Division of Indivisible Items: Fairness, Efficiency, and Strategyproofness

by

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Contents

[1]This chapter is based on a research article by Sun et al. [99]

[2]This chapter is based on a research article by Sun et al. [100]

List of Tables

List of Key Symbols

N	Then set of agents
E	The set of items
$\mathcal{V}$	The set of valuation functions
$\mathcal{I}$	A fair division instance
v_i	The valuation function of some agent i
$\mathbf{A}$	An allocation of items to agents
A_i	The set of items received by some agent i in an allocation
UW	The function of utilitarian welfare
EW	The function of egalitarian welfare
OPT_U	The maximum utilitarian welfare of some instance
OPT_E	The maximum egalitarian welfare of some instance
PoF_W	The price of fairness with respect to welfare function W
$\mathbf{v}$	The type profile of agents
$\mathbf{b}$	A reporting profile of agents
$\mathcal{M}$	A mechanism

Acknowledgments

This thesis is the outcome of four years of doctoral studies at the University of Warwick. Despite the inevitable challenges and difficulties, this journey was also enchanting, fulfilling, and memorable.

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Declarations

I hereby declare that this thesis is submitted to the University of Warwick in support of my application for the degree of Doctor of Philosophy. It has been composed by myself and has not been submitted for a degree at another university. I further declare that a paper titled “Connections between Fairness Criteria and Efficiency for Allocating Indivisible Chores”, drawn from Chapter 3, was co-authored with Prof. Bo Chen and Dr. Xuan Vinh Doan, and I contributed 80% of this work. A preliminary version of this paper appeared in the Proceedings of the 20th International Conference on Autonomous Agents and MultiAgent Systems [99]. A paper titled “Equitability and Welfare Maximization for Allocating Indivisible Items”, drawn from Chapter 4, was co-authored with Prof. Bo Chen and Dr. Xuan Vinh Doan, and I contributed 80% of this work. This paper has been accepted by Journal of Autonomous Agents and Multi-Agent Systems [100].

Abstract

This thesis theoretically studies fairness, efficiency, and strategyproofness, in the model of assigning a set of indivisible items to multiple agents. Fairness, with an interpretation of social justice, ensures that everyone is treated unbiasedly. Efficiency, a quantitative indicator, measures the utilization of the total resource. Strategyproofness, a desired property of the assignment protocol, inhibits the strategic behavior of misreporting information from participants. This work, first in Chapter 3, focuses on the allocation of chores (items with non-positive value) and studies two envy-based and two share-based fairness criteria. The analysis provides the connections between fairness criteria and also investigates, in the worst-case scenario, the efficiency loss when requiring allocations to be fair by establishing the corresponding *price of fairness*. This thesis, then in Chapter 4, studies two relaxations of *equitability*, a fairness notion that ensures agents the same level of value. This chapter cares about both cases of goods (items with non-negative value) and chores. The chapter first investigates the trade-off between efficiency and fairness and then provides the picture of the computational complexity of (i) deciding the existence of approximately equitable and welfare-maximizing allocation; (ii) computing a welfare maximizer among all approximately equitable allocation. Chapter 5 considers the setting where agents' preferences over items are their private information and not publicly known anymore. Agents are required to report their preferences so that assignment procedures can be carried on. Agents can and will report false information if they are able to receive additional value by doing so. This chapter proposes deterministic and randomised (group) strategyproof mechanisms in which each agent's (expected) value is maximized when she reports the true preference. Besides strategyproofness, the proposed mechanisms can output efficient allocations that capture a certain degree of fairness.

Chapter 1

Introduction

1.1 Introduction

The resource allocation problem is arguably a central matter of concern among Operations Research, Computer Science, and Economics. Resource allocation is to assign available resources to various users or agents, and each agent has his/her own evaluation of resources. Resource allocation problems, without any doubt, happen quite often in practice and are relevant to a wide range of applications, including but not limited to airport traffic management [49], scheduling and manufacturing [102, 98], and public transport [46]. Essentially, addressing these questions requires thinking of who gets what and how and why, which is, however, still very much a work in progress.

Resources can be theoretically modeled as items to be assigned to the participants or agents and, in general, are classified into two categories, divisible and indivisible items. A divisible item can be arbitrarily cut into several small pieces and assigned to multiple agents, and the problem regarding divisible items is also known as *cake-cutting*. On the contrary, each indivisible item has to be assigned as a whole to a single agent. Indivisible item assignment problems are prevalent in diverse real-world scenarios, including but not limited to higher education [4, 60, 74], healthcare [84, 87, 96], and business organizations [3, 9, 94]. These types of problems occur when assigning items that cannot be divided without losing their value, or when assigning tasks that cannot be broken down into smaller parts. Examples include assigning courses to professors at a university, as each course is typically assigned to a single professor and cannot be divided. Similarly, assigning shifts to nurses or doctors at a hospital presents an indivisible item assignment problem since each shift requires one person to perform their duties and cannot be divided. Moreover, assigning specialized tasks to employees in a workplace, such as conducting a specific experiment, can also be an indivisible items assignment problem since the task may require specific skills or knowledge that cannot be divided among multiple workers.

Besides whether it is indivisible or not, items can also be distinguished according to the type of impact on agents. Goods are the items resulting in positive value or benefit for agents, while chores are the items associated with non-negative value or cost. An alternative understanding of chores is that the value of chores represents the effort that the agents need to spend on completing the assigned items or tasks, and hence agents in the setting of chores tend to minimize their values. The examples mentioned above, such as the allocation of teaching workloads among faculty members, shifts among nurses, and specialized tasks among employees, represent assignment problems for indivisible chores. On the other hand, for the indivisible goods, real-life examples include real estate properties, such as houses, apartments, and land plots; works of art, such as paintings and sculptures; intellectual properties, such as patents, trademarks, and copyrights; and other indivisible assets, such as rare collectibles and vintage cars.

Fairness, or being fair, with the broadest sense interpretation that people should receive what they deserve, is a crucial social concept and remarkably matters in almost every resource allocation problem. The concepts of fair treatment of individuals and equitable distribution of resources are fundamental social ideals that are closely related to the concept of fairness. Fairness is essentially about ensuring that individuals receive what they deserve based on factors such as their contribution, need, or merit. Fairness is a fundamental value that underpins many societies around the world, and it is reflected in various aspects of our lives, from the legal system to the workplace and beyond. The importance of fairness lies in its ability to create a sense of trust and mutual respect within a community. When people feel that they are being treated fairly, they are more likely to trust others and cooperate with them, leading to more harmonious and productive relationships. Fairness is also critical for promoting social mobility and reducing inequality. When resources are allocated fairly, people have the opportunity to succeed based on their abilities and effort, rather than their social status or background. This can lead to a more equal and just society, where everyone has the opportunity to reach their full potential and contribute to the common good.

In addition to fairness, social welfare or efficiency, a competing criterion to fairness, is also an important factor that needs to be taken into account. From the perspective of system optimality, a community leader or a central decision maker (if one exists) tends to utilize resources as efficiently as possible. Traditionally, works study fairness and efficiency independently, such as quantitatively characterizing the fair/efficient outcomes and investigating the existence and computation aspects of such fair/efficient allocations. With more in-depth research, the trade-off between efficiency and fairness has been observed. In general, an efficient or a welfare-maximizing allocation highly likely leads to unfair outcomes, and reversely, requiring a fair outcome inevitably sacrifices the efficiency of the outcome. These observations make understanding the trade-off between efficiency and fairness interesting and worthwhile to explore. In the past decade, a line of research [30, 47, 28] quantitatively studies the efficiency loss under fairness constraints. Bertsimas et al. [30] and Caragiannis et al. [47] independently propose the notion of *price of fairness* (PoF) to quantify the extent of welfare loss when ensuring outcomes are fair. The PoF is defined as the ratio, in the worst-case scenario, between the unconstrained optimal social welfare and the optimal social welfare achieved by fair allocations. Successfully bounding the PoF ratio is capable of yielding a comprehensive understanding of the fairness and efficiency trade-off.

The aforementioned research regarding efficiency and fairness relies on the assumption of information completeness, which in our context, refers to that agents' preferences or valuations over items being publicly known or at least known by the central decision maker. Nevertheless, valuations, in practice, are agents' private information and are not accessed by the central decision maker. Although the central decision maker can collect information from agents by asking them to report their valuations, agents may not always tell the truth, especially in situations where misreporting can strictly benefit them. With the presence of false information, the assignment protocol supposed to output fair and efficient outcomes may lead to unreasonable solutions. This challenge requires the central authority to not only focus on the performance of outcomes but also to take the potential strategic behaviors of agents into account. One way to approach this challenge is to design *truthful* or *strategyproof* mechanisms, in which, compared to what can be achieved under truthful reporting, agents can never be strictly better off by reporting false information. Designing strategyproof mechanisms solely may not be extremely hard, whereas it is much more challenging to propose a strategyproof mechanism that can also output fair or efficient allocations.

1.2 Thesis Structure and Summary of Results

The thesis, first in Chapter 2 (Preliminaries), introduces the model of the allocation of indivisible items and defines the essential mathematical terminologies, such as the sets of agents, items, and valuation functions. Meanwhile, the formal definitions of all solution concepts, including fairness criteria and welfare functions, are presented. The last subsection (Section 2.4) of Chapter 2 contains terminologies and notations regarding the problem related to mechanism design.

The results of this thesis are presented in Chapters 3, 4, and 5. Chapter 3 focuses on the allocation of chores and investigates the connections among envy-based and share-based fairness criteria, and establishes their prices of fairness with respect to utilitarian welfare. Chapter 4 studies the notion of equitability in conjunction with system efficiency. It is known that equitable allocations do not always exist in the case of indivisible items, so two relaxations of equitability are considered. The last technique chapter (Chapter 5) addresses the problem of fair indivisible item allocations under the mechanism design framework, in which agents' preferences are their private information and are no longer publicly known. The main result for these three technique chapters is summarized as follows.

Chapter 3: Fairness Criteria for Allocating Indivisible Chores: Connections and Efficiencies

This chapter studies several fairness notions in allocating indivisible chores to agents who have additive and submodular cost functions. The fairness criteria we are concerned with are envy-free up to any item (EFX), envy-free up to one item (EF1), maximin share (MMS) fairness, and pairwise maximin share (PMMS) fairness, which are proposed as relaxations of envy-freeness in the additive setting. For allocations under each fairness criterion, we establish their approximation guarantee for other fairness criteria. Under the additive setting, results in this chapter show strong connections between these fairness criteria and, at the same time, reveal intrinsic differences between goods allocation and chores allocation. However, such strong relationships cannot be inherited by the submodular setting, under which PMMS and MMS are no longer relaxations of envy-freeness and, even worse, few non-trivial guarantees exist. We also investigate efficiency loss under these fairness constraints and establish their prices of fairness.

Chapter 4: Equitability and Welfare Maximization for Allocating Indivisible Items

This chapter considers fair allocations of indivisible goods and chores in conjunction with system efficiency, measured by two social welfare functions, namely utilitarian and egalitarian welfare. The fairness criteria we are concerned with are equitability up to any item (EQX) and equitability up to one item (EQ1). For the trade-off between fairness and efficiency, we investigate efficiency loss under these fairness constraints and establish the price of fairness. From the computational perspective, we provide a complete picture of the computational complexity of (i) deciding the existence of an EQX/EQ1 *and* welfare-maximizing allocation; (ii) computing a welfare maximizer among all EQX/EQ1 allocations.

Chapter 5: Allocating indivisible items to strategic agents

This chapter studies the problem of fairly allocating indivisible items to agents under the mechanism design framework, in which agents' preferences are privately known. To escape the impossibility presented by Amanatidis, Birmpas, Christodoulou and Markakis [5], we restrict to the setting where agents' valuations are *binary additive* and *restricted additive*, two subclasses of the additive valuation. We first show that no deterministic mechanism can be strategyproof, Pareto optimal, and equitability up to one item, even when assigning chores to two agents with binary additive valuations. If randomisation is allowed, in the setting of allocating chores to agents with restricted additive valuations, we design a group-strategyproof (in expectation) mechanism that is also *ex-ante* Pareto optimal, envy-free, equitable and *ex-post* envy-free to one item, equitability up to one item. We also extend to the setting where items are mixtures of goods and chores, and provide a strategyproof (in expectation) mechanism for two agents that also achieves ex-ante Pareto optimality and exact fairness and ex-post relaxed fairness.

Chapter 2

Preliminaries

In this chapter, we first define basic notations used throughout the thesis. Then, we formally present the underlying solution concepts, including various fairness criteria and social welfare functions. We also introduce terminologies and notations implemented in problems under the framework of mechanism design.

2.1 Basic Notations

Denote by $[k] = \{1, \ldots, k\}$ for any positive integer k. A fair division instance $\mathcal{I} = \langle N, E, \mathcal{V} \rangle$ is composed of a set $N = \{1, \ldots, n\}$ of n agents and a set $E = \{e_1, \ldots, e_m\}$ of m indivisible items. Each agent i is associated with a valuation function $v_i \in \mathcal{V}$ and $v_i : 2^E \to \mathbb{R}$. An item $e \in E$ can be either a good that brings non-negative values to agents or a chore resulting in non-positive values. Although an item can be a good for one agent and a chore for another, in most places (except for Section 5.3.2) of this thesis, we focus on the case where all items are goods or all items are chores. Formally, we say an item $e \in E$ is a good if $v_i(\{e\}) \geq 0$ for any $i \in [n]$, and is a chore if $v_i(\{e\}) \leq 0$ for any $i \in [n]$. Moreover, we call $\mathcal{I}$ a *fair-goods* (resp., *fair-chores*) instance if every item is a good (resp., a chore). Throughout the thesis, we assume that for all i, $v_i(\emptyset) = 0$ and $v_i(\cdot)$ is monotone, i.e., for any $S \subseteq T$, $v_i(S) \leq v_i(T)$ for the allocation of goods and $v_i(S) \geq v_i(T)$ in the case of chores. We say a (set) function $v(\cdot)$ is

- *Additive*, if $v(S) = \sum_{e \in S} v(\{e\})$ for any $S \subseteq E$.
- *Submodular*, if for any $S \subseteq T \subseteq E$ and $e \in E \setminus T$, $v(T \cup \{e\}) - v(T) \leq v(S \cup \{e\}) - v(S)$.

- *Subadditive*, if for any $S, T \subseteq E$, $v(S \cup T) \leq v(S) + v(T)$.

It is easy to see that additivity implies submodularity and that non-negative submodularity implies subadditivity. For simplicity, instead of $v_i(\{e\})$, we use $v_i(e)$ to represent the value of item e for agent i.

Every assignment of a set of items $S \subseteq E$ to agents results in an *allocation* $\mathbf{A} = (A_1, \ldots, A_n)$ that is an n-partition of S with $A_i \cap A_j = \emptyset$ for any $i \neq j$ and $\bigcup_{i \in N} A_i = S$. A subset $S \subseteq E$ also refers to a *bundle* of items. For any bundle S and $k \in \mathbb{N}^+$, we denote by $\Pi_k(S)$ the set of all k-partition of S and by $[k] = \{1, \ldots, k\}$. We also denote by $|S|$ the number of items of S.

2.2 Fairness Criteria

Fair allocation problems have been extensively studied through the years, and a plethora of fairness criteria have been proposed. Due to the different characteristics of allocation problems and different expectations of the allocation result, there is no single fairness criterion that can be universally implemented for all allocation problems. In the following, we formally present the (α-approximation) fairness criteria that we are concerned with in the thesis.

2.2.1 Envy-Freeness

In the fair division literature, one of the most compelling solutions is *envy-freeness* (EF) [64]. It portrays agents' attitudes toward the bundle of items received by others. In an envy-free allocation, the value of each agent is at least as high as the value that she assigns to any other agent's bundle.

Definition 2.2.1 (α-EF)**.** *In the case of goods, for any $\alpha \in (0, 1]$, an allocation $\mathbf{A} = (A_1, \ldots, A_n)$ is* α-EF *if for any $i, j \in [n]$, $v_i(A_i) \geq \alpha \cdot v_i(A_j)$ holds. In the case of chores, for any $\alpha \in [1, +\infty)$, the allocation $\mathbf{A}$ is* α-EF *if for any $i, j \in [n]$, $v_i(A_i) \geq \alpha \cdot v_i(A_j)$ holds. In particular,* 1-EF *is simply called* EF.

One weakness of the EF solutions is that an envy-free allocation does not always exist in the setting of indivisible items. For example, one can think of allocating an indivisible item to two agents, both of whom value the item at one. Every assignment inevitably results in envy. Even worse, from the computational complexity perspective, one can not, in polynomial time, determine whether an EF allocation exists or not [40]. Given these obstacles, researchers instead focus on EF relaxations with guaranteed existence. The notion of *envy-free up to one item* (EF1) [45, 48, 85] is a realistic relaxation of EF, and intuitively, in an EF1 allocation, agent i may envy agent j, but envy can be eliminated by removing one item.

Definition 2.2.2 (α-EF1)**.** *In the case of goods, for any $\alpha \in (0, 1]$, an allocation $\mathbf{A} = (A_1, \ldots, A_n)$ is* α-EF1 *if any $i, j \in [n]$, there exists $e \in A_j$ such that $v_i(A_i) \geq \alpha \cdot v_i(A_j \setminus \{e\})$. In the case of chores, for any $\alpha \in [1, +\infty)$, the allocation $\mathbf{A}$ is* α-EF1 *if for any $i, j \in [n]$, there exists $e \in A_i$ such that $v_i(A_i \setminus \{e\}) \geq \alpha \cdot v_i(A_j)$. In particular,* 1-EF1 *is simply called* EF1*.*

Another popular relaxation of EF is called *envy-free up to any item* (EFX) [48, 51, 93]. Different from EF1, EFX requires that envy can be eliminated by removing an arbitrary non-zero value item.

Definition 2.2.3 (α-EFX)**.** *In the case of goods, for any $\in (0, 1]$, an allocation $\mathbf{A} = (A_1, \ldots, A_n)$ is* α-EFX *if for any $i, j \in [n]$ and for every $e \in A_j$ with $v_i(e) > 0$, $v_i(A_i) \geq \alpha \cdot v_i(A_j \setminus \{e\})$ holds. In the case of chores, for any $\alpha \in [1, +\infty)$, the allocation $\mathbf{A}$ is* α-EFX *if for any $i, j \in [n]$ and for every $e \in A_i$ with $v_i(e) < 0$, $v_i(A_i \setminus \{e\}) \geq \alpha \cdot v_i(A_j)$ holds. In particular,* 1-EFX *is simply called* EFX*.*

2.2.2 Equitability

The notion of *equitability* [42, 57, 66] is originally studied in the cake-cutting problem (a divisible item). Informally, equitability ensures that each agent is equally happy with his/her bundle, or agents should receive the same level of value. Note that the aforementioned (relaxed) envy-freeness is defined in an intrapersonal manner, while equitability acts as an interpersonal fairness criterion.

Definition 2.2.4 (EQ)**.** *An allocation $\mathbf{A} = (A_1, \ldots, A_n)$ is equitable* (EQ) *if for any $i, j \in [n]$, $v_i(A_i) = v_j(A_j)$.*

Unfortunately, an EQ allocation does not always exist, and a typical example is assigning one indivisible item to two agents. Motivated by this impossibility result, we instead are concerned with its two relaxations, *equitable up to one item* (EQ1) and *equitable up to any item* (EQX) [66, 67, 73], both of which can be satisfiable under additive valuations.

Definition 2.2.5 (α-EQ1)**.** *In the case of goods, for any $\alpha \in (0, 1]$, an allocation $\mathbf{A} = (A_1, \ldots, A_n)$ is* α-EQ1 *if for any $i, j \in [n]$, there exists $e \in A_j$ such that $v_i(A_i) \geq \alpha \cdot v_j(A_j \setminus \{e\})$. In the case of chores, for any $\alpha \in [1, +\infty)$, the allocation*

$\mathbf{A}$ *is* α-EQ1 *if for any* $i, j \in [n]$*, there exists* $e \in A_i$ *such that* $v_i(A_i \setminus \{e\}) \geq \alpha \cdot v_j(A_j)$*. In particular,* 1-EQ1 *is simply called* EQ1*.*

Definition 2.2.6 (α-EQX)**.** *In the case of goods, for any* $\alpha \in (0, 1]$*, an allocation* $\mathbf{A} = (A_1, \ldots, A_n)$ *is* α-EQX *if for any* $i, j \in [n]$ *and for every* $e \in A_j$ *with* $v_j(e) > 0$ *,* $v_i(A_i) \geq \alpha \cdot v_j(A_j \setminus \{e\})$ *holds. In the case of chores, for any* $\alpha \in [1, +\infty)$*, the allocation* $\mathbf{A}$ *is* α-EQX *if for any* $i, j \in [n]$ *and for every* $e \in A_i$ *with* $v_i(e) < 0$*,* $v_i(A_i \setminus \{e\}) \geq \alpha \cdot v_j(A_j)$ *holds. In particular,* 1-EQX *is simply called* EQX*.*

In EQ1 or EQX allocations, the value of agent i can be smaller than the value of agent j, but agent i would receive a value no less than that of agent j if an item is removed. The specific way of removing an item depends on the nature of the underlying items, goods or chores. Moreover, it is straightforward to see that α-EQX is stricter than α-EQ1 for any α in both cases of goods and chores.

2.2.3 Proportionality and Other Share-Based Fairness

Note that both (relaxed) envy-freeness and equitability are defined in the way that after revealing the final allocation, an agent examines whether she is treated fairly based on her bundle and the bundles assigned to others. Another school of fairness is defined by requiring agents to receive an absolute value or share. The notion of *proportionality* is the first few share-based fairness criteria to be studied, and in a *proportional* (PROP) allocation, the value of each agent is at least $1/n$ of her value on all items.

Definition 2.2.7 (PROP)**.** *An allocation* $\mathbf{A} = (A_1, \ldots, A_n)$ *is* PROP *if for any* $i \in [n]$*,* $v_i(A_i) \geq 1/n \cdot v_i(E)$ *holds.*

In the allocation of indivisible items, PROP allocations do not always, and a non-existence example is assigning one item to two agents. Then the notion of *proportional up to one item* (PROP1) [14, 56], a relaxation of PROP, is proposed.

Definition 2.2.8 (PROP1)**.** *In the case of goods, an allocation* $\mathbf{A} = (A_1, \ldots, A_n)$ *is* PROP1 *if for any* $i \in [n]$*, there exists* $e \notin A_i$ *such that* $v_i(A_i \cup \{e\}) \geq 1/n \cdot v_i(E)$*. In the case of chores, the allocation* $\mathbf{A}$ *is* PROP1 *if for any* $i \in [n]$*, there exists* $e \in A_i$ *such that* $v_i(A_i \setminus \{e\}) \geq 1/n \cdot v_i(E)$ *holds.*

The *maximin share* (MMS) fairness, recently proposed by Budish [45], is another relaxation of PROP. The idea of MMS is a generalization of the "*cut and choose*" protocol from the cake-cutting problem. More specifically, imagine that agent i has the opportunity to partition S into k bundles, then the corresponding MMS of agent i is the maximum value that she can guarantee if she is to receive the bundle of the smallest value. The formal definition of MMS is presented as follows,

$$\mathsf{MMS}_i(k, S) = \max_{\mathbf{A} \in \Pi_k(S)} \min_{j \in [k]} v_i(A_j).$$

The MMS fairness requires that each agent receives value at least his/her maximin share.

Definition 2.2.9 (α-MMS)**.** *In the case of goods, for any $\alpha \in (0, 1]$, an allocation $\mathbf{A} = (A_1, \ldots, A_n)$ is α-*MMS *fair if for any $i \in [n]$, $v_i(A_i) \geq \alpha \cdot \mathsf{MMS}_i(n, E)$ holds. In the case of chores, for any $\alpha \in [1, +\infty)$, the allocation $\mathbf{A}$ is α-*MMS *fair if for any $i \in [n]$, $v_i(A_i) \geq \alpha \cdot \mathsf{MMS}_i(n, E)$ holds. In particular,* 1-MMS *fair is simply called* MMS *fair.*

We are also interested in another share-based fairness criterion, *pairwise maximin share* (PMMS), which is similar to MMS, but instead of partitioning E into n parts, agent i considers 2-partition of the union of his and another agent's bundle. The formal definition of PMMS is presented as follows.

Definition 2.2.10 (α-PMMS)**.** *In the case of goods, for any $\alpha \in (0, 1]$, an allocation $\mathbf{A} = (A_1, \ldots, A_n)$ is α-*PMMS *fair if for any $i, j \in [n]$, $v_i(A_i) \geq \alpha \cdot \mathsf{MMS}_i(2, A_i \cup A_j)$ holds. In the case of chore, for any $\alpha \in [1, +\infty)$, the allocation $\mathbf{A}$ is α-*PMMS *fair if for any $i, j \in [n]$, $v_i(A_i) \geq \alpha \cdot \mathsf{MMS}_i(2, A_i \cup A_j)$ holds. In particular,* 1-PMMS *fair is simply called* PMMS *fair.*

2.3 Economic Efficiency and Social Welfare

The fairness criteria mentioned above only are not capable of capturing the whole picture of this thesis, and especially, a significant task of this thesis is to characterize the trade-off between fairness and efficiency quantitatively. In order to investigate the fairness and efficiency trade-off, the notion(s) of efficiency needs to be introduced. In this section, we formally present the definition of *Pareto optimal* (PO) allocation and two social welfare functions, namely *utilitarian welfare* and *egalitarian welfare*.

2.3.1 Pareto Optimality and Social Welfare Functions

The notion of *Pareto optimality* characterizes a situation in which no agent can be better off without making at least another agent worse off. The mathematical definition is provided in the following.

Definition 2.3.1 (PO)**.** *An allocation* $\mathbf{A} = (A_1, \ldots, A_n)$ *is* PO *if there is no allocation* $\mathbf{B} = (B_1, \ldots, B_n)$ *that Pareto-dominates it, i.e., satisfies* $v_i(B_i) \geq v_i(A_i)$ *for all* $i \in [n]$ *and at least one inequality is strict.*

In welfare economics, the social welfare function is a microeconomic technique for evaluating the well-being at the aggregate level. The thesis considers two canonical welfare functions, *utilitarian welfare* (UW) and *egalitarian welfare* (EW). Utilitarian welfare is the sum of individuals' values, and egalitarian welfare is equal to the value of the worst-off agents. The formal definitions of welfare functions are provided as follows.

Definition 2.3.2. *Given an allocation* $\mathbf{A} = (A_1, \ldots, A_n)$*, the utilitarian welfare of* $\mathbf{A}$ *is defined as* $\mathsf{UW}(\mathbf{A}) = \sum_{i \in [n]} v_i(A_i)$ *and the egalitarian welfare of* $\mathbf{A}$ *is defined as* $\mathsf{EW}(\mathbf{A}) = \min_{i \in [n]} v_i(A_i)$*.*

Among all allocations, we are particularly interested in the one that has the maximum utilitarian/egalitarian welfare and such an allocation is also called a *welfare maximizer.*

Definition 2.3.3. *An allocation* $\mathbf{A}$ *is a utilitarian welfare maximizer* (UWM) *and an egalitarian welfare maximizer* (EWM) *if it has maximum utilitarian welfare and maximum egalitarian welfare, respectively.*

2.3.2 Price of Fairness

The *price of fairness* (PoF) quantifies the loss of economic welfare when enforcing allocation fairness. For allocation of goods, PoF is the supremum ratio over all problem instances between the maximum welfare of all allocations and maximum welfare of all fair allocations. In the case of chores, PoF is the supremum ratio over all problem instances between the maximum welfare of all fair allocations and maximum welfare of all allocations. The price of fairness has been applied to quantify the welfare loss under fairness requirements in both settings of goods [12, 28, 47] and chores [47, 77, 78]. Following previous studies [28, 47, 78], when we are concerned with the price of fairness, we assume that each agent i's valuation function is normalized; that is, $v_i(E) = 1$ for goods and $v_i(E) = -1$ for chores. Given an instance $\mathcal{I}$ and a welfare function $W \in \{\mathsf{EW}, \mathsf{UW}\}$, denote by $\mathsf{OPT}_W(\mathcal{I})$ the maximum welfare with respect to W over all allocations of instance $\mathcal{I}$. For ease of notations, when the instance $\mathcal{I}$ is clear from the context, we use OPT_E and OPT_U to refer $\mathsf{OPT}_{EW}(\mathcal{I})$

and $\mathsf{OPT}_{UW}(\mathcal{I})$, respectively. Moreover, given a fairness criterion F and an instance $\mathcal{I}$, denote by $F(\mathcal{I})$ the set of all allocations satisfying F.

Definition 2.3.4. *Given a fair-goods instance $\mathcal{I}$, for any fairness criterion F and welfare function W, the price of F with respect to W is defined as*

$$\mathsf{PoF}_W = \sup_{\mathcal{I}} \min_{\mathbf{A} \in F(\mathcal{I})} \frac{\mathsf{OPT}_W(\mathcal{I})}{W(\mathbf{A})}.$$

In the case of chores, swap the positions of the numerator and denominator.

In the above definition, we apply the following convention for the case where the maximum welfare of a fair-chores instance is equal to zero: if some fair allocation can achieve welfare zero, then the price of fairness is 1; otherwise, the price of fairness is infinite. The price of fairness with respect to fairness notion F is also called *price of F*, i.e., price of EQ1 and price of EF1.

2.4 Mechanism Design

Under the mechanism design setting, agents are rational and selfish and will behave strategically to gain benefit as much as possible. In particular, each agent privately observes her preference, which is not available to other agents. This is modeled by the fact that agent i privately knows her true valuation function v_i. Note that in Chapter 5, the terminology v_i always refers to the true valuation function of agent i. We refer v_i as the *type* or *private value* of agent i and denote by V_i the set of type of agent i and by $V = V_1 \times \cdots V_n$ the set of all profiles. A *type profile* includes the type of all agents and is represented as $\mathbf{v} = (v_1, \ldots, v_n)$. In the context of mechanism design (Chapter 5), allocations are sometimes written in the form of matrices. With a slight abuse of notations, an allocation or an *allocation matrix* $\mathbf{A} = (a_{ji})_{j \in [m], i \in [n]}$ is an $m \times n$ matrix, and the i-th column A_i determines the assignment of agent i. In a *deterministic* allocation $\mathbf{A}$, the entry a_{ji} is equal to either 0 or 1, and $a_{ji} = 1$ implies that item e_j is (entirely) assigned to agent i; otherwise, not. In a *randomised* allocation $\mathbf{A}$, the entry a_{ji} is the probability of assigning item e_j to agent i and takes value $0 \leq a_{ji} \leq 1$. We, with a slight abuse of notation, also use A_i, the i-th column of matrix $\mathbf{A}$, to represent the set of (fractional) items or the bundle received by agent i under allocation $\mathbf{A}$. Hence, the value of agent i in allocation $\mathbf{A}$ is equal to $v_i(\mathbf{A}) = v_i(A_i) = \sum_{j \in [m]} a_{ji} v_i(e_j)$. Note that by well-known Birkhoffvon Neumann (BvN) decomposition, a randomised allocation can also be interpreted as a probability distribution over a set of deterministic allocations. As a consequence, for a randomised allocation $\mathbf{A}$, the quantity $\sum_{j\in[m]} a_{ji}v_i(e_j)$ is also the expected value received by agent i. Moreover, for a randomised allocation $\mathbf{A} = (a_{ji})_{j\in[m],i\in[n]}$, the entry a_{ji} can be interpreted as the fraction of item e_j assigned to agent i, and thus, every randomised allocation implements a *fractional* allocation.

Denote by $\mathbb{A}$ the set of allocation matrices. A deterministic/randomised *mechanism* asks each agent to report his type and outputs an allocation based on the reporting. Due to the combinatorial explosion, it is irrealistic for agents to report their value on every subset $S \subseteq E$. We then restrict to the additive setting, and in particular, each agent i bids $b_i(e)$ for every $e \in E$ and his bids on a set $S \subseteq E$ can be computed as $\sum_{e\in S} b_i(e)$. A mechanism $\mathcal{M}$ takes a reporting profile $\mathbf{b} = (b_1, \ldots, b_n)$ as input and outputs an allocation $\mathcal{M}(\mathbf{b}) \in \mathbb{A}$. We remark that agent i may report or bid b_i different from his type if misreporting can increase her value. To simplify notations, we use a common notation in the game theory literature; that is, given a reporting profile and a set of agent $S \subseteq [n]$, denote by b_S the set of bids from agents in S, and $b_{-S} = b_{[n]\setminus S}$. Similar notations hold for agents' type and the set of agents' type.

2.4.1 Strategyproofness and Other Properties

We now introduce the notion of *strategyproofness*, also known as *truthfulness*. In a strategyproof (SP) mechanism, it is always the best response for an agent i to report his type, irrespective of what is reported by the rest of the agents. In other words, no agent can gain additional value (compared to the value received when reporting truthfully) by misreporting. In the following, we formally define the strategyproofness in both deterministic and randomised settings.

Definition 2.4.1 (SP)**.** *A deterministic mechanism $\mathcal{M}$ is strategyproof if $\forall i \in [n]$, $\forall v_i \in V_i$, $\forall b_i \in V_i$, $\forall b_{-i} \in V_{-i}$, the following holds*

$$v_i(\mathcal{M}(v_i, b_{-i})) \geq v_i(\mathcal{M}(b_i, b_{-i})).$$

Definition 2.4.2 (SPIE)**.** *A randomised mechanism $\widetilde{\mathcal{M}}$ is strategyproof in expectation (SPIE) if $\forall i \in [n]$, $\forall v_i \in V_i$, $\forall b_i \in V_i$, $\forall b_{-i} \in V_{-i}$, the following holds,*

$$\mathbb{E}_{\mathbf{A}\sim\widetilde{\mathcal{M}}(v_i,b_{-i})}[v_i(A_i)] \geq \mathbb{E}_{\mathbf{A}\sim\widetilde{\mathcal{M}}(b_i,b_{-i})}[v_i(A_i)].$$

Strategyproofness can prevent manipulations from a single individual agent, but it may not be able to prohibit the strategic behaviour of a group of agents. We also consider *groupstrategyproof* (GSP) mechanisms, in which no group of people can collude to misreport their valuations in a way that makes at least one member of the group better off without making any of the remaining members worse off.

Definition 2.4.3 (GSP)**.** *A deterministic mechanism $\mathcal{M}$ is said to be* GSP *if there does not exist a coalition $S \subseteq N$ and a reporting profile (b_S, b_{-S}) such that for all $i \in S$, $v_i(\mathcal{M}(b_S, b_{-S})) \geq v_i(\mathcal{M}(v_S, b_{-S}))$ and at least one strict inequality holds.*

Definition 2.4.4 (GSPIE)**.** *A randomised mechanism $\widetilde{\mathcal{M}}$ is group-strategyproof in expectation (GSPIE) if there does not exist a coliation $S \subseteq N$ and a reporting profile (b_S, b_{-S}) such that for all $i \in S$, $\mathbb{E}_{\mathbf{A} \sim \widetilde{\mathcal{M}}(b_S, b_{-S})}[v_i(A_i)] \geq \mathbb{E}_{\mathbf{A} \sim \widetilde{\mathcal{M}}(v_S, b_{-S})}[v_i(A_i)]$ and at least one strict inequality holds.*

We are also concerned with the notion of *anonymity*, requiring that the identity of an agent does not affect the assignment received by her. Throughout the thesis, given a permutation $\sigma : [n] \to [n]$, let $\sigma(t)$ be the t-th element of $\sigma([n])$.

Definition 2.4.5. *A mechanism $\mathcal{M}$ is said to be anonymous if for any permutation σ and two reporting profiles $\mathbf{b} = (b_1, \ldots, b_n)$ and $\mathbf{b}' = (b_{\sigma(1)}, \ldots, b_{\sigma(n)})$ with output allocation matrices $\mathbf{A}$ and $\mathbf{A}'$, it holds that $A_{\sigma(t)} = A'_t$ for each $t \in [n]$.*

The last property we care about is *non-bossiness* [97, 101]. A mechanism is non-bossy if an agent cannot change the allocation of other agents without changing her own assignment.

Definition 2.4.6. *A mechanism $\mathcal{M}$ is said to be non-bossy if for any two different reporting profiles $\mathbf{b} = (b_i, b_{-i})$ and $\mathbf{b}' = (b'_i, b_{-i})$ with $\mathcal{M}(b_i, b_{-i}) = \mathbf{A}$ and $\mathcal{M}(b'_i, b_{-i}) = \mathbf{A}'$, $A_i = A'_i$ implies $A_j = A'_j$ for any $j \neq i$.*

Chapter 3

Fairness Criteria for Allocating Indivisible Chores: Connections and Efficiencies[1]

3.1 Introduction

There emerges a tremendous demand for fair division when a set of indivisible resources, such as classrooms, bandwidths, and properties, are divided among a group of agents. Although this field has attracted the attention of researchers, a large proportion of results are established when the underlying resources are goods. Whereas, in some real-life division problems, such as assigning tasks among workers and allocating teaching load among teachers, the objects to be allocated is chores bringing non-positive value or cost to agents. Compared to goods, the fair division problem of chores is relatively under-developed. At first glance, fair chores allocation is similar to the corresponding problems on goods. However, in general, chores allocation is not covered by goods allocation and results established on goods do not necessarily hold on chores. Existing works have already pointed out the difference [36, 37, 44, 66, 67]. As an example, Freeman et al. [66] indicate that, when allocating goods, a *leximin*[2] allocation is PO and EQX, however, a leximin solution does not even guarantee EQX in chores allocation.

In this chapter, we restrict ourselves to the setting of indivisible chores and study the notions of EF1, EFX, MMS and PMMS, which have been well-studied

[1]This chapter is based on a research article by Sun et al. [99]

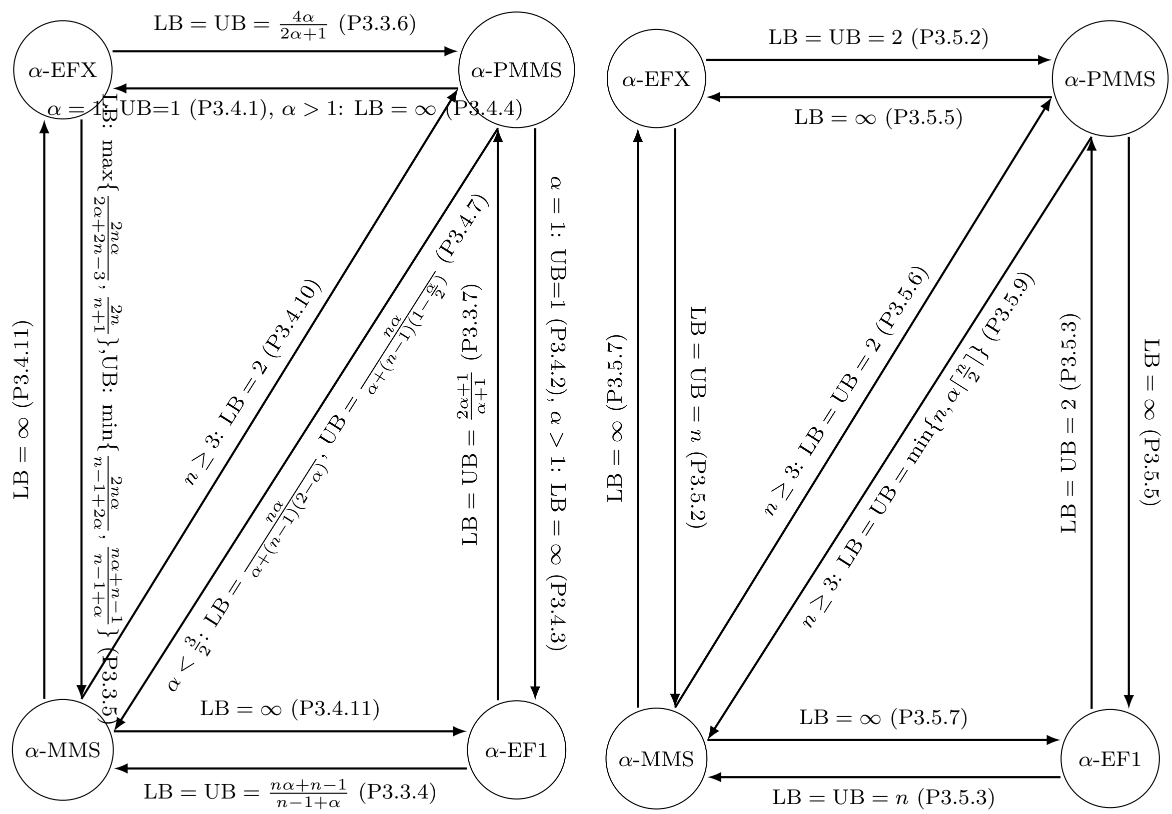
[2]A leximin solution selects the allocation that maximizes the utility of the least well-off agent, subject to maximizing the utility of the second least, and so on.

in the case of goods. Amanatidis et al. [7] compare the above four relaxations of envy-freeness and provide results on the approximation guarantee of one to another. A direct implication of these connections is that an approximation algorithm of one notion can be directly transferred to the approximation algorithm of another. Bei et al. [28] quantify the efficiency loss under fairness constraints, such as EF1, in the case of goods and provide a sequence of results on the price of fairness. The connections of these fairness criteria and the efficiency loss when enforcing allocation fairness have not been resolved in the case of chores. In this chapter, we fill these gaps by investigating the afore-mentioned four fairness notions on two aspects. On the one hand, we study the connections between these criteria and, in particular, we consider the following questions: *Does one fairness criterion implies another? To what extent can one criterion guarantee for another?* On the other hand, we study the trade-off between fairness and efficiency. Specifically, for each fairness criterion, we investigate its price of fairness with respect to the utilitarian welfare.

On the connections between fairness criteria, we summarize our main results in Figure 3.1 on the approximation guarantee of one fairness criterion for another. Figures 3.1a and 3.1b show connections when agents' valuation functions are additive and submodular, respectively. As shown in Figure 3.1a below, when agents have additive valuation functions, there exist evidently significant connections between these fairness notions. While some of our results show similarity to those in goods allocation [7], others also reveal their difference. Figure 3.1b provides the corresponding results under the submodular setting, which then show a sharp contrast to results under the additive setting. More specifically, except that PMMS can have a bounded approximation guarantee on MMS, no non-trivial guarantee exists between any other pair of fairness notions. After comparing each pair of fairness notions, we compare the utilitarian welfare of fair allocations with the maximum welfare of all allocations. To quantify the efficiency loss, we apply the idea of the price of fairness and our results are summarized in Table 3.1. As detailed later in the paper, most of the results summarized in Figure 3.1 and Table 3.1 are tight.

3.1.1 Related Works

The fair division problem has been studied for both indivisible goods [85, 32, 48] and indivisible chores [19, 15, 67]. Among various fairness notions, a prominent one is EF proposed by Foley [64]. But an EF allocation may not exist and even worse, checking the existence of an EF allocation is NP-complete [16]. For the relaxations of envy-freeness, the notion of EF1 originates from Lipton et al. [85] and



(a) Additive **(b)** Submodular

Note: Figure 3.1a and Figure 3.1b illustrate connections between fairness criteria under additive and submodular valuation functions, respectively. LB and UB stand for lower and upper bound, respectively. P$x.y$ points to Proposition $x.y$

Figure 3.1: Connections between fairness criteria

is formally defined by Budish [45]. Lipton et al. [85] provide an efficient algorithm for EF1 allocations of goods when agents have monotone valuation functions. When allocating chores, Aziz et al. [14] show that, in the additive setting, EF1 is achievable by allocating chores in a round-robin fashion. Another fairness notion that has been a subject of much attention in the last few years is MMS, proposed by Budish [45]. However, existence of an MMS allocation is not guaranteed either for goods [83] or for chores [19], even with additive valuation functions. Consequently, more efforts are on approximation of MMS in the additive setting, with Amanatidis et al. [8], Ghodsi et al. [71], Garg and Taki [70] on goods and Aziz et al. [19], Huang and Lu [79] on chores. Some other studies consider approximating MMS when agents have (a subclass of) submodular valuation functions. Barman and Krishnamurthy [23] consider the submodular setting and show that 0.21-approximation of MMS can be efficiently computed by the round-robin algorithm. Barman and Verma [24] show

<table>
<tr><th></th><th>EFX</th><th>EF1</th><th>PMMS</th><th>$\frac{3}{2}$-PMMS</th><th>2-MMS</th><th></th></tr>
<tr><td rowspan="2">$n = 2$</td><td>2
(P3.6.4)</td><td>$\frac{5}{4}$
(P3.6.1)</td><td>2
(P3.6.4)</td><td>$\frac{7}{6}$
(P3.6.3)</td><td rowspan="2">1
(L3.2.2)</td><td>additive</td></tr>
<tr><td>$[3, 4)$
(P3.7.1)</td><td>$[2, 4)$
(P3.7.2)</td><td>3
(P3.7.3)</td><td>$[\frac{4}{3}, \frac{8}{3})$
(P3.7.4)</td><td>submodular</td></tr>
<tr><td rowspan="2">$n \geq 3$</td><td colspan="4" rowspan="2">∞
(P3.6.5)</td><td>$[\frac{n+3}{6}, n)$
(P3.6.7)</td><td>additive</td></tr>
<tr><td>$[\frac{n+3}{6}, \frac{n^2}{2})$
(P3.7.6)</td><td>submodular</td></tr>
</table>

Note: Interval $[a, b)$ means that the lower bound is equal to a and upper bound is equal to b. P$x.y$ and L$x.y$ point to Proposition $x.y$ and Lemma $x.y$, respectively.

Table 3.1: Prices of fairness

that an MMS allocation is guaranteed to exist and can be computed efficiently if agents have submodular valuation functions with binary margin.

The notions of EFX and PMMS are introduced by Caragiannis et al. [48]. They consider goods allocation and establish that a PMMS allocation is also EFX when the valuation functions are additive. Beyond the simple case of $n = 2$, the existence of an EFX allocation has not been settled in general. However, significant progress has been made for some special cases. When $n = 3$, the existence of an EFX allocation of goods is proved by Chaudhury et al. [51]. Based on a modified version of leximin solutions, Plaut and Roughgarden [93] show that an EFX allocation is guaranteed to exist when all agents have identical valuations. The work most related to ours is by Amanatidis et al. [7], which is on goods allocation under additive setting, and provides connections among the above four relaxations of envy-freeness.

As for the price of fairness, Caragiannis et al. [47] show that, in the case of *divisible* goods, the price of proportionality is $\Theta(\sqrt{n})$ and the price of equitability is $\Theta(n)$. Bertsimas et al. [30] extend the study to other fairness notions, *maximin*[3] fairness and proportional fairness, and they provide a tight bound on the price of fairness for a broad family of problems. Bei et al. [27] focus on indivisible goods and concentrate on the fairness notions that are guaranteed to exist. They present an asymptotically tight upper bound of $\Theta(n)$ on the price of maximum Nash welfare [54], maximum egalitarian welfare [43] and leximin. They also consider the price of EF1 but leave a gap between the upper bound $O(n)$ and lower bound $\Omega(\sqrt{n})$.

[3]It maximizes the lowest utility level among all the agents.

This gap is later closed by Barman et al. [22] with the results that, for both EF1 and (1/2)-MMS, the price of fairness is $O(\sqrt{n})$. All the work reviewed above on the price of fairness is on the additive setting. On the other hand, the price of fairness has been studied in other multi-agent systems, such as machine scheduling [1] and kidney exchange [59].

3.1.2 Preliminaries

As this whole chapter focuses on the allocation of chores, to avoid frequently switching the terminologies of valuation functions and cost functions, in this chapter we instead assume that each agent i is associated with a cost function $c_i : 2^E \to \mathbb{R}_{\geq 0}$. The notion of cost function c_i is used only in this chapter. We then rewrite the definitions of the fairness criteria and the utilitarian welfare function by cost functions $c_i(\cdot)$.

Definition 3.1.1 (α-EF). *For any $\alpha \geq 1$, an allocation $\mathbf{A} = (A_1, \ldots, A_n)$ is* α-EF *if for any $i, j \in [n]$, $c_i(A_i) \leq \alpha \cdot c_i(A_j)$ holds.*

Definition 3.1.2 (α-EF1). *For any $\alpha \geq 1$, an allocation $\mathbf{A} = (A_1, \ldots, A_n)$ is* α-EF1 *if for any $i, j \in [n]$, there exists $e \in A_i$ such that $c_i(A_i \setminus \{e\}) \leq \alpha \cdot c_i(A_j)$.*

Definition 3.1.3 (α-EFX). *For any $\alpha \geq 1$, an allocation $\mathbf{A} = (A_1, \ldots, A_n)$ is* α-EFX *if for any $i, j \in [n]$ and for any $e \in A_i$ with $c_i(e) > 0$, any $c_i(A_i \setminus \{e\}) \leq \alpha \cdot c_i(A_j)$ holds.*

We remark that Plaut and Roughgarden [93] consider a stronger version of EFX by dropping the condition $c_i(e) > 0$. In this chapter, all results about EFX, except Propositions 3.4.1 and 3.4.6, still hold under the stronger version.

With cost functions, the maximin share of agent i on set S among k agents is defined as

$$\mathsf{MMS}_i(k, S) = \min_{A \in \Pi_k(S)} \max_{j \in [k]} c_i(A_j).$$

As for the (approximate) MMS fairness, we have the following definition.

Definition 3.1.4 (α-MMS). *For any $\alpha \geq 1$, an allocation $\mathbf{A} = (A_1, \ldots, A_n)$ is* α-MMS *fair if for any $i \in [n]$, $c_i(A_i) \leq \alpha \cdot \mathsf{MMS}_i(n, E)$ holds.*

Definition 3.1.5 (α-PMMS). *For any $\alpha \geq 1$, an allocation $\mathbf{A} = (A_1, \ldots, A_n)$ is* α-PMMS *fair if for any $i, j \in [n]$, $c_i(A_i) \leq \alpha \cdot \mathsf{MMS}_i(2, A_i \cup A_j)$ holds.*

Definition 3.1.6. *Given an allocation $\mathbf{A} = (A_1, \ldots, A_n)$, the social cost of $\mathbf{A}$ is $\sum_{i \in [n]} c_i(A_i)$.*

We remark that with cost functions, the utilitarian/egalitarian welfare-maximizing allocation now becomes the one that can minimize the underlying welfare.

3.2 Some Simple Observations

In this section, we first use a concrete example to provide the very first impression of the connections among fairness criteria. Following the example, we present two simple but non-trivial results, of which one is a lower bound of the maximin share and the other one is a bound of an allocation on the notions of PMMS and MMS.

Example 3.2.1. *Let us consider an example with three agents and a set* $E = \{e_1, \ldots, e_7\}$ *of seven chores. Agents have additive cost functions, displayed in the table below.*

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
Agent 1	2	3	3	0	4	2	1
Agent 2	3	1	3	2	5	0	5
Agent 3	1	5	10	2	3	1	3

It is not hard to verify that $\mathsf{MMS}_1(3, E) = 5, \mathsf{MMS}_2(3, E) = 7$, $\mathsf{MMS}_3(3, E) = 10$. *For instance, agent 2 can partition* E *into three bundles:* $\{e_1, e_3\}, \{e_2, e_7\}$, $\{e_4, e_5, e_6\}$, *so that the maximum cost of any single bundle for her is 7. Moreover, there is no other partition that can guarantee a better worst-case cost.*

We now examine allocation $\mathbf{A}$ *with* $A_1 = \{e_1, e_4, e_7\}, A_2 = \{e_2, e_3, e_6\}, A_3 = \{e_5\}$. *We can verify that* $c_i(A_i) \leq c_i(A_j)$ *for any* $i, j \in [3]$ *and thus allocation* $\mathbf{A}$ *is* EF *that is then also* EFX, EF1, MMS *and* PMMS. *For another allocation* $\mathbf{B}$ *with* $B_1 = \{e_1, e_5, e_7\}, B_2 = \{e_2, e_4, e_6\}, B_3 = \{e_3\}$, *agent 1 would still envy agent 2 even if chore* e_7 *is eliminated from her bundle, and hence, allocation* $\mathbf{B}$ *is neither exact* EF *nor* EFX. *One can verify that* $\mathbf{B}$ *is indeed* 7/3-EF *and* 2-EFX. *Moreover, allocation* $\mathbf{B}$ *is* EF1 *because agent 1 would not envy others if chore* e_5 *is eliminated from her bundle and agent 3 would not envy others if chore* e_3 *is eliminated from her bundle. As for the approximation guarantee on the notions of MMS and PMMS, it is not hard to verify that allocation* $\mathbf{B}$ *is* 7/5-MMS *and* 7/5-PMMS.

Next, we present some initial results, which reveal some intrinsic difference in allocating goods and allocating chores as far as approximation guarantee is concerned. First, we state a simple lemma concerning lower bounds of the maximin share.

