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Many-valued logics—implications and semantic consequences

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Abstract.

In this paper an application of the well-known matrix method to an extension of the classical logic to many-valued logic is discussed: we consider an $\mathfrak{n}$-valued propositional logic as a propositional logic language with a logical matrix over $\mathfrak{n}$ truth-values. The algebra of the logical matrix has operations expanding the operations of the classical propositional logic. Therefore we look over the Łukasiewicz, Post, Heyting and Rosser style expansions of the operations negation, conjunction, disjunction and with a special emphasis on implication.

In the frame of consequence operation, some notions of semantic consequence are examined. Then we continue with the decision problem and the logical calculi. We show that the cause of difficulties with the notions of semantic consequence is the weakness of the reviewed expansions of negation and implication. Finally, we introduce an approach to finding implications that preserve both the modus ponens and the deduction theorem with respect to our definitions of consequence.

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1 Introduction

The construction of propositional logics can follow several methodological ways. The algebraic method is fundamental and also prior to any others. One such an algebraic tool for constructing a logic is the logical matrix method. We begin with a brief survey of related notions and notations. After defining the notion of consequence operation, we show that the semantic consequence for the classical propositional logic generates a consequence operation. Later, we outline the conventional axiomatic treatment of logic for a given logic language. Here, we prove that the usual derivation notion is also a consequence operation.

After this, we discuss the n-valued propositional logics ($n > 2$). It is desirable to obtain an algebraic structure close to a Boolean algebra by expansion of the classical logical matrix to n values. Here, we define two notions of semantic consequence, and prove that both are consequence operations. Because the usual expansion of conjunction is the minimum unanimously, and the expansion of disjunction is the maximum in the same way, we deal only with the Łukasiewicz, Post, Heyting and Rosser style expansions of implication.

Finally, we look for a suitable implication for a general consequence notion such that both the modus ponens and the deduction theorem remain valid.

2 Logical matrices

Let U be any nonempty set. A mapping $o : U^m \to U$, defined on the Cartesian product of m copies of U, with values in U, is called an m-argument (or an m-ary) operation in U (for $m = 0, 1, \ldots$). By an algebra we mean a pair $\langle U, (o_1, o_2, \ldots, o_k)\rangle$ ($k \geq 1$), where U is a (nonempty) set, called the universe of the algebra, and each o_j is an m_j-argument operation over U. A tuple $(m_1, m_2, \ldots, m_k)$ associated to the operations is called the signature of the algebra.

We consider an arbitrary logic language $L = \langle V, (c_1, c_2, \ldots, c_k), F\rangle$, where V is the set of propositional variables; $c_1, c_2, \ldots, c_k$ are logical connectives; F is the set of formulas generated by the variables and the connectives in the standard way. At the same time, the set F of the formulas can also be regarded as the universe of an algebra with concatenation operations induced by the connectives. If we can connect m_j formulas with the connective c_j, the induced operation has m_j arguments, and the signature of the algebra freely generated by V is $(m_1, m_2, \ldots, m_k)$. This algebra is a logic language algebra.

A logic system is semantically determined, if we have an interpretation notion in the sense that every formula has some truth-value with respect to each such interpretation. A basic assumption in classical logics is the principle of compositionality: the truth-value of a compound formula is a function of the truth-values of its immediate subformulas (every formula represents a function into the set of truth-values). Hence the most essential semantical decision is the determination of the operations over the truth-value set which characterizes the connectives. Later, the algebraic structure of the truth-value set will play an important role.

Definition 1 [5] *By a logical matrix* M *for a logic language algebra* L *with a signature* $(\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_k)$ *we mean a triple*

$$\langle \mathsf{U}, (\mathsf{o}_1, \mathsf{o}_2, \ldots, \mathsf{o}_k), \mathsf{U}^* \rangle,$$

where $\langle \mathsf{U}, (\mathsf{o}_1, \mathsf{o}_2, \ldots, \mathsf{o}_k) \rangle$ *is an algebra with the signature* $(\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_k)$*, and* U^* *is a nonempty subset of* U*.* U *is the set of truth-values, the elements of* U^* *are called designated truth-values.*

After this, we define the semantics as a correspondence between the set of connectives and operations using the signature. This is followed by an interpretation $\mathrm{I} : \mathsf{V} \to \mathsf{U}$. The interpretation I can uniquely be extended to a homomorphism (called a valuation of formulas) from the set of formulas F to the universe U:

(a) $|\nu|_\mathrm{I} = \mathrm{I}(\nu)$ for $\nu \in \mathsf{V}$;

(b) $|\mathsf{c}_j(\alpha_1, \ldots, \alpha_{\mathfrak{m}_j})|_\mathrm{I} = \mathsf{o}_j(|\alpha_1|_\mathrm{I}, \ldots, |\alpha_{\mathfrak{m}_j}|_\mathrm{I})$ for every $\mathfrak{m}_j$-ary connective c_j and for all $\alpha_1, \ldots, \alpha_{\mathfrak{m}_j} \in \mathsf{F}$.

In every interpretation, a truth-value is assigned to a formula, depending on the truth-values assigned to the variables occurring in the formula. Thus, a formula expresses a truth-function $\mathsf{U}^n \to \mathsf{U}$ (an n-variable operation over U). If we want to handle every potential truth-function with the logic language, then the set of operations in the logical matrix should be functionally complete. We say that a set of operations is functionally complete, when every truth-function $\mathsf{U}^n \to \mathsf{U}$ can be expressed by a formula using only the logical connectives corresponding to these operations.

Now, a notion of partial interpretation $\mathrm{Ip} : \mathsf{V}' \to \mathsf{U}$ $(\mathsf{V}' \subseteq \mathsf{V})$ is convenient. If $\mathsf{V}' = \mathsf{V}$, the partial interpretation Ip is a (total) interpretation. And, if the domain of Ip contains all the variables occurring in a set X of formulas, then Ip is total with respect to X. Sometimes later, it is simpler to handle an (partial) interpretation Ip as a relation $\mathrm{Ip} \subseteq V \times U$, where for all pair (v_1, u_1) and (v_2, u_2) in Ip, if $u_1 \neq u_2$, then $v_1 \neq v_2$. In this notation, we can formalize an extension of the partial interpetation Ip to the variable $v \notin \mathrm{Dom}(\mathrm{Ip})$ with $\mathrm{Ip} \cup \{(v, u)\}$, where $u \in U$.

In order that a logic language and its matrix can become a logic system, the consequence notion and the decision problem are inevitable. In [7], Tarski developed an abstract theory of logical systems. He introduced a finitary closure operation on the sets of formulas, called consequence operation. Let $\mathcal{P}(F)$ denote the power set of F.

