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A COMBINATORIAL PROPERTY OF CARDINALS

PÉTER KOMJÁTH AND MIKLÓS LACZKOVICH

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Abstract. (GCH) For every cardinal $\kappa \geq \omega_2$ there exists $F : [\kappa]^{\leq 2} \to \{0,1\}$ such that for every $f : \kappa \to [\kappa]^{<\omega}$, $i < 2$, there are x, y such that $F(x,t) = i$ $(t \in f(y))$, $F(u,y) = i$ $(u \in f(x))$.

Let X be a nonempty set, and let F be a coloring of $[X]^{\leq 2} = \{H \subset X : |H| \leq 2\}$ with two colors; that is, let $F : [X]^{\leq 2} \to \{0,1\}$. If f is a map from X into $[X]^{<\omega} = \{Y \subseteq X : Y \text{ is finite}\}$, then we say that the pair (x,y) is a 0-pair, if $x, y \in X$, $F(x,t) = 0$ for every $t \in f(y)$, and $F(u,y) = 0$ for every $u \in f(x)$. The definition of 1-pairs is analogous.

For an infinite cardinal κ let $P(\kappa)$ denote the following statement. There exists a function $F : [\kappa]^{\leq 2} \to \{0,1\}$ such that for every map $f : \kappa \to [\kappa]^{<\omega}$ there exists a 0-pair and there exists a 1-pair.

We prove that $P(\kappa)$ fails for $\kappa \leq \omega_1$ (Theorem 1). We conjecture that $P(\kappa)$ is true whenever $\kappa \geq \omega_2$, but we can only prove this under GCH (Theorem 2). Nevertheless, our proof works in ZFC for all cardinals κ with $\kappa^{\aleph_2} = \kappa$ (Theorem 3).

We can show that for every cardinal λ with $\mathrm{cf}(\lambda) > \omega_1$ there is a cardinal preserving extension that adds a witness to $P(\omega_2)$ and makes $2^{\aleph_1} = \lambda$ (assuming GCH, Theorem 4).

Definitions and Notation. We use the standard axiomatic set theory notation and notions; see [1]. GCH stands for the Generalized Continuum Hypothesis. If S is a set and κ a cardinal, then we let

$$[S]^{\kappa} = \{x \subseteq S : |x| = \kappa\}, [S]^{<\kappa} = \{x \subseteq S : |x| < \kappa\}, [S]^{\leq 2} = [S]^1 \cup [S]^2.$$

For simplicity we use the notation $F(A,B) = 0$, etc. to denote that $F(x,y) = 0$ holds for every $x \in A$, $y \in B$.

Theorem 1. *$P(\kappa)$ fails for $\kappa \leq \omega_1$.*

Proof. Assume that we are given $F : [\kappa]^{\leq 2} \to \{0,1\}$. We will find a function $f : \kappa \to [\kappa]^{<\omega}$ either with no 0-pairs or else with no 1-pairs.

Case 1. $\kappa = \omega$.

If there is an element a such that $F(x,a) = 1$ holds for every x , we can choose $f(x) = \{a\}$ (for $x < \omega$), this f has no 0-pairs. We can assume, therefore, that

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for every $x < \omega$ the element $g(x)$ satisfies $F\big(x, g(x)\big) = 0$. Now the choice $f(x) = \big\{g(0), \ldots, g(x-1)\big\}$ witnesses the failure of $P(\omega)$, that is, this function has no 1-pair as for $x < y$ we have $g(x) \in f(y)$ and $F\big(x, g(x)\big) = 0$.

Case 2. $\kappa = \omega_1$.

If, for every $\alpha < \omega_1$, there is some $h(\alpha) < \omega_1$ with $F\big(\alpha, \{h(\alpha)\}\big) = 1$ (that is, $F\big(\beta, h(\alpha)\big) = 1$ holds for every $\beta < \alpha$), then the function $f(\alpha) = \{h(\alpha)\}$ has no 0-pair. We can assume, therefore, that there exist a countable set $X = \{\gamma_0, \gamma_1, \ldots\}$ such that for every $\alpha \in \omega_1 - X$ there is some γ_n with $F(\alpha, \gamma_n) = 0$. Decompose $\omega_1 - X$ as $\omega_1 - X = X_0 \cup X_1 \cup \cdots$ where

$$X_n = \{\alpha \in \omega_1 - X : F(\alpha, \gamma_n) = 0\}.$$

Further, let $Y_n = X_n - (X_0 \cup \cdots \cup X_{n-1})$. Again, we can assume, by the argument at the beginning of the proof, that for every x there is some $g(x)$ with $F\big(x, g(x)\big) = 0$. Define for $\alpha \in Y_n \cup \{\gamma_n\}$

$$f(\alpha) = \big\{\gamma_0, \ldots, \gamma_n, g(\gamma_0), \ldots, g(\gamma_n)\big\}.$$

