HARMONIC AND MINIMAL UNIT VECTOR FIELDS ON RIEMANNIAN SYMMETRIC SPACES

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ABSTRACT. We present new examples of harmonic and minimal unit vector fields on Riemannian symmetric spaces. These examples are constructed from cohomogeneity one actions with a reflective singular orbit. The radial unit vector field associated to such a reflective submanifold is harmonic and minimal.

1. Introduction

Let M be a compact connected orientable Riemannian manifold admitting smooth unit vector fields. Any such vector field ξ can be considered as an embedding of M into its unit tangent sphere bundle T_1M equipped with the Sasaki metric. We then define the energy $\mathrm{E}(\xi)$ of ξ as the energy of the map $\xi : M \to T_1M$ and the volume $\mathrm{Vol}(\xi)$ of ξ as the volume of the submanifold $\xi(M)$ of T_1M. This gives two functionals on the space of all smooth unit vector fields on M. The first variation of these functionals has been studied in [7] and [16], where the critical point condition has been derived in terms of differential forms. The critical point condition does not require compactness or orientability and hence can be investigated on arbitrary Riemannian manifolds. A unit vector field on a connected Riemannian manifold is said to be *harmonic* if it satisfies the critical point condition of the energy functional, and it is said to be *minimal* if it satisfies the critical point condition of the volume functional. A lot of examples of harmonic or minimal unit vector fields have been constructed. We refer to [6] for an overview and a list of relevant references, as well as for a brief historical account.

A connected closed submanifold F of a complete Riemannian manifold M is said to be *reflective* if the geodesic reflection of M in F is a well-defined global isometry. Since the reflective submanifold F is a connected component of the fixed point set of an isometry, F is necessarily a totally geodesic

submanifold of M. Now assume that M is a Riemannian symmetric space and $o \in F$. Then the orthogonal complement of T_oF in T_oM represents the tangent space of another reflective submanifold $F^\perp$ of M. Every complete totally geodesic submanifold of M is also a symmetric space and hence the rank of $F^\perp$ is well-defined. The reflective submanifolds of simply connected irreducible Riemannian symmetric spaces of compact type have been classified by Leung in [12] and [13].

The purpose of this paper is to give new examples of harmonic and minimal unit vector fields on semisimple symmetric spaces. These vector fields will be constructed from isometric cohomogeneity one actions with a reflective singular orbit. Namely, if F is a reflective submanifold such that the rank of $F^\perp$ equals one, then F determines a cohomogeneity one action on the symmetric space M such that F is one of the orbits. Then the principal orbits can be viewed as tubes about F provided that the codimension of F is greater than one. The geodesics emanating perpendicularly from F intersect each principal orbit of this action orthogonally. The unit tangent vectors of these geodesics yield the *radial unit vector field* ξ associated to F, which is defined on the open and dense subset formed by the union of principal orbits. Our main result is:

THEOREM. *Let M be a Riemannian symmetric space of compact or non-compact type, and let F be a reflective submanifold of M such that its codimension is greater than one and the rank of $F^\perp$ is equal to one. Then the radial unit vector field ξ associated to F is harmonic and minimal.*

This result provides a set of new examples of minimal and harmonic unit vector fields, since there are several cohomogeneity one actions with a reflective singular orbit. An explicit list for irreducible Riemannian symmetric spaces of noncompact type can be found in [2] for rank one and in [3] for higher rank. A classification of cohomogeneity one actions, or, more general, of hyperpolar actions, on irreducible symmetric spaces of compact type has been obtained by Kollross [11]. Such actions have also been discussed in [15]. The classification of cohomogeneity one actions on irreducible symmetric spaces of noncompact type is still unknown, but those without a singular orbit have been classified in [4].

In Section 2 of this paper, we summarize some basic facts about harmonic and minimal unit vector fields. In Section 3, we review the necessary facts about symmetric spaces. Moreover, we prove that each reflective submanifold F of a Riemannian symmetric space M of compact type determines a hyperpolar action on M such that F is an orbit of this action and the cohomogeneity of the action coincides with the rank of $F^\perp$. Finally, in Section 4, we present the proof of the main result.

2. Harmonic and minimal unit vector fields

Let M be an m-dimensional connected Riemannian manifold with Riemannian metric g. We denote by ∇ the Levi Civita connection and by R the Riemannian curvature tensor with the convention $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$. The Sasaki metric on the unit tangent sphere bundle $T_1 M$ of M will be denoted by g_S.

Let $\mathfrak{X}_1(M)$ be the set of all smooth unit vector fields on M and assume that $\mathfrak{X}_1(M)$ is nonempty. Every $\xi \in \mathfrak{X}_1(M)$ can be regarded as an embedding $\xi : M \to T_1 M$, and the pullback $\xi^* g_S$ of the Sasaki metric g_S onto M via ξ is given by

$$\xi^* g_S(X, Y) = g(X, Y) + g(\nabla_X \xi, \nabla_Y \xi).$$

This shows that ξ is an isometric embedding if and only if ξ is a parallel vector field.

We now define the tensor fields A_ξ and L_ξ on M by

$$A_\xi X = -\nabla_X \xi \qquad \text{and} \qquad L_\xi X = X + A_\xi^t A_\xi X \ ,$$

where A_ξ^t denotes the transpose of A_ξ. Assume that M is compact and orientable. Then the energy $\mathrm{E}(\xi)$ and the volume $\mathrm{Vol}(\xi)$ of ξ are defined by

$$\mathrm{E}(\xi) = \frac{1}{2} \int_M \mathrm{tr}(L_\xi) dv = \frac{m}{2} \mathrm{Vol}(M) + \frac{1}{2} \int_M |\nabla \xi|^2 dv$$

$$\mathrm{Vol}(\xi) = \int_M \sqrt{\det(L_\xi)} dv,$$

where dv is the volume form of M with respect to g. Therefore we get two functionals E and Vol on the set $\mathfrak{X}_1(M)$ of unit vector fields on M. Then $\xi \in \mathfrak{X}_1(M)$ is called *harmonic*, respectively *minimal*, if it is a critical point of E, respectively Vol. It was proved in [7] that ξ is minimal if and only if $\xi(M)$ is a minimal submanifold of $T_1 M$.

