime} \rightarrow \Lambda_{1} \otimes \Lambda_{1}^{\prime}, \xi} \\
& B_{\Lambda_{1}, \alpha \rightarrow \Lambda_{2}, p ; \mu}^{P_{k_{1}}(N) \rightarrow \mathbb{C} S_{k_{1}}} B_{\Lambda_{1}, \beta \rightarrow \Lambda_{2}, p ; \nu}^{P_{k_{1}}(N) \rightarrow \mathbb{C} S_{k_{1}}} B_{\Lambda_{1}^{\prime}, \alpha^{\prime} \rightarrow \Lambda_{2}^{\prime}, p^{\prime} ; \mu^{\prime}}^{P_{k_{2}}(N) \rightarrow \mathbb{C} S_{k_{2}}} P_{\Lambda_{1}^{\prime}, \beta^{\prime} \rightarrow \Lambda_{2}^{\prime}, p^{\prime} ; \nu^{\prime}}^{P_{k_{2}}(N) \rightarrow \mathbb{C} S_{k_{2}}} B_{\Lambda_{1}^{\prime}, \alpha^{\prime \prime} \rightarrow \Lambda_{2}^{\prime \prime}, p^{\prime \prime} ; \mu^{\prime \prime}}^{P_{k}(N) \rightarrow \mathbb{C} S_{k}} B_{\Lambda_{1}^{\prime}, \beta^{\prime \prime} \rightarrow \Lambda_{2}^{\prime}, p^{\prime \prime} ; \nu^{\prime \prime}}^{P_{k}(N) \rightarrow \mathbb{C} S_{k}} . \tag{5.242}
\end{align*}
$$

Introducing the following diagram representation of these branching coefficients,

$$
B_{\Lambda_{1}, \alpha \rightarrow \Lambda_{2}, p ; \mu}^{P_{k}(N) \rightarrow \mathbb{C} S_{k}}=\left.\begin{array}{r}
p  \tag{5.243}\\
\Lambda_{2} \\
\mu_{1} \\
\Lambda_{1} \\
\alpha
\end{array}\right|_{\alpha},
$$

we can write (5.242) as the following diagram


From (5.244) we can read off the claimed result, that the extremal correlator vanishes if the Kronecker coefficient of the operator irreducible representations does $C\left(\Lambda_{1}, \Lambda_{1}^{\prime}, \Lambda_{1}^{\prime \prime}\right)=0$. Analogous results for extremal correlators in general quiver gauge theories are described in (13).

### 5.7 Discussion

In this chapter we investigated the effects of permutation symmetry on the state space and dynamics of quantum mechanical systems of $N \times N$ matrix variables. After a brief review of the matrix harmonic oscillator and introduction of notation in section 5.1, we explored the $S_{N}$ invariant Hilbert space $\mathcal{H}_{\text {inv }}$ of generic matrix quantum mechanics systems at large $N$ in section 5.2. We found a one-to-one correspondence between $S_{N}$ invariant states of degree $k$ and elements in the symmetrised partition algebra $S P_{k}(N)$. Two bases of $S P_{k}(N)$ were discussed: the diagram basis and the representation basis. A construction of the latter was explained in section 5.3 in terms of diagonalising commuting algebraic charges.

Having discussed the $S_{N}$ invariant state space, we moved on to interesting $S_{N}$ invariant Hamiltonians. The general permutation invariant harmonic matrix oscillator was described and solved (diagonalised) in section 5.4. This was achieved with the introduction of oscillators labelled by representation theoretic quantities, as in 5.147). In section 5.5 we described a set of algebraic Hamiltonians for matrix quantum mechanics that preserve the $S_{N}$ invariant subspace of the Hilbert space. These Hamiltonians, given by equations (5.201),
(5.205) and (5.207) realise the three dynamical scenarios illustrated on the left hand side of figure 5.1a, the right hand side of figure 5.1a, and figure 5.1b respectively. The representation basis introduced in section 5.2 .3 diagonalises all of these algebraic Hamiltonians. We provided a lattice interpretation of the matrix oscillators in section 5.5.6. The diagram basis is the most efficient basis for describing inner and outer products. As a consequence extremal correlators, defined in 5.225), which are analogues of three-point extremal correlators in $\mathcal{N}=4$ SYM are simple in the diagram basis. The extremal correlators satisfy representation theoretic selection rules, based on Kronecker coefficients, which were derived in the representation basis. The selection rules are based on exact expressions for extremal correlators, involving Kronecker coefficients and Littlewood-Richardson coefficients, given in equation (5.242).

## Chapter 6

## Discussion and conclusion

In this thesis we have developed results on the theme of permutation invariant Gaussian matrix models. In chapter 3 we specialised the PIGM models consisting of general matrices to those consisting of symmetric matrices with vanishing diagonal. In this case the 13 parameters of the general matrix model are reduced to just four: one linear and three quadratic. Observables of this model are permutation invariant polynomials in the matrix variables. They are in 1-1 correspondence with undirected, loop-less multi-graphs (3.62). We found an analytic formula in $N$ for the two-point function of the matrix $M_{i j}$ (3.74), in combination with Wick's theorem this can be used to calculate the expectation value of higher-order observables. This model was used to demonstrate approximate permutation invariant Gaussianity in an ensemble of financial correlation matrices constructed from correlations between price movements in foreign exchange market data. We concluded this chapter by constructing vectors of observables to act as low-dimensional representations of the matrices in the ensemble. The observable vectors performed well in anomaly detection tasks, comparing favourably with both the original matrices and the standard dimensionality reduction technique of principal component analysis.

In chapter 4 we began by establishing a correspondence between permutation invariant matrix observables and equivalence classes of partition algebra elements. This relied on the Schur-Weyl duality between the symmetric group $S_{N}$ and the partition algebra $P_{k}(N)$ acting on $V_{N}$ the natural representation of the symmetric group. It is analogous to a similar organisation of $U(N)$ invariants by the conjugacy classes of $S_{k}$ due to the classic instance of Schur-Weyl duality. The description of PIMOs in terms of the partition algebra was used to prove the large $N$ factorisation of these observables under the simplest $O(N)$ inner product 4.61).

Finally, in chapter 5 we considered permutation invariant matrix quantum mechanics, in
which matrix oscillators $\left(a^{\dagger}\right)_{j}^{i}$ took the place of matrix elements $M_{i j}$. We first gave a description of the permutation invariant state space. The partition algebra again played a key role in the construction of these states through (5.48). We then introduced a representation basis for the invariant state space $\mathcal{H}_{k}$, this basis could be understood as a basis that diagonalises a set of algebraic commuting charges. These charges were constructed in section 5.3.1 and used to define classes of Hamiltonians with a variety of ground state behaviours illustrated in figure 5.1 . We concluded by calculating two- and three-point correlators, the former enjoy a similar large $N$ factorisation property proven for the matrix observables in chapter $4(5.224)$, the latter were shown to obey selection rules based on the Kronecker coefficients for tensor products of irreducible representations of $S_{N}$.

A natural future direction following this work would be to extend the results to permutation invariant models for tensor variables $T_{i j k}$ transforming as $V_{N} \otimes V_{N} \otimes V_{N}$. This is further motivated by the application of these models in type-driven compositional distributional semantics in which three-index tensors are used to represent transitive verbs [46]. Explicit machine-learning algorithms for constructing ensembles of these three-index tensors, as well as two-matrix ensembles, from natural language data have been designed [129, 130, 131]. In chapters 4 and 5 frequent use was made of Schur-Weyl dual algebras. These principles can be exploited in a number of natural generalisations of the results in these chapters, including to tensor models: this is being developed in as yet unpublished work [132.

The discussion of observables lies at a rich intersection of representation theory, combinatorics, graph theory and group theory. The explicit formulae and theoretical perspective developed here for the enumeration of graphs can potentially be useful in other applications of graphs within theoretical physics: for example an interesting recent application of graphs is in jet algorithms [133].

An interesting future direction of the factorisation property proven in chapter 4 would be to investigate the large $N$ factorisation properties of the inner product of permutation invariant observables arising from the most general $S_{N}$ invariant action. Progress in this direction has already been made in that the $S_{N}$ invariant two-point function of the fundamental fields $M_{i j}$ is known (2.115, [43]). The form of the simplest $O(N)$ invariant two-point function of the fundamental fields, given in (4.55), allowed us to write a simple expression for the associated two-point function of PIMOs of general order $k$, equation 4.56). In contrast, the form of the $S_{N}$ invariant two-point function of the fundamental fields involves many more terms and is much more complicated.

Throughout our default position was to assume $N \geq 2 k$, known as the stable limit. This lead to considerable simplifications, including the construction of a basis for the $S_{N}$ invariant subspace $\mathcal{H}_{\text {inv }}$, a simplification related to the existence of a kernel free map from
$P_{k}(N)$ to $\operatorname{End}\left(V_{N}^{\otimes k}\right)$. However, it would be interesting to uncover any finite $N$ effects appearing in the permutation invariant quantum mechanical matrix systems of chapter 5. At finite $N$ the diagrams in $P_{k}(N)$ provide an over complete basis of operators. That is, there are some linear relations between operators. The precise form of these relations can be found using the orbit basis. The question remains of how to use this knowledge in order to construct a representation theoretic basis for $2 k<N$. This would involve a detailed study of the Artin-Wedderburn decomposition in (5.70) below the stable limit. The detailed study includes putting constraints on the irreducible representations appearing in the decomposition below the stable limit, as well as computing the dimension of the multiplicity spaces.

A very interesting avenue towards applications of the Hilbert spaces and Hamiltonians considered in chapter 5 is to find systems where the permutation invariant sectors described using partition algebras are naturally selected by the physics. For example, in a BoseEinstein condensate composed of $N$ identical bosons, excited by vibrational modes between pairs of particles, oscillators $\left(a^{\dagger}\right)_{i}^{j}$ exciting the pair $(i, j)$ of particles with $i, j \in\{1, \cdots, N\}$ would naturally be subject to the kind of $S_{N}$ invariance we have considered here. This would provide links between the theoretical application of partition algebras as considered here with the phenomenological modelling of Bose-Einstein physics, e.g. along the lines of [134.

## Appendix A

## Inner product calculations

Here we present some inner products which are useful in arriving at equations for the physical variables $S^{\text {phys } ; V_{[N]}}$ and $S_{a}^{\text {phys } V_{[N-1,1]}}$ obtained in (3.42) and (3.43) of section 3.2 Using the inner product on the orthonormal basis of the natural representation

$$
\begin{equation*}
\left(e_{i}, e_{j}\right)=\delta_{i j} . \tag{A.1}
\end{equation*}
$$

We calculate the following inner products used to determine the Clebsch coefficients from $V_{N} \otimes V_{N}$ to the physical $V_{[N]}$ and $V_{[N-1,1]}$ irreducible representations $\left(S^{\text {diag; } V_{[N]}}, S^{V_{[N]} ; 1}\right), \quad\left(S^{\text {diag; } ; V_{[N]}}, S^{V_{[N]} ; 2}\right), \quad\left(S_{a_{1}}^{\text {diag; } V_{[N-1,1]}}, S_{a_{2}}^{V_{[N-1,1]} ; 1,2}\right), \quad\left(S_{a_{1}}^{\text {diag; } V_{[N-1,1]}}, S_{a_{2}}^{V_{[N-1,1]} ; 3}\right)$.