This f has no 1-pair. □

Theorem 2 (GCH)**.** *$P(\kappa)$ holds for every $\kappa \geq \omega_2$.*

Proof. We first consider the case $\kappa = \omega_2$.

For $S \subseteq [\omega]^{\aleph_0}$, $j < 2$ let $T_j(S) \subseteq \omega_2$ be a stationary set such that $T_j(S)$ and $T_{j'}(S')$ are disjoint if either $S \neq S'$ or $j \neq j'$. (Exist as by GCH $\aleph_2^{\aleph_0} = \aleph_2$ and by Solovay's theorem.)

Enumerate the systems consisting of $\aleph_1$ disjoint finite subsets of ω_2 as

$$\big\{\{A_i^\alpha : i < \omega_1\} : \alpha < \omega_2\big\}.$$

We require that $\sup\big(\bigcup\{A_i^\alpha : i < \omega_1\}\big) < \alpha$ by formally allowing that the systems are not defined for some values of α.

We define $F(x, \alpha)$ by transfinite recursion on α (for the values $x < \alpha$).

Assume that $F(x, \beta)$ is defined for $x < \beta < \alpha$. If $\alpha \in T_j(S)$ for some j and S, then, of course, we make $F(S, \{\alpha\}) = j$. Beyond this, we make sure that the following property holds:

$(*)$ for $\beta \leq \beta_1 < \cdots < \beta_n < \alpha$, $j < 2$, there are $\aleph_1$ indices $i < \omega_1$ that $F\big(A_i^\beta, \{\beta_1, \ldots, \beta_n, \alpha\}\big) = j$.

We show that this function F works.

Assume that $f(\alpha)$ is a finite subset of ω_2. We find a 0-pair, the case of getting a 1-pair is similar.

Lemma 1. *There is a countable S such that for every countable $S' \supseteq S$ there are stationary many α such that $F\big(S', \{\alpha\}\big) = 0$ and $f(\alpha) \cap (S' - S) = \emptyset$.*

Proof. Otherwise, for every countable S there are a countable $S' \supseteq S$ and a closed, unbounded set C with the following property; if $\alpha \in C$ and $F\big(S', \{\alpha\}\big) = 0$, then $f(\alpha) \cap (S' - S) \neq \emptyset$. We define by induction the countable sets $S_0, S_1, \ldots$ and closed, unbounded sets, $C_0, C_1, \ldots$ such that $S_0 = \emptyset$ and for S_n the sets S_{n+1}, C_n are as described above. Put $S = \bigcup\{S_n : n < \omega\}$ and $C = \bigcap\{C_n : n < \omega\}$. S is countable, while C is closed, unbounded. Pick $\alpha \in T_j(S) \cap C$ (such an α exists as $T_j(S)$ is stationary). Then $f(\alpha) \cap (S_{n+1} - S_n) \neq \emptyset$ holds for every $n < \omega$ which is impossible, as $f(\alpha)$ is finite. □

From now on fix an S as in Lemma 1.

Lemma 2. *There is a countable $S' \supseteq S$ such that if t is finite, $t \cap S' = \emptyset$, then there are stationary many α such that $f(\alpha) \cap \alpha \subseteq S'$ and $F\big(S \cup t, \{\alpha\}\big) = 0$.*

Proof. For $\alpha \in T_0(S)$, $F(S, \{\alpha\}) = 0$. On this set, the function $f(\alpha) \cap \alpha$ is regressive, so by Fodor's lemma, there is a stationary $H \subseteq T_0(S)$ and a countable $S' \supseteq S$ that $f(\alpha) \cap \alpha \subseteq S'$ holds for $\alpha \in H$. If the statement of Lemma 2 fails, we can inductively choose the disjoint finite sets t_ξ and closed, unbounded sets C_ξ such that for $\alpha \in H \cap C_\xi$ we have $F(t_\xi, \{\alpha\}) \neq 0$. If $C = \bigcap\{C_\xi : \xi < \omega_1\}$, then for the $\aleph_2$ $\alpha \in H \cap C$ we have that $F(t_\xi, \{\alpha\}) \neq 0$ for every $\xi < \omega_1$ which contradicts property (*) of the construction. □

Fix an $S' \supseteq S$ as in Lemma 2.

