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LIMIT DISTRIBUTION OF DEGREES IN RANDOM FAMILY TREES

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Abstract

In a one-parameter model for evolution of random trees, which also includes the Barabási–Albert random tree [1], almost sure behavior and the limiting distribution of the degree of a vertex in a fixed position are examined. A functional central limit theorem is also given. Results about Pólya urn models are applied in the proofs.

1 Introduction

Evolving random graphs and random trees have been widely examined recently, see e.g. [1, 3, 11]. One of the simplest dynamics is the following. At each step one new vertex is born, and it attaches with one edge to one of the old vertices. The probability that a given old vertex is chosen is proportional to a fixed linear function of its actual degree. The asymptotic degree distribution is well-known. These trees have the so-called scale free property: the proportion of vertices of degree d converges to c_d almost surely as the number of vertices goes to infinity, and $c_d \sim c \cdot d^{-\gamma}$ $(d \to \infty)$ with some positive constants c and γ (see e.g. [1, 11]).

Instead of the degree distribution, we focus on the degree of a vertex in a given position in the tree, as the number of vertices goes to infinity. Denote the degree of a fixed vertex (e.g. the root of the tree, or the jth child of the root, or the kth child of the jth child of the root, etc.) after n steps by X_n. We will see that $n^{-\delta}X_n$ converges to a positive random variable almost surely, with some $\delta > 0$, and we will have some information on the structure of this random variable. In addition, we will prove a functional central limit theorem on the degree sequence (X_n), based on the results of Gouet [6].

We will characterize the distribution of the limit for the Albert–Barabási tree, where the probability that a given old vertex is chosen is proportional to its actual degree. We will also examine some generalizations, namely, random sprouts and multitrees [8, 9].

2 Random trees

2.1 Notations

Trees are connected graphs without cycles. We assign a vertex, this is the root of the tree. We consider trees growing at discrete time steps. More precisely, we start from the root, and add one new vertex with one edge at each step. When examining the neighbours of a certain vertex, we keep count of the order they were born. Thus our graph is a *rooted ordered tree* (also known as rooted planar tree or family tree). See for example [3].

We will use the following commonly known terminology and notation for rooted ordered trees [3, 11]. The vertices are individuals, and the edges of the tree represent the parent-child relations. When a new vertex with one edge is added to the graph, we say that it becomes the child of its only neighbour, its parent.

We will label the vertices with sequences of positive integers, based on the parent-child relations. We set

$$\mathbb{Z}_+ = \{1, 2, \dots\}, \quad \mathbb{Z}_+^0 = \{\emptyset\}, \quad \mathcal{N} = \bigcup_{n=0}^{\infty} \mathbb{Z}_+^n.$$

The label of the root is $\emptyset$. The jth child of the root is labelled with j. Similarly, the jth child of the vertex labelled with $x = (x_1, \dots, x_k) \in \mathcal{N}$ is labelled with $(x_1, \dots, x_k, j)$. To put it in an other way, the vertex with label $x = (x_1, \dots, x_k) \in \mathcal{N}$ is the x_kth child of the vertex with label $(x_1, \dots, x_{k-1})$, which is the x_{k-1}st child of its parent, and so on.

Note that trees can be represented by the set of the labels of their vertices; the labels give all information about the edges. In the sequel we identify vertices with their labels, and trees with the set of labels. The set of finite rooted ordered trees is denoted by $\mathcal{G}$. We say that a vertex with label $x = (x_1, \dots, x_k) \in \mathcal{N}$ belongs to the kth generation of $G \in \mathcal{G}$. The degree of a vertex x in G will be denoted by $\deg(x, G)$.

2.2 The random tree model with linear weight function

We consider randomly growing trees. At each step we add a new vertex, which attaches to a randomly chosen, already existing vertex with one edge. The probability that a vertex of degree d gets the new edge is proportional to a fixed linear function of d.