The normalised representation theory states are given by

$$
\begin{align*}
S^{V_{[N]} ; 1} & =\frac{1}{N} \sum_{i, j=1}^{N} e_{i} \otimes e_{j},  \tag{A.3}\\
S^{V_{[N]} ; 2} & =\frac{1}{\sqrt{N-1}} \sum_{a=1}^{N-1} E_{a} \otimes E_{a},  \tag{A.4}\\
S^{\text {diag; } V_{[N]}} & =\frac{1}{\sqrt{N}} \sum_{i=1}^{N} e_{i} \otimes e_{i},  \tag{A.5}\\
S_{a}^{V_{[N-1,1] ; 1,2}} & =\frac{1}{\sqrt{2 N}} \sum_{i=1}^{N}\left(e_{i} \otimes E_{a}+E_{a} \otimes e_{i}\right),  \tag{A.6}\\
S_{a}^{V_{[N-1,1] ; 3}} & =\frac{1}{2} \sqrt{\frac{N}{N-2}} \sum_{b, c=1}^{N-1} \sum_{i=1}^{N} C_{a, i} C_{b, i} C_{c, i}\left(E_{b} \otimes E_{c}+E_{c} \otimes E_{b}\right), \tag{A.7}
\end{align*}
$$

$$
\begin{equation*}
S_{a}^{\mathrm{diag} ; V_{[N-1,1]}}=E_{a} \otimes E_{a} \tag{A.8}
\end{equation*}
$$

The non-zero inner products are given by the following

$$
\begin{align*}
\left(S^{\text {diag; } \left.V_{[N]}, S^{V_{[N]} ; 1}\right)}=\right. & N^{-\frac{3}{2}} \sum_{i, j, k=1}^{N}\left(e_{i} \otimes e_{i}, e_{j} \otimes e_{k}\right)=N^{-\frac{3}{2}} \sum_{i, j, k=1}^{N} \delta_{i j} \delta_{i k}=\sum_{i=1}^{N} N^{-\frac{3}{2}} \\
& =\frac{1}{\sqrt{N}},  \tag{A.9}\\
\left(S^{\text {diag; } \left.V_{[N]}, S^{V_{[N]} ; 2}\right)}\right. & =\frac{1}{\sqrt{N} \sqrt{N-1}} \sum_{i=1}^{N} \sum_{a=1}^{N-1}\left(e_{i} \otimes e_{i}, E_{a} \otimes E_{a}\right) \\
& =\frac{1}{\sqrt{N} \sqrt{N-1}} \sum_{i, j, k=1}^{N} \sum_{a=1}^{N-1} C_{a, j} C_{a, k}\left(e_{i} \otimes e_{i}, e_{j} \otimes e_{k}\right) \\
& =\frac{1}{\sqrt{N} \sqrt{N-1}} \sum_{i, j, k=1}^{N} \sum_{a=1}^{N-1} C_{a, j} C_{a, k} \delta_{i j} \delta_{i k} \\
& =\frac{1}{\sqrt{N} \sqrt{N-1}} \sum_{j=1}^{N} \sum_{a=1}^{N-1} C_{a, j} C_{a, j} \\
& =\sum_{a=1}^{N-1} \frac{1}{\sqrt{N} \sqrt{N-1}} \\
& =\frac{\sqrt{N-1}}{\sqrt{N}}, \tag{A.10}
\end{align*}
$$

$$
\left(S_{a_{1}}^{\mathrm{diag} ; V_{[N-1,1]}}, S_{a_{2}}^{V_{[N-1,1]} 1,2}\right)=\frac{1}{\sqrt{2 N}} \sum_{i, j, k=1}^{N} C_{a_{1}, j} C_{a_{2}, k}\left(e_{j} \otimes e_{j},\left(e_{i} \otimes e_{k}+e_{k} \otimes e_{i}\right)\right)
$$

$$
=\frac{1}{\sqrt{2 N}} \sum_{i, j, k=1}^{N} C_{a_{1}, j} C_{a_{2}, k}\left(\delta_{i j} \delta_{j k}+\delta_{j k} \delta_{i j}\right)
$$

$$
=\sqrt{\frac{2}{N}} \sum_{i, j=1}^{N} C_{a_{1}, j} C_{a_{2}, j} \delta_{i j}
$$

$$
\begin{equation*}
=\sqrt{\frac{2}{N}} \delta_{a_{1} a_{2}} \tag{A.11}
\end{equation*}
$$

$$
\begin{align*}
\left(S_{a_{1}}^{\left.\mathrm{diag} ; V_{[N-1,1]}, S_{a_{2}}^{V_{[N-1,1]} ; 3}\right)}\right. & =\frac{1}{2} \sqrt{\frac{N}{N-2}} \sum_{b, c=1}^{N-1} \sum_{i, j, k, l=1}^{N} C_{a_{1}, i} C_{b, i} C_{c, i} C_{a_{2}, j} C_{b, k} C_{c, l}\left(e_{j} \otimes e_{j},\left(e_{k} \otimes e_{l}+e_{l} \otimes e_{k}\right)\right) \\
& =\sqrt{\frac{N}{N-2}} \sum_{b, c=1}^{N-1} \sum_{i, j, k, l=1}^{N} C_{a_{1}, i} C_{b, i} C_{c, i} C_{a_{2}, j} C_{b, k} C_{c, l} \delta_{j k} \delta_{j l} \\
& =\sqrt{\frac{N}{N-2}} \sum_{b, c=1}^{N-1} \sum_{i, j=1}^{N} C_{a_{1}, i} C_{b, i} C_{c, i} C_{a_{2}, j} C_{b, j} C_{c, j} \\
& =\sqrt{\frac{N}{N-2}} \sum_{i, j=1}^{N} C_{a_{1}, i} C_{a_{2}, j}\left(\delta_{i j}-\frac{1}{N}\right)\left(\delta_{i j}-\frac{1}{N}\right) \\
& =\sqrt{\frac{N}{N-2}} \sum_{i, j=1}^{N} C_{a_{1}, i} C_{a_{2}, j}\left(\delta_{i j}\left(1-\frac{2}{N}\right)+\frac{1}{N^{2}}\right) \\
& =\sqrt{\frac{N}{N-2}}\left[\sum_{i=1}^{N} C_{a_{1}, i} C_{a_{2}, i}\left(1-\frac{2}{N}\right)+\frac{1}{N^{2}} \sum_{i, j=1}^{N} C_{a_{1}, i} C_{a_{2}, j}\right] \\
& =\sqrt{\frac{N}{N-2}}\left(1-\frac{2}{N}\right) \delta_{a_{1}, a_{2}} . \tag{A.12}
\end{align*}
$$

## Appendix B

## Matrix units and Fourier inversion from inner product

In this appendix we list the results used in section 5.2.3 on representation bases. The proofs of many of these results are to be found within the appendices of [3] (the paper on which chapter 5 is based). We start by discussing non-degenerate bilinear forms on algebras and how they define dual elements through (B.10). The existence of dual elements is used in proving the orthogonality of matrix elements of irreducible representations of $P_{k}(N)$, which we merely state in (B.12). Orthogonality is essential for the construction of matrix units of $P_{k}(N)$ using the Fourier inversion formula (B.14). Matrix units for $S P_{k}(N)$ are constructed using branching coefficients, as in (B.18). These results represent minor modifications to the constructions in [90], which defines a non-degenerate bilinear using the trace in the regular representation of $P_{k}(N)$. Throughout chapter 5, the physical trace relevant to the inner product (5.54) and two point function, is a trace in $V_{N}^{\otimes k}$. The two traces are related in $\overline{\mathrm{B} .7}$ ), through a so-called $\Omega$-factor, and the basic formulae following from this are similarly affected by minor changes.

## B. 1 Schur-Weyl duality and non-degenerate bilinear forms

The construction of matrix units for $P_{k}(N)$ relies on the existence of a non-degenerate bilinear form on $P_{k}(N)$. The bilinear form used in [90] is defined using the trace in the regular representation of $P_{k}(N)$. In this chapter the physical trace, associated with inner products, is a trace in $V_{N}^{\otimes k}$ including a transposition as in equation (5.54). Here we state, without proof the result that this trace also defines a non-degenerate bilinear form and give the relation between the two traces (B.7).

Let $\mathcal{B}=\left\{b_{1}, \ldots, b_{B(2 k)}\right\}$ be a basis for $P_{k}(N)$. The regular representation of $P_{k}(N)$ is defined by the left action of $P_{k}(N)$ on itself. The matrix representation of $b_{i}$ is defined by the structure constants $C_{i j}^{k}$

$$
\begin{equation*}
b_{i} b_{j}=\sum_{k=1}^{B(2 k)} C_{i j}^{k} b_{k} \tag{B.1}
\end{equation*}
$$

Consequently, the trace in the regular representation can be written as

$$
\begin{equation*}
\operatorname{tr}\left(\mathrm{b}_{\mathrm{i}}\right)=\sum_{\mathrm{j}=1}^{\mathrm{B}(2 \mathrm{k})} \mathrm{C}_{\mathrm{ij}}^{\mathrm{j}}=\sum_{\mathrm{j}=1}^{\mathrm{B}(2 \mathrm{k})} \operatorname{Coeff}\left(\mathrm{b}_{\mathrm{j}}, \mathrm{~b}_{\mathrm{i}} \mathrm{~b}_{\mathrm{j}}\right) \tag{B.2}
\end{equation*}
$$

where Coeff $\left(b_{j}, d\right)$ is the coefficient of $b_{j}$ in the expansion of $d \in P_{k}(N)$ in the basis $\mathcal{B}$.
For $N \geq 2 k, P_{k}(N)$ is semi-simple (see [61, Theorem 3.27]) and therefore,

$$
\begin{equation*}
G_{i j} \equiv \operatorname{tr}\left(\mathrm{~b}_{\mathrm{i}} \mathrm{~b}_{\mathrm{j}}\right) \tag{B.3}
\end{equation*}
$$

is an invertible matrix. We say that the trace in the regular representation defines a non-degenerate bilinear form on $P_{k}(N)$ (see [61, Equation 5.9]). It will be useful to use the following equivalent definition of non-degeneracy in what follows. A bilinear form on $P_{k}(N)$ is non-degenerate if there exists no non-zero element $d \in P_{k}(N)$ such that

$$
\begin{equation*}
\operatorname{tr}\left(\mathrm{b}_{\mathrm{i}} \mathrm{~d}\right)=0, \quad \forall \mathrm{i}=1, \ldots, \mathrm{~B}(2 \mathrm{k}) \tag{B.4}
\end{equation*}
$$

We can relate the two traces

$$
\begin{equation*}
\operatorname{Tr}_{V_{N}^{\otimes k}}(d)=\sum_{\Lambda_{1}} \operatorname{Dim} V_{\Lambda_{1}}^{S_{N}} \chi^{\Lambda_{1}}(d)=\sum_{\Lambda_{1}} \frac{\operatorname{Dim} V_{\Lambda_{1}}^{S_{N}}}{\operatorname{Dim} V_{\Lambda_{1}}^{P_{k}(N)}} \operatorname{tr}\left(\mathrm{p}_{\Lambda_{1}} \mathrm{~d}\right) \tag{B.5}
\end{equation*}
$$

where $p_{\Lambda_{1}} \in P_{k}(N)$ are projection operators and the sum is over all irreducible representations of $P_{k}(N)$. It is convenient to define

$$
\begin{equation*}
\Omega=\sum_{\Lambda_{1}} \frac{\operatorname{Dim} V_{\Lambda_{1}}^{S_{N}}}{\operatorname{Dim} V_{\Lambda_{1}}^{P_{k}(N)}} p_{\Lambda_{1}} \tag{B.6}
\end{equation*}
$$

such that equation B.5 becomes

$$
\begin{equation*}
\operatorname{Tr}_{V_{N}^{\otimes k}}(d)=\operatorname{tr}(\Omega \mathrm{d}) \tag{B.7}
\end{equation*}
$$

The bilinear form $(-,-): P_{k}(N) \times P_{k}(N) \rightarrow \mathbb{C}$ given by

$$
\begin{equation*}
\left(b_{i}, b_{j}\right)=\operatorname{Tr}_{V_{N}^{\otimes k}}\left(b_{i} b_{j}\right) \tag{B.8}
\end{equation*}
$$

is non-degenerate. It immediately follows (use proof by contradiction again) that the bilinear form given by

$$
\begin{equation*}
\left\langle b_{i}, b_{j}\right\rangle=\operatorname{Tr}_{V_{N}^{\otimes k}}\left(b_{i} b_{j}^{T}\right) \equiv g_{i j}, \tag{B.9}
\end{equation*}
$$

is non-degenerate and $g_{i j}$ is invertible. The inverse matrix is used to define elements dual to $b_{i}$ which we denote $b_{i}^{*}$

$$
\begin{equation*}
b_{i}^{*}=\sum_{j=1}^{B(2 k)}\left(g^{-1}\right)_{i j} b_{j} . \tag{B.10}
\end{equation*}
$$

Dual elements satisfy

$$
\begin{equation*}
\left\langle b_{i}^{*}, b_{j}\right\rangle=\delta_{i j} . \tag{B.11}
\end{equation*}
$$

## B. 2 Orthogonality of matrix elements

The matrix elements $D_{\alpha \beta}^{\Lambda_{1}}\left(b_{i}\right)$ of irreducible representations of $P_{k}(N)$ are orthogonal. This is a generalisation of the corresponding orthogonality theorem for group algebras (see section 3.15 in [68]). The definition of dual elements given in the previous subsection is such that

$$
\begin{equation*}
\sum_{i=1}^{B(2 k)} D_{\alpha \beta}^{\Lambda_{1}}\left(b_{i}\right) D_{\rho \sigma}^{\Lambda_{j}^{\prime}}\left(\left(b_{i}^{*}\right)^{T}\right)=\frac{1}{\operatorname{Dim} V_{\Lambda_{1}}^{S_{N}}} \delta_{\beta \rho} \delta_{\alpha \sigma} \delta^{\Lambda_{1} \Lambda_{1}^{\prime}} . \tag{B.12}
\end{equation*}
$$

and we can always choose irreducible representations satisfying

$$
\begin{equation*}
D_{\alpha \beta}^{\Lambda_{1}}\left(d^{T}\right)=D_{\beta \alpha}^{\Lambda_{1}}(d), \quad \text { for } d \in P_{k}(N), \tag{B.13}
\end{equation*}
$$

where $d^{T}$ is as in (5.53).