Lemma 3. *There is a finite t and there are stationary many α such that $F(S', \{\alpha\}) = 0$, $f(\alpha) \cap \alpha \subseteq S \cup t$, $f(\alpha) \cap (S' - S) = \emptyset$.*

Proof. By applying Lemma 1 to our particular pair S, S' we get stationary many α with the first and the last property. By Fodor's lemma, there is a finite t such that for a stationary subset, the second property holds, as well. □

In order to conclude the proof of $P(\omega_2)$ we observe that by Lemma 2 there are $\aleph_2$ many elements β such that $f(\beta) \cap \beta \subseteq S'$ and $F(S \cup t, \{\beta\}) = 0$. We choose a set B consisting of ω_1 of them such that the sets $\{f(\beta) - \beta : \beta \in B\}$ are disjoint. By Lemma 3 there are $\aleph_2$ elements α for which $F(S', \{\alpha\}) = 0$, $f(\alpha) \cap \alpha \subseteq S \cup t$, $f(\alpha) \cap (S' - S) = \emptyset$. If α is one of them which is large enough, then

$$F\big(f(\beta) - \beta, f(\alpha) - \alpha\big) = 0$$

and then $\{\alpha, \beta\}$ is a 0-pair.

We now consider the case when $\kappa > \omega_2$. Let τ be an infinite, regular cardinal. Let $(H, <)$ be an ordered set of cardinality τ^{++} in which A is a co-initial subset of ordinal τ and B is a cofinal set of ordinal τ^+ and every initial- and end-segment has cardinal τ^{++}. Call a subset *up-big* if it has τ^{++} elements in every end-segment, and *down-big* if it has τ^{++} elements in every initial-segment. It is *big* if it is up-big and down-big.

Let $H' \subseteq H$ be a big subset and $f : H' \to [H]^{<\omega}$ a function. For $s \subseteq H$, $x \in H$ let $s > x$ denote that every element of s is greater than x, and likewise for $s < x$.

Lemma 4. *There is an $a \in H$ such that $\{x \in H' : f(x) > a\}$ is up-big.*

Proof. Otherwise, for every $a \in A$ there is some $b(a) \in B$ with

$$\big|\{x \in H' : x > b(a), f(x) > a\}\big| \leq \tau^+.$$

There is a $b \in B$ with $b > b(a)$ for $a \in A$, and then H' can have only at most τ^+ elements above b, a contradiction. □

Lemma 5. *There is a $b \in H$ such that $\{x \in H' : f(x) < b\}$ is down-big.*

Proof. Otherwise, for every $b \in B$ there is some $a(b) \in A$ with

$$\big|\{x \in H' : x < a(b), f(x) < b\}\big| \leq \tau^+.$$

There is an $a \in A$ which assumes the value of $a(b)$ for τ^+ many $b \in B$ and we get that H' has only at most τ^+ elements below a, a contradiction. □

Let $X \supseteq H$ be some set of cardinal κ and assume that either $\mathrm{cf}(\kappa) > \tau^{++}$ or else κ is singular and $\mathrm{cf}(\kappa) < \tau$. Given κ this can be arranged by choosing either $\tau = \omega$ or $\tau = \omega_3$. We are going to construct a function $F : [X]^2 \to [X]^{<\omega}$ witnessing $P(\kappa)$.

Lemma 6. *There is a family of functions $f_\alpha : H_\alpha \to [X]^{<\omega}$ for $\alpha < \kappa$ such that H_α is always big and every $f : H \to [X]^{<\omega}$ restricts to some f_α.*

Proof. In the first case this is obvious as by the GCH and $\mathrm{cf}(\kappa) > \tau^{++}$ the number of all $f : H \to [X]^{<\omega}$ functions is κ, so we can take $H_\alpha = H$ and let $\{f_\alpha : \alpha < \kappa\}$ enumerate the $H \to [X]^{<\omega}$ functions.