To formulate this, let $\beta > -1$ be fixed, and let $G_n \in \mathcal{G}$ $(n \in \mathbb{N})$ be a sequence of random finite rooted ordered trees, such that $G_1 = \{\emptyset\}$, and the following holds for all $n, k \in \mathbb{N}$, $x = (x_1, \dots, x_k) \in G_n$ and $d = \deg(x, G_n)$.

$$\mathbb{P}\left(G_{n+1} = G_n \cup \{d+1\} \middle| G_n\right) = \frac{d+\beta}{S_n} \qquad \text{if } x = \emptyset,$$
$$\mathbb{P}\left(G_{n+1} = G_n \cup \{(x_1, \dots, x_k, d)\} \middle| G_n\right) = \frac{d+\beta}{S_n} \qquad \text{if } x \neq \emptyset,$$

where $S_n = 2n - 2 + n\beta = \sum_{v \in G_n} \left(\deg(v, G_n) + \beta\right)$. The condition $\beta > -1$ guarantees that the given probabilities are positive.

We say that the *weight* of a vertex of degree d is $d + \beta$. In other words, each vertex has weight $1 + \beta$ when it is born, and its weight increases by 1 every time when it gives birth to a child.

3 Main results

Our goal is to describe the almost sure behavior and the limiting distribution of the degree of a fixed vertex x in G_n as $n \to \infty$.
For the root, the moments of the limiting distribution are calculated in [10]. It is proved that

$$\frac{\deg\left(\emptyset, G_n\right)}{n^{1/(2+\beta)}} \to \zeta_\emptyset \tag{1}$$

as $n \to \infty$, with probability 1, with a positive random variable $\zeta_\emptyset$. Moreover, the following holds for every integer $k \geq 1$.

$$\mathbb{E}\zeta_\emptyset^k = k! \cdot \frac{\Gamma\left(1+\frac{\beta}{2+\beta}\right)}{\Gamma\left(1+\frac{k+\beta}{2+\beta}\right)} \cdot \binom{k+\beta}{k} = \frac{\Gamma\left(1+\frac{\beta}{2+\beta}\right)}{\Gamma\left(\beta+1\right)} \cdot \frac{\Gamma\left(k+\beta+1\right)}{\Gamma\left(1+\frac{k+\beta}{2+\beta}\right)}. \tag{2}$$

Our main result is the following.

Theorem 1. *Let $k \in \mathbb{Z}_+$ and $x = \left(x_1, \dots, x_k\right) \in \mathcal{N}$ be fixed. The following holds in the random tree model of Section* 2.2*.*

$$\frac{\deg\left(x, G_n\right)}{n^{1/(2+\beta)}} \to \zeta_x$$

with probability 1*, with some positive random variable ζ_x. The distribution of ζ_x is the same as the distribution of $\zeta_\emptyset \cdot \xi_1 \cdot \ldots \cdot \xi_k$, where*

- *$\zeta_\emptyset, \xi_1, \dots, \xi_k$ are independent random variables;*
- *$\zeta_\emptyset$ is defined by equation* (1)*;*
- *ξ_1 has distribution Beta $\left(1+\beta, x_1-1\right)$ if $x_1 > 1$; $\xi_1 \equiv 1$ if $x_1 = 1$;*
- *ξ_s has distribution Beta $\left(1+\beta, x_s\right)$ for $2 \leq s \leq k$.*

Remark 1. For the Albert–Barabási tree [1] we have $\beta = 0$; the probability that a given vertex of degree d gets the new edge is proportional to d. In this particular case we can determine the distribution of $\zeta_\emptyset$. From equation (2) we get that

$$\mathbb{E}\left(\left(2^{-1/2}\zeta_\emptyset\right)^{2k}\right) = \frac{\Gamma\left(2k+1\right)}{2^k\Gamma\left(k+1\right)} = (2k-1)!!.$$

These are the even moments of the standard normal distribution. Thus the kth moment of $\zeta_\emptyset^2/2$ is equal to $(2k-1)!!$. It is easy to verify that

$$\sum_{k=1}^{\infty}((2k-1)!!)^{-\frac{1}{2k}} = \infty.$$

Hence using Carleman's condition (see e.g. § 4 Chapter XV in Feller [4]) we obtain that the distribution of $\zeta_\emptyset^2/2$ is is equal to the distribution of the square of a standard normal random variable. $2^{-1/2}\zeta_\emptyset$ is nonnegative, thus it is identically distributed with the absolute value of a standard normal random variable.