Definition 2 *The consequence operation* $Cn : \mathcal{P}(F) \to \mathcal{P}(F)$ *in* L *is an operation which satisfies the following conditions for any* $X, Y \subseteq F$ *and* $\alpha, \beta \in F$:

(1) $X \subseteq Cn(X) \subseteq F$*;*

(2) if $X \subseteq Cn(Y)$ *then* $Cn(X) \subseteq Cn(Y)$*;*

(3) if $\alpha \in Cn(X)$ *then there exists a finite set* Y *such that* $Y \subseteq X$ *and* $\alpha \in Cn(Y)$*.*

Note that $Cn(Cn(X)) \subseteq Cn(X)$ holds for every consequence operation, because $Cn(X) \subseteq Cn(X)$ and *(2)*.

Let α be a formula and let X be a set of formulas. The decision problem is to decide whether $\alpha \in Cn(X)$.

To sum it up, by a propositional logic we mean a quadruple

$$\langle L, M, In, Pr \rangle ,$$

where L is a logic language algebra, M is a logical matrix for L, In is the set of interpretations of L, Pr is a consequence operation.

Example 3 *A classical two-valued propositional logic (CPL) is a quadruple*

$$\left\langle L, M, In, Pr' \right\rangle ,$$

where

(a) L *is a language algebra* $\langle V, (\neg, \wedge, \vee), F \rangle$ *with signature* $(1, 2, 2)$*.*

(b) M *is a logical matrix* $\langle \{0, 1\}, (\neg', \wedge', \vee'), \{1\} \rangle$*, where the values* 0 *and* 1 *are truth-values, 1 stands for true, 0 stands for false. The operation* $\wedge'$ *is the classical conjunction (minimum),* $\vee'$ *is the classical disjunction*

(maximum), and $\neg'$ *is the classical negation. This operation set is functionally complete. (We remark, if we use the definition* $x \supset' y \rightleftharpoons \neg' x \vee' y$ *in* M*, the set* $\{\neg', \supset'\}$ *is also functionally complete.)*

The structure

$$\left\langle \{0,1\}, \left\{\neg', \wedge', \vee'\right\}, 0, 1 \right\rangle$$

yields a Boolean algebra. The set $\{0,1\}$ *is the universe of the Boolean algebra, the operations* $\wedge'$ *and* $\vee'$ *are lattice operations, the unary operation* $\neg'$ *is the complementation, and 1 is the unit, 0 is the zero element.*

(c) $In = \{I \mid I : V \to \{0,1\}$ *is an interpretation of* $L\}$.

(d) Pr' *is the usual semantic consequence:* $\alpha \in Pr'(X)$ *if and only if* $|\alpha|_I = 1$, *whenever* $|\beta|_I = 1$ *for every formula* β *in* X.

Next, we verify that Pr' is a consequence operation.

Proposition 4 Pr' *satisfies the conditions (1)-(3) in Definition 2.*

Proof.

(1) is obvious.

(2) Let In_X be the set of interpretations, where $|\beta|_I = 1$ for every formula β in X. If elements of X are consequences of Y, then $In_Y \subseteq In_X$. Whereas $In_X \subseteq In_{Pr'(X)}$, thus $In_Y \subseteq In_{Pr'(X)}$.

(3) If $\alpha \in Pr'(X)$, then $In_X \cap In_{\neg\alpha} = \emptyset$. Because of compactness theorem in CPL, if $In_X \cap In_{\neg\alpha} = \emptyset$, then there exists a finite set Y such that $Y \subseteq X$ and $In_Y \cap In_{\neg\alpha} = \emptyset$ also. Thus, Y is a finite subset of X and $\alpha \in Pr'(Y)$. □

3 Axiomatic treatment of logics

Another method to construct logics is the axiomatic (syntax-based) way. Let L be a logic language with the set F of formulas.

Definition 5 *A finite subset* A *of formulas is called an axiom system.*

Definition 6 *A rule over* F *is a nonempty relation*

$$r \subseteq \{(\alpha_1, \ldots, \alpha_m, \alpha) \mid \alpha_1, \ldots, \alpha_m, \alpha \in F\}.$$

Definition 7 *Let* A *be an axiom system,* R *a set of rules and* X *any set of formulas. A formula* α *is derived from* X *if there is a finite sequence of formulas* $\alpha_1, \ldots, \alpha_k$ *such that*

(1) $\alpha_k = \alpha$*, and*

(2) for each i $(1 \leq i \leq k)$*, either* $\alpha_i \in X \cup A$*, or there exist indices* $i_1, \ldots, i_l$ *smaller than* i *such that* $(\alpha_{i_1}, \ldots, \alpha_{i_l}, \alpha_i) \in r$ *for some rule* $r \in R$.

Proposition 8 $Pr^* : X \to \{\alpha \mid \alpha$ *is derived from* $X\}$ *satisfies conditions (1)-(3) in Definition 2.*

Proof.

(1) is obvious.

(3) can be seen easily. If $\alpha \in Pr^*(X)$, the derivation of α is a finite sequence of formulas. Let Y be the set of formulas of X occuring in this derivation. Clearly, α can be derived from Y, as well.

(2) If α can be derived from X, because of *(3)*, there is a finite $Z \subseteq X$ such that α can be derived from Z. But every element of Z can be derived from Y, i.e. from some finite subset of Y. If we concatenate the derivations of the elements of Z from Y and furthermore, we add the derivation of α from Z to it, then the result is a derivation of α from Y. Herewith, condition *(2)* holds. □

Informally, a propositional logic is axiomatically given by

$$\langle L, A, R, Pr^* \rangle ,$$

if its language algebra L is specified, an axiom system A is fixed, a finite set R of derivation rules is specified and Pr^* is the consequence operation.

An axiomatically given propositional logic (calculus) $\langle L, A, R, Pr^* \rangle$ is said to be (strongly) adequate for a propositional logic $\langle L, M, In, Pr \rangle$ if their consequence operations are the same.

Example 9 *By a classical propositional calculus we mean a quadruple*

$$\langle L^*, A, R, Pr^* \rangle ,$$

where

(a) L^* *is the free language algebra* $\langle V, (\neg, \supset), F\rangle$ *with signature* $(1, 2)$*;*

(b) the axiom system A *consists of the axioms*

$$\{\alpha \supset (\beta \supset \alpha),\ (\alpha \supset (\beta \supset \gamma)) \supset ((\alpha \supset \beta) \supset (\alpha \supset \gamma)),\\ (\neg\alpha \supset \beta) \supset ((\neg\alpha \supset \neg\beta) \supset \alpha)\};$$

(c) the set R *contains the single derivation rule*

$$\frac{\alpha, \alpha \supset \beta}{\beta};$$

(d) and $\mathsf{Pr}^* : X \to \{\alpha \mid \alpha$ *is derived from* $X\}$ *is the consequence operation.*

The classical propositional calculus $\langle \mathsf{L}^*, \mathsf{A}, \mathsf{R}, \mathsf{Pr}^*\rangle$ *is adequate for the classical propositional logic* $\langle \mathsf{L}^*, \mathsf{M}^*, \mathsf{In}, \mathsf{Pr}'\rangle$*, where* M^* *is a logical matrix for* L^**.*

4 Propositional many-valued logics

By the literature [1], [2] and [3], a non-classical logic may be an extended logic and/or a deviant logic. "Extended logics expand classical logic by additional logical constructs. For example, in modal logic modal operators are added to classical logic to express modal notions. In contrast, deviant logics are rivals to classical logic that give up some classical principles. In many-valued logics, we allow for many truth-values instead of two truth-values (we give up the principle of bivalence)". This deviation leads to the extension of the operations of the classical two-valued logic. An operation is extended if, whenever the arguments are classical truth-values, the result has the same truth-value as it does in classical logic. "In this sense, classical logic can be thought of as a special case of many-valued logic."