In the second case decompose X as an increasing union $X = \bigcup\{X_\xi : \xi < \mathrm{cf}(\kappa)\}$ with $|X_\xi| < \kappa$. Assume that $f : H \to [X]^{<\omega}$. For every $a \in A$ there is some $\xi < \mathrm{cf}(\kappa)$ such that

$$\big|\{x < a : f(x) \subseteq X_\xi\}\big| = \tau^{++}.$$

For every $b \in B$ there is some $\xi < \mathrm{cf}(\kappa)$ such that

$$\big|\{x > b : f(x) \subseteq X_\xi\}\big| = \tau^{++}.$$

By cardinality considerations, there is a ξ that is good for τ many $a \in A$ and τ^+ many $b \in B$ and so $H' = \{x \in H : f(x) \subseteq X_\xi\}$ is big. To finish the proof we only have to remark that given H', $\xi < \mathrm{cf}(\kappa)$ the number of these functions is less than κ, so we have altogether κ many such functions. □

We now describe the definition of F. Let $<_w$ be a well ordering of H into order type τ^{++}.

For x, $y \in H$ we set

$$F(x,y) = \begin{cases} 0, & x < y, x <_w y, \\ 1, & x < y, y <_w x. \end{cases}$$

For $\alpha < \kappa$ choose the elements y_α, $z_\alpha \in X - \bigcup\{\mathrm{Ran}(f_\beta) : \beta \le \alpha\}$ different from each other. Choose also a_α, $b_\alpha \in H$ in such a way that $\{x \in H_\alpha : f_\alpha(x) \cap H > a_\alpha\}$ is up-big and $\{x \in H_\alpha : f_\alpha(x) \cap H < b_\alpha\}$ is down-big.

We now define F for some further pairs:

$$F(x, y_\alpha) = \begin{cases} 0, & x \in f_\alpha(z), z \in H_\alpha, f_\alpha(z) \cap H > a_\alpha, \\ 0, & x > a_\alpha, x \in H, \\ 1, & x < a_\alpha, x \in H; \end{cases}$$

$$F(x, z_\alpha) = \begin{cases} 0, & x > b_\alpha, x \in H, \\ 1, & x \in f_\alpha(z), z \in H_\alpha, f_\alpha(z) \cap H < b_\alpha, \\ 1, & x < b_\alpha, x \in H. \end{cases}$$

If x is an element of $X - H$ different from all the points y_α, z_α, we choose arbitrarily a $u \in H$ and set $F(y,x) = 0$ for $y \in H$, $y > u$, and $F(y,x) = 1$ for $y \in H$, $y \le u$. So far, we have defined $F(x,y)$ if $x \in H$ and $y \in X$, and for some other pairs, as well. For the remaining pairs we can extend F arbitrarily. We notice that for every $x \in X - H$, $F(x,y) = 0$ if $y \in H$ is large enough and $F(x,y) = 1$ if $y \in H$ is small enough. Moreover, for every $x \in X$ for all but τ^+ elements $y \in H$ it is true that if y is large enough, then $F(x,y) = 0$, and if it is small enough, then $F(x,y) = 1$.

Assume now that $f : X \to [X]^{<\omega}$. There is some $\alpha < \kappa$ that $f|H_\alpha = f_\alpha$. All but τ^+ many large enough $x \in H_\alpha$ have $F(x, f(y_\alpha)) = 0$. If such an x has even $f_\alpha(x) \cap H > a_\alpha$, then we also have that $F(f(x), y_\alpha) = 0$ and we are done. □

Theorem 3 (ZFC)**.** *$P(\kappa)$ holds if $\kappa^{\aleph_2} = \kappa$.*

Proof. In the above proof (for $\kappa > \aleph_2$) we needed an instance of GCH in Lemma 6 to show that for $|H| = \aleph_2$, $|X| = \kappa$ we have no more than κ functions from H to $[X]^{<\omega}$. But the number of these functions is $\kappa^{\aleph_2}$. □

As mentioned in the introduction, we could not establish $P(\omega_2)$ in ZFC alone. We can, nevertheless, show that $P(\omega_2)$ can consistently hold with any reasonable value of $2^{\aleph_1}$.