## B. 3 Matrix units for $P_{k}(N)$

The orthogonality of matrix elements (B.12) can be used so show

$$
\begin{equation*}
Q_{\alpha \beta}^{\Lambda_{1}}=\sum_{i} \operatorname{Dim}\left(V_{\Lambda_{1}}^{S_{N}}\right) D_{\beta \alpha}^{\Lambda_{1}}\left(\left(b_{i}^{*}\right)^{T}\right) b_{i}, \tag{B.14}
\end{equation*}
$$

multiply like a generalised matrix algebra. That is,

$$
\begin{equation*}
Q_{\alpha \beta}^{\Lambda_{1}} Q_{\rho \sigma}^{\Lambda_{1}^{\prime}}=\delta^{\Lambda_{1} \Lambda_{1}^{\prime}} \delta_{\beta \rho} Q_{\alpha \sigma}^{\Lambda_{1}} . \tag{B.15}
\end{equation*}
$$

Elements $d \in P_{k}(N)$ act on $Q_{\alpha \beta}^{\Lambda_{1}}$ from the left and right as

$$
\begin{align*}
& d Q_{\alpha \beta}^{\Lambda_{1}}=D_{\alpha \sigma}^{\Lambda_{1}}\left(d^{T}\right) Q_{\sigma \beta}^{\Lambda_{1}},  \tag{B.16}\\
& Q_{\alpha \beta}^{\Lambda_{1}} d=Q_{\alpha \sigma}^{\Lambda_{1}} D_{\sigma \beta}^{\Lambda_{1}}\left(d^{T}\right) . \tag{B.17}
\end{align*}
$$

respectively.

## B. 4 Matrix units for $S P_{k}(N)$ and normalisation constants

The matrix units for $S P_{k}(N)$ are constructed from $Q_{\alpha \beta}^{\Lambda_{1}}$ using Branching coefficients.

$$
\begin{equation*}
Q_{\Lambda_{2}, \mu \nu}^{\Lambda_{1}}=\sum_{\alpha, \beta, p} Q_{\alpha \beta}^{\Lambda_{1}} B_{\Lambda_{1}, \alpha \rightarrow \Lambda_{2}, p ; \mu}^{P_{k}(N) \rightarrow \mathbb{C}\left[S_{k}\right]} B_{\Lambda_{1}, \beta \rightarrow \Lambda_{2}, p ; \nu}^{P_{k}(N) \rightarrow \mathbb{C}\left[S_{k}\right]} . \tag{B.18}
\end{equation*}
$$

Branching coefficients are understood as follows. The partition algebra $P_{k}(N)$ has a subalgebra (isomorphic to) $\mathbb{C}\left[S_{k}\right]$ (for example, see equation (2.38)). For any given irreducible representation $V_{\Lambda_{1}}^{P_{k}(N)}$ there exists a basis where the action of $\mathbb{C}\left[S_{k}\right] \subset P_{k}(N)$ is manifest and irreducible. That is, we consider the decomposition

$$
\begin{equation*}
V_{\Lambda_{1}}^{P_{k}(N)} \cong \bigoplus_{\Lambda_{2} \vdash k} V_{\Lambda_{2}}^{\mathbb{C}\left[S_{k}\right]} \otimes V_{\Lambda_{1} \Lambda_{2}}^{P_{k}(N) \rightarrow \mathbb{C}\left[S_{k}\right]} \tag{B.19}
\end{equation*}
$$

On the LHS we have a basis

$$
\begin{equation*}
E_{\alpha}^{\Lambda_{1}}, \quad \alpha \in\left\{1, \ldots \operatorname{Dim}\left(V_{\Lambda_{1}}^{P_{k}(N)}\right)\right\}, \tag{B.20}
\end{equation*}
$$

where the representation of $d \in P_{k}(N)$ is irreducible,

$$
\begin{equation*}
d\left(E_{\alpha}^{\Lambda_{1}}\right)=\sum_{\beta} D_{\beta \alpha}^{\Lambda_{1}}(d) E_{\beta}^{\Lambda_{1}} . \tag{B.21}
\end{equation*}
$$

The RHS has a basis

$$
\begin{align*}
E_{\Lambda_{2}, p}^{\Lambda_{1}, \mu}, & p \in\left\{1, \ldots, \operatorname{Dim}\left(V_{\Lambda_{1}}^{\mathbb{C}\left[S_{k}\right]}\right)\right\},  \tag{B.22}\\
& \mu \in\left\{1, \ldots, \operatorname{Dim}\left(V_{\Lambda_{1} \Lambda_{2}}^{P_{k}(N) \rightarrow \mathbb{C}\left[S_{k}\right]}\right)\right\},
\end{align*}
$$

where $\mu$ is a multiplicity label for $V_{\Lambda_{2}}^{\mathbb{C}\left[S_{k}\right]}$ in the decomposition. We demand that the representation of $\tau \in \mathbb{C}\left[S_{k}\right]$ is irreducible in this basis,

$$
\begin{equation*}
\tau\left(E_{\Lambda_{2}, p}^{\Lambda_{1}, \mu}\right)=\sum_{q} D_{q p}^{\Lambda_{2}}(\tau) E_{\Lambda_{2}, q}^{\Lambda_{1}, \mu}, \tag{B.23}
\end{equation*}
$$

where $D_{q p}^{\Lambda_{2}}(\tau)$ is an irreducible representation of $\tau \in \mathbb{C}\left[S_{k}\right]$. The change of basis coefficients are called Branching coefficients

$$
\begin{equation*}
E_{\Lambda_{2}, p}^{\Lambda_{1}, \mu}=\sum_{\alpha} B_{\Lambda_{1}, \alpha \rightarrow \Lambda_{2}, p ; \mu}^{P_{k}(N) \rightarrow \mathbb{C}\left[S_{k}\right]} E_{\alpha}^{\Lambda_{1}} \tag{B.24}
\end{equation*}
$$

The matrix unit property

$$
\begin{equation*}
Q_{\Lambda_{2}, \mu \nu}^{\Lambda_{1}} Q_{\Lambda_{2}^{\prime}, \mu^{\prime} \nu^{\prime}}^{\Lambda_{1}^{\prime}}=\delta^{\Lambda_{1} \Lambda_{1}^{\prime}} \delta^{\Lambda_{2} \Lambda_{2}^{\prime}} \delta_{\nu \mu^{\prime}} Q_{\Lambda_{2}, \mu \nu^{\prime}}^{\Lambda_{1}} \tag{B.25}
\end{equation*}
$$

of the $S P_{k}(N)$ basis follows from that of the $P_{k}(N)$ units together with orthogonality of $E_{\Lambda_{2}, p}^{\Lambda_{1}, \mu}$.
$\operatorname{Tr}_{V_{N}^{\otimes k}}\left(Q_{\alpha \beta}^{\Lambda_{1}}\right)$ is given by

$$
\begin{equation*}
\operatorname{Tr}_{V_{N}^{\otimes k}}\left(Q_{\alpha \beta}^{\Lambda_{1}}\right)=\sum_{\Lambda_{1}^{\prime} \vdash N} \operatorname{Dim} V_{\Lambda_{1}^{\prime}}^{S_{N}} \chi^{\Lambda_{1}^{\prime}}\left(Q_{\alpha \beta}^{\Lambda_{1}}\right)=\sum_{\Lambda_{1}^{\prime} \vdash N} \operatorname{Dim} V_{\Lambda_{1}^{\prime}}^{S_{N}} \delta_{\alpha \beta} \delta^{\Lambda_{1} \Lambda_{1}^{\prime}}=\operatorname{Dim} V_{\Lambda_{1}}^{S_{N}} \delta_{\alpha \beta} \tag{B.26}
\end{equation*}
$$

Consequently,

$$
\begin{align*}
\operatorname{Tr}_{V_{N}^{\otimes k}}\left(Q_{\Lambda_{2}, \mu \nu}^{\Lambda_{1}}\right) & =\sum_{\alpha, \beta, p} B_{\Lambda_{1}, \alpha \rightarrow \Lambda_{2}, p ; \mu}^{P_{k}(N) \rightarrow S_{k}} B_{\Lambda_{1}, \beta \rightarrow \Lambda_{2}, p ; \nu}^{P_{k}(N) \rightarrow S_{k}} \operatorname{Tr}_{V_{N}^{\otimes k}}\left(Q_{\alpha \beta}^{\Lambda_{1}}\right) \\
& =\sum_{\alpha, \beta, p} B_{\Lambda_{1}, \alpha \rightarrow \Lambda_{2}, p ; \mu}^{P_{k}(N) \rightarrow S_{k}} B_{\Lambda_{1}, \beta \rightarrow \Lambda_{2}, p ; \nu}^{P_{k}(N) \rightarrow S_{k}} \delta_{\alpha \beta} \operatorname{Dim} V_{\Lambda_{1}}^{S_{N}}  \tag{B.27}\\
& =\sum_{p} \delta_{p p} \delta_{\mu \nu} \operatorname{Dim} V_{\Lambda_{1}}^{S_{N}}=\delta_{\mu \nu} \operatorname{Dim} V_{\Lambda_{1}}^{S_{N}} \operatorname{Dim} V_{\Lambda_{2}}^{S_{k}},
\end{align*}
$$

where the last two equalities hold if and only if the branching coefficients are non-zero.

Finally, we note that this construction gives $S_{k}$ invariant elements, i.e.

$$
\begin{equation*}
\tau Q_{\Lambda_{2}, \mu \nu}^{\Lambda_{1}} \tau^{-1}=Q_{\Lambda_{2}, \mu \nu}^{\Lambda_{1}} \quad \text { for } \tau \in S_{k} \tag{B.28}
\end{equation*}
$$

## Appendix C

## Orbit basis

In section 5.2 we described two bases for the partition algebra $P_{k}(N)$ : a diagram basis and a representation basis. Here we describe another basis, in terms of combinatorially explicit linear combinations of the diagrams from section 5.2.2. This basis is called the orbit basis [23]. We also show that it is orthogonal for any $N$ and $k$. This makes it a suitable basis to describe permutation invariant matrix quantum mechanics in the $N<2 k$ regime, a preliminary discussion of which concludes this appendix. A possible future direction is to use the orbit basis to describe how the representation basis is modified in this regime.

As in the diagram basis, the orbit basis is indexed by the set partitions $\Pi_{2 k}$ of $\left\{1, \ldots, k, 1^{\prime}, \ldots, k^{\prime}\right\}$. These are partially ordered under the relation

$$
\begin{equation*}
\pi \preceq \pi^{\prime} \quad \text { if every block of } \pi \text { is contained within a block of } \pi^{\prime} \tag{C.1}
\end{equation*}
$$

in this case we say that $\pi$ is a refinement of $\pi^{\prime}$ or equivalently that $\pi^{\prime}$ is a coarsening of $\pi$. Since we are already familiar with the diagram basis of $P_{k}(N)$ we express the orbit basis in terms of the diagram basis using the above partial ordering

$$
\begin{equation*}
d_{\pi}=\sum_{\pi \preceq \pi^{\prime}} x_{\pi^{\prime}} \tag{C.2}
\end{equation*}
$$

with $\left\{x_{\pi} \mid \pi \in \Pi_{2 k}\right\}$. The diagram basis element $d_{\pi}$ is a sum of all orbit basis elements labelled by set partitions equal to or coarser than $\pi$, for example

$$
\begin{equation*}
]_{0}^{\bullet}=0_{0}^{0}+0_{0}^{0}+0_{0}^{\infty}+0_{0}^{0} \tag{C.3}
\end{equation*}
$$

We will continue to distinguish the diagram and orbit bases by drawing diagram basis elements with black vertices and labelling them with the letter $d$, and drawing orbit basis
elements with white vertices and labelling them with the letter $x$. The transition matrix determined by (C.2) is $\zeta_{2 k}$ and is called the zeta matrix of the partially ordered set $\Pi_{2 k}$. It is upper triangular, with ones on the diagonal and hence invertible.