Let U_n be a set of truth-values $\{0, 1, 2, \ldots, n-1\}$ $(n \geq 2)$. Formally, we can define a propositional many-valued logic (MVPL) as a quadruple

$$\langle \mathsf{L}, \mathsf{M}, \mathsf{In}, \mathsf{Pr}^n\rangle,$$

where

(a) $\mathsf{L} = \langle V, \mathsf{Con}, F\rangle$ is a language algebra with a signature σ.

(b) $\mathsf{M} = \langle U_n, \mathsf{Op}, U_n^*\rangle$ is a logical matrix for L, where $\langle U_n, \mathsf{Op}\rangle$ is an algebra over U_n with the signature σ, as well. Moreover, let $S \in U_n$. Then $U_n^* = \{S+1, \ldots, n-1\}$ is the set of the designated truth-values, and $0, 1, \ldots, S$ are called non-designated ones.

(c) $\mathrm{In} = \{\mathrm{I} \mid \mathrm{I} : \mathsf{V} \to \mathsf{U}_n$ is an interpretation of $\mathsf{L}\}$.

(d) $\mathsf{Pr}^n : \mathcal{P}(\mathsf{F}) \to \mathcal{P}(\mathsf{F})$ should be a consequence operation.

Now, we look for a consequence operation.

Let $\mathsf{L} = \langle \mathsf{V}, \mathsf{Con}, \mathsf{F} \rangle$ be a language algebra and let $\mathsf{M} = \langle \mathsf{U}_n, \mathsf{Op}, \mathsf{U}_n^* \rangle$ be a logical matrix for the language L.

Definition 10 *A formula α is a weak semantic consequence of a set X of formulas, denoted as*

$$\mathsf{X} \models_{\mathsf{S}} \alpha,$$

if for any interpretation in which the truth-value of every formula $\beta \in \mathsf{X}$ is designated, the truth-value of α is also designated. If X is the empty set, we have no constraint for the interpretations. Thus, α is said to be an S-tautology ($\emptyset \models_{\mathsf{S}} \alpha$) if the truth-value of α is designated for every interpretation.

We can give a more rigorous notion of the consequence relation if we also take the extent of truth-values of formulas into consideration.

Definition 11 *A formula α is a strong semantic consequence of a set X of formulas, denoted as*

$$\mathsf{X} \models_{\mathsf{S}^*} \alpha,$$

if for any interpretation in which the truth-value of every formula $\beta \in \mathsf{X}$ is designated, the truth-value of α is also designated with at least the same truth-value as the minimum of the truth-values of formulas in X in the underlying interpretation.

We need some further notions and a lemma to discuss simply the characteristic of the consequence relation.

Definition 12 *Let a partial interpretation* Ip *be a total interpretation with respect to the set $\mathsf{X} \cup \{\alpha\}$ of formulas. X is appropriate for α in* Ip *if the truth-value of α is not less than the minimum of the truth-values of formulas in X, whenever this minimum is designated.*

Definition 13 X *is finitely bad for α with respect to a partial interpretation* Ip *if for all finite subsets Y of X there exists an extension of* Ip *in which Y is not appropriate for α.*

Lemma 14 *If* X *is finitely bad for* α *with respect to a partial interpretation* Ip *and the variable* ν *has no value in* Ip *yet, then there is some* $j \in U_n$ *such that* X *is also finitely bad for* α *with respect to the partial interpretation* $\mathrm{Ip} \cup \{(\nu, j)\}$.

Proof. Otherwise, X is not finitely bad for α with respect to any partial interpretation $\mathrm{Ip} \cup \{(\nu, i)\}$ $(i \in U_n)$. So for all i, a finite subset Y_i of X would exist such that Y_i would be appropriate for α in all of the total extensions of $\mathrm{Ip} \cup \{(\nu, i)\}$. Thus, $\cup_{i=0}^{n-1} Y_i$ is a finite set and appropriate for α in all of the total extensions of Ip. It means that X is not finitely bad for α with respect to a partial interpretation Ip. It is a contradiction. □

Proposition 15 $\mathrm{Pr}_S^n : X \to \{\alpha \mid X \models_S \alpha\}$ *and* $\mathrm{Pr}_{S^*}^n : X \to \{\alpha \mid X \models_{S^*} \alpha\}$ *are consequence operations.*

Proof.

(1) is obvious.

(2) For every interpretation I', where $q = \min_{\alpha \in X} |\alpha|_{I'} > S$, $|\gamma|_{I'} \geq q$ holds for any $\gamma \in \mathrm{Pr}_S^n(X)$. Now, let I be an interpretation, where $|\beta|_I > S$ for every formula β in Y, and let p be $\min_{\beta \in Y} |\beta|_I$. According to condition $X \subseteq \mathrm{Pr}_S^n(Y)$, we get $|\alpha|_I \geq p > S$ for every $\alpha \in X$. Thus, I is an interpretation, where $|\gamma|_I \geq p$ holds for all $\gamma \in \mathrm{Pr}_S^n(X)$. It means, we have $\mathrm{Pr}_S^n(X) \subseteq \mathrm{Pr}_S^n(Y)$.

(3) Now, let α be a strong consequence of X. Then α is also a weak consequence of X. Let us define a special kind of negation:

$$\neg x \rightleftharpoons \begin{cases} 0 & \text{if } x \in U_n^* \ , \\ (n-1) & \text{otherwise} \end{cases}$$

Moreover, let In_X contain all the interpretations in which every formula in X has a designated truth-value.

It is clear that $X \models_S \alpha$ if and only if $In_X \cap In_{\neg\alpha} = \emptyset$. Because of the compactness theorem in MVPL (see in [4]), if $In_X \cap In_{\neg\alpha} = \emptyset$, then there exists a finite set Y_0 such that $Y_0 \subseteq X$ and $In_{Y_0} \cap In_{\neg\alpha} = \emptyset$.

Thus, Y_0 is a finite subset of X and $Y_0 \models_S \alpha$. It means, that for any interpretation in which the truth-value of every formula in Y_0 is designated, the truth-value of α is also designated. At the same time, the truth-value of α may be less than the minimum of the truth-values of formulas in Y_0 in several (however finite number of) interpretations.