Theorem 4 (GCH)**.** *Assume that $\mathrm{cf}(\lambda) > \omega_1$. Then it is consistent that $2^{\aleph_1} = \lambda$ and $P(\omega_2)$ holds.*

Proof. With a preliminary forcing we can assume that CH and $2^{\aleph_1} = \lambda$ already hold in the ground model. We add a "generic" coloring $F : [\omega_2]^{\leq 2} \to \{0, 1\}$ and show that it works.

Let $p \in P$ if it is of the form $p = (s, h)$ where $s \in [\omega_2]^{\leq 2}$, $h : [s]^2 \to \{0, 1\}$. Extension is defined as $p' = (s', h') \leq p = (s, h)$ if $s' \supseteq s$, $h = h'|[s]^{\leq 2}$. If G is a generic filter, then we let $F = \bigcup\{h : (s, h) \in G\}$. We claim that F witnesses $P(\omega_2)$.

Assume that $1 \Vdash f : \omega_2 \to [\omega_2]^{<\omega}$. We show that there is a 0-pair (the other case is similar). The following argument is in $V[G]$.

Claim. *There exists $T \in [\omega_2]^{\aleph_0}$ such that for every $T' \in [\omega_2]^{\aleph_0}$ with $T' \supseteq T$ there is some $\alpha < \omega_2$ such that $F(T', \alpha) = 0$ and $f(\alpha) \cap (T' - T) = \emptyset$.*

Proof of Claim. Assume otherwise. Then, for every countable T there is some countable $T' \supseteq T$ such that whenever $F(T', \alpha) = 0$ then necessarily $f(\alpha) \cap (T' - T) \neq \emptyset$. Define inductively $T_0 \subseteq T_1 \subseteq \cdots$ by $T_0 = \emptyset$, $T_{n+1} = T'_n$. Set $T_\omega = \bigcup\{T_n : n < \omega\}$. We eventually get that if $F(T_\omega, \alpha) = 0$ then $f(\alpha) \cap (T_{n+1} - T_n) \neq \emptyset$ holds for $n = 0, 1, \ldots$ which contradicts the finiteness of $f(\alpha)$. And indeed such an α exists, as we can easily force it. □

Assume therefore that $p \Vdash T$ is as in the statement of the Claim. For every $\alpha < \omega_2$ let $p_\alpha \leq p$ be a condition forcing that $F(T, \alpha) = 0$. Applying the Δ-system lemma we find $\aleph_2$ conditions $\{p_\alpha : \alpha \in Z\}$ such that $p_\alpha = (T' \cup T_\alpha, h_\alpha)$ where $T' \supseteq T$, and $h_\alpha|[T']^{\leq 2} = h$. By the Claim there is some β such that $F(T', \beta) = 0$ and $f(\beta) \cap (T' - T) = \emptyset$, moreover, this is forced by some $\overline{p} = (\overline{s}, \overline{h}) \leq (T', h)$. For some $\alpha \in Z$ we have $T_\alpha \cap \overline{T} = \emptyset$ and we can consider $q = (\overline{T} \cup T_\alpha, h')$ where h' is 0 on $T_\alpha \times (\overline{T} - T')$. Then $q \leq \overline{p}, p_\alpha$ and (α, β) will be a 0-pair as $f(\alpha) \subseteq T \cup T_\alpha$ and $f(\beta) \subseteq \big((T' - T) \cup (\overline{T} - T')\big)$. □

We notice that even this can slightly be extended to show that if $\kappa \leq \lambda$, $\mathrm{cf}(\kappa) > \omega$, $\mathrm{cf}(\lambda) > \omega_1$, then it is consistent that $2^{\aleph_0} = \kappa$, $2^{\aleph_1} = \lambda$, and $P(\omega_2)$ holds. One only has to add κ many Cohen reals simultaneously.

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Department of Computer Science, Eötvös University, P.O. Box 120, 1518, Budapest, Hungary

E-mail address: kope@cs.elte.hu

Department of Analysis, Eötvös University, P.O. Box 120, 1518, Budapest, Hungary

E-mail address: laczk@cs.elte.hu