Lemma 3.2.1. *When agents have subadditive cost functions, for any $i \in N$ and $S \subseteq E$, we have*

$$\mathsf{MMS}_i(k,S) \geq \frac{1}{k}c_i(S), \forall k \in [n]; \qquad \mathsf{MMS}_i(k,S) \geq c_i(e), \forall e \in S, \forall k \in [n].$$

Proof. Let $\mathbf{T} = (T_1, \ldots, T_k)$ be the k-partition of S defining $\mathsf{MMS}_i(k,S)$; that is $\max_{T_j} c_i(T_j) = \mathsf{MMS}_i(k,S)$. We start with the lower bound $k^{-1}c_i(S)$. Without loss of generality, assume $c_i(T_1) \geq c_i(T_2) \geq \cdots \geq c_i(T_k)$ and as a result, we have $c_i(T_1) = \mathsf{MMS}_i(k,S)$. Then, the following holds

$$kc_i(T_1) \geq \sum_{j=1}^{k} c_i(T_j) \geq c_i(\bigcup_{j=1}^{k} T_j) = c_i(S),$$

where the second transition is due to subadditivity. Due to $c_i(T_1) = \mathsf{MMS}_i(k,S)$, we have $\mathsf{MMS}_i(k,S) \geq k^{-1}c_i(S)$. As for the lower bound $c_i(e)$, for any given chore $e \in S$, there must exist a bundle $T_{j'}$ containing e. Due to the monotonicity of cost function, we have $c_i(T_{j'}) \geq c_i(e)$, which combines $\mathsf{MMS}_i(k,S) = c_1(T_1) \geq c_1(T_{j'})$, implying $\mathsf{MMS}_i(k,S) \geq c_i(e)$. □

Based on the lower bounds in Lemma 3.2.1, we provide a trivial approximation guarantee for PMMS and MMS.

Lemma 3.2.2. *When agents have subadditive cost functions, any allocation is 2-*PMMS *and n-*MMS*.*

Proof. Let $\mathbf{A} = (A_1, \ldots, A_n)$ be an arbitrary allocation without any specified properties. We first show it's already an n-MMS allocation. By Lemma 3.2.1, for each agent i, we have $c_i(E) \leq n \cdot \mathsf{MMS}_i(n,E)$. Then, due to the monotonicity of the cost function, $c_i(A_i) \leq c_i(E) \leq n \cdot \mathsf{MMS}_i(n,E)$ holds.

Next, by a similar argument, we prove the result about 2-PMMS. By Lemma 3.2.1, $c_i(A_i \cup A_j) \leq 2\mathsf{MMS}_i(2, A_i \cup A_j)$ holds for any $i,j \in N$. Again, due to the monotonicity of the cost function, we have $c_i(A_i) \leq c_i(A_i \cup A_j)$ that implies $c_i(A_i) \leq 2\mathsf{MMS}_i(2, A_i \cup A_j)$. Therefore, allocation $\mathbf{A}$ is also 2-PMMS, completing the proof. □

As can be seen from the proof of Lemma 3.2.2, in allocating chores, if one assigns all chores to one agent, then the allocation still has a bounded approximation for PMMS and MMS. However, when allocating goods, if an agent receives nothing but his maximin share is positive, then clearly the corresponding allocation has an infinite approximation guarantee for PMMS and MMS.

3.3 Guarantees from Envy-Based Relaxations

Let us start with EF. According to definitions, for any $\alpha \geq 1$, α-EF is stronger than α-EFX and α-EF1. Following propositions present the approximation guarantee of α-EF for MMS and PMMS.

Proposition 3.3.1. *When agents have additive cost functions, for any $\alpha \geq 1$, an α-EF allocation is also $\frac{n\alpha}{n-1+\alpha}$-MMS fair, and this result is tight.*

Proof. We first prove the upper bound and focus on agent i. Let $\mathbf{A} = (A_1, \ldots, A_n)$ be an α-EF allocation, then according to its definition, $c_i(A_i) \leq \alpha \cdot c_i(A_j)$ holds for any $j \in N$. By summing up j over $N \backslash \{i\}$, we have $(n-1)c_i(A_i) \leq \alpha \cdot \sum_{j \in N \backslash \{i\}} c_i(A_j)$ and as a result, $(n-1+\alpha)c_i(A_i) \leq \alpha \cdot \sum_{j \in N} c_i(A_j) = \alpha \cdot c_i(E)$ where the last transition is by the additivity of cost functions. On the other hand, from Lemma 3.2.1, it holds that $\mathsf{MMS}_i(n, E) \geq \frac{1}{n} c_i(E)$, implying the ratio

$$\frac{c_i(A_i)}{\mathsf{MMS}_i(n, E)} \leq \frac{n\alpha}{n-1+\alpha}.$$

Regarding tightness, consider the following instance with n agents and n^2 chores denoted as $\{e_1, \ldots, e_{n^2}\}$. Agents have an identical cost profile and for every $i \in [n]$, $c_i(e_j) = \alpha$ for $1 \leq j \leq n$ and $c_i(e_j) = 1$ for $n+1 \leq j \leq n^2$. Consider allocation $\mathbf{B} = (B_1, \ldots, B_n)$ with $B_i = \{e_{(i-1)n+1}, \ldots, e_{in}\}$ for any $i \in N$. It is not hard to verify that allocation $\mathbf{B}$ is α-EF. As for $\mathsf{MMS}_1(n, E)$, since in total we have n chores with cost α each and $(n-1)n$ chores with cost 1 each, then in the partition defining $\mathsf{MMS}_1(n, E)$, each bundle contains exactly one chore with cost α and $n-1$ chores with cost 1. Consequently, we have $\mathsf{MMS}_1(n, E) = n - 1 + \alpha$ and the approximation ratio is equal to

$$\frac{c_1(B_1)}{\mathsf{MMS}_1(n, E)} = \frac{n\alpha}{n-1+\alpha},$$

which completes the proof. □

Proposition 3.3.2. *When agents have additive cost functions, for any $\alpha \geq 1$, an α-EF allocation is also $\frac{2\alpha}{1+\alpha}$-PMMS fair, and this result is tight.*

Proof. We first prove the upper bound. Let $\mathbf{A} = (A_1, A_2, \ldots, A_n)$ be an α-EF allocation, then according to the definition, for any $i, j \in N$, $c_i(A_i) \leq \alpha \cdot c_i(A_j)$ holds. By additivity, we have $c_i(A_i \cup A_j) = c_i(A_i) + c_i(A_j) \geq (1 + \alpha^{-1}) \cdot c_i(A_i)$, and consequently, $c_i(A_i) \leq \frac{\alpha}{\alpha+1} \cdot c_i(A_i \cup A_j)$ holds. On the other hand, from Lemma 3.2.1, we know $c_i(A_i \cup A_j) \leq 2 \cdot \mathsf{MMS}_i(2, A_i \cup A_j)$, and therefore the following holds

$$c_i(A_i) \leq \frac{2\alpha}{\alpha+1} \cdot \mathsf{MMS}_i(2, A_i \cup A_j).$$

As for tightness, consider an instance with n agents and $2n$ chores denoted as $\{e_1, e_2, \ldots, e_{2n}\}$. Agents have identical cost profile and for every $i \in [n]$, $c_i(e_1) = c_i(e_2) = \alpha$ and $c_i(e_j) = 1$ for $3 \leq j \leq 2n$. Now, consider an allocation $\mathbf{B} = (B_1, \ldots, B_n)$ where $B_i = \{e_{2i-1}, e_{2i}\}$ for any $i \in N$. It is not hard to verify that allocation $\mathbf{B}$ is α-EF and except for agent 1, no one else will violate the condition of PMMS. For any $j \geq 2$, one can calculate $\mathsf{MMS}_1(2, B_1 \cup B_j) = 1 + \alpha$, yielding the ratio $\frac{c_1(B_1)}{\mathsf{MMS}_1(2, B_1 \cup B_j)} = \frac{2\alpha}{1+\alpha}$, as required. □

Proposition 3.3.2 indicates that the approximation guarantee of α-EF for PMMS is independent of the number of agents. However, according to Proposition 3.3.1, its approximation guarantee for MMS is affected by the number of agents. Moreover, this guarantee ratio converges to α as n goes to infinity.

We remark that none of EFX, EF1, PMMS and MMS has a bounded guarantee for EF. We show this by a simple example. Consider an instance of two agents and one chore, and the chore has a positive cost for both agents. Assigning the chore to an arbitrary agent results in an allocation that satisfies EFX, EF1, PMMS and MMS, simultaneously. However, since one agent has a positive cost for his own bundle and zero cost for other agent's bundle, such an allocation has an infinite approximation guarantee for EF.

Next, we consider approximation of EFX and EF1.

Proposition 3.3.3. *When agents have additive cost functions, an* α-EFX *allocation is* α-EF1 *for any* $\alpha \geq 1$. *On the other hand, an* EF1 *allocation is not* β-EFX *for any* $\beta \geq 1$.

Proof. We first show the positive part. Let $\mathbf{A} = (A_1, A_2, \ldots, A_n)$ be an α-EFX allocation, then according to its definition, $\forall i, j \in N, \forall e \in A_i$ with $c_i(e) > 0$, $c_i(A_i \setminus \{e\}) \leq \alpha \cdot c_i(A_j)$ holds. This implies $\mathbf{A}$ is also α-EF1.

For the impossibility result, consider an instance with n agents and $2n$ chores denoted as $\{e_1, e_2, \ldots, e_{2n}\}$. Agents have identical cost profile. The cost function of agent 1 is: $c_1(e_1) = p, c_1(e_j) = 1, \forall j \geq 2$ where $p \gg 1$. Now consider an allocation $\mathbf{B} = (B_1, \ldots, B_n)$ with $B_i = \{e_{2i-1}, e_{2i}\}, \forall i \in N$. It is not hard to see allocation $\mathbf{B}$ is EF1 and except for agent 1, no one else will envy the bundle of others. Thus, we only concern agent 1 when calculating the approximation guarantee for EFX. By removing chore e_2 from bundle B_1, $\frac{c_1(B_1 \setminus \{e_2\})}{c_1(B_j)} = \frac{p}{2}$ holds for any $j \in N \setminus \{1\}$, and the ratio $\frac{p}{2} \to \infty$ as $p \to \infty$. $\square$

Next, we consider the approximation guarantee of EF1 for MMS. In allocating goods, Amanatidis et al. [7] present a tight result that an α-EF1 allocation is $O(n)$-MMS. In contrast, in allocating chores, α-EF1 can have a much better guarantee for MMS.

Proposition 3.3.4. *When agents have additive cost functions, for any $\alpha \geq 1$ and $n \geq 2$, an α-EF1 allocation is also $\frac{n\alpha+n-1}{n-1+\alpha}$-MMS, and this result is tight.*

Proof. We first prove the upper bound. Let $\mathbf{A} = (A_1, \ldots, A_n)$ be an α-EF1 allocation and the approximation guarantee for MMS is determined by agent i. We can further assume $c_i(A_i) > 0$; otherwise agent i meets the condition of MMS and we are done. Let $\bar{e}$ be the chore with largest cost for agent i in bundle A_i, i.e., $\bar{e} \in \arg\max_{e \in A_i} c_i(e)$.

By the definition of α-EF1, for any $j \in N \setminus \{i\}$, $c_i(A_i \setminus \{\bar{e}\}) \leq \alpha \cdot c_i(A_j)$ holds. Then, by summing up over $j \in N \setminus \{i\}$ and adding a term $\alpha c_i(A_i)$ on both sides, the following holds,

$$\alpha \cdot \sum_{j \in N} c_i(A_j) \geq (n-1+\alpha) c_i(A_i) - (n-1) c_i(\bar{e}). \tag{3.1}$$

From Lemma 3.2.1, we have $\text{MMS}_i(n, E) \geq \max\{\frac{1}{n} c_i(E), c_i(\bar{e})\}$, and by additivity, it holds that

$$n\alpha \mathsf{MMS}_i(n, E) \geq (n-1+\alpha) c_i(A_i) - (n-1) \mathsf{MMS}_i(n, E). \tag{3.2}$$

Inequality (3.2) is equivalent to $\frac{c_i(A_i)}{\mathsf{MMS}_i(n,E)} \leq \frac{n\alpha+n-1}{n-1+\alpha}$, as required.

As for tightness, consider the following instance with n agents and a set $E = \{e_1, \ldots, e_{n^2-n+1}\}$ of $n^2 - n + 1$ chores. Agents have an identical cost profile and for every $i \in [n]$, $c_i(e_1) = \alpha + n - 1$, $c_i(e_j) = \alpha$ for any $2 \leq j \leq n$ and $c_i(e_j) = 1$ for $j \geq n+1$. Now, consider an allocation $\mathbf{B} = \{B_1, \ldots, B_n\}$ with $B_1 = \{e_1, \ldots, e_n\}$ and $B_j = \{e_{n+(n-1)(j-2)+1}, \cdots, e_{n+(n-1)(j-1)}\}$ for any $j \geq 2$. Then, we have $c_i(B_j) = n-1$ for any $i \in [n]$ and $j \geq 2$. Accordingly, except for agent 1, no one else will violate the condition of α-EF1 and MMS. As for agent 1, since $c_1(B_1 \setminus \{e_1\}) = (n-1)\alpha = \alpha c_1(B_j), \forall j \geq 2$, then we can claim that allocation $\mathbf{B}$ is α-EF1. To calculate $\mathsf{MMS}_1(n, E)$, consider an allocation $\mathbf{T} = (T_1, \ldots, T_n)$ with $T_1 = \{e_1\}$ and $T_j = \{B_j \cup \{e_j\}\}$ for any $2 \leq j \leq n$. It is not hard to verify that $c_1(T_j) = \alpha + n - 1$ for any $j \in N$. Therefore, we have $\mathsf{MMS}_1(n, E) = \alpha + n - 1$

implying the ratio $\frac{c_1(B_1)}{\mathsf{MMS}_1(n,E)} = \frac{n\alpha+n-1}{n-1+\alpha}$, completing the proof. $\square$

We now study α-EFX in terms of its approximation guarantee for MMS and provide upper and lower bounds for general $\alpha \geq 1$ or $n \geq 2$.

Proposition 3.3.5. *When agents have additive cost functions, for any $\alpha \geq 1$ and $n \geq 2$, an α-*EFX *allocation is* $\min\left\{\frac{2n\alpha}{n-1+2\alpha}, \frac{n\alpha+n-1}{n-1+\alpha}\right\}$-MMS *fair, while it is not guaranteed to be β-*MMS *fair for any* $\beta < \max\left\{\frac{2n\alpha}{2\alpha+2n-3}, \frac{2n}{n+1}\right\}$.

Proof. We first prove the upper bound. Let $\mathbf{A} = (A_1, \ldots, A_n)$ be an α-EFX allocation with $\alpha \geq 1$ and the approximation guarantee for MMS is determined by agent i. The upper bound $\frac{n\alpha+n-1}{n-1+\alpha}$ directly follows from Propositions 3.3.3 and 3.3.4. In what follows, we prove the upper bound $\frac{2n\alpha}{n-1+2\alpha}$. We assume $c_i(A_i) > 0$; otherwise agent i meets the condition of MMS and we are done. Let e^* be the chore in bundle A_i having the minimum non-zero cost for agent i, and the existence of e^* is guaranteed by $c_i(A_i) > 0$. Next, we divide the proof into two cases.

Case 1: $|A_i| = 1$. Then e^* is the only item in A_i and thus $c_i(A_i) = c_i(e^*)$. By Lemma 3.2.1, $c_i(e^*) \leq \mathsf{MMS}_i(n, E)$ holds, and thus, $c_i(A_i) \leq \mathsf{MMS}_i(n, E)$.

Case 2: $|A_i| \geq 2$. By the definition of α-EFX, for any $j \in N \setminus \{i\}$, we have

$$c_i(A_i \setminus \{e^*\}) \leq \max_{e \in A_i} c_i(A_i \setminus \{e\}) \leq \alpha \cdot c_i(A_j).$$

If e^* is the only item with non-zero cost in A_i, then $c_i(A_i) \leq \mathsf{MMS}_i(n, E)$ based on Lemma 3.2.1, and we are done. If in A_i, there are at least two items with non-zero cost for agent i, by the definition of e^*, we have $c_i(e^*) \leq \frac{1}{2} c_i(A_i)$. Then, the following holds,

$$\alpha \cdot c_i(A_j) \geq c_i(A_i) - c_i(e^*) \geq \frac{1}{2} c_i(A_i), \qquad \forall j \in N \setminus \{i\}. \tag{3.3}$$

By summing up j over $N \setminus \{i\}$ and adding a term $\alpha c_i(A_i)$ on both sides of inequality (3.3), the following holds

$$\alpha \cdot c_i(E) = \alpha \cdot \sum_{j \in N \setminus \{i\}} c_i(A_j) + \alpha \cdot c_i(A_i) \geq \frac{n-1+2\alpha}{2} c_i(A_i). \tag{3.4}$$

On the other hand, from Lemma 3.2.1, we know $\mathsf{MMS}_i(n, E) \geq \frac{1}{n} c_i(E)$, which together with inequality (3.4) yields the ratio

$$\frac{c_i(A_i)}{\mathsf{MMS}_i(n, M)} \leq \frac{2n\alpha}{n-1+2\alpha}.$$

Regarding the lower bound $\frac{2n}{n+1}$, consider an instance with n agents and a set $E = \{e_1, e_2, ..., e_{2n}\}$ of $2n$ chores. Agents have identical cost profile and $c_i(e_j) = \lceil \frac{j}{2} \rceil$ for any i, j. It is not hard to verify that for any $i \in [n]$, $\mathsf{MMS}_i(n, E) = n + 1$. Then, consider the allocation $\mathbf{B} = (B_1, ..., B_n)$ with $B_1 = \{e_{2n-1}, e_{2n}\}$ and $B_i = \{e_{i-1}, e_{2n-i}\}$ for any $i \geq 2$. Accordingly, we have $c_i(B_j) = n$ for any $i \in [n]$ and $j \geq 2$. Thus, except for agent 1, no one else will violate the condition of MMS and EFX. As for agent 1, since $c_1(B_1 \setminus \{e_{2n}\}) = c_1(B_1 \setminus \{e_{2n-1}\}) = n$, envy can be eliminated by removing any single chore . Hence, the allocation $\mathbf{B}$ is EFX and its approximation guarantee for MMS equals to $\frac{c_1(B_1)}{\mathsf{MMS}_1(n,E)} = \frac{2n}{n+1}$, as required.

Next, for lower bound $\frac{2n\alpha}{2\alpha+2n-3}$, let us consider an instance with n agents and a set $E = \{e_1, ..., e_{2n^2-2n}\}$ of $2n^2 - 2n$ chores. We focus on agent 1 with cost function $c_1(e_j) = 2\alpha$ for $1 \leq j \leq n$ and $c_1(e_j) = 1$ for $j \geq n + 1$. Consider the allocation $\mathbf{B} = (B_1, ..., B_n)$ with $B_1 = \{e_1, ..., e_n\}, B_2 = \{e_{n+1}, ..., e_{3n-2}\}$ and $B_j = \{e_{3n-1+(j-3)(2n-1)}, \ldots, e_{3n-2+(j-2)(2n-1)}\}$ for any $j \geq 3$. Accordingly, bundle B_2 contains $2n - 2$ chores and B_j contains $2n - 1$ chores for any $j \geq 3$. For any agent $i \geq 2$, her cost functions is $c_i(e) = 0$ for $e \in B_i$ and $c_i(e) = 1$ for $e \in E \setminus B_i$. Consequently, except for agent 1, no one else violates the condition of MMS and α-EFX. As for agent 1, his cost on B_2 is the smallest over all bundles and $c_1(B_1 \setminus \{e_1\}) = 2\alpha(n - 1) = \alpha c_1(B_2)$, as a result, the allocation $\mathbf{B}$ is α-EFX. For $\mathsf{MMS}_1(n, E)$, it happens that E can be evenly divided into n bundles of the same cost (for agent 1), so we have $\mathsf{MMS}_1(n, E) = 2\alpha + 2n - 3$ implying the ratio $\frac{c_1(B_1)}{\mathsf{MMS}_1(n,E)} = \frac{2n\alpha}{2\alpha+2n-3}$, completing the proof. □

The performance bound in Proposition 3.3.5 is almost tight since $\frac{n\alpha+n-1}{n-1+\alpha} - \frac{2n\alpha}{2\alpha+2n-3} < \frac{n-1}{n-1+\alpha} < 1$. In addition, we highlight that the upper and lower bounds provided in Proposition 3.3.5 are tight in two interesting cases: (i) $\alpha = 1$ and (ii) $n = 2$.

On the approximation of EFX and EF1 for PMMS, we have the following propositions.

Proposition 3.3.6. *When agents have additive cost functions, for any $\alpha \geq 1$, an α-*EFX *allocation is also $\frac{4\alpha}{2\alpha+1}$-*PMMS *fair, and this guarantee is tight.*

Proof. We first prove the upper bound. Let $\mathbf{A} = (A_1, A_2, \ldots, A_n)$ be an α-EFX allocation and the approximation guarantee for PMMS is determined by agent i. We can assume $c_i(A_i) > 0$; otherwise agent i meets the condition of PMMS and we are done. Let e^* be the chore in A_i having the minimum cost for agent i, i.e., $e^* \in \arg\min_{e \in A_i} c_i(e)$. Then, we divide the proof into two cases.

Case 1: $|A_i| = 1$. Then chore e^* is the unique element in A_i, and thus

$c_i(e^*) = c_i(A_i)$. By Lemma 3.2.1, $c_i(e^*) \leq \mathsf{MMS}_i(2, A_i \cup A_j)$ holds for any $j \in N \setminus \{i\}$. As a result, we have $c_i(A_i) \leq \mathsf{MMS}_i(2, A_i \cup A_j), \forall j \in N \setminus \{i\}$.

Case 2: $|A_i| \geq 2$. Since $e^* \in \arg\min_{e \in A_i} c_i(e)$ and $|A_i| \geq 2$, we have $c_i(e^*) \leq \frac{1}{2} c_i(A_i)$, and equivalently, $c_i(A_i \setminus \{e^*\}) = c_i(A_i) - c_i(e^*) \geq \frac{1}{2} c_i(A_i)$. Then, based on the definition of α-EFX allocation, for any $j \in N \setminus \{i\}$, the following holds

$$\alpha \cdot c_i(A_j) \geq c_i(A_i \setminus \{e^*\}) \geq \frac{1}{2} \cdot c_i(A_i). \tag{3.5}$$

Combining Lemma 3.2.1 and Inequality (3.5), for any $j \in N \setminus \{i\}$, we have

$$\mathsf{MMS}_i(2, A_i \cup A_j) \geq \frac{1}{2}(c_i(A_i) + c_i(A_j)) \geq \frac{2\alpha + 1}{4\alpha} c_i(A_i).$$

Therefore, for any $j \in N \setminus \{i\}$, $c_i(A_i) \leq \frac{4\alpha}{2\alpha+1} \cdot \mathsf{MMS}_i(2, A_i \cup A_j)$ holds, as required.

As for the tightness, consider an instance with n agents and a set $E = \{e_1, \ldots, e_{2n}\}$ of $2n$ chores. Agents have identical cost profile and for every $i \in [n]$, $c_i(e_1) = c_i(e_2) = 2\alpha$ and $c_i(e_j) = 1$ for $3 \leq j \leq 2n$. Consider the allocation $\mathbf{B} = (B_1, \ldots, B_n)$ with $B_i = \{e_{2i-1}, e_{2i}\}, \forall i \in N$. It is not hard to verify that, except for agent 1, no one else would violate the condition of EFX and PMMS. For agent 1, by removing any single chore from his bundle, the remaining cost is α times of the cost on others' bundle. Thus, allocation $\mathbf{B}$ is α-EFX. Notice that for any $j \geq 2$, bundle $B_1 \cup B_j$ contains exactly two chores with cost 2α and two chores with cost 1, then $\mathsf{MMS}_1(2, B_1 \cup B_j) = 2\alpha + 1$, implying for any $j \neq 1$, $\frac{c_1(B_1)}{\mathsf{MMS}_1(2, B_1 \cup B_j)} = \frac{4\alpha}{2\alpha+1}$, as required. $\square$

Proposition 3.3.7. *When agents have additive cost functions, for any $\alpha \geq 1$, an α-EF1 allocation is also $\frac{2\alpha+1}{\alpha+1}$-PMMS fair, and this guarantee is tight.*

Proof. We first prove the upper bound part. Let $\mathbf{A} = (A_1, \ldots, A_n)$ be an α-EF1 allocation and the approximation guarantee for PMMS is determined by agent i. We can assume $c_i(A_i) > 0$; otherwise agent i meets the condition of PMMS and we are done. To study PMMS, we fix another agent $j \in N \setminus \{i\}$, and let $e^* \in A_i$ be the chore such that $c_i(A_i \setminus \{e^*\}) \leq \alpha \cdot c_i(A_j)$. We divide our proof into two cases.

Case 1: $c_i(e^*) > c_i(A_i \cup A_j \setminus \{e^*\})$. Consider $\{\{e^*\}, A_i \cup A_j \setminus \{e^*\}\}$, a 2-partition of $A_i \cup A_j$. Since $c_i(e^*) > c_i(A_i \cup A_j \setminus \{e^*\})$, we can claim that this partition defining $\mathsf{MMS}_i(2, A_i \cup A_j)$, and accordingly, $\mathsf{MMS}_i(2, A_i \cup A_j) = c_i(e^*)$ holds. From Lemma 3.2.1 and the definition of α-EF1, the following holds

$$c_i(e^*) \geq \frac{1}{2}(c_i(A_i) + c_i(A_j)) \geq \frac{1}{2} c_i(A_i) + \frac{1}{2\alpha} \cdot c_i(A_i \setminus \{e^*\}). \tag{3.6}$$

Then, based on (3.6) and the fact $\mathsf{MMS}_i(2, A_i \cup A_j) = c_i(e^*)$, we have

$$\frac{c_i(A_i)}{\mathsf{MMS}_i(2, A_i \cup A_j)} \le \frac{2\alpha + 1}{\alpha + 1}.$$

Case 2: $c_i(e^*) \le c_i(A_i \cup A_j \setminus \{e^*\})$. By the definition of α-EF1, we have $c_i(A_i \setminus \{e^*\}) \le \alpha \cdot c_i(A_j)$. As a consequence,

$$c_i(A_i) = c_i(e^*) + c_i(A_i \setminus \{e^*\}) \le 2c_i(A_i \setminus \{e^*\}) + c_i(A_j) \le (2\alpha + 1) \cdot c_i(A_j), \quad (3.7)$$

where the first inequality transition is due to $c_i(e^*) \le c_i(A_i \cup A_j \setminus \{e^*\})$. Using Inequality (3.7) and additivity of cost function, we have $c_i(A_i) \le \frac{2\alpha+1}{2\alpha+2} \cdot c_i(A_i \cup A_j)$. By Lemma 3.2.1, we have $\mathsf{MMS}_i(2, A_i \cup A_j) \ge \frac{1}{2} c_i(A_i \cup A_j)$ and then, the following holds,

$$\frac{c_i(A_i)}{\mathsf{MMS}_i(2, A_i \cup A_j)} \le \frac{2\alpha + 1}{\alpha + 1}.$$

As for tightness, consider the following instance of n agents and a set $E = \{e_1, \ldots, e_{n+1}\}$ of $n + 1$ chores. Agents have an identical cost profile and for every $i \in [n]$, $c_i(e_1) = \alpha + 1, c_i(e_2) = \alpha$ and $c_i(e_j) = 1$ for $j \ge 3$. Then, consider the allocation $\mathbf{B} = (B_1, \ldots, B_n)$ with $B_1 = \{e_1, e_2\}$ and $B_j = \{e_{j+1}\}, \forall j \ge 2$. It is not hard to verify that allocation $\mathbf{B}$ satisfying α-EF1, and moreover, the guarantee for PMMS is determined by agent 1. Notice that for any $j \ge 2$, the combined bundle $B_1 \cup B_j$ contains three chores with cost $\alpha + 1, \alpha, 1$, respectively. Thus, for any $j \ge 2$, we have $\mathsf{MMS}_1(2, B_1 \cup B_j) = \alpha + 1$, implying the ratio $\frac{c_1(B_1)}{\mathsf{MMS}_1(2, B_1 \cup B_j)} = \frac{2\alpha+1}{\alpha+1}$. □

In addition to the approximation guarantee for PMMS, Proposition 3.3.7 also has a direct implication in approximating PMMS algorithmically. It is known that an EF1 allocation can be found efficiently by allocating chores in a *round-robin* fashion — each of the agent $1, \ldots, n$ in turn picks her most preferred one from the remaining items , and repeat until all chores are assigned [13]. Therefore, Proposition 3.3.7 with $\alpha = 1$ leads to the following corollary, which is the only algorithmic result for PMMS (in chores allocation), to the best of our knowledge.

Corollary 3.3.1. *When agents have additive cost functions, the round-robin algorithm outputs a* $\frac{3}{2}$-PMMS *allocation in polynomial time.*

3.4 Guarantees from Share-Based Relaxations

Note that PMMS implies EFX in goods allocation according to Caragiannis et al. [48]. This implication also holds in allocating chores as stated in our proposition below.

Proposition 3.4.1. *When agents have additive cost functions, a* PMMS *allocation is also* EFX*.*

Proof. Let $\mathbf{A} = (A_1, \ldots, A_n)$ be a PMMS allocation. For the sake of contradiction, assume $\mathbf{A}$ is not EFX and agent i violates the condition of EFX, which implies $c_i(A_i) > 0$.

As agent i violates the condition of EFX, there must exist an agent $j \in N$ and $e^* \in A_i$ with $c_i(e^*) > 0$ such that $c_i(A_i \setminus \{e^*\}) > c_i(A_j)$. Note chore e^* is well-defined owing to $c_i(A_i) > 0$. Now, consider the 2-partition $\{A_i \setminus \{e^*\}, A_j \cup \{e^*\}\} \in \Pi_2(A_i \cup A_j)$. By $c_i(A_i \setminus \{e^*\}) > c_i(A_j)$, the following holds:

$$\begin{aligned} c_i(A_i) &> \max\{c_i(A_i \setminus \{e^*\}), c_i(A_j \cup \{e^*\})\} \\ &\geq \min_{\mathbf{B} \in \Pi_2(A_i \cup A_j)} \max\{c_i(B_1), c_i(B_2)\} \geq c_i(A_i), \end{aligned} \tag{3.8}$$

where the last transition is by the definition of PMMS. Inequality (3.8) is a contradiction, and therefore, $\mathbf{A}$ must be an EFX allocation. □

Since EFX implies EF1, Proposition 3.4.1 directly leads to the following result.

Proposition 3.4.2. *When agents have additive cost functions, a* PMMS *allocation is also* EF1*.*

For approximate version of PMMS, when allocating goods it is shown in Amanatidis et al. [7] that for any α, α-PMMS can imply $\frac{\alpha}{2-\alpha}$-EF1. However, in the case of chores, our results indicate that α-PMMS has no bounded guarantee for EF1.

Proposition 3.4.3. *When agents have additive cost functions, an* α-PMMS *fair allocation with* $1 < \alpha \leq 2$ *is not necessarily* β-EF1 *for any* $\beta \geq 1$*.*

Proof. It suffices to show an α-PMMS allocation with $\alpha \in (1, 2)$ can not have a bounded guarantee for the notion of EF1. Consider an instance with n agents and $n+1$ chores $e_1 \ldots, e_{n+1}$. Agents have identical cost profile and for any i, we let

$c_i(e_1) = (\alpha - 1)^{-1}$, $c_i(e_2) = 1$ and $c_i(e_j) = \epsilon$ for $3 \leq j \leq n+1$ where $\epsilon > 0$ is arbitrarily small. Then, consider an allocation $\mathbf{B} = (B_1, \ldots, B_n)$ with $B_1 = \{e_1, e_2\}$ and $B_j = \{e_{j+1}\}$ for $2 \leq j \leq n$. Consequently, except for agent 1, other agents violate neither EF1 nor α-PMMS. As for agent 1, notice that $(\alpha - 1)^{-1} > 1 + \epsilon$ and thus, for any $j \geq 2$, the combined bundle $B_1 \cup B_j$ admits $\mathsf{MMS}_1(2, B_1 \cup B_j) = \frac{1}{\alpha - 1}$ implying $\frac{c_1(B_1)}{\mathsf{MMS}_1(2, B_1 \cup B_j)} = \alpha$. Thus, allocation $\mathbf{B}$ is α-PMMS. For the guarantee on EF1, as $c_1(B_j) = \epsilon$ for any $j \geq 2$, then removing the chore with the largest cost from B_2 still yields the ratio $\frac{c_1(B_1 \setminus \{e_1\})}{c_1(B_j)} = \frac{1}{\epsilon} \to \infty$ as $\epsilon \to 0$. □

Since for any $\alpha \geq 1$, α-EFX is stricter than α-EF1, the impossibility result on EF1 in Proposition 3.4.3 is also true for EFX.

Proposition 3.4.4. *When agents have additive cost functions, an* α-PMMS *allocation with* $1 < \alpha \leq 2$ *is not necessarily a* β-EFX *allocation for any* $\beta \geq 1$.

We now study the approximation guarantee of PMMS for MMS. Since these two notions coincide when there are only two agents, we consider the situation where $n \geq 3$. We first provide a tight bound for $n = 3$ and then give an almost tight bound for general n.

Proposition 3.4.5. *When agents have additive cost functions, for* $n = 3$*, a* PMMS *allocation is also* $\frac{4}{3}$-MMS*, and moreover, this bound is tight.*

Proof. We first prove the upper bound. Let $\mathbf{A} = (A_1, A_2, A_3)$ be a PMMS allocation and we focus on agent 1. For the sake of contradiction, we assume $c_1(A_1) > \frac{4}{3}\mathsf{MMS}_1(3, E)$. We can also assume that bundles A_1, A_2, A_3 do not contain chores with zero cost for agent 1 since the existence of such a chore does not affect approximation ratio of allocation A on PMMS or MMS. To this end, we let $c_1(A_2) \leq c_1(A_3)$ (the other case is symmetric).

We first show that A_1 must be the bundle yielding the largest cost for agent 1. Otherwise, if $c_1(A_1) \leq c_1(A_2) \leq c_1(A_3)$, then by additivity $c_1(A_1) \leq 3^{-1}c_1(E) \leq \mathsf{MMS}_1(3, E)$, contradicting $c_1(A_1) > 4/3\mathsf{MMS}_1(3, E)$. Or if $c_1(A_2) < c_1(A_1) \leq c_1(A_3)$, since A_1 and A_2 is a 2-partition of $A_1 \cup A_2$, then $c_1(A_1)$ is at least $\mathsf{MMS}_1(2, A_1 \cup A_2)$. On the other hand, since $\mathbf{A}$ is a PMMS allocation, we know $c_1(A_1) \leq \mathsf{MMS}_1(2, A_1 \cup A_2)$, and thus, $c_1(A_1) = \mathsf{MMS}_1(2, A_1 \cup A_2)$ holds. Based on assumption $c_1(A_1) > 4/3\mathsf{MMS}_1(3, E)$ and Lemma 3.2.1, we have $c_1(A_1) > 4/3\mathsf{MMS}_1(3, E) \geq 4/9 \cdot c_1(E)$, then $c_1(A_3) \geq c_1(A_1) > 4/9 \cdot c_1(E)$ which yields $c_1(A_2) < 9^{-1}c_1(E)$ owning to the additivity. As a result, the difference between $c_1(A_1)$ and $c_1(A_2)$ is lower bounded $c_1(A_1) - c_1(A_2) > 3^{-1}c_1(E)$. Due to $c_1(A_1) = \mathsf{MMS}_1(2, A_1 \cup A_2)$, we can claim that every single chore in A_1 has cost strictly greater than $3^{-1}c_1(E)$; otherwise, $\exists e \in A_1$ with $c_1(e) \leq 3^{-1}c_1(E)$, then reassigning chore e to A_2 yields a 2-partition $\{A_1 \setminus \{e\}, A_2 \cup \{e\}\}$ with $\max\{c_1(A_1 \setminus \{e\}), c_1(A_2 \cup \{e\})\} < c_1(A_1) = \mathsf{MMS}_1(2, A_1 \cup A_2)$, contradicting to the definition of maximin share. Since every single chore in A_1 has cost strictly greater than $\frac{1}{3}c_1(E)$, then A_1 can only contain a single chore; otherwise, $c_1(A_3) \geq c_1(A_1) \geq 3^{-1}|A_1|c_1(E) \geq 2/3c_1(E)$, implying $c_1(A_3 \cup A_1) \geq 4/3 \cdot c_1(E)$, contradiction. However, if $|A_1| = 1$, according to the second point of Lemma 3.2.1, $c_1(A_1) > 4/3\mathsf{MMS}_1(3, E)$ can never hold. Therefore, it must hold that $c_1(A_1) \geq c_1(A_3) \geq c_1(A_2)$, which then implies $c_1(A_1) = \mathsf{MMS}_1(2, A_1 \cup A_3) = \mathsf{MMS}_1(2, A_1 \cup A_2)$ as a consequence of PMMS.

Next, we prove our statement by carefully checking the possibilities of $|A_1|$. According to Lemma 3.2.1, if $|A_1| = 1$, then $c_1(A_1) \leq \mathsf{MMS}_1(3, E)$. Thus, we can further assume $|A_1| \geq 2$. We first consider the case $|A_1| \geq 3$. Since $c_1(A_1) > 4/3\mathsf{MMS}_1(3, E) \geq 4/9 \cdot c_1(E)$, by additivity, we have $c_1(A_2) + c_1(A_3) < 5/9 \cdot c_1(E)$ and moreover, $c_1(A_2) < 5/18 \cdot c_1(E)$ due to $c_1(A_2) \leq c_1(A_3)$. Then the cost difference between bundle A_1 and A_2 satisfies $c_1(A_1) - c_1(A_2) > 6^{-1}c_1(E)$. This allow us to claim that every single chore in A_1 has cost strictly greater than $6^{-1}c_1(E)$; otherwise, reassigning a chore with cost no larger than $6^{-1}c_1(E)$ to A_2 yields another 2-partition of $A_1 \cup A_2$ in which the cost of larger bundle is strictly smaller than $\mathsf{MMS}_1(2, A_1 \cup A_2)$, a contradiction. In addition, since $c_1(A_1) = \mathsf{MMS}_1(2, A_1 \cup A_2)$, we claim $c_1(A_2) \geq c_1(A_1 \setminus \{e\}), \forall e \in A_1$; otherwise, $\exists e' \in A_1$ such that $c_1(A_2) < c_1(A_1 \setminus \{e'\})$, then reassigning e' to A_2 yields another 2-partition of $A_1 \cup A_2$ of which both two bundles' cost are strictly smaller than $\mathrm{MMS}_1(2, A_1 \cup A_2)$, a contradiction. Thus, for any $e \in A_1$, we have $c_1(A_2) \geq c_1(A_1 \setminus \{e\}) \geq 6^{-1}c_1(E) \cdot |A_1 \setminus \{e\}| \geq 3^{-1}c_1(E)$, where the last transition is due to $|A_1| \geq 3$. However, the cost of bundle A_2 is $c_1(A_2) < 5/18 \cdot c_1(E)$, a contradiction.

The remaining work is to rule out the possibility of $|A_1| = 2$. Let $A_1 = \{e_1^1, e_2^1\}$ with $c_1(e_1^1) \leq c_1(e_2^1)$ (the other case is symmetric). Note that $c_1(A_1) > 4/3\mathsf{MMS}_1(3, E) \geq 4/9 \cdot c_1(E)$, then $c_1(e_2^1) > 2/9 \cdot c_1(E)$. Let $S_2^* \in \arg\max_{S \subseteq A_2}\{c_1(S) : c_1(S) < c_1(e_1^1)\}$ (can be empty set) be the largest subset of A_2 with cost strictly smaller than $c_1(e_1^1)$. Due to $c_1(A_1) = \mathsf{MMS}_1(2, A_1 \cup A_2)$, then swapping S_2^* and e_1^1 would not produce a 2-partition in which the cost of both bundles are strictly smaller than $c_1(A_1)$, and thus $c_1(A_2 \setminus S_2^* \cup \{e_1^1\}) \geq c_1(A_1)$, equivalent to

$$c_1(A_2 \setminus S_2^*) \geq c_1(e_2^1) > \frac{2}{9}c_1(E). \tag{3.9}$$

Then, by $c_1(A_1) - c_1(A_2) > 6^{-1}c_1(E)$ and $c_1(A_2 \setminus S_2^*) \geq c_1(e_2^1)$, we have $c_1(e_1^1) - c_1(S_2^*) > 6^{-1}c_1(E)$, which allows us to claim that every single chore in $A_2 \setminus S_2^*$

has cost strictly greater than $6^{-1}c_1(E)$; otherwise, we can find another subset of A_2 whose cost is strictly smaller than e_1^1 but larger than $c_1(S_2^*)$, contradicting the definition of S_2^*. As a result, bundle $A_2 \setminus S_2^*$ must contain a single chore; if not, $c_1(A_2) > 6^{-1}c_1(E)\cdot|A_2 \setminus S_2^*| \geq 3^{-1}c_1(E)$, which implies $c_1(A_1 \cup A_2 \cup A_3) > 10/9 \cdot c_1(E)$ due to $c_1(A_1) > 4/9 \cdot c_1(E)$ and $c_1(A_3) \geq c_1(A_2) > 3^{-1}c_1(E)$. Thus, bundle $A_2 \setminus S_2^*$ only contains one chore, denoted by e_1^2. So we can decompose A_2 as $A_2 = \{e_1^2\} \cup S_2^*$ where $c_1(e_1^2) \geq c_1(e_2^1) > 2/9 \cdot c_1(E)$.

Next, we analyse the possible composition of bundle A_3. To have an explicit discussion, we introduce two more notions Δ_1, Δ_2 as follows

$$\begin{aligned} c_1(A_1) &= \frac{4}{9}c_1(E) + \Delta_1, \\ c_1(A_2) &= \frac{2}{9}c_1(E) + c_1(S_2^*) + \Delta_2. \end{aligned} \tag{3.10}$$

Recall $c_1(A_1) > 4/9 \cdot c_1(E)$ and $c_1(e_1^2) \geq c_1(e_2^1) > 2/9 \cdot c_1(E)$, so both $\Delta_1, \Delta_2 > 0$. Similarly, let $S_3^* \in \arg\min_{S \subseteq A_3}\{c_1(S) : c_1(S) < c_1(e_1^1)\}$, then we claim $c_1(A_3 \setminus S_3^*) \geq c_1(e_2^1)$; otherwise, swapping S_3^* and e_1^1 yields a 2-partition of $A_1 \cup A_3$ in which the cost of both bundles are strictly smaller than $c_1(A_1) = \mathsf{MMS}_1(2, A_1 \cup A_3)$, contradicting the definition of maximin share. By additivity of cost functions and Equation (3.10), we have $c_1(A_3) = 3/9 \cdot c_1(E) - c_1(S_2^*) - \Delta_1 - \Delta_2$, and accordingly $c_1(A_1) - c_1(A_3) = 9^{-1}c_1(E) + c_1(S_2^*) + 2\Delta_1 + \Delta_2$. This combing $c_1(A_3 \setminus S_3^*) \geq c_1(e_2^1)$ yields

$$c_1(e_1^1) - c_1(S_3^*) \geq \frac{1}{9}c_1(E) + c_1(S_2^*) + 2\Delta_1 + \Delta_2. \tag{3.11}$$

Based on Inequality (3.11), we can claim that every single chore in $A_3 \setminus S_3^*$ has cost at least $9^{-1}c_1(E) + c_1(S_2^*) + 2\Delta_1 + \Delta_2$; otherwise, contradicting the definition of S_3^*. Recall $c_1(A_3) = 3/9 \cdot c_1(E) - c_1(S_2^*) - \Delta_1 - \Delta_2$, then due to the constraint on the cost of single chore in $A_3 \setminus S_3^*$, we have $|A_3 \setminus S_3^*| \leq 2$. Meanwhile, $c_1(A_3 \setminus S_3^*) \geq c_1(e_2^1)$ implying that bundle $A_3 \setminus S_3^*$ can not be empty. In the following, we separate our proof by discussing two possible cases: $|A_3 \setminus S_3^*| = 1$ and $|A_3 \setminus S_3^*| = 2$.

Case 1: $|A_3 \setminus S_3^*| = 1$. Let $A_3 \setminus S_3^* = \{e_1^3\}$. Therefore, the whole set E is composed by four single chores and two subsets S_2^*, S_3^*, i.e., $E = \{e_1^1, e_2^1, e_1^2, S_2^*, e_1^3, S_3^*\}$. Then, we let $\mathbf{T} = (T_1, T_2, T_3)$ be the allocation defining $\mathsf{MMS}_1(3, E)$ and without loss of generality, let $c_1(T_1) = \mathsf{MMS}_1(3, E)$. Next, to find contradictions, we analyse bounds on both $\mathsf{MMS}_1(3, E)$ and $c_1(A_1)$. Since $\min\{c_1(e_1^2), c_1(e_1^3)\} \geq c_1(e_2^1) \geq 2^{-1}c_1(A_1)$, we claim that $c_1(A_1) \leq 2^{-1}c_1(E)$; otherwise $c_1(A_1) + c_1(e_1^2) + c_1(e_1^3) > c_1(E)$. Notice that E contains three chores with the cost at least $2/9 \cdot c_1(E)$ each, if any two of them are in the same bundle under $\mathbf{T}$, then $\mathrm{MMS}_1(3, E) > 4/9 \cdot c_1(E)$

and consequently, $\frac{c_1(A_1)}{\mathsf{MMS}_1(3,E)} < \frac{9}{8}$, a contradiction. Or if each of $\{e_2^1, e_1^2, e_1^3\}$ is contained in a distinct bundle, then the bundle also containing chore e_1^1 has cost at least $c_1(A_1)$ as a result of $\min\{c_1(e_1^2), c_1(e_1^3)\} \geq c_1(e_2^1)$ and $A_1 = \{e_1^1, e_2^1\}$. Thus, $\mathsf{MMS}_1(3,E) \geq c_1(A_1)$ holds, contradicting $c_1(A_1) > 4/3\mathsf{MMS}_1(3,E)$.

Case 2: $|A_3 \setminus S_3^*| = 2$. Let $A_3 \setminus S_3^* = \{e_1^3, e_2^3\}$ and accordingly, the whole set can be decomposed as $E = \{e_1^1, e_2^1, e_1^2, S_2^*, e_1^3, e_2^3, S_3^*\}$. Note the upper bound $c_1(A_1) \leq 2^{-1}c_1(E)$ still holds since $\min\{c_1(A_3 \setminus S_3^*), c_1(e_1^2)\} \geq c_1(e_2^1)$. Then, we analyse the possible lower bound of $\mathrm{MMS}_1(3,E)$. If chores e_2^1, e_1^2 are in the same bundle of $\mathbf{T}$, then $\mathsf{MMS}_1(3,E) > 4/9 \cdot c_1(E)$ holds and so $\frac{c_1(A_1)}{\mathsf{MMS}_1(3,E)} < \frac{9}{8}$, a contradiction. Thus, chores e_2^1, e_1^2 are in different bundles in $\mathbf{T}$. Then, if both chores e_1^3, e_2^3 are in the bundle containing e_2^1 or e_1^2, then we also have $\mathsf{MMS}_1(3,E) > 4/9 \cdot c_1(E)$ implying $\frac{c_1(A_1)}{\mathsf{MMS}_1(3,E)} < \frac{9}{8}$, a contradiction. Therefore, only two possible cases; that is, both e_1^3, e_2^3 are in the bundle different from that containing e_2^1 or e_1^2; or the bundle having e_2^1 or e_1^2 contains at most one of e_1^3, e_2^3.

Subcase 1: both e_1^3, e_2^3 are in the bundle different from that containing e_2^1 or e_1^2; Recall $c_1(e_1^1) > 1/6 \cdot c_1(E) + c_1(S_2^*)$ and the fact $\min\{c_1(e_2^1), c_1(e_1^2), c_1(e_1^3 \cup e_2^3)\} > 2/9 \cdot c_1(E)$, the bundle also containing e_1^1 has cost strictly greater than $7/18 \cdot c_1(E)$. Thus, $\mathsf{MMS}_1(3,E) > 7/18 \cdot c_1(E)$, which combines $c_1(A_1) \leq 2^{-1}c_1(E)$ implying $\frac{c_1(A_1)}{\mathrm{MMS}_1(3,E)} < \frac{9}{7} < \frac{4}{3}$, a contradiction.

Subcase 2: bundle having e_2^1 or e_1^2 contains at most one of e_1^3, e_2^3. Recall $c_1(e_1^2) \geq c_1(e_2^1)$ and $\min\{c_1(e_1^3), c_1(e_2^3)\} \geq 9^{-1}c_1(E) + c_1(S_2^*) + 2\Delta_1 + \Delta_2$, thus in allocation $\mathbf{T}$ there always exist a bundle with cost at least $9^{-1}c_1(E) + c_1(S_2^*) + 2\Delta_1 + \Delta_2 + c_1(e_2^1)$ and results in the ratio

$$\frac{c_1(A_1)}{\mathsf{MMS}_1(3,E)} \leq \frac{c_1(e_1^1) + c_1(e_2^1)}{\frac{1}{9}c_1(E) + c_1(S_2^*) + 2\Delta_1 + \Delta_2 + c_1(e_2^1)}. \tag{3.12}$$

In order to satisfy our assumption of $\frac{c_1(A_1)}{\mathsf{MMS}_1(3,E)} > \frac{4}{3}$, the RHS of Inequality (3.12) must be strictly greater than 4/3, which implies the following

$$c_1(e_1^1) > \frac{2}{9}c_1(E) + 2c_1(S_1^*) + 4\Delta_1 + 2\Delta_2. \tag{3.13}$$

However, based on the first equation of (3.10) and $c_1(e_1^1) \leq c_1(e_2^1)$, we have $c_1(e_1^1) \leq 2/9 \cdot c_1(E) + 2^{-1}\Delta_1 < 2/9 \cdot c_1(E) + 2c_1(S_1^*) + 4\Delta_1 + 2\Delta_2$ due to $\Delta_1, \Delta_2 > 0$. This contradicts Inequality (3.13). Therefore, $\frac{c_1(A_1)}{\mathsf{MMS}_1(3,E)} > \frac{4}{3}$ can never hold under *Case 2*. Up to here, we complete the proof of the upper bound.

Next, as for tightness, consider an instance with three agents and a set $E =$

$\{e_1, ..., e_6\}$ of six chores. Agents have identical cost functions. The cost function of agent 1 is as follows: $c_1(e_j) = 2, \forall j = 1, 2, 3$ and $c_1(e_j) = 1, \forall j = 4, 5, 6$. It is easy to see that $\mathsf{MMS}_1(3, E) = 3$. Then, consider an allocation $\mathbf{B} = \{B_1, B_2, B_3\}$ with $B_1 = \{e_1, e_2\}, B_2 = \{e_3\}$ and $B_3 = \{e_4, e_5, e_6\}$. It is not hard to verify that allocation $\mathbf{B}$ is PMMS and due to $c_1(B_1) = 4$, we have the ratio $\frac{c_1(B_1)}{\mathsf{MMS}_1(3,E)} = \frac{4}{3}$. $\square$

For general n, we use the connections between PMMS, EFX and MMS to find the approximation guarantee of PMMS for MMS. According to Proposition 3.4.1, a PMMS allocation is also EFX, and by Proposition 3.3.5, EFX implies $\frac{2n}{n+1}$-MMS. As a result, we can claim that PMMS also implies $\frac{2n}{n+1}$-MMS. With the following proposition we show that this guarantee is almost tight.

Proposition 3.4.6. *When agents have additive cost functions, for $n \geq 4$, a* PMMS *allocation is* $\frac{2n}{n+1}$-MMS *fair but not necessarily* β-MMS *fair for any* $\beta < \frac{2n+2}{n+3}$.

Proof. The positive part directly follows from Propositions 3.4.1 and 3.3.5. As for the lower bound, consider an instance with n (odd) agents and a set $E = \{e_1, \ldots, e_{2n}\}$ of $2n$ chores. We focus on agent 1 and his cost function is $c_1(e_j) = (n + 1)/2$ for $1 \leq j \leq n$ and $c_1(e_j) = 1$ for $n + 1 \leq j \leq 2n$. Consider the allocation $\mathbf{B} = (B_1, \ldots, B_n)$ with $B_1 = \{e_1, e_2\}$, $B_n = \{e_{n+1}, \ldots, e_{2n}\}$ and $B_j = \{e_{j+1}\}$ for any $j = 2, \ldots, n - 1$. For agents $i \geq 2$, her cost function is $c_i(e) = 0$ for any $e \in B_i$ and $c_i(e) = 1$ for any $e \in E \setminus B_i$, and thus agent i has zero cost under allocation $\mathbf{B}$. As a result, except for agent 1, other agents violate neither MMS nor PMMS. For agent 1, we have $c_1(B_1) \leq \mathsf{MMS}_1(2, B_1 \cup B_j)$ holds for any $j \geq 2$, which implies allocation $\mathbf{B}$ is PMMS. For $\mathsf{MMS}_1(n, E)$, it happens that E can be evenly divided into n bundles of the same cost (for agent 1), so we have $\mathsf{MMS}_1(n, E) = (n + 3)/2$ yielding the ratio $\frac{c_1(B_1)}{\mathsf{MMS}_1(n,E)} = \frac{2n+2}{n+3}$. $\square$

Next, we investigate the approximation guarantee of approximate PMMS for MMS. Let us start with an example of six chores $E = \{e_1, \ldots, e_6\}$ and three agents. We focus on agent 1 and the cost function of agent 1 is $c_1(e_j) = 1$ for $j = 1, 2, 3$ and $c_1(e_j) = 0$ for $j = 4, 5, 6$, thus clearly, $\mathsf{MMS}_1(3, E) = 1$. Consider an allocation $\mathbf{A} = (A_1, A_2, A_3)$ with $A_1 = \{e_1, e_2, e_3\}$. It is not hard to verify that allocation $\mathbf{A}$ is a 3/2-PMMS allocation and also a 3-MMS allocation. Combining the result in Lemma 3.2.2, we observe that allocation $\mathbf{A}$ only has a trivial guarantee on the notion of MMS. Motivated by this example, we focus on α-PMMS allocations with $\alpha < \frac{3}{2}$.