Now, let the partial interpretation Ip be total with respect to $Y_0 \cup \{\alpha\}$ such that Y_0 is not appropriate for α in Ip. We prove that there is a finite subset Y of X such that $Y_0 \cup Y$ is appropriate for α in all of the total extensions of Ip.

Let $v_1, v_2, \dots$ be a list of variables not occuring in Ip. Assume that the opposite of what we are trying to prove is true: X is finitely bad for α with respect to Ip.

In view of Lemma 14, for all k there is some $j_k \in U_n$ such that X is also finitely bad for α with respect to $Ip_k = Ip_{k-1} \cup \{(v_k, j_k)\}$. In the total interpretation $\cup_{k=1}^{\infty} Ip_k$, there is an index k for all $\gamma \in X$ such that Ip_k is total with respect to γ. Since X is finitely bad for α with respect to Ip_k, $|\gamma|_{Ip_k} > |\alpha|_{Ip_k}$. It means that X is not appropriate for α in the interpretation $\cup_{k=1}^{\infty} Ip_k$, so α is not a strong consequence of X, a contradiction.

Our indirect assumption is false, so there is a finite subset Y of X such that $Y_0 \cup Y$ is appropriate for α in all extensions of Ip.

To sum it up, if we have some interpretations, in which the truth-value of α is less than the minimum of the truth-values of formulas in Y_0, when it is designated, then we have no more than finitely many such interpretations. For every such interpretation, there exists a finite subset Y of X such that $Y_0 \cup Y$ is appropriate for α in all extensions of the interpretation. Adding the union of finite number of the finite subsets to Y_0 we get a finite set and in every interpretation, if the minimum of the truth-values of formulas in this set is designated, the minimum is not greater than the truth-value of α. □

5 Problems with n-valued operations

In the classical logic, the consequence notion leads to the decision problem through the deduction theorem. The deduction theorem requires the classical syllogism, modus ponens.

In a many-valued logic with the weak consequence relation, the modus ponens is valid if we have an operation $\supset$ with $\alpha \supset \beta, \alpha \models_S \beta$, i.e. if $\alpha \supset \beta$ and α have designated values, then β has a designated value, too.

The Łukasiewicz implication is defined by

$$x_1 \supset_L x_2 \rightleftharpoons \begin{cases} n-1 & \text{if } x_1 \leq x_2, \\ (n-1) - x_1 + x_2 & \text{if } x_1 > x_2, \end{cases}$$

or by Table 1. Designated values are marked by an asterisk.

$\supset_L$	0	1	$\cdots$	S	$S+1^*$	$\cdots$	$n-3^*$	$n-2^*$	$n-1^*$
0	$n-1$	$n-1$	$\cdots$	$n-1$	$n-1$	$\cdots$	$n-1$	$n-1$	$n-1$
1	$n-2$	$n-1$	$\cdots$	$n-1$	$n-1$	$\cdots$	$n-1$	$n-1$	$n-1$
2	$n-3$	$n-2$	$\cdots$	$n-1$	$n-1$	$\cdots$	$n-1$	$n-1$	$n-1$
$\vdots$			$\cdots$			$\cdots$			
$S-1$	$n-S$	$n-S+1$	$\cdots$	$n-1$	$n-1$	$\cdots$	$n-1$	$n-1$	$n-1$
S	$n-S-1$	$n-S$	$\cdots$	$n-1$	$n-1$	$\cdots$	$n-1$	$n-1$	$n-1$
$S+1^*$	$n-S-2$	$n-S-1$	$\cdots$	$n-2$	$n-1$	$\cdots$	$n-1$	$n-1$	$n-1$
$\vdots$			$\cdots$			$\cdots$			
$n-3^*$	2	3	$\cdots$	$S+2$	$S+3$	$\cdots$	$n-1$	$n-1$	$n-1$
$n-2^*$	1	2	$\cdots$	$S+1$	$S+2$	$\cdots$	$n-2$	$n-1$	$n-1$
$n-1^*$	0	1	$\cdots$	S	$S+1$	$\cdots$	$n-3$	$n-2$	$n-1$

Table 1: Łukasiewicz implication

We can see that if $S < n-2$, then $S+1 \supset_L S$ and $S+1$ are designated, but S is not. The modus ponens is not valid in such many-valued logics and moreover, it is not valid when the consequence relation is the second one.

The Post implication is defined by

$$x_1 \supset_P x_2 \rightleftharpoons \begin{cases} n-1 & \text{if } x_1 \le x_2, \\ x_2 & \text{if } x_1 > x_2,\ x_1 > S, \\ (n-1) - x_1 + x_2 & \text{if } x_1 > x_2,\ x_1 \le S, \end{cases}$$

or by Table 2.

The modus ponens is valid in the case of Post implication:

Proposition 16

$$\alpha \supset_P \beta, \alpha \models_{S^*} \beta.$$

Proof. If $|\alpha \supset_P \beta| > S$ and $|\alpha| > S$ in an interpretation, then either $|\alpha| > |\beta|$ or $|\alpha| \le |\beta|$. In the first case, $S < |\alpha \supset_P \beta| = |\beta|$ and $\min(|\alpha|, |\alpha \supset_P \beta|) = |\beta|$. In the second case, $|\alpha \supset_P \beta| = n-1$, so $\min(|\alpha|, |\alpha \supset_P \beta|) = |\alpha| \le |\beta|$. □

Now, we must verify whether the deduction theorem is valid. The deduction theorem would state that $X, \alpha \models_S \beta$ if and only if $X \models_S \alpha \supset_P \beta$. It is easy to realize, this theorem is not valid: if $X, \alpha \models_S \beta$, $X \models_S \alpha \supset_P \beta$ does not necessarily follow.

Actually, let $n-1 \le 2S$ and $\gamma, \alpha \models_S \beta$. There is no constraint for the truth-value of β in the interpretations where α is not designated. So $|\gamma| =$

$\supset_P$	0	1	$\cdots$	S	$S+1^*$	$\cdots$	$n-3^*$	$n-2^*$	$n-1^*$
0	$n-1$	$n-1$	$\cdots$	$n-1$	$n-1$	$\cdots$	$n-1$	$n-1$	$n-1$
1	$n-2$	$n-1$	$\cdots$	$n-1$	$n-1$	$\cdots$	$n-1$	$n-1$	$n-1$
2	$n-3$	$n-2$	$\cdots$	$n-1$	$n-1$	$\cdots$	$n-1$	$n-1$	$n-1$
$\vdots$			$\cdots$			$\cdots$			
$S-1$	$n-S$	$n-S+1$	$\cdots$	$n-1$	$n-1$	$\cdots$	$n-1$	$n-1$	$n-1$
S	$n-S-1$	$n-S$	$\cdots$	$n-1$	$n-1$	$\cdots$	$n-1$	$n-1$	$n-1$
$S+1^*$	0	1	$\cdots$	S	$n-1$	$\cdots$	$n-1$	$n-1$	$n-1$
$\vdots$			$\cdots$			$\cdots$			
$n-3^*$	0	1	$\cdots$	S	$S+1$	$\cdots$	$n-1$	$n-1$	$n-1$
$n-2^*$	0	1	$\cdots$	S	$S+1$	$\cdots$	$n-3$	$n-1$	$n-1$
$n-1^*$	0	1	$\cdots$	S	$S+1$	$\cdots$	$n-3$	$n-2$	$n-1$

Table 2: Post implication

$n-1$, $|\alpha| = S$ and $|\beta| = 0$ might hold in an interpretation. In this case γ is designated, but $|\alpha \supset_P \beta| = (n-1) - S \leq S$ is not. Thus, $\gamma \models_S \alpha \supset_P \beta$ is not valid.