The inverse of $\zeta_{2 k}$ is given in [27]. It is the matrix $\mu_{2 k}$

$$
\begin{equation*}
x_{\pi}=\sum_{\pi \preceq \pi^{\prime}} \mu_{2 k}\left(\pi, \pi^{\prime}\right) d_{\pi^{\prime}} . \tag{C.4}
\end{equation*}
$$

If $\pi \preceq \pi^{\prime}$ and $\pi^{\prime}$ consists of $l$ blocks such that the $i$ th block of $\pi^{\prime}$ is the union of $b_{i}$ blocks of $\pi$ then

$$
\begin{equation*}
\mu_{2 k}\left(\pi, \pi^{\prime}\right)=\prod_{i=1}^{l}(-1)^{b_{i}-1}\left(b_{i}-1\right)! \tag{C.5}
\end{equation*}
$$

For example, this gives the following expansion of the orbit basis element labelled by $\pi=\{1|2| 3 \mid 4\}$


The orbit basis is orthogonal with respect to the inner product (5.54). We will prove,

$$
\left\langle x_{\pi} \mid x_{\pi^{\prime}}\right\rangle=\left\{\begin{array}{l}
\left|G_{\pi}\right| N_{(|\pi|)} \quad \text { if }\left[x_{\pi^{\prime}}\right]=\left[x_{\pi}\right]  \tag{C.7}\\
0 \\
\text { otherwise }
\end{array}\right.
$$

where $\pi, \pi^{\prime}$ are set partitions of $\left\{1, \ldots, k, 1^{\prime}, \ldots, k^{\prime}\right\}, N_{(l)}=N(N-1) \ldots(N-l+1)$ is the falling factorial, $|\pi|$ is the number of blocks in $\pi$, and $\left|G_{\pi}\right|$ is the order of the subgroup of $S_{k}$ that leaves $x_{\pi}$ invariant. As was the case in the diagram basis, we note that

$$
\begin{equation*}
\left|\left[x_{\pi^{\prime}}\right]\right\rangle=\left|x_{\pi^{\prime}}\right\rangle \tag{C.8}
\end{equation*}
$$

and use the RHS ket labels for the sake of notational efficiency.
First consider the simpler proposition

$$
\begin{equation*}
\operatorname{Tr}_{V_{N}^{\otimes k}}\left(x_{\pi} x_{\pi^{\prime}}^{T}\right)=N_{(|\pi|)} \delta_{\pi \pi^{\prime}} . \tag{C.9}
\end{equation*}
$$

The proof of this follows from the definition (see section 5.2 in [27]) of $x_{\pi}$ acting on $V_{N}^{\otimes k}$

$$
\left(x_{\pi}\right)_{i_{1} \ldots i_{k}}^{i_{1} \prime \ldots i_{k^{\prime}}}= \begin{cases}1 & \text { if } i_{a}=i_{b} \text { if and only if a and } \mathrm{b} \text { are in the same block of } \pi  \tag{C.10}\\ 0 & \text { otherwise }\end{cases}
$$

The trace is equal to

$$
\begin{equation*}
\operatorname{Tr}_{V_{N}^{\otimes k}}^{\otimes k}\left(x_{\pi} x_{\pi^{\prime}}^{T}\right)=\sum_{\substack{i_{1} \ldots i_{k} \\ i_{1} \ldots \ldots i_{k^{\prime}}}}\left(x_{\pi}\right)_{i_{1} \ldots i_{k}}^{i_{i_{1} \ldots i_{k}}}\left(x_{\pi^{\prime}}\right)_{i_{1} \ldots i_{k}}^{i_{1^{\prime} \ldots i_{k^{\prime}}}} \tag{C.11}
\end{equation*}
$$

Equation C.10 implies

$$
\left(x_{\pi}\right)_{i_{1} \ldots i_{k}}^{i_{1^{\prime}} \ldots i_{k^{\prime}}}\left(x_{\pi^{\prime}}\right)_{i_{1} \ldots i_{k}}^{i_{1^{\prime}} \ldots i_{k^{\prime}}}= \begin{cases}1 & \text { if } i_{a}=i_{b} \text { if and only if } a \text { and } b \text { are in the same }  \tag{C.12}\\ \text { block of } \pi \text { and the same block of } \pi^{\prime} \\ 0 & \text { otherwise }\end{cases}
$$

If $\pi \neq \pi^{\prime}$ two situations exist. Consider the set of all pairs $(a, b)$ for $a, b=1, \ldots, k, 1^{\prime}, \ldots, k^{\prime}$ such that $a$ and $b$ are in the same block of $\pi$. Since $\pi \neq \pi^{\prime}$ at least one of these pairs are such that $a$ and $b$ are in different blocks of $\pi^{\prime}$. The second case is the reverse. Consider the set of all $(a, b)$ such that $a$ and $b$ are in the same block of $\pi^{\prime}$. Then $\pi^{\prime} \neq \pi$ implies that there exists at least one pair such that $a$ and $b$ are not in the same block of $\pi$. In that case, there are no choices of $i_{a}, i_{b}$ which satisfy the first criteria in C.12. For example, take $a, b$ to be in the same block of $\pi$ but different blocks of $\pi^{\prime}$. The matrix elements $\left(x_{\pi}\right)_{i_{1} \ldots i_{k}}^{i_{1} \ldots i_{k}}$ vanish if $i_{a} \neq i_{b}$ while the matrix elements $\left(x_{\pi^{\prime}}\right)_{i_{1} \ldots i_{k}}^{i_{1^{\prime} \ldots}}$ vanish unless $i_{a} \neq i_{b}$. Therefore, the product identically vanishes,

$$
\begin{equation*}
\left(x_{\pi}\right)_{i_{1} \ldots i_{k}}^{i_{1} \ldots i_{k^{\prime}}}\left(x_{\pi^{\prime}}\right)_{i_{1} \ldots i_{k}}^{i_{1} \ldots i_{k^{\prime}}}=\delta_{\pi \pi^{\prime}}\left(x_{\pi}\right)_{i_{1} \ldots i_{k}}^{i_{1} \prime \ldots i_{k^{\prime}}} \tag{C.13}
\end{equation*}
$$

and

$$
\begin{equation*}
\operatorname{Tr}_{V_{N}^{\otimes k}}^{\otimes}\left(x_{\pi} x_{\pi^{\prime}}^{T}\right)=\sum_{\substack{i_{1} \ldots i_{k} \\ i_{1} \ldots i_{k^{\prime}}}}\left(x_{\pi}\right)_{i_{1} \ldots i_{k}}^{i_{1} \ldots i_{k^{\prime}}} \delta_{\pi \pi^{\prime}}=\delta_{\pi \pi^{\prime}} N(N-1) \ldots(N-|\pi|+1) \tag{C.14}
\end{equation*}
$$

The last equality is a consequence of (C.10). For example, consider the set partition $12 \mid 1^{\prime} 2^{\prime}$. The trace of $x_{12 \mid 1^{\prime} 2^{\prime}}$ is

$$
\begin{equation*}
\operatorname{Tr}_{V_{N}^{\otimes 2}}\left(x_{12 \mid 1^{\prime} 2^{\prime}}\right)=\sum_{i_{1}=i_{2} \neq i_{3}, i_{3}=i_{4}}=N(N-1) \tag{C.15}
\end{equation*}
$$

since we have $N$ choices of indices for $i_{1}$ and $(N-1)$ choices for $i_{3}$ (for every choice of $i_{1}$ ).

The general case is analogous,

$$
\begin{equation*}
\operatorname{Tr}_{V_{N}^{\otimes k}}\left(x_{\pi}\right)=N_{(|\pi|)} . \tag{C.16}
\end{equation*}
$$

We have $N$ choices for the indices of the first block of $\pi, N-1$ choices for the indices of the second block and so on.

The inner product of two orbit basis elements of $S P_{k}(N)$ is given by 5.54

$$
\begin{equation*}
\left\langle x_{\pi} \mid x_{\pi^{\prime}}\right\rangle=\sum_{\gamma \in S_{k}} \operatorname{Tr}_{V_{N}^{\otimes k}}\left(\gamma x_{\pi} \gamma^{-1} x_{\pi^{\prime}}^{T}\right) . \tag{C.17}
\end{equation*}
$$

We re-write

$$
\begin{equation*}
\sum_{\gamma \in S_{k}} \gamma x_{\pi} \gamma^{-1}=\left|G_{\pi}\right| \sum_{\lambda \in[\pi]} x_{\lambda}, \tag{C.18}
\end{equation*}
$$

where the sum on the RHS is over the distinct elements in the $S_{k}$ orbit of $x_{\pi}$. Substituting this into the trace gives

$$
\left\langle x_{\pi} \mid x_{\pi^{\prime}}\right\rangle=\left|G_{\pi}\right| \sum_{\lambda \in[\pi]} \operatorname{Tr}_{V_{N}^{\otimes k}}\left(x_{\lambda} x_{\pi^{\prime}}\right)=\left|G_{\pi}\right| \sum_{\lambda \in[\pi]} N_{(|\pi|)} \delta_{\lambda \pi^{\prime}}=\left\{\begin{array}{l}
\left|G_{\pi}\right| N_{(|\pi|)} \quad \text { if }\left[x_{\pi^{\prime}}\right]=\left[x_{\pi}\right]  \tag{C.19}\\
0 \quad \text { otherwise },
\end{array}\right.
$$

where $\left[x_{\pi}\right]$ denotes $S_{k}$ symmetrisation as in equation (5.47).
For the majority of this chapter 5 we assume $N \geq 2 k$ in order to take advantage of the many simplifications that occur in this limit. However, utilising results from the partition algebra literature we are able to say something about what happens below this limit, in which we expect to encounter finite $N$ effects.

In the limit $N \geq 2 k$ the map from the partition algebra to $\operatorname{End}_{S_{N}}\left(V_{N}^{\otimes k}\right)$ is bijective. When $N<2 k$ this map acquires a non-trivial kernel (but remains surjective). Accordingly, we expect a reduction in the size of the state space $\mathcal{H}_{\text {inv }}$. This reduction is most easily expressed in the orbit basis of $P_{k}(N)$. Theorem 5.17 (a) in [27] states that if $N \in \mathbb{Z} \geq 1$ and $\left\{x_{\pi} \mid \pi \in \Pi_{2 k}\right\}$ is the orbit basis for $P_{k}(N)$ then for $k \in \mathbb{Z}_{\geq 1}$, the representation $\Phi_{k, N}: P_{k}(N) \rightarrow \operatorname{End}\left(V_{N}^{\otimes k}\right)$ has the following image and kernel

$$
\begin{align*}
\operatorname{im}\left(\Phi_{k, N}\right) & =\operatorname{End}_{S_{N}}\left(V_{N}^{\otimes k}\right)=\operatorname{span}_{\mathbb{C}}\left\{\Phi_{k, N}\left(x_{\pi}\right) \mid \pi \in \Pi_{2 k} \text { has } \leq N \text { blocks }\right\}, \\
\operatorname{ker}\left(\Phi_{k, N}\right) & =\operatorname{span}_{\mathbb{C}}\left\{x_{\pi} \mid \pi \in \Pi_{2 k} \text { has }>N \text { blocks }\right\} . \tag{C.20}
\end{align*}
$$

Due to the bosonic symmetry of our theory we are actually interested in the map from the symmetrised partition algebra $S P_{k}(N)$, defined in 5.47), to $\operatorname{End}\left(V_{N}^{\otimes k}\right)$. To this end we note that the definition of the kernel of $\Phi_{k, N}$ given in C.20) is $S_{k}$ invariant. If one
element of an $S_{k}$ orbit is in the kernel then C.20 tells us that the entire orbit belongs to the kernel - the action of $S_{k}$ does not change the number of blocks in a partition $\pi$. The image and kernel of this map are the following

$$
\begin{align*}
\operatorname{im}\left(\Phi_{k, N}\right) & =\operatorname{span}_{\mathbb{C}}\left\{\left.[b]=\frac{1}{k!} \sum_{\gamma \in S_{k}} \gamma b \gamma^{-1} \right\rvert\, b=\Phi_{k, N}\left(x_{\pi}\right), \forall \pi \in \Pi_{2 k} \text { with } \leq N \text { blocks }\right\} \\
\operatorname{ker}\left(\Phi_{k, N}\right) & =\operatorname{span}_{\mathbb{C}}\left\{\left[x_{\pi}\right] \mid \pi \in \Pi_{2 k}, \pi \text { has }>N \text { blocks }\right\} \tag{C.21}
\end{align*}
$$

Therefore a state basis is given by $\left|\left[x_{\pi}\right]\right\rangle$ for $\pi$ having $N$ or fewer blocks, this basis is orthogonal for all $N$, including for $N<2 k$.