Proposition 3.4.7. *When agents have additive cost functions, for $n \geq 3$ and $1 < \alpha < \frac{3}{2}$, an* α-PMMS *allocation is* $\frac{n\alpha}{\alpha+(n-1)(1-\frac{\alpha}{2})}$-MMS *fair, but not necessarily* ($\frac{n\alpha}{\alpha+(n-1)(2-\alpha)} - \epsilon$)-MMS *fair for any $\epsilon > 0$.*

Before we can prove the above proposition, we need the following two lemmas.

Lemma 3.4.8. *For any $i \in N$ and $S \subseteq E$, suppose $\mathsf{MMS}_i(2,S)$ is defined by a 2-partition $\mathbf{T} = (T_1, T_2)$ with $c_i(T_1) = \mathsf{MMS}_i(2,S)$. If the number of chores in T_1 is at least two, then $\frac{c_i(S)}{\mathsf{MMS}_i(2,S)} \geq \frac{3}{2}$.*

Proof. For the sake of contradiction, we assume $\frac{c_i(S)}{\mathsf{MMS}_i(2,S)} < \frac{3}{2}$. Since $c_i(T_1) = \mathsf{MMS}_i(2,S)$, we have $c_i(T_1) > 2/3 \cdot c_i(S)$, and accordingly, $c_i(T_2) < 3^{-1}c_i(S)$ due to additivity. Thus, $c_i(T_1) - c_i(T_2) > 3^{-1}c_i(S)$ holds, and we claim that each single chore in T_1 has cost strictly larger than $3^{-1}c_i(S)$ for agent i; otherwise, by moving the chore with the smallest cost in T_1 to T_2, one can find a 2-partition in which the cost of larger bundle is smaller than $c_i(T_1)$, contradiction. Based on our claim, we have $|T_1| = 2$. Notice that for any $e \in T_1$, $c_i(e) > c_i(T_2)$ holds. As a result, moving one chore from T_1 to T_2 results in a 2-partition, in which the cost of larger bundle is strictly smaller than $c_i(T_1)$, contradicting the construction of allocation $\mathbf{T}$. □

Lemma 3.4.9. *For any $i \in N$ and $S_1, S_2 \subseteq E$, if $\mathsf{MMS}_i(2, S_1 \cup S_2) > \mathsf{MMS}_i(2, S_1)$, then $\mathsf{MMS}_i(2, S_1 \cup S_2) \leq \frac{1}{2}c_i(S_1) + c_i(S_2)$.*

Proof. Suppose $\mathsf{MMS}_i(2, S_1)$ is defined by partition (T_1, T_2) and we have $\mathsf{MMS}_i(2, S_1) = c_i(T_1)$. We distinguish two cases according to the value of $c_i(T_1)$. If $c_i(T_1) = 2^{-1}c_i(S_1)$, then consider $(T_1 \cup S_2, T_2)$, a 2-partition of $S_1 \cup S_2$. Clearly, $\mathsf{MMS}_i(2, S_1 \cup S_2) \leq c_i(T_1 \cup S_2) = 2^{-1}c_i(S_1) + c_i(S_2)$. If $c_i(T_1) > 2^{-1}c_i(S_1)$, since $\mathsf{MMS}_i(2, S_1 \cup S_2) > \mathsf{MMS}_i(2, S_1)$, we can claim that $c_i(T_1) - c_i(T_2) < c_i(S_2)$; otherwise, considering partition $\{T_1, T_2 \cup S_2\}$, we have $\mathsf{MMS}_i(2, S_1 \cup S_2) \leq c_i(T_1) = \mathsf{MMS}_i(2, S_1)$, a contradiction. Now let us consider $\{T_2 \cup S_2, T_1\}$, another 2-partition of $S_1 \cup S_2$. According to our claim, we have $c_i(T_2 \cup S_2) > c_i(T_1)$, and thus, $\mathsf{MMS}_i(2, S_1 \cup S_2) \leq c_i(T_2 \cup S_2) < 2^{-1}c_i(S_1) + c_i(S_2)$, where the last inequality is due to $c_i(T_2) = c_i(S_1) - c_i(T_1) < 2^{-1}c_i(S_1)$. □

Proof of Proposition 3.4.7. We first prove the upper bound. Let $\mathbf{A} = (A_1, ..., A_n)$ be an α-PMMS allocation and we focus our analysis on agent i. Let

$$\alpha^{(i)} = \max_{j \neq i} \frac{c_i(A_i)}{\mathsf{MMS}_i(2, A_i \cup A_j)}$$

and $j^{(i)}$ be the index such that $\mathsf{MMS}_i(2, A_i \cup A_{j^{(i)}}) \leq \mathsf{MMS}_i(2, A_i \cup A_j)$ for any $j \in N \setminus \{i\}$ (tie breaks arbitrarily). By such a construction, clearly, $\alpha = \max_{i \in N} \alpha^{(i)}$ and $c_i(A_i) = \alpha^{(i)} \cdot \mathsf{MMS}_i(2, A_i \cup A_{j^{(i)}})$. Then, we split our proof into two different cases.

Case 1: $\exists j \neq i$ such that $\mathsf{MMS}_i(2, A_i \cup A_j) = \mathsf{MMS}_i(2, A_i)$. Then $\alpha^{(i)} = \frac{c_i(A_i)}{\mathsf{MMS}_i(2,A_i)}$ holds. Suppose $\mathsf{MMS}_i(2, A_i)$ is defined by the 2-partition (T_1, T_2) with $c_i(T_1) = \mathsf{MMS}_i(2, A_i)$. If $|T_1| \geq 2$, by Lemma 3.4.8, we have $\alpha^{(i)} = \frac{c_i(A_i)}{\mathsf{MMS}_i(2,A_i)} \geq \frac{3}{2}$, contradicting $\alpha^{(i)} \leq \alpha < 3/2$. As a result, we can further assume $|T_1| = 1$. Then, by Lemma 3.2.1, we have $\mathsf{MMS}_i(n, E) \geq c_i(T_1)$ and accordingly, $\frac{c_i(A_i)}{\mathsf{MMS}_i(n,E)} \leq \frac{c_i(A_i)}{c_i(T_1)} = \alpha^{(i)} \leq \alpha$. For $1 < \alpha < 3/2$ and $n \geq 3$, it is not hard to verify that $\alpha \leq \frac{n\alpha}{\alpha+(n-1)(1-\frac{\alpha}{2})}$, completing the proof for this case.

Case 2: $\forall j \neq i$, $\mathsf{MMS}_i(2, A_i \cup A_j) > \mathsf{MMS}_i(2, A_i)$ holds. According to Lemma 3.4.9, for any $j \neq i$, the following holds

$$\mathsf{MMS}_i(2, A_i \cup A_j) \leq \frac{1}{2} c_i(A_i) + c_i(A_j). \tag{3.14}$$

Due to the construction of $\alpha^{(i)}$, for any $j \neq i$, we have $c_i(A_i) \leq \alpha^{(i)} \cdot \mathsf{MMS}_i(2, A_i \cup A_j)$. Combining Inequality (3.14), we have $c_i(A_j) \geq \frac{2-\alpha^{(i)}}{2\alpha^{(i)}} c_i(A_i)$ for any $j \neq i$. Thus, the following holds,

$$\frac{c_i(A_i)}{\mathsf{MMS}_i(n, E)} \leq \frac{n c_i(A_i)}{c_i(E)} \leq \frac{n c_i(A_i)}{c_i(A_i) + (n-1)\frac{2-\alpha^{(i)}}{2\alpha^{(i)}} c_i(A_i)}. \tag{3.15}$$

The last expression in (3.15) is monotonically increasing in $\alpha^{(i)}$, and accordingly, we have

$$\frac{c_i(A_i)}{\mathsf{MMS}_i(n, E)} \leq \frac{n\alpha}{\alpha + (n-1)(1 - \frac{\alpha}{2})}.$$

As for the lower bound, consider an instance of n (even) agents and a set $E = \{e_1, ..., e_{n^2}\}$ of n^2 chores. Agents have identical cost functions and for any i, we let $c_i(e_j) = \alpha$ for $1 \leq j \leq n$ and $c_i(e_j) = 2 - \alpha$ for $n + 1 \leq j \leq n^2$. Consider the allocation $\mathbf{B} = (B_1, ..., B_n)$ with $B_i = \{e_{(i-1)n+1}, ..., e_{ni}\}$ for any $i \in [n]$. Since $\alpha > 1$, it is not hard to verify that, except for agent 1, no one else violates the condition of PMMS, and accordingly, the approximation guarantee for PMMS is determined by agent 1. For agent 1, since n is even, $\mathsf{MMS}_1(2, B_1 \cup B_j) = n$ holds for any $j \geq 2$, and due to $c_1(B_1) = n\alpha$, we can claim that the allocation $\mathbf{B}$ is α-PMMS. Moreover, it is not hard to verify that $\mathsf{MMS}_1(n, E) = \alpha + (n-1)(2-\alpha)$ and so $\frac{c_1(B_1)}{\mathsf{MMS}_1(n,E)} = \frac{n\alpha}{\alpha+(n-1)(2-\alpha)}$, completing the proof. $\square$

The motivating example right before Proposition 3.4.7, unfortunately, only works for the case of $n = 3$. When n becomes larger, an α-PMMS allocation with $\alpha \geq \frac{3}{2}$ is still possible to provide a non-trivial approximation guarantee on the notion of MMS.

We remain to consider the approximation guarantee of MMS for other fairness criteria. Notice that all of EFX, EF1 and PMMS can have non-trivial guarantee for MMS (i.e., better than n-MMS). However, the converse is not true and even the exact MMS does not provide any substantial guarantee for the other three criteria.

Proposition 3.4.10. *When agents have additive cost functions, for any $n \geq 3$, an* MMS *allocation is not necessarily β-*PMMS *fair for any $1 \leq \beta < 2$.*

Proof. Consider an instance with n agents and $p + 2n - 1$ chores denoted as $\{e_1, \ldots, e_{2n+p-1}\}$ where $p \in \mathbb{N}^+$ and $p \gg 1$. We focus on agent 1 and his cost function is: $c_1(e_j) = 1$ for any $1 \leq j \leq n+p$ and $c_1(e_j) = p$ for any $j \geq n+p+1$. Consider allocation $\mathbf{B} = (B_1, \ldots, B_n)$ with $B_1 = \{e_1, \ldots, e_{p+1}\}$, $B_i = \{e_{p+i}\}, \forall i = 2, \ldots, n-2$, $B_{n-1} = \{e_{n+p-1}, e_{n+p}\}$ and $B_n = \{e_{n+p+1}, \ldots, e_{2n+p-1}\}$. For any agent $i \geq 2$, her cost function is $c_i(e) = 0$ for any $e \in B_i$ and $c_i(e) = 1$ for any $e \notin B_i$. Consequently, except for agent 1, other agents violate neither MMS nor PMMS, and accordingly the approximation guarantee for PMMS and MMS is determined by agent 1. For $\mathrm{MMS}_1(n, E)$, it happens that E can be evenly divided into n bundles of the same cost (for agent 1), so we have $\mathrm{MMS}_1(n, E) = p + 1$. Accordingly, $c_1(B_1) = \mathrm{MMS}_1(n, E)$ holds and thus, allocation $\mathbf{B}$ is MMS. As for the approximation guarantee on PMMS, consider the combined bundle $B_1 \cup B_2$ and it is not hard to verify that $\mathrm{MMS}_1(2, B_1 \cup B_2) = \lceil \frac{p+2}{2} \rceil$ implying $\frac{c_1(B_1)}{\mathrm{MMS}_1(2, B_1 \cup B_2)} = \frac{p+1}{\lceil \frac{p+2}{2} \rceil} \to 2$ as $p \to \infty$. □

Proposition 3.4.11. *When agents have additive cost functions, an* MMS *allocation is not necessarily β-*EF1 *or β-*EFX *for any $\beta \geq 1$.*

Proof. By Proposition 3.3.3, the notion β-EFX is stricter than β-EF1, and thus, we only need to show the unbounded guarantee on EF1. Again, we consider the instance given in the proof of Proposition 3.4.10. As stated in that proof, $\mathbf{B}$ is an MMS allocation, and except for agent 1, no one else will violate the condition of PMMS. Note that PMMS is stricter than EF1, then no one else will violate the condition of EF1. As for agent 1, each chore in B_1 has the same cost for him, so we can remove any single chore in B_1 and check its performance in terms of EF1. When comparing to bundle B_2, we have $\frac{c_1(B_1 \setminus \{e_1\})}{c_1(B_2)} = p \to \infty$ as $p \to \infty$. □

3.5 Guarantees beyond the Additive Setting

The results in previous sections demonstrate the strong connections between the four (additive) relaxations of envy-freeness in the setting of additive cost functions.

Under this umbrella, what would also be interesting is that whether there still exists certain connections when agents' cost functions are no longer additive. In this section, we also study the connections among fairness criteria, and instead of additive cost functions, we assume that agents have submodular and monotone cost functions, which have also been widely considered in fair division literature [50, 71].

As a starting point, we consider EF, the strongest notion in the setting of additive, and see whether it can still provide guarantee on other fairness notions. According to the definitions, the notion of EF is, clearly, still stricter than EFX and EF1 if cost functions are monotone. Then, we study the approximation guarantee of EF on MMS and PMMS. As shown by our results below, in contrast to the results under additive setting, PMMS and MMS are no longer the relaxations of EF, and even worse, the notion of EF does not provide any substantial guarantee on PMMS and MMS.

Proposition 3.5.1. *When agents have submodular cost functions, an* EF *allocation is not necessarily* β_1-MMS *fair or* β_2-PMMS *fair for any* $1 \le \beta_1 < n$, $1 \le \beta_2 < 2$.

Proof. It suffices to show that there exists an EF allocation with approximation ratio n and 2 for MMS and PMMS, respectively. Consider an instance with n (even) agents and a set E of chores with $|E| = n^2$. Chores are placed in the form of $n \times n$ matrix $E = (e_{ij})$. All agents have an identical cost function $c(S) = \sum_{i=1}^{n} \min\{|E_i \cap S|, 1\}$ for any $S \subseteq E$, where E_i is the set of all elements in the i-th row of matrix E, i.e., $E_i = \{e_{i1}, \ldots, e_{in}\}$. Since capped cardinality function $|E_i \cap S|$ of $S \subseteq E$ is monotone and submodular for any fixed i ($1 \le i \le n$), it follows that $c(\cdot)$ is also monotone and submodular.[4]

Next, we prove that this instance permits an EF allocation, with which the approximation ratio for MMS and PMMS is n and 2, respectively. Consider an allocation $\mathbf{B} = (B_1, ..., B_n)$ where for any j, bundle B_j contains all elements in the j-th column of matrix E, i.e., $B_j = (e_{1j}, e_{2j}, ..., e_{nj})$. One can compute that $c(B_j) = \sum_{i=1}^{n} \min(|E_i \cap B_j|, 1) = n$ holds for any $j \in [n]$, which implies that allocation $\mathbf{B}$ is EF. Next, we check the approximation guarantee of $\mathbf{B}$ on MMS. With a slight abuse of notation, we let $\mathbf{E}$ be the allocation defined by n-partition $E_1, ..., E_n$, i.e., $\mathbf{E} = (E_1, ..., E_n)$. It is not hard to see that for any $i \in N, c(E_i) = 1$. Then we claim that allocation $\mathbf{E}$ defines MMS for all agents; otherwise, there exists another allocation in which each bundle has cost strictly smaller than 1, and this

[4]More generally, if $f(\cdot)$ is submodular, then $g(f(\cdot))$ is also submodular for any $g(\cdot)$ that is non-decreasing and concave. Furthermore, conical combination (with sum as a special case) of submodular functions is also submodular.

never happens because $c(e) = 1$ for any $e \in E$ and $c(\cdot)$ is monotone. Therefore, for any $i \in N, \mathrm{MMS}_i(n, E) = 1$, which implies $\frac{c(B_i)}{\mathsf{MMS}_i(n,E)} = n$, as required.

Next, we argue that the allocation $\mathbf{B}$ is 2-PMMS. Fix $i, j \in N$ and $j \neq i$. Notice that the combined bundle $B_i \cup B_j$ contains two columns of chores, so we can consider another allocation $\mathbf{B}' = (B_i', B_j')$ with $B_i' = \left\{e_{1i}, \ldots, e_{\frac{n}{2}i}, e_{1j}, ..., e_{\frac{n}{2}j}\right\}$ and $B_j' = \{e_{\frac{n}{2}+1i}, ..., e_{ni}, e_{\frac{n}{2}+1j}, ..., e_{nj}\}$. The idea of $\mathbf{B}'$ is to split each column into two parts with equal size and one part staring from the first row to $(n/2)$-th row while the other one containing the remaining half. By the definition of cost function $c(\cdot)$, we know $c(B_i') = c(B_j') = n/2$, which implies $\mathsf{MMS}_i(2, B_i \cup B_j) \leq \max\{c(B_i'), c(B_j')\} = 2^{-1}c_i(B_i)$. Therefore, $\mathbf{B}$ is a 2-PMMS fair allocation. □

In the aspect of worst-case analysis, combining Lemma 3.2.2 and Proposition 3.5.1, EF can only have a trivial guarantee (n and 2, respectively) on MMS and PMMS, which is a sharp contrast to the results in additive setting where EF is strictly stronger than these two notions. As we mentioned above, EF is stricter than EFX and EF1, then we can directly argue that neither EFX nor EF1 can have better guarantees than trivial ones, namely, 2-PMMS and n-MMS.

Proposition 3.5.2. *When agents have submodular cost functions, an* EFX *allocation is not necessarily* β_1-MMS *fair or* β_2-PMMS *fair for any* $1 \leq \beta_1 < n$, $1 \leq \beta_2 < 2$.

Proposition 3.5.3. *When agents have submodular cost functions, an* EF1 *allocation is not necessarily* β_1-MMS *fair or* β_2-PMMS *fair for any* $1 \leq \beta_1 < n$, $1 \leq \beta_2 < 2$.

As for the connections between EFX and EF1, the statement of Proposition 3.3.3 is still true in the case of submodular cost functions.

Proposition 3.5.4. *When agents have submodular cost functions, an* α-EFX *allocation is also* α-EF1 *for any* $\alpha \geq 1$. *On the other hand, an* EF1 *allocation is not necessarily a* β-EFX *for any* $\beta \geq 1$.

Proof. The positive part follows directly from definitions of EFX and EF1. As for the impossibility result, the instance in the proof of Proposition 3.3.3 is established for the additive case. Since an additive function is also submodular, we also have such an impossibility result here. □

Next, we study the notion of PMMS in terms of its approximation guarantee on EFX and EF1. Recall the results of Propositions 3.4.1 and 3.4.2, the notion of PMMS is stricter than EFX and EF1 in the additive setting. However, in the submodular case, this relationship does not hold any more, and even worse, PMMS provides non-trivial guarantee on neither EFX nor EF1.

Proposition 3.5.5. *When agents have submodular cost functions, a* PMMS *allocation is not necessarily a* β-EF1 *or* β-EFX *allocation for any* $\beta \geq 1$.

Proof. By Proposition 3.5.4, for any $\beta \geq 1$, β-EFX is stronger than β-EF1, and thus it suffices to show the approximation guarantee for EF1 is unbounded. In what follows, we provide an instance that has a PMMS allocation with only trivial guarantee on EF1.

Consider an instance with two agents and a set $E = \{e_1, e_2, e_3\}$ of chores. Agents have identical cost function $c(S) = \min\{|S|, 2\}$. Since $|S|$ is monotone and submodular, it follows that $c(\cdot)$ is also monotone and submodular (see Footnote 4).

Next, we prove this instance having a PMMS allocation whose guarantee for EF1 is unbounded. Since in total, we have three chores, and thus in any 2-partition there always exists an agent receiving at least two chores. Thus, we can claim that $\mathsf{MMS}_i(2, E) = 2$ for any $i \in [2]$. Then, consider an allocation $\mathbf{B} = (B_1, B_2)$ with $B_1 = E$ and $B_2 = \emptyset$. Allocation $\mathbf{B}$ is PMMS since, for any $i \in [2]$, $\max\{c(B_1), c(B_2)\} = \mathsf{MMS}_i(2, E) = 2$ holds. However, bundle B_2 is empty and so $c_1(B_2) = c(B_2) = 0$. Then, no matter which chore is removed from bundle B_1, agent 1 still has a positive cost, which implies an unbounded approximation guarantee for the notion of EF1. □

The approximation guarantee of an MMS allocation for EFX, EF1 and PMMS can be directly derived from the results in the additive setting. According to Propositions 3.4.10 and 3.4.11, in the additive setting, the MMS fairness does not provide non-trivial guarantee on all other three notions. Since additive functions belong to the class of submodular functions, we directly have the following two results.

Proposition 3.5.6. *When agents have submodular cost functions, an* MMS *fair allocation is not necessarily* β-PMMS *fair for any* $1 \leq \beta < 2$.

Proposition 3.5.7. *When agents have submodular cost functions, an* MMS *fair allocation is not necessarily a* β-EF1 *or* β-EFX *allocation for any* $\beta \geq 1$.

At this stage, what remains is the approximation guarantee of PMMS on MMS. Before presenting the main result, we provide a lemma, which states that the quantity of MMS is monotonically non-decreasing on the set of chores to be assigned.

Lemma 3.5.8. *Given a monotone function $c(\cdot)$ defined on ground set E, for any subsets $S \subseteq T \subseteq E$ and for any $k \in \mathbb{N}^+$, if quantities $\mathsf{MMS}(k, S)$ and $\mathsf{MMS}(k, T)$ are computed based on function $c(\cdot)$, then $\mathsf{MMS}(k, S) \leq \mathsf{MMS}(k, T)$.*

Proof. Let $\{T_1, \ldots, T_k\}$ be the k-partition of set T that defines $\mathsf{MMS}(k, T) = c(T_1) \geq c(T_j)$ for any $j \in [k]$. We then consider $\{T_1 \cap S, T_2 \cap S, \ldots, T_k \cap S\}$, which is, clearly, a k-partition of S due to $S \subseteq T$. According to the definition of MMS, we have

$$\mathsf{MMS}(k, S) \leq \max_{j \in [k]}\{c(T_j \cap S)\} \leq \max_{j \in [k]}\{c(T_j)\} = \mathsf{MMS}(k, T),$$

where the second inequality transition is because $c(\cdot)$ is monotone. □

Proposition 3.5.9. *When agents have submodular cost functions, for any $1 \leq \alpha \leq 2$, an α-PMMS allocation is also $\min\{n, \alpha\lceil \frac{n}{2} \rceil\}$-MMS fair, and this guarantee is tight.*

Proof. We first prove the upper bound. According to Lemma 3.2.2, any allocation is n-MMS and so what remains is to prove the upper bound of $\alpha\lceil \frac{n}{2} \rceil$. Fix agent i with cost function $c_i(\cdot)$. Suppose n-partition $\{T_1, \ldots, T_n\}$ defines $\mathsf{MMS}_i(n, E)$ and w.l.o.g, we assume $c_i(T_1) \geq c_i(T_2) \geq \cdots \geq c_i(T_n)$, i.e., $c_i(T_1) = \mathsf{MMS}_i(n, E)$. Then, we let 2-partition $\{Q_1, Q_2\}$ defines $\mathsf{MMS}_i(2, E)$ and $c_i(Q_1) \geq c_i(Q_2)$, i.e., $c_i(Q_1) = \mathsf{MMS}_i(2, E)$. Let $\mathbf{A}$ be an arbitrary α-PMMS allocation, and accordingly, for any $j \neq i$, we have $c_i(A_i) \leq \alpha \cdot \mathsf{MMS}_i(2, A_i \cup A_j)$. Since $A_i \cup A_j$ is a subset of E, according to Lemma 3.5.8, we have $\mathsf{MMS}_i(2, A_i \cup A_j) \leq \mathsf{MMS}_i(2, E)$. We then construct an upper bound of $\mathsf{MMS}_i(2, E)$ through partition $\{T_1, \ldots, T_n\}$.

Let us consider a 2-partition $\{B_1, B_2\}$ of E with $B_1 = \{T_1, T_2, \ldots, T_{\lceil \frac{n}{2} \rceil}\}$, $B_2 = \{T_{\lceil \frac{n}{2} \rceil + 1}, \ldots, T_n\}$. Then, the following holds:

$$\begin{aligned}\max\{c_i(B_1), c_i(B_2)\} &= \max\{c_i(\cup_{j=1}^{\lceil \frac{n}{2} \rceil} T_j), c_i(\cup_{j=\lceil \frac{n}{2} \rceil + 1}^{n} T_j)\} \\ &\leq \max\{\sum_{j=1}^{\lceil \frac{n}{2} \rceil} c_i(T_j), \sum_{j=\lceil \frac{n}{2} \rceil + 1}^{n} c_i(T_j)\} \leq \lceil \frac{n}{2} \rceil \cdot c_i(T_1),\end{aligned}$$

where the first inequality transition is due to subadditivity of $c_i(\cdot)$ and the second inequality transition is because $c_i(T_1) \geq c_i(T_2) \cdots \geq c_i(T_n)$. Recall $c_i(Q_1) = \mathsf{MMS}_i(2, E) \leq \max\{c_i(B_1), c_i(B_2)\}$, and accordingly we have $\mathsf{MMS}_i(2, E) \leq \lceil \frac{n}{2} \rceil \cdot c_i(T_1) = \lceil \frac{n}{2} \rceil \cdot \mathsf{MMS}_i(n, E)$. Therefore, for any $j \neq i$, the following holds:

$$\frac{c_i(A_i)}{\mathsf{MMS}_i(n, E)} \leq \frac{\alpha \cdot \mathsf{MMS}_i(2, A_i \cup A_j)}{\mathsf{MMS}_i(n, E)} \leq \frac{\alpha \cdot \mathsf{MMS}_i(2, E)}{\mathsf{MMS}_i(n, E)} \leq \alpha \cdot \lceil \frac{n}{2} \rceil.$$

As for the lower bound, it suffices to show that for any $\alpha \in [1, 2]$, there exists an α-PMMS allocation with approximation guarantee $\alpha\lceil\frac{n}{2}\rceil$ of MMS when $\alpha\lceil\frac{n}{2}\rceil \leq n$. Let us consider an instance with n (even) agents and a set E of chores with $|E| = n(n+1)$. Since $\alpha \leq 2$ and n is even, clearly we have $\alpha\lceil\frac{n}{2}\rceil \leq n$. Chores are placed in $n \times (n+1)$ matrix $E = (e_{ij})$. For $j \in [n+1]$, denote by P_j the j-th column, i.e., $P_j = \{e_{1j}, e_{2j}, \ldots, e_{nj}\}$. We concentrate on allocation $\mathbf{A}$ with $A_1 = P_1 \cup \cdots \cup P_{\lfloor\alpha\frac{n}{2}\rfloor} \cup P_n$, $A_j = \left\{e_{j,\lfloor\alpha\frac{n}{2}\rfloor+1}, \ldots, e_{j,n-1}, e_{j,n+1}\right\}$ for any $2 \leq j \leq n-1$, and $A_n = \left\{e_{n,\lfloor\alpha\frac{n}{2}\rfloor+1}, \ldots, e_{n,n-1}, e_{n,n+1}\right\} \cup \left\{e_{1,\lfloor\alpha\frac{n}{2}\rfloor+1}, \ldots, e_{1,n-1}, e_{1,n+1}\right\}$. For any $2 \leq i \leq n$, agent i has additive cost function $c_i(\cdot)$ with $c_i(e) = 0$ for any $e \in A_i$, and $c_i(e) = 1$ for any $e \in E \setminus A_i$. Then, for every $2 \leq i \leq n$, agent i has an additive, clearly monotone and submodular, cost function, and violates neither PMMS nor MMS due to $c_i(A_i) = 0$. Consequently, the approximation guarantee of $\mathbf{A}$ on both PMMS and MMS are determined by agent 1.

As for the cost function $c_1(\cdot)$ of agent 1, for any $S \subseteq E$, we let

$$c_1(S) = \sum_{j=1}^{n-1} \min\{|S \cap P_j|, 1\} + \delta \cdot \min\{|S \cap P_n|, 1\} + (1-\delta) \cdot \min\{|S \cap P_{n+1}|, 1\},$$

where $\delta = \alpha\frac{n}{2} - \lfloor\alpha\frac{n}{2}\rfloor$. Function $c_1(\cdot)$ is clearly monotone. As in the proof of Proposition 3.5.1 (see Footnote 4), $c_1(\cdot)$ as a conical combination of submodular functions is also submodular.

We argue $\mathbf{A}$ is an α-PMMS allocation with approximation guarantee $\alpha\frac{n}{2}$ with respect to MMS. In fact, under allocation $\mathbf{A}$, one can compute $c_1(A_1) = \lfloor\alpha\frac{n}{2}\rfloor + \delta = \alpha\frac{n}{2}$ and $c_1(A_1 \cup A_j) = c_1(E) = n$ for any $2 \leq j \leq n$. Then, for any $j \geq 2$, due to Lemma 3.2.1, it holds that $\mathsf{MMS}_1(2, A_1 \cup A_j) \geq n/2$, which then imply $c_1(A_1) \leq \alpha\mathsf{MMS}_1(2, A_1 \cup A_j)$. Thus, allocation $\mathbf{A}$ is α-PMMS. As for the quantity of $\mathsf{MMS}_1(n, E)$, consider partition $\{B_i\}_{i=1}^n$ with $B_i = P_i$ for $1 \leq i \leq n-1$ and $B_n = P_n \cup P_{n+1}$. It is not hard to verify $c_1(B_i) = 1$ for any $i \in [n]$. According to Lemma 3.2.1, we have $\mathsf{MMS}_1(n, E) \geq n^{-1}c_1(E) = 1$. Hence, partition $\{B_i\}_{i=1}^n$ defines $\mathsf{MMS}_1(n, E) = 1$, and accordingly, the approximation guarantee of $\mathbf{A}$ for MMS is $\alpha\frac{n}{2}$, equivalent to $\alpha\lceil\frac{n}{2}\rceil$ since n is even. □

We remark that all statements in this section are still true if agents have subadditive cost functions. Results in this section show that although PMMS (or MMS) are relaxations of EF under additive setting, EF allocations do not always provide non-trivial approximation guarantee for PMMS (or MMS) in the submodular setting. This motivates new submodular fairness notions which is not only a relaxation of EF but also inherit the spirit of PMMS (or MMS).

3.6 Price of Fairness under Additive Setting

After having compared the fairness criteria among themselves, in this section we study the efficiency of these fairness criteria in terms of the price of fairness with respect to the social cost.

3.6.1 Two Agents

We start with the case of two players. Our first result concerns EF1.

Algorithm 1

Input: An instance $\mathcal{I}$ with two agents.
Output: An EF1 allocation of instance $\mathcal{I}$.

Partition $E = E_0 \cup E_1 \cup E_2$ where $E_1 = \{e \in E \mid c_1(e) < c_2(e)\}$ and $E_2 = \{e \in E \mid c_1(e) > c_2(e)\}$ (assume $c_1(E_1) \leq c_2(E_2)$ and the other case is symmetric).
Order chores such that $\frac{c_1(e_1)}{c_2(e_1)} \leq \frac{c_1(e_2)}{c_2(e_2)} \leq \cdots \leq \frac{c_1(e_m)}{c_2(e_m)}$, tie breaks arbitrarily. For chore e with $c_1(e) = 0$, put it at the front and chore e with $c_2(e) = 0$ at back.
Find index s such that $c_1(e_s) < c_2(e_s)$ and $c_1(e_{s+1}) \geq c_2(e_{s+1})$.
if $s = 0$ **then**
  Run a round-robin algorithm: let each of the agent $1, 2$ picks her most preferred item in that order, and repeat until all chores are assigned.
  return the output
else
  Let $\mathbf{O}$ be the allocation with $O_1 = L(s)$ and $O_2 = R(s+1)$.
  if allocation $\mathbf{O}$ is EF1 **then**
    return allocation $\mathbf{O}$.
  else
    find the maximum index $f \geq s$ such that $c_2(R(f+2)) > c_2(L(f))$.
    return allocation $\mathbf{A}$ with $A_1 = L(f+1)$ and $A_2 = R(f+2)$.
  end if
end if

Proposition 3.6.1. *When $n = 2$ and agents have additive cost functions, the price of* EF1 *is* $5/4$.

Proof. For the upper bound part, we analyze the allocation returned by Algorithm 1. In this proof, we denote $L(k) = \{e_1, \ldots, e_k\}$ and $R(k) = \{e_k, \ldots, e_m\}$. We first show that Algorithm 1 is well-defined and can always output an EF1 allocation. Note that $\mathbf{O}$ is the optimal allocation for the underlying instance due to the order of chores. We consider the possible value of index s. Because of the normalized cost function, trivially, $s < m$ holds. If $s = 0$, Algorithm 1 outputs the allocation returned by round-robin (Step 6) and clearly, it's EF1. If the optimal allocation **O** is EF1 (Step 9), we are done. Moreover, we claim that if $s = m - 1$, then **O** must be EF1. The reason is that for agent 1, his cost $c_1(O_1) \leq c_2(O_2) \leq c_1(O_2)$ where the first transition is by the assumption in line 1 of Algorithm 1, and thus he does not envy agent 2. For agent 2, since he only receives a single chore in optimal allocation due to $s = m - 1$, clearly, he does not violate the condition of EF1, either. Thus, allocation **O** is EF1 in the case of $s = m-1$. Next, we study the remaining case (Steps 11–13) that can only happen when $1 \leq s \leq m - 2$. We first show that the index f is well-defined. It suffices to show $c_2(R(s + 2)) > c_2(L(s))$. For the sake of contradiction, assume $c_2(R(s+2)) \leq c_2(L(s))$. This is equivalent to $c_2(O_2 \setminus \{e_{s+1}\}) \leq c_2(O_1)$, which means agent 2 satisfying EF1 in allocation **O**. Due to the assumption (Step 1), $c_1(O_1) \leq c_2(O_2) \leq c_1(O_2)$ holds, and thus, agent 1 is EF under the allocation **O**. Consequently, the allocation **O** is EF1, contradiction. Then, we prove allocation **A** (Step 13) is EF1. According to the order of chores, it holds that

$$\frac{c_1\,(L(f))}{c_2(L(f))} \leq \frac{c_1(R(f+2))}{c_2(R(f+2))}.$$

Since $c_2(R(f + 2)) > c_2(L(f)) \geq 0$, this implies,

$$\frac{c_1(L(f))}{c_1(R(f+2))} \leq \frac{c_2(L(f))}{c_2(R(f+2))}.$$

By the definition of index f, we have $c_2(R(f + 2)) > c_2(L(f))$ and therefore $c_1(L(f)) < c_1(R(f+2))$ which is equivalent to $c_1(A_1 \setminus \{e_{f+1}\}) < c_1(A_2)$. Thus, agent 1 is EF1 under allocation **A**. As for agent 2, if $f = m - 2$, then $A_2 = 1$ and clearly, agent 2 does not violate the condition of EF1. We can further assume $f \leq m - 3$. Since f is the maximum index satisfying $f \geq s$ and $c_2(R(f+2)) > c_2(L(f))$, it must hold that $c_2(R(f+3)) \leq c_2(L(f+1))$, which is equivalent to $c_2(A_2 \setminus \{e_{f+2}\}) \leq c_2(A_1)$ and so agent 2 is also EF1 under allocation **A**.

Next, we show the social cost of the allocation returned by Algorithm 1 is at most 1.25 times the optimal social cost. If $s = 0$, both agents have the same cost profile, then any allocations have the optimal social cost and we are done in this case. If allocation **O** is EF1, then clearly, we are done. The remaining case is of Steps 11–13 of Algorithm 1. Since $c_1(O_1) \leq c_2(O_2) \leq c_1(O_2)$, we have $c_1(O_1) \leq 2^{-1}$. Notice that **O** is not EF1, then $c_2(O_2) > 2^{-1}$ must hold; otherwise, $c_2(O_2) \leq c_2(O_1)$ and allocation **O** is EF, contradiction. Therefore, under the case where allocation **O** is not EF1, we must have $c_1(O_1) \leq 2^{-1}$ and $c_2(O_2) > 2^{-1}$. Due to $f + 2 \geq s + 1$

and the order of chores, it holds that

$$\frac{c_1(R(f+2))}{c_2(R(f+2))} \geq \frac{c_1(O_2)}{c_2(O_2)}.$$

This implies $c_1(R(f+2)) \geq \frac{c_1(O_2)}{c_2(O_2)} c_2(R(f+2))$, and equivalently,

$$c_1(A_1) = c_1(L(f+1)) \leq 1 - \frac{c_1(O_2)}{c_2(O_2)} c_2(R(f+2)).$$

Again, by the construction of f, we have

$$c_2(A_2) = c_2(R(f+2)) > c_2(L(f)) \geq c_2(L(s)) = c_2(O_1).$$

Therefore, we derive the following upper bound,

$$\begin{aligned} c_1(A_1) + c_2(A_2) &\leq 1 - \left(\frac{c_1(O_2)}{c_2(O_2)} - 1\right) c_2(A_2) \leq 1 - \left(\frac{c_1(O_2)}{c_2(O_2)} - 1\right) c_2(O_1) \\ &= 1 - \left(\frac{1 - c_1(O_1)}{c_2(O_2)} - 1\right) (1 - c_2(O_2)), \end{aligned} \tag{3.16}$$

where the second inequality is due to $\frac{c_1(O_2)}{c_2(O_2)} \geq 1$ and $c_2(A_2) \geq c_2(O_1)$. Based on (3.16), we have an upper bound on the price of EF1 as follows:

$$\text{Price of EF1} \leq \frac{1 - \left(\frac{1 - c_1(O_1)}{c_2(O_2)} - 1\right)(1 - c_2(O_2))}{c_1(O_1) + c_2(O_2)}. \tag{3.17}$$

Recall $0 \leq c_1(O_1) \leq 2^{-1} < c_2(O_2) \leq 1$. The partial derivative of the fraction in (3.17) with respect to $c_1(O_1)$ is equal to the following:

$$\frac{1}{(c_1(O_1) + c_2(O_2))^2} \left(\frac{1}{c_2(O_2)} - 2\right).$$

It is not hard to see this derivative has a negative value for any $2^{-1} < c_2(O_2) \leq 1$. Thus, the fraction in (3.17) takes maximum value only when $c_1(O_1) = 0$ and hence,

$$\text{Price of EF1} \leq \frac{3 - \frac{1}{c_2(O_2)}}{c_2(O_2)} - 1.$$

Similarly, by taking the derivative with respect to $c_2(O_2)$, the maximum value of this expression happens only when $c_2(O_2) = \frac{2}{3}$, then one can easily compute the maximum value of the RHS of Inequality (3.17) is 1.25. Therefore, the price of EF1 $\leq$ 1.25.

As for the lower bound, consider an instance with a set $E = \{e_1, e_2, e_3\}$ of three chores. The cost function of agent 1 is $c_1(e_1) = 0$ and $c_1(e_2) = c_1(e_3) = 1/2$. For agent 2, his cost is $c_2(e_1) = 1/3 - 2\epsilon$ and $c_2(e_2) = c_2(e_3) = 1/3 + \epsilon$ where $\epsilon > 0$ is arbitrarily small. An optimal allocation assigns chore e_1 to agent 1 and the remaining chores to agent 2, which yields the optimal social cost $2/3 + \epsilon$. However, this allocation is not EF1 since agent 2 envies agent 1 even after removing one chore from his bundle. To achieve EF1, agent 2 can not receive both of chores e_2 and e_3, and so, agent 1 must receive one of chore e_2 and e_3. Therefore, the best EF1 allocation can be assigning chore e_1 and e_2 to agent 1 and chore e_3 to agent 2 resulting in the social cost $5/6 + \epsilon$. Thus, the price of EF1 is at least $\frac{\frac{5}{6}+\epsilon}{\frac{2}{3}+2\epsilon} \to \frac{5}{4}$ as $\epsilon \to 0$, completing the proof. □

According to Propositions 3.3.4 and 3.3.7, EF1 implies 2-MMS and (3/2)-PMMS. In the following, we pay special attention to notions of 2-MMS and (3/2)-PMMS, of which the existence is guaranteed, and provide tight results on the price of fairness.

Proposition 3.6.2. *When $n = 2$ and agents have additive cost functions, the price of 2-*MMS *is 1.*

Proof. The proof directly follows from Lemma 3.2.2. □

Proposition 3.6.3. *When $n = 2$ and agents have additive cost functions, the price of $\frac{3}{2}$-*PMMS *is* $7/6$.

Proof. We first prove the upper bound. Given an instance $\mathcal{I}$, let $\mathbf{O} = (O_1, O_2)$ be an optimal allocation of $\mathcal{I}$. If the allocation $\mathbf{O}$ is already 3/2-PMMS, we are done. For the sake of contradiction, we assume that agent 1 violates the condition of 3/2-PMMS in allocation $\mathbf{O}$, i.e., $c_1(O_1) > 3/2\mathsf{MMS}_1(2, E)$. Suppose $O_1 = \{e_1, \ldots, e_h\}$ and the index satisfies the following rule; $\frac{c_1(e_1)}{c_2(e_1)} \geq \frac{c_1(e_2)}{c_2(e_2)} \geq \cdots \geq \frac{c_1(e_h)}{c_2(e_h)}$. In this proof, for simplicity, we write $L(k) := \{e_1, ..., e_k\}$ for any $1 \leq k \leq h$ and $L(0) = \emptyset$. Then, let s be the index such that $c_1(O_1 \setminus L(s)) \leq 3/2\mathsf{MMS}_1(2, E)$ and $c_1(O_1 \setminus L(s-1)) > 3/2\mathsf{MMS}_1(2, E)$. In the following, we divide our proof into two cases.

Case 1: $c_1(L(s)) \leq 1/2 \cdot c_1(O_1)$. Consider allocation $\mathbf{A} = (A_1, A_2)$ with $A_1 = O_1 \setminus L(s)$ and $A_2 = O_2 \cup L(s)$. We first show allocation $\mathbf{A}$ is 3/2-PMMS. For agent 1, due to the construction of index s, he does not violate the condition of 3/2-PMMS. As for agent 2, we claim that $c_2(A_2) = 1 - c_2(O_1 \setminus L(s-1)) + c_2(e_s) < 1/4 + c_2(e_s)$ because $c_2(O_1 \setminus L(s-1)) \geq c_1(O_1 \setminus L(s-1)) > 3/2\mathsf{MMS}_1(2, E) \geq \frac{3}{4}$ where the first inequality transition is due to the fact that O_1 is the bundle assigned to agent 1 in the optimal allocation. If $c_2(e_s) < 1/2$, then clearly, $c_2(A_2) < 3/4 \le 3/2\mathsf{MMS}_2(2,E)$. If $c_2(e_s) \ge 1/2$, then $c_2(e_s) = \mathsf{MMS}_1(2,E)$ and accordingly, it is not hard to verify that $c_2(A_2) \le 3/2\mathsf{MMS}_1(2,E)$. Thus, $\mathbf{A}$ is a 3/2-PMMS allocation.

Next, based on allocation $\mathbf{A}$, we derive an upper bound on the price of 3/2-PMMS. First, by the order of index, $\frac{c_1(L(s))}{c_2(L(s))} \ge \frac{c_1(O_1)}{c_2(O_1)}$ holds, implying $c_2(L(s)) \le \frac{c_2(O_1)}{c_1(O_1)} c_1(L(s))$. Since $A_1 = O_1 \setminus L(s)$ and $A_2 = O_2 \cup L(s)$, we have the following:

$$\begin{aligned}
\text{Price of } \frac{3}{2}\text{-PMMS} &\le 1 + \frac{c_2(L(s)) - c_1(L(s))}{c_1(O_1) + c_2(O_2)} \le 1 + \frac{c_1(L(s))(\frac{c_2(O_1)}{c_1(O_1)} - 1)}{c_1(O_1) + c_2(O_2)} \\
&= 1 + \frac{\frac{c_1(L(s))}{c_1(O_1)}(1 - c_2(O_2) - c_1(O_1))}{c_1(O_1) + c_2(O_2)} \\
&\le 1 + \frac{\frac{1}{2} - \frac{1}{2}(c_1(O_1) + c_2(O_2))}{c_1(O_1) + c_2(O_2)} \le 1 - \frac{1}{2} + \frac{1}{2} \times \frac{4}{3} = \frac{7}{6},
\end{aligned}$$

where the second inequality due to $c_2(L(s)) \le \frac{c_2(O_1)}{c_1(O_1)} c_1(L(s))$; the third inequality due to the condition of Case 1; and the last inequality is because $c_1(O_1) > 3/2\mathsf{MMS}_1(2,E) \ge 3/4$.

Case 2: $c_1(L(s)) > 1/2 \cdot c_1(O_1)$. We first derive a lower bound on $c_1(e_s)$. Since $c_1(e_s) = c_1(O_1 \setminus L(s-1)) + c_1(L_s) - c_1(O_1)$, combine which with the condition of Case 2 implying $c_1(e_s) > c_1(O_1 \setminus L(s-1)) - 1/2 \cdot c_1(O_1)$, and consequently we have $c_1(e_s) > 3/2\mathsf{MMS}_1(2,E) - 1/2 \cdot c_1(O_1) \ge 1/4$ where the last transition is due to $\mathsf{MMS}_1(2,E) \ge 1/2$ and $c_1(O_1) \le 1$. Then, we consider two subcases.

If $0 \le c_2(e_s) - c_1(e_s) \le 1/8$, consider an allocation $\mathbf{A} = (A_1, A_2)$ with $A_1 = O_1 \setminus \{e_s\}$ and $A_2 = O_2 \cup \{e_s\}$. We first show the allocation $\mathbf{A}$ is 3/2-PMMS. For agent 1, since $c_1(e_s) > 1/4$, $c_1(A_1) = c_1(O_1) - c_1(e_s) < 3/4 \le 3/2\mathsf{MMS}_1(2,E)$. As for agent 2, $c_2(A_2) = c_2(O_2) + c_2(e_s) \le 1 - c_1(O_1) + c_2(e_s) < 1/4 + c_2(e_s)$. If $c_2(e_s) < 1/2$, then clearly, $c_2(A_2) \le 3/4 < 3/2\mathsf{MMS}_2(2,E)$ holds. If $c_2(e_s) \ge 1/2$, we have $c_2(e_s) = \mathsf{MMS}_2(2,E)$ and accordingly, it is not hard to verify that $c_2(A_2) \le 3/2\mathsf{MMS}_2(2,E)$. Thus, the allocation $\mathbf{A}$ is 3/2-PMMS. Next, based on the allocation $\mathbf{A}$, we derive an upper bound regarding the price of 3/2-PMMS,

$$\text{Price of } \frac{3}{2}\text{-PMMS} \le \frac{c_1(O_1) - c_1(e_s) + c_2(O_2) + c_2(e_s)}{c_1(O_1) + c_2(O_2)} \le 1 + \frac{1}{8} \times \frac{4}{3} = \frac{7}{6},$$

where the second inequality due to $0 \le c_2(e_s) - c_1(e_s) \le 1/8$ and $c_1(O_1) > 3/4$.

If $c_2(e_s) - c_1(e_s) > 1/8$, consider an allocation $\mathbf{A}' = (A'_1, A'_2)$ with $A'_1 = \{e_s\}$ and $A'_2 = E \setminus \{e_s\}$. We first show that the allocation $\mathbf{A}'$ is 3/2-PMMS. For agent 1, due to Lemma 3.2.1, $c_1(e_s) \le \mathsf{MMS}_1(2,E)$ holds. As for agent 2, since $c_2(e_s) \ge$

$c_1(e_s) > 1/4$, we have $c_2(A'_2) = c_2(E) - c_2(e_s) < 3/4 \leq 3/2\mathsf{MMS}_2(2, E)$. Thus, the allocation $\mathbf{A}'$ is 3/2-PMMS. In the following, we first derive an upper bound for $c_2(O_1 \setminus \{e_s\}) - c_1(O_1 \setminus \{e_s\})$, then based on the bound, we provide the target upper bound for the price of fairness. Since $c_1(O_1) > 3/4$ and $c_2(e_s) - c_1(e_s) > 1/8$, we have $c_2(O_1 \setminus \{e_s\}) - c_1(O_1 \setminus \{e_s\}) = c_2(O_1) - c_1(O_1) - (c_2(e_s) - c_1(e_s)) < 1/8$, and then, the following holds,

$$\text{Price of } \frac{3}{2}\text{-PMMS} \leq 1 + \frac{c_2(O_1 \setminus \{e_s\}) - c_1(O_1 \setminus \{e_s\})}{c_1(O_1) + c_2(O_2)} \leq 1 + \frac{1}{8} \times \frac{4}{3} = \frac{7}{6},$$

which completes the proof of the upper bound.

Regarding lower bound, consider an instance I with two agents and a set $E = \{e_1, e_2, e_3, e_4\}$ of four chores. The cost function for agent 1 is: $c_1(e_1) = 3/8, c_1(e_2) = 3/8 + \epsilon, c_1(e_3) = 1/8 - \epsilon, c_1(e_4) = 1/8$ where $\epsilon > 0$ is arbitrarily small. For agent 2, her cost function is: $c_2(e_1) = c_2(e_2) = 1/2$, $c_2(e_3) = c_2(e_4) = 0$. It is not hard to verify that $\mathsf{MMS}_i(2, E) = \frac{1}{2}$ for any $i = 1, 2$. In the utilitarian welfare-maximizing allocation, the assignment is; e_1, e_2 to agent 1 and e_3, e_4 to agent 2, resulting in $\mathsf{OPT}_U(\mathcal{I}) = 3/4 + \epsilon$. Observe that to satisfy 3/2-PMMS, agent 1 cannot receive both chores e_1, e_2, and accordingly, the minimum social cost of a 3/2-PMMS allocation is 7/8 by assigning e_1 to agent 1 and the rest chores to agent 2. Based on this instance, when $n = 2$, the price of 3/2-PMMS is at least $\frac{\frac{7}{8}}{\frac{6}{8}+\epsilon} \to \frac{7}{6}$ as $\epsilon \to 0$. □

The above two propositions confirm an intuition — if one relaxes the fairness condition, then less efficiency will be sacrificed. We also remark that if we have an *oracle* for the maximin share, then our constructive proof of Proposition 3.6.3 can be transformed into an efficient algorithm for finding a (3/2)-PMMS allocation whose cost is at most 7/6 times the optimal social cost. Moving to other fairness criteria, we have the following uniform bound.

Proposition 3.6.4. *When $n = 2$ and agents have additive cost functions, the price of* PMMS, MMS*, and* EFX *are all 2.*

Proof. We first show results on the upper bound. When $n = 2$, PMMS is identical with MMS and can imply EFX, so it suffices to show that the price of PMMS is at most 2. Given an instance $\mathcal{I}$, let allocation $\mathbf{O} = (O_1, O_2)$ be the utilitarian welfare-maximizing allocation and w.l.o.g, we assume $c_1(O_1) \leq c_2(O_2)$. If $c_2(O_2) \leq 1/2$, then we have $c_1(O_1) \leq 1 - c_1(O_1) = c_1(O_2)$ and $c_2(O_2) \leq 1 - c_2(O_2) = c_2(O_1)$. So allocation $\mathbf{O}$ is an EF and accordingly is PMMS, which yields that the price of PMMS equals to one. Thus, we can further assume $c_2(O_2) > 1/2$ and hence the optimal social cost is larger than $\frac{1}{2}$.