$\supset_H$	0	1	$\cdots$	S	$S+1^*$	$\cdots$	$n-3^*$	$n-2^*$	$n-1^*$
0	$n-1$	$n-1$	$\cdots$	$n-1$	$n-1$	$\cdots$	$n-1$	$n-1$	$n-1$
1	0	$n-1$	$\cdots$	$n-1$	$n-1$	$\cdots$	$n-1$	$n-1$	$n-1$
2	0	1	$\cdots$	$n-1$	$n-1$	$\cdots$	$n-1$	$n-1$	$n-1$
$\vdots$			$\cdots$			$\cdots$			
$S-1$	0	1	$\cdots$	$n-1$	$n-1$	$\cdots$	$n-1$	$n-1$	$n-1$
S	0	1	$\cdots$	$n-1$	$n-1$	$\cdots$	$n-1$	$n-1$	$n-1$
$S+1^*$	0	1	$\cdots$	S	$n-1$	$\cdots$	$n-1$	$n-1$	$n-1$
$\vdots$			$\cdots$			$\cdots$			
$n-3^*$	0	1	$\cdots$	S	$S+1$	$\cdots$	$n-1$	$n-1$	$n-1$
$n-2^*$	0	1	$\cdots$	S	$S+1$	$\cdots$	$n-3$	$n-1$	$n-1$
$n-1^*$	0	1	$\cdots$	S	$S+1$	$\cdots$	$n-3$	$n-2$	$n-1$

Table 3: Heyting implication

The Heyting implication is often used in a many-valued logic: $x_1 \supset_H x_2$ is the greatest element in U_n such that $x_1 \wedge (x_1 \supset_H x_2) \leq x_2$, that is for every $x_1, x_2 \in U_n$,

$$x_1 \supset_H x_2 \rightleftharpoons \begin{cases} n-1 & \text{if } x_1 \leq x_2, \\ x_2 & \text{if } x_1 > x_2, \end{cases}$$

or see Table 3.

Proposition 17

$$\alpha \supset_H \beta, \alpha \models_{S^*} \beta.$$

Proof. If $|\alpha \supset_H \beta| > S$ and $|\alpha| > S$ hold in an interpretation, then by the definition of the Heyting implication

(1) if $|\alpha| > |\beta|$, $|\alpha \supset_H \beta| = |\beta|$, thus $|\beta| > S$ and $\min(|\alpha \supset_H \beta|, |\alpha|) = |\beta|$, and

(2) if $|\alpha| \leq |\beta|$, thus $|\beta| > S$ and, because $|\alpha \supset_H \beta| = n-1$, thus $\min(|\alpha \supset_H \beta|, |\alpha|) = |\alpha| \leq |\beta|$.

□

It is easy to see, the deduction theorem is not valid: if $X, \alpha \models_S \beta$, it does not necessarily follow that $X \models_S \alpha \supset_H \beta$.

Actually, let $\gamma, \alpha \models_S \beta$. There is no constraint for the truth-value of β in the interpretations where α is not designated. So it can happen that $|\gamma| = n-1$, $|\alpha| = S$ and $|\beta| = 0$ hold in an interpretation. In this case γ is designated, but $|\alpha \supset_H \beta| = 0$ is not. So, $\gamma \models_S \alpha \supset_H \beta$ is not valid.

$\supset_R$	0	1	$\cdots$	S	$S+1^*$	$\cdots$	$n-3^*$	$n-2^*$	$n-1^*$
0	$n-1$	$n-1$	$\cdots$	$n-1$	$n-1$	$\cdots$	$n-1$	$n-1$	$n-1$
1	$n-1$	$n-1$	$\cdots$	$n-1$	$n-1$	$\cdots$	$n-1$	$n-1$	$n-1$
2	$n-1$	$n-1$	$\cdots$	$n-1$	$n-1$	$\cdots$	$n-1$	$n-1$	$n-1$
$\vdots$			$\cdots$			$\cdots$			
$S-1$	$n-1$	$n-1$	$\cdots$	$n-1$	$n-1$	$\cdots$	$n-1$	$n-1$	$n-1$
S	$n-1$	$n-1$	$\cdots$	$n-1$	$n-1$	$\cdots$	$n-1$	$n-1$	$n-1$
$S+1^*$	0	1	$\cdots$	S	$S+1$	$\cdots$	$n-3$	$n-2$	$n-1$
$\vdots$			$\cdots$			$\cdots$			
$n-3^*$	0	1	$\cdots$	S	$S+1$	$\cdots$	$n-3$	$n-2$	$n-1$
$n-2^*$	0	1	$\cdots$	S	$S+1$	$\cdots$	$n-3$	$n-2$	$n-1$
$n-1^*$	0	1	$\cdots$	S	$S+1$	$\cdots$	$n-3$	$n-2$	$n-1$

Table 4: Rosser implication

In [6], another implication has been used by Rosser:

$$x_1 \supset_R x_2 \rightleftharpoons \begin{cases} n-1 & \text{if } x_1 \leq S, \\ x_2 & \text{if } x_1 > S. \end{cases}$$

Table 4 shows the truth-table of this implication. In this paper we name this implication after Rosser.

Proposition 18

$$\alpha \supset_R \beta, \alpha \models_{S^*} \beta.$$

Proof. If $|\alpha \supset_R \beta| > S$ and $|\alpha| > S$ hold in an interpretation, then by the definition of the Rosser implication $|\alpha \supset_R \beta| = |\beta|$, thus $|\beta| > S$ and $\min(|\alpha \supset_R \beta|, |\alpha|) \leq |\beta|$. □

Proposition 19 *If* $X, \alpha \models_S \beta$*, then* $X \models_S \alpha \supset_R \beta$.

Proof. Suppose $X, \alpha \models_S \beta$. In every interpretation where every formula from X is designated, either α is also deignated or not. In the first case, according to the condition, β is designated, and because $|\alpha \supset_R \beta| = |\beta|$, $\alpha \supset_R \beta$ is designated, too. In the second case, according to the definition of the Rosser implication, we have $|\alpha \supset_R \beta| = n - 1$. This is a designated value. Therefore, $X \models \alpha \supset_R \beta$. □

We can not prove that if $\gamma, \alpha \models_{S^*} \beta$, then $\gamma \models_{S^*} \alpha \supset_R \beta$. If $|\alpha| = |\beta| = S + 1$ and $|\gamma| = n - 1$ in an interpretation, then $|\alpha \supset_R \beta| = |\beta| = S + 1$, thus $\alpha \supset_R \beta$ is designated, but if $S + 1 < n - 1$, then $|\gamma| > |\alpha \supset_R \beta|$.