The first variation of the functionals E and Vol, that is, the critical point condition for these functionals, has been investigated in [7] and [16] in terms of differential forms. It turns out that $\xi \in \mathfrak{X}_1(M)$ is a harmonic unit vector field if and only if the one-form

$$\nu_\xi(X) = \mathrm{tr}(Y \mapsto (\nabla_Y A_\xi^t) X)$$

vanishes on $\xi^\perp$. Here we denote by $\xi^\perp$ the $(m-1)$-dimensional distribution on M that is perpendicular to the one-dimensional distribution spanned by the vector field ξ. And $\xi \in \mathfrak{X}_1(M)$ is a minimal unit vector field if and only if the one-form

$$\omega_\xi(X) = \mathrm{tr}(Y \mapsto (\nabla_Y K_\xi) X)$$

vanishes on $\xi^\perp$, where

$$K_\xi = -\sqrt{\det(L_\xi)} \, L_\xi^{-1} A_\xi^t.$$

Clearly, compactness and orientability are not required for formulating the two critical point conditions. We thus say in general that $\xi \in \mathfrak{X}_1(M)$ is harmonic, respectively minimal, if the one-form ν_ξ, respectively ω_ξ, vanishes on $\xi^\perp$.

From now on, we assume that the integral curves of ξ are geodesics and that the distribution $\xi^\perp$ is integrable. In this situation, the critical point conditions can be reformulated as follows (see [5] for details). We denote by $\mathfrak{F}_\xi$ the foliation determined by $\xi^\perp$. Then ξ is a harmonic unit vector field if and only if

$$(2.1) \qquad dh^{\mathfrak{F}_\xi}(X) = \mathrm{ric}^M(\xi, X)$$

for all vector fields X tangent to $\mathfrak{F}_\xi$, where $h^{\mathfrak{F}_\xi}$ denotes the mean curvature function of the leaves of $\mathfrak{F}_\xi$ and ric^M is the Ricci curvature of M. If M is an Einstein manifold, this shows that ξ is a harmonic unit vector field if and only if the leaves of $\mathfrak{F}_\xi$ have constant mean curvature. At each point $p \in M$, the endomorphism $A_\xi(p)$ restricted to $\xi^\perp(p)$ is the shape operator at p of the leaf $\mathfrak{F}_\xi(p)$ of $\mathfrak{F}_\xi$ through p, and hence its eigenvalues are the principal curvatures of $\mathfrak{F}_\xi(p)$ at p. We denote by M_c the open and dense subset of M on which the multiplicities of the principal curvatures are locally constant. On M_c we can find smooth functions $\lambda_1, \ldots, \lambda_{m-1}$ and orthonormal vector fields $E_1, \ldots, E_{m-1}$ tangent to $\mathfrak{F}_\xi$ such that $A_\xi E_i = \lambda_i E_i$ for all $i = 1, \ldots, m-1$. Then ξ is a minimal unit vector field if and only if

$$(2.2) \qquad \sum_{i=1}^{m-1} \frac{1}{1+\lambda_i^2}(d\lambda_i(E_j) - (1 - \lambda_i\lambda_j)g(R(\xi, E_i)E_i, E_j)) = 0$$

holds for all $j = 1, \ldots, m-1$.

3. Isometric actions on symmetric spaces induced by reflective submanifolds

In this section we summarize some basic facts about symmetric spaces of compact type and prove two auxiliary results. For the general theory of symmetric spaces and the structure theory of semisimple Lie algebras, we refer to [9].

Let M be a Riemannian symmetric space of compact type and G the identity component of the full isometry group of M. Then G is a compact Lie group such that its Lie algebra $\mathfrak{g}$ is semisimple. We fix a point $o \in M$ and denote by K the isotropy subgroup of G at o. As is well-known, M can be identified with the homogeneous space G/K. Consider the involutive automorphism $\sigma : G \to G$ defined by $\sigma(g) = s_o g s_o$ for $g \in G$, where s_o denotes the geodesic symmetry of M at o. The tangent linear map $T_e\sigma$ of σ at the identity element e is an involutive automorphism of the Lie algebra $\mathfrak{g}$. The ± 1-eigenspaces of $T_e\sigma$ induce the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, where $\mathfrak{k}$ is

the Lie algebra of K. We identify the tangent space T_oM of M at o with the subspace $\mathfrak{p}$ in the usual way. Then the Riemannian curvature tensor R of M at o satisfies

$$(3.1) \qquad\qquad R(v_1, v_2)v_3 = -\big[\, [v_1, v_2], v_3\,\big]$$

for all $v_1, v_2, v_3 \in \mathfrak{p}$, where $[\cdot, \cdot]$ denotes the bracket operation in $\mathfrak{g}$.

It is well-known that the Killing form B of $\mathfrak{g}$ is negative definite. Let $\langle\cdot, \cdot\rangle_o$ be the inner product on $T_oM = \mathfrak{p}$ which is induced from the Riemannian metric of M. In what follows we always assume that $\langle\cdot, \cdot\rangle_o$ coincides with the restriction of $-B$ to $\mathfrak{p} \times \mathfrak{p}$.

For each $X \in \mathfrak{g}$ the adjoint transformation $\mathrm{ad}(X) : \mathfrak{g} \to \mathfrak{g}$ is the endomorphism on $\mathfrak{g}$ defined by $\mathrm{ad}(X)Y = [X, Y]$ for all $Y \in \mathfrak{g}$. Let $\mathfrak{a}$ be a maximal Abelian subspace of $\mathfrak{p} = T_oM$. Then the dimension of $\mathfrak{a}$ is equal to the rank of M. For each linear form α in the dual space $\mathfrak{a}^*$ of $\mathfrak{a}$ we define

$$\mathfrak{g}_\alpha = \{Y \in \mathfrak{g} \mid \mathrm{ad}(X)^2 Y = -\alpha(X)^2 Y \text{ for all } X \in \mathfrak{a}\}.$$