We next show that there exist a PMMS allocation whose social cost is at most 1. W.l.o.g, we assume $\mathsf{MMS}_1(2, E) \leq \mathsf{MMS}_2(2, E)$ (the other case is symmetric). Let $\mathbf{T} = (T_1, T_2)$ be the allocation defining $\mathsf{MMS}_1(2, E)$ and $c_1(T_1) \leq c_1(T_2) = \mathsf{MMS}_1(2, E)$. If $c_2(T_2) \leq c_2(T_1)$, then allocation $\mathbf{T}$ is EF (also PMMS), and thus it hold that $c_1(T_1) \leq 1/2$ and $c_2(T_2) \leq 1/2$. Therefore, the social cost of allocation $\mathbf{T}$ is no more than one, which implies that the price of PMMS is at most two. If $c_2(T_2) > c_2(T_1)$, then consider the allocation $\mathbf{T}' = (T_2, T_1)$. Since $c_1(T_1') = c_1(T_2) = \mathsf{MMS}_1(2, E)$ and $c_2(T_2') = c_2(T_1) < c_2(T_2)$, then $\mathbf{T}'$ is a PMMS allocation. Owing to $\mathsf{MMS}_1(2, E) \leq \mathsf{MMS}_2(2, E)$, we claim that $c_2(T_1) \leq c_1(T_1)$; otherwise, we have $\mathsf{MMS}_1(2, E) = c_1(T_2) > c_2(T_2) > c_2(T_1)$, and equivalently, allocation $\mathbf{T}'$ is a 2-partition where the cost of both bundles for agent 2 is strictly smaller than $\mathsf{MMS}_1(2, E)$, contradicting $\mathsf{MMS}_1(2, E) \leq \mathsf{MMS}_2(2, E)$. By $c_2(T_1) \leq c_1(T_1)$, the social cost of allocation $\mathbf{T}'$ satisfies $c_2(T_1) + c_1(T_2) \leq 1$ and so the price of PMMS is at most two.

Regarding the tightness, consider an instance $\mathcal{I}$ with two agents and a set $E = \{e_1, e_2, e_3\}$ of three chores. The cost function of agent 1 is : $c_1(e_1) = 1/2$, $c_1(e_2) = 1/2 - \epsilon$ and $c_1(e_3) = \epsilon$ where $\epsilon > 0$ is arbitrarily small. For agent 2, his cost is $c_2(e_1) = 1/2$, $c_2(e_2) = \epsilon$ and $c_2(e_3) = 1/2 - \epsilon$. An utilitarian welfare-maximizing allocation assigns chores e_1, e_2 to agent 2, and e_3 to agent 1, and consequently, the optimal social cost equals to $1/2 + 2\epsilon$. We first concern the tightness on the notion of PMMS (or MMS, these two are identical when $n = 2$). In any PMMS allocations, it must be the case that an agent receives chore e_1 and the other one receives chores e_2 and e_3, and thus the social cost of PMMS allocations is one. Therefore, the price of PMMS and of MMS are at least $\frac{1}{\frac{1}{2}+\epsilon} \to 2$ as $\epsilon \to 0$. As for EFX, similarly, it must be the case that in any EFX allocations, the agent receiving chore e_1 cannot receive any other chores. Thus, it not hard to verify that the social cost of EFX allocations is also one and the price of EFX is at least $\frac{1}{\frac{1}{2}+\epsilon} \to 2$ as $\epsilon \to 0$. □

3.6.2 More than Two Agents

Note that the existence of an MMS allocation is not guaranteed in general, even when agents have additive valuation functions [19, 83], and the existence of PMMS or EFX allocation is still open in chores when $n \geq 3$. Nonetheless, we are still interested in the prices of fairness in case such a fair allocation does exist.

Proposition 3.6.5. *When agent have additive cost functions, for $n \geq 3$, the price* *of* EF1, EFX, PMMS *and* (3/2)-PMMS *are all infinity.*

Proof. In this proof, $\epsilon > 0$ is arbitrarily small. Based on our results on the connections between fairness criteria, we have the relationship: PMMS→EFX→EF1→(3/2)-PMMS, where $A \to B$ refers to that notion A is stricter than notion B. Therefore, it suffices to give a proof for (3/2)-PMMS.

Consider an instance with n agents and $m \geq 5$ chores. The cost function of agent 1 is $c_1(e_1) = 1-4\epsilon$, $c_1(e_j) = 0$ for $j = 2, \ldots, m-4$, and $c_1(e_j) = \epsilon$ for $j \geq m-3$. For agent 2, his cost is $c_2(e_1) = 1 - 4/m$, $c_2(e_j) = 0$ for $j = 2, \ldots, m-4$, and $c_2(e_j) = 1/m$ for $j \geq m-3$. The cost function of agent 3 is: $c_3(e_1) = \epsilon$, $c_3(e_j) = 1/m$ for $j = 2, \ldots, m-1$, and $c_3(e_m) = 1/m - \epsilon$. For any $i \geq 4$, the cost function of agent i is $c_i(e_j) = 1/m$ for any $j \in [m]$. An utilitarian welfare-maximizing allocation assigns $e_{m-3}, e_{m-2}, e_{m-1}, e_m$ to agent 1 and e_1 to agent 3. For each of rest chore, it is assigned to the agent having zero cost on it. Accordingly, the optimal social cost is 5ϵ. However, for any optimal allocation $\mathbf{O}$, we have $\mathsf{MMS}_1(2, O_1 \cup O_2) = 2\epsilon$, implying $c_1(O_1) > 3/2\mathsf{MMS}_1(2, O_1 \cup O_2)$. Thus, agent 1 violates (3/2)-PMMS. In order to achieve (3/2)-PMMS, at least one of $e_{m-3}, e_{m-2}, e_{m-1}, e_m$ has to be assigned to someone other than agent 1, and so the social cost of a (3/2)-PMMS allocation is at least $1/m + 3\epsilon$, resulting in an unbounded price of (3/2)-PMMS when $\epsilon \to 0$. □

In the context of goods allocation, Barman et al. [22] present an asymptotically tight price of EF1, $O(\sqrt{n})$. However, as shown by Proposition 3.6.5, when allocating chores, the price of EF1 is infinite, which shows a sharp contrast between goods and chores allocation.

We are now left with MMS fairness. Let us first provide upper and lower bounds on the price of MMS.

Proposition 3.6.6. *When agents have additive cost functions, for $n \geq 3$, the price of* MMS *is at most n^2 and at least $n/2$.*

Proof. We first prove the upper bound part. For any instance $\mathcal{I}$, if the minimum social cost of $\mathcal{I}$ is no more than $1/n$, then by Lemma 3.2.1, every optimal allocation is MMS fair. Thus, we can further assume that the minimum social cost of $\mathcal{I}$ is larger than $1/n$. Note that the maximum social cost of an allocation is n and thus the upper bound of n^2 is straightforward.

For the lower bound, consider an instance I with n agents and $n+1$ chores $E = \{e_1, \ldots, e_{n+1}\}$. For agent $i = 2, \ldots, n$, $c_i(e_1) = c_i(e_2) = 1/2$ and $c_i(e_j) = 0$ for any $j \geq 3$. As for agent 1, $c_1(e_1) = 1/n$, $c_1(e_2) = \epsilon$, $c_1(e_3) = 1/n - \epsilon$ and $c_1(e_j) = 1/n$ for any $j \geq 4$ where $\epsilon > 0$ is arbitrarily small. It is not hard to verify that $\mathsf{MMS}_1(n, E) = 1/n$ and $\mathsf{MMS}_i(n, E) = 1/2$ for $i \geq 2$. In any optimal allocation $\mathbf{O} = (O_1, \dots, O_n)$, the first two chores are assigned to agent 1 and each of the remaining chores is assigned to agents having cost zero. Thus, we have the minimum social cost is equal to $1/n + \epsilon$. However, in an utilitarian welfare-maximizing allocation $\mathbf{O}$, we have $c_1(O_1) > \mathsf{MMS}_1(n, E) = 1/n$. In order to achieve MMS, agent 1 can not receive both chores e_1 and e_2, and so at least one of them has to be assigned to the agent other than agent 1. As a result, the social cost of an MMS allocation is at least $1/2 + \epsilon$, which implies that the price of MMS is at least $n/2$ as $\epsilon \to 0$. $\square$

As mentioned earlier, the existence of MMS fair allocation is not guaranteed. So we also provide an asymptotically tight price of 2-MMS, whose existence is guaranteed for any instance with additive cost functions.

Proposition 3.6.7. *When agents have additive cost functions, for $n \geq 3$, the price of 2-*MMS *is at least $(n+3)/6$ and at most n, asymptotically tight $\Theta(n)$.*

Proof. We first prove the upper bound. By Proposition 3.3.4, we know that an EF1 allocation is also $\frac{2n-1}{n}$-MMS (also 2-MMS). As we mentioned earlier, the round-robin algorithm always output EF1 allocations. Consequently, given any instance $\mathcal{I}$, the allocation returned by round-robin is also 2-MMS. In the following, we incorporate the idea of expectation in probability theory and show that there exists an order of round-robin such that the output allocation has social cost at most 1.

Let σ be a uniformly random permutation of $\{1, \dots, n\}$ and $\mathbf{A}(\sigma) = (A_1(\sigma), \dots, A_n(\sigma))$ be the allocation returned by round-robin based on the order σ. Clearly, each element $A_i(\sigma)$ is a random variable. Since σ is chosen uniformly random, the probability of agent i on j-th position is $1/n$. Fix an agent i, we assume $c_i(e_1) \leq c_i(e_2) \leq \cdots \leq c_i(e_m)$. If agent i is in j-th position of the order, then his cost is at most $c_i(e_j) + c_i(e_{n+j}) + \cdots + c_i(e_{\lfloor \frac{m-j}{n} \rfloor n+j})$. Accordingly, his expected cost is at most $\sum_{j=1}^{n} \frac{1}{n} \sum_{l=0}^{\lfloor \frac{m-j}{n} \rfloor} c_i(e_{ln+j})$. Thus, we have an upper bound of the expected social cost,

$$\mathbb{E}[\mathsf{UW}(\mathbf{A}(\sigma))] \leq \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{n} \sum_{l=0}^{\lfloor \frac{m-j}{n} \rfloor} c_i(e_{ln+j}) = \frac{1}{n} \sum_{i=1}^{n} c_i(E) = 1.$$

Therefore, there exists an order such that the social cost of the output is at most 1. Note that for any instance $\mathcal{I}$, if the minimum social cost of $\mathcal{I}$ is at most $1/n$, then any optimal allocations are also MMS. Thus, we can further assume that the minimum social cost of $\mathcal{I}$ is larger than $1/n$, and accordingly, the price of 2-MMS is at most n.

For the lower bound, consider an instance $\mathcal{I}$ with n agents and a set $E = \{e_1, \ldots, e_{n+3}\}$ of $n+3$ chores. The cost function of agent 1 is: $c_1(e_1) = c_1(e_2) = 1/n - \epsilon$, $c_1(e_3) = 3\epsilon$, $c_1(e_4) = c_1(e_5) = \epsilon$, $c_1(e_6) = 1/n - 3\epsilon$ where $\epsilon > 0$ is arbitrarily small, and $c_1(e_j) = 1/n$ for any $j > 6$ (if exists). For agent $i = 2, \ldots, n$, his cost is: $c_i(e_j) = 1/3$ for any $j \in [3]$ and $c_i(e_j) = 0$ for $j \geq 4$. It is not hard to verify that $\mathsf{MMS}_1(n, E) = 1/n$ and $\mathsf{MMS}_i(n, E) = 1/3$ for any $i \geq 2$. In a social cost-minimizing allocation $\mathbf{O} = (O_1, \ldots, O_n)$, the first three chores are assigned to agent 1 and all rest chores are assigned to agents having cost zero on them. Thus, we have the minimum social cost is equal to $2/n + \epsilon$. However, note that $2/n + \epsilon = c_1(O_1) > 2\mathsf{MMS}_1(n, E)$ holds, and so agent 1 violates 2-MMS. In order to achieve a 2-MMS fair allocation, agent 1 can not receive all first three chores, and so at least one of them has to be assigned to the agent other than agent 1. As a result, the social cost of a 2-MMS fair allocation is at least $1/3 + 1/n + 2\epsilon$, yielding that the price of 2-MMS is at least $n/6 + 1/2$. Combing lower and upper bound, the price of 2-MMS is $\Theta(n)$ □

3.7 Price of Fairness beyond Additive Setting

In this section, we study the price of fairness when agents have submodular cost functions. Note that for those fairness notions whose prices of fairness are unbounded in the additive setting, the efficiency loss would still be unbounded in the submodular setting. As a consequence, for most notions, it suffices to study its price of fairness in the case of two agents. Recall that, when studying a specific fairness notion, we only consider instances for which allocations satisfying the underlying fairness notion do exist. All results established in this section remain true if agents have subadditive cost functions.

Proposition 3.7.1. *When $n = 2$ and agents have submodular cost functions, if an* EFX *allocation exists, the price of* EFX *is at least 3 and at most 4.*

Proof. We first prove the upper bound. For an instance $\mathcal{I}$, let $\mathbf{O} = (O_1, O_2)$ be an optimal allocation, and without loss of generality, we assume $c_1(O_1) \leq c_2(O_2)$. Since $c_2(\cdot)$ is submodular and also subadditive, then $c_i(O_i) + c_i(O_{3-i}) \geq c_i(E)$ holds for $i \in [2]$. If $c_1(O_1) \leq c_2(O_2) \leq 1/2$, then $c_i(O_{3-i}) \geq c_i(E) - c_i(O_i) \geq 1/2 \geq c_i(O_i)$ holds for $i \in [2]$. Accordingly, allocation $\mathbf{O}$ is already EFX and we are done. Thus, without loss of generality, we can further assume $c_2(O_2) > 1/2$. Note that the social cost of an allocation is at most 2, and so the price of EFX is at most 4.

As for the lower bound, let us consider an instance with a set $E = \{e_1, e_2, e_3\}$ of three chores. The cost function of agent 1 is: $c_1(e_1) = 1/2, c_1(e_2) = 1/2 - \epsilon, c_1(e_3) = \epsilon$ and for any $S \subseteq E, c_1(S) = \sum_{e \in S} c_1(e)$ where $\epsilon > 0$ is arbitrarily small. The cost function of agent 2 is: $c_2(e_1) = 1 - \epsilon$, $c_2(e_2) = 3\epsilon$, $c_2(e_3) = 1 - 2\epsilon$ and for any $S \subseteq E$, $c_2(S) = \min\{\sum_{e \in S} c_2(e), 1\}$. Function $c_1(\cdot)$ is additive and hence clearly monotone and submodular. For function $c_2(\cdot)$, since $\sum_{e \in S} c_2(e)$ is additive (also monotone and submodular) on S, it follows that $c_2(\cdot)$ is also monotone and submodular (see Footnote 4).

For this instance, the social cost-minimizing allocation is $\mathbf{O} = (O_1, O_2)$ with $O_1 = \{e_1, e_3\}$ and $O_2 = \{e_2\}$, having a social cost $1/2+4\epsilon$. But due to $c_1(O_1 \setminus \{e_3\}) = 1/2 > 1/2 - \epsilon = c_1(O_2)$, agent 1 violates EFX in $\mathbf{O}$. In an EFX allocation, agent 2 can not receive the whole E or $\{e_1, e_3\}$ or $\{e_1, e_2\}$. Thus, the EFX allocation with the smallest social cost is $A_1 = \{e_2, e_3\}$ and $A_2 = \{e_1\}$, yielding social cost $3/2 - \epsilon$. As a consequence, the price of EFX is at least $\frac{3/2-\epsilon}{1/2+4\epsilon} \to 3$ as $\epsilon \to 0$. □

Proposition 3.7.2. *When $n = 2$ and agents have submodular cost functions, if an* EF1 *allocation exists, the price of* EF1 *is at least 2 and at most 4.*

Proof. For the upper bound part, similar to the proof of Proposition 3.7.1, we can without loss of generality assume $c_1(O_1) \leq c_2(O_2)$ and $c_2(O_2) > 1/2$; otherwise, $\mathbf{O}$ is already EF1. Note that the social cost of an allocation is at most 2, and so the price of EF1 is at most 4.

As for the lower bound, let us consider an instance $\mathcal{I}$ with a set $E = \{e_1, e_2, e_3\}$ of three chores. The cost function of agent 1 is: $c_1(e_1) = 1/3 + \epsilon$, $c_1(e_2) = 1/3$, $c_1(e_3) = 1/3 - \epsilon$ and for any $S \subseteq E$, $c_1(S) = \sum_{e \in S} c_1(e)$ where $\epsilon > 0$ is arbitrarily small. The cost function of agent 2 is: $c_2(e_1) = 1 - \epsilon$, $c_2(e_2) = 1 - \epsilon$, $c_2(e_3) = \epsilon$ and for any $S \subseteq E$, $c_2(S) = \min\{\sum_{e \in S} c_2(e), 1\}$. Function $c_1(\cdot)$ is additive and clearly monotone and submodular. For function $c_2(\cdot)$, since $\sum_{e \in S} c_2(e)$ is additive (also monotone and submodular) on S, it follows that $c_2(\cdot)$ is also monotone and submodular (see Footnote 4).

For this instance, the social cost-minimizing allocation is $\mathbf{O} = (O_1, O_2)$ with $O_1 = \{e_1, e_2\}$ and $O_2 = \{e_3\}$, having a social cost $2/3+2\epsilon$. But since $\min_{e \in O_1} c_1(O_1 \setminus \{e\}) = 1/3 > 1/3 - \epsilon = c_1(O_2)$, agent 1 violates EF1 under allocation $\mathbf{O}$. In an EF1 allocation, agent 2 can not receive all chores and can not receive both e_1, e_2, either. Thus, the EF1 allocation with the minimum social cost is $\mathbf{A} = (A_1, A_2)$ with $A_1 = \{e_2\}$ and $A_2 = \{e_1, e_3\}$, and $\mathsf{UW}(\mathbf{A}) = 4/3$. As a consequence, the price of EF1 is at least $\frac{4/3}{3/2+2\epsilon} \to 2$ as $\epsilon \to 0$. □

Proposition 3.7.3. *When $n = 2$ and agents have submodular cost functions, if an* PMMS *fair allocation exists, the price of* PMMS *is 3.*

Proof. According to Lemma 3.2.1, $\mathsf{MMS}_i(2, E) \geq 1/2$ holds for any $i \in [2]$. Given an instance $\mathcal{I}$ and an allocation $\mathbf{O}$ with minimum social cost, we can assume that allocation $\mathbf{O}$ is not MMS and without loss of generality, agent 2 violates the condition of MMS. Let $\mathbf{A}$ be an MMS fair allocation. Due to $c_2(A_2) \leq \mathsf{MMS}_2(2, E) < c_2(O_2)$, we have

$$\frac{c_1(A_1) + c_2(A_2)}{c_1(O_1) + c_2(O_2)} < \frac{c_1(A_1) + \mathsf{MMS}_2(2, E)}{\mathsf{MMS}_2(2, E)} \leq 3,$$

where the last inequality transition is due to $c_1(A_1) \leq 1$ and $\mathsf{MMS}_2(2, E) \geq 1/2$.

As for the lower bound, let us consider an instance $\mathcal{I}$ with a set $E = \{e_1, e_2, e_3\}$ of chores. The cost function of agent 1 is: $c_1(e_1) = 1/2$, $c_1(e_2) = 1/2 - \epsilon$, $c_1(e_3) = \epsilon$ and for $S \subseteq E, c_1(S) = \sum_{e \in S} c_1(e)$. The cost function of agent 2 is: $c_2(e_1) = 1 - 2\epsilon$, $c_2(e_2) = 10\epsilon$, $c_2(e_3) = 1 - 3\epsilon$, $c_2(e_1 \cup e_2) = 1$, $c_2(e_1 \cup e_3) = 1$, $c_2(e_2 \cup e_3) = 1 - \epsilon$, $c_2(E) = 1$ where $\epsilon > 0$ is arbitrarily small. Function $c_1(\cdot)$ is additive and hence monotone and submodular. It is not hard to verify $c_2(\cdot)$ is monotone. Suppose $c_2(\cdot)$ is not submodular, and accordingly, there exists $S \subsetneq T \subseteq E$ and $e \in E \setminus T$ such that $c_2(T \cup \{e\}) - c_2(T) > c_2(S \cup \{e\}) - c_2(S)$. Since $c_2(\cdot)$ is monotone, we have $c_2(S \cup \{e\}) - c_2(S) \geq 0$ implying $c_2(T \cup \{e\}) - c_2(T) > 0$. If $|T| = 2$, the only possibility is $T = e_2 \cup e_3$ and adding e_1 to T has margin ϵ. But for any $S \subsetneq T$ the margin of adding e_1 to S is larger than ϵ, a contradiction. If $|T| = 1$, then $c_2(S \cup \{e\}) - c_2(S) = c_2(e)$ that is the largest margin of adding item e to a subset, a contradiction. Thus, function $c_2(\cdot)$ is also submodular.

For this instance, the partition $\{\{e_1\}, \{e_2, e_3\}\}$ defines $\mathsf{MMS}_1(2, E) = 1/2$, and $\{\{e_1\}, \{e_2, e_3\}\}$ defines $\mathsf{MMS}_2(2, E) = 1 - \epsilon$. The social cost-minimizing allocation is $\mathbf{O} = (O_1, O_2)$ with $O_1 = \{e_1, e_3\}$ and $O_2 = \{e_2\}$, and has social cost $\mathsf{UW}(\mathbf{O}) = 1/2 + 11\epsilon$. But $c_1(O_1) = 1/2 + \epsilon > \mathsf{MMS}_1(2, E)$, and thus $\mathbf{O}$ is not MMS. Observe that in an MMS allocation, agent 2 can only receive either a single chore or $\{e_2, e_3\}$. The MMS allocation with minimum social cost is $\mathbf{A}$ with $A_1 = \{e_2, e_3\}$ and $A_2 = \{e_1\}$ whose social cost is equal to $3/2 - 2\epsilon$. As a consequence, the price of MMS is at least $\frac{3/2 - 2\epsilon}{1/2 + 11\epsilon} \to 3$ as $\epsilon \to 0$. ☐

Proposition 3.7.4. *When $n = 2$ and agents have submodular cost functions, if a* (3/2)-PMMS *fair allocation exists, the price of* (3/2)-PMMS *is at least* 4/3 *and at most* 8/3.

Proof. We first prove the upper bound. According to Lemma 3.2.1, $\mathsf{MMS}_i(2, E) \geq 1/2$ holds for any $i \in [2]$. Given an instance $\mathcal{I}$, let $\mathbf{O} = (O_1, O_2)$ be an social cost-minimizing allocation of $\mathcal{I}$, and without loss of generality, we assume $c_1(O_1) \leq c_2(O_2)$. Moreover, we can assume $c_2(O_2) > 3/4$; otherwise $\mathbf{O}$ is already a (3/2)-PMMS fair allocation and we are done. Note that the social cost of an allocation is at most 2, and so the price of (3/2)-PMMS is at most 8/3.

As for the lower bound, let us consider an instance with a set $E = \{e_1, e_2, e_3, e_4\}$ of four chores. Throughout this proof, $\epsilon > 0$ is arbitrarily small. The cost function of agent 1 is: $c_1(e_1) = 3/8$, $c_1(e_2) = 3/8 + \epsilon$, $c_1(e_3) = 1/8 - \epsilon$, $c_1(e_4) = 1/8$ and for $S \subseteq E, c_1(S) = \sum_{e \in S} c_1(e)$. The cost profile of agent 2 is: $c_2(e_1) = c_2(e_2) = 1 - \epsilon$, $c_2(e_3) = c_2(e_4) = \epsilon$ and for $S \subseteq E, c_2(S) = \min\{\sum_{e \in S} c_2(e), 1\}$. Function $c_1(\cdot)$ is additive and hence monotone and submodular. For function $c_2(\cdot)$, since $\sum_{e \in S} c_2(e)$ is additive (also monotone and submodular) on S, it follows that $c_2(\cdot)$ is also monotone and submodular (see Footnote 4).

For the quantity of MMS, the partition $\{\{e_1, e_4\}, \{e_2, e_3\}\}$ defines $\mathsf{MMS}_1(2, E) = 1/2$, and any allocation defines $\mathsf{MMS}_2(2, E) = 1$. The social cost-minimizing allocation is $\mathbf{O}$ with $O_1 = \{e_1, e_2\}$ and $O_2 = \{e_3, e_4\}$ whose social cost is equal to $\mathsf{UW}(\mathbf{O}) = 3/4 + 3\epsilon$. But due to $c_1(O_1) = 3/4 + \epsilon > 3/2\mathsf{MMS}_1(2, E)$, agent 1 violates (3/2)-PMMS fairness under $\mathbf{O}$. Note agent 1 can not receive both e_1 and e_2, one can check that the (3/2)-PMMS allocation with minimum social cost assigns all chores to agent 2, yielding a social cost exactly 1. As a consequence, the price of (3/2)-PMMS is at least $\frac{1}{3/4+3\epsilon} \to \frac{4}{3}$ as $\epsilon \to 0$. □

Proposition 3.7.5. *When $n = 2$ and agents have submodular cost functions, the price of* 2-MMS *is 1.*

Proof. According to Lemma 3.2.2, the allocation with minimum social cost is also 2-MMS fairness, completing the proof. □

Proposition 3.7.6. *When $n \geq 3$ and agents have submodular cost functions, the price of 2-*MMS *is at least $(n + 3)/6$ and at most $n^2/2$.*

Proof. The lower bound directly follows from the instance constructed in Proposition 3.6.7. As for the upper bound, given any social cost-minimizing allocation $\mathbf{O}$, if $\max_{i \in [n]} c_i(O_i) \leq 2/n$, then due to $\mathsf{MMS}_i(n, E) \geq 1/n$ from Lemma 3.2.1, we have $c_i(O_i) \leq 2\mathsf{MMS}_i(n, E)$ for any $i \in [n]$. This implies allocation $\mathbf{O}$ is 2-MMS fairness and we are done. Thus, we can without loss of generality assume that $\max_{i \in [n]} c_i(O_i) > 2/n$. Note the social cost of an allocation is at most n due to the normalization cost functions, so the price of 2-MMS is at most $n^2/2$. □

3.8 Conclusions

In this chapter, we are concerned with fair allocations of indivisible chores among agents under the setting of both additive and submodular (subadditive) cost functions. First, under the additive setting, we have established pairwise connections between several (additive) relaxations of the envy-free fairness in allocating, which look at how an allocation under one fairness criterion provides an approximation guarantee for fairness under another criterion. Some of our results in that part are in sharp contrast to what is known in allocating indivisible goods, reflecting the difference between goods and chores allocation. We have also extended to the submodular setting and investigated the connections between these fairness criteria. Our results have shown that, under the submodular setting, the interesting connections we have established under the additive setting almost disappear and few non-trivial approximation guarantees exist. Then we have studied the trade-off between fairness and efficiency, for which we have established the price of fairness for all these fairness notions in both additive and submodular settings. We hope our results have provided an almost complete picture for the connections between these chores fairness criteria together with their individual efficiencies relative to social optimum.

Chapter 4

Equitability and Welfare Maximization for Allocating Indivisible Items[1]

4.1 Introduction

Fairness and efficiency are two fascinating goals in resource allocation problems, and there exists a subtle competition between them. It is known that there is a trade-off between fairness and efficiency, i.e., optimization on one notion may lead to bad performance on the other. The PoF results established in Chapter 3 are evidence supporting that achieving envy-based and share-based fairness notions inevitably yields efficiency loss. In this chapter, we carry on this line of research and focus on another canonical fairness notion, *equitability.* In an *equitable* (EQ) allocation, agents should receive the same level of value. Equitability acts as an interpersonal fairness criterion, while envy-based fairness criteria considered in Chapter 3 are proposed in an intrapersonal manner. As shown by experiments, the interpersonal criterion acts as cognitive fairness more often than the intrapersonal criterion when facing distribution problems [62, 63, 76]. When assigning indivisible items, the existence of EQ allocations is not guaranteed, which then motivates us to study two of its relaxations: EQ1 and EQX. As shown by Freeman et al. [66] and by Freeman et al. [67], EQ1 and EQX allocations always exist in both settings of goods and chores if agents' valuations are additive.

We, in this chapter, care about both utilitarian and egalitarian welfare. More-

[1]This chapter is based on a research article by Sun et al. [100]

over, as we have argued in Chapter 3, results established in the case of goods do not necessarily hold for chores, and vice versa. Accordingly, it is worthwhile to investigate both cases of goods and chores, and as we will see later on, this chapter's results reveal both similarities and differences between the allocations of goods and chores. Besides quantifying efficiency loss under fairness constraints, we also consider algorithmic problems regarding the relationship between fairness and efficiency. In particular, we are interested in determining whether there exists a fair allocation that achieves optimal social welfare. On the one hand, a positive answer dramatically narrows down the search space of desired allocations, making it possible to compute and choose such an allocation in practice efficiently. On the other hand, a negative answer implies that some realistic relaxations on one of or both criteria are needed. The main tasks of this chapter are to establish the prices of EQ1 and of EQX, and to decide whether there exists an EQ1/EQX allocation that also achieves the maximum utilitarian or egalitarian welfare. Furthermore, we study the computational complexity of computing a welfare maximizer among all EQ1/EQX allocations.

The results on the price of fairness are summarized in Table 4.1. We highlight a subset of these next. In chores allocation, the price of EQX and of EQ1 with respect to utilitarian and egalitarian welfare are both infinite. In goods allocation, the price of EQX and of EQ1 with respect to egalitarian welfare are both 1. For utilitarian welfare, if there are two agents, the price of EQX is 3/2 and the price of EQ1 is at least 6/5 and at most $(\sqrt{2}+1)/2$. For general n agents, the price of EQX and of EQ1 are both at least $n-1$ and at most $3n$, asymptotically tight $\Theta(n)$.

	EQX	EQ1	
Utilitarian	$n=2$: $\frac{3}{2}$ (T4.2.4)	$n=2$: $\left[\frac{6}{5}, \frac{\sqrt{2}+1}{2}\right]$ (T4.2.5)	Goods
	$n \geq 3$: $[n-1, 3n]$ (T4.2.6)	$n \geq 3$: $[n-1, 3n]$ (T4.2.6)	
	∞ (T4.2.3)	∞ (T4.2.3)	Chores
Egalitarian	1 (T4.2.1)	1 (T4.2.1)	Goods
	∞ (T4.2.2)	∞ (T4.2.2)	Chores

Note: Interval $[a, b]$ means that the lower bound is equal to a and upper bound is equal to b. T$x.y$ points to Theorem $x.y$.

Table 4.1: Prices of fairness

After quantifying the welfare loss under fair allocations, we investigate relaxed equitability and welfare maximization from the algorithmic perspective. When concerning egalitarian welfare in goods allocation, results on the price of fairness show that there exist EQX and EQ1 allocations that achieve optimal egalitarian welfare. We then prove that, on the contrary, when assigning chores, deciding the existence of an EQX (resp., EQ1) allocation that also maximizes the egalitarian welfare is strongly NP-hard for general n and NP-hard for fixed $n \geq 2$ (resp., $n \geq 3$). For optimization problems, we show that computing an EQX (or EQ1) allocation with the maximum egalitarian welfare is strongly NP-hard for general n and NP-hard for fixed $n \geq 2$ in both cases of goods and chores. Moreover, in the case of fixed n, we design pseudo-polynomial time algorithms that output an EQX or EQ1 allocation with the maximum egalitarian welfare.

On the other hand, when focusing on utilitarian welfare, the computational complexity in allocating goods and chores is identical. In particular, for general n, every decision or optimization problem is strongly NP-complete and strongly NP-hard, respectively. For fixed n, our results are summarized in Table 4.2. The first column of Table 4.2 contains the (decision/optimization) problem descriptors. We denote by "E($W \times F$)" the problem of deciding whether there exists an F allocation that also maximizes W among all allocations, and denote by "C(W/F)" the problem of computing an F allocation that maximizes W among all F allocations. The notion of W refers to the welfare function, and the first row of Table 4.2 introduces the welfare function under consideration.

	UW Goods/Chores		EW Goods	EW Chores	
	$n = 2$	$n \geq 3$	$n \geq 2$	$n = 2$	$n \geq 3$
E($W \times$EQ1)	P (T4.3.11)	NP-complete (T4.3.9) pseudo-poly (T4.4.5)	P (T4.2.1)	?	NP-hard (T4.3.14) pseudo-poly (T4.4.3 & 4.4.6)
C(W/EQ1)	NP-hard (T4.3.12) pseudo-poly (T4.4.4)		NP-hard (T4.3.15) pseudo-poly (T4.4.1 & 4.4.4)		
E($W \times$EQX)	NP-complete (T4.3.7) pseudo-poly (T4.4.5)		P (T4.2.1)	NP-hard (T4.3.13) pseudo-poly (T4.4.3 & 4.4.6)	
C(W/EQX)	NP-hard (T4.3.8) pseudo-poly (T4.4.4)		NP-hard (T4.3.15) pseudo-poly (T4.4.2 & 4.4.4)		

Note: The problem descriptors in the first column are defined in detailed at the beginning of Section 4.3. Abbreviations "UW" and "EW" refer to utilitarian welfare and egalitarian welfare, respectively. Abbreviation "T$x.y$" points to Theorem $x.y$. The complexity of E(EW×EQ1) for allocating chores to two agents is open.

Table 4.2: Computational complexity for fixed n

4.1.1 Related Works

The notion of equitable allocation is originally studied in the cake-cutting problem (a divisible item) and its existence has been proved to be guaranteed by Dubins and Spanier [61]. For indivisible items, Gourvès et al. [73] relax equitability based on the up to one item scheme and prove the existence of this relaxed equitability in the matroid context. The notions of EQX and EQ1 are formally defined by Freeman et al. [66] that study these two fairness criteria together with Pareto efficiency and envy-freeness. They answer the existence and computational complexity of a sequence of related problems. In chores allocation, Freeman et al. [67] also consider EQX and EQ1 together with Pareto efficiency and answer corresponding existence and computation problems. Additional work on equitable allocations imposes connectivity constraints; each item is placed in a vertex of a graph and the bundle received by agents must be connected. Bouveret et al. [39] consider assigning chores in the path, star, and complete graph and establish results on the complexity of the existence of equitable and other fair allocations. None of the above-mentioned work studies these two fairness notions together with another important objective, social welfare.

Study on fairness together with social welfare is considerably intensified recently [30, 31, 47]. To quantify the efficiency loss under fairness requirements, Caragiannis et al. [47] introduce the price of fairness and study the notion of envy-freeness, proportionality, and equitability in divisible and indivisible goods and chores. In the case of indivisible goods, Bei et al. [28] consider fairness notions whose existences are guaranteed and provide characterizations on their price of fairness. They present lower bound $\Omega(\sqrt{n})$ and upper bound $O(n)$ on the price of envy-free up to one item, and this gap is then closed by Barman et al. [22], who show that the price of envy-free up to one item and of (1/2)-approximate maximin share are both $\Theta(\sqrt{n})$. When assigning indivisible chores, Section 3 provides tight results on several fairness notions that are proposed as relaxation of envy-freeness. The notion of the price of fairness is also applied to more practical topics such as kidney exchange [59] and machine scheduling [1, 33]. One of the papers closest to ours is Aziz et al. [17], which focuses on the notion of (relaxed) envy-freeness and proportionality. The authors study in the setting of goods the computational complexity of computing fair and welfare-maximizing allocations. They also briefly discuss the adaptability of their approach to other notions of fairness. In addition, the relationship between fairness and social welfare has been recently investigated in the online setting [35, 72, 105].

4.2 Results on Price of Fairness

We start with studying the price of fairness for every possible pair of welfare function and fairness criterion, which also answers the question of the existence of a nearly equitable allocation that approximates a welfare maximizer. Intuitively, if the price of fairness with respect to fairness criterion F and welfare function W is 1, then in any instance $\mathcal{I}$, there exists an F allocation that achieves $\mathsf{OPT}_W(\mathcal{I})$.

4.2.1 With respect to Egalitarian Welfare

First we are concerned with egalitarian welfare in both cases of goods and chores, and provide tight results that also reveal differences between goods and chores allocation. Freeman et al. [66] state that leximin implies EQX in allocating goods when agents have a strictly positive value on every item, while leximin fails to guarantee EQX when some items are valued at 0. Below, we prove that EQX (or EQ1) is compatible with optimal egalitarian welfare, even dropping the requirement of strictly positive values of all items.

Theorem 4.2.1. *When allocating goods, the price of* EQX *and of* EQ1 *with respect to egalitarian welfare are both equal to 1.*

Proof. Since EQX is stricter than EQ1, it suffices to show the statement holds for EQX. We explicitly construct such an allocation $\mathbf{A} = (A_1, \ldots, A_n)$ as follows.

> $\mathbf{A}$ first maximizes the egalitarian welfare among all allocations. If there is a tie, $\mathbf{A}$ minimizes the number of agents who receive the value $\mathrm{EW}(\mathbf{A})$, and subject to that, maximizes the total number of items assigned to all agents who receive the value $\mathrm{EW}(\mathbf{A})$.

By construction, it is straightforward to see that $\mathbf{A}$ is an egalitarian-welfare maximizing allocation. If allocation $\mathbf{A}$ is EQX, then clearly the theorem statement holds.

Next, we focus on the case where $\mathbf{A}$ is not EQX. Without loss of generality, assume $v_1(A_1) \le \cdots \le v_n(A_n)$. Note that if $v_1(A_1) \ge v_j(A_j \setminus \{e\})$ holds for all j and $e \in A_j$ with $v_j(e) > 0$, then allocation $\mathbf{A}$ is EQX, a contradiction. Thus, agent 1 must violate EQX. Let $\mathcal{J}$ be the set of agents such that agent 1 violates EQX when compared to agent $j \in \mathcal{J}$. For each $j \in \mathcal{J}$, order goods as $A_j = \{e_{j_1}, \ldots, e_{j_{|A_j|}}\}$ with $v_j(e_{j_k}) \le v_j(e_{j_{k+1}})$. We claim that $v_j(e_{j_1}) > 0$ holds; otherwise, reassigning e_{j_1} to agent 1 results in another allocation satisfying one of the following properties: (1)

the egalitarian welfare is larger than $\text{EW}(\mathbf{A})$; (2) the number of agents receiving the value $\text{EW}(\mathbf{A})$ is one less than that of $\mathbf{A}$; (3) the total number of items assigned to all agents with the value $\text{EW}(\mathbf{A})$ is one more than that of $\mathbf{A}$. For each $j \in \mathcal{J}$, since agent 1 violates EQX when comparing to agent j, it holds that $v_1(A_1) < v_j(A_j \setminus \{e_{j_1}\})$, and accordingly, there exists an index $l \geq 1$ such that $v_j(A_j \setminus \{e_{j_1} \cup \cdots \cup e_{j_l}\}) > v_1(A_1)$ and $v_j(A_j \setminus \{e_{j_1} \cup \cdots \cup e_{j_{l+1}}\}) \leq v_1(A_1)$. We then refer bundle $\{e_{j_1}, \ldots, e_{j_l}\}$ as S_j, and clearly, for each $j \in \mathcal{J}$, one can construct the corresponding non-empty set S_j. Moreover, we claim that $v_1(S_j) = 0$ for each $j \in \mathcal{J}$; otherwise, reassigning S_j to agent 1 results in another allocation that either has egalitarian welfare larger than $\text{EW}(\mathbf{A})$ or has one less agent receiving the value $\text{EW}(\mathbf{A})$ compared to that of $\mathbf{A}$. We then consider the allocation $\mathbf{A}'$ with $A'_1 = (\bigcup_{j \in \mathcal{J}} S_j) \cup A_1$, $A'_j = A_j \setminus S_j$ for $j \in \mathcal{J}$ and $A'_j = A_j$ for other j. Due to the construction of $\mathcal{J}$ and $\{S_j\}_{j \in \mathcal{J}}$, allocation $\mathbf{A}'$ achieves the optimal egalitarian welfare, and moreover, the number of agents receiving the value $\text{EW}(\mathbf{A})$ in $\mathbf{A}'$ is the same as that of $\mathbf{A}$. However, in allocation $\mathbf{A}'$, the total number of items assigned to all agents who receive value $\text{EW}(\mathbf{A})$ is $\sum_{j \in \mathcal{J}} |S_j|$ more than that of $\mathbf{A}$, which contradicts the definition of allocation $\mathbf{A}$ as S_j is non-empty for each $j \in \mathcal{J}$. Therefore, allocation $\mathbf{A}$ must satisfy EQX. □

We remark that the allocation $\mathbf{A}$ constructed in the proof of Theorem 4.2.1 is not necessarily leximin. To see this, consider Example 1 in Freeman et al. [66]. There are three agents and a set $E = \{e_1, \ldots, e_6\}$ of six goods. The goods e_1, e_2, e_3 are valued at 1 by agent 1 and 0 by agents 2 and 3. The goods e_4, e_5, e_6 are valued at 1 by agents 2 and 3 and at 0 by agent 1. In the constructed allocation, one of e_1, e_2, e_3 is assigned to agent 2 or agent 3, so that the total number of items received by the agents with value 1 is maximized. However, in leximin allocations, all e_1, e_2, e_3 must be assigned to agent 1.

Theorem 4.2.2. *When allocating chores, the price of* EQX *and of* EQ1 *with respect to egalitarian welfare are both infinite.*

Proof. Note that EQX is stricter than EQ1, and it suffices to prove the statement for EQ1. Let us consider a fair-chores instance with $n \geq 2$ agents and a set $E = \{e_1, \ldots, e_{m+1}\}$ of $m+1$ chores with $m \geq n$. The valuations are shown in Table 4.3, where $V > 0$ is arbitrarily large.

Items	e_1	e_2	$\cdots$	e_m	e_{m+1}
$v_1(\cdot)$	$-\frac{1}{V}$	$-\frac{1}{V}$	$\cdots$	$-\frac{1}{V}$	$-\frac{V-m}{V}$
$v_i(\cdot)$ for $i \geq 2$	$-\frac{2V-1}{2mV}$	$-\frac{2V-1}{2mV}$	$\cdots$	$-\frac{2V-1}{2mV}$	$-\frac{1}{2V}$

Table 4.3: The fair-chores instance for Theorem 4.2.2

Since V is arbitrarily large, we have $-m/V > -1/m + 1/(2mV)$, and so, the unique EWM assigns the first m items to agent 1 and e_{m+1} to any agent i with $i \geq 2$, yielding the maximum egalitarian welfare $-m/V$. But in this allocation, agent 1 violates EQ1 because she still receives less value even eliminating one chore from her bundle. Then, to achieve EQ1, agent 1 cannot receive all the first m items, and thus, the egalitarian welfare of an EQ1 allocation is at most $-1/m + 1/(2mV)$, based on which the price of EQ1 is at least $V/m^2 - 1/(2m^2)$ that approaches to positive infinity as $V \to +\infty$. $\square$

According to Theorems 4.2.1 and 4.2.2, in goods allocation, both EQX and EQ1 are compatible with EWM, while in chores allocation, achieving EQX or EQ1 sacrifices most of the egalitarian welfare.

4.2.2 With respect to Utilitarian Welfare

On utilitarian welfare, we establish the price of fairness for each of EQX and EQ1. A sharp contrast between goods and chores is also revealed by the results below. Specifically, EQX/EQ1 allocations can provide a bounded welfare guarantee relative to the optimal one in goods allocation, while in the case of chores, the price of fairness is infinite.

Theorem 4.2.3. *When allocating chores, the price of* EQX *and of* EQ1 *with respect to utilitarian welfare are both infinite.*

Proof. Note that EQX is stricter than EQ1, and so it suffices to prove that the statement holds for EQ1. Again, we consider the instance constructed in the proof of Theorem 4.2.2. Since V is arbitrarily large, we have $-1/V > -1/m + 1/(2mV)$ and $-1/(2V) > -1 + m/V$, and as a consequence, in a UWM allocation, the first m items are assigned to agent 1 and e_{m+1} is assigned to agent i with $i \geq 2$, yielding the maximum utilitarian welfare $-(2m+1)/(2V)$. But in such an allocation, agent 1 violates EQ1 because she still receives less value even eliminating one chore from her bundle. Then, to achieve EQ1, agent 1 cannot receive all the first m items, and thus, the utilitarian welfare of an EQ1 allocation is at most $-1/m+1/(2mV)$, based on which the price of EQ1 is at least $(2V-1)/(2m^2+m) \to +\infty$ as $V \to +\infty$. $\square$

When moving to the case of goods, we distinguish between two cases: $n = 2$ and general $n \geq 3$, and provide (asymptotically) tight results on the price of fairness.

Theorem 4.2.4. *When allocating goods to two agents, the price of* EQX *with respect to utilitarian welfare is equal to* $3/2$.

Proof. In the proof of Theorem 4.2.1, we show that, for any instance $\mathcal{I}$, there exists an EQX allocation achieving the maximum egalitarian welfare. Let $\mathbf{A}$ be such an EQX allocation, and without loss of generality, assume $v_1(A_1) \leq v_2(A_2)$. Thus, the maximum egalitarian welfare is $v_1(A_1)$ and the maximum utilitarian welfare is at most $1 + v_1(A_1)$. Consider another allocation $\mathbf{A}'$ with $A'_1 = A_2$ and $A'_2 = A_1$. Since $\mathbf{A}$ achieves the maximum egalitarian welfare, we must have $v_1(A_1) \geq \min\{v_1(A_2), v_2(A_1)\}$, and accordingly, $v_1(A_1) \geq v_2(A_1)$ implies $v_1(A_1) + v_2(A_2) \geq 1$ due to the normalized valuations; $v_1(A_1) \geq v_1(A_2)$ also implies $v_1(A_1) + v_2(A_2) \geq 1$ due to $v_1(A_1) \leq v_2(A_2)$ and the normalized valuations. Hence, we have $\text{UW}(\mathbf{A}) \geq \max\{2v_1(A_1), 1\}$. Recall that the maximum utilitarian welfare of $\mathcal{I}$ is at most $1 + v_1(A_1)$, then for any $0 \leq v_1(A_1) \leq 1$, we have

$$\text{Price of EQX} \leq \frac{1 + v_1(A_1)}{\max\{2v_1(A_1), 1\}} \leq \frac{3}{2}.$$

For the lower bound, let us consider a fair-goods instance with two agents and a set $E = \{e_1, e_2, e_3\}$ of three goods. The valuations are shown in Table 4.4, where $\epsilon > 0$ is arbitrarily small. A utilitarian welfare-maximization allocation assigns e_1

Items	e_1	e_2	e_3
$v_1(\cdot)$	$\frac{1}{2}$	$\frac{1}{2} - \epsilon$	ϵ
$v_2(\cdot)$	$\frac{1}{2}$	ϵ	$\frac{1}{2} - \epsilon$

Table 4.4: The fair-goods instance for Theorem 4.2.4

to an arbitrary agent and e_2, e_3 to agent 1 and agent 2, respectively, which leads to an optimal utilitarian welfare $3/2 - 2\epsilon$. But in such an allocation, the agent who does not receive e_1 violates EQX, and so, in any EQX allocations, one agent only receives e_1 and the other agent receives the remaining two goods, yielding utilitarian welfare exactly 1. Therefore, the price of EQX is at least $3/2 - 2\epsilon \to 3/2$ as $\epsilon \to 0$. □

Before we state our result on the price of EQ1 in Theorem 4.2.5, we first present Algorithm 2, which uses Algorithm 3 as a subroutine and outputs an EQ1 allocation with utilitarian welfare guarantee at least $2/(\sqrt{2}+1)$ times the maximum one. Intuitively, given a fair-goods instance $\mathcal{I}$, Algorithm 2 first checks whether a specific partial allocation, in which some items are assigned to the agent having the larger value, can be extended to an EQ1 allocation. If yes, it implements Algorithm 3, a subroutine where in each turn, let the agent with the smallest current value pick the item of the highest value from the remaining, and makes the partial allocation a complete EQ1 allocation with the maximum utilitarian welfare. If the answer is no, to achieve EQ1, Algorithm 2 orders the items in a way similar to that of Algorithm 1, and carefully reassigns some items from the bundle of the agent with the larger value to the other agent, while avoiding as much welfare loss as possible. For simplicity, in the description of Algorithm 2 and its proof we write $L(k) := \{e_1, \ldots, e_k\}$ and $R(k) := \{e_k, \ldots, e_m\}$ for any $k \in [m]$.

Algorithm 2

Input: A fair-goods instance $\mathcal{I} = \langle [2], E, \mathcal{V} \rangle$.
Output: Allocation $\mathbf{A}$ of instance $\mathcal{I}$.
1: Partition $E = E_0 \cup E_1 \cup E_2$ where $E_1 = \{e \in E \mid v_1(e) > v_2(e)\}$ and $E_2 = \{e \in E \mid v_1(e) < v_2(e)\}$ (assume $v_1(E_1) \leq v_2(E_2)$ and the other case is symmetric).
2: **if** $v_1(E_1 \cup E_0) \geq \min_{e \in E_2} v_2(E_2 \setminus \{e\})$ **then**
3: $\quad \mathbf{A} \leftarrow \text{Greedy}((E_1, E_2), \mathcal{I})$;
4: **else**
5: $\quad$ Order goods such that $\frac{v_1(e_1)}{v_2(e_1)} \leq \frac{v_1(e_2)}{v_2(e_2)} \leq \cdots \leq \frac{v_1(e_m)}{v_2(e_m)}$, break ties arbitrarily. For good e with $v_1(e) = 0$, put it at the front and good e with $v_2(e) = 0$ at back;
6: $\quad$ Add two virtual items e_0, e_{m+1} with $v_i(e_0) = v_i(e_{m+1}) = 0, \forall i = 1, 2$;
7: $\quad$ Let index s be the one such that $v_2(e_s) > v_1(e_s)$ and $v_2(e_{s+1}) \leq v_1(e_{s+1})$.
8: $\quad$ Find the maximum index $f < s$ such that $v_2(L(f)) \leq v_1(R(f+2))$;
9: $\quad A_1 \leftarrow R(f+2), A_2 \leftarrow L(f+1)$;
10: **end if**
11: **return** $\mathbf{A}$

Algorithm 3 Greedy $(\mathbf{A}', \mathcal{I})$

Input: An instance $\mathcal{I} = \langle [n], E, \mathcal{V} \rangle$ and a partial allocation $\mathbf{A}'$ of $\mathcal{I}$;
Output: A complete allocation $\mathbf{A}$ of instance $\mathcal{I}$;
Initialize $E \leftarrow E \setminus \bigcup_{i \in [n]} A'_i$ and $A_i \leftarrow A'_i$ for $i \in [n]$.
while $E \neq \emptyset$ **do**
$\quad i \leftarrow \arg\min_{i \in [n]} |v_i(A_i)|$, break ties arbitrarily;
$\quad e \leftarrow \arg\max_{e' \in E} v_i(e')$;
$\quad$ Update $A_i \leftarrow A_i \cup \{e\}$ and $E \leftarrow E \setminus \{e\}$;
end while
return allocation $\mathbf{A}$.

Lemma 4.2.1. *Algorithm 2 always terminates and returns an* EQ1 *allocation.*

Proof. If the condition of Step 2 in Algorithm 2 holds, then clearly Algorithm 2 terminates and allocation $\mathbf{A}$ is returned by Greedy with partial assignment (E_1, E_2). For any $i \in [2]$, let $e^{(i)}$ be the last good received by agent i in Greedy, and if $A_i = E_i$, let $e^{(i)} = \emptyset$. The following proof considers two cases. If $A_2 \cap E_0 = \emptyset$, according to

Greedy, we have $v_2(A_2) \geq v_1(A_1 \setminus \{e^{(1)}\})$, which implies agent 2 satisfies EQ1. As for agent 1, from the Step 2 of Algorithm 2, we have $v_1(A_1) = v_1(E_1 \cup E_0) \geq \min_{e \in E_2} v_2(E_2 \setminus \{e\})$, and thus agent 1 also meets the condition of EQ1. For the case of $A_2 \cap E_0 \neq \emptyset$, we let $\mathbf{A}^{(i)}$ be the allocation right after agent i receiving item $e^{(i)}$. Then, according to Greedy, when agent 1 just receives $e^{(1)}$, it holds that $v_2(A_2) \geq v_2(A_2^{(1)}) \geq v_1(A_1^{(1)} \setminus \{e^{(1)}\}) = v_1(A_1 \setminus \{e^{(1)}\})$, which implies that agent 2 satisfies EQ1. Similarly, we have $v_1(A_1) \geq v_1(A_1^{(2)}) \geq v_2(A_2^{(2)} \setminus \{e^{(2)}\}) = v_2(A_2 \setminus \{e^{(2)}\})$. Then, agent 1 satisfies EQ1 under allocation $\mathbf{A}$ as well.

When Algorithm 2 goes to Step 5, since valuations are normalized and $v_2(e_0) = 0$, the existence of f is guaranteed. According to the order and index s, we have $E_2 = L(s)$ and $E_0 \cup E_1 = R(s+1)$. Moreover, we can assume without loss of generality $s \geq 2$ because the condition of Step 2 would hold if $s = 1$. For index f, if $f = s-1$, then $v_2(E_2 \setminus \{e_s\}) = v_2(L(s-1)) \leq v_1(R(s+1)) = v_1(E_1 \cup E_0)$ holds, and this relationship satisfies the condition of Step 2, a contradiction. Thus, it must hold that $f \leq s-2$. We then prove that allocation $\mathbf{A}$ constructed by Step 9 is EQ1. For agent 1, she would not violate EQ1 since $v_1(A_1) = v_1(R(f+2)) \geq v_2(L(f)) = v_2(A_2 \setminus \{e_{f+1}\})$. As for agent 2, since $f+1 \leq s-1$ and $f+1$ is not chosen by Step 8, then we have $v_2(L(f+1)) > v_1(R(f+3))$, equivalent to $v_2(A_2) > v_1(A_1 \setminus \{e_{f+2}\})$. Thus, agent 2 also satisfies EQ1. □

Theorem 4.2.5. *When allocating goods to two agents, the price of* EQ1 *with respect to utilitarian welfare is at least* $6/5$*, and at most* $(\sqrt{2}+1)/2$.