Thus applying Theorem 1 for the Albert–Barabási tree we get the following. The distribution of ζ_x is identical to the distribution of $\zeta_\emptyset \cdot \xi_1 \cdot \ldots \cdot \xi_k$, where the random variables in the product are independent; $\zeta_\emptyset$ is identically distributed with $\sqrt{2}$ times the absolute value of a standard normal random variable; ξ_1 has distribution $Beta\left(1, x_1 - 1\right)$ if $x_1 > 1$; $\xi_1 \equiv 1$ if $x_1 = 1$; ξ_s has distribution $Beta\left(1, x_s\right)$ for $2 \le s \le k$.

Remark 2. Plane oriented random trees (PORT) are also rooted ordered trees. In this case the tree is embedded in the plane, and the left-to-right order of the children of the vertices is relevant (see e.g. [3]). The out-degree of vertex v in tree G is the number of its children, and it will be denoted by $\deg^+(v, G)$. It is equal to the degree of the vertex minus one, except for the root, where it is the same as the degree.
Now the probability that a vertex of out-degree d gets the new edge is proportional to $d + \gamma$ for some $\gamma > 0$ [2, 11]. We can get similar results for the limit distribution of

$$\frac{\deg\left(x, G_n\right)}{n^{1/(1+\gamma)}}$$

as $n \to \infty$ by straightforward modifications of the proofs. Note that the case $\gamma = \beta + 1$ corresponds to the random tree model with linear weights of parameter β.

Proof. We prove the theorem by induction on k.
For $k = 1$, let $j \in \mathbb{N}$ be fixed. Vertex $x = j$ is the jth child of the root.
For $j = 1$ we have one edge, for symmetry reasons it is clear that ζ_1 and $\zeta_\emptyset$ are identically distributed.
For $j > 1$, assume that vertex j appears in the Nth step, that is, $j \in G_N \setminus G_{N-1}$. N is a random positive integer. After the birth of vertex j, we divide the weight of the vertices into two parts, a "black" and a "white" one. $W_n(x)$ and $B_n(x)$ denote the "white" and "black" weight of vertex x in G_n, respectively, for $n \ge N$. The total weight of a vertex is equal to its actual degree plus β, thus we have

$$\deg\left(x, G_n\right) + \beta = W_n(x) + B_n(x) \quad \left(n \ge N, x \in G_n\right). \tag{3}$$

We set

$$\begin{aligned} W_N(j) &= 1 + \beta, & B_N(j) &= 0; \\ W_N(\emptyset) &= 1 + \beta, & B_N(\emptyset) &= j - 1; \\ W_N(x) &= 0, & B_N(x) &= \deg\left(x, G_N\right) + \beta \quad \left(x \in G_N \setminus \{\emptyset, j\}\right). \end{aligned}$$

This is possible, because vertex j has weight $1 + \beta$ when it is born, and the root has degree j at the same time. All the other weights are coloured black.
Later on, when a new vertex appears, it gets black weight $1 + \beta$, its total weight is black. When an old vertex, x, gets a new edge, its weight increases by 1. The colour of this increment will be randomly chosen; the probability that the increment is white is the ratio of the white part to the total weight of x. We formulate this in the following way. For $n \ge N$, $k \in \mathbb{N}$, $x = \left(x_1, \ldots, x_k\right) \in G_n$ let $d = \deg\left(x, G_n\right)$ if $x \ne \emptyset$, and $d = 1 + \deg\left(x, G_n\right)$ if $x = \emptyset$. We set