Proposition 20 *If* $X \models_{S^*} \alpha \supset_R \beta$*, then* $X, \alpha \models_{S^*} \beta$.

Proof. Let I be an interpretation in which every formula from X and α are designated. According to the condition, $\alpha \supset_R \beta$ is designated with truth-value at least $\min_{\gamma \in X}\{|\gamma|\}$. If α is designated, $|\beta| = |\alpha \supset_R \beta|$, thus β is also designated with truth-value at least $\min_{\gamma \in X}\{|\gamma|\} \geq \min_{\gamma \in X}\{|\gamma|, |\alpha|\}$. □

Finally, let $f : U \times U \to U$ such that $f(x_1, x_2) \leq S$ for all $x_1, x_2 \in U$, when $x_1 > S$ and $x_2 \leq S$. Then the implication defined below admits the modus ponens and the deduction theorem:

$$x_1 \supset_*^f x_2 \rightleftharpoons \begin{cases} n - 1 & \text{if } x_1 \leq S \text{ or } x_1 \leq x_2, \\ x_2 & \text{if } x_1 > x_2 > S, \\ f(x_1, x_2) & \text{otherwise.} \end{cases}$$

Proposition 21

$$\alpha \supset_*^f \beta, \alpha \models_{S^*} \beta.$$

$\supset^f_*$	0	1	$\cdots$	S	$S+1^*$	$\cdots$	$n-3^*$	$n-2^*$	$n-1^*$
0	$n-1$	$n-1$	$\cdots$	$n-1$	$n-1$	$\cdots$	$n-1$	$n-1$	$n-1$
1	$n-1$	$n-1$	$\cdots$	$n-1$	$n-1$	$\cdots$	$n-1$	$n-1$	$n-1$
2	$n-1$	$n-1$	$\cdots$	$n-1$	$n-1$	$\cdots$	$n-1$	$n-1$	$n-1$
$\vdots$			$\cdots$			$\cdots$			
$S-1$	$n-1$	$n-1$	$\cdots$	$n-1$	$n-1$	$\cdots$	$n-1$	$n-1$	$n-1$
S	$n-1$	$n-1$	$\cdots$	$n-1$	$n-1$	$\cdots$	$n-1$	$n-1$	$n-1$
$S+1^*$	0	1	$\cdots$	S	$n-1$	$\cdots$	$n-1$	$n-1$	$n-1$
$\vdots$			$\cdots$			$\cdots$			
$n-3^*$	0	1	$\cdots$	S	$S+1$	$\cdots$	$n-1$	$n-1$	$n-1$
$n-2^*$	0	1	$\cdots$	S	$S+1$	$\cdots$	$n-3$	$n-1$	$n-1$
$n-1^*$	0	1	$\cdots$	S	$S+1$	$\cdots$	$n-3$	$n-2$	$n-1$

Table 5: The implication with $f(x_1, x_2) = x_2$

Proof. If $|\alpha \supset^f_* \beta| > S$ and $|\alpha| > S$ in an interpretation, then by the definition of the new implication either $|\alpha \supset^f_* \beta| = n-1$, or $|\alpha \supset^f_* \beta| = |\beta|$. In the first case $|\beta| \geq |\alpha| > S$ and $\min(|\alpha \supset^f_* \beta|, |\alpha|) \leq |\beta|$. In the second case $|\alpha| > |\beta| > S$ and $\min(|\alpha \supset^f_* \beta|, |\alpha|) = |\beta|$. □

Proposition 22 *If* $X, \alpha \models_{S^*} \beta$*, then* $X \models_{S^*} \alpha \supset^f_* \beta$*.*

Proof. Suppose $X, \alpha \models_S \beta$. In every interpretation where every formula from X is designated, either α is also deignated or not. In the first case, according to the condition, β is designated, and

(1) either $S < |\alpha| \leq |\beta|$ and $|\alpha \supset^f_* \beta| = n-1$, so $\min_{\gamma \in X}\{|\gamma|\} \leq |\alpha \supset^f_* \beta|$;

(2) or $|\alpha| > |\beta| > S$ and $|\alpha \supset^f_* \beta| = |\beta|$, thus $\alpha \supset^f_* \beta$ is also designated moreover, $\min_{\gamma \in X}\{|\gamma|\} \leq |\beta| = |\alpha \supset^f_* \beta|$ because $\min_{\gamma \in X}\{|\gamma|, |\alpha|\} \leq |\beta| < |\alpha|$.

In the second case, $|\alpha \supset^f_* \beta| = n-1$, which is a designated value. Therefore, $X \models_{S^*} \alpha \supset^f_* \beta$. □

Proposition 23 *If* $X \models_{S^*} \alpha \supset^f_* \beta$*, then* $X, \alpha \models_{S^*} \beta$*.*

Proof. Let I be an interpretation in which every formula from X and α are designated. According to the condition, $\alpha \supset^f_* \beta$ is designated with truth-value at least $\min_{\gamma \in X}\{|\gamma|\}$. Because α is designated, if $|\alpha| \leq |\beta|$, then $|\beta|$ is designated with truth-value at least $\min_{\gamma \in X}\{|\gamma|, |\alpha|\}$. In the case $|\alpha| > |\beta|$, $|\alpha \supset_*^f \beta| = |\beta|$, thus β is designated with truth-value at least

$$\min_{\gamma \in X}\{|\gamma|\} \geq \min_{\gamma \in X}\{|\gamma|, |\alpha|\},$$

as well. □

Finally, all what was proved about the examined implications at this section we summarized in the following table:

	$\supset_L$	$\supset_P$	$\supset_H$	$\supset_R$	$\supset_*^f$
modus ponens	-	+	+	+	+
deduction theorem with $\models_S$	-	-	-	+	+
deduction theorem with $\models_{S^*}$	-	-	-	-	+

6 Suitable implication for a given consequence

It is desirable that both the modus ponens and the deduction theorem hold with respect to the underlying consequence. Now, we look for a suitable implication for a generally given consequence notion such that both the modus ponens and the deduction theorem are valid.

Now, let $\psi : U \times U \to \{0, 1\}$ be an arbitrary classical truth-valued function with the following properties:

(a) $\psi(x, x) = 1$ for all $x \in U$,

(b) if $\psi(x, y) \wedge \psi(y, z) = 1$, then $\psi(x, z) = 1$ for all $x, y, z \in U$.