Clearly, $\mathfrak{g}_\alpha = \mathfrak{g}_{-\alpha}$ is valid. A dual vector $\alpha \in \mathfrak{a}^*$ is said to be a restricted root if $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq 0$. Let $\triangle$ be the set of all restricted roots. We equip $\mathfrak{a}^*$ with a lexicographic ordering and denote by $\triangle^+$ the corresponding set of positive restricted roots. Then we get the orthogonal decomposition

$$\mathfrak{g} = \mathfrak{g}_0 + \sum_{\alpha \in \triangle^+} \mathfrak{g}_\alpha$$

with respect to the Killing form B. For each $\alpha \in \triangle \cup \{0\}$ we define $\mathfrak{k}_\alpha = \mathfrak{k} \cap \mathfrak{g}_\alpha$ and $\mathfrak{p}_\alpha = \mathfrak{p} \cap \mathfrak{g}_\alpha$. Obviously, $\mathfrak{p}_0 = \mathfrak{a}$ holds. Moreover, $\mathfrak{g}_\alpha$ is the direct sum of the subspaces $\mathfrak{k}_\alpha$ and $\mathfrak{p}_\alpha$, and $\mathfrak{k}_\alpha$ and $\mathfrak{p}_\alpha$ have the same dimension for all $\alpha \neq 0$. Then we obtain the orthogonal decompositions

$$(3.2) \qquad\qquad \mathfrak{k} = \mathfrak{k}_0 + \sum_{\alpha \in \triangle^+} \mathfrak{k}_\alpha \quad \text{and} \quad \mathfrak{p} = \mathfrak{a} + \sum_{\alpha \in \triangle^+} \mathfrak{p}_\alpha$$

with respect to the Killing form B. For each $\alpha \in \triangle$ we define $X_\alpha \in \mathfrak{a}$ by $B(X_\alpha, X) = \alpha(X)$ for all $X \in \mathfrak{a}$. This induces the associated root system $\mathcal{D} = \{X_\alpha \mid \alpha \in \triangle\}$ in the maximal Abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$. For each $\alpha \in \triangle$ we fix a unit vector $Y_\alpha \in \mathfrak{p}_\alpha$. Then there exists a unique vector $Z_\alpha \in \mathfrak{k}_\alpha$ such that $B(Z_\alpha, Z_\alpha) = -1$ and

$$(3.3) \qquad\qquad [X, Y_\alpha] = \alpha(X)Z_\alpha \quad \text{and} \quad [X, Z_\alpha] = -\alpha(X)Y_\alpha$$

for all $X \in \mathfrak{a}$. Note that for these vectors the equality

$$(3.4) \qquad\qquad [Z_\alpha, Y_\alpha] = X_\alpha$$

holds.

For each $u \in T_oM$ the selfadjoint endomorphism R_u on T_oM defined by $R_u w = R(w, u)u$ for all $w \in T_oM$ is called the Jacobi operator with respect

to u. From (3.1) it follows that $R_u = -\mathrm{ad}(u)^2$. The following statement can easily be deduced from the above relations.

LEMMA 1.   *Let M be a compact symmetric space of rank one and let $u, w \in T_oM$ be orthonormal. If $R_u w = aw$ holds for some real number $a > 0$, then $R_w u = au$.*

Proof. Obviously, we have to consider only the case when M has nonconstant sectional curvature. Then we get the decompositions

$$\mathfrak{k} = \mathfrak{k}_0 + \mathfrak{k}_\alpha + \mathfrak{k}_{2\alpha} \quad \text{and} \quad \mathfrak{p} = \mathfrak{a} + \mathfrak{p}_\alpha + \mathfrak{p}_{2\alpha},$$

where $\mathfrak{a}$ is one-dimensional. Without loss of generality, we can assume that $\mathfrak{a} = \mathbb{R}u$ holds for the selected unit vector $u \in \mathfrak{p} = T_oM$. Hence, we obtain the relations $X_\alpha = \langle X_\alpha, u \rangle_o u = -B(X_\alpha, u)u = -\alpha(u)u$ and $X_{2\alpha} = -2\alpha(u)u$. It is clear that $\mathfrak{p}_\alpha$ and $\mathfrak{p}_{2\alpha}$ are eigenspaces of R_u with corresponding positive eigenvalues $\alpha(u)^2$ and $4\,\alpha(u)^2$, respectively. Using the relations (3.1), (3.3) and (3.4), it can be easily verified that $R_{Y_\alpha} u = \alpha(u)^2 u$ and $R_{Y_{2\alpha}} u = 4\,\alpha(u)^2 u$ holds for all unit vectors $Y_\alpha \in \mathfrak{p}_\alpha$ and $Y_{2\alpha} \in \mathfrak{p}_{2\alpha}$. □

Let F be a reflective submanifold of M, which means that F coincides with a connected component of the fixed point set of an involutive isometry r of M. Then F is a totally geodesic submanifold of M and therefore a compact symmetric space. The isometry r is called the reflection of M in F. Without loss of generality we may assume that $o \in F$. Then the equality $s_o r = r s_o$ holds. Let $F^\perp$ be the connected component of the fixed point set of the involutive isometry $s_o r$ which contains the point o. The submanifold $F^\perp$ will be called the complementary reflective submanifold of F (at o). Then we get the direct sum $\mathfrak{p} = T_oM = T_oF + T_oF^\perp$, where the tangent spaces T_oF and $T_oF^\perp$ are Lie triple systems in $\mathfrak{p}$. Consider now the involutive automorphism ρ on G defined by $\rho(g) = rgr$ for all $g \in G$. Then the induced involutive automorphism $T_e\rho$ on $\mathfrak{g}$ commutes with $T_e\sigma$, and hence $\mathfrak{k}$ and $\mathfrak{p}$ are invariant subspaces of $T_e\rho$.

LEMMA 2.   *The restriction of $T_e\rho$ to $\mathfrak{p} = T_oM$ coincides with the tangent linear map $T_o r$ of the reflection r at o.*

Proof. Denote by $\exp$ the Lie exponential map from $\mathfrak{g}$ into G and by Exp_o the exponential map from T_oM into M. Let $Y \in T_oF \subset \mathfrak{p}$, and define the isometries $\varphi = \exp(tY)$ and $\psi = r\exp(tY)r$ of M for some $t > 0$. Obviously, $\varphi(o) = \mathrm{Exp}_o(tY) = \psi(o)$ holds. Since the tangent linear map $T_o\varphi$ coincides with the parallel displacement of T_oM along the geodesic segment γ defined by $\gamma(\tau) = \mathrm{Exp}_o(\tau Y)$ for $\tau \in [0, t]$, it is easy to see that $T_o\varphi = T_o\psi$. The above facts imply that $\exp(tY) = r\exp(tY)r$ for all $t \in \mathbb{R}$. Analogously, for $\tilde{Y} \in T_oF^\perp$, we obtain $s_o r \exp(t\tilde{Y})r s_o = \exp(t\tilde{Y})$. From the above relations

it follows that $T_e\rho(Y) = Y$ holds for all $Y \in T_oF$ and $T_e\rho(\tilde{Y}) = -\tilde{Y}$ for all $\tilde{Y} \in T_oF^\perp$, which completes the proof. $\qquad\square$

Let L be the identity component of the closed subgroup $G_\rho = \{g \in G \mid \rho(g) = g\}$ of G. The ± 1-eigenspaces of $T_e\rho$ induce the Cartan decomposition $\mathfrak{g} = \mathfrak{l} + \mathfrak{n}$, where $\mathfrak{l}$ is the Lie algebra of L. We denote by $\lambda : L \times M \to M$ the natural isometric action of L on M, and call λ the isometric action associated to the reflective submanifold F. Each orbit $L(p)$, $p \in M$, of this action is a connected closed submanifold of M.