The original statement C.20 applies to multi-matrix theories in which observables are constructed from distinct matrices - in this case there is no bosonic $S_{k}$ symmetry to account for. If a state in this theory is null then all states generated by the action of $S_{k}$ on this state will also be null. The equivalent of (5.48) for the multi-matrix case is

$$
\begin{equation*}
|d\rangle=\sum_{\substack{i_{1}, \ldots, i_{k} \\ i_{1^{\prime}}, \ldots, i_{k^{\prime}}}}(d)_{i_{1} \ldots i_{k}}^{i_{1} \ldots i_{k^{\prime}}}\left(a_{1}^{\dagger}\right)_{i_{1^{\prime}}}^{i_{1}} \ldots\left(a_{k}^{\dagger}\right)_{i_{k^{\prime}}}^{i_{k}}|0\rangle=\operatorname{Tr}_{V_{N} \otimes k}\left[d\left(a_{1}^{\dagger} \otimes \cdots \otimes a_{k}^{\dagger}\right)\right]|0\rangle \tag{C.22}
\end{equation*}
$$

in which we have $k$ distinct oscillators and each element $d$ in the full partition algebra $P_{k}(N)$ corresponds to a unique state, instead of $S_{k}$ equivalence classes $[d] \in S P_{k}(N)$. We illustrate with the following examples that under the map C.22 elements $d \in P_{k}(N)$ that are in the kernel of $\Phi_{k, N}$ label zero vectors in the Hilbert space $\mathcal{H}$. For $k=2$ and $N=1$ we see

$$
\begin{align*}
|\emptyset \emptyset\rangle & =|\boldsymbol{d}|\rangle-|\vec{\bullet}\rangle \\
& =\left[\sum_{i, j}\left(a_{1}^{\dagger}\right)_{i}^{i}\left(a_{2}^{\dagger}\right)_{j}^{j}-\sum_{i}\left(a_{1}^{\dagger}\right)_{i}^{i}\left(a_{2}^{\dagger}\right)_{i}^{i}\right]|0\rangle \\
& =\left[\left(a_{1}^{\dagger}\right)_{1}^{1}\left(a_{2}^{\dagger}\right)_{1}^{1}-\left(a_{1}^{\dagger}\right)_{1}^{1}\left(a_{2}^{\dagger}\right)_{1}^{1}\right]|0\rangle \\
& =0 \tag{C.23}
\end{align*}
$$

in the first line we have used (C.4) to express the orbit basis element in terms of the diagram basis. Similarly, taking $k=2$ and $N=2$ we have

$$
\begin{align*}
& \left|\begin{array}{ll}
\circ & 0 \\
0 & 0
\end{array}\right\rangle=\left|\begin{array}{ll}
\bullet & \bullet \\
\bullet & \bullet
\end{array}\right\rangle-\left|\begin{array}{ll}
\bullet & \bullet
\end{array}\right\rangle-\left|\begin{array}{ll}
\bullet & \bullet \\
\bullet & \bullet
\end{array}\right\rangle-\left|\begin{array}{ll}
\bullet & \bullet \\
\bullet & \bullet
\end{array}\right\rangle+2|\bullet . \quad . \quad\rangle \\
& =\left[\sum_{i, j, k}\left(a_{1}^{\dagger}\right)_{j}^{i}\left(a_{2}^{\dagger}\right)_{k}^{k}-\sum_{i, j}\left(a_{1}^{\dagger}\right)_{i}^{i}\left(a_{2}^{\dagger}\right)_{j}^{j}-\sum_{i, j}\left(a_{1}^{\dagger}\right)_{j}^{i}\left(a_{2}^{\dagger}\right)_{j}^{j}-\sum_{i, j}\left(a_{1}^{\dagger}\right)_{j}^{i}\left(a_{2}^{\dagger}\right)_{i}^{i}+2 \sum_{i}\left(a_{1}^{\dagger}\right)_{i}^{i}\left(a_{2}^{\dagger}\right)_{i}^{i}\right]|0\rangle \\
& =0 \text {. } \tag{C.24}
\end{align*}
$$

We can split the first term by imposing different restrictions on the ranges of the sum

$$
\begin{equation*}
\sum_{i, j, k}=\sum_{i=j=k}+\sum_{\substack{i=j \\ j \neq k}}+\sum_{\substack{i=k \\ k \neq j}}+\sum_{\substack{j=k \\ k \neq i}}+\sum_{i \neq j \neq k} \tag{C.25}
\end{equation*}
$$

Similarly, we can split the second, third and fourth terms

$$
\begin{equation*}
\sum_{i, j}=\sum_{i=j}+\sum_{i \neq j} . \tag{C.26}
\end{equation*}
$$

The terms in (C.24) cancel due to the equivalence of coarsening diagrams and restricting summation ranges - adding edges to a diagram $d \in P_{k}(N)$ is equivalent to evaluating the original diagram $d$ over a restricted summation range. Another way of saying this is that (C.2) and C.25) encode identical expansions, in fact the five terms in each expansion give equivalent contributions. Orbit basis elements label states in which the oscillator indices are summed over the restricted range $i_{1} \neq i_{2} \neq \cdots \neq i_{m}$ where $m$ is the number of blocks in the orbit basis element. From this perspective it is easy to see that these states must be zero when $N<m$ as there are not enough distinct values in $[1, N]$ to satisfy the inequality defining the summation range. Contrastingly, the diagram basis produces states corresponding to sums with unrestricted indices. Although at finite $N$ there is a stark difference between states in the orbit and diagram bases, at large $N$ the two descriptions are equivalent.

Elements of $S P_{k}(N)$ are $S_{k}$ orbits on $P_{k}(N)$ and so states in $\mathcal{H}_{\text {inv }}^{(k)}$ are linear combinations of states in $\mathcal{H}$. If a state in $\mathcal{H}$ is labelled by a partition algebra element in the kernel of $\Phi_{k, N}$, the state in $\mathcal{H}_{\text {inv }}^{(k)}$ generated by the action of $S_{k}$ on this zero $\mathcal{H}$ state will also be zero. It is clear that if an element $d \in P_{k}(N)$ produces a zero vector under (C.22) then the equivalence class $[d] \in S P_{k}(N)$ containing that element $d \in P_{k}(N)$ also produces a zero vector under the map to $\mathcal{H}_{\text {inv }}^{(k)}$

$$
\begin{equation*}
|d\rangle=\sum_{\substack{i_{1}, \ldots, i_{k} \\ i_{1}, \ldots, i_{k^{\prime}}}}([d]]_{i_{1} \ldots i_{k}}^{i_{1} \ldots i_{k}}\left(a^{\dagger}\right)_{i_{1^{\prime}}}^{i_{1}} \ldots\left(a^{\dagger}\right)_{i_{k^{\prime}}}^{i_{k}}|0\rangle=\operatorname{Tr}_{V_{N}^{\otimes k}}\left([d]\left(a^{\dagger}\right)^{\otimes k}\right)|0\rangle . \tag{C.27}
\end{equation*}
$$

We can also check that for suitably low values of $N$ the norm of the orbit basis states vanishes. For $x_{\pi_{1}}={ }_{\circ}{ }^{\circ}{ }^{\circ}$ we expect

$$
\begin{equation*}
\left.\left\langle x_{\pi_{1}} \mid x_{\pi_{1}}\right\rangle\right|_{N<4}=\left.g_{x_{\pi_{1}} x_{\pi_{1}}}\right|_{N<4}=0 . \tag{C.28}
\end{equation*}
$$

Indeed, substituting (C.6 into this expression gives

$$
\begin{aligned}
\left\langle x_{\pi_{1}} \mid x_{\pi_{1}}\right\rangle & =\left\langle d_{\pi_{1}} \mid d_{\pi_{1}}\right\rangle-\left\langle d_{\pi_{1}} \mid d_{\pi_{2}}\right\rangle-\left\langle d_{\pi_{2}} \mid d_{\pi_{1}}\right\rangle+\left\langle d_{\pi_{2}} \mid d_{\pi_{2}}\right\rangle+\cdots-12\left\langle d_{\pi_{14}} \mid d_{\pi_{15}}\right\rangle+36\left\langle d_{\pi_{15}} \mid d_{\pi_{15}}\right\rangle \\
& =N(N-1)(N-2)(N-3)
\end{aligned}
$$

which is zero for $N<4$.
 diagram basis expansion

The norm of this state is

$$
\begin{align*}
\left\langle x_{\pi_{2}} \mid x_{\pi_{2}}\right\rangle & =\left\langle\begin{array}{cc|c}
\bullet & \bullet & \bullet \\
\bullet & \bullet
\end{array}\right\rangle-\left\langle\begin{array}{cc|c}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet
\end{array}\right)-\cdots+4\langle\bullet \cdot||\cdot\rangle \\
& =N(N-1)(N-2) \tag{C.30}
\end{align*}
$$

which does vanish for $N<3$. For a general orbit basis state $x_{\pi}$ we expect the norm to be some polynomial in $N$ which vanishes for any $N<|\pi|$.

## Appendix D

## Computing low degree matrix units

In this appendix we find the full set of matrix units for $k=2$. As discussed in 5.3.3, we use the following elements of $S P_{2}(N)$ to distinguish the full set of labels on matrix units $Q_{\Lambda_{2}, \mu \nu}^{\Lambda_{1}}$. The irreducible representation $\Lambda_{1} \vdash N$ is distinguished by

$$
\begin{equation*}
\bar{T}_{2}{ }^{(2)}=\frac{(N-2)(N-3)-4}{2} \mathfrak{\bullet}+\mathfrak{\bullet}+\mathbf{\bullet}+\underset{+}{\bullet}+\mathbf{C}+N!\mathbf{~} \tag{D.1}
\end{equation*}
$$

while $\Lambda_{2} \vdash k$ is distinguished by

$$
\begin{equation*}
t_{2}^{(2)}=\mathcal{X} \tag{D.2}
\end{equation*}
$$

and multiplicity labels $\mu, \nu$ are distinguished by acting with

$$
\begin{equation*}
\bar{T}_{2,1}^{(2)}=!\cdot+\stackrel{\bullet}{\bullet}, \tag{D.3}
\end{equation*}
$$

on the left and right. It will be useful to know that $\bar{T}_{2,1}^{(2)}$ is related to

$$
\begin{equation*}
\bar{T}_{2}^{(1)}=\frac{N(N-3)}{2}!+\stackrel{\bullet}{\bullet}, \tag{D.4}
\end{equation*}
$$

since

$$
\begin{equation*}
\left.\bar{T}_{2}^{(1)} \otimes 1+1 \otimes \bar{T}_{2}^{(1)}=\stackrel{\bullet}{\bullet}+\mathfrak{\bullet}+N(N-3)\right\rfloor! \tag{D.5}
\end{equation*}
$$