Proof. We start from the upper bound and consider the allocation $\mathbf{A}$ returned by the Algorithm 2. Based on Lemma 4.2.1, it suffices to show that the UW($\mathbf{A}$) is at least $2/(\sqrt{2}+1)$ times the maximum utilitarian welfare. If allocation $\mathbf{A}$ is returned by Step 3 of Algorithm 2, according to Greedy, we have $E_i \subseteq A_i$ for any $i \in [2]$, which implies UW($\mathbf{A}$) equals to the optimal utilitarian welfare. We then consider the case where $\mathbf{A}$ is created by Step 9. Denote by $\mathbf{O} = (O_1, O_2)$ an allocation with maximum utilitarian welfare. Clearly, $E_i \subseteq O_i$ for any $i \in [2]$. Then, due to index order in Step 5 and $f \leq s-2$, it holds that

$$\frac{v_1(A_2)}{v_2(A_2)} \leq \frac{v_1(O_2)}{v_2(O_2)},$$

which then implies the following

$$v_1(A_1) \geq 1 - v_2(A_2)\frac{v_1(O_2)}{v_2(O_2)}.$$

Recall $f \leq s-2$, then we have $O_1 \subseteq R(f+3)$, which leads to $v_2(A_2) = v_2(L(f+1)) > v_1(R(f+3)) \geq v_1(O_1)$. Then, we have the following:

$$v_1(A_1) + v_2(A_2) \geq 1 + (1 - \frac{v_1(O_2)}{v_2(O_2)})v_2(A_2) \geq 1 + (1 - \frac{v_1(O_2)}{v_2(O_2)})v_1(O_1),$$

and equivalently,

$$\frac{v_1(O_1) + v_2(O_2)}{v_1(A_1) + v_2(A_2)} \leq \frac{v_1(O_1) + v_2(O_2)}{1 + v_1(O_1)(1 - \frac{1-v_1(O_1)}{v_2(O_2)})}.$$

We observe that the right hand side is actually a function with two variables $v_1(O_1), v_2(O_2)$, and for simplicity, let $v_1(O_1) = x$, $v_2(O_2) = y$. As for the domain, since the condition of Step 2 is not satisfied, then $v_2(O_2) \geq v_2(E_2) > v_1(E_1 \cup E_0) \geq v_1(O_1)$, implying $y > x$. Also, $v_2(O_2) > v_1(O_2) = 1 - v_1(O_1)$ due to normalized valuations. Therefore, we have $y > \max\{x, 1-x\}$. Let $f(x,y)$ corresponds to the right hand side of the above inequality, then we have its derivatives with respect to y,

$$\frac{\partial f(x,y)}{\partial y} = \frac{(1+x)y^2 + 2(x^2 - x)y + x^3 - x^2}{(1 + x(1 - \frac{1-x}{y}))^2 y^2}.$$

We then let $g(x,y) = (1+x)y^2 + 2(x^2 - x)y + x^3 - x^2$, and its partial derivative with respect to y is

$$\frac{\partial g(x,y)}{\partial y} = 2(1+x)y + 2x^2 - 2x.$$

The root of equation $\partial g / \partial y = 0$ is $y = \frac{x - x^2}{1+x}$ and inequality $\frac{x-x^2}{1+x} < x$ consistently holds for any $x \in [0,1)$. Thus, for any $x \in [0,1)$, function $g(x,y)$ is monotonically increase in y. Accordingly, we have the following:

$$\min_{\max\{x,1-x\} < y \leq 1} g(x,y) = \begin{cases} 2x^2(2x-1), & \text{if } x \geq \frac{1}{2} \\ (2x-1)(x-1), & \text{if } x < \frac{1}{2} \end{cases}$$

It is not hard to verify that $g(x,y) \geq 0$ consistently holds, which implies that, for any $x \in [0,1)$, $f(x,y)$ is a monotonically increasing function of y. Then, to find the maxima of $f(x,y)$, we substitute $y = 1$, then by simple calculation, the maximum value of $f(x,1)$ is equal to $(\sqrt{2}+1)/2$ which happens when $x = \sqrt{2} - 1$, completing the proof for upper bound.

As for the lower bound, let us consider a fair-goods instance with two agents and a set $E = \{e_1, e_2, e_3\}$ of three items. The valuations are shown in Table 4.5, where $\epsilon > 0$ is arbitrarily small. Suppose $\mathbf{O}$ is a UWM, then we have $O_1 = \{e_2, e_3\}$

Items	e_1	e_2	e_3
$v_1(\cdot)$	0	$\frac{1}{2}$	$\frac{1}{2}$
$v_2(\cdot)$	$\frac{1}{2}-2\epsilon$	$\frac{1}{4}+\epsilon$	$\frac{1}{4}+\epsilon$

Table 4.5: The fair-goods instance for Theorem 4.2.5

and $O_2 = \{e_1\}$ yielding $\text{UW}(\mathbf{O}) = 3/2 - 2\epsilon$. But agent 2 violates EQ1 in $\mathbf{O}$ due to $v_2(O_2) < v_1(O_1 \setminus \{e\})$ for any $e \in O_1$. Thus, in any EQ1 allocations, agent 1 can not receive both e_2, e_3, which means the welfare loss of EQ1 allocations is at least $1/4 - \epsilon$. And one can verify that $\mathbf{A}'$ with $A'_1 = \{e_3\}$ and $A'_2 = \{e_1, e_2\}$ is an EQ1 allocation with $\text{UW}(\mathbf{A}') = 5/4 - \epsilon$. Therefore, regarding utilitarian welfare, we have

$$\text{Price of EQ1} \geq \frac{\frac{3}{2} - 2\epsilon}{\frac{5}{4} - \epsilon} \to \frac{6}{5} \text{ as } \epsilon \to 0,$$

which completes the proof. $\square$

We now consider the case of general $n \geq 3$ and provide asymptotically tight results. Before stating the main result, we first present several lemmas. The following lemma provides a sufficient condition for extending a partial allocation into a complete EQX allocation.

Lemma 4.2.2. *Given a fair-goods instance $\mathcal{I}$ and a partial allocation $\mathbf{A}'$ of $\mathcal{I}$, if allocation $\mathbf{A}'$ is* EQX *and for any i with $A'_i \neq \emptyset$, $\min_{e \in A'_i} v_i(e) \geq \max_{e \in E \setminus \bigcup_{i \in [n]} A'_i} v_i(e)$ holds, then allocation $\mathbf{A}$ returned by* Greedy $(\mathbf{A}', \mathcal{I})$ *is* EQX.

Proof. For the sake of contradiction, we assume that $\mathbf{A}$ is not EQX. Without loss of generality we assume $v_1(A_1) \leq \cdots \leq v_n(A_n)$ and agent 1 violates EQX when comparing to agent k, i.e., $\max_{e \in A_k : v_k(e) > 0} v_k(A_k \setminus \{e\}) > v_1(A_1)$. Consider two cases.

Case 1: $A_k \setminus A'_k \neq \emptyset$. Denote by $e^{(k)}$ the last item received by agent k in Greedy and $\mathbf{A}^{(k)}$ the partial allocation right before agent k receiving item $e^{(k)}$, i.e., $A_k = A_k^{(k)} \cup \{e^{(k)}\}$. Then, since $A_1^{(k)} \subseteq A_1$, we have $v_1(A_1) \geq v_1(A_1^{(k)})$. According to the choice of Step 3 in Algorithm 3, it holds that $v_1(A_1^{(k)}) \geq v_k(A_k^{(k)})$, and hence $v_1(A_1) \geq v_k(A_k^{(k)})$. We show that $e^{(k)}$ is the item with the smallest positive value for agent k in bundle A_k. If $A'_k = \emptyset$, since an agent always picks the single item with largest value, item $e^{(k)}$ chosen the last must have a value no larger than any other item in A_k. If $A'_k \neq \emptyset$, the condition $\min_{e \in A'_k} v_k(e) \geq \max_{e \in E \setminus \bigcup_{i \in [n]} A'_i} v_k(e)$ together with the way of picking items can also guarantee $v_k(e^{(k)}) \leq v_k(e)$ for any $e \in A_k$. Accordingly, we have $\max_{e \in A_k} v_k(A_k \setminus \{e\}) = v_k(A_k^{(k)}) \leq v_1(A_1)$, contradicting the assumption that agent 1 violates EQX when comparing to agent k.

Case 2: $A_k \setminus A'_k = \emptyset$. In this case, we have $A_k = A'_k$ as $A'_k \subseteq A_k$. Since $A'_1 \subseteq A_1$ and allocation $\mathbf{A}'$ is EQX, it holds that $v_1(A_1) \geq v_1(A'_1) \geq \max_{e \in A_k : v_k(e) > 0} v_k(A_k \setminus \{e\})$, which again contradicts the assumption that agent 1 violates EQX when comparing to agent k. $\square$

In the following, we propose algorithm ALG_3, which efficiently computes an EQX allocation that also has an absolute welfare guarantee. The ALG_3 first assigns one *large item*, with value at least $1/(3n)$, to as many agents as possible, and at the same time, maximizes the welfare of the partial allocation. This is achieved by computing a maximum-weight matching of a bipartite graph. Then, it carefully assigns a bundle to every unmatched agent so that each of them receives value at least $1/(3n)$, while maintaining the partial allocation being EQX. At last, the remaining goods are assigned to agents by running algorithm Greedy. In what follows we formally prove that ALG_3 can efficiently output an EQX allocation with the desired utilitarian welfare guarantee.

Lemma 4.2.3. *Algorithm 4 returns in polynomial time an* EQX *allocation* $\mathbf{A}$ *with utilitarian welfare* $\mathrm{UW}(\mathbf{A}) \geq 1/3$.

Proof. We first consider the case where allocation $\mathbf{A}$ is returned in Step 4 of ALG_3. For this case, allocation $\mathbf{B}$ is the partial allocation established based on matching μ and hence $|B_i| \leq 1$ for $i \in [n]$. Observe that allocation $\mathbf{A}$ is returned by Greedy($\mathbf{B}, \mathcal{I}$), and thus, $B_i \subseteq A_i$ for any $i \in [n]$. Consequently, we have $\mathrm{UW}(\mathbf{A}) \geq \mathrm{UW}(\mathbf{B}) \geq 1/3$. What remains to be shown is that allocation $\mathbf{A}$ is EQX. Notice that $|B_i| = 1$ for each matched agent $i \in N_0$ and $B_i = \emptyset$ for $i \in [n] \setminus N_0$. Thus, the partial allocation $\mathbf{B}$ is EQX. For each matched agent $i \in N_0$, if $\exists\ e \in E \setminus E_0$ such that $v_i(e) > v_i(B_i) \geq 1/(3n)$, then by matching i to the corresponding vertex of e and keeping other matched pair in μ, one can find another matching μ' with a weight larger than that of μ, a contradiction. Thus, for every i with $B_i \neq \emptyset$, it holds that $v_i(B_i) \geq \max_{e \in E \setminus E_0} v_i(e)$. According to Lemma 4.2.2 and the fact that $\mathbf{B}$ is EQX, we conclude that allocation $\mathbf{A}$ is EQX.

Now consider the case where allocation $\mathbf{A}$ is *not* returned in Step 4 of ALG_3 and hence $\mathrm{UW}(\mathbf{B}) < 1/3$. Clearly, not all agents are matched in μ. Since μ is a maximum-weight matching, for any $i \in [n] \setminus N_0$ and $e \in E \setminus E_0$, we have $v_i(e) < 1/(3n)$. Moreover, even if $v_i(e') \geq 1/(3n)$ for some $e' \in E_0$ and $e' = \mu(i')$, it must hold that $v_i(e') \leq v_{i'}(e')$. Accordingly, for each $i \in [n] \setminus N_0$, we have $v_i(E_0) \leq \mathrm{UW}(\mathbf{B})$ and thus, $v_i(E \setminus E_0) > 1 - \mathrm{UW}(\mathbf{B}) > 2/3$. We then prove that every agent $i \in [n] \setminus N_0$ can receive a bundle S_i in the while-loop of ALG_3, which is equivalent to showing

Algorithm 4 *ALG*$_3$

Input: A fair-goods instance $\mathcal{I} = \langle [n], E, \mathcal{V} \rangle$;
Output: An EQX allocation $\mathbf{A}$ with $\mathrm{UW}(\mathbf{A}) \geq \frac{1}{3}$;

Construct weighted bipartite graph $G = ([n] \cup [m], [n] \times [m])$ where agents vertices on one side and goods vertices on the other side. The edge (i, j) from agent i's vertex to the vertex of good e_j exists only if $v_i(e_j) \geq \frac{1}{3n}$, and the weight of (i, j) is $v_i(e_j)$.
Compute the maximum weight matching μ of G and denote by $\mu(i)$ the good matched to agent i. If agent i is unmatched, let $\mu(i) = \emptyset$. Construct the partial allocation $\mathbf{B}$ with $B_i = \mu(i)$ for every $i \in [n]$, and let $N_0 = \{i \in [n] \mid B_i \neq \emptyset\}$ be the set of matched agents and $E_0 = \bigcup_{i \in N_0} B_i$ the set of matched goods.
if $\mathrm{UW}(\mathbf{B}) \geq \frac{1}{3}$ **then**
  Compute $\mathbf{A} \leftarrow$ Greedy $(\mathbf{B}, \mathcal{I})$.
else
  $N_1 \leftarrow [n] \setminus N_0$ and $E_1 \leftarrow E \setminus E_0$.
  while $N_1 \neq \emptyset$ **do**
    For each $i \in N_1$, set the subset $S_i \subseteq E_1$ as the one with minimum cardinality such that $v_i(S_i) \geq \frac{1}{3n}$ and $v_i(S_i \setminus \{e\}) < \frac{1}{3n}$ for any $e \in S_i$. If there is a tie, choose the one with the largest value among all candidates. (We will show that S_i always exists.)
    Let $\Phi = \{i \in N_1 \mid |S_i| \leq |S_j| \text{ for any } j \in N_1\}$. Find $i^* \in \arg\max_{i \in \Phi} v_i(S_i)$ (break ties arbitrarily) and make the assignment $B_{i^*} \leftarrow S_{i^*}$.
    Update $E_1 \leftarrow E_1 \setminus S_{i^*}$ and $N_1 \leftarrow N_1 \setminus \{i^*\}$.
  end while
  Compute $\mathbf{A} \leftarrow$ Greedy $(\mathbf{B}, \mathcal{I})$.
end if
return allocation $\mathbf{A}$.

that the value of the remaining items for agent i is always at least $1/(3n)$. Consider an arbitrary point where the while-loop starts with remaining items E' and agents N'. For each $j \in [n] \setminus (N_0 \cup N')$ and $i \in N'$, it must hold that $v_i(S_j) < 2/(3n)$; otherwise, at the time when agent j receives a bundle, there exists a subset $S^* \subsetneq S_j$ such that $v_i(S^*) > 1/(3n)$ and $v_i(S^* \setminus \{e\}) < 1/(3n)$ for all $e \in S^*$, and thus, instead of agent j, the algorithm assigns a bundle to agent i, a contradiction. Consequently, for each $i \in N'$, we have

$$\begin{aligned} v_i(E') &= v_i(E) - v_i(E_0) - v_i\Big(\bigcup_{j \in [n] \setminus (N_0 \cup N')} S_j\Big) \\ &> \frac{2}{3} - \frac{2}{3n} \cdot (n - |N_0| - |N'|) > \frac{2}{3n}, \end{aligned}$$

where the last transition is due to $|N'| \geq 1$. Thus, every agent $i \in [n] \setminus N_0$ is able to receive a bundle that satisfies the conditions described in Step 8.

With a slight abuse of notations, we now let $\mathbf{B}$ be the partial allocation when the while-loop ends. Clearly, we have $v_i(B_i) \geq 1/(3n)$ for every $i \in [n]$, which implies $\mathrm{UW}(\mathbf{B}) \geq 1/3$. Since allocation $\mathbf{A}$ is computed by Greedy with input $\mathbf{B}$, it then holds that $\mathrm{UW}(\mathbf{A}) \geq \mathrm{UW}(\mathbf{B}) \geq 1/3$. In order to prove that allocation $\mathbf{A}$ is EQX, we use Lemma 4.2.2. We first claim that allocation $\mathbf{B}$ satisfies EQX. For each agent $i \in [n]$, when comparing to agent $j \in N_0$, it does not violate EQX as $|B_j| = 1$, and when comparing to agent $j \in [n] \backslash N_0$, agent i also satisfies EQX because $v_i(B_i) \geq 1/(3n) > \max_{e \in B_j} v_j(B_j \setminus \{e\})$. Thus, allocation $\mathbf{B}$ is EQX. As for the remaining condition, for any $i \in N_0$, it holds that $v_i(B_i) = \min_{e \in B_i} v_i(e) \geq \max_{e \in E \setminus \bigcup_{i \in [n]} B_i} v_i(e)$ because μ computed in Step 2 is the maximum-weight matching. For an agent $i \in [n] \setminus N_0$, if $\exists e' \in E \setminus \bigcup_{i \in [n]} B_i$ such that $v_i(e') > v_i(e)$ for some $e \in B_i$, then at the moment when agent i is chosen in Step 9 and bundle S_i is constructed, bundle $S_i \cup \{e'\} \setminus \{e\}$ is also a candidate and has the same size as $|S_i|$ but with value strictly larger than $v_i(S_i)$. Thus, instead of S_i, bundle $S_i \cup \{e'\} \setminus \{e\}$ would be assigned to agent i, a contradiction. Therefore, based on Lemma 4.2.2, we can conclude that allocation $\mathbf{A}$ is EQX and has welfare $\mathrm{UW}(\mathbf{A}) \geq 1/3$.

As for time complexity of ALG_3, since the maximum-weight matching can be computed efficiently, ALG_3 clearly finishes in polynomial time. □

Now, we are at the stage to present and prove the main statement on the price of EQX and of EQ1 with respect to utilitarian welfare.

Theorem 4.2.6. *When allocating goods to n agents, the price of* EQ1 *and of* EQX *with respect to utilitarian welfare are at least $n-1$ and at most $3n$, asymptotically tight $\Theta(n)$.*

Proof. We start from the upper bound. According to Lemma 4.2.3, there exists an EQX allocation with welfare at least 1/3. Due to the normalized valuations, the optimal utilitarian welfare is at most n, and thus, the price of EQX is at most $3n$. This upper bound also holds for the price of EQ1 as the notion of EQX is stricter than EQ1.

As for the lower bound, it suffices to prove that the statement holds for EQ1. Let us consider a fair-goods instance $\mathcal{I}$ with n agents and a set $E = \{e_1, \ldots, e_{(n-2)p+1}\}$ of $(n-2)p+1$ goods where $p \in \mathbb{N}^+$ is arbitrarily large. For agent $i \in [n-2]$, her valuation function is: $v_i(e_j) = 1/p$ if $(i-1)p + 1 \leq j \leq ip$ and $v_i(e_j) = 0$ for other j. Both agents $n-1$ and n value $e_{(n-2)p+1}$ at 1 and other items at 0. In a utilitarian welfare-maximizing allocation $\mathbf{O}$, bundle $O_i = \{e_{(i-1)p+1}, \ldots, e_{ip}\}$ is assigned to agent i for all $i \in [n-2]$. One of agents $n-1$ and n receives $e_{(n-2)p+1}$, which leaves the other receiving an empty set. The optimal utilitarian welfare equals $\text{UW}(\mathbf{O}) = n-1$. Note that the agent receiving an empty set violates EQ1 in $\mathbf{O}$ when comparing to every agent $i \in [n-2]$. To achieve EQ1, each agent $i \in [n-2]$ can receive at most one of the goods on which she has non-zero value, and accordingly, the welfare of an EQ1 allocation is at most $1 + (n-2)/p$. Therefore, regarding utilitarian welfare, we have

$$\text{Price of EQ1} \geq \frac{n-1}{1+\frac{n-2}{p}} \to n-1 \text{ as } p \to \infty,$$

which completes the proof. □

According to Caragiannis et al. [47], when assigning indivisible goods, the price of EQ is infinite for general n and two for the case of $n = 2$. From our results, the prices of EQX and EQ1 are smaller than that of EQ in both cases of general n and two agents. Moreover, the price of EQX is greater than that of EQ1 when $n = 2$. Such observations confirm an intuition in stating that if one relaxes the fairness requirement, then less welfare would be sacrificed.

Equitability aims to reduce the difference between agents' value, while utilitarianism may lead to unbalanced outcomes. This intuition suggests that approximately equitable allocations may have a poor performance in guaranteeing utilitarian welfare, and Theorems 4.2.3 and 4.2.6 can be evidence for this intuition. Egalitarianism also aims for "balancing" agents' value, and accordingly, equitability is highly likely to have a considerable guarantee on egalitarian welfare. However, while Theorem 4.2.1 confirms this guess in goods allocation, Theorem 4.2.2 states that the price of EQ1 with respect to egalitarian welfare is infinite in the case of chores. We believe the reason behind is that in the case of chores, agents' value are balanced around zero, and the price of fairness as the ratio of two negligible numbers can be enormous.

4.3 Results on Computational Complexity

Results on the price of fairness not only quantify the efficiency loss under fairness constraints but also can derive answers to the existence of a welfare maximizer that is also fair. In particular, our results on the price of fairness suggest that relaxed equitability is not always compatible with utilitarian welfare in either goods or chores and not always compatible with egalitarian welfare in chores allocation. These impossibilities motivate two crucial follow-up algorithmic problems, i.e., given an instance, whether one can efficiently determine the existence of fair and welfare-maximizing allocations and whether one can efficiently compute the allocation with maximum welfare among the set of fair allocations. In this section, we settle the computational complexity of these two algorithmic problems.

We introduce some notation for ease of reference to a problem. For a fairness criterion $F \in \{\text{EQX}, \text{EQ1}\}$ and welfare objective $W \in \{\text{UW}, \text{EW}\}$, denote by "E($W \times F$)" the problem of deciding whether there exists an F allocation that also maximizes W, and denote by "C(W/F)" the problem of computing an F allocation that maximizes W among all F allocations. For example, E(UW×EQX) refers to the problem of deciding the existence of an EQX allocation that also maximizes utilitarian welfare, and C(EW/EQ1) denotes the problem of computing an EQ1 allocation that maximizes the egalitarian welfare among all EQ1 allocations.

To establish (strong) NP-hardness, we provide polynomial time reductions from two well-known problems; that is, 3-Partition and Partition, described as below. According to Garey and Johnson [69], the problem 3-Partition is strongly NP-complete, and the problem Partition is NP-complete.

3-Partition: given a non-empty finite set $B = \{b_i : i \in I = \{1, \ldots, 3m\}\}$ of $3m$ positive integers and another positive integer T such that $T/4 < b_i < T/2$ for any $i \in [3m]$ and $\sum_{i \in I} b_i = mT$, can I be partitioned into m disjoint subsets $I_1, \ldots, I_m$ such that $\sum_{i \in I_k} b_i = T$ for any $k \in [m]$?

Partition: given a non-empty finite set $P = \{p_i : i \in I = \{1, \ldots, m\}\}$ of m positive integers such that $\sum_{i \in I} p_i = 2T$, can I be partitioned into two disjoint subsets I_1, I_2 such that $\sum_{i \in I_1} p_i = \sum_{i \in I_2} p_i = T$?

4.3.1 Non-equivalence between Goods and Chores

As we already mentioned, the results for goods and chores are not mirror images of one another, which is supported by our results on the price of fairness. Before diving deep into studying the complexity, we argue that even being restricted to the algorithmic problems we are concerned with, the chores problem may not be equivalent to the corresponding goods version, neither do the other direction.

Proposition 4.3.1. *For any fairness criterion $F \in \{\text{EQX}, \text{EQ1}\}$, there is no mapping $f : [-1, 0] \to \mathbb{R}_+ \cup \{0\}$ such that a fair-chores instance $\mathcal{I}^c = \langle [n], E, \mathcal{V} \rangle$ admits an F and utilitarian welfare-maximizing allocation if and only if the fair-goods instance $\mathcal{I}^g = \langle [n], E, f(\mathcal{V}) \rangle$ admits an F and utilitarian welfare-maximizing allocation.*

Proof. For the sake of contradiction, assume such a mapping f exists. We first consider instance $\mathcal{I}_1^c = \langle [2], E^1, \mathcal{V}^1 \rangle$ with two agents and three items $E^1 = \{e_1^1, e_2^1, e_3^1\}$. The valuation function of agent 1 is: $v_1^1(e_1^1) = v_1^1(e_2^1) = -1/2$ and $v_1^1(e_3^1) = -1$ and of agent 2 is: $v_2^1(e_1^1) = v_2^1(e_2^1) = -1$ and $v_2^1(e_3^1) = 0$. It is not hard to verify that $\mathcal{I}_1^c$ admits an unique UWM, in which agent 1 violates fairness criterion F. Accordingly, no UWM of $\mathcal{I}_1^c$ satisfies F. We then consider the corresponding fair-goods instance $\mathcal{I}_1^g = \langle [2], E^1, f(\mathcal{V}^1) \rangle$, in which the value of agent i on good $e \in E^1$ is $f(v_i^1(e))$. Due to the definition of mapping f, it must hold that instance $\mathcal{I}_1^g$ does not admit F and utilitarian welfare-maximizing allocations, either. Next, we discuss the several cases classified by the possible relationship among $f(-1), f(-1/2)$ and $f(0)$.

If $f(-1) \geq \max\{f(-1/2), f(0)\}$, then assigning e_1^1, e_2^1 to agent 2 and e_3^1 to agent 1 results in an F allocation that is also a UWM, a contradiction.

If $f(-1/2) > f(-1) \geq f(0)$, we consider another fair-chores instance $\mathcal{I}_2^c = \langle [2], E^2, \mathcal{V}^2 \rangle$ with six items $E^2 = \{e_1^2, \ldots, e_6^2\}$ and its corresponding fair-goods instance $\mathcal{I}_2^g = \langle [2], E^2, f(\mathcal{V}^2) \rangle$. Under instance $\mathcal{I}_2^c$, function $v_1^2(\cdot)$ is: $v_1^2(e_j^2) = -1/2$ for $j = 1, 2, 3, 6$ and $v_1^2(e_j^2) = -1$ for $j = 4, 5$, and $v_2^2(\cdot)$ is: $v_2^2(e_j^2) = -1$ for $j \in [4]$ and $v_2^2(e_j^2) = 0$ for $j = 5, 6$. Notice $\mathcal{I}_2^c$ admits an F and utilitarian welfare-maximizing allocation $\mathbf{A}$ with $A_1 = \{e_1^2, e_2^2, e_3^2\}$ and $A_2 = E^2 \setminus A_1$. Due to the construction of f, instance $\mathcal{I}_2^g$ should also admits an F and utilitarian welfare-maximizing allocation. However, in any UWM of instance $\mathcal{I}_2^g$, since $f(-1/2) > f(-1) \geq f(0)$, agent 2 violates fairness criterion F, a contradiction.

If $f(-1/2) \geq f(0) > f(-1)$, one can verify that $\mathcal{I}_1^g$ does not admit an allocation that is both F and UWM only if $f(-1/2) > f(0) > f(-1)$ holds. Again, we consider instances $\mathcal{I}_2^c$ and $\mathcal{I}_2^g$. Under the relationship of $f(-1/2) > f(0) > f(-1)$, in any UWM of instance $\mathcal{I}_2^g$, agent 2 does not satisfy F, while allocation $\mathbf{A}$ (defined above) is a UWM of instance $\mathcal{I}_2^c$ and satisfies F, a contradiction.

If $f(0) > f(-1/2) \geq f(-1)$, for instance $\mathcal{I}_1^g$, assigning e_1^1, e_2^1 to agent 1 and e_3^1 to agent 2 leads to a UWM that also satisfies F. This contradicts the fact that $\mathcal{I}_1^c$ does not have an F and utilitarian welfare-maximizing allocation.

If $f(0) > f(-1) > f(-1/2)$, we consider another fair-chores instance $\mathcal{I}_3^c = \langle [2], E^3, \mathcal{V}^3 \rangle$ with five items $E^3 = \{e_1^3, \ldots, e_5^3\}$ and its corresponding fair-goods instance $\mathcal{I}_3^g = \langle [2], E^3, f(\mathcal{V}^3) \rangle$. In instance $\mathcal{I}_3^c$, function $v_1^3(\cdot)$ is: $v_1^3(e_j^3) = 0$ for $j \in [3]$ and $v_1^3(e_j^3) = -1$ for $j = 4, 5$ and function $v_2^3(\cdot)$ is: $v_2^3(e_j^3) = -1/2$ for $j \in [4]$ and $v_2^3(e_5^3) = 0$. It is not hard to verify that any UWM of $\mathcal{I}_3^c$ satisfies fairness criterion F. But in a UWM of $\mathcal{I}_3^g$, agent 1 has value $3f(0) + f(-1)$ and agent 2 has value $f(0)$. Hence, agent 2 violates F in the unique UWM of $\mathcal{I}_3^g$, and consequently, instance $\mathcal{I}_3^g$

does not admit a UWM that also satisfies F, a contradiction.

Overall, no possible relationship among $f(-1), f(-1/2)$ and $f(0)$ can make such a mapping f exist, completing the proof. $\square$

The above assertion indicates that a fair-chores instance can not be transformed to a fair-goods instance by a unified mapping on valuations so that two instances admit the same answer to the question of whether there exists a UWM that also satisfies the notion of F. This statement, together with the price of fairness with respect to egalitarian welfare, motivates us to study the complexity of the existence and optimization problem from both goods and chores perspectives.

4.3.2 Computational Complexity with Variable Number of Agents

We first consider the case of general n agents and study the utilitarian welfare. For the problem of deciding the existence of an EQX and utilitarian welfare-maximizing allocation, we establish the strong NP-hardness in both cases of goods and chores.

Theorem 4.3.1. *For both goods and chores, the decision problem* E(UW×EQX) *is strongly NP-complete.*

Proof of Theorem 4.3.1 for chores. The decision problem is in NP as both utilitarian welfare maximization and EQX can be examined in polynomial time. We then derive a reduction from the problem 3-Partition.

Given an arbitrary instance of 3-Partition, we construct a fair-chores instance as follows. There are $m+1$ agents and a set $E = \{e_1, \ldots, e_{3m+2}\}$ of $3m+2$ items. The valuations are shown in Table 4.6. It is not hard to verify that an allo-

Items	e_1	e_2	$\cdots$	e_{3m-1}	e_{3m}	e_{3m+1}	e_{3m+2}
$v_i(\cdot)$ for $i \le m$	$-b_1$	$-b_2$	$\cdots$	$-b_{3m-1}$	$-b_{3m}$	$-(m+1)T$	$-(m+1)T$
$v_{m+1}(\cdot)$	$-T$	$-T$	$\cdots$	$-T$	$-T$	$-T$	$-T$

Table 4.6: The fair-chores instance for Theorem 4.3.1

cation is a UWM if and only if the first $3m$ chores are assigned to the first m agents and the last two chores are assigned to agent $m+1$.

Suppose we have a "yes" instance of 3-Partition, and let $I_1, \ldots, I_m$ be a solution. Then, consider the allocation $\mathbf{A}$ with $A_i = \bigcup_{j \in I_i} e_j$ for any $i \in [m]$ and $A_{m+1} = \{e_{3m+1}, e_{3m+2}\}$. It is straightforward to see that allocation $\mathbf{A}$ has the maximum utilitarian welfare. As for the fairness requirement, since $v_i(A_i) = -T$ for any $i \in [m]$ and $v_{m+1}(A_{m+1}) = -2T$, none of the first m agents would violate the condition of EQX. Moreover, for any $e \in A_{m+1}$, we have $v_{m+1}(A_{m+1} \setminus \{e\}) = v_i(A_i)$ for any $i \in [m]$. Thus, allocation $\mathbf{A}$ is a UWM and satisfies EQX.

Now we prove the other direction. Suppose we have a "no" instance of 3-Partition. Notice that in any UWM, agent $m+1$ receives value $-2T$. Because it is a "no" 3-Partition instance, assigning the first $3m$ chores to the first m agents always results in an allocation in which at least one agent receives value strictly larger than $-T$. Through the comparison between agent $m+1$ and the agent receiving the largest value, agent $m+1$ violates the condition of EQX in any UWM, completing the proof. □

Proof of Theorem 4.3.1 for goods. The problem is in NP as both utilitarian welfare maximization and EQX can be tested in polynomial time. We then derive a reduction from 3-Partition.

Given an arbitrary instance of 3-Partition, we construct a fair-goods instance as follows. There are $m+1$ agents and a set $E = \{e_1, \ldots, e_{3m+2}\}$ of $3m+2$ goods. The valuation functions are shown in Table 4.7. It is not hard to verify that

Items	e_1	e_2	$\cdots$	e_m	e_{m+1}	e_{m+2}
$v_i(\cdot)$ for $i \leq m$	b_1	b_2	$\cdots$	b_{3m}	0	0
v_{m+1}	$\frac{(m-2)b_1}{m}$	$\frac{(m-2)b_2}{m}$	$\cdots$	$\frac{(m-2)b_{3m}}{m}$	T	T

Table 4.7: The fair-goods instance for Theorem 4.3.1

an allocation is a UWM if and only if the first $3m$ goods are assigned to the first m agents and the last two goods are allocated to agent $m+1$.

Suppose we have a "yes" instance of 3-Partition, and let $I_1, \ldots, I_m$ be a solution. Then, consider allocation $\mathbf{A}$ with $A_i = \bigcup_{j \in I_i} e_j$ for any $i \in [m]$ and $A_{m+1} = \{e_{3m+1}, e_{3m+2}\}$. Clearly, allocation $\mathbf{A}$ is a UWM. For agents' value, we have $v_i(A_i) = T$ for every $i \in [m]$ and $v_{m+1}(A_{m+1}) = 2T$. Notice that $v_i(A_i) = v_{m+1}(A_{m+1} \setminus \{e\})$ holds for any $i \in [m]$ and $e \in A_{m+1}$, and consequently, $\mathbf{A}$ is a UWM that satisfies EQX.

Now we prove the other direction. Suppose we have a "no" instance of 3-Partition. Recall that in any UWM, agent $m+1$ only receives the last two goods and has value $2T$. Because it is a "no" instance, assigning the first $3m$ goods to the first m agents always results in an allocation in which at least one agent receives value strictly smaller than T. In any UWM, when comparing to agent $m+1$, the agent receiving the least value violates the condition of EQX, completing the proof. □

For the notion of EQ1, the strong NP-hardness on the decision problem is also established for both goods and chores.

Theorem 4.3.2. *For both goods and chores, the decision problem* E(UW×EQ1) *is strongly NP-complete.*

Proof. The arguments in the proof of Theorem 4.3.1 can be carried over to the decision problem E(UW×EQ1). □

Theorems 4.3.1 and 4.3.2 indicate that although the chores and goods versions of the decision problem (with respect to utilitarian welfare) are not equivalent in general, there surprisingly exist some similarities; that is, neither of them has pseudo-polynomial time algorithms. When concerning egalitarian welfare, results are different under these two settings. According to Theorem 4.2.1, the price of EQX and of EQ1 with respect to egalitarian welfare are both 1 in goods allocations, which implies that the existence of a goods allocation that is an EWM and also satisfies EQX (or EQ1) is guaranteed. Then, the answer to both problems E(EW×EQX) and E(EW×EQ1) is "yes" in the case of goods; however, finding the egalitarian welfare-maximizing allocation that satisfies EQX (or EQ1) is computationally hard as shown in our Theorem 4.3.15 later. On the other hand, when assigning chores, as shown by the results below, deciding the existence of an EQX and egalitarian welfare-maximizing allocation is computationally intractable.

Theorem 4.3.3. *When allocating chores, the decision problem* E(EW×EQX) *is strongly NP-hard.*

Proof. Given an arbitrary instance of 3-Partition, we construct a fair-chores instance as follows. There are $m+1$ agents and a set $E = \{e_1, \ldots, e_{3m+2}\}$ of $3m+2$ chores. The valuation functions are shown in Table 4.8. Consider an allocation

Items	e_1	e_2	$\cdots$	e_{3m}	e_{3m+1}	e_{3m+2}
$v_i(\cdot)$ for $i \leq m$	$-b_1$	$-b_2$	$\cdots$	$-b_{3m}$	$-\frac{(5m+1)T+1}{2}$	$-\frac{(5m+1)T+1}{2}$
$v_{m+1}(\cdot)$	$-2T$	$-2T$	$\cdots$	$-2T$	$-T$	-1

Table 4.8: The fair-chores instance for Theorem 4.3.3

$\hat{\mathbf{A}}$ with $\hat{A}_i = \{e_{3i-2}, e_{3i-1}, e_{3i}\}$ for $i \in [m]$ and $\hat{A}_{m+1} = \{e_{3m+1}, e_{3m+2}\}$. Due to $T/4 < b_i < T/2$, we have $\text{EW}(\hat{\mathbf{A}}) > -3T/2$, so that the maximum egalitarian welfare of this instance is larger than $-3T/2$. Thus, in any EWM, agent $m+1$ must receive exactly the last two chores, resulting in value $-T-1$ for her.

Suppose we have a "yes" instance of 3-Partition, and let $I_1, \ldots, I_m$ be a solution. Then, consider allocation $\mathbf{A}$ with $A_i = \bigcup_{j \in I_i} e_j$ for $i \in [m]$ and

$A_{m+1} = \{e_{3m+1}, e_{3m+2}\}$. One can compute $\mathrm{EW}(\mathbf{A}) = -T - 1$. Since agent $m+1$ must receive value $-T-1$ in an EWM, allocation $\mathbf{A}$ clearly achieves the maximum egalitarian welfare. As for fairness constraint, since for any $i \in [m]$, $v_i(A_i) = -T$ and $v_{m+1}(A_{m+1}) = -T-1$, none of the first m agents would violate EQX. For agent $m+1$, it holds that $v_{m+1}(A_{m+1} \setminus \{e\}) \geq v_i(A_i) = -T, \forall i \in [m], \forall e \in A_{m+1}$. Thus, allocation $\mathbf{A}$ is an EWM and also satisfies EQX.

We now prove the other direction. Suppose we have a "no" instance of 3-Partition. Recall that in any EWM, agent $m+1$ must receive value exactly $-T-1$. Because it is a "no" instance, assigning the first $3m$ items to the first m agents always results in an allocation in which at least one agent receives value larger than $-T$. Through the comparison between agent $m+1$ and the agent receiving the largest value, one can easily verify that no EWM can also be EQX, completing the proof. □

For the problem in which the quantity of optimal egalitarian welfare is involved, since computing an EWM is NP-hard [32], which is different from the utilitarian welfare, one may not able to find the maximum egalitarian welfare in polynomial time. As a consequence, we are unclear whether verifying a "yes" instance of E(EW×EQX) can be done in polynomial time, based on which we only state hardness in Theorem 4.3.3. For the same reason, we also only state hardness in the following theorem.

Theorem 4.3.4. *When allocating chores, the decision problem* E(EW×EQ1) *is strongly NP-hard.*

Proof. Given an arbitrary instance of 3-Partition, we construct a fair-chores instance as follows. There are $m+1$ agents and a set $E = \{e_1, \ldots, e_{3m+2}\}$ of $3m+2$ items. The valuation functions are shown in Table 4.9. We first consider an allo-

Items	e_1	e_2	$\cdots$	e_{3m}	e_{3m+1}	e_{3m+2}
$v_i(\cdot)$ for $i \leq m$	$-b_1$	$-b_2$	$\cdots$	$-b_{3m}$	$-4mT$	$-4mT$
$v_{m+1}(\cdot)$	$-\frac{9m-2}{3m}T$	$-\frac{9m-2}{3m}T$	$\cdots$	$-\frac{9m-2}{3m}T$	$-T$	$-T$

Table 4.9: The fair-chores instance for Theorem 4.3.4

cation that assigns three of $\{e_1, \ldots, e_{3m}\}$ to each agent $i \in [m]$, and assigns e_{3m+1} and e_{3m+2} to agent $m+1$. Due to $b_j < T/2$ for any $j \in [3m]$, the value of agent i is larger than $-3T/2$ for each $i \in [m]$. The value of agent $m+1$ is equal to $-2T$, and hence the egalitarian welfare of this allocation is equal to $-2T$. Due to the value of e_{3m+1} and e_{3m+2}, no allocation can achieve egalitarian welfare larger than

$-2T$, which then implies that the maximum egalitarian welfare of the constructed fair-chores instance is equal to $-2T$.

Suppose we have a "yes" instance of 3-Partition, and let $I_1, \ldots, I_n$ be the solution. Consider an allocation $\mathbf{A}$ with $A_i = \bigcup_{j \in I_i} e_j$ for any $i \in [m]$ and $A_{m+1} = \{e_{3m+1}, e_{3m+2}\}$. It is not hard to see that $v_i(A_i) = -T$ for any $i \in [m]$, and $v_{m+1}(A_{m+1}) = -2T$, then we have $\mathrm{EW}(\mathbf{A}) = -2T$, which implies that the allocation $\mathbf{A}$ is an EWM. As for the fairness constraint, none of the first m agents would violate EQ1, and for agent $m+1$, we have $v_{m+1}(A_{m+1} \setminus \{e\}) = v_i(A_i) = -T$ for any $i \in [m]$ and $e \in A_{m+1}$. Thus, the allocation $\mathbf{A}$ is also EQ1.

We now prove the other direction. Suppose we have a "no" instance of 3-Partition. Recall that in any EWM, agent $m+1$ must receive exactly items e_{3m+1} and e_{3m+2}, and must have value exactly $-2T$. Due to "no" instance, there exists an agent receiving a value strictly larger than $-T$ from the assignment of the first $3m$ chores. Through the comparison between agent $m+1$ and the agent receiving the largest value, one can verify that no EWM can satisfy EQ1, completing the proof. □

The above results show that, except for concerning egalitarian welfare in the allocation of goods, all other decision problems are strongly NP-hard in the case of general n agents. These results directly imply the NP-hardness of computing the allocation with maximum utilitarian welfare among all EQX (or EQ1) allocations. In the following, we use *Karp reduction* [81], a many-to-one reduction, to give a relatively simple argument for the hardness of the computation problem. The main difference between Karp reduction and Turing reduction is that in Turing reduction, one can use the oracle as many times as needed, while in Karp reduction, the oracle can be invoked only once at the end. Except for the following one, other reductions in this thesis are Turing reduction.

Theorem 4.3.5. *For both goods and chores, problems* C(UW/EQX) *and* C(UW/EQ1) *are strongly NP-hard.*

Proof. To prove the strong NP-hardness of C(UW/EQX), we use the strong NP-hardness of E(UW×EQX) established by Theorem 4.3.1. For the sake of contradiction, if there exists a (pseudo) polynomial time algorithm *ALG* for problem C(UW/EQX), then we can compute the maximum UW among all EQX allocations. Notice that an UWM (without EQX requirement) can be found in polynomial time by assigning each item to the agent who has the largest value on it, and so, one can efficiently find the optimal utilitarian welfare. By comparing the optimal utilitarian welfare with the output from *ALG*, the problem E(UW×EQX) is then solved in (pseudo) polynomial time, a contradiction. Based on Theorem 4.3.2, similar arguments can be applied to prove the strong NP-hardness of C(UW/EQ1). □

Theorem 4.3.6. *When allocating goods and chores, problems* C(EW/EQX) *and* C(EW/EQ1) *are strongly NP-hard.*

Proof. It suffices to prove strong NP-completeness of the *decision version* of C(EW/EQ1) (resp., C(EW/EQX)): given an instance $\mathcal{I} = \langle [n], E, \mathcal{V} \rangle$ and a threshold value W, does there exist an EQ1 (resp., EQX) allocation with egalitarian welfare at least W?

For both goods and chores, we provide a reduction from 3-Partition. Given an arbitrary instance of 3-Partition, we construct a fair-goods (resp. fair-chores) instance with m agents and a set $E = \{e_1, \ldots, e_{3m}\}$ of $3m$ goods (resp. chores). Agents have identical valuation functions and in the fair-goods instance, $v_i(e_j) = b_j$ for any $i \in [m]$ and $j \in [3m]$; in the fair-chores instance, $v_i(e_j) = -b_j$ for any $i \in [m]$ and $j \in [3m]$. The threshold value is defined as: $W = T$ in the fair-goods instance; $W = -T$ in the fair-chores instance. Then, in the both cases of goods and chores, there exists an EQ1 (or EQX) allocation with egalitarian welfare at least W if and only if the 3-Partition instance is a "yes" instance. □

Results of this section completely answer the computational complexity of the decision and computation problem we are concerned with for general n agents. Most problems are computationally intractable, which means that one can not efficiently determine the existence of a fair and welfare-maximizing allocation or compute the allocation with maximum welfare among all fair allocations. In addition, our results provide more insights into the similarities between goods and chores. According to Theorems 4.3.1 and 4.3.2, for both EQX and EQ1, together with utilitarian welfare, problems for goods and chores share the same complexity. When concerning egalitarian welfare, we show that decision problems for chores are strongly NP-hard and one can directly answer "yes" in the case of goods, but this may not indicate goods and chores have different algorithmic features. Instead of linking to algorithmic property, such an "inconsistency" is due to distinct structural properties, i.e., optimal egalitarian welfare is compatible with EQX/EQ1 for goods but not for chores.

4.3.3 Computational Complexity with Fixed Number of Agents

The above complexity results are for general n, but in practice, the number of agents is usually fixed. Theoretically, even a system with two agents, $n = 2$, can already yield valuable and non-trivial results, especially in machine scheduling [1, 2], an application of chores allocation. This observation motivates us to resolve the complexity for the case where n is fixed.

We carry out the analysis for the case of small and fixed n from utilitarian welfare and EQX, and show NP-completeness for both goods and chores.

Theorem 4.3.7. *For both goods and chores, the decision problem* E(UW×EQX) *is NP-complete, even for two agents.*

Proof of Theorem 4.3.7 for chores. To prove NP-hardness, we derive a reduction from the problem PARTITION. Given an arbitrary instance of PARTITION, we construct a fair-chores instance with two agents and a set $E = \{e_1, \ldots, e_{m+4}\}$ of $m + 4$ chores. The valuation functions are shown in Table 4.10, where $0 < \epsilon < 1$. It is not

Items	e_1	e_2	$\cdots$	e_m	e_{m+1}	e_{m+2}	e_{m+3}	e_{m+4}
$v_1(\cdot)$	$-p_1$	$-p_2$	$\cdots$	$-p_m$	$-T$	$-T$	$-\epsilon$	$-\epsilon$
$v_2(\cdot)$	$-p_1$	$-p_2$	$\cdots$	$-p_m$	$-\epsilon$	$-\epsilon$	$-T$	$-T$

Table 4.10: The fair-chores instance for Theorem 4.3.7

hard to verify that an allocation is a UWM if and only if chores e_{m+1} and e_{m+2} are assigned to agent 2 and e_{m+3}, e_{m+4} are assigned to agent 1.

Suppose we have a "yes" instance of PARTITION, and let I_1, I_2 be a solution. Then, consider an allocation $\mathbf{A}$ in which A_1 contains bundle $\bigcup_{j \in I_1} e_j$ and e_{m+3}, e_{m+4}, and $A_2 = E \setminus A_1$. Clearly, allocation $\mathbf{A}$ is a UWM. Moreover, for agents' value, we have $v_1(A_1) = v_2(A_2) = -T - 2\epsilon$. Thus, allocation $\mathbf{A}$ is also EQX.

We now prove the other direction. Suppose we have a "no" instance of PARTITION. Let $S = \{e_1, \ldots, e_m\}$ be the set of first m items, and recall $\Pi_2(S)$ is the set of 2-partitions of S. We denote by $\Delta = \min_{\mathbf{B} \in \Pi_2(S)} |v_1(B_1) - v_2(B_2)|$, and claim that $\Delta \geq 1$ because $p_i \in \mathbb{N}$ for any $i \in [m]$ and it's a "no" instance. Notice that any utilitarian welfare-maximizing allocation $\mathbf{O}$ has the form of $O_1 = B_1 \cup \{e_{m+3}, e_{m+4}\}$ and $O_2 = B_2 \cup \{e_{m+1}, e_{m+2}\}$ where $\{B_1, B_2\} \in \Pi_2(S)$. Without loss of generality we assume $v_2(O_2) > v_1(O_1)$, and accordingly, due to the definition of Δ and ϵ, we have $v_1(O_1 \setminus \{e_{m+3}\}) - v_2(O_2) \leq \epsilon - \Delta < 0$. So agent 1 violates EQX in allocation $\mathbf{O}$, and therefore, no UWM can satisfy EQX, completing the proof. □

Proof of Theorem 4.3.7 for goods. To prove NP-hardness, we derive a reduction from the problem PARTITION. Given an arbitrary instance of PARTITION, we construct a fair-goods instance with two agents and a set $E = \{e_1, \ldots, e_{m+2}\}$ of $m + 2$ goods. The valuation functions are shown in Table 4.11, where $0 < \epsilon < 1$. Given these

Items	e_1	e_2	$\cdots$	e_m	e_{m+1}	e_{m+2}
$v_1(\cdot)$	p_1	p_2	$\cdots$	p_m	0	ϵ
$v_2(\cdot)$	p_1	p_2	$\cdots$	p_m	ϵ	0

Table 4.11: The fair-goods instance for Theorem 4.3.7

valuation functions, an allocation is a UWM if and only if e_{m+1} and e_{m+2} are assigned to agent 2 and agent 1, respectively.

Suppose we have a "yes" instance of Partition, and let I_1, I_2 be a solution. Then, consider allocation $\mathbf{A}$ in which A_1 contains bundle $\bigcup_{j \in I_i} e_j$ and good e_{m+2}, and $A_2 = E \setminus A_1$. Clearly, allocation $\mathbf{A}$ is a UWM. Then, since $v_1(A_1) = v_2(A_2) = T + \epsilon$, allocation $\mathbf{A}$ is also EQX.

We now prove the other direction. Suppose we have a "no" instance of Partition. Then, let $S = \{e_1, \ldots, e_m\}$ be the set of first m goods and $\Pi_2(S)$ be the set of 2-partition of S. We denote by $\Delta = \arg\min_{\mathbf{B} \in \Pi_2(S)} |v_1(B_1) - v_2(B_2)|$, and we claim that $\Delta \geq 1$ because $p_i \in \mathbb{N}, \forall i \in [m]$ and it's a "no" Partition instance. Let $\mathbf{O}$ be an arbitrary UWM, and without loss of generality assume $v_1(O_1) > v_2(O_2)$. According to the construction of Δ, it must hold that $0 < \Delta - \epsilon \leq v_1(O_1 \setminus \{e_{m+2}\}) - v_2(O_2)$ implying that agent 2 violates the condition of EQX. Therefore, no UWM can also be EQX, completing the proof. □

The theorem below follows from Theorem 4.3.7.

Theorem 4.3.8. *For both goods and chores, the problem* C(UW/EQX) *is NP-hard, even with two agents.*

We then move to EQ1, a notion weaker than EQX, and show that E(UW×EQ1) is NP-complete when $n \geq 3$, but in P when only two agents are involved.

Theorem 4.3.9. *For both goods and chores, the decision problem* E(UW×EQ1) *is NP-complete, even for three agents.*

Proof of Theorem 4.3.9 for chores. The problem is in NP as both utilitarian welfare-maximization and EQ1 can be examined in polynomial time. To prove NP-hardness, we derive a reduction from problem Partition. Given an arbitrary instance of Partition, we construct a fair-chores instance for three agents with a set $E = \{e_1, \ldots, e_{m+3}\}$ of $m + 3$ chores. The valuation functions are shown in Table 4.12. An allocation of the constructed instance is a UWM if and only if the first $m + 2$ chores are assigned to the first two agents and the last chore is assigned to agent 3.

Items	e_1	e_2	$\cdots$	e_m	e_{m+1}	e_{m+2}	e_{m+3}
$v_i(\cdot)$ for $i = 1, 2$	$-p_1$	$-p_2$	$\cdots$	$-p_m$	$-5T$	$-5T$	$-2mT - 9T$
$v_3(\cdot)$	$-2T$	$-2T$	$\cdots$	$-2T$	$-10T$	$-10T$	$-T$

Table 4.12: The fair-chores instance for Theorem 4.3.9

Suppose we have a "yes" instance of PARTITION, and let I_1, I_2 be a solution. Then, consider allocation $\mathbf{A}$, in which A_1 contains bundle $\bigcup_{j \in I_1} e_j$ and e_{m+1}; A_2 contains bundle $\bigcup_{j \in I_2} e_j$ and e_{m+2}; $A_3 = \{e_{m+3}\}$. Clearly, allocation $\mathbf{A}$ is a UWM. As for agents' value, it holds that $v_1(A_1) = v_2(A_2) < v_3(A_3)$, and moreover, $v_i(A_i \setminus \{e_{m+i}\}) = v_3(A_3)$ for any $i \in [2]$. Thus, allocation $\mathbf{A}$ is a UWM that also satisfies EQ1.