PROPOSITION 1.   *The orbit $L(o)$ through o of the action of L on M is the reflective submanifold F, and the cohomogeneity of this action is equal to the rank of the complementary reflective submanifold $F^\perp$.*

Proof. Recall that $T_e\rho$ leaves $\mathfrak{p}$ invariant. From Lemma 2 we see that $T_oF = \mathfrak{p} \cap \mathfrak{l}$ and $T_oF^\perp = \mathfrak{p} \cap \mathfrak{n}$. Each $X \in \mathfrak{g}$ has a unique decomposition $X = X_{\mathfrak{k}} + X_{\mathfrak{p}}$ with respect to the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. The tangent space $T_oL(o)$ of the orbit $L(o)$ at o is spanned by the tangent vectors $\dot{\omega}_X(0)$ of the smooth curves $\omega_X : \mathbb{R} \to M$ defined by $\omega_X(t) = \lambda(\exp(tX), o)$, $X \in \mathfrak{l}$. Clearly, the tangent vector $\dot{\omega}_X(0)$ coincides with the component $X_{\mathfrak{p}}$. Since σ and ρ commute, we have the decomposition $\mathfrak{l} = (\mathfrak{k} \cap \mathfrak{l}) + (\mathfrak{p} \cap \mathfrak{l})$, which implies $T_oL(o) = \mathfrak{p} \cap \mathfrak{l} = T_oF$. Moreover, $\lambda(\exp(tY), o) = \mathrm{Exp}_o(tY)$ holds for all Y in the Lie triple system $\mathfrak{p} \cap \mathfrak{l}$. It follows from the above facts that the orbit $L(o)$ is totally geodesic in M and coincides with F.

Let H be the identity component of the closed subgroup $G_{\rho\sigma} = \{g \in G \mid \rho\sigma(g) = g\}$ of G. An analogous argument as above shows that the orbit $H(o)$ of the isometric action of H on M coincides with the complementary reflective submanifold $F^\perp$. It can be easily seen that $\sigma(L) = L$, $\sigma(H) = H$, and $L \cap K$ and $H \cap K$ have the same identity component. Hence $(L, L \cap K)$ and $(H, H \cap K)$ are two Riemannian symmetric pairs, and the corresponding symmetric spaces can be identified with F and $F^\perp$, respectively. The codimension of the principal orbits of the action of L is equal to the cohomogeneity of the isometric action of $H \cap K$ on $F^\perp$. Since $H \cap K$ is the isotropy subgroup of H at o of the symmetric space $F^\perp = H/H \cap K$, the cohomogeneity is equal to the rank of $F^\perp$. $\qquad\square$

It is worthwhile to mention that the maximal dimensional flat totally geodesic submanifolds of $F^\perp$ which pass through o intersect all the orbits of L orthogonally. This means that λ is a so-called hyperpolar action (see [8] and [11] for more details about such actions).

REMARK 1.   Using the duality between symmetric spaces of compact type and of noncompact type, one can transfer all the results of Section 3 to Riemannian symmetric spaces of noncompact type.

4. The new examples of harmonic and minimal unit vector fields

Let M be a connected Riemannian symmetric space of compact type and let F be a reflective submanifold of M. In what follows m and n denotes the dimension of M and F, respectively. Furthermore, we shall use the notation introduced in Sections 2 and 3. In order to prove the theorem in the introduction, we shall first prove the following result.

PROPOSITION 2.   *Let F be a reflective submanifold of M such that the rank of the complementary reflective submanifold $F^\perp$ is equal to one. Let $H \in T_oF^\perp$ be a unit vector. Then there exist orthonormal vectors $E_1, \ldots, E_{m-1}, E_m = H$ in T_oM such that $E_1, \ldots, E_n$ is a basis of T_oF, $E_{n+1}, \ldots, E_{m-1}, E_m$ is a basis of $T_oF^\perp$, and*

$$(4.1) \qquad R_H E_i = a_i E_i \quad , \quad R_{E_i} H = a_i H$$

holds for some real numbers $a_i \geq 0$, $i \in \{1, \ldots, m-1\}$.

Proof. Let $\mathfrak{a}$ be a maximal Abelian subspace of $\mathfrak{p}$ with $H \in \mathfrak{a}$. First we show that $\mathfrak{a}$ is an invariant subspace of the tangent linear map $T_o r$ of the reflection r of M in F. We define $\mathfrak{b} = \{X \in \mathfrak{a} \mid \langle H, X \rangle_o = 0\}$ and $W = \{w \in T_oF^\perp \mid \langle H, w \rangle_o = 0\}$. From (3.1) we see that $R_H X = 0$ holds for all $X \in \mathfrak{a}$. Since $F^\perp$ is a symmetric space of rank one, the subspace W is generated by all $Y \in W$ such that $R_H Y = \kappa_Y Y$ with some positive real number κ_Y. This implies that $\mathfrak{b}$ and W are orthogonal to each other and $\mathfrak{b} \subset T_oF$. Therefore we obtain the decomposition $\mathfrak{a} = (\mathfrak{a} \cap T_oF) + (\mathfrak{a} \cap T_oF^\perp) = \mathfrak{a}^T + \mathfrak{a}^\perp$ with $\mathfrak{a}^T = \mathfrak{b}$ and $\mathfrak{a}^\perp = \mathbb{R}H$.