Then, define the consequence as below:

Definition 24 *A formula α is a formal semantic consequence of a set X of formulas, denoted as $X \models \alpha$, if*

$$\bigvee_{\gamma \in X} \psi(|\gamma|_I, |\alpha|_I) = 1 \textit{ for any interpretation } I,$$

where $\bigvee_{\gamma \in X} \psi(|\gamma|_I, |\alpha|_I)$ denotes the supremum of $\{\psi(|\gamma|_I, |\alpha|_I) \mid \gamma \in X\}$.

Proposition 25 $\mathsf{Pr} : X \to \{\alpha \mid X \models \alpha\}$ *satisfies conditions (1)-(3) in Definition 2.*

Proof.

(1) Now to prove the condition (1), let $\alpha \in X$. Since in any interpretation $\psi(|\alpha|, |\alpha|) = 1$, therefore $\bigvee_{\gamma \in X} \psi(|\gamma|, |\alpha|) = 1$, so $X \models \alpha$. It means that $X \subseteq \mathsf{Pr}(X)$.

(2) Next, let $X \subseteq \mathsf{Pr}(Y)$ for some $X, Y \subseteq F$. We show that $\mathsf{Pr}(X) \subseteq \mathsf{Pr}(Y)$.

- For any $\alpha \in \mathsf{Pr}(X)$, since

$$\bigvee_{\gamma \in X} \psi(|\gamma|_I, |\alpha|_I) = 1, \text{ so } \bigvee_{\gamma \in \mathsf{Pr}(Y)} \psi(|\gamma|_I, |\alpha|_I) = 1.$$

  It means $\mathsf{Pr}(X) \subseteq \mathsf{Pr}(\mathsf{Pr}(Y))$.

- Now, we show that $\mathsf{Pr}(\mathsf{Pr}(Y)) = \mathsf{Pr}(Y)$. Since $\mathsf{Pr}(Y) \subseteq \mathsf{Pr}(\mathsf{Pr}(Y))$ by the property (1), it is enough to prove, that $\alpha \in \mathsf{Pr}(Y)$ for all $\alpha \in \mathsf{Pr}(\mathsf{Pr}(Y))$.

  Obviously $Y \subseteq \mathsf{Pr}(Y)$. Let Y' denote the set $\mathsf{Pr}(Y) \setminus Y$ and let $\alpha \in \mathsf{Pr}(\mathsf{Pr}(Y))$. Then

$$\bigvee_{\gamma \in \mathsf{Pr}(Y)} \psi(|\gamma|_I, |\alpha|_I) = \bigvee_{\gamma \in Y' \cup Y} \psi(|\gamma|_I, |\alpha|_I) = 1.$$

  If

$$\bigvee_{\gamma \in Y'} \psi(|\gamma|_I, |\alpha|_I) = 0, \text{ then } \bigvee_{\gamma \in Y} \psi(|\gamma|_I, |\alpha|_I) = 1.$$

  And if

$$\bigvee_{\gamma \in Y'} \psi(|\gamma|_I, |\alpha|_I) = 1,$$

  then there exists a $\gamma' \in Y'$ for which $\psi(|\gamma'|_I, |\alpha|_I) = 1$. But $\gamma' \in \mathsf{Pr}(Y)$ also holds, thus

$$\bigvee_{\gamma \in Y} \psi(|\gamma|_I, |\gamma'|_I) = 1$$

  must hold, i.e. there exists a $\gamma'' \in Y$ for which $\psi(|\gamma''|_I, |\gamma'|_I) = 1$. Using the property (b) of ψ we get $\psi(|\gamma''|_I, |\alpha|_I) = 1$, that is

$$\bigvee_{\gamma \in Y} \psi(|\gamma|_I, |\alpha|_I) = 1.$$

  Thus in both cases, we get $\alpha \in \mathsf{Pr}(Y)$.

– Since $X \subseteq \mathsf{Pr}(Y)$, thus $\mathsf{Pr}(X) \subseteq \mathsf{Pr}(\mathsf{Pr}(Y))$, and thereby $\mathsf{Pr}(X) \subseteq \mathsf{Pr}(Y)$ must hold.

(3) We prove compactness by reducing the problem to the compactness of first-order logic with equality. First, let us define the language of our encoding:

- We have a single binary predicate symbol $\widehat{\psi}$.
- For each many-valued operation o, we have a corresponding function symbol $\hat{o}$ with the same arity.
- For each variable v, we have a corresponding constant c_v.
- For each truth-value u, we have an additional constant $\hat{u}$.

Given this language, we might fix the interpretation of our symbols by defining a set Σ of the following axioms:

(i) $\forall x(x = \hat{u}_1 \vee x = \hat{u}_2 \vee \cdots \vee x = \hat{u}_n)$ if $U = \{u_1, u_2, \ldots, u_n\}$

(ii) $\hat{u} \neq \widehat{u'}$ for each $u, u' \in U$ with $u \neq u'$

(iii) $\widehat{\psi}(\hat{u}, \widehat{u'})$ if $\psi(u, u') = 1$ and $u, u' \in U$

(iv) $\neg\widehat{\psi}(\hat{u}, \widehat{u'})$ if $\psi(u, u') = 0$ and $u, u' \in U$

(v) $\hat{o}(\hat{a_1}, \hat{a_2}, \ldots, \hat{a_k}) = \hat{u}$ if o is an operator with arity k, $a_1, a_2, \ldots, a_k, u \in U$, and $o(a_1, a_2, \ldots, a_k) = u$

Since U is finite, Σ is a finite set of first-order formulas as well. It is easy to see that if $\hat{I}$ is a first-order model of Σ, then there is a corresponding many-valued interpretation I which assigns the same values to variables as did $\hat{I}$ to the corresponding constants.

Let $\hat{\alpha}$ denote the encoding of a formula α in this language, i.e. the first-order formula we get from α by substituting each symbol with the corresponding first-order symbol. By our definitions, if I and $\hat{I}$ are corresponding many-valued and first-order interpretations, $|\alpha|_I = u$ if and only if $|\hat{\alpha}|_{\hat{I}} = \hat{u}$. Thus, for each α, β, we have $\psi(|\alpha|_I, |\beta|_I)$ holds if and only if $\widehat{\psi}(\hat{\alpha}, \hat{\beta})$ holds in $\hat{I}$.

Now, by our assumptions, $X \models \alpha$ if and only if $\bigvee_{\gamma \in X} \psi(|\gamma|_I, |\alpha|_I)$ holds for all interpretation I. This, on the other hand, holds if and only if the set $\Gamma = \{\neg\, \psi(|\gamma|_I, |\alpha|_I) \mid \gamma \in X\}$ is not satisfied under any interpretation I. Consider the first-order set

$$\hat{\Gamma} = \Sigma \cup \{\neg\, \widehat{\psi}(\hat{\gamma}, \hat{\alpha}) \mid \gamma \in X\}$$

From our considerations it follows that $\hat{\Gamma}$ is unsatisfiable if and only if the original Γ is unsatisfiable.