As we will now explain, the eigenvalues of $\bar{T}_{2,1}^{(2)}$ uniquely determine the labels $\mu, \nu$ by left and right action respectively. For fixed $\Lambda_{1}, \Lambda_{2}$ the multiplicity labels correspond to basis
elements for $V_{\Lambda_{1}, \Lambda_{2}}^{P_{2}(N) \rightarrow \mathbb{C}\left[S_{2}\right]}$, appearing in the decomposition

$$
\begin{align*}
V_{N} \otimes V_{N} \cong & \left(V_{[N]}^{S_{N}} \otimes V_{[2]}^{S_{2}} \otimes V_{[N],[2]}^{P_{2}(N) \rightarrow \mathbb{C}\left[S_{2}\right]}\right) \oplus\left(V_{[N-1,1]}^{S_{N}} \otimes V_{[2]}^{S_{2}} \otimes V_{[N-1,1],[2]}^{P_{2}(N) \rightarrow \mathbb{C}\left[S_{2}\right]}\right) \oplus \\
& \left(V_{[N-1,1]}^{S_{N}} \otimes V_{[1,1]}^{S_{2}} \otimes V_{[N-1,1],[1,1]}^{P_{2}(N) \rightarrow \mathbb{C}\left[S_{2}\right]}\right) \oplus\left(V_{[N-2,2]}^{S_{N}} \otimes V_{[2]}^{S_{2}} \otimes V_{[N-2,2],[2]}^{P_{2}(N) \rightarrow \mathbb{C}\left[S_{2}\right]}\right) \oplus \\
& \left(V_{[N-2,1,1]}^{S_{N}} \otimes V_{[1,1]}^{S_{2}} \otimes V_{[N-2,1,1],[1,1]}^{P_{2}(N) \rightarrow \mathbb{C}\left[S_{2}\right]}\right) . \tag{D.6}
\end{align*}
$$

On the right hand side, $\bar{T}_{2,1}^{(2)}$ acts on the vector spaces $V_{\Lambda_{1} \Lambda_{2}}^{P_{2}(N) \rightarrow \mathbb{C}\left[S_{2}\right]}$ with dimensions

$$
\begin{align*}
& \operatorname{Dim} V_{[N],[2]}^{P_{2}(N) \rightarrow \mathbb{C}\left[S_{2}\right]}=2, \quad \operatorname{Dim} V_{[N-1,1],[2]}^{P_{2}(N) \rightarrow \mathbb{C}\left[S_{2}\right]}=2, \quad \operatorname{Dim} V_{[N-1,1],[1,1]}^{P_{2}(N) \rightarrow \mathbb{C}\left[S_{2}\right]}=1 \\
& \operatorname{Dim} V_{[N-2,2],[2]}^{P_{2}(N) \rightarrow \mathbb{C}\left[S_{2}\right]}=1, \quad \operatorname{Dim} V_{[N-2,1,1],[1,1]}^{P_{2}(N) \rightarrow \mathbb{C}\left[S_{2}\right]}=1 . \tag{D.7}
\end{align*}
$$

We will find that $\bar{T}_{2,1}^{(2)}$ has precisely as many distinct eigenvalues (in each subspace) as the corresponding dimension.

To confirm that this is the case, note that $\bar{T}_{2,1}^{(2)}$ acts on $V_{N}^{\otimes 2}$ as

$$
\begin{equation*}
\bar{T}_{2,1}^{(2)}\left(e_{i_{1}} \otimes e_{i_{2}}\right)=\bar{T}_{2}^{(1)} e_{i_{1}} \otimes e_{i_{2}}+e_{i_{1}} \otimes \bar{T}_{2}^{(1)} e_{i_{2}}-N(N-3) e_{i_{1}} \otimes e_{i_{2}} \tag{D.8}
\end{equation*}
$$

It follows that the eigenvalues are directly related to the eigenvalues of $\bar{T}_{2}^{(1)}$ defined in (5.118). These are known by the decompsition

$$
\begin{equation*}
V_{N} \cong V_{[N]}^{S_{N}} \oplus V_{[N-1,1]}^{S_{N}} \tag{D.9}
\end{equation*}
$$

where $\bar{T}_{2}^{(1)}$ acts on each summand by a normalised character. Using this on the left hand side of (D.6) gives
$V_{N} \otimes V_{N} \cong\left(V_{[N]}^{S_{N}} \otimes V_{[N]}^{S_{N}}\right) \oplus\left(V_{[N]}^{S_{N}} \otimes V_{[N-1,1]}^{S_{N}}\right) \oplus\left(V_{[N-1,1]}^{S_{N}} \otimes V_{[N]}^{S_{N}}\right) \oplus\left(V_{[N-1,1]}^{S_{N}} \otimes V_{[N-1,1]}^{S_{N}}\right)$.
Consequently, the three distinct eigenvalues of $\bar{T}_{2,1}^{(2)}$ are (one for each summand, but the vectors in the second and third space have the same eigenvalue)

$$
\begin{align*}
2 \frac{\chi^{[N]}\left(T_{2}\right)}{\operatorname{Dim} V_{[N]}^{S_{N}}}-N(N-3) & =N(N-1)-N(N-3)=2 N,  \tag{D.11}\\
2 \frac{\chi^{[N-1,1]}\left(T_{2}\right)}{\operatorname{Dim} V_{[N-1,1]}^{S_{N}}}-N(N-3) & =N(N-3)-N(N-3)=0,  \tag{D.12}\\
\frac{\chi^{[N]}\left(T_{2}\right)}{\operatorname{Dim} V_{[N]}^{S_{N}}}+\frac{\chi^{[N-1,1]}\left(T_{2}\right)}{\operatorname{Dim} V_{[N-1,1]}^{S_{N}}}-N(N-3) & =\frac{1}{2} N(N-1)+\frac{1}{2} N(N-3)-N(N-3)=N . \tag{D.13}
\end{align*}
$$

By decomposing (D.10) into $S_{N} \times S_{k}$ representations we will see that the multiplicities in (D.6) are uniquely associated with one of the above eigenvalues. We start by considering the multiplicities of $V_{[N]}^{S_{N}} \otimes V_{[2]}^{S_{2}}$. The representation $V_{[N]}^{S_{N}}$ occurs in the decomposition (D.10) as subspaces

$$
\begin{equation*}
V_{[N]}^{S_{N}} \cong V_{[N]}^{S_{N}} \otimes V_{[N]}^{S_{N}} \quad \text { and } \quad V_{[N]}^{S_{N}} \subset V_{[N-1,1]}^{S_{N}} \otimes V_{[N-1,1]}^{S_{N}} \tag{D.14}
\end{equation*}
$$

The first subspace has eigenvalue $2 N$, while the second subspace has eigenvalue 0 with respect to $\bar{T}_{2,1}^{(2)}$. Therefore, the two multiplicities are distinguished. Next we consider multiple occurances of $V_{[N-1,1]}^{S_{N}}$. The two spaces

$$
\begin{equation*}
\left(V_{[N]}^{S_{N}} \otimes V_{[N-1,1]}^{S_{N}}\right) \oplus\left(V_{[N-1,1]}^{S_{N}} \otimes V_{[N]}^{S_{N}}\right) \tag{D.15}
\end{equation*}
$$

combine into representations of $S_{N} \times S_{2}$ as

$$
\begin{equation*}
\left(V_{[N-1,1]}^{S_{N}} \otimes V_{[2]}^{S_{2}}\right) \oplus\left(V_{[N-1,1]}^{S_{N}} \otimes V_{[1,1]}^{S_{2}}\right) . \tag{D.16}
\end{equation*}
$$

Both of these spaces have eigenvalue $N$ with respect to $\bar{T}_{2,1}^{(2)}$, but they are distinguished by their $S_{2}$ representation (or equivalently eigenvalue of $t_{2}^{(2)}$ ). The symmetric part of $V_{[N-1,1]}^{S_{N}} \otimes V_{[N-1,1]}^{S_{N}}$ has a subspace

$$
\begin{equation*}
V_{[N-1,1]}^{S_{N}} \otimes S_{[2]}^{S_{2}} \subset V_{[N-1,1]}^{S_{N}} \otimes V_{[N-1,1]}^{S_{N}} \tag{D.17}
\end{equation*}
$$

with eigenvalue 0 . We have found that the two subspaces $V_{[N-1,1]}^{S_{N}} \otimes V_{[2]}^{S_{2}}$ are distinguished by the eigenvalues $N$ and 0 with respect to $\bar{T}_{2,1}^{(2)}$. The last two terms in (D.6) are multiplicity free and uniquely determined by their eigenvalue with respect to $T_{2}^{(2)}$,

In the Sage code, we simultaneously diagonalised all the operators by considering a linear combination

$$
\begin{equation*}
T=a \bar{T}_{2}^{(2)}+b t_{2}^{(2)}+c \bar{T}_{2,1}^{(2), L}+f \bar{T}_{2,1}^{(2), R} \tag{D.18}
\end{equation*}
$$

with $a, b, c, f \in \mathbb{R}$ such that there is no eigenvalue degeneracy in $T$. The superscript $L$ means left action and $R$ means right action. An eigenbasis for $T$ will be a simultaneous eigenbasis for $\left\{\bar{T}_{2}{ }^{(2)}, t_{2}^{(2)}, \bar{T}_{2,1}^{(2), L}, \bar{T}_{2,1}^{(2), R}\right\}$, which corresponds to a basis of matrix units. In the implementation, these operators act on $P_{2}(N)$, as opposed to $S P_{2}(N)$. The projection to $S P_{2}(N)$ was implemented by adding a fifth operator $P^{S P_{2}(N)}$ to $T$. The action of $P^{S P_{2}(N)}$ on $d \in P_{2}(N)$ is $d \mapsto[d]$. It commutes with all of the previous operators. This was useful in practice, since elements in $S P_{2}(N)$ will have eigenvalue 1 with respect to $P^{S P_{2}(N)}$ (the orthogonal complement has eigenvalue 0 ).

The matrix units for $k=2$ are given below. The multiplicity labels have been chosen to
correspond to eigenvalues of $\bar{T}_{2,1}^{(2), L}$ and $\bar{T}_{2,1}^{(2), R}$ as follows

$$
\begin{align*}
& 1 \leftrightarrow 2 N, \\
& 2 \leftrightarrow 0  \tag{D.19}\\
& 3 \leftrightarrow N
\end{align*}
$$

The elements below have not gone through the final step of being normalised.

$$
\begin{align*}
& \left(Q_{[2]}^{[N]}\right)_{11}=\text { • • , }  \tag{D.20}\\
& \left(Q_{[2]}^{[N]}\right)_{21}=-\frac{1}{N} \bullet \bullet+\stackrel{\bullet}{\bullet}  \tag{D.21}\\
& \left(Q_{[2]}^{[N-1,1]}\right)_{33}=-\frac{4}{N} \bullet \bullet+\mathfrak{\bullet} \cdot+\bullet \bullet+\bullet \bullet+\bullet \bullet,  \tag{D.22}\\
& \left(Q_{[1,1]}^{[N-1,1]}\right)_{33}=-\downarrow \bullet+\bullet .+. \bullet-\bullet .,  \tag{D.23}\\
& \left(Q_{[2]}^{[N-1,1]}\right)_{23}=\frac{4}{N^{2}} \bullet \bullet-\frac{2}{N} \bullet \bullet-\frac{1}{N} \bullet \bullet+\bullet \bullet-\frac{1}{N} \bullet \bullet-\frac{1}{N} \bullet \bullet+\bullet \bullet-\frac{1}{N} \bullet \bullet \text { • } \bullet  \tag{D.24}\\
& \left(Q_{[2]}^{[N]}\right)_{12}=-\frac{1}{N} \bullet \bullet+\stackrel{\bullet}{ } \text { • , }  \tag{D.25}\\
& \left(Q_{[2]}^{[N-1,1]}\right)_{32}=\frac{4}{N^{2}} \bullet \bullet-\frac{2}{N} \bullet \bullet-\frac{1}{N} \bullet \bullet+\mathfrak{\bullet}-\frac{1}{N} \bullet \bullet-\frac{1}{N} \bullet \bullet+\bullet \bullet-\frac{1}{N} \bullet \bullet \text { • }  \tag{D.26}\\
& \left(Q_{[2]}^{[N]}\right)_{22}=\frac{1}{N^{2}} \bullet \bullet-\frac{1}{N} \bullet \bullet-\frac{1}{N} \curvearrowleft \bullet+\stackrel{\bullet}{\bullet},  \tag{D.27}\\
& \left(Q_{[2]}^{[N-1,1]}\right)_{22}=-\frac{4}{N^{3}} \bullet \bullet+\frac{2}{N^{2}} \bullet \bullet+\frac{1}{N^{2}} \text { • • }-\frac{1}{N} \bullet \bullet+\frac{1}{N^{2}} \bullet \bullet+\frac{2}{N^{2}} \bullet \bullet
\end{align*}
$$