The Jacobi operator satisfies $T_o r R_v = R_{T_o r(v)} T_o r$ for $v \in T_oM = \mathfrak{p}$. This implies

$$(4.2) \qquad T_o r R_H = R_H T_o r \qquad \text{and} \qquad T_o r R_X = R_X T_o r$$

for all $X \in \mathfrak{a}^T$. It follows from (3.1), (3.3) and (4.2) that any nonzero element of $\mathfrak{p}_\alpha$ resp. $T_o r(\mathfrak{p}_\alpha)$, $\alpha \in \triangle$, is an eigenvector of the Jacobi operator R_H resp. R_X, $X \in \mathfrak{a}^T$, with eigenvalue $\alpha(H)^2$ resp. $\alpha(X)^2$.

We now consider the set $\mathcal{D} = \{X_\alpha \mid \alpha \in \triangle\}$ of the root vectors X_α, $\alpha \in \triangle$. Let Λ denote the lattice in $\mathfrak{a}$ that is generated by the vectors $(2\pi/B(X_\alpha, X_\alpha))X_\alpha$, $\alpha \in \triangle$. Without loss of generality we may assume that M is simply connected. Then the closed geodesics in the flat totally geodesic submanifold $\mathrm{Exp}_o(\mathfrak{a})$ passing through o are determined by the lattice Λ (see Theorem 8.5 of Chapter VII in [9]). Consequently, $r(\mathrm{Exp}_o(\mathfrak{a})) = \mathrm{Exp}_o(\mathfrak{a})$ implies $T_o r(\Lambda) = \Lambda$ and $T_o r(\mathcal{D}) = \mathcal{D}$. Now assume that $T_o r(X_\alpha) = X_\beta$ for $\alpha, \beta \in \triangle$. Then we get $\alpha(H)^2 = \beta(H)^2$ and $\alpha(X)^2 = \beta(X)^2$ for all $X \in \mathfrak{a}^T$. It can be seen easily that (4.2) and $T_o r R_{X_\alpha} = R_{X_\beta} T_o r$ implies $T_o r(\mathfrak{p}_\alpha) = \mathfrak{p}_\beta$. Clearly, if $X_\alpha \in \mathfrak{a}^T$ or $X_\alpha \in \mathfrak{a}^\perp$, then $\mathfrak{p}_\alpha$ is invariant under $T_o r$. Motivated

by the previous discussion we now introduce the following notations:

$$\triangle^T = \{\alpha \in \triangle \mid X_\alpha \in \mathfrak{a}^T\},$$
$$\triangle^N = \{\alpha \in \triangle \mid X_\alpha \in \mathfrak{a}^\perp\},$$
$$\triangle^G = \{\alpha \in \triangle \mid X_\alpha \notin \mathfrak{a}^T \text{ and } X_\alpha \notin \mathfrak{a}^\perp\}.$$

Case 1: $\alpha \in \triangle^T$. Then $\alpha(H) = B(X_\alpha, H) = 0$ and hence

(4.3) $$R_H Y_\alpha = 0 \quad \text{and} \quad R_{Y_\alpha} H = 0$$

for all unit vectors $Y_\alpha \in \mathfrak{p}_\alpha$. Moreover, $\mathfrak{p}_\alpha \subset T_o F$.

Case 2: $\alpha \in \triangle^N$. Then $X_\alpha = -\alpha(H)H$, and (3.1), (3.3) and (3.4) imply

(4.4) $$R_H Y_\alpha = \alpha(H)^2 Y_\alpha \quad \text{and} \quad R_{Y_\alpha} H = \alpha(H)^2 H$$

for all unit vectors $Y_\alpha \in \mathfrak{p}_\alpha$. Moreover, since $\mathfrak{p}_\alpha$ is an invariant subspace under $T_o r$, we obtain the direct sum $\mathfrak{p}_\alpha = (\mathfrak{p}_\alpha \cap T_o F) + (\mathfrak{p}_\alpha \cap T_o F^\perp) = \mathfrak{p}_\alpha^T + \mathfrak{p}_\alpha^\perp$.

Case 3: $\alpha \in \triangle^G$. Then $T_o r(X_\alpha) = X_\beta$ for some $\beta \in \triangle$ with $\beta \neq \pm\alpha$, $T_o r(\mathfrak{p}_\alpha) = \mathfrak{p}_\beta$ and $T_o r(\mathfrak{p}_\beta) = \mathfrak{p}_\alpha$. We shall sometimes denote β also by $r(\alpha)$. It is easy to show that

(4.5) $$\alpha(H) + \beta(H) = 0 \quad \text{and} \quad X_\beta - X_\alpha = 2\alpha(H)H.$$

The subspace $\mathfrak{q}_{\alpha,\beta} = \mathfrak{p}_\alpha + \mathfrak{p}_\beta$ is an invariant subspace under $T_o r$. Therefore we obtain the decomposition

$$\mathfrak{q}_{\alpha,\beta} = (\mathfrak{q}_{\alpha,\beta} \cap T_o F) + (\mathfrak{q}_{\alpha,\beta} \cap T_o F^\perp) = \mathfrak{q}_{\alpha,\beta}^T + \mathfrak{q}_{\alpha,\beta}^\perp.$$

We now consider the unit vectors $Y_\alpha \in \mathfrak{p}_\alpha$ and $Z_\alpha \in \mathfrak{k}_\alpha$ which satisfy the equalities in (3.3). By Lemma 2, the restriction of the involution $T_e \rho$ to $\mathfrak{p}$ coincides with $T_o r$. Hence the relation $[X_\alpha, \mathfrak{p}_\alpha] = \mathfrak{k}_\alpha$ implies $T_e \rho(\mathfrak{k}_\alpha) = \mathfrak{k}_\beta$. The vectors $Y_\beta = T_e \rho(Y_\alpha)$ and $Z_\beta = T_e \rho(Z_\alpha)$ satisfy $[H, Y_\beta] = -\alpha(H) Z_\beta$, $[H, Z_\beta] = \alpha(H) Y_\beta$ and $[Z_\beta, Y_\beta] = X_\beta$. The vector $\tilde{Y}_{\alpha,\beta} = Y_\alpha - Y_\beta = Y_\alpha - T_o r(Y_\alpha) \in \mathfrak{q}_{\alpha,\beta}^\perp \subset T_o F^\perp$ satisfies $\|\tilde{Y}_{\alpha,\beta}\| = \sqrt{2}$, and the above relations imply $R(\tilde{Y}_{\alpha,\beta}, H)H = \alpha(H)^2 \tilde{Y}_{\alpha,\beta}$. It follows from Lemma 1 that

(4.6) $$R_H \tilde{E}_{\alpha,\beta} = \alpha(H)^2 \tilde{E}_{\alpha,\beta} \quad \text{and} \quad R_{\tilde{E}_{\alpha,\beta}} H = \alpha(H)^2 H$$

holds for the unit vector $\tilde{E}_{\alpha,\beta} = (1/\sqrt{2})\tilde{Y}_{\alpha,\beta}$.