We now prove the other direction. Suppose we have a "no" instance of PARTITION. Since for any $j \in [m]$, $p_j < 2T$, then agent 3 receives value exactly $-T$ in any UWM. Based on the assignment of chores e_{m+1} and e_{m+2}, we discuss two cases: (i) allocations with exactly one $-5T$ chore for each $i \in [2]$; (ii) other assignments. If case (ii) happens, without loss of generality assume agent 1 receives two $-5T$ chores. Then, when comparing agent 1 and agent 3, agent 1 still receives less value no matter which chore is removed from his bundle, violating the condition of EQ1. Consequently, to make an allocation be both EQ1 and utilitarian welfare-maximizing, case (i) must happen. Moreover, since it is a "no" instance of PARTITION, assigning the first m chores to the first two agents always results in allocations in which there exists one agent with value strictly less than $-T$. Then, when comparing to agent 3, the agent receiving the least value violates EQ1. Thus, no UWM can satisfy EQ1, completing the proof. □

Proof of Theorem 4.3.9 for goods. The problem is in NP as both utilitarian welfare maximization and EQ1 can be tested in polynomial time. To prove NP-hardness, we derive a reduction from the problem of PARTITION. Given an arbitrary instance of PARTITION, we construct a fair-goods instance for three agents with a set $E = \{e_1, \ldots, e_{m+2}\}$ of $m+2$ goods. The valuation functions are shown in Table 4.13. An allocation is a UWM if and only if the first m goods are assigned to the first two

Items	e_1	e_2	$\cdots$	e_m	e_{m+1}	e_{m+2}
$v_i(\cdot)$ for $i = 1, 2$	p_1	p_2	$\cdots$	p_m	0	0
$v_3(\cdot)$	0	0	$\cdots$	0	T	T

Table 4.13: The fair-goods instance for Theorem 4.3.9

agents and the last two goods are assigned to agent 3.

Suppose we have a "yes" instance of Partition, and let I_1, I_2 be a solution. Then, consider allocation $\mathbf{A}$ with $A_1 = \bigcup_{j \in I_1} e_j$, $A_2 = \bigcup_{j \in I_2} e_j$ and $A_3 = \{e_{m+1}, e_{m+2}\}$. It is straightforward to verify that allocation $\mathbf{A}$ is a UWM. As for agents' value, we have $v_1(A_1) = v_2(A_2) = T$ and $v_3(A_3) = 2T$, and moreover, $v_3(A_3 \setminus \{e\}) = T$ holds for any $e \in A_3$. Thus, allocation $\mathbf{A}$ is also EQ1.

We now prove the other direction. Suppose we have a "no" instance of Partition. Recall that in any UWM, agent 3 receives value $2T$. Since it is a "no" Partition instance, assigning the first m goods to the first two agents always results in an allocation in which one agent receives value strictly smaller than T. In an UWM, when comparing to agent 3, the agent receiving the least value violates the condition of EQ1. Therefore, no UWM is EQ1, completing the proof. □

The theorem below follows from Theorem 4.3.9.

Theorem 4.3.10. *For both goods and chores, problem* C(UW/EQ1) *is NP-hard, even for three agents.*

Next, we provide a polynomial-time algorithm that can determine the existence of an EQ1 and utilitarian welfare-maximizing allocation in the case of two agents. In the algorithm, we first guarantee the allocation achieving maximum utilitarian welfare by assigning each item to the agent who values it the most, and then for unassigned items (if exist), in each round, allocate one of them to the agent with a smaller absolute value via Algorithm 3.

Algorithm 5 Computing the UWM that satisfies EQ1 if one exists

Input: An instance $\mathcal{I} = \langle [2], E, \mathcal{V} \rangle$.

Let $E = E_0 \cup E_1 \cup E_2$ with $E_1 = \{e \in E \mid v_1(e) > v_2(e)\}$ and $E_2 = \{e \in E \mid v_1(e) < v_2(e)\}$.
$\mathbf{A} \leftarrow \text{Greedy}((E_1, E_2), \mathcal{I})$;
if allocation $\mathbf{A} = (A_1, A_2)$ is EQ1 **then**
  return yes and $\mathbf{A}$.
else
  return no.
end if

Theorem 4.3.11. *For both goods and chores allocation with two agents, there exists a polynomial time algorithm that solves problem* E(UW×EQ1)

Proof of Theorem 4.3.11 for chores. The proof uses Algorithm 5. It is not hard to verify that assigning E_i to agent i for all i is a necessary and sufficient condition for the output having maximum utilitarian welfare.

Denote by $\mathbf{A} = (A_1, A_2)$ with $A_1 = \{e_1^{(1)}, \ldots, e_{k_1}^{(1)}\}$ and $A_2 = \{e_1^{(2)}, \ldots, e_{k_2}^{(2)}\}$. Clearly, we have $k_1 + k_2 = m$. Term $e_j^{(1)}$ refers to the j-th chore assigned by Algorithm 5 to agent 1 and $e_l^{(2)}$ is the l-th chore assigned to agent 2. If the algorithm terminates at Step 4, clearly we find a UWM that satisfies EQ1. We now consider the case where algorithm terminates at Step 6, and show if so, no UWM is EQ1. We first claim that, in this case, either $E_0 \subseteq A_1$ or $E_0 \subseteq A_2$ holds. It suffices to show that if both $A_1 \cap E_0$ and $A_2 \cap E_0$ are non-empty set, Algorithm 5 terminates at Step 4. To prove this claim, we discuss two cases. If $|v_1(A_1)| > |v_2(A_2)|$, since $A_1 \cap E_0 \neq \emptyset$, we have $e_{k_1}^{(1)} \in E_0$. Then by algorithm, it must hold that $|v_1(A_1 \setminus \{e_{k_1}^{(1)}\})| \leq |v_2(A_2)|$; otherwise, agent 1 would not receive chore $e_{k_1}^{(1)}$. Accordingly, allocation $\mathbf{A}$ is EQ1, and Algorithm 5 terminates at Step 4. On the other hand, if $|v_1(A_1)| < |v_2(A_2)|$, then since $A_2 \cap E_0 \neq \emptyset$, similarly, $|v_1(A_1)| \geq |v_2(A_2 \setminus \{e_{k_2}^{(2)}\})|$ holds and $\mathbf{A}$ is EQ1. Up to here, the claim is proved.

Next, if $E_0 \subseteq A_1$, then either (i) $|v_1(A_1)| < |v_2(A_2)|$ or (ii) $|v_1(A_1)| > |v_2(A_2)|$ and $|v_1(A_1) \setminus \{e_{k_1}^{(1)}\}| < |v_2(A_2)|$ holds. The latter one indicates that allocation $\mathbf{A}$ is EQ1, a contradiction. As for the first possible case, recall that terminating at Step 6 implies allocation $\mathbf{A}$ is not EQ1, and consequently, even when agent 2 only receives E_2, she still violates the condition of EQ1. Notice that in any UWM, all E_2 must be assigned to agent 2, and so, no UWM satisfies EQ1. If $E_0 \subseteq A_2$, similarly, there are two possible cases: (i) $|v_1(A_1)| > |v_2(A_2)|$; (ii) $|v_1(A_1)| < |v_2(A_2)|$ but $|v_1(A_1)| \geq |v_2(A_2 \setminus \{e_{k_2}^{(2)}\})|$. If case (ii) happens, allocation $\mathbf{A}$ is EQ1, a contradiction. The only possibility is case (i), which then implies that even when agent 1 only receives bundle E_1, he still violates the condition of EQ1. Therefore, no UWM is EQ1, completing the proof. □

Proof of Theorem 4.3.11 for goods. The proof uses Algorithm 5. It is not hard to verify that assigning E_i to agent i for all i is the necessary and sufficient condition for guaranteeing the output being an UWM.

Denote by $\mathbf{A} = (A_1, A_2)$ the with $A_1 = \{e_1^{(1)}, \ldots, e_{k_1}^{(1)}\}$ and $A_2 = \{e_1^{(2)}, \ldots, e_{k_2}^{(2)}\}$. Clearly, we have $k_1 + k_2 = m$. Term $e_j^{(i)}$ refers to the j-th goods assigned by algorithm to agent i. If Algorithm 5 terminates at Step 4, clearly we find a UWM that satisfies EQ1. We now consider the case where Algorithm 5 terminates at Step 6, and claim in this case, either $E_0 \subseteq A_1$ or $E_0 \subseteq A_2$. It suffices to show that if both $A_1 \cap E_0$ and $A_2 \cap E_0$ are not empty set, then Algorithm 5 terminates at Step 4. To prove this claim, we discuss two cases. If $v_1(A_1) > v_2(A_2)$, since $A_1 \cap E_0 \neq \emptyset$, we have $e_{k_1}^{(1)} \in E_0$. Then by algorithm, it mush hold that $v_1(A_1 \setminus \{e_{k_1}^{(1)}\}) \leq v_2(A_2)$; otherwise, agent 1 would not receive good $e_{k_1}^{(1)}$. Accordingly, allocation $\mathbf{A}$ is EQ1, and Algorithm 5 terminate at Step 4. On the other hand, if $v_1(A_1) < v_2(A_2)$, then since $A_2 \cap E_0 \neq \emptyset$, similarly, $v_1(A_1) \geq v_2(A_2 \setminus \{e^{(2)}_{k_2}\})$ holds and allocation $\mathbf{A}$ is EQ1. The claim is proved.

Next, if $E_0 \subseteq A_1$, then either (i) $v_1(A_1) < v_2(A_2)$ or (ii) $v_1(A_1) > v_2(A_2)$ and $v_1(A_1 \setminus \{e^{(1)}_{k_1}\}) \leq v_2(A_2)$. The latter one indicates that allocation $\mathbf{A}$ is EQ1, a contradiction. As for the first possible case, recall that terminating at Step 6 means that allocation $\mathbf{A}$ is not EQ1, and as a consequence, even when agent 1 receives all E_0, he still violates the condition of EQ1. Notice that in any UWM, agent 1 receives value at most $v_1(E_0 \cup E_1) = v_1(A_1)$, and so, no UWM is EQ1. If $E_0 \subseteq A_2$, similarly, there are two possible cases: (i) $v_1(A_1) > v_2(A_2)$; (ii) $v_1(A_1) < v_2(A_2)$ and $v_1(A_1) \geq v_2(A_2 \setminus \{e^{(2)}_{k_2}\})$. If case (ii) happens, then allocation $\mathbf{A}$ is EQ1, a contradiction. The only possibility is case (i), which then implies that even when agent 2 receives both E_0 and E_2, she still violates that condition of EQ1. Thus, no UWM is EQ1, completing the proof. □

We remark that the goods version of Algorithm 5 has been proposed by Aziz et al. [17] to answer E(UW×EQ1) on allocating goods to two agents. For completeness, we have presented the algorithm and proof for both cases of chores and goods.

In the case of two agents, given that E(UW×EQ1) is polynomial-time solvable, the remaining question is whether one can solve C(UW/EQ1) in polynomail time.

Theorem 4.3.12. *For both goods and chores, the problem* C(UW/EQ1) *is NP-hard, even for two agents.*

Proof of Theorem 4.3.12 for chores. It suffices to show prove NP-completeness of the decision version of C(UW/EQ1): given an instance $\mathcal{I} = \langle [n], E, \mathcal{V} \rangle$ and a threshold value W, does there exist an EQ1 allocation with utilitarian welfare at least W? We then derive a reduction from the problem of PARTITION.

Given an arbitrary instance of PARTITION, we construct an instance of fair-chores with two agents and a set $E = \{e_1, \ldots, e_{m+1}\}$ of $m+1$ chores. The valuation functions are shown in Table 4.14, where $0 < \epsilon < 0.1$. The thresholds value is

Items	e_1	e_2	$\cdots$	e_m	e_{m+1}
$v_1(\cdot)$	$-p_1$	$-p_2$	$\cdots$	$-p_m$	$-3T - \epsilon$
$v_2(\cdot)$	$-(1-\epsilon)p_1$	$-(1-\epsilon)p_2$	$\cdots$	$-(1-\epsilon)p_n$	$-(1-\epsilon)(3T+\epsilon)$

Table 4.14: The fair-chores instance for Theorem 4.3.12

defined as $W = -5T + (4T-1)\epsilon + \epsilon^2$.

Suppose we have a "yes" instance of Partition, and let I_1, I_2 be a solution. Then, consider allocation $\mathbf{A}$ with $A_1 = \cup_{j \in I_1} e_j$ and $A_2 = E \setminus A_1$. For agents' value, we have $v_1(A_1) > v_2(A_2)$ and $v_2(A_2 \setminus \{e_{m+1}\}) > v_1(A_1)$, and thus, $\mathbf{A}$ is an EQ1 allocation with utilitarian welfare

$$\mathrm{UW}(\mathbf{A}) = -T - (1-\epsilon)T - (1-\epsilon)(3T+\epsilon) = -5T + (4T-1)\epsilon + \epsilon^2 = W.$$

Therefore, we find an EQ1 allocation $\mathbf{A}$ with utilitarian welfare W.

We now prove the other direction. Suppose we have a "no" instance of Partition. Denote by $\mathbf{B} = (B_1, B_2)$ an EQ1 allocation and it must be in the form of $\{S_1 \cup \{e_{m+1}\}, S_2\}$ where $\{S_1, S_2\}$ is a 2-partition of set $\{e_1, \ldots, e_m\}$. We highlight that S_1 is not necessarily assigned to agent 1. Let $-\sum_{e \in S_1} v_1(e) = \Delta \geq 0$, and accordingly, $-\sum_{e \in S_2} v_1(e) = 2T - \Delta \geq 0$. To meet the condition of EQ1, chore e_{m+1} has to be eliminated when comparing; otherwise, the agent who receives bundle $S_1 \cup \{e_{m+1}\}$ would violate the condition of EQ1, even when $S_1 = \emptyset$. As a consequence, the condition of EQ1 is equivalent to $\Delta \leq T$. Moreover, since it's a "no" instance of Partition, we have $\Delta < T$. Given the form of $\{S_1 \cup \{e_{m+1}\}, S_2\}$, there are two possible allocations: assigning bundle $S_1 \cup \{e_{m+1}\}$ to agent 1; assigning bundle $S_1 \cup \{e_{m+1}\}$ to agent 2.

If $B_1 = S_1 \cup \{e_{m+1}\}$ and $B_2 = S_2$, then such an assignment results in welfare

$$\begin{aligned}\mathrm{UW}(\mathbf{B}) &= -\Delta - 3T - \epsilon - (1-\epsilon)(2T - \Delta) \\ &= -5T + (2T - \Delta - 1)\epsilon < W.\end{aligned}$$

If $B_1 = S_2$ and $B_2 = S_1 \cup \{e_{m+1}\}$, then allocation $\mathbf{B}$ has welfare

$$\begin{aligned}\mathrm{UW}(\mathbf{B}) &= -2T + \Delta - (1-\epsilon)\Delta - (1-\epsilon)(3T+\epsilon) \\ &= -5T + (3T + \Delta - 1)\epsilon + \epsilon^2 < W,\end{aligned}$$

where the last transition is due to $\Delta < T$. Therefore, in both cases, no EQ1 allocations can have utilitarian welfare at least W, completing the proof. □

Proof of Theorem 4.3.12 for goods. It suffices to prove that the decision version of C(UW/EQ1) is NP-complete. We prove it by deriving a reduction from the problem of Partition.

Given an arbitrary instance of Partition, we construct an instance of the decision version of C(UW/EQ1) with a set $E = \{e_1, \ldots, e_{m+3}\}$ goods. The valuation functions are shown in Table 4.15, where $0 < \epsilon < 1/T$. The threshold value is defined as $W = 5T$.

Items	e_1	e_2	$\cdots$	e_m	e_{m+1}	e_{m+2}	e_{m+3}
$v_1(\cdot)$	p_1	p_2	$\ldots$	p_m	T	0	0
$v_2(\cdot)$	$(1+\epsilon)p_1$	$(1+\epsilon)p_2$	$\ldots$	$(1+\epsilon)p_m$	0	$(1-\epsilon)T$	T

Table 4.15: The fair-goods instance for Theorem 4.3.12

We denote by $S = \{e_1, \ldots, e_m\}$. Suppose we have a "yes" instance of PARTITION, and let I_1, I_2 be a solution. Then, consider an allocation $\mathbf{A}$, in which A_1 contains bundles $\bigcup_{j \in I_1} e_j$ and good e_{m+1} and $A_2 = E \setminus A_1$. It holds that $v_1(A_1) = 2T = v_2(A_2 \setminus \{e_{m+3}\})$, which implies that allocation $\mathbf{A}$ is EQ1. Also, we have $\text{UW}(\mathbf{A}) = 5T$ implying that $\mathbf{A}$ is an EQ1 allocation with utilitarian welfare $5W$.

We now prove the other direction. Suppose we have a "no" instance of PARTITION. For an arbitrary EQ1 allocation $\mathbf{B}$, its utilitarian welfare is no less than $5T$ only if e_{m+1} is assigned to agent 1 and e_{m+2}, e_{m+3} are assigned to agent 2. Moreover, it is not hard to verify that in order to gain utilitarian welfare $5T$, agent 2 has to receive value at least $(1+\epsilon)T$ from bundle S. Then we discuss two cases according to the largest p_i. Let $i^* \in \arg\max_{i \in [m]} p_i$.

If $p_{i^*} > T$, then due to $p_{i^*} \in \mathbb{N}^+$ and it is an "no" instance, we have $p_{i^*} - T \geq 1$. Then, good e_{i^*} must be assigned to agent 2 so that the utilitarian welfare is no less than $5T$. Then, in such an allocation $\mathbf{B}$, agent 1 has value $v_1(B_1) \leq 3T - p_{i^*} \leq 2T - 1 < 2T - \epsilon T \leq \min_{e \in B_2} v_2(B_2 \setminus \{e\})$. Thus, $\mathbf{B}$ is not EQ1, which implies that the utilitarian welfare of any EQ1 allocation is smaller than $5T$.

If $p_{i^*} < T$ ($p_{i^*} \neq T$ because of "no" instance), suppose S_1, S_2 be a 2-partition of S such that allocation $\mathbf{B}$ is composed of $B_1 = S_1 \cup \{e_{m+1}\}$ and $B_2 = S_2 \cup \{e_{m+2}, e_{m+3}\}$ and has utilitarian welfare no less than $5T$. Recall that to meet the requirement on welfare, $v_2(S_2) \geq (1+\epsilon)T$ must hold, and moreover, owing to "no" PARTITION instance, it holds that $v_2(S_2) > (1+\epsilon)T$. Accordingly, we have $v_1(S_1) < T$, derived by $\sum_{i \in [m]} p_i = 2T$. As a consequence, $v_1(B_1) < 2T \leq \min_{e \in B_2} v_2(B_2 \setminus \{e\})$ where the last transition is because, in this case, e_{m+3} is the most valuable item in bundle B_2 for agent 2 given $p_{i^*} < T$. This inequality contradicts the fact that allocation $\mathbf{B}$ is EQ1, completing the proof. □

We have completely settled the computational complexity of algorithmic decision and optimization problems when utilitarian welfare is concerned. Most of results are computationally intractable. In addition, the above algorithmic problems (regarding utilitarian welfare) on goods and chores have the same computational complexity, which somewhat reveals similarities between these two settings.

Next, we investigate egalitarian welfare-maximizing allocations and provide the computational complexity in the case of chores.

Theorem 4.3.13. *When allocating chores, the problem* E(EW×EQX) *is NP-hard, even for two agents.*

Proof. To prove NP-hardness, we derive a reduction from the problem PARTITION. Given an arbitrary instance of PARTITION, we construct a fair-chores instance with two agents and a set $E = \{e_1, \dots, e_{m+2}\}$ of $m + 2$ items. The valuation functions are shown in Table 4.16, where $0 < \epsilon < 0.1$. Since $\sum_{i\in[m]} p_i = 2T$, it is not hard

Items	e_1	e_2	$\cdots$	e_m	e_{m+1}	e_{m+2}
$v_1(\cdot)$	$-p_1$	$-p_2$	$\cdots$	$-p_m$	$-\epsilon$	$-2T$
$v_2(\cdot)$	$-p_1$	$-p_2$	$\cdots$	$-p_m$	$-2T$	$-\epsilon$

Table 4.16: The fair-chores instance for Theorem 4.3.13

to verify that in an EWM, chores e_{m+1} and e_{m+2} must be assigned to agent 1 and agent 2, respectively, and moreover, the maximum egalitarian welfare is at most $-T - \epsilon$.

Suppose we have a "yes" instance of PARTITION, and let I_1, I_2 be a solution. Then, consider an allocation $\mathbf{A}$, in which A_1 contains bundle $\bigcup_{j\in I_1} e_j$ and chore e_{m+1}, and $A_2 = E \setminus A_1$. For agents' value, we have $v_1(A_1) = v_2(A_2) = -T - \epsilon$, and so allocation $\mathbf{A}$ is an EWM that satisfies EQX.

We now prove the other direction. Suppose we have a "no" instance of PARTITION. Let $S = \{e_1, \dots, e_m\}$ be the set of the first m chores, and $\Pi_2(S)$ be the set of 2-partition of S. We then denote by $\Delta = \min_{\mathbf{B}\in\Pi_2(S)} |v_1(B_1) - v_2(B_2)|$, and claim that $\Delta \geq 1$ because $p_i \in \mathbb{N}, \forall i \in [m]$ and it's a "no" instance. Notice that every egalitarian welfare-maximizing allocation $\mathbf{O}$ is in the form of $O_1 = B_1 \cup \{e_{m+1}\}$ and $O_2 = B_2 \cup \{e_{m+2}\}$ where $\{B_1, B_2\} \in \Pi_2(S)$. Without loss of generality we assume $v_1(O_1) < v_2(O_2)$, and based on the definition of Δ and ϵ, we have $v_1(O_1 \setminus \{e_{m+1}\}) - v_2(O_2) \leq \epsilon - \Delta < 0$, based on which, agent 1 violates EQX in allocation $\mathbf{O}$. Therefore, no EWM is EQX, completing the proof. □

Theorem 4.3.14. *When allocating chores, the problem* E(EW×EQ1) *is NP-hard, even for three agents.*

Proof. To prove NP-hardness, we derive a reduction from PARTITION. Given an arbitrary instance of PARTITION, we construct a fair-chores instance with three agents and a set $E = \{e_1, \ldots, e_{m+2}\}$ of $m + 2$ items. The valuation functions are shown in Table 4.17. Consider an allocation of arbitrarily assigning the first

Items	e_1	e_2	$\cdots$	e_m	e_{m+1}	e_{m+2}
$v_i(\cdot)$ for $i = 1, 2$	$-p_1$	$-p_2$	$\cdots$	$-p_m$	$-\frac{3m}{2}T$	$-\frac{3m}{2}T$
$v_3(\cdot)$	$-3T$	$-3T$	$\cdots$	$-3T$	$-T$	$-T$

Table 4.17: The fair-chores instance for Theorem 4.3.14

m chores to the first two agents, and assigning chores e_{m+1} and e_{m+2} to agent 3. Since $\sum_{j=1}^{m} p_j = 2T$, the egalitarian welfare of that allocation is $-2T$. Due to the value of e_{m+1} and e_{m+2}, no allocation can achieve egalitarian welfare larger than $-2T$, which then implies that the maximum egalitarian welfare of the constructed fair-chores instance is equal to $-2T$.

Suppose we have a "yes" instance of Partition, and let I_1, I_2 be a solution. Consider an allocation $\mathbf{A}$ with $A_1 = \bigcup_{j \in I_1} e_j$, $A_2 = \bigcup_{j \in I_2} e_j$ and $A_3 = \{e_{m+1}, e_{m+2}\}$. It is not hard to see that $v_1(A_1) = v_2(A_2) = -T$ and $v_3(A_3) = -2T$, and accordingly, the allocation $\mathbf{A}$ is an EWM. Since $v_3(A_3 \setminus \{e\}) = v_i(A_i)$ holds for any $e \in A_3$ and $i = 1, 2$, the allocation $\mathbf{A}$ is also EQ1.

We now prove the other direction. Suppose we have a "no" instance of Partition. Recall that in any EWM, agent 3 must receive exactly items e_{m+1} and e_{m+2} and have value exactly $-2T$. Due to "no" instance, one of the first two agents receives a value strictly larger than $-T$ from the assignment of the first m chores. Through the comparision between agent 3 and the agent receiving the largest value, one can verify that no EWM can satisfy EQ1, completing the proof. □

Theorem 4.3.15. *When allocating goods and chores, problems* C(EW/EQ1) *and* C(EW/EQX) *are NP-hard, even for two agents.*

Proof. It suffices to show NP-completeness of the decision version of problems C(EW/EQ1) and C(EW/EQX) (see the proof of Theorem 4.3.6). For both goods and chores, we provide a reduction from Partition. Given an arbitrary instance of Partition, we construct a fair-goods (resp., fair-chores) instance with two agents and a set $E = \{e_1, \ldots, e_m\}$ of m goods (resp., chores). Agents have identical valuation functions and in the fair-goods instance, $v_i(e_j) = p_j$ for $i \in [2]$ and $j \in [m]$; in the fair-chores instance, $v_i(e_j) = -p_j$ for $i \in [2]$ and $j \in [m]$. The threshold value is defined as $W = T$ in the fair goods instance; $W = -T$ in the fair-chores instance. Then, in both cases of goods and chores, there exists an EQ1 (or EQX) allocation with egalitarian welfare at least W if and only if the Partition instance is a "yes" instance. □

Concerning all complexity results in Section 4, we remark that, except for Theorem 4.3.12, the established (strong) NP-hardness still holds even when agents' valuations are normalized to a constant. The results of this section leaves the following two interesting open questions.

Open question 1: For fixed n, what is the time complexity of computing an EQ1 allocation maximizing UW among all EQ1 allocation when agents' valuations are normalized?

Open question 2: For chores and two agents, what is the time complexity of deciding whether there exists an EWM that is also EQ1?

4.4 Pseudo-Polynomial-Time Algorithms for Fixed Number of Agents

From the results established in the previous sections, for general n, all decision and computation problems are strongly NP-hard. And for fixed n, it is still unknown whether problems are pseudo-polynomial time solvable. In this section, we design pseudo-polynomial time algorithms that can output the approximately equitable allocation with the maximum welfare for both goods and chores. Similar to the algorithms in Aziz et al. [17], our pseudo-polynomial time algorithms mainly rely on dynamic programming with the subproblem of assigning the first k items. Once the assignment of k-th item has been settled, we augment k by one and analyse the assignment of $(k+1)$-th item upon the allocation of the first k items. Throughout this section, we assume agents' values are integers and $V = \max_{i\in[n]} \sum_{j\in[m]} |v_i(e_j)|$.

When considering utilitarian welfare, Aziz et al. [17] already provide an algorithm to compute an EQ1/EQX allocation with the maximum utilitarian welfare in the case of goods. For each $i \in [n]$, their algorithm uses t_i and s_i to represent the lower bound of agent i's value and a specific item in agent i's bundle, respectively. In particular, it applies dynamic programming to compute $G_k(t_1, \ldots, t_n, s_1, \ldots, s_n)$ for all k, t_i and s_i, and sets it as True if there exists an allocation of $e_1, \ldots, e_k$ such that for all i, the value of agent i is at least t_i and item s_i is in agent i's bundle; otherwise, False. The desired EQ1/EQX allocation can be found by visiting all $G_m(t_1, \ldots, t_n, s_1, \ldots, s_n)$. Since for each i, parameter $t_i \in \{0, 1, \ldots, V\}$ has $V + 1$ possible values and parameter $s_i \in E$ has m possible values, the worst-case running time of their algorithm is at least $O(m^n V^n)$.

In what follows, we borrow the idea from Aziz et al. [17] and design pseudo-polynomial time algorithms for optimization/decision problems regarding egalitarian welfare. Our algorithms rely on dynamic programmings subroutines, Algorithms 6 and 7, which compute $B(k, \mathbf{t}, \mathbf{s})$ for all possible $k, \mathbf{t}$ and $\mathbf{s}$. For every $B(k, \mathbf{t}, \mathbf{s})$, the parameter k refers to the number of items; vector $\mathbf{t} = (t_1, \ldots, t_n)$ represents agents' values; vector $\mathbf{s} = (s_1, \ldots, s_n)$ represents specific items in bundles of individual agents. To make the case of $k = 0$ well-defined, we assume that every agent is endowed with a dummy item e_0 of zero value (in total there are n dummy items) and moreover, the dummy item of an agent cannot be reassigned to others. We remark that for all i, s_i takes value from $\{e_0, e_1, \ldots, e_m\}$. Note that Algorithms 6 and 7 work for both fair-goods and fair-chores instances. After computing $B(k, \mathbf{t}, \mathbf{s}) \in \{\text{True}, \text{False}\}$ for every k, $\mathbf{t}$ and $\mathbf{s}$, the desired EQ1 or EQX allocations can be found by visiting all $B(m, \mathbf{t}, \mathbf{s})$ and backtracking the specific one.

Algorithm 6 Dynamic programming for C(EW/EQ1)

Input: An instance $\mathcal{I} = \langle [n], E, \mathcal{V} \rangle$.
Output: Tuple $B(k, \mathbf{t}, \mathbf{s})$ for all k, $\mathbf{t}$ and $\mathbf{s}$.

1: Initialize $B(k, \mathbf{t}, \mathbf{s}) = \text{False}$ for all k, $\mathbf{t}$ and $\mathbf{s}$.
2: Let $B(0, \mathbf{t}, \mathbf{s}) = \text{True}$ if $s_i = e_0$ and $t_i = 0$ for all i.
3: **for** $k = 1, \ldots, m$ **do**
4:   **for** all $t_1, \ldots, t_n$ and $s_1, \ldots, s_n$ **do**
5:     **for** $i = 1, \ldots, n$ **do**
6:       **if** $s_i = e_k$ **then**
7:         Set $B(k, t_1, \ldots, t_i, \ldots, t_n, s_1, \ldots, s_i, \ldots, s_n) = \text{True}$ if $B(k-1, t_1, \ldots, t_i - v_i(e_k), \ldots, t_n,\ s_1, \ldots, s_i', \ldots, s_n) = \text{True}$ for some $s_i' \in \bigcup_{r=0}^{k-1} e_r$.
8:       **end if**
9:       **if** $s_i \neq e_k$ **then**
10:         Set $B(k, t_1, \ldots, t_i, \ldots, t_n, s_1, \ldots, s_i, \ldots, s_n) = \text{True}$ if $B(k-1, t_1, \ldots, t_i - v_i(e_k), \ldots, t_n,\ s_1, \ldots, s_i, \ldots, s_n) = \text{True}$
11:       **end if**
12:     **end for**
13:   **end for**
14: **end for**

The following lemma holds in both cases of goods and chores.

Lemma 4.4.1. *Given a $B(k, \mathbf{t}, \mathbf{s})$ returned by Algorithm 6, $B(k, \mathbf{t}, \mathbf{s}) = $ True if and only if there exists an allocation of $e_1, \ldots, e_k$ such that for all i, the value of agent i is equal to t_i and $s_i \in \bigcup_{r=0}^{k-1} e_r$ is in agent i's bundle.*

Proof. We first prove the "only if" part by mathematical induction. In the case of $k = 0$, Step 2 of Algorithm 6 sets $B(0, \mathbf{t}, \mathbf{s}) = \text{True}$ if $s_i = e_0$ and $t_i = 0$ for all i.

Note that when $k = 0$, each agent i receives only e_0 with value equal to 0. Thus, the statement holds for the case of $k = 0$. We now assume that the statement holds for $k = 0, \dots, h$ and show that it also holds for the case of $k = h + 1$. Fix $\mathbf{t}$ and $\mathbf{s}$ with $B(h+1, \mathbf{t}, \mathbf{s}) =$ True and suppose that Algorithm 6 makes $B(h+1, \mathbf{t}, \mathbf{s}) =$ True when the for-loop in Step 5 being $i = i^*$.

If $s_{i^*} = e_{h+1}$, then Step 7 sets $B(h+1, \mathbf{t}, \mathbf{s}) =$ True and thus $B(h, t_1, \dots, t_{i^*} - v_{i^*}(e_{h+1}), \dots, t_n, s_1, \dots, s'_{i^*}, \dots, s_n) =$ True for some $s'_{i^*} \in \bigcup_{r=0}^{h} e_r$. As the statement holds for $k = h$, there exists an allocation $\mathbf{P}$ of $e_1, \dots, e_h$ such that (i) $v_j(P_j) = t_j$ for $j \neq i^*$ and $v_{i^*}(P_{i^*}) = t_{i^*} - v_{i^*}(e_{h+1})$; (ii) $s_j \in \bigcup_{r=0}^{h} e_r$ and $s_j \in P_j$ for all $j \neq i^*$; $s'_{i^*} \in \bigcup_{r=0}^{h} e_r$ and $s'_{i^*} \in P_{i^*}$. Then consider allocation $\mathbf{P}^*$ with $P^*_j = P_j$ for $j \neq i^*$ and $P^*_{i^*} = P_{i^*} \cup \{e_{h+1}\}$. One can verify that $\mathbf{P}^*$ is an allocation of $e_1, \dots, e_{h+1}$ with $v_i(P^*_i) = t_i$ and $s_i \in P^*_i$ for all i.

If $s_{i^*} \neq e_{h+1}$, then Step 10 sets $B(h+1, \mathbf{t}, \mathbf{s}) =$ True and thus $B(h, t_1, \dots, t_{i^*} - v_{i^*}(e_{h+1}), \dots, t_n, s_1, \dots, s_{i^*}, \dots, s_n) =$ True. As the statement holds for $k = h$, there exists an allocation $\mathbf{Q}$ of $e_1, \dots, e_h$ such that (i) $v_j(Q_j) = t_j$ for $j \neq i^*$ and $v_{i^*}(Q_{i^*}) = t_{i^*} - v_{i^*}(e_{h+1})$; (ii) $s_j \in \bigcup_{r=0}^{h} e_r$ and $s_j \in Q_j$ for all j. Then consider allocation $\mathbf{Q}^*$ with $Q^*_j = Q_j$ for $j \neq i^*$ and $Q^*_{i^*} = Q_{i^*} \cup \{e_{h+1}\}$. One can verify that $\mathbf{Q}^*$ is an allocation of $e_1, \dots, e_{h+1}$ with $v_i(Q^*_i) = t_i$ and $s_i \in Q^*_i$ for all i. Up to here, the statement also holds when $k = h + 1$.

Overall, by mathematical induction, the "only if" part is proved.

Now let us prove the "if" part, again with mathematical induction. In the case of $k = 0$, Step 2 of Algorithm 6 sets $B(0, \mathbf{t}, \mathbf{s}) =$ True if $s_i = e_0$ and $t_i = 0$ for all i. Accordingly, if $B(0, \mathbf{t}', \mathbf{s}') =$ False, then either $t'_q \neq 0$ or $s'_q \neq e_0$ holds for some q. Note that in the case of $k = 0$, no allocation can make agent q receive non-dummy item s'_q or a non-zero value. Thus, the statement holds for the case of $k = 0$. We now assume that the statement holds for $k = 0, \dots, h$ and show that it also holds for the case of $k = h + 1$. Fix $\mathbf{t}$ and $\mathbf{s}$ with $B(h + 1, \mathbf{t}, \mathbf{s}) =$ False. For a contradiction, assume that $\mathbf{P}$ is an allocation of $e_1, \dots, e_{h+1}$ with $v_j(P_j) = t_j$ and $s_j \in P_j$ for all j. Without loss of generality, we assume $e_{h+1} \in P_{i^*}$. Construct an allocation $\mathbf{P}'$ of $e_1, \dots, e_h$ with $P'_j = P_j$ for $j \neq i^*$ and $P'_{i^*} = P_{i^*} \setminus \{e_{h+1}\}$.

If $s_{i^*} = e_{h+1}$, due to the construction of $\mathbf{P}$ and $\mathbf{P}'$ and the fact that the statement holds for $k = h$, one can verify that allocation $\mathbf{P}'$ makes $B(h, t_1, \dots, t_{i^*} - v_{i^*}(e_{h+1}), \dots, t_n, s_1, \dots, s'_{i^*}, \dots, s_n\}) =$ True, where $s'_{i^*} \in \bigcup_{r=0}^{h} e_r$. Accordingly, when the for-loop in Step 5 is $i = i^*$, Step 7 will set $B(h + 1, \mathbf{t}, \mathbf{s}) =$ True, a contradiction.

If $s_{i^*} \neq e_{h+1}$, since $s_{i^*} \in P_{i^*}$ and $s_{i^*} \neq e_{h+1}$, we have $s_{i^*} \in P'_{i^*}$. Due to the construction of $\mathbf{P}$ and $\mathbf{P}'$ and the fact that the statement holds for $k = h$, one can verify that allocation $\mathbf{P}'$ makes $B(h, t_1, \ldots, t_{i^*} - v_{i^*}(e_{h+1}), \ldots, t_n, s_1, \ldots, s_{i^*}, \ldots, s_n) =$ True. As a result, when the for-loop in Step 5 is $i = i^*$, Step 10 will set $B(h+1, \mathbf{t}, \mathbf{s}) =$ True, another contradiction. Therefore, the statement also holds when $k = h + 1$.

Therefore, with mathematical induction, we have also proved the "if" part of the lemma. □

Consequently, we have the following theorem.

Theorem 4.4.1. *Given an instance* $\mathcal{I} = \langle [n], E, \mathcal{V} \rangle$*, one can compute an* EQ1 *allocation with the maximum egalitarian welfare in time* $O(m^{n+2}V^n)$.

Proof. Note that Algorithm 6 can return $B(m, \mathbf{t}, \mathbf{s})$ for all $\mathbf{t}$ and $\mathbf{s}$. By visiting the entire $B(m, \mathbf{t}, \mathbf{s})$, we can find the set Γ, of which the construction depends on the underlying items. The full construction of Γ is presented below, and the second condition for both goods and chores is to verify the underlying fairness notion.

$$\Gamma = \begin{cases} \{(\mathbf{t}, \mathbf{s}) | B(m, \mathbf{t}, \mathbf{s}) = \text{True and } t_i + v_j(s_j) \geq t_j \text{ for all } i, j\}, & \text{for goods;} \\ \{(\mathbf{t}, \mathbf{s}) | B(m, \mathbf{t}, \mathbf{s}) = \text{True and } t_i - v_i(s_i) \geq t_j \text{ for all } i, j\}, & \text{for chores.} \end{cases}$$

Given an arbitrary EQ1 allocation $\mathbf{A}' = (A'_1, \ldots, A'_n)$, construct $(\mathbf{t}', \mathbf{s}')$ as follows: for all i, $t'_i = v_i(A'_i)$ and $s'_i \in A'_i$ is the item with the largest absolute value for agent i. Then, we have $(\mathbf{t}', \mathbf{s}') \in \Gamma$ due to the property of EQ1 and the construction of $\mathbf{t}'$ and $\mathbf{s}'$. Accordingly, by visiting all element of Γ, we are able to find the element $(\mathbf{t}^*, \mathbf{s}^*) \in \Gamma$, of which $\mathbf{t}^*$ represents the agents' values in the EQ1 allocation maximizing egalitarain welfare over all EQ1 allocations. In particular, one can pursue the $(\mathbf{t}^*, \mathbf{s}^*)$ with $\min_{i \in [n]} t^*_i \geq \min_{i \in [n]} t_i$ for all $(\mathbf{t}, \mathbf{s}) \in \Gamma$. The specific EQ1 allocation can be found by backtracking $B(m, \mathbf{t}^*, \mathbf{s}^*)$ in the following way: assigning e_m to agent i_m if the value of $B(m, \mathbf{t}^*, \mathbf{s}^*)$ is set to True by $B(m-1, \mathbf{t}^{m-1}, \mathbf{s}^{m-1}) =$ True and at that time the for-loop in Step 5 is $i = i_m$; then assigning e_{m-1} to agent i_{m-1} if the value of $B(m-1, \mathbf{t}^{m-1}, \mathbf{s}^{m-1})$ is set to True by $B(m-2, \mathbf{t}^{m-2}, \mathbf{s}^{m-2}) =$ True and at that time the for-loop in Step 5 is $i = i_{m-1}$; repeat this process until all items are assigned. If in some step, the choice of $B(h, \mathbf{t}^h, \mathbf{s}^h)$ is not unique, then arbitrarily pick one.

As for the time complexity, the running time of Algorithm 6 is $O(m^{n+2}V^n)$, and visiting the entire $B(m, \mathbf{t}, \mathbf{s})$ and backtracking takes time $O(m^n V^n)$. Therefore, the running time of the algorithm is $O(m^{n+2}V^n)$. □

Moving on to our consideration of EQX, the dynamic programming for problem C(EW/EQX) is shown as Algorithm 7. Different from Algorithm 6, we now use s_i to represent the item with least non-zero absolute value in agent i's bundle if agent i receives a non-zero value. We establish that the EQX allocation with the maximum egalitarian welfare can be found by visiting the entire $B(m, \mathbf{t}, \mathbf{s})$ and backtracking specific ones.

Algorithm 7 Dynamic programming for C(EW/EQX)

Input: An instance $\mathcal{I} = \langle [n], E, \mathcal{V} \rangle$.
Output: Tuple $B(k, \mathbf{t}, \mathbf{s})$ for all k, $\mathbf{t}$ and $\mathbf{s}$.

Initialize $B(k, \mathbf{t}, \mathbf{s}) =$ False for all k, $\mathbf{t}$ and $\mathbf{s}$.
Let $B(0, \mathbf{t}, \mathbf{s}) =$ True if $s_i = e_0$ and $t_i = 0$ for all i.
for $k = 1, \ldots, m$ **do**
  for all $t_1, \ldots, t_n$ and $s_1, \ldots, s_n$ **do**
    for $i = 1, \ldots, n$ **do**
      if $s_i = e_k$ **and** $t_i = v_i(e_k) \neq 0$ **then**
        Set $B(k, t_1, \ldots, t_i, \ldots, t_n, s_1, \ldots, s_i, \ldots, s_n) =$ True if $B(k - 1, t_1, \ldots, t_{i-1}, 0, t_{i+1} \ldots, t_n, s_1, \ldots, s_{i-1}, e_0, s_{i+1}, \ldots, s_n) =$ True.
      end if
      if $s_i = e_k$ **and** $t_i \neq v_i(e_k) \neq 0$ **then**
        Set $B(k, t_1, \ldots, t_i, \ldots, t_n, s_1, \ldots, s_i, \ldots, s_n) =$ True if $B(k - 1, t_1, \ldots, t_i - v_i(e_k), \ldots, t_n, s_1, \ldots, s'_i, \ldots, s_n) =$ True for some $s'_i \in \bigcup_{r \in [k-1]} e_r$ with $|v_i(s_i)| \leq |v_i(s'_i)|$.
      end if
      if $e_0 = s_i \neq e_k$ **and** $t_i = v_i(e_k) = 0$ **then**
        Set $B(k, t_1, \ldots, t_i, \ldots, t_n, s_1, \ldots, s_i, \ldots, s_n) =$ True if $B(k - 1, t_1, \ldots, t_{i-1}, 0, t_{i+1} \ldots, t_n, s_1, \ldots, s_{i-1}, e_0, s_{i+1}, \ldots, s_n) =$ True.
      end if
      if $e_0 \neq s_i \neq e_k$ **and** $t_i \neq v_i(e_k)$ **then**
        Set $B(k, t_1, \ldots, t_i, \ldots, t_n, s_1, \ldots, s_i, \ldots, s_n) =$ True if $B(k - 1, t_1, \ldots, t_i - v_i(e_k), \ldots, t_n, s_1, \ldots, s_i, \ldots, s_n) =$ True and either $0 < |v_i(s_i)| \leq |v_i(e_k)|$ or $v_i(e_k) = 0$.
      end if
    end for
  end for
end for

The following lemma holds in both cases of goods and chores.

Lemma 4.4.2. *Given a $B(k, \mathbf{t}, \mathbf{s})$ returned by Algorithm 7, $B(k, \mathbf{t}, \mathbf{s}) =$ True if and only if there exists an allocation such that for all i:*

(i) the value of agent i is equal to t_i;

(ii) if $s_i = e_0$, then $t_i = 0$; if $s_i \neq e_0$, item $s_i \in \bigcup_{r \in [k]} e_r$ is in agent i's bundle and

moreover, either s_i is the unique item of non-zero value in agent i's bundle or $0 < |v_i(s_i)| \leq |v_i(e)|$ for every item e of non-zero value in agent i's bundle.

Proof. We first prove the "only if" part with mathematical induction. In the case of $k = 0$, Step 2 of Algorithm 7 sets $B(0, \mathbf{t}, \mathbf{s}) = \text{True}$ if $s_i = e_0$ and $t_i = 0$ for all i. Note that when $k = 0$, each agent i receives only e_0 with value equal to 0. Thus, the statement holds for the case of $k = 0$. We now assume that the statement holds for $k = 0, \dots, h$ and show that it also holds for the case of $k = h + 1$. Fix $\mathbf{t}$ and $\mathbf{s}$ with $B(h+1, \mathbf{t}, \mathbf{s}) = \text{True}$ and suppose that Algorithm 7 makes $B(h+1, \mathbf{t}, \mathbf{s}) = \text{True}$ when for-loop in Step 5 is $i = i^*$.

If $s_{i^*} = e_{h+1}$ and $t_{i^*} = v_{i^*}(e_{h+1}) \neq 0$, then Step 7 sets $B(h + 1, \mathbf{t}, \mathbf{s}) = \text{True}$ and $B(h, t_1, \dots, t_{i^*-1}, 0, t_{i^*+1}, \dots, t_n, s_1, \dots, s_{i^*-1}, e_0, s_{i^*+1}, \dots, s_n) = \text{True}$. As the statement holds for $k = h$, there exists an allocation $\mathbf{P}$ of $e_1, \dots, e_h$ such that $v_j(P_j) = t_j$ for $j \neq i^*$ and $v_{i^*}(P_{i^*}) = 0$, and moreover, satisfies property (ii) described in the statement. We now consider $\mathbf{P}'$ with $P'_j = P_j$ for $j \neq i^*$ and $P'_{i^*} = P_{i^*} \cup \{e_{h+1}\}$. Since $v_{i^*}(P'_{i^*}) = v_{i^*}(P_{i^*}) + v_{i^*}(e_{h+1}) = t_{i^*}$ and $e_{h+1} \in P'_{i^*}$ is the unique item of non-zero value in P'_{i^*}, $\mathbf{P}'$ is an allocation of $e_1, \dots, e_{h+1}$ that satisfies the properties (i) and (ii) described in the statement regarding $\mathbf{t}$ and $\mathbf{s}$.

If $s_{i^*} = e_{h+1}$ and $t_{i^*} \neq v_{i^*}(e_{h+1}) \neq 0$, then Step 10 sets $B(k, \mathbf{t}, \mathbf{s}) = \text{True}$ and $B(h, t_1, \dots, t_{i^*} - v_{i^*}(e_{h+1}), \dots, t_n, s_1, \dots, s'_{i^*}, \dots, s_n) = \text{True}$, where $s'_{i^*} \in \bigcup_{r \in [h]} e_r$ satisfies $|v_{i^*}(s'_{i^*})| \geq |v_{i^*}(e_{h+1})|$. As the statement holds for $k = h$, there exists an allocation $\mathbf{Q}$ of $e_1, \dots, e_h$ such that $v_j(Q_j) = t_j$ for $j \neq i^*$ and $v_{i^*}(Q_{i^*}) = t_{i^*} - v_{i^*}(e_{h+1})$, and moreover, satisfies property (ii) described in the statement regarding $s_1, \dots, s'_{i^*}, \dots, s_n$. We now consider $\mathbf{Q}'$ with $Q'_j = Q_j$ for $j \neq i^*$ and $Q'_{i^*} = Q_{i^*} \cup \{e_{h+1}\}$. Since $v_{i^*}(Q'_{i^*}) = v_{i^*}(Q_{i^*}) + v_{i^*}(e_{h+1}) = t_{i^*}$ and $Q'_j = Q_j$ for $j \neq i^*$, then we have $v_j(Q'_j) = t_j$ for all j. As for property (ii), note that $|v_{i^*}(s'_{i^*})| \geq |v_{i^*}(e_{h+1})| \neq 0$ and s'_{i^*} is the item with the least non-zero absolute value for agent i^* in Q_{i^*}, thus e_{h+1} is the item with the least non-zero absolute value in Q'_{i^*}. Then, property (ii) regarding $\mathbf{s}$ described in the statement is also satisfied by $\mathbf{Q}'$.

If $e_0 = s_{i^*} \neq e_{h+1}$ and $t_{i^*} = v_{i^*}(e_{h+1}) = 0$, then Step 13 sets $B(k, \mathbf{t}, \mathbf{s}) = \text{True}$ and also sets $B(h, t_1, \dots, t_{i^*-1}, 0, t_{i^*+1}, \dots, t_n, s_1, \dots, s_{i^*-1}, e_0, s_{i^*+1}, \dots, s_n) = \text{True}$. As the statement holds for $k = h$, there exists an allocation $\mathbf{R}$ of $e_1, \dots, e_h$ such that $v_j(R_j) = t_j$ for $j \neq i^*$ and $v_{i^*}(R_{i^*}) = 0$, and moreover, satisfies property (ii) described in the statement. We now consider $\mathbf{R}'$ with $R'_j = R_j$ for $j \neq i^*$ and $R'_{i^*} = R_{i^*} \cup \{e_{h+1}\}$. Note that $v_{i^*}(R'_{i^*}) = 0$, then one can verify that $\mathbf{R}'$ is an allocation of $e_1, \dots, e_{h+1}$ that satisfies the properties (i) and (ii) described in the statement regarding $\mathbf{t}$ and $\mathbf{s}$.

If $e_0 \neq s_{i^*} \neq e_{h+1}$ and $t_{i^*} \neq v_{i^*}(e_{h+1})$, then Step 16 sets $B(k, \mathbf{t}, \mathbf{s}) =$ True and $B(h, t_1, \ldots, t_{i^*} - v_{i^*}(e_{h+1}), \ldots, t_n, s_1, \ldots, s_{i^*}, \ldots, s_n) =$ True and either $0 < |v_{i^*}(s_{i^*})| \leq |v_{i^*}(e_{h+1})|$ or $v_{i^*}(e_{h+1}) = 0$. As the statement holds for $k = h$, there exists an allocation $\mathbf{S}$ of $e_1, \ldots, e_h$ such that $v_j(S_j) = t_j$ for $j \neq i^*$ and $v_{i^*}(S_{i^*}) = t_{i^*} - v_{i^*}(e_{h+1})$, and moreover, satisfies property (ii) described in the statement regarding $\mathbf{s}$. We now consider $\mathbf{S}'$ with $S'_j = S_j$ for $j \neq i^*$ and $S'_{i^*} = S_{i^*} \cup \{e_{h+1}\}$. Since $v_{i^*}(S'_{i^*}) = v_{i^*}(S_{i^*}) + v_{i^*}(e_{h+1}) = t_{i^*}$ and $S'_j = S_j$ for $j \neq i^*$, then we have $v_j(S'_j) = t_j$ for all j. As for property (ii), note that either $|v_{i^*}(s_{i^*})| \leq |v_{i^*}(e_{h+1})|$ or $v_{i^*}(e_{h+1}) = 0$ and s_{i^*} is the item with the least non-zero absolute value for agent i^* in S_{i^*}, then one can verify that property (ii) regarding $\mathbf{s}$ described in the statement is also satisfied by $\mathbf{S}'$. Up to here, the statement also holds when $k = h + 1$.

Overall, with mathematical induction, we have proved the "only if" part of the lemma.

Now let us prove the "if" part, again with mathematical induction. In the case of $k = 0$, Step 2 of Algorithm 7 sets $B(0, \mathbf{t}, \mathbf{s}) =$ True if $s_i = e_0$ and $t_i = 0$ for all i. Accordingly, if $B(0, \mathbf{t}', \mathbf{s}') =$ False, then either $t'_q \neq 0$ or $s'_q \neq e_0$ holds for some q. Note that in the case of $k = 0$, no allocation can make agent q receive non-dummy item s'_q or non-zero value. Thus, the statement holds for the case of $k = 0$. We now assume that the statement holds for $k = 0, \ldots, h$ and show that the statement also holds for the case of $k = h + 1$. Fix $\mathbf{t}$ and $\mathbf{s}$ with $B(h + 1, \mathbf{t}, \mathbf{s}) =$ False, and for a contradiction, assume $\mathbf{P}$ is an allocation of $e_1, \ldots, e_{h+1}$ satisfying properties (i) and (ii) described in the statement regarding $\mathbf{t}$ and $\mathbf{s}$. Without loss of generality, we assume $e_{h+1} \in P_{i^*}$. Construct allocation $\mathbf{P}'$ with $P'_j = P_j$ for $j \neq i^*$ and $P'_{i^*} = P_{i^*} \setminus \{e_{h+1}\}$. We then split the proof into four cases based on the possibilities of s_{i^*} and t_{i^*}.