Then, by the compactness of first-order logic, we know that there is a finite $\hat{\Gamma}' \subseteq \hat{\Gamma}$ such that $\hat{\Gamma}'$ is unsatisfiable. Since Σ is finite, we might assume $\Sigma \subseteq \hat{\Gamma}'$. Now, let X' the finite set $\{\gamma \in X \mid \neg\hat{\psi}(\hat{\gamma}, \hat{\alpha}) \in \hat{\Gamma}'\}$.

We know that the corresponding set $\Gamma' = \{\neg\psi(|\gamma|_I, |\alpha|_I) \mid \gamma \in X'\}$ is not satisfied by any I either. Therefore, $X' \models \alpha$ must hold where X' is a finite subset of X. □

Proposition 26 *Let* $\supset$ *be an implication operation over* U. *If*

$$\psi(x_1, x_2) \vee \psi(y, x_2) = \psi(y, x_1 \supset x_2)$$

for all $x_1, x_2, y \in U$, *then* $\supset$ *admits the modus ponens and the deduction theorem.*

Proof. First, we prove the modus ponens, i.e. we show, that $\{\alpha, \alpha \supset \beta\} \models \beta$ holds for any formulas α, β. For all $\alpha, \beta \in F$ and for all $I \in In$ we get

$$\psi(|\alpha|_I, |\beta|_I) \vee \psi(|\alpha \supset \beta|_I, |\beta|_I).$$

For all $x_1, x_2 \in U$, by applying the proposed equality

$$\psi(x_1, x_2) \vee \psi(x_1 \supset x_2, x_2) = \psi(x_1 \supset x_2, x_1 \supset x_2) = 1.$$

To prove the deduction theorem, we have to show for any α, β, X

$$X, \alpha \models \beta \text{ if and only if } X \models \alpha \supset \beta.$$

Again, applying our assumptions to both sides, we get for all $I \in In$

$$\bigvee_{\gamma \in X} \psi(|\gamma|_I, |\beta|_I) \vee \psi(|\alpha|_I, |\beta|_I) = \bigvee_{\gamma \in X} (\psi(|\gamma|_I, |\beta|_I) \vee \psi(|\alpha|_I, |\beta|_I))$$

if and only if for all $I \in In$

$$\bigvee_{\gamma \in X} \psi(|\gamma|_I, |\alpha|_I \supset |\beta|_I).$$

From our assumption with $y = |\gamma|_I, x_1 = |\alpha|_I, x_2 = |\beta|_I$, we get

$$\psi(|\gamma|_I, |\beta|_I) \vee \psi(|\alpha|_I, |\beta|_I) = \psi(|\gamma|_I, |\alpha|_I \supset |\beta|_I),$$

from which the desired equivalence immediately follows. □

In the remaining part of the section we apply this proposition to the earlier defined semantic consequences.

Example 27 *By Definition 10,*

$$X \models_S \alpha \text{ if and only if } \min_{\gamma\in X}\{|\gamma|_I\} \leq S \vee S < |\alpha|_I \text{ for all } I \in In.$$

Thus, for this case we get $\psi(x, y) = (x \leq S \vee S < y)$. *To find a suitable implication, it is enough to satisfy*

$$(x_1 \leq S \vee S < x_2) \vee (y \leq S \vee S < x_2) \text{ if and only if } (y \leq S \vee S < x_1 \supset x_2)$$

for all $x_1, x_2, y \in U$. *Let* $f, h : U \times U \to U$ *such that for all* $x_1 > S$ *and* $x_2 \leq S$ $f(x_1, x_2) \leq S$ *and if* $x_1 \leq S$ *or* $x_2 > S$, *then* $h(x_1, x_2) > S$. *Then, as we have seen above, the implication defined below admits the modus ponens and the deduction theorem:*

$$x_1 \supset_*^{f,h} x_2 \rightleftharpoons \begin{cases} h(x_1, x_2) & \text{if } x_1 \leq S \text{ or } x_2 > S, \\ f(x_1, x_2) & \text{otherwise.} \end{cases}$$

Example 28 *By Definition 11,*

$$X \models_{S^*} \alpha \text{ if and only if } \min_{\gamma\in X}\{|\gamma|_I\} \leq S \vee \min_{\gamma\in X}\{|\gamma|_I\} \leq |\alpha|_I \text{ for all } I \in In.$$

For this case we get $\psi(x, y) = x \leq S \vee x \leq y$. *Thus to find a suitable implication, it is enough to satisfy*

$$(x_1 \leq S \vee x_1 \leq x_2) \vee (y \leq S \vee y \leq x_2) \text{ if and only if } (y \leq x_1 \supset x_2 \vee y \leq S)$$

for all $x_1, x_2, y \in U$. *The possible values for* $x_1 \supset x_2$ *might be deduced as follows:*

- $x_1 \leq S \vee x_1 \leq x_2$: *since the right side must also hold, even for* $y = n-1$, *we get* $x_1 \supset x_2 = n - 1$, *which is indeed a good choice.*
- $x_1 \geq x_2 > S$: *for* $y = x_2$ *we get* $x_2 \leq x_1 \supset x_2$, *and for* $y = x_2 + 1$ $x_1 \supset x_2 < x_2 + 1$. *Thus only* $x_1 \supset x_2 = x_2$ *is possible, and it indeed satisfies the equality in this case.*
- $x_1 > S \geq x_2$: *for* $y > S$ *we get* $x_1 \supset x_2 \leq S$. *In this case any value smaller than* S *satisfies the equality.*

Let $f : U \times U \to U$ *be such that for all* $x_1 > S$ *and* $x_2 \leq S$ $f(x_1, x_2) \leq S$. *Then, as we have seen above, the implication defined below admits the modus ponens and the deduction theorem:*

$$x_1 \supset_*^{f} x_2 \rightleftharpoons \begin{cases} n-1 & \text{if } x_1 \leq S \text{ or } x_1 \leq x_2, \\ x_2 & \text{if } x_1 > x_2 > S, \\ f(x_1, x_2) & \text{otherwise.} \end{cases}$$

7 Summary

In this paper we demonstrated that both semantic and syntactic consequences of classical logic generate consequence operators. We proved similar propositions about the weak and strong semantic consequences in the many-valued logic. After this, we investigated the Lukasiewicz, Post, Heyting and Rosser style many-valued implications whether the modus ponens rule and the deduction theorem are valid beside of our consequence relations. By the strong consequence, the deduction theorem is not valid with none of them. However, the implication family $\supset_*^{\mathrm{f}}$ defined in our paper found to comply with the modus ponens and the deduction theorem by the strong consequence as well.

In the last section, we introduced a general formal consequence relation and showed, that it also leads to a consequence operator. The weak and strong consequence definitions are realizations of this general consequence notion. It would be profitable to consider what additional realizations are possible. By this general consequence, we also gave a suitable implication which admits the modus ponens and the deduction theorem as well.

The problem of a syntactic treatment of logical consequences in the many-valued logic could be an exciting topic of future research.

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