On the other hand, using again the Lie bracket relations and (4.5), we get

$$R(H, \tilde{E}_{\alpha,\beta})\tilde{E}_{\alpha,\beta} = \alpha(H)^2 H + \frac{1}{2}\alpha(H)([Z_\alpha, Y_\beta] - [Z_\beta, Y_\alpha]),$$

which implies

(4.7) $$[Z_\alpha, Y_\beta] - [Z_\beta, Y_\alpha] = 0.$$

Next, $Y_{\alpha,\beta} = Y_\alpha + Y_\beta = Y_\alpha + T_o r(Y_\alpha) \in \mathfrak{q}_{\alpha,\beta}^T \subset T_o F$ satisfies $R(Y_{\alpha,\beta}, H)H = \alpha(H)^2 Y_{\alpha,\beta}$, and by a straightforward calculation we get

$$R(H, Y_{\alpha,\beta})Y_{\alpha,\beta} = \alpha(H)(X_\beta - X_\alpha) - \alpha(H)\left([Z_\alpha, Y_\beta] - [Z_\beta, Y_\alpha]\right).$$

Hence, by means of (4.5) and (4.7),

$$(4.8) \qquad R_H E_{\alpha,\beta} = \alpha(H)^2 E_{\alpha,\beta} \quad \text{and} \quad R_{E_{\alpha,\beta}} H = \alpha(H)^2 H$$

holds for the unit vector $E_{\alpha,\beta} = (1/\sqrt{2})Y_{\alpha,\beta}$.

Finally, the tangent spaces of F and $F^\perp$ at o can be expressed as the direct sums of orthogonal subspaces in the following way:

$$T_o F = \mathfrak{a}^T + \sum\nolimits_{\alpha \in \triangle^T} \mathfrak{p}_\alpha + \sum\nolimits_{\alpha \in \triangle^N} \mathfrak{p}_\alpha^T + \sum\nolimits_{\alpha \in \triangle^G} \mathfrak{q}_{\alpha, r(\alpha)}^T,$$

$$T_o F^\perp = \mathfrak{a}^\perp + \sum\nolimits_{\alpha \in \triangle^N} \mathfrak{p}_\alpha^\perp + \sum\nolimits_{\alpha \in \triangle^G} \mathfrak{q}_{\alpha, r(\alpha)}^\perp.$$

From (4.3), (4.4), (4.6) and (4.8) we then easily get an orthonormal basis $E_1, \ldots, E_m$ of $T_o M$ which satisfies the required conditions in Proposition 2. $\square$

REMARK 2.   The above proof shows that if M is a simply connected symmetric space of compact type and F is a reflective submanifold of M with codimension one, then $M = N \times S^n$ and $F = N \times S^{n-1}$, where N is also a symmetric space of compact type (possibly a point).

It is easy to see that Proposition 2 remains valid for symmetric spaces of noncompact type. Clearly, in this case the eigenvalues a_i in (4.1) are nonpositive.

We now recall some basic facts about isometric actions. Let M be a connected complete Riemannian manifold and L a connected closed subgroup of the isometry group of M. Assume that L acts on M with cohomogeneity one, that is, the principal orbits have codimension one. It is well-known (see [14] for the compact case and [1] for the general case) that the space of orbits of this action is homeomorphic (with respect to the induced quotient topology) to the real line $\mathbb{R}$, to the circle S^1, to the half-open interval $[0, \infty)$, or to the closed interval $[0, 1]$. In the first two cases the orbits form a Riemannian foliation on M, in the third case there exists exactly one singular orbit, and in the last case there exist exactly two singular orbits. In the latter two cases each principal orbit can be viewed as a tube about a singular orbit. The set of principal orbits always forms an open and dense subset of M.

Let M be a Riemannian symmetric space of compact or of noncompact type, and let F be a reflective submanifold of M such that the codimension of F is greater than one and the rank of the complementary reflective submanifold $F^\perp$ is equal to one. Then, by Proposition 1, the isometric action $\lambda : L \times M \to M$ associated to F is of cohomogeneity one.

Assume that M is a symmetric space of compact type. Then L has exactly two singular orbits and F coincides with one of them. Let δ be the maximum of the distances between F and the points of M, and denote by F^δ the set of all points in M whose distance from F is equal to δ. Then F^δ is the other singular orbit, and the principal orbits of L are tubes about F, or equivalently, about F^δ. Consider now the connected open domain $U = M \setminus (F \cup F^\delta)$ in M. Take a point p of U whose distance from F is equal to t with $0 < t < \delta$. There exists a unique geodesic $\omega : \mathbb{R} \to M$ parametrized by arc length such that $\omega(0) \in F$ and $\omega(t) = p$ hold. Clearly, the unit tangent vector $\dot{\omega}(0)$ is perpendicular to F, and the closed geodesic ω intersects all the orbits of L orthogonally. (Note that ω is closed, since all geodesics in a compact symmetric space of rank one are closed, and $F^\perp$ has rank one by assumption.) By defining $\xi(p) = \dot{\omega}(t)$ at each point p of U, we obtain a smooth vector field ξ on U, which we call the radial unit vector field associated to F. Note that this ξ is invariant under the action of L.

Suppose that M is a symmetric space of noncompact type, which means that M is diffeomorphic to $\mathbb{R}^m$. Then F is the only singular orbit of L, and the other orbits coincide with the tubes about F. As above, we can define the radial unit vector field ξ on the connected domain $U = M \setminus F$.

We are now in a position to formulate our main result.