$$\begin{aligned} &\mathbb{P}\left(G_{n+1} = G_n \cup \left\{\left(x_1, \ldots, x_k, d\right)\right\}, W_{n+1}(x) = W_n(x) + 1 \,\middle|\, G_n\right) = \frac{d + \beta}{S_n} \cdot \frac{W_n(x)}{d + \beta} = \frac{W_n(x)}{S_n}, \\ &\mathbb{P}\left(G_{n+1} = G_n \cup \left\{\left(x_1, \ldots, x_k, d\right)\right\}, B_{n+1}(x) = B_n(x) + 1 \,\middle|\, G_n\right) = \frac{d + \beta}{S_n} \cdot \frac{B_n(x)}{d + \beta} = \frac{B_n(x)}{S_n}. \end{aligned} \tag{4}$$

Of course, if $W_{n+1}(x) = W_n(x) + 1$, then $B_{n+1}(x) = B_n(x)$, and if $B_{n+1}(x) = B_n(x) + 1$, then $W_{n+1}(x) = W_n(x)$; the total weight is increased by 1 in both cases.
Note that vertices, except the root and its jth child, never get white weight, which implies that

$$W_n(x) = 0, \ \ B_n(x) = \deg(x, G_n) + \beta \quad \left(n \geq N, x \in G_n \setminus \{\emptyset, j\}\right).$$

On the other hand, the total weight of vertex j is white when it is born, thus we have

$$W_n(j) = \deg(j, G_n) + \beta, \ \ B_n(j) = 0 \quad (n \geq N). \tag{5}$$

We coloured the weights in such a way that

$$W_N(j) = W_N(\emptyset) = 1 + \beta.$$

From formula (4) it follows that the probability that the white weight of a vertex increases depends only on its actual white weight, independently of the structure of G_n; and S_n is deterministic. Thus, for symmetry reasons, we have that if

$$\frac{W_n(\emptyset)}{n^{1/(2+\beta)}} \to \zeta_j^0$$

almost surely for some positive random variable ζ_j^0, then

$$\frac{W_n(j)}{n^{1/(2+\beta)}} \to \zeta_j$$

almost surely as well, where ζ_j^0 and ζ_j are identically distributed. Thus we will focus on the behavior of the root.
Recall that $W_N(\emptyset) = 1 + \beta, B_N(\emptyset) = j - 1$. From equation (4) it follows that

$$\begin{aligned} &\mathbb{P}\left(W_{n+1}(\emptyset) = W_n(\emptyset) + 1 \,\middle|\, \deg(\emptyset, G_{n+1}) = \deg(\emptyset, G_n) + 1\right) = \frac{W_n(\emptyset)}{W_n(\emptyset) + B_n(\emptyset)}, \\ &\mathbb{P}\left(B_{n+1}(\emptyset) = B_n(\emptyset) + 1 \,\middle|\, \deg(\emptyset, G_{n+1}) = \deg(\emptyset, G_n) + 1\right) = \frac{B_n(\emptyset)}{W_n(\emptyset) + B_n(\emptyset)}. \end{aligned} \tag{6}$$

This corresponds to a Pólya–Eggenbegger urn model. We start with an urn with a white and b black balls. At each step, we draw a ball with uniform distribution, and put it back together with c balls of the same colour. It is well known (see e.g. Theorem 3.2. in [8] or Section 4.2. in [7]) that the proportion of the white balls converges almost surely to a $Beta(a/c, b/c)$ distributed random variable. Moreover, saying that the number of the balls corresponds to “white" and “black" weights, this remains the same for positive, not necessarily integer weights. In this case, at each step, we choose a random colour. The probability that white is chosen is equal to the actual proportion of the white weight in the urn. Then the weight of the selected colour is increased by c. As equations (6) show, this happens at the steps when the root gives birth to a child. Thus we can apply these results with $a = 1 + \beta$, $b = j - 1$ and $c = 1$, and we get that

$$\frac{W_n(\emptyset)}{W_n(\emptyset) + B_n(\emptyset)} \to \xi_j^0$$

almost surely as $n \to \infty$, and ξ_j^0 has distribution $Beta(1 + \beta, j - 1)$.