Case 1: $s_{i^*} = e_{h+1}$ and $t_{i^*} = v_{i^*}(e_{h+1})$. As $s_{i^*} \neq e_0$, we must have $t_{i^*} = v_{i^*}(P_{i^*}) \neq 0$, and accordingly, $v_{i^*}(e_{h+1}) \neq 0$ holds. Note that $v_{i^*}(P'_{i^*}) = t_{i^*} - v_{i^*}(e_{h+1}) = 0$. As the statement holds for the case of $k = h$, it is not hard to verify that allocation $\mathbf{P}'$ makes $B(h, t_1, \ldots, t_{i^*-1}, 0, t_{i^*+1}, \ldots, t_n, s_1, \ldots, s_{i^*-1}, e_0, s_{i^*+1}, \ldots, s_n) =$ True. Thus, when the for-loop in Step 5 of Algorithm 7 is $i = i^*$, Step 7 sets $B(h + 1, \mathbf{t}, \mathbf{s}) =$ True, a contradiction.

Case 2: $s_{i^*} = e_{h+1}$ and $t_{i^*} \neq v_{i^*}(e_{h+1})$. For this case, we must have $v_{i^*}(e_{h+1}) \neq 0$; otherwise, item s_{i^*} has zero value for agent i^*, contradicting the property (ii) satisfied by $\mathbf{P}$. Note that $v_{i^*}(P'_{i^*}) = t_{i^*} - v_{i^*}(e_{h+1}) \neq 0$, and accordingly, bundle P'_{i^*} contains items of non-zero value for agent i^*. Denote by e' the item with the least non-zero absolute value in P'_{i^*}. As $\mathbf{P}$ satisfies property (ii) regarding $\mathbf{s}$ and $e' \in P'_{i^*} \subseteq P_{i^*}$, we have $|v_{i^*}(s_{i^*})| \leq |v_{i^*}(e')|$. As for agent i^*'s value, we have $v_{i^*}(P'_{i^*}) = t_{i^*} - v_{i^*}(e_{h+1})$. Consequently, one can verify that allocation $\mathbf{P}'$ makes $B(h, t_1, \ldots, t_{i^*} - v_{i^*}(e_{h+1}), \ldots, t_n, s_1, \ldots, e', \ldots, s_n) = \text{True}$. Thus, when the for-loop in Step 5 of Algorithm 7 is $i = i^*$, Step 10 sets $B(h+1, \mathbf{t}, \mathbf{s}) = \text{True}$, a contradiction.

Case 3: $e_0 = s_{i^*} \neq e_{h+1}$. Note that $s_{i^*} = e_0$ implies $t_{i^*} = v_{i^*}(P_{i^*}) = 0$, and consequently, $v_{i^*}(e_{h+1}) = 0$ holds as $e_{h+1} \in P_{i^*}$. Also, it is evident that $v_{i^*}(P'_{i^*}) = 0$. As the statement holds for the case of $k = h$, it is not hard to verify that $\mathbf{P}'$ makes $B(h, t_1, \ldots, t_{i^*-1}, 0, t_{i^*+1} \ldots, t_n,\ s_1, \ldots, s_{i^*-1}, e_0, s_{i^*+1}, \ldots, s_n) = \text{True}$. Thus, when the for-loop in Step 5 of Algorithm 7 is $i = i^*$, Step 13 sets $B(h+1, \mathbf{t}, \mathbf{s}) = \text{True}$, a contradiction.

Case 4: $e_0 \neq s_{i^*} \neq e_{h+1}$. Note that both items s_{i^*} and e_{h+1} are in P_{i^*}, then $|t_{i^*}| = |v_{i^*}(P_{i^*})| \geq |v_{i^*}(s_{i^*}) + v_{i^*}(e_{h+1})|$. Since agent i^* has a non-zero value on item s_{i^*}, it must hold that $|t_{i^*}| \neq |v_{i^*}(e_{h+1})|$. Since s_{i^*} is the item with the least non-zero absolute value for agent i^* in P_{i^*}, then s_{i^*} is also the item with the least non-zero absolute value in P'_{i^*}, and moreover, either $|v_{i^*}(s_{i^*})| \leq |v_{i^*}(e_{h+1})|$ or $v_{i^*}(e_{h+1}) = 0$ holds. For agent i's value, we have $v_{i^*}(P'_{i^*}) = t_{i^*} - v_{i^*}(e_{h+1})$. Consequently, one can verify that allocation $\mathbf{P}'$ makes $B(h, t_1, \ldots, t_{i^*} - v_{i^*}(e_{h+1}), \ldots, t_n, s_1, \ldots, s_{i^*}, \ldots, s_n) = \text{True}$ with either $0 < |v_{i^*}(s_{i^*})| \leq |v_{i^*}(e_{h+1})|$ or $v_{i^*}(e_{h+1}) = 0$. Thus, when the for-loop in Step 5 of Algorithm 7 is $i = i^*$, Step 16 sets $B(h+1, \mathbf{t}, \mathbf{s}) = \text{True}$, another contradiction. Up to here, the statement also holds when $k = h + 1$.

Overall, with mathematical induction, we have also proved the "if" part of the lemma. □

Consequently, we obtain the following theorem.

Theorem 4.4.2. *Given an instance $\mathcal{I} = \langle [n], E, \mathcal{V} \rangle$, one can compute an* EQX *allocation with the maximum egalitarian welfare in time $O(m^{n+2}V^n)$.*

Proof. Note that Algorithm 7 can return $B(m, \mathbf{t}, \mathbf{s})$ for all $\mathbf{t}$ and $\mathbf{s}$. By visiting the entire $B(m, \mathbf{t}, \mathbf{s})$, we can find the set Γ, of which the construction depends on the underlying items. The full construction of Γ is presented as follows.

$$\Gamma = \begin{cases} \{(\mathbf{t}, \mathbf{s}) | B(m, \mathbf{t}, \mathbf{s}) = \text{True and } t_i + v_j(s_j) \geq t_j \text{ for all } i, j\}, & \text{for goods;} \\ \{(\mathbf{t}, \mathbf{s}) | B(m, \mathbf{t}, \mathbf{s}) = \text{True and } t_i - v_i(s_i) \geq t_j \text{ for all } i, j\}, & \text{for chores.} \end{cases}$$

Given an arbitrary EQX allocation $\mathbf{A}' = (A'_1, \ldots, A'_n)$, construct $(\mathbf{t}', \mathbf{s}')$ as follows: for all i, $t'_i = v_i(A'_i)$ and if $v_i(A'_i) = 0$, then $s'_i = e_0$ and otherwise, $s'_i \in A'_i$ is the item with the least non-zero absolute value for agent i. Then, we have $(\mathbf{t}', \mathbf{s}') \in \Gamma$ due to the property of EQX and the construction of $\mathbf{t}'$ and $\mathbf{s}'$. Accordingly, by visiting all element of Γ, we are able to find the element $(\mathbf{t}^*, \mathbf{s}^*) \in \Gamma$, of which $\mathbf{t}^*$ represents the agents' value in the EQX allocation maximizing egalitarain welfare over all EQX allocations. In particular, one can pursue the $(\mathbf{t}^*, \mathbf{s}^*)$ with $\min_{i\in[n]} t_i^* \geq \min_{i\in[n]} t_i$ for all $(\mathbf{t}, \mathbf{s}) \in \Gamma$. The specific EQX allocation can be found by backtracking $B(m, \mathbf{t}^*, \mathbf{s}^*)$ in the following way: assigning e_m to agent i_m if the value of $B(m, \mathbf{t}^*, \mathbf{s}^*)$ is set to True by $B(m-1, \mathbf{t}^{m-1}, \mathbf{s}^{m-1}) =$ True and at that time the for-loop in Step 5 is $i = i_m$; then assigning e_{m-1} to agent i_{m-1} if the value of $B(m-1, \mathbf{t}^{m-1}, \mathbf{s}^{m-1})$ is set to True by $B(m-2, \mathbf{t}^{m-2}, \mathbf{s}^{m-2}) =$ True and at that time the for-loop in Step 5 is $i = i_{m-1}$; repeat this process until all items are assigned. If in some step, the choice of $B(h, \mathbf{t}^h, \mathbf{s}^h)$ is not unique, then arbitrarily pick one.

As for the time complexity, the running time of Algorithm 6 is $O(m^{n+2}V^n)$, and visiting the entire $B(m, \mathbf{t}, \mathbf{s})$ and backtracking takes time $O(m^n V^n)$. Therefore, the running time of the algorithm is $O(m^{n+2}V^n)$. □

Theorems 4.4.1 and 4.4.2 indicate that both problems of C(EW/EQ1) and C(EW/EQX) can be solved in pseudo-polynomial time. Accordingly, in the case of fixed n, problems C(EW/EQ1) and C(EW/EQX) are weakly NP-hard. Note that for goods, optimal egalitarian welfare is compatible with EQX/EQ1. To address the decision problems of E(EW×EQ1) and E(EW×EQX) for chores, we remark that in Algorithms 6 and 7, tuples $B(m, \mathbf{t}, \mathbf{s})$ actually record all possible values that agents can receive in nearly equitable allocations. Moreover, by eliminating the conditions for parameters $\{s_i\}_{i=1}^n$, it is possible to compute the maximum egalitarian welfare in time $O(mV^n)$ via a dynamic program similar to Algorithm 6. Therefore, in the case of chores, the problems of E(EW×EQ1) and E(EW×EQX) are also weakly NP-hard.

Theorem 4.4.3. *In the allocations of chores,* E(EW×EQ1) *and* E(EW×EQX) *can be answered in time $O(m^{n+2}V^n)$.*

In some resource allocation scenarios, the number of items m to be allocated can be larger than V, and allocating items to agents with dichotomous preferences[2] is a typical example of $m \gg V$. We then further explore the possibility of algorithms whose running time is smaller than $O(m^{n+2}V^n)$ in the case of $m \gg V$. The main strategy is to decrease the running time related to input size m, which

[2]If agent i has dichotomous preference on the set of goods E, then $v_i(e) \in \{0, 1\}$. For a detailed discussion of dichotomous preferences, we refer the reader to Bogomolnaia and Moulin [34] and Bogomolnaia et al. [38].

may inevitably increase the running time caused by V. Below we design pseudo-polynomial time algorithms that can compute the EQ1/EQX allocation with the maximum utilitarian/egalitarian welfare in time $O(mV^{2n+1})$. The algorithms rely on dynamic programming subroutines, Algorithm 8, in which for each $i \in [n]$, we use t_i to represent the value of agent i and p_i to represent the absolute value of a special item in agent i's bundle. The special item is in particular used to examine whether the underlying fairness notion is satisfied or not. To take an example, when considering EQ1, p_i would represent the absolute value of the item with the largest absolute value in agent i's bundle. The dynamic programming subroutines therefore admit two n-dimensional vectors $\mathbf{t} = (t_1, \ldots, t_n)$ with $t_i \in \{-V, \ldots, 0, \ldots, V\}$ and $\mathbf{p} = (p_1, \ldots, p_n)$ with $p_i \in \{0, \ldots, V\}$. For any $\mathbf{t}$ and $\mathbf{p}$, the dynamic programming examines that for each $k \in [m]$ whether the assignments of the first k items can satisfy the constraints regarding $\mathbf{t}$ and $\mathbf{p}$, and returns a tuple $B(k, \mathbf{t}, \mathbf{p})$ that takes value from $\{\text{True}, \text{False}\}$. Informally, given fixed k, $\mathbf{t}$ and $\mathbf{p}$, if there exists an assignment of $e_1, \ldots, e_k$ such that for each $i \in [n]$, agent i receives value t_i and the value of the special item in agent i's bundle is equal to p_i, then the dynamic programming sets $B(k, \mathbf{t}, \mathbf{p}) = \text{True}$; otherwise, $B(k, \mathbf{t}, \mathbf{p}) = \text{False}$.

Lemma 4.4.3. *Given a $B(k, \mathbf{t}, \mathbf{p})$ returned by Algorithm 8, $B(k, \mathbf{t}, \mathbf{p}) =$ True if and only if there exists an allocation $\mathbf{A} = (A_1, \ldots, A_n)$ of $e_1, \ldots, e_k$ such that for all i:*

(i) when considering EQ1*, if $v_i(A_i) = 0$ then $p_i = 0$, and otherwise $p_i = \max_{e \in A_i} |v_i(e)|$; when considering* EQX*, if $v_i(A_i) = 0$ then $p_i = 0$, and otherwise $p_i = \min_{e \in A_i : v_i(e) \neq 0} |v_i(e)|$;*

(ii) $v_i(A_i) = t_i$.

Proof. We first prove the "only if" part by mathematical induction. In the case of $k = 0$, Step 2 of Algorithm 8 sets $B(0, \mathbf{t}, \mathbf{p}) = \text{True}$ only if $t_i = p_i = 0$ for all i. Note that when $k = 0$, each agent receives nothing and has value 0. Thus, the statement holds for the case of $k = 0$. We now assume that the statement holds for $k = 0, \ldots, h$, and show that it also holds for the case of $k = h + 1$. Fix $\mathbf{t}$ and $\mathbf{p}$ with $B(h+1, \mathbf{t}, \mathbf{p}) = \text{True}$ and suppose that Algorithm 8 makes $B(h+1, \mathbf{t}, \mathbf{p}) = \text{True}$ when the for-loop in Step 5 is $i = i^*$. By Steps 7 and 10, it holds that $B(h, t_1, \ldots, t_{i^*} - v_{i^*}(e_{h+1}), \ldots, t_n, p_1, \ldots, p'_{i^*}, \ldots, p_n) = \text{True}$ for some p'_{i^*}. As the statement holds for $k = h$, there exists an allocation $\mathbf{R}$ of $e_1, \ldots, e_h$ satisfying properties (i) and (ii) regarding $(t_1, \ldots, t_{i^*} - v_{i^*}(e_{h+1}), \ldots, t_n)$ and $(p_1, \ldots, p'_{i^*}, \ldots, p_n)$. We now consider

Algorithm 8 Dynamic programming subroutines

Input: An instance $\mathcal{I} = \langle [n], E, \mathcal{V} \rangle$ and a fairness notion $F \in \{\text{EQ1}, \text{EQX}\}$.
Output: Tuple $B(k, \mathbf{t}, \mathbf{p})$ for all k, $\mathbf{t}$ and $\mathbf{p}$.

1: Initialize $B(k, \mathbf{t}, \mathbf{p}) = \text{False}$ for all k, $\mathbf{t}$ and $\mathbf{p}$.
2: Let $B(0, \mathbf{t}, \mathbf{p}) = \text{True}$ if $t_i = 0$ and $p_i = 0$ for all i.
3: **for** $k = 1, \ldots, m$ **do**
4:   **for** all $t_1, \ldots, t_n$ and $p_1, \ldots, p_n$ **do**
5:     **for** $i = 1, \ldots, n$ **do**
6:       **if** $F = \text{EQ1}$ **then**
7:         Set $B(k, t_1, \ldots, t_i, \ldots, t_n, p_1, \ldots, p_i, \ldots, p_n) = \text{True}$ if $B(k-1, t_1, \ldots, t_i - v_i(e_k), \ldots, t_n,\ p_1, \ldots, p_i', \ldots, p_n) = \text{True}$ holds for some p_i' with $\max\{|v_i(e_k)|, p_i'\} = p_i$.
8:       **end if**
9:       **if** $F = \text{EQX}$ **then**
10:         Set $B(k, t_1, \ldots, t_i, \ldots, t_n, p_1, \ldots, p_i, \ldots, p_n) = \text{True}$ if $B(k-1, t_1, \ldots, t_i - v_i(e_k), \ldots, t_n,\ p_1, \ldots, p_i', \ldots, p_n) = \text{True}$ holds for some p_i' satisfying the following condition,

$$\begin{cases} \min\{|v_i(e_k)|, p_i'\} = p_i, & \text{if } v_i(e_k) \neq 0 \text{ and } p_i' \neq 0; \\ \max\{|v_i(e_k)|, p_i'\} = p_i, & \text{otherwise.} \end{cases}$$

11:       **end if**
12:     **end for**
13:   **end for**
14: **end for**

the allocation $\mathbf{R}^*$ of $e_1, \ldots, e_{h+1}$ with $R_j^* = R_j$ for $j \neq i^*$ and $R_{i^*}^* = R_{i^*} \cup \{e_{h+1}\}$. It is not hard to see that $v_j(R_j^*) = t_j$ for any $j \in [n]$.

As for property (i), since each agent $j \neq i^*$ receives an identical bundle in allocations $\mathbf{R}$ and $\mathbf{R}^*$ and the parameter p_j is consistent in the two tuples under consideration, it suffices to prove that property (i) regarding agent i^* is satisfied by the allocation $\mathbf{R}^*$. When considering EQ1, we split the proof into two cases:

Case 1: $p_{i^*}' = 0$. As the allocation $\mathbf{R}$ satisfies property (i) regarding $(p_1, \ldots, p_{i^*}', \ldots, p_n)$, the equality $v_{i^*}(R_{i^*}) = 0$ holds. Based on Step 7, we have $p_{i^*} = \max\{|v_{i^*}(e_{h+1})|, p_{i^*}'\} = |v_{i^*}(e_{h+1})|$. If $v_{i^*}(e_{h+1}) = 0$, then it holds that $v_{i^*}(R_{i^*}^*) = 0$ and $p_{i^*} = 0$, which implies that the allocation $\mathbf{R}^*$ satisfies property (i). If $|v_{i^*}(e_{h+1})| > 0$, we have $p_{i^*} = \max_{e \in R_{i^*}^*} |v_{i^*}(e)|$, and thus property (i) is also satisfied by $\mathbf{R}^*$.

Case 2: $p'_{i^*} > 0$. In this case, the following holds,

$$p_{i^*} = \max\left\{|v_{i^*}(e_{h+1})|, p'_{i^*}\right\} = \max\left\{|v_{i^*}(e_{h+1})|, \max_{e \in R_{i^*}} |v_{i^*}(e)|\right\} = \max_{e \in R^*_{i^*}} |v_{i^*}(e)|,$$

where the first equality is due to Step 7 and the second equality is owing to the construction of $\mathbf{R}$. Thus, property (i) regarding $\mathbf{p}$ is also satisfied by $\mathbf{R}^*$.

Therefore, $\mathbf{R}^*$ is an allocation of $e_1, \ldots, e_{h+1}$ that satisfies properties (i) and (ii) regarding $\mathbf{t}$ and $\mathbf{p}$.

When considering EQX, we also split the proof into two cases.

Case 1: $v_{i^*}(e_{h+1}) = 0$. According to Step 10, we have $p'_{i^*} = p_{i^*}$ since $p'_{i^*} \geq 0$ always holds. If $p'_{i^*} = 0$, then we have $v_{i^*}(R_{i^*}) = 0$, which implies $v_{i^*}(R^*_{i^*}) = 0$ due to $R^*_{i^*} = R_{i^*} \cup \{e_{h+1}\}$ and $v_{i^*}(e_{h+1}) = 0$. Note that $p_{i^*} = p'_{i^*} = 0$, and thus property (i) is satisfied by $\mathbf{R}^*$. If $p'_{i^*} \neq 0$, then we have the following

$$p_{i^*} = p'_{i^*} = \min_{e \in R_{i^*} : v_{i^*}(e) \neq 0} |v_{i^*}(e)| = \min_{e \in R^*_{i^*} : v_{i^*}(e) \neq 0} |v_{i^*}(e)|,$$

where the second equality is due to the construction of $\mathbf{R}$ and the third equality is due to $v_{i^*}(e_{h+1}) = 0$. Thus, the property (i) is also satisfied.

Case 2: $v_{i^*}(e_{h+1}) \neq 0$. If $p'_{i^*} = 0$, then according to Step 10, it holds that $p_{i^*} = |v_{i^*}(e_{h+1})|$. Since $p'_{i^*} = 0$ and $R^*_{i^*} = R_{i^*} \cup \{e_{h+1}\}$, item e_{h+1} is the unique non-zero value item in $R^*_{i^*}$ for agent i. Thus, we have $p_{i^*} = |v_{i^*}(e_{h+1})| = \min_{e \in R^*_{i^*} : v_{i^*}(e) \neq 0} |v_{i^*}(e)|$ and property (i) is satisfied. If $p'_{i^*} \neq 0$, then the following holds

$$\begin{aligned} p_{i^*} = \min\{|v_{i^*}(e_{h+1})|, p'_{i^*}\} &= \min\left\{|v_{i^*}(e_{h+1})|, \min_{e \in R_{i^*} : v_{i^*}(e) \neq 0} |v_{i^*}(e)|\right\} \\ &= \min_{e \in R^*_{i^*} : v_{i^*}(e) \neq 0} |v_{i^*}(e)|, \end{aligned}$$

where the first equality is due to Step 10; the second equality is due to the construction of $\mathbf{R}$; the third equality is due to $v_{i^*}(e_{h+1}) \neq 0$ and $R^*_{i^*} = R_{i^*} \cup \{e_{h+1}\}$. As a consequence, property (i) is also satisfied. Therefore, when considering EQX, $\mathbf{R}^*$ is an allocation of $e_1, \ldots, e_{h+1}$ that satisfies properties (i) and (ii) regarding $\mathbf{t}$ and $\mathbf{p}$.

Overall, by mathematical induction, the "only if" part is proved.

Now let us prove the "if" part, again with mathematical induction. In the case of $k = 0$, Step 2 of Algorithm 8 sets $B(0, \mathbf{t}, \mathbf{p}) =$ True if $t_i = p_i = 0$ for all i. Accordingly, if $B(0, \mathbf{t}', \mathbf{p}') =$ False, then either $t'_q \neq 0$ or $p'_q \neq 0$ holds for some q. Note that the value of an agent can only be zero in the case of $k = 0$. Thus, the statement holds for the case of $k = 0$. We now assume the statement holds for $k = 0, \ldots, h$ and show that it also holds for the case of $k = h + 1$. Fix $\mathbf{t}$ and $\mathbf{p}$ with $B(h + 1, \mathbf{t}, \mathbf{p}) =$ False. For a contradiction, assume that $\mathbf{S}$ is an allocation of $e_1, \ldots, e_{h+1}$ satisfying the properties (i) and (ii) regarding $\mathbf{t}$ and $\mathbf{p}$. Without loss of generality, we assume $e_{h+1} \in S_{i^*}$. Construct an allocation $\mathbf{S}'$ of $e_1, \ldots, e_h$ with $S'_j = S_j$ for $j \neq i^*$ and $S'_{i^*} = S_{i^*} \setminus \{e_{h+1}\}$. In the following, we show that $\mathbf{S}'$ is an allocation that makes $B(h, t_1, \ldots, t_{i^*} - v_{i^*}(e_{h+1}), \ldots, t_n, p_1, \ldots, \widetilde{p}_{i^*}, \ldots, p_n) =$ True for some $\widetilde{p}_{i^*}$ (will be specified later on) satisfying the condition in Steps 7 and 10, which then results in $B(h + 1, \mathbf{t}, \mathbf{p}) =$ True, a desired contradiction. Note that $v_j(S'_j) = v_j(S_j) = t_j$ for $j \neq i^*$ and $v_{i^*}(S'_{i^*}) = t_{i^*} - v_{i^*}(e_{h+1})$, then the property (ii) regarding $(t_1, \ldots, t_{i^*} - v_{i^*}(e_{h+1}), \ldots, t_n)$ is satisfied by $\mathbf{S}'$.

As for property (i), since each agent $j \neq i^*$ receives an identical bundle in allocations $\mathbf{S}$ and $\mathbf{S}'$ and the parameter p_j is consistent in the two tuples under consideration, it suffices to prove that property (i) regarding agent i^* is satisfied by the allocation $\mathbf{S}'$.

When considering EQ1, if $v_{i^*}(S'_{i^*}) = 0$, then let $\widetilde{p}_{i^*} = 0$, and otherwise $\widetilde{p}_{i^*} = \max_{e \in S'_{i^*}} |v_{i^*}(e)|$. It is not hard to verify that the allocation $\mathbf{S}'$ makes $B(h, t_1, \ldots, t_{i^*} - v_{i^*}(e_{h+1}), \ldots, t_n, p_1, \ldots, \widetilde{p}_{i^*}, \ldots, t_n) =$ True. If $\widetilde{p}_{i^*} = 0$, we clearly have $\max\{|v_{i^*}(e_{h+1})|, \widetilde{p}_{i^*}\} = p_{i^*}$. If $\widetilde{p}_{i^*} > 0$, the following holds,

$$\max\{|v_{i^*}(e_{h+1})|, \widetilde{p}_{i^*}\} = \max\left\{|v_{i^*}(e_{h+1})|, \max_{e \in S'_{i^*}} |v_{i^*}(e)|\right\} = \max_{e \in S_{i^*}} |v_{i^*}(e)| = p_{i^*},$$

where the second equality is due to $S_{i^*} = S'_{i^*} \cup \{e_{h+1}\}$. Thus, Step 7 always sets $B(h + 1, \mathbf{t}, \mathbf{p}) =$ True when the for-loop in Step 5 is $i = i^*$, a contradiction.

When considering EQX, we define $\widetilde{p}_{i^*}$ as follows,

$$\widetilde{p}_{i^*} = \begin{cases} 0, & \text{if } v_{i^*}(S'_{i^*}) = 0; \\ \min\limits_{e \in S'_{i^*} : v_{i^*}(e) \neq 0} |v_{i^*}(e)|, & \text{otherwise.} \end{cases}$$

Then one can verify that the allocation $\mathbf{S}'$ makes $B(h, t_1, \ldots, t_{i^*} - v_{i^*}(e_{h+1}), \ldots, t_n, p_1, \ldots, \widetilde{p}_{i^*}, \ldots, p_n) =$ True because the statement holds for $k = h$. In the following, we split the proof into two cases and for each case prove that the condition in Step 10 is satisfied.

Case 1: $v_{i^*}(e_{h+1}) = 0$. Since $S_{i^*} = S'_{i^*} \cup \{e_{h+1}\}$ and $v_{i^*}(e_{h+1}) = 0$, it holds that $p_{i^*} = \max\{|v_{i^*}(e_{h+1})|, \widetilde{p}_{i^*}\}$ no matter whether $\widetilde{p}_{i^*} = 0$ or not.

Case 2: $v_{i^*}(e_{h+1}) \neq 0$. If $\widetilde{p}_{i^*} = 0$, then e_{h+1} is the unique item in S_{i^*}

having non-zero value for agent i^*. Accordingly, we have $p_{i^*} = |v_{i^*}(e_{h+1})| = \max\{|v_{i^*}(e_{h+1})|, \widetilde{p}_{i^*}\}$, and thus the condition of Step 10 is satisfied. If $\widetilde{p}_{i^*} \neq 0$, then the following holds,

$$p_{i^*} = \min_{e \in S_{i^*}: v_{i^*}(e) \neq 0} |v_{i^*}(e)| = \min\left\{|v_{i^*}(e_{h+1})|, \min_{e \in S'_{i^*}: v_{i^*}(e) \neq 0} |v_{i^*}(e)|\right\}$$
$$= \min\{|v_{i^*}(e_{h+1})|, \widetilde{p}_{i^*}\},$$

where the second equality is due to $S_{i^*} = S'_{i^*} \cup \{e_{h+1}\}$; the third equality comes from the definition of $\widetilde{p}_{i^*}$. Thus, in both cases, Step 10 sets $B(h+1, \mathbf{t}, \mathbf{p}) = \text{True}$ when the for-loop in Step 5 is $i = i^*$, a contradiction,

Therefore, with mathematical induction, we have also proved the "if" part of the lemma. □

Theorem 4.4.4. *Given an instance $\mathcal{I} = \langle [n], E, \mathcal{V} \rangle$, one can compute an $F \in \{\text{EQ1}, \text{EQX}\}$ allocation with the maximum $W \in \{\text{EW}, \text{UW}\}$ welfare in time $O(mV^{2n+1})$.*

Proof. Note that Algorithm 8 can return $B(m, \mathbf{t}, \mathbf{p})$ for all $\mathbf{t}$ and $\mathbf{p}$. By visiting the entire $B(m, \mathbf{t}, \mathbf{p})$, we can find the set Γ, of which the construction depends on the underlying items. The full construction of Γ is presented below, and the second condition for both goods and chores is to examine whether the underlying fairness notion is satisfied or not.

$$\Gamma = \begin{cases} \{(\mathbf{t}, \mathbf{p}) | B(m, \mathbf{t}, \mathbf{p}) = \text{True and } t_i + p_j \geq t_j \text{ for all } i, j\}, & \text{for goods;} \\ \{(\mathbf{t}, \mathbf{p}) | B(m, \mathbf{t}, \mathbf{p}) = \text{True and } t_i + p_i \geq t_j \text{ for all } i, j\}, & \text{for chores.} \end{cases}$$

For an arbitrary F allocation $\mathbf{A}' = (A'_1, \ldots, A'_n)$, consider two vectors $\mathbf{t}'$ and $\mathbf{p}'$ with for all i, $t'_i = v_i(A'_i)$, and for p'_i, we distinguish between EQ1 and EQX: for EQ1, if $v_i(A'_i) = 0$ then let $p'_i = 0$, and otherwise let $p'_i = \max_{e \in A'_i} |v_i(e)|$; for EQX, if $v_i(A'_i) = 0$, then let $p'_i = 0$ and otherwise let $p'_i = \min_{e \in A'_i: v_i(e) \neq 0} |v_i(e)|$.

We first claim that $(\mathbf{t}', \mathbf{p}') \in \Gamma$. Based on Lemma 4.4.3, the allocation $\mathbf{A}'$ makes $B(m, \mathbf{t}', \mathbf{p}') = \text{True}$. Moreover, due to the construction of $\mathbf{t}'$ and $\mathbf{p}'$ and the fact that $\mathbf{A}'$ is an F allocations, the second condition is also satisfied. As a consequence, it holds that $(\mathbf{t}', \mathbf{p}') \in \Gamma$. Then, by visiting all elements of Γ, we are able to find the element $(\mathbf{t}^*, \mathbf{p}^*) \in \Gamma$, of which $\mathbf{t}^*$ represents the agents' values in the F allocation maximizing W over all F allocations. In particular, when $W = \text{UW}$ (resp., $W = \text{EW}$), one can pursue the $(\mathbf{t}^*, \mathbf{p}^*)$ with $\sum_{i \in [n]} t^*_i \geq \sum_{i \in [n]} t_i$ (resp., $\min_{i \in [n]} t^*_i \geq \min_{i \in [n]} t_i$) for all $(\mathbf{t}, \mathbf{p}) \in \Gamma$. The specific F allocation that maximizes W over all F allocations can be constructed by backtracking $B(m, \mathbf{t}^*, \mathbf{p}^*)$ in the following way: assigning e_m to agent i_m if the value of $B(m, \mathbf{t}^*, \mathbf{p}^*)$ is set to True by $B(m-1, \mathbf{t}^{m-1}, \mathbf{p}^{m-1}) =$ True and at that time the for-loop in Step 5 is $i = i_m$; then assigning e_{m-1} to agent i_{m-1} if the value of $B(m-1, \mathbf{t}^{m-1}, \mathbf{p}^{m-1})$ is set to True by $B(m-2, \mathbf{t}^{m-2}, \mathbf{p}^{m-2}) =$ True and at that time the for-loop in Step 5 is $i = i_{m-1}$; repeat this process until all items are assigned. If in some step, the choice of $B(h, \mathbf{t}^h, \mathbf{p}^h)$ is not unique, then arbitrarily pick one.

As for the time complexity, the running time of Algorithm 8 is $O(mV^{2n+1})$, and visiting all $B(m, \mathbf{t}, \mathbf{p})$ and backtracking the specific one takes time $O(mV^{2n})$. Therefore, the running time of the algorithm is $O(mV^{2n+1})$. $\square$

For the decision problems, note that the maximum utilitarian welfare can be computed in linear time, and accordingly, problems E(UW×EQ1) and E(UW×EQX) can be answered in time $O(mV^{2n+1})$.

Theorem 4.4.5. *When allocating goods and chores, problems* E(UW×EQ1) *and* E(UW×EQX) *can be answered in time* $O(mV^{2n+1})$*.*

When considering egalitarian welfare, recall that for goods, decision problems E(EW×EQ1) and E(EW×EQX) trivially have answer "yes". For the allocation of chores, the maximum egalitarian welfare can be computed in time $O(mV^n)$ via a dynamic program similar to Algorithm 8 (eliminating $\mathbf{p}$ and the corresponding conditions). Consequently, for chores, the problems of E(EW×EQ1) and E(EW×EQX) can also be answered in time $O(mV^{2n+1})$.

Theorem 4.4.6. *When allocating chores, problems* E(EW×EQ1) *and* E(EW×EQX) *can be answered in time* $O(mV^{2n+1})$*.*

We remark that, when considering egalitarian welfare, we have presented two pseudo-polynomial algorithms, one using Algorithms 6 and 7 as subroutines, and the other using Algorithm 8 as subroutines. The two types of algorithms are mutually non-dominating in terms of running time. Specific resource allocation problems determine which one is more efficient.

4.5 Conclusions

In this chapter, we have conducted an analysis on both indivisible goods and indivisible chores and studied two notions of relaxed equitability, namely EQX and EQ1, together with efficiency measured by two welfare allocations, utilitarian and egalitarian welfare. On every pairwise fairness and welfare combination, we have provided (almost) tight results on the price of fairness. In the case of chores, to achieve relaxed equitability, almost all efficiency would be sacrificed, while the prices of fairness in goods allocation are all bounded. Particularly, with two agents, fairness can be achieved at the cost of at most half of welfare in goods allocations; however, in the case of chores, fair allocations cannot have a bounded guarantee on welfare. Our results on the price of fairness somewhat reflect the differences between goods and chores.

From the results on the price of fairness, relaxed equitability is not always compatible with optimal social welfare, which motivates us to investigate whether one can efficiently determine the existence of a fair and welfare-maximizing allocation and compute the one with maximum welfare among fair allocations. We have depicted a complete picture of the computational complexity of all decision and optimization problems. In particular, when utilitarian welfare is concerned, all decision and optimization problems are strongly NP-complete or strongly NP-hard for general n agents. For the case of fixed n, except for E(UW×EQ1) with two agents, other problems are still intractable in polynomial time. On the positive side, we are able to propose a pseudo-polynomial time algorithm that output the fair allocation with the maximum utilitarian welfare. For problem E(UW×EQ1) with two agents, a polynomial time algorithm exists. When focusing on egalitarian welfare, EQX and EQ1 are compatible with optimal egalitarian welfare in goods allocations. On the contrary, in the case of chores, deciding the existence of EQX (resp., EQ1) and egalitarian welfare-maximizing allocation is strongly NP-hard for general n and weakly NP-complete for fixed $n \geq 2$ (resp., $n \geq 3$). Both goods and chores versions for optimization problems are strongly NP-hard for general n and weakly NP-hard for fixed n. Our results indicate that although goods and chores yield different results in the problem where the numerical value of input matters, for example, price of fairness, they may have similar features in terms of computational complexity.

Chapter 5

Allocating Indivisible Items to Strategic Agents

5.1 Introduction

Chapters 3 and 4 are concerned with questions of fairly dividing indivisible items in the environment of complete information, i.e., agents' valuation functions are publicly known. Whereas in practice, this kind of information, such as the preference of each agent, is only known by himself. A canonical example is an auction [53, 89, 103], the problem of deciding how to allocate objects to potential bidders while the value of objects to a bidder can only be accessed by the bidder himself. Agents are selfish and will behave untruthfully by reporting the false value if this serves their own ends, which then deteriorates the auction outcomes. To circumvent the challenge of incomplete information, it is desirable to have a mechanism that can take bidders' self-interest into account so that they can never be better off by behaving strategically. With the emergence of the Internet as the platform of computation, the requirement of ensuring *truthfulness*, also known as *strategyproofness*, has also drawn the attention of the computer science community. Computer scientists observe that with the presence of false information or input, even the most efficient algorithm may lead to unreasonable solutions unless it is designed to cope with strategic behaviours. Nisan and Ronen [90] then initiated the study of algorithmic mechanism design, pursuing the efficient algorithm that provides incentives to agents for being truthful. Within the scope of algorithmic mechanism design, a line of research [10, 52, 58] uses the money to give incentives to agents and achieve the expected algorithmic outcomes. However, in some real-life resource allocation scenarios, such as allocating course seats to students in the university, and organs to patients, money transfer is impossible and sometimes illegal. This then leaves the question of when limiting or even prohibiting the usage of money, can strategyproofness still be achieved or even together another (algorithmic) objective?

The study of mechanism design without money is arguably initiated by Procaccia and Tennenholtz [95], which studies how to select the location of a public facility in a real line. After that, mechanism design without money draws the attention of the fair division community. In the allocations of divisible items (cake-cutting), some significant progress has been achieved [26, 29, 55], while in the setting of indivisible items, positive results are seldom established, even when agents' valuations are assumed to be additive [5, 6]. To escape from the strong impossibility results, one possible way is to shrink the domain of agents' valuation functions. In the allocations of goods, Halpern et al. [75], Babaioff et al. [21], Barman and Verma [25] design strategyproof mechanisms achieving envy-based fairness by assuming that the marginal value of a single item is *binary*, i.e., either 0 or 1. When allocating chores, Aziz et al. [18] ask agents to report their ranking, an ordinal preference, over items and design strategyproof mechanisms that achieve good approximations of share-based fairness.

Compared to what has been achieved in goods allocation, mechanism design without money for fair division in the settings of (i) items are all chores, (ii) items are mixtures of goods and chores, is not well explored. In particular, it is still unknown whether there exist strategyproof and (approximately) fair mechanisms that can elicit agents' cardinal preferences. Note that strategyproofness in itself is not difficult to achieve, and for instance, one can totally ignore the agents' reporting and assign all items to a predetermined agent. However, the outcome of such a mechanism can be highly unfair and inefficient. This then motivates us to ask the challenging question:

> *When items are chores or are mixtures of goods and chores, whether it is possible to find strategyproof mechanisms (without money) that can output fair and efficient outcomes?*

5.1.1 Related Works

The search for a mechanism that satisfies truthfulness, also known as strategyproofness, efficiency, and other prescribed distributional objectives, dates back to work by Hylland and Zeckhauser [80]. Then, Gale [68] raises the question of whether we can find a "nice" mechanism that can satisfy strategyproofness, efficiency, and other distributional properties, simultaneously. The follow-up paper by Zhou [106] answers Gale's conjecture and proves that in the problem of assigning n objects to n agents (with von Neumann-Morgenstern utility), no mechanism can be strategyproof, Pareto optimal and symmetric[1] when $n \geq 3$. Pápai [92] then studies the problem of allocating a single indivisible item to several agents without the money transfer, and examines possible extra properties of strategyproof mechanisms can have. It indicates that along with strategyproofness, three other criteria, namely Pareto efficiency, non-dictatorship[2], and non-bossiness[3], can not be satisfied simultaneously, while any two of the additional criteria can be satisfied together with strategyproofness. Pápai [91] considers the case where agents can receive more than one object. She shows that an allocation rule is strategyproof, *non-bossy*, and satisfies citizen sovereignty if and only if it is a *sequential dictatorship*.

In recent two decades, a great deal of results has been established by the computer science community. Bezáková and Dani [32] consider the setting of two players and indicate that no truthful mechanism can achieve exact max-min fairness. Markakis and Psomas [86] provide the lower bound of the value of the agent in the worst-case scenario and also show that no deterministic mechanism can achieve a (2/3)-approximation of the worst-case bound. For the share-based fairness notions, such as MMS fairness, Amanatidis et al. [6] provide a truthful deterministic mechanism (on goods) with $O(m)$-approximation on MMS fairness and complement their result by showing that no mechanism can achieve better than (1/2)-MMS. In the case of chores, Aziz et al. [18] assume that agents' valuations are ordinal and present deterministic and randomized mechanisms that output $O(\log(m/n))$-MMS and $O(\log(\sqrt{n}))$-MMS allocations, respectively.

As for envy-based fairness, Amanatidis et al. [5] show that no deterministic strategyproof mechanism can always achieve envy-free up to one item (EF1), even with two agents. Thereafter, in order to achieve positive results regarding envy-based fairness notions, the assumption that the marginal of every single item is binary for any agent has been incorporated. Halpern et al. [75] show that when agents' valuations are binary additive, the rule of maximizing Nash welfare with a lexicographic tie-breaking is group strategyproofness (a stronger notion requiring no group of the agent can misreport to increase values), EF1 and PO. Then, Babaioff

[1] A notion weaker than anonymity. In a symmetry mechanism, agents with same bid should receive identical (expected) utility.

[2] A mechanism is dictatorial if there exists an agent i who receives the item whenever reporting the item as desirable. Such an agent i is called dictator.

[3] When the preference of some agent i is changed, if the bundle received by agent i is unchanged, then nobody else's assignment should change.

et al. [21] consider a broader preference domain, submodular, monotone and binary margin (the matroid rank function), and indicate that the Lorenz dominating[4] rule is strategyproof, EF1 and PO when agents' valuations are matroid rank functions. A follow-up paper [25] strengthens those results and shows that the mechanism proposed by Babaioff et al. [21] is indeed group strategyproof.

5.2 Results under Deterministic Setting

In this section, we are concerned with the allocations of chores and focus on deterministic mechanisms. Recall that the existing works for goods already show strong impossibility results when agents have additive valuations. We will then start with the special case, the absolute margin value of an item being either 0 or 1, in which some positive results have been achieved on the allocations of goods [21, 25, 75]. In particular, we study binary additive valuations, in which $v_i(e) \in \{0, -1\}$ for any $i \in [n]$ and $e \in E$, and moreover, for any subset $S \subseteq E$, agent i has a value $v_i(S) = \sum_{e \in S} v_i(e)$. While the binary additive valuation, at first glance, seems quite simple, this preference domain already leads to significant and challenging allocation questions [34, 38, 88].

5.2.1 A Strategyproof and Efficient Mechanism

One important class of existing deterministic mechanisms, *sequential picking*, originated by Kohler and Chandrasekaran [82], has already been widely studied and implemented in the fair division of indivisible items [6, 41, 104]. A crucial subclass of sequential picking is that; agents are ordered in advance, and each agent i picks a number $t_i \geq 0$ items in his turn. With $\sum_{i \in [n]} t_i = m$, this mechanism is capable of assigning all items to agents. The advantage of sequential picking is that this mechanism is strategyproof once $\{t_i\}_{i=1}^m$ is pre-determined (not related to the bids). The truthfulness comes from the fact that each agent i only has one chance to pick, and her optimal strategy is receiving top t_i items from the remaining, which leaves no incentive to misreport. For simplicity, we call this subclass of sequential picking, *SEQ-pick*. Although each SEQ-pick mechanism is strategyproof, we then show that none of them is Pareto optimal.

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[4]For two ascending-ordered vectors $\mathbf{a}$ and $\mathbf{b}$, the Lorenz domination partial order is defined as $\mathbf{a} \succ_{Lorenz} \mathbf{b}$ if for every k, the sum of the first k entries of $\mathbf{a}$ is at least as large as that of $\mathbf{b}$. A Lorenz dominating allocation is the allocation whose valuation vector Lorenz dominates the vector of every other allocation.

Proposition 5.2.1. *In the allocations of chores, no SEQ-pick mechanism is Pareto optimal, even when agents have binary additive valuation functions.*

Proof. Without loss of generality, we assume that agents are ordered $1, \ldots, n$ and agent 1 is the first to pick. Note that every SEQ-pick mechanism can be characterized by a sequence of number $t_1, \ldots, t_n$ with $\sum_{i \in [n]} t_i = m$. The sequence is pre-determined and is not affected by agents' bids. Fix such an sequence $\{t_i\}_{i=1}^n$. If $t_1 < m$, then consider an instance $\mathcal{I}_1$ where the type of agent 1 is $v_1(e) = 0$ for all $e \in E$ and the type of agent $i \geq 2$ is $v_i(e) = -1$ for all $e \in E$. Instance $\mathcal{I}_1$ admits an allocation in which all agents have value zero. But in the allocation returned by a SEQ-pick with $t_1 < m$, some agents receive negative value, and consequently, the returned allocation is not Pareto optimal. Note that the Pareto optimality requires $a_1 = m$. However, any SEQ-pick mechanism with $a_1 = m$ can not output Pareto optimal allocation for the instance $\mathcal{I}_2$ in which the type of agent 1 is $v_1(e) = -1$ for all $e \in E$ and the type of agent $i \geq 2$ is $v_i(e) = 0$ for all $e \in E$. Therefore, no SEQ-pick mechanism can always return PO allocations for both $\mathcal{I}_1$ and $\mathcal{I}_2$. □

Proposition 5.2.1 indicates that the existing approach fails to guarantee Pareto optimality, even when agents' valuations are binary additive. Next, we present a deterministic mechanism (see Algorithm 9) that is both strategyproof and Pareto optimal. The mechanism assigns the items based on a pre-determined order of agents. According to the order, the first $n-1$ agents, in turn, receive the set of items for which they bid zero, then the remaining items are assigned to the last agent.

Algorithm 9 SP-PO(σ)

Input: A fair-chores instance $\mathcal{I} = \langle [n], E, \mathcal{V} \rangle$ with binary additive valuations and a permutation σ of $[n]$.

```
Every agent i bids b_i.
for t = σ(1), σ(2), ..., σ(n − 1) do
    A_t ← {e ∈ E | b_i(e) = 0} and E ← E \ A_t.
end for
A_σ(n) ← E.
return Allocation A
```

Theorem 5.2.1. *The mechanism* SP-PO *is strategyproof and Pareto efficient.*

Proof. We first show the strategyproofness of SP-PO. Without loss of generality, we assume that the permutation σ is defined as $\sigma(k) = k$ for $k \in [n]$. For an agent $i \leq n-1$, she has no incentive to misreport as if she reports her type, she would receive a value zero, the maximum achievable value. As for agent n, her bundle is entirely determined by other agents' bids b_{-n}. In other words, agent n can not change her bundle by reporting differently once b_{-n} has been fixed, and hence, she has no incentive to lie. Hence, every SP-PO mechanism is strategyproof.

We now prove that SP-PO can always return Pareto optimal allocations. Let $\mathbf{A}$ be the returned allocation. Given a type profile $\{v_i\}_{i=1}^n$, we construct the set $E_c = \{e \in E \mid v_i(e) = -1, \forall i \in [n]\}$; that is, each item $e \in E_c$ (if $E_c \neq \emptyset$) results in value -1 for all agents. As agents will bid truthfully in SP-PO, according to Steps 3 and 5, we have $v_i(A_i) = 0$ for all $i \in [n-1]$, $v_n(A_n) = -|E_c|$ and moreover $E_c \subseteq A_n$. Suppose $\mathbf{A}$ is Pareto-dominated by another allocation $\mathbf{B}$, and accordingly, $v_n(B_n) \geq -|E_c| + 1$ must hold. As a consequence, there exists an agent $i \neq n$ with $B_i \cap E_c \neq \emptyset$, which implies $v_i(B_i) \leq -1$, which together with $v_i(A_i) = 0$ contradicts the fact that $\mathbf{A}$ is Pareto-dominated by the allocation $\mathbf{B}$. □

5.2.2 Incorporate Additional Properties

As we have a positive answer on the existence of SP and PO deterministic mechanisms, we now investigate whether SP and PO mechanisms can additionally satisfy other distributional properties. For the notion of anonymity, we have a simple observation; that is, anonymity can not be satisfied by deterministic mechanisms, even without the strategyproofness requirement. Such an impossibility is due to the nature of indivisibility. One can think of an example: allocating three items e_1, e_2, e_3 to two agents with identical valuation functions $v_i(e_j) = -1$ for any i, j. Suppose that a deterministic mechanism $\mathcal{M}$ takes the type profile as input and returns the allocation $\mathbf{A}$, then it must hold that $A_1 \neq A_2$ as there exist only three items. Given a permutation σ with $\sigma(1) = 2$ and $\sigma(2) = 1$, implementing σ on the type profile results in another profile $(v_{\sigma(1)}, v_{\sigma(2)})$, identical to (v_1, v_2), and hence, $\mathcal{M}$ should also return $\mathbf{A}$. However, for any agent i, the bundle she receives in $\mathcal{M}(v_1, v_2)$ is different from that in $\mathcal{M}(v_{\sigma(1)}, v_{\sigma(2)})$.

We next examine whether fairness is compatible with SP and PO. Within this scope, we have an impossibility result on the notion of EQ1.

Theorem 5.2.2. *In the allocations of chores, no deterministic mechanism is* SP*,* PO *and* EQ1*, even when agents have binary additive valuation functions.*

Proof. For a contradiction, let $\mathcal{M}$ be a deterministic SP, PO and EQ1 mechanism. Consider an instance $\mathcal{I}$ with two agents and a set E of four items e_1, e_2, e_3, e_4. For a reporting profile $\mathbf{b} = (b_1, b_2)$ with $b_i(e_j) = -1$ for all i, j, the mechanism $\mathcal{M}$ has to assign each agent two items so that the final allocation can be EQ1 when agents bid truthfully. Without loss of generality, we assume $\mathcal{M}(\mathbf{b}) = \mathbf{A}$ with $A_1 = \{e_1, e_2\}$ and $A_2 = \{e_3, e_4\}$. We now consider another reporting profile $\mathbf{b}' = (b'_1, b'_2)$ where $b'_2 = b_2$ and $b'_1(e_1) = b'_1(e_2) = 0$ and $b'_1(e_j) = -1$ for $j = 3, 4$. Let $\mathcal{M}(\mathbf{b}') = \mathbf{A}'$. Note that the allocation $\mathbf{A}'$ needs to be Pareto optimal when b'_i is the type of each agent i, and accordingly, items e_1, e_2 have to be assigned to agent 1. Since the allocation $\mathbf{A}'$ also satisfies EQ1, each agent i should be assigned one item from $\{e_3, e_4\}$.

We now assume that for each $i \in [2]$, b'_i is the type for agent i, i.e., $b'_i = v_i$. Then, if both agents report truthfully, $\mathcal{M}$ returns the allocation $\mathbf{A}'$ and $v_i(A'_1) = -1$ for all $i = 1, 2$. However, if agent 1 misreports b_1, then the outcome becomes $\mathcal{M}(b_1, b'_2) = \mathbf{A}$, and the value of agent 1 in the allocation $\mathbf{A}$ is equal to $v_1(A_1) = 0$. Therefore, agent 1 has incentive to lie, contradicting the strategyproofness of $\mathcal{M}$. $\square$

5.3 Results under randomized Setting

Having provided impossibilities of deterministic mechanisms, we thereafter incorporate lotteries and study randomized mechanisms. As shown by our results in this section, implementing lotteries in mechanisms allows us to escape from impossibility results. Recall that a randomized mechanism $\widetilde{\mathcal{M}}$ returns a randomized allocation, which can also be interpreted as a distribution over a set of deterministic allocations. Before presenting the main results, we formally define the *ex-ante* and *ex-post* fairness/efficiency of a randomized allocation.

Definition 5.3.1. *(ex-ante and ex-post) Given a fairness or efficiency property P, a randomized allocation $\widetilde{\mathbf{A}}$ is ex-ante P if its corresponding fractional allocation is P, and is ex-post P, if every deterministic allocation in its support is P.*

Definition 5.3.2. *Given properties P_1, P_2, a randomized mechanism $\widetilde{\mathcal{M}}$ is ex-ante P_1 and ex-post P_2 if it always returns ex-ante P_1 and ex-post P_2 randomized allocations.*

Note that SPIE and GSPIE (see Definitions 2.4.2 and 2.4.4) are ex-ante in the sense that agents may regret behaving truthfully after the realization of the final assignment. For Pareto optimality, we remark that ex-ante PO allocation is also ex-post PO[5]. Instead, differently from Pareto optimality, not every ex-ante fair

[5]Denote by $\widetilde{\mathbf{A}}$ an ex-ante Pareto optimal randomized allocation with support $\mathbf{A}^1, \ldots, \mathbf{A}^k$ and probability p_i on the deterministic allocation $\mathbf{A}^i$. Suppose $\widetilde{\mathbf{A}}$ is not ex-post Pareto optimal, then

solution can have a good ex-post fairness guarantee. To take an example, consider a mechanism that picks an agent uniformly at random and assigns all items to that agent. The returned randomized allocation is ex-ante envy-free and equitable but has poor performance on ex-post EF and EQ. Whereas we pursue a mechanism that guarantees both ex-ante and ex-post fairness, so-called "best of both world" [20, 65]. In summary, our task is to design randomized mechanisms that are (group) strategyproof, ex-ante Pareto optimal, and can be best of both world from the perspective of fairness.

5.3.1 A Strategyproof, Efficient, and Fair Randomized Mechanism

The main result of this section is a randomized mechanism for chores when agents have *restricted additive* valuation functions that have been studied by both Bezáková and Dani [32] and Asadpour et al. [11] in designing the approximation algorithm for maximin fairness. The restricted additive preference domain is slightly larger than the domain of binary additive. Specifically, a valuation function is said to be *restricted additive* if each item e_j has an inherent value $v(e_j) \neq 0$ and the value of item e_j for every agent i is either 0 or $v(e_j)$, i.e., $v_i(e_j) \in \{0, v(e_j)\}$. Given a subset $S \subseteq E$, the value of agent i on S is equal to $v_i(S) = \sum_{e_j \in S} v_i(e_j)$. We consider the setting where the inherent value is common knowledge, and hence, the central decision marker can access the $v(e_j)$ for all $j \in [m]$. As a consequence, for any pair of i, j, the reported value on e_j by agent i can only be either 0 or $v(e_j)$. Denote by $\mathbf{b} = (b_1, \ldots, b_n)$ a reported profile where $b_i(\cdot)$ refers to the valuation function reported by agent i.