THEOREM. *Let M be a Riemannian symmetric space of compact or of noncompact type, and let F be a reflective submanifold of M such that its codimension is greater than one and the rank of the complementary reflective submanifold $F^\perp$ is equal to one. Then the radial unit vector field ξ associated to F is harmonic and minimal.*

Proof. Assume that M is of compact type. Let $E_1, \ldots, E_{m-1}, E_m = H$ be an orthonormal basis of T_oM which satisfies the conditions in Proposition 2. The second equation of (4.1) shows that

$$(4.9) \qquad \langle R(H, E_i)E_i, E_j \rangle_o = 0$$

for all $i, j = 1, \ldots, m-1$. Let $\gamma : \mathbb{R} \to M$ be the closed geodesic in M defined by $\gamma(\tau) = \mathrm{Exp}_o(\tau H)$ for $\tau \in \mathbb{R}$, and let $P_1, \ldots, P_{m-1}$ be the parallel vector fields along γ with $P_i(0) = E_i$, $i = 1, \ldots, m-1$. Let us consider the principal orbit $L(\gamma(t))$ for some $0 < t < \delta$, and denote by $A_{\dot{\gamma}(t)}$ the shape operator of $L(\gamma(t))$ with respect to $\dot{\gamma}(t)$. Using the first equation of (4.1), Jacobi field theory implies

$$(4.10) \qquad \begin{aligned} A_{\dot{\gamma}(t)}P_j(t) &= \sqrt{a_j}\tan(\sqrt{a_j}t)P_j(t), \\ A_{\dot{\gamma}(t)}P_k(t) &= -\sqrt{a_k}\cot(\sqrt{a_k}t)P_k(t), \end{aligned}$$

for $j \in \{1, \ldots, n\}$ and $k \in \{n+1, \ldots, m-1\}$. Since M is a symmetric space, the parallel transport along geodesics preserves the curvature, and hence (4.9)

implies

$$(4.11) \qquad g\big(R\big(\xi(\gamma(t)), P_i(t)\big)P_i(t), P_j(t)\big) = 0$$

for all $i, j \in \{1, \dots, m-1\}$.

The leaves of the foliation $\mathfrak{F}_\xi$ determined by $\xi^\perp$ coincide with the principal orbits of L. Therefore the functions $h^{\mathfrak{F}_\xi}$ and λ_i $(i = 1, \dots, m-1)$ are constant on the leaves of $\mathfrak{F}_\xi$. Then (4.10) and (4.11) imply that (2.1) and (2.2) hold. This means that the radial unit vector field ξ is harmonic and minimal.

If M is a symmetric space of noncompact type, the assertion is obtained by a slight modification of the above argument. $\square$

REMARK 3.   In the table below, we present all simply connected irreducible Riemannian symmetric spaces M of compact type which admit an isometric cohomogeneity one action with a reflective singular orbit F. Here we list the pairs F and $F^\perp$ of the relevant reflective submanifolds of M. If a cohomogeneity one action has two reflective singular orbits, then the associated radial unit vector fields differ just by sign. These pairs of reflective submanifolds are written in the same lines of the table. The table has been obtained by using the lists of reflective submanifolds provided by Leung in [12] and [13].

In some cases the reflective submanifold F is in fact a subcovering of the space listed here, but we omit these subtle details in the table. A corresponding table for the noncompact case can be easily derived by using the well-known concept of duality between symmetric spaces of compact type and of noncompact type (see also [2] for rank one and [3] for higher rank).

REMARK 4.   There are cohomogeneity one actions on irreducible symmetric spaces with a totally geodesic but nonreflective singular orbit. A complete list for the noncompact case can be found in [3], which by duality yields also a complete list for the compact case. It would be interesting to see whether the corresponding radial unit vector field of such an action is harmonic or minimal.

An example of such an action is the one of $SU(3)$ on $G_2/SO(4)$. This action has two totally geodesic singular orbits, namely $\mathbb{C}P^2$ and $SU(3)/SO(3)$, but both are nonreflective. There are only few other examples where both singular orbits are totally geodesic and nonreflective. We mention that there are also cohomogeneity one actions on compact symmetric spaces for which both singular orbits are not totally geodesic. An example for such an action is the one of $Sp(n)Sp(1)$ on the Grassmannian $G_2^+(\mathbb{R}^{4n})$, $n \geq 2$. Moreover, the two singular orbits of many cohomogeneity one actions on spheres (see [10] for the classification) are not totally geodesic, but the sphere is rather exceptional in this respect.