$\left(Q_{[2]}^{[N-2,2]}\right)_{22}=-\left(\frac{1}{N^{2}-N}\right) \bullet \bullet+\left(\frac{1}{N^{2}-N}\right) \bullet \bullet+\frac{1}{2 N} \bullet \bullet-\frac{1}{N} \downarrow \bullet$
$+\frac{1}{2 N} \cdot \stackrel{\bullet}{ }+\left(\frac{1}{N^{2}-N}\right) \bullet \bullet-\left(\frac{1}{N^{2}-N}\right) \curvearrowleft-\frac{1}{N}$ • $+\ldots-\frac{1}{N} \cdot+\frac{1}{2 N} \cdot\left(\frac{-\frac{1}{2} N+1}{N}\right)$. $\left.\left.-\frac{1}{N} \cdot\right\rfloor+\frac{1}{2 N} \cdot\right\rfloor+\left(\frac{-\frac{1}{2} N+1}{N}\right)!\cdot$,
$\left(Q_{[1,1]}^{[N-2,1,1]}\right)_{22}=\frac{1}{N} \bullet \bullet-\frac{1}{N} \bullet \bullet-\frac{1}{N} \bullet \bullet+\searrow+\frac{1}{N} \bullet \bullet-\bullet .$.
For example,

$$
\begin{equation*}
\left(Q_{[2]}^{[N]}\right)_{11}\left(Q_{[2]}^{[N]}\right)_{11}=N^{2}\left(Q_{[2]}^{[N]}\right)_{11} \tag{D.31}
\end{equation*}
$$

and the properly normalised matrix unit is given by

$$
\begin{equation*}
\frac{\left(Q_{[2]}^{[N]}\right)_{11}}{N^{2}} . \tag{D.32}
\end{equation*}
$$