Algorithm 10 $\widetilde{\mathcal{M}}^*$

Input: A fair-chores instance $\mathcal{I}$ with restricted additive valuation functions.

1: Every agent i bids $b_i(\cdot)$.
2: Let $Q = \{q \in [m] \mid \exists i \text{ such that } b_i(e_q) = 0\}$ and $\bar{Q} = [m] \setminus Q$.
3: For every $q \in Q$, uniformly randomly assign e_q to an agent who reports zero value on it.
4: Let $\bar{Q} = \{l_1, l_2, \ldots, l_k\}$ with reporting inherent values $b(e_{l_1}) \geq \cdots \geq b(e_{l_k})$.
5: Let σ be a uniformly random permutation of $\{1, 2, \ldots, n\}$. According to σ, assign the chore with the largest inherent value from the remaining to an agent each time, until all chores are assigned. If there is a tie on the largest value, pick e_{l_j} with the smallest j.
6: **return** A randomized allocation $\widetilde{\mathbf{A}}$

assume $\mathbf{A}^1$ is Pareto dominated by the deterministic allocation $\mathbf{A}'$, which implies the fractional allocation $p_1\mathbf{A}' + \sum_{j=2}^{k} p_j \mathbf{A}^j$ Pareto dominates $\widetilde{\mathbf{A}}$, a contradiction.

Intuitively, $\widetilde{\mathcal{M}}^*$ (see Algorithm 10) first assigns items for which some agent bids zero, and then allocates the remaining items in a round-robin fashion; that is, agents, in turn, receive one item. In what follows, we first show that the mechanism $\widetilde{\mathcal{M}}^*$ is GSPIE and ex-ante PO, and then present distributional properties, such as fairness satisfied by $\widetilde{\mathcal{M}}^*$.

Proposition 5.3.1. *The mechanism $\widetilde{\mathcal{M}}^*$ is* GSPIE*.*

The proof of Proposition 5.3.1 relies on following lemmas.

Lemma 5.3.2. *Step 5 of Algorithm 10 assigns the expected value $\frac{1}{n}v_i(\bigcup_{q\in\bar{Q}} e_q)$ to every agent $i \in [n]$.*

Proof. Fix i. Since σ is a uniformly random permutation of $\{1, 2, \dots, n\}$, for any pair of i, j, the probability of agent i on position $\sigma(j)$ is $1/n$. And according to Step 5, if agent i is in position $\sigma(j)$, then she receives value exactly $v_i(e_{l_j} \cup e_{l_{n+j}} \cup \dots \cup e_{l_{\lfloor \frac{|\bar{Q}|-j}{n} \rfloor n+j}})$. Thus, her expected value from the assignment of $\cup_{q\in\bar{Q}} e_q$ is

$$\sum_{j=1}^{n} \frac{1}{n} v_i(\bigcup_{p=0}^{\lfloor \frac{|\bar{Q}|-j}{n} \rfloor} e_{l_{pn+j}}) = \frac{1}{n} v_i(\bigcup_{j=1}^{n} \bigcup_{p=0}^{\lfloor \frac{|\bar{Q}|-j}{n} \rfloor} e_{l_{pn+j}}) = \frac{1}{n} v_i(\bigcup_{q\in\bar{Q}} e_q),$$

where the first equation is due to the additivity of $v_i(\cdot)$. □

The lemma below states that none of the agents in a coalition $S \subseteq N$ can be strictly better off by misreporting $b_i(e) = v(e) \neq 0$ for some item e, while her true value of e is $v_i(e) = 0$.

Lemma 5.3.3. *Given a subset $S \subseteq N$ and a reporting profile (b_S, b_{-S}), $b_S \neq v_S$ implies*

$$\mathbb{E}_{\mathbf{A}\sim\widetilde{\mathcal{M}}^*(b_S,b_{-S})}[v_i(A_i)] \leq \mathbb{E}_{\mathbf{A}\sim\widetilde{\mathcal{M}}^*(b'_S,b_{-S})}[v_i(A_i)],$$

for each $i \in S$ where b'_S is defined as: for each $i \in S$, if $v_i(e_q) = 0$, then $b'_i(e_q) = 0$; otherwise, $b'_i(e_q) = b_i(e_q)$.

Proof. Let $Q, \bar{Q}$ and $Q', \bar{Q}'$ be the corresponding index sets constructed in Step 2 of Algorithm 10 with reporting profiles (b_S, b_{-S}) and (b'_S, b_{-S}), respectively. For every $i \in S$, let $P_i = \{e \in E \mid b_i(e) = 0\}$ and $P'_i = \{e \in E \mid b'_i(e) = 0\}$. For each $i \in S$ and $e \in P_i$, if $v_i(e) = 0$, due to the construction of b'_i, we have $b'_i(e) = 0$ that implies $e \in P'_i$. If $v_i(e) \neq 0$, again from the definition of b'_i, we have $b'_i(e) = b_i(e) = 0$, implying $e \in P'_i$. Accordingly, $P_i \subseteq P'_i$ holds for all $i \in S$, based on which we have $Q \subseteq Q'$, equivalent to $\bar{Q}' \subseteq \bar{Q}$.

Note that the expected value received by an agent comes from exactly two parts, the assignment of Step 3 and 5 of Algorithm 10. Define $C_i = \{e \in E \mid v_i(e) \neq 0\}$. If $C_i \cap P_i \neq \emptyset$ (resp. $C_i \cap P_i' \neq \emptyset$), agent i would receive a negative expected value from the assignment of $C_i \cap P_i$ (resp. $C_i \cap P_i'$) under the reporting profile (b_S, b_{-S}) (resp. (b_S', b_{-S})). Recall that $P_i \subseteq P_i'$ and thus $C_i \cap P_i \subseteq C_i \cap P_i'$. For each $e \in C_i \cap P_i'$, we have $b_i'(e) = 0$ and due to the construction of $b_i'(\cdot)$, $b_i(e) = 0$ holds, which then implies $e \in C_i \cap P_i$. Thus, we claim that $C_i \cap P_i = C_i \cap P_i'$ for each $i \in S$. Then, for each $e \in C_i \cap P_i$, as the situation of $b_i'(e) = v(e_j)$ but $b_i(e) = 0$ can never happen, the number of agents reporting zero on e_j in (b_S, b_{-S}) can not exceed that number under the reporting profile (b_S', b_{-S}). For each $e \in C_i \cap P_i$, mechanism $\widetilde{\mathcal{M}}^*$ uniformly randomly assigns e to an agent reporting zero on it, then the probability of assigning e to agent i with input (b_S, b_{-S}) is no less than the probability of assigning e to agent i when agents bid (b_S', b_{-S}). Consequently, the expected value received by each agent $i \in S$ from the assignment of Step 3 would not decrease when the set of agents S deviate their bids from b_S to b_S'.

According to Lemma 5.3.2, the assignment of Step 5 results in the expected value $\frac{1}{n} v_i(\cup_{q \in \bar{Q}'} e_q)$ and $\frac{1}{n} v_i(\cup_{q \in \bar{Q}} e_q)$ to each agent $i \in S$ under the reporting profiles (b_S', b_{-S}) and (b_S, b_{-S}), respectively. Recall that $\bar{Q}' \subseteq \bar{Q}$, we have $\frac{1}{n} v_i(\cup_{q \in \bar{Q}'} e_q) \geq \frac{1}{n} v_i(\cup_{q \in \bar{Q}} e_q)$ because the underlying items are chores. Therefore, we can conclude that the expected value received by each agent $i \in S$ under (b_S', b_{-S}) is no less than that under the reporting profile (b_S, b_{-S}). $\square$

Lemma 5.3.4. *Given a subset $S \subseteq N$ and bids b_{-S}, the summation of the expected value received by agents in S is maximized when every agent $i \in S$ reports truthfully.*

Proof. For a contradiction, assume that the set of agents S can misreport $b_S \neq v_S$ such that

$$\sum_{i \in S} \mathbb{E}_{\mathbf{A} \sim \widetilde{\mathcal{M}}^*(b_S, b_{-S})}[v_i(A_i)] > \sum_{i \in S} \mathbb{E}_{\mathbf{A} \sim \widetilde{\mathcal{M}}^*(v_S, b_{-S})}[v_i(A_i)].$$

We then construct another set of bids b_S' as follows: for each $i \in S$, if $v_i(e) = 0$, then $b_i'(e) = 0$; otherwise $b_i'(e) = b_i(e)$. Then, according to Lemma 5.3.3, we have $\mathbb{E}_{\mathbf{A} \sim \widetilde{\mathcal{M}}^*(b_S', b_{-S})}[v_i(A_i)] \geq \mathbb{E}_{\mathbf{A} \sim \widetilde{\mathcal{M}}^*(b_S, b_{-S})}[v_i(A_i)]$ for all $i \in S$. Consequently, we can further assume that for each $i \in S$ if $v_i(e) = 0$, then $b_i(e) = 0$. In other words, bids b_S only contain one type of misreporting; that is, some agent $i \in S$ bids $b_i(e) = 0$ while the true value is $v_i(e) = v(e) < 0$.

Let $Q, \bar{Q}$ and $Q^b, \bar{Q}^b$ be the corresponding index sets constructed in Step 2 of Algorithm 10 when agents bid (v_S, b_{-S}) and (b_S, b_{-S}), respectively. For every $i \in S$, construct $P_i = \{e \in E \mid v_i(e) = 0\}$ and $P_i^b = \{e \in E \mid b_i(e) = 0\}$. Based on what we just assumed, for $i \in S$ and $e \in P_i$, we have $e \in P_i^b$, which then implies $Q \subseteq Q^b$, equivalent to $\bar{Q}^b \subseteq \bar{Q}$.

If $\bar{Q}^b = \bar{Q}$, then we have

$$\begin{aligned}\sum_{i\in S} \mathbb{E}_{\mathbf{A}\sim\widetilde{\mathcal{M}}^*(v_S,b_{-S})}[v_i(A_i)] = \sum_{i\in S}\frac{1}{n}v_i(\bigcup_{q\in\bar{Q}} e_q) = \sum_{i\in S}\frac{1}{n}v_i(\bigcup_{q\in\bar{Q}^b} e_q) \\ \geq \sum_{i\in S} \mathbb{E}_{\mathbf{A}\sim\widetilde{\mathcal{M}}^*(b_S,b_{-S})}[v_i(A_i)],\end{aligned}$$

where the first equation is due to the fact that the assignment of $\bigcup_{q\in Q} e_q$ results in value zero on every agent $i \in S$ under the reporting profile (v_S, b_{-S}) and the inequality is due to that items are chores. The above inequality derives a contradiction.

If $\bar{Q}^b \subsetneq \bar{Q}$, then item e_q with $q \in \bar{Q} \setminus \bar{Q}^b$ must be assigned to some agents in S under (b_S, b_{-S}) because agents in $N \setminus S$ always report b_{-S} and the only possibility is that some agents in S bid zero on e_q in (b_S, b_{-S}). Moreover, for every $i \in S$ and $e \in E$, if $v_i(e) = 0$, then $b_i(e) = 0$ and hence, e must be assigned in Step 3. As a result, for each e_q with $q \in \bar{Q}$, every agent $i \in S$ has value $v_i(e_q) = v(e_q) < 0$. When agents in S report truthfully, the assignment of Step 3 results in the value zero for all $i \in S$ and accordingly, in this case, the expected value received by agent $i \in S$ entirely comes from the assignment of Step 5. Then, we have the following

$$\begin{aligned}\sum_{i\in S} \mathbb{E}_{\mathbf{A}\sim\widetilde{\mathcal{M}}^*(v_S,b_{-S})}[v_i(A_i)] &= \sum_{i\in S}\frac{1}{n}v_i(\bigcup_{q\in\bar{Q}^b} e_q) + \sum_{i\in S}\frac{1}{n}v_i(\bigcup_{q\in\bar{Q}\setminus\bar{Q}^b} e_q) \\ &= \sum_{i\in S}\frac{1}{n}v_i(\bigcup_{q\in\bar{Q}^b} e_q) + \sum_{i\in S}\frac{1}{n}v_i(\bigcup_{q\in\bar{Q}\setminus\bar{Q}^b} e_q) \\ &= \sum_{i\in S}\frac{1}{n}v_i(\bigcup_{q\in\bar{Q}^b} e_q) + \sum_{i\in S}\frac{1}{n}v(\bigcup_{q\in\bar{Q}\setminus\bar{Q}^b} e_q) \\ &\geq \sum_{i\in S}\frac{1}{n}v_i(\bigcup_{q\in\bar{Q}^b} e_q) + \sum_{q\in\bar{Q}\setminus\bar{Q}^b} v(e_q) \\ &\geq \sum_{i\in S} \mathbb{E}_{\mathbf{A}\sim\mathcal{M}^*(b_S,b_{-S})}[v_i(A_i)],\end{aligned}$$

where the first inequality is due to $|S| \leq n$; the second inequality is due to the fact that e_q with $q \in \bar{Q}\setminus\bar{Q}^b$ must be assigned to the agent in S under the reporting profile (b_S, b_{-S}). The above inequality leads to a contradiction, completing the proof. □

Proof of Proposition 5.3.1. For a contradiction, assume there exists a coalition

$S \subseteq N$ and a reporting profile $(b_S, b_{-S}) \in V$ such that for all $i \in S$, it holds that

$$\mathbb{E}_{\mathbf{A} \sim \widetilde{\mathcal{M}}^*(b_S, b_{-S})}[v_i(A_i)] \geq \mathbb{E}_{\mathbf{A} \sim \widetilde{\mathcal{M}}^*(v_S, b_{-S})}[v_i(A_i)],$$

and at least one of agents in S is strictly better off under (b_S, b_{-S}). As a result, we have the following,

$$\sum_{i \in S} \mathbb{E}_{\mathbf{A} \sim \widetilde{\mathcal{M}}^*(b_S, b_{-S})}[v_i(A_i)] > \sum_{i \in S} \mathbb{E}_{\mathbf{A} \sim \widetilde{\mathcal{M}}^*(v_S, b_{-S})}[v_i(A_i)],$$

which contradicts Lemma 5.3.4. □

After establishing the group strategyproofness, we now prove that $\widetilde{\mathcal{M}}^*$ can always output a Pareto optimal allocation.

Proposition 5.3.5. *The mechanism* $\widetilde{\mathcal{M}}^*$ *is ex-ante* PO*.*

Proof. Denote by $\mathbf{A}$ the fractional allocation matrix returned by $\widetilde{\mathcal{M}}^*$. It suffices to show that allocation $\mathbf{A}$ achieves the maximum utilitarian welfare among all fractional allocations. Under allocation $\mathbf{A}$, each agent i receives the expected value $\frac{1}{n}\sum_{q \in \bar{Q}} v(e_q)$ and accordingly, the utilitarian welfare of $\mathbf{A}$ is equal to $\sum_{q \in \bar{Q}} v(e_q)$. Since all items must be assigned, the utilitarian welfare of any (fractional) allocations is at most $\sum_{q \in \bar{Q}} v(e_q)$. Therefore, we conclude that $\widetilde{\mathcal{M}}^*$ is ex-ante PO. □

Next, we present other distributional properties satisfied by the allocation returned by $\widetilde{\mathcal{M}}^*$.

Proposition 5.3.6. *The mechanism* $\widetilde{\mathcal{M}}^*$ *is non-bossy and anonymous.*

Proof. We start with non-bossiness. Fix i and consider two different reporting profiles (b_i, b_{-i}) and (b'_i, b_{-i}) with outputs $\widetilde{\mathcal{M}}^*(b_i, b_{-i}) = \mathbf{A}$ and $\widetilde{\mathcal{M}}^*(b'_i, b_{-i}) = \mathbf{A}'$. Denote by Q, $\bar{Q}$ the index sets constructed in Step 2 of Algorithm 10 with the reporting profile (b_S, b_{-S}). For a contradiction, assume $A_i = A'_i$ but $A_j \neq A'_j$ for some $j \neq i$. Suppose the probability of assigning an item e_q to agent j differs in $\mathbf{A}$ and $\mathbf{A}'$. If $q \in Q$, then either $b_i(e_q) = 0$ and $b'_i(e_q) = v(e_q)$ or $b_i(e_q) = v(e_q)$ and $b'_i(e_q) = 0$ holds, which then implies that the probability of assigning e_q to agent i in $\mathbf{A}$ is not identical to that in $\mathbf{A}'$, i.e., $A_i \neq A'_i$, a contradiction. If $q \in \bar{Q}$, then it must be the case that $b_i(e_q) = v(e_q)$ and $b'_i(e_q) = 0$. The probability of assigning e_q to agent i varies from $1/n$ to 1 by deviating from (b_S, b_{-S}) to (b'_S, b_{-S}), which implies $A_i \neq A'_i$. Therefore, we can conclude that $\widetilde{\mathcal{M}}^*$ is non-bossy.

As for the anonymity, let σ^* be a permutation of $\{1, 2, \ldots, n\}$. Denote by $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ the original type profile and $\mathbf{v}^* = (v_{\sigma^*(1)}, v_{\sigma^*(2)}, \ldots, v_{\sigma^*(n)})$ the type profile after permutation. Then we have two allocation matrices $\widetilde{\mathcal{M}}^*(\mathbf{v}) = \mathbf{A} = (a_{ji})_{j\in[m],i\in[n]}$ and $\widetilde{\mathcal{M}}^*(\mathbf{v}^*) = \mathbf{A}' = (a'_{ji})_{j\in[m],i\in[n]}$. Then, it suffices to show that $A'_i = A_{\sigma^*(i)}$, which is then equivalent to show $a'_{ji} = a_{j\sigma^*(i)}$ for all $j \in [m]$. Denote by $Q^*, \bar{Q}^*$ the index sets constructed by Step 2 of Algorithm 10 under reporting $\mathbf{v}^*$. Without loss of generality, fix a pair i, j. If $j \in Q^*$ and $a'_{ji} = 0$, then we clearly have $a_{j\sigma^*(i)} = 0$. If $j \in Q$ and $a'_{ji} > 0$, we have $a_{ji} = a_{j\sigma^*(i)}$ as the number of agent reporting zero on e_j in $\mathbf{v}$ is same as that in $\mathbf{v}^*$. If $j \in \bar{Q}^*$, then $a'_{ji} = a_{j\sigma^*(i)} = 1/n$ holds. Therefore, we can conclude that $A'_i = A_{\sigma^*(i)}$ holds for any $i \in [n]$, completing the proof. □

In what follows, we examine the fairness guarantee of allocations returned by $\widetilde{\mathcal{M}}^*$. In particular, we show that the mechanism $\widetilde{\mathcal{M}}^*$ achieves ex-ante EF, EQ, and PROP as well as ex-post EF1, EQ1, and PROP1, i.e., best of the both world.

Proposition 5.3.7. *The mechanism $\widetilde{\mathcal{M}}^*$ is ex-ante* EF*,* EQ *and* PROP*, and ex-post* EF1*,* EQ1 *and* PROP1*.*

Proof. Denote by $Q, \bar{Q}$ the index sets constructed by Step 2 of Algorithm 10 under $\mathbf{v}$ and $\mathbf{A} = (A_1, \dots, A_n)$ the returned allocation (matrix). Note that agents can receive negative expected value from the assignment of $\bigcup_{q\in\bar{Q}} e_q$ and the expected value is determined by the probability of assigning each item to that agent. According to $\widetilde{\mathcal{M}}^*$, the probability of assigning e_q with $q \in \bar{Q}$ to each agent i is $1/n$. Thus, for any i, j, we have $v_i(A_i) = v_i(A_j) = 1/n \cdot \sum_{q\in\bar{Q}} v(e_q) = v_j(A_j)$. As a consequence, the fractional allocation implemented by $\mathbf{A}$ is EF and EQ, and therefore, the mechanism $\widetilde{\mathcal{M}}^*$ is ex-ante EF and EQ. According to Definitions 2.2.1 and 2.2.7, it is not hard to verify that ex-ante EF allocations are also ex-ante PROP.

As for the ex-post fairness guarantee, let $\mathbf{A}^* = (A^*_1, \dots, A^*_n)$ be a deterministic allocation in the support of $\widetilde{\mathcal{M}}^*(\mathbf{v})$. Since only e_q with $q \in \bar{Q}$ can result in non-zero value to agents, we have $v_i(A^*_i) = v_i(\bigcup_{q\in\bar{Q}} e_q \cap A^*_i)$ for any $i \in [n]$. Moreover, as the assignment of $\bigcup_{q\in Q} e_q$ does not affect values, we can further assume $Q = \emptyset$. Then, it sufficient to consider the reduced instance with the set of items $E = \bigcup_{q\in\bar{Q}} e_q$ and valuation functions $v_i(e) = v(e) < 0$ for every $i \in [n]$ and $e \in E$. Suppose $\mathbf{A}^*$ is returned by $\mathcal{M}^*$ with a permutation σ^* in Step 5. Let the number of items be $m = kn + d$ with $k, d \in \mathbb{N}$ and $0 < d \leq n$, and so the assignment in Step 5 has $k + 1$ rounds. Given two agents $i, j \in [n]$, without loss of generality, we assume $\sigma^*(i) < \sigma^*(j)$. If agents i, j receive same amount of items, then in every single round, agent i receives value no less than that of agent j, implying $v_i(A^*_i) \geq v_j(A^*_j)$. As for agent j, her value in round $l \leq k - 1$ is no less than the value received by agent i in round $l + 1$. So by eliminating the last chore received by agent j, the value of agent j is no less than that of agent i. If agent i receives one more item, then still the value received by agent j in round l is no less than the value received by agent i in round $l+1$ and so $v_j(A_j^*) \geq v_i(A_i^*)$. As for agent i, if the last item she receives is removed, she would have value no less than that of agent j. Therefore, the allocation $\mathbf{A}^*$ is EQ1. Recall that $v_i(\cdot) = v_j(\cdot)$ for any $i, j \in [n]$, so an EQ1 allocation is clearly also EF1. Therefore, $\widetilde{\mathcal{M}}^*$ is ex-post EF1 and EQ1. According to Definitions 2.2.2 and 2.2.8, it is not hard to verify that ex-post EF1 allocations are also ex-post PROP1. □

The notions of (relaxed) envy-freeness, equitability, and proportionality are proposed more from the individual perspective, and it may have a bad performance regarding fairness from the system level. To take an example, Theorem 4.2.2 indicates that when assigning chores, EQ1 allocations can not provide a bounded guarantee of egalitarian welfare. Below we show that $\widetilde{\mathcal{M}}^*$ is capable of providing a good approximation on ex-post maximin fairness.

Proposition 5.3.8. *The mechanism $\widetilde{\mathcal{M}}^*$ is ex-post 2-approximation of max-min fairness.*

Proof. Let the allocation $\mathbf{A}^* = (A_1^*, \ldots, A_n^*)$ be a deterministic allocation in the support of $\widetilde{\mathcal{M}}^*(\mathbf{v})$, and $\mathbf{A}^*$ is returned by $\widetilde{\mathcal{M}}^*$ with the permutation σ^* in Step 5. Without loss of generality, assume $\sigma^*(i) = i$ for all $i \in [n]$. Similar to the proof of Proposition 5.3.7, we can further assume $E = \cup_{q \in \bar{Q}} e_q$ and agents have identical valuation functions: $v_i(e) = v(e) < 0$ for every $i \in [n]$ and $e \in E$. Denote by OPT_E the maximum egalitarian welfare of all deterministic allocations. Let $m = kn + d$ with $k, d \in \mathbb{N}$ and $0 < d \leq n$, then each agent $i \in [d]$ receives $k+1$ items and each agent $i \geq d+1$ receives k items. Moreover, agent d receives the last item in Step 5. In the following, we first show agent d receiving the minimum value in allocation $\mathbf{A}^*$.

It is equivalent to show $v_d(A_d^*) \leq v_i(A_i^*)$ for any $i \in [n]$. For $i < d$, both agents d and i receive $k+1$ items and, in each round, agent d receives value at most the value of agent i, which implies $v_d(A_d^*) \leq v_i(A_i^*)$. As for $i > d$, agent i receives k items and the value received by agent i in round l is no less than the value of agent d in round $l+1$. Since items are chores, we also have $v_d(A_d^*) \leq v_i(A_i^*)$.

We then show $v_d(A_d^* \setminus \{e_m\}) \geq \mathsf{OPT}_E$ where e_m is the last item received by agent d. For a contradiction, assume $v_d(A_d^* \setminus \{e_m\}) < \mathsf{OPT}_E$. Since $\mathbf{A}^*$ is EQ1, we have $v_i(A_i^*) \leq \max_{e \in A_d^*} v_d(A_d^* \setminus \{e\}) = v_d(A_d^* \setminus \{e_m\})$, then by summing up $i \in [n]$, we have an upper bound of the utilitarian welfare,

$$\sum_{i\in[n]} v_i(A_i^*) \le n \cdot v_d(A_d^* \setminus \{e_m\}) + v_d(e_m) < n \cdot \mathsf{OPT}_E,$$

where the last inequality is due to our assumption $v_d(A_d^* \setminus \{e_m\}) < \mathsf{OPT}_E$ and $v_d(e_m) \le 0$. However, since agents have identical valuation functions, the utilitarian welfare of $\mathbf{A}^*$ should be at least $n \cdot \mathsf{OPT}_E$, implying a contradiction. Thus, we have $v_d(A_d^* \setminus \{e_m\}) \ge \mathsf{OPT}_E$. Since agents have identical valuations and items are chores, we then have $v_i(e_m) \ge \mathsf{OPT}_E$. Therefore, the following holds,

$$v_d(A_d^*) = v_d(A_d^* \setminus \{e_m\}) + v_d(e_m) \ge 2 \cdot \mathsf{OPT}_E,$$

which completes the proof. □

We remark that the 2-approximation of maximin fairness is actually the limitation of mechanism $\widetilde{\mathcal{M}}^*$ when the number of agents is large.

Proposition 5.3.9. *The mechanism $\widetilde{\mathcal{M}}^*$ is not ex-post $(2 - \frac{1}{n})$-approximation of max-min fairness.*

Proof. Let us consider the instance with n agents and a set $E = \{e_1, \dots, e_{(n-1)n+1}\}$ of $(n-1)n+1$ chores. Agents have identical valuation functions: $v_i(e_j) = v(e_j)$ for any $i, j \in [n]$. The inherent values are: $v(e_j) = -1$ for $j \le (n-1)n$ and $v(e_{n(n-1)+1}) = -n$. We say an item e an β-*chore* if its inherent value is equal to $v(e) = -\beta$. Thus, there are a number $n(n-1)$ of 1-chore and one n-chore. Denote by OPT_E the maximum egalitarian welfare of all deterministic allocations. It is easy to see that $\mathsf{OPT}_E \le -n$ as there must be an agent receiving the n-chore. Consider the allocation $\mathbf{B}$ in which every agent $i \le n-1$ receives a number n of 1-chore and agent n receives the unique n-chore. One can verify that $\min_{i\in[n]} v_i(B_i) = -n$ and thus $\mathsf{OPT}_E = -n$.

Let $\mathbf{A}^* = (A_1^*, \dots, A_n^*)$ be a deterministic allocation in the support of $\widetilde{\mathcal{M}}^*(\mathbf{v})$, and $\mathbf{A}^*$ is returned by $\widetilde{\mathcal{M}}^*$ with the permutation σ^* in Step 5. Without loss of generality, assume $\sigma^*(i) = i$ for any $i \in [n]$. Then, for each $2 \le i \le n$, agent i receives a number $n-1$ of 1-chore, while agent 1 receives a number $n-1$ of 1-chore and also the unique n-chore. Thus, we have $v_1(A_1^*) = -2n+1$ that is equal to the egalitarian welfare of $\mathbf{A}^*$. Therefore, the approximation of $\mathbf{A}^*$ on the max-min fairness is at least $2 - 1/n$ that approaches to 2 as the number of agents becomes large. □

Theorem 5.3.1. *The mechanism* $\widetilde{\mathcal{M}}^*$ *is* GSPIE*, anonymous and non-bossy mechanism and also satisfies ex-ante* PO*,* EF*,* PROP*,* EQ *and ex-post* EF1*,* PROP1*,* EQ1 *and 2-approximation of max-min fairness.*

5.3.2 A Strategyproof, Efficient, and Fair Randomized Mechanism for Allocating Mixed Items to Two Agents

In this section, we extend to the scenario where an item can be a good for one agent and a chore for another, i.e., the setting of mixed items or instances. We are also interested in the restricted additive valuation functions. In the setting of mixed items, a valuation function $v_i(\cdot)$ is called an *M-restricted additive* function if $v_i(e_j) \in \{-c(e_j), 0, v(e_j)\}$ with $c(e_j), v(e_j) > 0$ for any i, j. Specifically, for an agent i and an item e_j, there are three possible situations: (i) e_j is a chore for agent i and $v_i(e_j) = -c(e_j)$ with $c(e_j) > 0$; (ii) e_j is a good for agent i and $v_i(e_j) = v(e_j) > 0$; (iii) e_j is valued at zero by agent i. We remark that $c(e_j)$ and $v(e_j)$ are not required to be identical. Similarly, to the previous section, we also assume that $c(\cdot)$ and $v(\cdot)$ are common knowledge, available to the central decision maker, which then makes the bid of each agent i be $b_i(e_j) \in \{-c(e_j), 0, v(e_j)\}$ for all $j \in [m]$.

Note that in the mixed setting, the fairness criteria, particularly those with an "up to one item" scheme, such as EF1 and PROP1, are slightly different from the definition in the case where all items are goods or all items are chores. The formal definitions of PROP1 and EF1 in the mixed setting are presented below.

Definition 5.3.3. *Given an allocation* $\mathbf{A}$*, we say agent* i *envies agent* j *by more than one item if agent* i *envies agent* j *and* $v_i(A_i \setminus \{e\}) < v_i(A_j \setminus \{e\})$ *for any* $e \in A_i \cup A_j$*. An allocation* $\mathbf{A}$ *of a mixed instance is* EF1 *if for any* $i, j \in [n]$*, agent* i *does not envy agent* j *by more than one item.*

Intuitively, an agent satisfies EF1 if, from her perspective, envy can be eliminated by either removing a chore from her bundle or removing a good from the bundle of another agent. Similarly, this idea is also applied to the notion of PROP1 in the mixed setting.

Definition 5.3.4. *An allocation* $\mathbf{A}$ *of a mixed instance is* PROP1 *if for each agent* $i \in [n]$*, at least one of the following three situations happen: (i)* $v_i(A_i) \geq n^{-1}v_i(E)$*; (ii)* $v_i(A_i \cup \{e\}) \geq n^{-1}v_i(E)$ *for some* $e \in E \setminus A_i$*; (iii)* $v_i(A_i \setminus \{e\}) \geq n^{-1}v_i(E)$ *for some* $e \in A_i$*.*

It has been proved that, in the setting of (additive) mixed items, an EF1 allocation is also PROP1.

Proposition 5.3.10 (Proposition 2 in [13]). *For additive valuation functions, an* EF1 *allocation of a mixed instance satisfies* PROP1*.*

In what follows, we first present a randomized mechanism (see Algorithm 11) for allocating mixed items to two agents with M-restricted additive valuations, and then prove that the proposed mechanism achieves strategyproofness, PO, and both ex-ante and ex-post fairness. The high-level idea of $\widetilde{\mathcal{M}}^2$ is to partition, according to the bids, the items into four subsets. The set of items with non-identical bids is assigned to the agent with higher bids. For the items with identical bids, if both agents bid zero, then this part of items are uniformly randomly assigned to agents, and the remaining items are assigned in a round-robin way.

Algorithm 11 $\widetilde{\mathcal{M}}^2$

Input: A mixed instance $\mathcal{I} = \langle [2], E, \mathcal{V} \rangle$ with M-restricted additive valuation functions.

1: Every agent i bids $b_i(\cdot)$.
2: Partition $E = E_0 \cup E_1 \cup E_2 \cup E_3$ where $E_0 = \{e \in E \mid b_1(e) = b_2(e) = 0\}, E_1 = \{e \in E \mid b_1(e) > b_2(e)\}, E_2 = \{e \in E \mid b_1(e) < b_2(e)\}$ and $E_3 = \{e \in E \mid b_1(e) = b_2(e) \neq 0\}$.
3: For every $e \in E_0$, uniformly randomly pick an agent to whom assign e.
4: For each $i \in [2]$, assign $A_i \leftarrow E_i$.
5: Let σ be a uniformly random permutation of $\{1, 2\}$ and $\sigma(j)$ be its j-th element. Run Round-robin on remaining items E_3 based on σ.
6: **return** Allocation $\mathbf{A}$

Proposition 5.3.11. *With the reported profile (b_1, b_2), for any $i \in [2]$, agent i receives expected value $\frac{1}{2}v_i(E_0) + v_i(E_i) + \frac{1}{2}v_i(E_3)$ in $\widetilde{\mathcal{M}}^2$.*

Proof. Fix i. As the entire bundle E_{3-i} is assigned to agent $3-i$, the value of agent i comes from the assignment of E_0, E_i and E_3. Note that the probability of assigning E_0 to agent i is $1/2$, the assignment of E_0 results in expected value $2^{-1}v_i(E_0)$ for agent i. The whole bundle E_i is assigned to agent i with probability 1, which leads to (expected) value $v_i(E_i)$ for agent i. As for the bundle E_3, it is assigned by Round-robin based on an uniformly random permutation, and according to Lemma 5.3.2, this part results in the expected value $2^{-1}v_i(E_3)$ for agent i. Therefore, the total expected value received by agent i is $2^{-1}v_i(E_0) + v_i(E_i) + 2^{-1}v_i(E_3)$. □

Lemma 5.3.12. *Given an agent i and two bids $b_i, b_i^k \in V_i$ only differing on item e_k, if $b_i(e_k) = v_i(e_k)$, then for every b_{-i} that together with b_i and b_i^k being an M-*

restricted additive profile, the following holds

$$\mathbb{E}_{\mathbf{A}\sim\widetilde{\mathcal{M}}^2(b_i,b_{-i})}[v_i(A_i)] \geq \mathbb{E}_{\mathbf{A}\sim\widetilde{\mathcal{M}}^2(b_i^k,b_{-i})}[v_i(A_i)].$$

Proof. We remark that the reporting space of agent $3-i$ can be simplified to $\{-c(e_k), 0, v(e_k)\}$; Other bids result in either "Error" (if (b_1, b_2) is not M-restricted additive) or the same outcome as reporting from $\{-c(e_k), 0, v(e_k)\}$. Denote by $\{E_i\}_{i=0}^3$ and $\{E_i^k\}_{i=0}^3$ the corresponding sets constructed by $\widetilde{\mathcal{M}}^2$ with the reporting profiles b_i, b_{-i} and b_i^k, b_{-i}, respectively. We let Δ be the difference between expected value of agent i when reporting b_i and b_i^k, and formally,

$$\Delta = \mathbb{E}_{\mathbf{A}\sim\widetilde{\mathcal{M}}^2(b_i,b_{-i})}[v_i(A_i)] - \mathbb{E}_{\mathbf{A}\sim\widetilde{\mathcal{M}}^2(b_i^k,b_{-i})}[v_i(A_i)].$$

Then it suffices to prove $\Delta \geq 0$. The remaining proof is given by carefully checking all possible cases.

Case 1: $b_i(e_k) = v_i(e_k) = v(e_k)$.

Subcase 1.1: $b_i^k(e_k) = 0$ and $b_{3-i}(e_k) = v(e_k)$. For this subcase, we have $E_{3-i}^k = E_{3-i} \cup \{e_k\}$, $E_3^k \cup \{e_k\} = E_3$ and moreover $E_0^k = E_0$, $E_i^k = E_i$. According to Proposition 5.3.11, we have $\Delta = 2^{-1}[v_i(\bigcup_{e\in E_3} e) - v_i(\bigcup_{e\in E_3^k} e)] = 2^{-1}v_i(e_k) = 2^{-1}v(e_k) > 0$.

Subcase 1.2: $b_i^k(e_k) = 0$ and $b_{3-i}(e_k) = 0$. In this subcase, we have $E_i^k \cup \{e_k\} = E_i$, $E_0^k = E_0 \cup \{e_k\}$ and moreover $E_3^k = E_3$, $E_{3-i}^k = E_{3-i}$. Again from Proposition 5.3.11, we can compute $\Delta = 2^{-1}v(e_k) > 0$.

Subcase 1.3: $b_i^k(e_k) = 0$ and $b_{3-i}(e_k) = -c(e_k)$. Then, we have $E_l^k = E_l$ for all $l = 0, 1, 2, 3$, and thus, $\Delta = 0$.

Subcase 1.4: $b_i^k(e_k) = -c(e_k)$ and $b_{3-i}(e_k) = v(e_k)$. The composition of $\{E_i^k\}_{i=0}^3$ and of $\{E_i\}_{i=0}^3$ are same as that in subcase 1.1, and thus, we also have $\Delta > 0$.

Subcase 1.5: $b_i^k(e_k) = -c(e_k)$ and $b_{3-i}(e_k) = 0$. The composition of $\{E_i^k\}_{i=0}^3$ and of $\{E_i\}_{i=0}^3$ are same as that in subcase 1.1, and thus, we also have $\Delta > 0$.

Subcase 1.6: $b_i^k(e_k) = -c(e_k)$ and $b_{3-i}(e_k) = -c(e_k)$. For this subcase, we have $E_i^k \cup \{e_k\} = E_i$, $E_3^k = E_3 \cup \{e_k\}$ and $E_0^k = E_0$, $E_{3-i}^k = E_{3-i}$. Then, according to Proposition 5.3.11, we have $\Delta = 2^{-1}v_i(e_k) = 2^{-1}v(e_k) > 0$.

Case 2: $b_i(e_k) = v_i(e_k) = 0$. According to $\widetilde{\mathcal{M}}^2$, different bids on e_k only change the composition of the bundle on whether including e_k or not, and in particular, do not affect the assignment of other items. Based on Proposition 5.3.11 and

$v_i(e_k) = 0$, we have $\Delta = 0$ for this case.

Case 3: $b_i(e_k) = v_i(e_k) = -c(e_k)$.

Subcase 3.1: $b_i^k(e_k) = v(e_k)$ and $b_{3-i}(e_k) = v(e_k)$. In this subcase, we have $E_3^k = E_3 \cup \{e_k\}$, $E_{3-i}^k \cup \{e_k\} = E_{3-i}$ and $E_i^k = E_i$, $E_0^k = E_i$. Then we compute $\Delta = (-2)^{-1} v_i(e_k) = 2^{-1} c(e_k) > 0$.

Subcase 3.2: $b_i^k(e_k) = v(e_k)$ and $b_{3-i}(e_k) = 0$. For this subcase, it holds that $E_i^k = E_i \cup \{e_k\}$, $E_{3-i}^k \cup \{e_k\} = E_{3-i}$ and $E_0^k = E_0$, $E_3^k = E_3$. Then, we have $\Delta = -v_i(e_k) = c(e_k) > 0$.

Subcase 3.3: $b_i^k(e_k) = v(e_k)$ and $b_{3-i}(e_k) = -c(e_k)$. In this situation, we have $E_3^k \cup \{e_k\} = E_3$, $E_i^k = E_i \cup \{e_k\}$ and $E_0^k = E_0$, $E_{3-i}^k = E_{3-i}$. Then according to Proposition 5.3.11, we can compute $\Delta = (-2)^{-1} v_i(e_k) = 2^{-1} c(e_k) > 0$.

Subcase 3.4: $b_i^k(e_k) = 0$ and $b_{3-i}(e_k) = v(e_k)$. In this subcase, we have $E_l^k = E_l$ for all $l = 0, 1, 2, 3$, and thus, $\Delta = 0$.

Subcase 3.5: $b_i^k(e_k) = 0$ and $b_{3-i}(e_k) = 0$. In this situation, we have $E_0^k = E_0 \cup \{e_k\}$, $E_{3-i}^k \cup \{e_k\} = E_{3-i}$ and $E_i^k = E_i$, $E_3^k = E_3$. Then we compute $\Delta = (-2)^{-1} v_i(e_k) = 2^{-1} c(e_k) > 0$.

Subcase 3.6: $b_i^k(e_k) = 0$ and $b_{3-i}(e_k) = -c(e_k)$. In this subcase, we have $E_l^k = E_l$ for any $l = 0, 1, 2, 3$, and thus, $\Delta = 0$.

Therefore, for all possible values of $v_i(e_k)$ and of $b_{3-i}(e_k)$, it always holds that $\Delta \geq 0$. $\square$

Lemma 5.3.12 states that every single deviation from the type of the agent does not bring extra expected value, and hence, a sequence of deviations can never make the agent better-off. We below establish the strategyproofness of $\widetilde{\mathcal{M}}^2$.

Proposition 5.3.13. *The mechanism $\widetilde{\mathcal{M}}^2$ is* SPIE.

Proof. Let b_1 be an arbitrary bid that differs from v_1 on a set $\{e_{p_1}, \ldots, e_{p_k}\}$ of $k > 0$ items. We let $b_1^0 = b_1$, then for every $l \in [k]$ construct b_1^l as follows: $b_1^l(e) = b_1^{l-1}(e)$ for all $e \neq e_{p_l}$ and $b_1^l(e_{p_l}) = v_1(e_{p_l})$. Then it is easy to see $b_1^k = v_1$. According to Lemma 5.3.12, the following holds,

$$\mathbb{E}_{\mathbf{A} \sim \widetilde{\mathcal{M}}^2(b_1^k, b_2)}[v_1(A_1)] \geq \mathbb{E}_{\mathbf{A} \sim \widetilde{\mathcal{M}}^2(b_1^{k-1}, b_2)}[v_1(A_1)] \geq \cdots \geq \mathbb{E}_{\mathbf{A} \sim \widetilde{\mathcal{M}}^2(b_1^0, b_2)}[v_1(A_1)].$$

By the construction, the above inequality is equivalent to

$$\mathbb{E}_{\mathbf{A} \sim \widetilde{\mathcal{M}}^2(v_1, b_2)}[v_1(A_1)] \geq \mathbb{E}_{\mathbf{A} \sim \widetilde{\mathcal{M}}^2(b_1, b_2)}[v_1(A_1)],$$

which completes the proof. $\square$

We now present the additional properties, such as efficiency and fairness satisfied by $\widetilde{\mathcal{M}}^2$.

Proposition 5.3.14. *The mechanism* $\widetilde{\mathcal{M}}^2$ *is ex-ante* PO.

Proof. Denote by $\mathbf{A}$ the fractional allocation (matrix) returned by $\widetilde{\mathcal{M}}^2$. It is sufficient to show that allocation $\mathbf{A}$ achieves the maximum utilitarian welfare among all fractional allocations. As two agents have identical valuations on bundle $E_0 \cup E_3$, the utilitarian welfare from the assignment of $E_0 \cup E_3$ should be identical among all allocations. For the bundle E_i with $i = 1, 2$, in allocation $\mathbf{A}$ every item of E_i has been assigned to the agent having a larger value, and therefore, other (fractional) allocations can never achieve an utilitarian welfare larger than $\mathsf{UW}(\mathbf{A})$. $\square$

Proposition 5.3.15. *The mechanism* $\widetilde{\mathcal{M}}^2$ *is ex-ante* EF*,* PROP *and ex-post* EF1*,* PROP1*.*

Proof. Denote by $\{E_i^v\}_{i=0}^3$ the corresponding bundles constructed by $\widetilde{\mathcal{M}}^2$ when agents' reporting profile being (v_1, v_2), and by $\mathbf{A} = (A_1, A_2)$ the returned fractional allocation. Based on Proposition 5.3.11, we have $v_i(A_i) = 2^{-1}v_i(E_0^v) + v_i(E_i^v) + 2^{-1}v_i(E_3^v)$. The fractional bundle A_{3-i} contains fractional items from $E_0^v \cup E_{3-i}^v \cup E_3^v$. The probability of assigning every $e \in E_0^v \cup E_3^v$ to agent $3-i$ is $1/2$ and the probability of assigning every $e \in E_{3-i}^v$ to agent $3 - i$ is 1. Therefore, we have $v_i(A_{3-i}) = 2^{-1}v_i(E_0^v) + v_i(E_{3-i}^v) + 2^{-1}v_i(E_3^v)$. According to Step 2 of $\widetilde{\mathcal{M}}^2$ and the fact that agents' valuations are M-restricted additive, it holds that $v_i(E_i^v) \geq 0 \geq v_i(E_{3-i}^v)$, which then implies $v_i(A_i) \geq v_i(A_{3-i})$. The ex-ante PROP follows from ex-ante EF.

As for the ex-post fairness guarantee, since ex-post EF1 implies ex-post PROP1, it sufficient to show that the mechanism $\widetilde{\mathcal{M}}^2$ is ex-post EF1. Let $\mathbf{B} = (B_1, B_2)$ be a support of the randomized allocation $\mathbf{A}$ with $B_i = E_0^i \cup E_i^v \cup E_3^i$ for $i = 1, 2$, where $E_0^i \subseteq E_0^v$ and $E_3^i \subseteq E_3$. Then, we have $v_i(B_i) = v_i(E_i^v) + v_i(E_3^i)$ and $v_i(B_{3-i}) = v_i(E_{3-i}^v) + v_i(E_3^v \setminus E_3^i)$. According to Step 2 of $\widetilde{\mathcal{M}}^2$ and the fact that agents' valuations are M-restricted additive, it holds that $v_i(E_i^v) \geq 0 \geq v_i(E_{3-i}^v)$. As a consequence, it suffices to show that the assignment of E_3^v in Step 5 results in an EF1 allocation. Without loss of generality, we assume $\sigma(i) = i$ for $i = 1, 2$. Suppose that round-robin executes in total k rounds. If two agents receive the same amount of items, then in every round, agent 1 receives a value no less than that of agent j, and thus agent 1 does not violate EF1. As for agent j, her value in round $l \leq k - 1$ is no less than the value received by agent i in round $l + 1$. So by eliminating the first item received by agent i, agent j does not envy agent i. If agent i receives one more item, she does not envy agent j if the last item she receives is eliminated. For agent j, she does not envy agent i if the first item received by agent i is removed. Therefore, allocation $\mathbf{B}$ is EF1. □

Theorem 5.3.2. $\widetilde{\mathcal{M}}^2$ *is* SPIE*, ex-ante* PO*,* EF*,* PROP *and ex-post* EF1 *and* PROP1*.*

The proposed mechanism $\widetilde{\mathcal{M}}^2$ does not provide significant guarantees on equitability (EQ). One reason can be that (relaxed) EQ is incompatible with Pareto efficiency in the mixed setting, even without the requirement of strategyproofness. To take an example, one can consider a mixed instance of two agents and two items. Agent 1 values each item at 1, and agent 2 values each item at -1. The PO allocation assigns both items to agent 1, which violates the notion of EQ or EQ1.

5.4 Conclusions

In this chapter, we have studied the model of dividing indivisible items from the mechanism design perspective. We are particularly interested in the settings where (i) items are chores; (ii) items are mixtures of goods and chores. When randomness is not allowed, we have first shown that the (truthful) subclass of picking sequence fails to guarantee Pareto optimality, even when agents' valuations are binary additive. Then, we have provided a truthful and Pareto optimal deterministic mechanism. When requiring fairness, we end up with an impossibility result; no deterministic mechanism can achieve SP, PO and EQ1, even when agents' valuations are binary additive. Such an impossibility motivates us to incorporate randomness into the mechanism, which, in the restricted additive setting, leads us to a "nice" mechanism satisfying truthfulness, Pareto optimality, best-of-both-world fairness, anonymity, and non-bossiness. The last result of this chapter is a randomized, strategyproof, Pareto optimal, and fair mechanism for two agents in the setting where items are mixtures of goods and chores.

Chapter 6

Conclusion

This thesis is concerned with the problem of allocating indivisible items to a set of agents, a classical model of resource allocation with a wide range of practical applications. From the point of view of the central decision maker, the ultimate goal is to make better use of resources so that the outcome can be as efficient as possible. When an allocation is to optimize efficiency, it can harm the interests of certain participants and produce unfair outcomes, not acceptable to the participants. Only when the result of the allocation is fair, all participants can accept such a result, thus ensuring a certain degree of sustainability. This makes fairness a necessary factor that needs to be taken into account when making decisions. Traditionally, system efficiency and fairness are studied separately, with questions being: how to utilize the resource in a more efficient way, what is the appropriate definition of fairness, and how to achieve a fair allocation. As the research progresses, it has been found that efficiency and fairness are not completely unrelated or independent. In particular, optimization on one notion may lead to bad performance on the other, i.e., a trade-off between efficiency and fairness. Although the existing work proposes the framework of the price of fairness to quantify the efficiency loss due to fairness constraints, in the problem of allocating indivisible items, the price of fairness regarding envy-based, share-based, and equitability-based fairness constraints has not been well-studied.

This thesis in Chapter 3 has provided a picture of the PoF ratios regarding envy-based and share-based fairness notions for chores and also a collection of results on the connections among these fairness criteria under consideration. Chapter 4 has presented the PoF ratios with respect to equitability-based fairness and also a sequence of algorithmic results on related decision and optimization problems. These two chapters have addressed two closely related theoretical questions on the allocation of indivisible items, and contribute to the fair division literature. In addition to efficiency and fairness, strategyproofness is another widely-studied notion in various resource allocation problems. Strategyproofness ensures that participants do not misreport their valuations to gain additional benefits, and without strategyproofness, the assignment protocol can result in unreasonable solutions. Most existing works on the strategyproof algorithms provide negative answers on designing a strategyproof, fair, and efficient mechanism under some specific settings. This thesis, in Chapter 5, suggests the power of randomisation and provides randomised strategyproof mechanisms that can return fair and efficient allocations, which also makes theoretical contributions to the literature on fairly allocating indivisible items.

The research outcomes presented in this thesis not only provide theoretical insights to the existing literature but also hold practical significance. One key implication is that understanding the trade-off between efficiency and fairness enables decision-makers to make informed choices regarding resource allocation. By comprehending the price of fairness, decision-makers can evaluate the potential costs and benefits of incorporating fairness constraints into resource allocation decisions to ensure that resources are allocated in a way that promotes fairness while maximizing overall welfare and efficiency. Moreover, the proposed polynomial time algorithms for computing the best possible fair solutions in worst-case scenarios can act as practical guidelines for decision-makers when facing assignment problems. Another practical implication of the research is related to the machine scheduling problem. Specifically, the model of indivisible chores analyzed in this study is identical to the unrelated machine scheduling setting. By treating the items as jobs and the agents as machines, the non-positive valuations in the assignment model naturally correspond to the processing time of jobs on machines in the machine scheduling model. Consequently, the algorithms proposed in this research can be directly applied to machine scheduling scenarios. Finally, it is worth noting that the mechanism design framework is frequently encountered in real-world scenarios, where individuals may be reluctant to disclose their true preferences. In the absence of accurate preferences, the central decision maker would struggle to suggest reasonable solutions, much less to allocate resources in a fair and effective manner. Therefore, decision-makers must guarantee that participants report their preferences honestly, enabling the proposal of a valid and feasible assignment procedure. The mechanisms suggested in this thesis introduce protocols that discourage participants from providing false information, as truthful reporting maximizes their valuations. These mechanisms serve as valuable guidelines and instructions for decision-makers when faced with such assignment problems in practice.

The content of this thesis leaves some open questions for future research. For the results in Chapter 4, areas that allure immediate exploration include: (a) bound tightness on the price of EQX in the case of general n and the price of EQ1 with respect to utilitarian welfare; (b) whether Theorem 4.3.12 still holds when agents' valuations are normalized to a constant; (c) for the allocations of chores with two agents, what is the time complexity of deciding the existence of an EQ1 allocation that also achieves the maximum egalitarian welfare? As for Chapter 5, an immediate direction for future work is to allow agents to have a broader preference domain, such as additive. It is unknown whether there exists a randomised mechanism that can achieve SPIE, ex-ante PO and fairness when the valuations are additive. Another interesting question is about the deterministic mechanisms. Our results have shown that one can never find an SP, PO, and EQ1 mechanism for chores with binary additive valuations. It is still possible, in the setting of chores and binary additive agents, to achieve deterministic SP and PO mechanisms that are compatible with other fairness notions, such as EF1. Note that the goods version of the question: whether SP, PO, and EF1 are compatible in the setting of binary additive valuations, has been answered by [75], whose mechanisms, however, do not work for chores.

Besides the above-mentioned open question, our results also suggest a direction worthwhile to be explored. Given the unboundedness of the PoF in our consideration of fair and efficient allocation of chores, it is desirable to improve the current lens of the PoF to see a refined picture of the efficiency loss of a fair allocation of chores. One possible way could be to add a positive parameter intrinsic to the problem instance to both the numerator and denominator of our current definition of the PoF ratio.

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