M	F	$F^\perp$	Remarks
S^n	*point*	S^n	
	S^k, S^{n-k-1}	S^{n-k}, S^{k+1}	$1 \le k \le n-2$
$\mathbb{C}P^n$	$\mathbb{C}P^k$, $\mathbb{C}P^{n-k-1}$	$\mathbb{C}P^{n-k}$, $\mathbb{C}P^{k+1}$	$0 \le k \le n-1$
	$\mathbb{R}P^n$	$\mathbb{R}P^n$	
$\mathbb{H}P^n$	$\mathbb{H}P^k$, $\mathbb{H}P^{n-k-1}$	$\mathbb{H}P^{n-k}$, $\mathbb{H}P^{k+1}$	$0 \le k \le n-1$
	$\mathbb{C}P^n$	$\mathbb{C}P^n$	
$\mathbb{O}P^2$	*point*, S^8	$\mathbb{O}P^2$, S^8	
	$\mathbb{H}P^2$	$\mathbb{H}P^2$	
$G_k^+(\mathbb{R}^n)$	$G_{k-1}^+(\mathbb{R}^{n-1})$, $G_k^+(\mathbb{R}^{n-1})$	S^{n-k}, S^k	$1 < k < n-k$
			$(k,n) \ne (2,2l)$,
			$l > 2$
$G_2^+(\mathbb{R}^{2n})$	S^{2n-2}, $G_2^+(\mathbb{R}^{2n-1})$	S^{2n-2}, S^2	$n \ge 3$
	$\mathbb{C}P^{n-1}$	$\mathbb{C}P^{n-1}$	
$G_3^+(\mathbb{R}^6)$	$G_2^+(\mathbb{R}^5) = G_3^+(\mathbb{R}^5)$	S^3	
	$SU(3)/SO(3) \times S^1$	$\mathbb{R}P^3$	
$G_k^+(\mathbb{R}^{2k})$	$G_{k-1}^+(\mathbb{R}^{2k-1}) = G_k^+(\mathbb{R}^{2k-1})$	S^k	$k \ge 4$
$G_k(\mathbb{C}^n)$	$G_{k-1}(\mathbb{C}^{n-1})$, $G_k(\mathbb{C}^{n-1})$	$\mathbb{C}P^{n-k}$, $\mathbb{C}P^k$	$1 < k < n-k$
			$(k,n) \ne (2,2l)$,
			$l > 2$
$G_2(\mathbb{C}^{2n})$	$\mathbb{C}P^{2n-2}$, $G_2(\mathbb{C}^{2n-1})$	$\mathbb{C}P^{2n-2}$, $\mathbb{C}P^2$	$n \ge 3$
	$\mathbb{H}P^{n-1}$	$\mathbb{H}P^{n-1}$	
$G_k(\mathbb{C}^{2k})$	$G_{k-1}(\mathbb{C}^{2k-1}) = G_k(\mathbb{C}^{2k-1})$	$\mathbb{C}P^k$	$k \ge 3$
$G_k(\mathbb{H}^n)$	$G_{k-1}(\mathbb{H}^{n-1})$, $G_k(\mathbb{H}^{n-1})$	$\mathbb{H}P^{n-k}$, $\mathbb{H}P^k$	$1 < k < n-k$
$G_k(\mathbb{H}^{2k})$	$G_{k-1}(\mathbb{H}^{2k-1}) = G_k(\mathbb{H}^{2k-1})$	$\mathbb{H}P^k$	$k \ge 2$
$SU(n)/SO(n)$	$SU(n-1)/SO(n-1) \times S^1$	$\mathbb{R}P^{n-1}$	$n = 3$ or $n \ge 5$
$SU(6)/Sp(3)$	$S^5 \times S^1$	$\mathbb{H}P^2$	
	$SU(3)$	$\mathbb{C}P^3$	
$SU(2n)/Sp(n)$	$SU(2n-2)/Sp(n-1) \times S^1$	$\mathbb{H}P^{n-1}$	$n \ge 4$
$SO(2n)/U(n)$	$SO(2n-2)/U(n-1)$	$\mathbb{C}P^{n-1}$	$n \ge 5$
$Sp(n)/U(n)$	$Sp(n-1)/U(n-1) \times S^2$	$\mathbb{C}P^{n-1}$	$n \ge 3$
$SU(3)$	$U(2)$	$\mathbb{C}P^2$	
	$SU(3)/SO(3)$	$\mathbb{R}P^3$	
$SU(4) = Spin(6)$	$U(3)$	$\mathbb{C}P^3$	
	$Spin(5)$	S^5	
$SU(n)$	$U(n-1)$	$\mathbb{C}P^{n-1}$	$n \ge 5$
$Spin(n)$	$Spin(n-1)$	S^{n-1}	$n = 5$ or $n \ge 7$
$Sp(n)$	$Sp(n-1)Sp(1)$	$\mathbb{H}P^{n-1}$	$n \ge 3$
F_4	$Spin(9)$	$\mathbb{O}P^2$	
$E_6/SU(6)SU(2)$	$F_4/Sp(3)SU(2)$	$\mathbb{H}P^3$	
$E_6/Spin(10)U(1)$	$\mathbb{O}P^2$	$\mathbb{O}P^2$	
E_6/F_4	$S^9 \times S^1$	$\mathbb{O}P^2$	
	$SU(6)/Sp(3)$	$\mathbb{H}P^3$	
$F_4/Sp(3)SU(2)$	$G_4^+(\mathbb{R}^9)$	$\mathbb{H}P^2$	

References

[1] L. Bérard-Bergery, *Sur de nouvelles variétés riemanniennes d'Einstein*, Inst. Élie Cartan, vol. 6, Univ. Nancy, Nancy, 1982, pp. 1–60. MR **85b:**53048

[2] J. Berndt and M. Brück, *Cohomogeneity one actions on hyperbolic spaces*, J. Reine Angew. Math. **541** (2001), 209–235. MR **2002j:**53059

[3] J. Berndt and H. Tamaru, *Cohomogeneity one actions on noncompact symmetric spaces with a totally geodesic singular orbit*, to appear in Tôhoku Math. J.

[4] ———, *Homogeneous codimension one foliations on noncompact symmetric spaces*, J. Differential Geom. **63** (2003), 1–40.

[5] E. Boeckx and L. Vanhecke, *Harmonic and minimal radial vector fields*, Acta Math. Hungar. **90** (2001), 317–331. MR **2003h:**53089

[6] ———, *Isoparametric functions and harmonic and minimal unit vector fields*, Global differential geometry: the mathematical legacy of Alfred Gray (Bilbao, 2000), Contemp. Math., vol. 288, Amer. Math. Soc., Providence, R.I., 2001, pp. 20–31. MR **2003g:**53038

[7] O. Gil-Medrano and E. Llinares-Fuster, *Minimal unit vector fields*, Tôhoku Math. J. (2) **54** (2002), 71–84. MR **2002j:**53086

[8] E. Heintze, R. Palais, C.-L. Terng, and G. Thorbergsson, *Hyperpolar actions and k-flat homogeneous spaces*, J. Reine Angew. Math. **454** (1994), 163–179. MR **96b:**53062

[9] S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Pure and Applied Mathematics, vol. 80, Academic Press Inc., New York, 1978. MR **80k:**53081

[10] W. Y. Hsiang and H. B. Lawson Jr., *Minimal submanifolds of low cohomogeneity*, J. Differential Geom. **5** (1971), 1–38. MR 45#7645

[11] A. Kollross, *A classification of hyperpolar and cohomogeneity one actions*, Trans. Amer. Math. Soc. **354** (2002), 571–612. MR **2002g:**53091

[12] D. S. P. Leung, *On the classification of reflective submanifolds of Riemannian symmetric spaces*, Indiana Univ. Math. J. **24** (1974), 327–339; Errata: ibid. **24** (1975), 1199. MR 51#4115; MR 51#13935

[13] ———, *Reflective submanifolds. III. Congruency of isometric reflective submanifolds and corrigenda to the classification of reflective submanifolds*, J. Differential Geom. **14** (1979), 167–177. MR **83e:**53052

[14] P. S. Mostert, *On a compact Lie group acting on a manifold*, Ann. of Math. (2) **65** (1957), 447–455. MR 19,44b

[15] L. Verhóczki, *Special cohomogeneity one isometric actions on irreducible symmetric spaces of types I and II*, Beiträge Algebra Geom. **44** (2003), 57–74. MR 1990984

[16] G. Wiegmink, *Total bending of vector fields on Riemannian manifolds*, Math. Ann. **303** (1995), 325–344. MR **97a:**53050

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