## Bibliography

[1] George Barnes, Adrian Padellaro, and Sanjaye Ramgoolam. Permutation invariant Gaussian 2-matrix models. arXiv:2104.03707 [hep-th], April 2021. arXiv: 2104.03707.
[2] George Barnes, Adrian Padellaro, and Sanjaye Ramgoolam. Hidden symmetries and large n factorisation for permutation invariant matrix observables. Journal of High Energy Physics, 2022(8):1-33, 2022.
[3] George Barnes, Adrian Padellaro, and Sanjaye Ramgoolam. Permutation symmetry in large n matrix quantum mechanics and partition algebras. Physical Review D, 106(10), nov 2022.
[4] George Barnes, Sanjaye Ramgoolam, and Michael Stephanou. Permutation invariant gaussian matrix models for financial correlation matrices. 2023.
[5] William Fulton and Joe Harris. Representation theory: a first course, volume 129. Springer Science \& Business Media.
[6] Steve Corley, Antal Jevicki, and Sanjaye Ramgoolam. Exact correlators of giant gravitons from dual $N=4$ SYM theory. Adv. Theor. Math. Phys., 5:809-839, 2002.
[7] Yusuke Kimura and Sanjaye Ramgoolam. Branes, anti-branes and brauer algebras in gauge-gravity duality. $J H E P, 11: 078,2007$.
[8] T. W. Brown, P. J. Heslop, and S. Ramgoolam. Diagonal multi-matrix correlators and BPS operators in $N=4$ SYM. Journal of High Energy Physics, 2008(02):030030, February 2008. arXiv: 0711.0176.
[9] Rajsekhar Bhattacharyya, Storm Collins, and Robert de Mello Koch. Exact MultiMatrix Correlators. JHEP, 03:044, 2008.
[10] Rajsekhar Bhattacharyya, Robert de Mello Koch, and Michael Stephanou. Exact Multi-Restricted Schur Polynomial Correlators. JHEP, 06:101, 2008.
[11] Thomas William Brown, P. J. Heslop, and S. Ramgoolam. Diagonal free field matrix correlators, global symmetries and giant gravitons. JHEP, 04:089, 2009.
[12] Yusuke Kimura and Sanjaye Ramgoolam. Enhanced symmetries of gauge theory and resolving the spectrum of local operators. Phys. Rev. D, 78:126003, 2008.
[13] Jurgis Pasukonis and Sanjaye Ramgoolam. Quivers as Calculators: Counting, Correlators and Riemann Surfaces. JHEP, 04:094, 2013.
[14] David Berenstein. Extremal chiral ring states in the $A d S / C F T$ correspondence are described by free fermions for a generalized oscillator algebra. Phys. Rev. D, 92(4):046006, 2015.
[15] Pawel Caputa, Robert de Mello Koch, and Pablo Diaz. A basis for large operators in $N=4$ SYM with orthogonal gauge group. JHEP, 03:041, 2013.
[16] Yusuke Kimura, Sanjaye Ramgoolam, and Ryo Suzuki. Flavour singlets in gauge theory as Permutations. JHEP, 12:142, 2016.
[17] Hai Lin and Yuwei Zhu. Entanglement and mixed states of Young tableau states in gauge/gravity correspondence. Nucl. Phys. B, 972:115572, 2021.
[18] F. Aprile, J. M. Drummond, P. Heslop, H. Paul, F. Sanfilippo, M. Santagata, and A. Stewart. Single particle operators and their correlators in free $\mathcal{N}=4$ SYM. JHEP, 11:072, 2020.
[19] Christopher Lewis-Brown and Sanjaye Ramgoolam. BPS operators in $\mathcal{N}=4 S O(N)$ super Yang-Mills theory: plethysms, dominoes and words. JHEP, 11:035, 2018.
[20] Sanjaye Ramgoolam. Schur-Weyl duality as an instrument of Gauge-String duality. AIP Conf. Proc., 1031(1):255-265, 2008.
[21] Sanjaye Ramgoolam. Permutations and the combinatorics of gauge invariants for general N. arXiv:1605.00843 (hep-th], May 2016. arXiv: 1605.00843.
[22] Paul Martin. Temperley-lieb algebras for non-planar statistical mechanics - the partition algebra construction. Journal of Knot Theory and Its Ramifications, 03(01):5182, March 1994.
[23] V.F.R. Jones. The Potts model and the symmetric group. Subfactors: Proceedings of the Taniguchi Symposium on Operator Algebras, pages 259-267, 1994.
[24] Paul Martin. The Structure of the Partition Algebras. Journal of Algebra, 183(2):319-358, July 1996.
[25] Tom Halverson. Characters of the partition algebras. Journal of Algebra, 238(2):502533, 2001.
[26] John Enyang. Jucys-murphy elements and a presentation for partition algebras. Journal of Algebraic Combinatorics, 37(3):401-454, may 2012.
[27] Georgia Benkart and Tom Halverson. Partition algebras and the invariant theory of the symmetric group. 92017.
[28] Tom Halverson and Theodore N. Jacobson. Set-partition tableaux and representations of diagram algebras, 2019.
[29] Gerard 't Hooft. A planar diagram theory for strong interactions. Nucl. Phys. B, 72:461, 1974.
[30] Michael R. Douglas and Stephen H. Shenker. Strings in Less Than One-Dimension. Nucl. Phys. B, 335:635, 1990.
[31] E. Brezin and V. A. Kazakov. Exactly Solvable Field Theories of Closed Strings. Phys. Lett. B, 236:144-150, 1990.
[32] David J. Gross and Alexander A. Migdal. Nonperturbative Two-Dimensional Quantum Gravity. Phys. Rev. Lett., 64:127, 1990.
[33] David J. Gross and Washington Taylor. Two-dimensional QCD is a string theory. Nucl. Phys. B, 400:181-208, 1993.
[34] Juan Martin Maldacena. The Large N limit of superconformal field theories and supergravity. Adv. Theor. Math. Phys., 2:231-252, 1998.
[35] Vijay Balasubramanian, Micha Berkooz, Asad Naqvi, and Matthew J. Strassler. Giant gravitons in conformal field theory. JHEP, 04:034, 2002.
[36] Edward Witten. Anti-de Sitter space and holography. Adv. Theor. Math. Phys., 2:253-291, 1998.
[37] S. S. Gubser, Igor R. Klebanov, and Alexander M. Polyakov. Gauge theory correlators from noncritical string theory. Phys. Lett. B, 428:105-114, 1998.
[38] David Berenstein. A Toy model for the AdS / CFT correspondence. JHEP, 07:018, 2004.
[39] John McGreevy, Leonard Susskind, and Nicolaos Toumbas. Invasion of the giant gravitons from Anti-de Sitter space. JHEP, 06:008, 2000.
[40] Akikazu Hashimoto, Shinji Hirano, and N. Itzhaki. Large branes in AdS and their field theory dual. JHEP, 08:051, 2000.
[41] Marcus T. Grisaru, Robert C. Myers, and Oyvind Tafjord. SUSY and goliath. JHEP, 08:040, 2000.
[42] Dimitrios Kartsaklis, Sanjaye Ramgoolam, and Mehrnoosh Sadrzadeh. Linguistic matrix theory. Ann. Inst. H. Poincare D Comb. Phys. Interact., 6(3):385-426, 2019.
[43] Sanjaye Ramgoolam. Permutation Invariant Gaussian Matrix Models. Nuclear Physics B, 945:114682, August 2019. arXiv: 1809.07559.
[44] J. Firth. A synopsis of linguistic theory 1930-1955. In Studies in Linguistic Analysis. Philological Society, Oxford, 1957. reprinted in Palmer, F. (ed. 1968) Selected Papers of J. R. Firth, Longman, Harlow.
[45] Zellig S. Harris. Mathematical structures of language. In Interscience tracts in pure and applied mathematics. Wiley, 1968.
[46] Bob Coecke, Mehrnoosh Sadrzadeh, and Stephen Clark. Mathematical foundations for a compositional distributional model of meaning. Lambek Festschrift Linguistic Analysis, 36, 032010.
[47] Marco Baroni and Roberto Zamparelli. Nouns are vectors, adjectives are matrices: Representing adjective-noun constructions in semantic space. In Proceedings of the 2010 Conference on Empirical Methods in Natural Language Processing, EMNLP '10, page 1183-1193. Association for Computational Linguistics, 2010.
[48] Eugene P. Wigner. Characteristic vectors of bordered matrices with infinite dimensions. Annals of Mathematics, 62(3):548-564, 1955.
[49] Freeman J. Dyson. A brownian-motion model for the eigenvalues of a random matrix. Journal of Mathematical Physics, 3(6):1191-1198, 1962.
[50] C. W. J. Beenakker. Random-matrix theory of quantum transport. Reviews of Modern Physics, 69(3):731-808, jul 1997.
[51] Thomas Guhr, Axel Müller-Groeling, and Hans A. Weidenmüller. Random-matrix theories in quantum physics: common concepts. Physics Reports, 299(4-6):189-425, jun 1998.
[52] Alan Edelman and Yuyang Wang. Random Matrix Theory and Its Innovative Applications, pages 91-116. Springer US, Boston, MA, 2013.
[53] Igor R. Klebanov. String theory in two dimensions, 2003.
[54] P. Ginsparg and Gregory Moore. Lectures on 2d gravity and 2d string theory (tasi 1992), 1993.
[55] Sanjaye Ramgoolam, Mehrnoosh Sadrzadeh, and Lewis Sword. Gaussianity and typicality in matrix distributional semantics. Ann. Inst. H. Poincare D Comb. Phys. Interact., 9(1):1-45, 2022.
[56] Manuel Accettulli Huber, Adriana Correia, Sanjaye Ramgoolam, and Mehrnoosh Sadrzadeh. Permutation invariant matrix statistics and computational language tasks. 22022.
[57] Gijs Wijnholds, Mehrnoosh Sadrzadeh, and Stephen Clark. Representation learning for type-driven composition. In Proceedings of the 24th Conference on Computational Natural Language Learning, pages 313-324, Online, November 2020. Association for Computational Linguistics.
[58] Dimitrios Kartsaklis, Sanjaye Ramgoolam, and Mehrnoosh Sadrzadeh. Linguistic matrix theory. arXiv:1703.10252 [hep-th], March 2017. arXiv: 1703.10252.
[59] Morton Hamermesh. Group theory and its application to physical problems. Courier Corporation, 2012.
[60] Bruce E. Sagan. The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions. Springer Science \& Business Media, March 2013.
[61] Tom Halverson and Arun Ram. Partition algebras. European Journal of Combinatorics, 26(6):869-921, 2005. Combinatorics and Representation Theory.
[62] F. D. Murnaghan. The analysis of the kronecker product of irreducible representations of the symmetric group. American Journal of Mathematics, 60(3):761-784, 1938.
[63] Francis D. Murnaghan. On the analysis of the kronecker product of irreducible representations of sn. Proceedings of the National Academy of Sciences of the United States of America, 41(7):515-518, 1955.
[64] Emmanuel Briand, Rosa Orellana, and Mercedes Rosas. The stability of the kronecker product of schur functions. Journal of Algebra, 331(1):11-27, apr 2011.
[65] Zajj Daugherty and Rosa Orellana. The quasi-partition algebra. Journal of Algebra, 404:124-151, 2014.
[66] Wikipedia contributors. Partition of a set - Wikipedia, the free encyclopedia, 2021. [Online; accessed 1-December-2021].
[67] A Zee. Quantum Field Theory in a nutshell. Princeton University Press, 2010.
[68] Hamermesh. Group Theory and Its Application to Physical Problem. Addison-Wesley.
[69] Vladimir A Marčenko and Leonid Andreevich Pastur. Distribution of eigenvalues for some sets of random matrices. Mathematics of the USSR-Sbornik, 1(4):457, 1967.
[70] Laurent Laloux, Pierre Cizeau, Jean-Philippe Bouchaud, and Marc Potters. Noise dressing of financial correlation matrices. Phys. Rev. Lett., 83:1467-1470, Aug 1999.
[71] Laurent Laloux, Pierre Cizeau, Marc Potters, and Jean-Philippe Bouchaud. Random matrix theory and financial correlations. International Journal of Theoretical and Applied Finance, 03(03):391-397, 2000.
[72] Vasiliki Plerou, Parameswaran Gopikrishnan, Bernd Rosenow, Luis A Nunes Amaral, Thomas Guhr, and H Eugene Stanley. Random matrix approach to cross correlations in financial data. Physical Review E, 65(6):066126, 2002.
[73] Jean-Phillipe Bouchaud and Marc Potters. Financial applications of random matrix theory: a short review. The Oxford Handbook of Random Matrix Theory, 2009.
[74] Joël Bun, Jean-Philippe Bouchaud, and Marc Potters. Cleaning large correlation matrices: Tools from random matrix theory. Physics Reports, 666:1-109, 2017.
[75] Marcos M López de Prado. Machine learning for asset managers. Cambridge University Press, 2020.
[76] M Potters, J-P Bouchaud, and L Laloux. Financial applications of random matrix theory: Old laces and new pieces. Acta Physica Polonica B, 36(9):2767, 2005.
[77] T.W. Anderson. An Introduction to Multivariate Statistical Analysis, Third Edition. Wiley-Interscience, 2003.
[78] OEIS Foundation Inc. The On-Line Encyclopedia of Integer Sequences, 2023. Published electronically at http://oeis.org.
[79] Integral. Truefx. https://www.truefx.com/, 2021. Accessed: 2021-10-15.
[80] Yacine Aït-Sahalia, Jianqing Fan, and Dacheng Xiu. High-frequency covariance estimates with noisy and asynchronous financial data. Journal of the American Statistical Association, 105(492):1504-1517, 2010.
[81] Paul Malliavin and Maria Elvira Mancino. Fourier series method for measurement of multivariate volatilities. Finance and Stochastics, 6(1):49-61, 2002.
[82] Paul Malliavin and Maria Elvira Mancino. A Fourier transform method for nonparametric estimation of multivariate volatility. The Annals of Statistics, 37(4):1983 2010, 2009.
[83] Hamid Ghorbani. Mahalanobis distance and its application for detecting multivariate outliers. Facta Univ Ser Math Inform, 34(3):583-95, 2019.
[84] FXSSI. Fxssi. https://fxssi.com/high-impact-forex-news, 2022. Accessed: 2022-12-08.
[85] FXSTREET. Fxstreet. https://www.fxstreet.com/economic-calendar, 2022. Accessed: 2022-07-01.
[86] I.T. Jolliffe. Principal Component Analysis. Springer Series in Statistics. Springer, 2002.
[87] Garrett Birkhoff. Lattice theory, volume 25. American Mathematical Soc., 1940.
[88] JH Wedderburn. On hypercomplex numbers. Proceedings of the London Mathematical Society, 2(1):77-118, 1908.
[89] Emil Artin. Zur theorie der hyperkomplexen zahlen. In Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg, volume 5, pages 251-260. Springer, 1927.
[90] Arun Ram. Dissertation, chapter 1 representation theory, 1990.
[91] Yusuke Kimura. Multi-matrix models and Noncommutative Frobenius algebras obtained from symmetric groups and Brauer algebras. Commun. Math. Phys., 337(1):140, 2015.
[92] Paolo Mattioli and Sanjaye Ramgoolam. Permutation centralizer algebras and multimatrix invariants. Physical Review D, 93(6), Mar 2016.
[93] Yusuke Kimura. Noncommutative frobenius algebras and open-closed duality, 2017.
[94] Joseph Ben Geloun and Sanjaye Ramgoolam. Tensor models, kronecker coefficients and permutation centralizer algebras. Journal of High Energy Physics, 2017(11), Nov 2017.
[95] Joseph Ben Geloun and Sanjaye Ramgoolam. Quantum mechanics of bipartite ribbon graphs: Integrality, lattices and kronecker coefficients, 2021.
[96] Madan Lal Mehta. Random matrices. Elsevier, 2004.
[97] Franck Gabriel. Combinatorial theory of permutation-invariant random matrices i: partitions, geometry and renormalization, 2016.
[98] Michael R. Douglas. Conformal field theory techniques in large N Yang-Mills theory. In NATO Advanced Research Workshop on New Developments in String Theory, Conformal Models and Topological Field Theory, 51993.
[99] David Berenstein. A Toy model for the AdS / CFT correspondence. JHEP, 07:018, 2004.
[100] Sameer Murthy. Unitary matrix models, free fermion ensembles, and the giant graviton expansion. 22022.
[101] David Berenstein. Large N BPS states and emergent quantum gravity. JHEP, 01:125, 2006.
[102] Justin Kinney, Juan Martin Maldacena, Shiraz Minwalla, and Suvrat Raju. An Index for 4 dimensional super conformal theories. Commun. Math. Phys., 275:209254, 2007.
[103] Christopher Lewis-Brown and Sanjaye Ramgoolam. Quarter-BPS states, multisymmetric functions and set partitions. JHEP, 03:153, 2021.
[104] Troels Harmark and Marta Orselli. Spin Matrix Theory: A quantum mechanical model of the AdS/CFT correspondence. JHEP, 11:134, 2014.
[105] Stefano Baiguera, Troels Harmark, and Yang Lei. Spin Matrix Theory in near $\frac{1}{8}$-BPS corners of $\mathcal{N}=4$ super-Yang-Mills. JHEP, 02:191, 2022.
[106] A. Jevicki and B. Sakita. The Quantum Collective Field Method and Its Application to the Planar Limit. Nucl. Phys. B, 165:511, 1980.
[107] Laurence G. Yaffe. Large n Limits as Classical Mechanics. Rev. Mod. Phys., 54:407, 1982.
[108] David Berenstein and Shannon Wang. BPS coherent states and localization. Journal of High Energy Physics, 2022(8), aug 2022.
[109] Adolfo Holguin and Shannon Wang. Giant Gravitons, Harish-Chandra integrals, and BPS states in symplectic and orthogonal $\mathcal{N}=4$ SYM. 52022.
[110] Hai Lin. Coherent state excitations and string-added coherent states in gauge-gravity correspondence. 62022.
[111] Christopher Lewis-Brown and Sanjaye Ramgoolam. BPS operators in $\mathcal{N}=4 S O(N)$ super Yang-Mills theory: plethysms, dominoes and words. JHEP, 11:035, 2018.
[112] Garreth Kemp and Sanjaye Ramgoolam. BPS states, conserved charges and centres of symmetric group algebras. JHEP, 01:146, 2020.
[113] Ian Grant Macdonald. Symmetric Functions and Hall Polynomials. Clarendon Press, 1998.
[114] Stephen Doty, Aaron Lauve, and George H Seelinger. Canonical idempotents of multiplicity-free families of algebras. L'Enseignement Mathématique, 64(1):23-63, 2019.
[115] Yusuke Kimura. Quarter BPS classified by brauer algebra. Journal of High Energy Physics, 2010(5), may 2010.
[116] Robert de Mello Koch, Matthias Dessein, Dimitrios Giataganas, and Christopher Mathwin. Giant graviton oscillators. Journal of High Energy Physics, 2011(10), oct 2011.
[117] Vincent De Comarmond, Robert de Mello Koch, and Katherine Jefferies. Surprisingly simple spectra. Journal of High Energy Physics, 2011(2):1-29, feb 2011.
[118] T.W Brown. Permutations and the loop. Journal of High Energy Physics, 2008(06):008-008, jun 2008.
[119] Robert de Mello Koch and Sanjaye Ramgoolam. A double coset ansatz for integrability in AdS/CFT. Journal of High Energy Physics, 2012(6), jun 2012.
[120] Robert de Mello Koch, Jelena Smolic, and Milena Smolic. Giant gravitons ${ }^{-}$with strings attached (II). Journal of High Energy Physics, 2007(09):049-049, sep 2007.
[121] Vijay Balasubramanian, David Berenstein, Bo Feng, and Min xin Huang. D-branes in yang-mills theory and emergent gauge symmetry. Journal of High Energy Physics, 2005(03):006-006, mar 2005.
[122] Michel Lassalle. An explicit formula for the characters of the symmetric group. Mathematische Annalen, 340:383-405, 2007.
[123] X. G. Wen and Q. Niu. Ground-state degeneracy of the fractional quantum hall states in the presence of a random potential and on high-genus riemann surfaces. Phys. Rev. B, 41:9377-9396, May 1990.
[124] Alexei Kitaev. Anyons in an exactly solved model and beyond. Annals of Physics, 321(1):2-111, jan 2006.
[125] Vijay Balasubramanian, Bartomiej Czech, Klaus Larjo, and Joan Simón. Integrability vs. information loss: a simple example. Journal of High Energy Physics, 2006(11):001-001, nov 2006.
[126] Sanjaye Ramgoolam and Eric Sharpe. Combinatoric topological string theories and group theory algorithms. Journal of High Energy Physics, 2022(10), oct 2022.
[127] B Eden, Paul S Howe, E Sokatchev, and Peter C West. Extremal and next-toextremal n-point correlators in four-dimensional scft. Physics Letters B, 494(1-2):141-147, 2000.
[128] C. Bowman, M. De Visscher, and Rosa C. Orellana. The partition algebra and the kronecker coefficients. Transactions of the American Mathematical Society, 367:36473667, 2012.
[129] Edward Grefenstette, Georgiana Dinu, Yao-Zhong Zhang, Mehrnoosh Sadrzadeh, and Marco Baroni. Multi-step regression learning for compositional distributional semantics, 2013.
[130] Tamara Polajnar, Laura Rimell, and Stephen Clark. Using sentence plausibility to learn the semantics of transitive verbs, 2014.
[131] Gijs Wijnholds, Mehrnoosh Sadrzadeh, and Stephen Clark. Representation learning for type-driven composition. In Proceedings of the 24th Conference on Computational Natural Language Learning, pages 313-324, Online, November 2020. Association for Computational Linguistics.
[132] George Barnes, Adrian Padellaro, and Sanjaye Ramgoolam. Permutation invariant tensor models and partition algebras. in preparation.
[133] Patrick T. Komiske, Eric M. Metodiev, and Jesse Thaler. Energy flow polynomials: a complete linear basis for jet substructure. Journal of High Energy Physics, 2018(4), apr 2018.
[134] Pedro C. Diniz, Eduardo A. B. Oliveira, Aristeu R. P. Lima, and Emanuel A. L. Henn. Ground state and collective excitations of a dipolar bose-einstein condensate in a bubble trap. Scientific Reports, 10(1), mar 2020.


[^0]:    ${ }^{1}$ This platform is an ECN i.e. an Electronic Communication Network
    ${ }^{2} 1 \mathrm{pip}=10^{-4}$

[^1]:    ${ }^{3}$ The four observables that differ by more than one standard deviation are $\mathcal{O}_{3}, \mathcal{O}_{17}, \mathcal{O}_{19}$ and $\mathcal{O}_{22}$.

[^2]:    ${ }^{1}$ Note, this is a different four-parameter subspace to that considered in chapter 3

[^3]:    ${ }^{2}$ Immediate comparison gives a parameter limit for the coupling matrices, as opposed to their inverses as in equation 4.41. In this case, they are identical.

[^4]:    ${ }^{3}$ We thank Franck Gabriel for this observation.

[^5]:    ${ }^{1}$ We assume the eigenvalues are positive such that the spectrum of the Hamiltonian is bounded from below. Therefore, we may write the eigenvalues as squares without loss of generality.