From this result, equations (1), (3) and the condition $\beta > -1$ it follows that

$$\frac{W_n(\emptyset)}{n^{1/(2+\beta)}} = \frac{W_n(\emptyset)}{W_n(\emptyset) + B_n(\emptyset)} \cdot \frac{\deg(\emptyset, G_n) + \beta}{n^{1/(2+\beta)}} \to \xi_j^0 \cdot \zeta_\emptyset = \zeta_j^0$$

almost surely as $n \to \infty$. ξ_j^0 and $\zeta_\emptyset$ are independent, because the almost sure convergence of the proportion of the white weight does not depend on the behavior of the total weight of the root. As we have seen before, this implies that

$$\frac{W_n(j)}{n^{1/(2+\beta)}} \to \zeta_j$$

almost surely as $n \to \infty$, where ζ_j^0 and ζ_j are identically distributed. Thus, using equation (5) we get that

$$\frac{\deg(j, G_n)}{n^{1/(2+\beta)}} = \frac{W_n(j) - \beta}{n^{1/(2+\beta)}} \to \zeta_j$$

almost surely as $n \to \infty$, and ζ_j has the same distribution as $\zeta_\emptyset \cdot \xi_j$, where $\zeta_\emptyset$ is defined by equation (1), ξ_j has distribution $Beta\left(1+\beta, j-1\right)$, and finally, $\zeta_\emptyset$ and ξ_j are independent. This completes the proof for the case $k = 1$.

Assume that the statement is proved for some $k \geq 1$.

We fix $x = (x_1, \ldots, x_k) \in \mathcal{N}$ and $x_{k+1} \in \mathbb{Z}_+$. The induction step can be verified by a slight modification of the previous argument. The only difference is that x has degree x_{k+1} when its x_{k+1}st child is born, namely, one edge is attached to its parent, $(x_1, \ldots, x_{k-1})$, and $x_{k+1} - 1$ to its already existing children. This means that $a = 1 + \beta$ and $b = x_{k+1}$ in the urn model, and the last factor ξ_{k+1} has distribution $Beta\left(1+\beta, x_{k+1}\right)$. □

4 Generalizations

4.1 Random recursive sprouts

For a fixed positive integer m consider the following generalization of the random tree model. At each step, we choose one vertex; the probability that a vertex of degree d is chosen is proportional to $d + \beta$, where $\beta > -1$ is fixed. Then m new vertices are added to the graph; each of them is attached with one edge to the chosen vertex. That is, the chosen vertex gets m children at the same time. This model is a particular case of the random recursive sprouts in Section 8.2.5. of [8].

By straightforward modifications of the proof of Theorem 1 one can achieve a similar result. The only essential difference is that we put m balls of the drawn colour in the urn model, that is, $c = m$. Finally we get that in this case ξ_1 has distribution $Beta\left(\frac{1+\beta}{m}, \frac{m\lceil x_1/m \rceil - 1}{m}\right)$ if m or x_1 is greater than 1; $\xi_1 \equiv 1$ if $x_1 = m = 1$; ξ_s has distribution $Beta\left(\frac{1+\beta}{m}, \frac{m\lceil x_s/m \rceil}{m}\right)$ for $2 \leq s \leq k$.

4.2 Multitrees.

We consider the model of random multitrees [9]. $m > 1$ is a positive integer. An m-multicherry is a hypergraph on $m + 1$ vertices. One of them, called center, is distinguished. The other m vertices

form the base, and they are contained in an m-hyperedge. The center is connected to every other vertex with a single edge.
Random m-multitrees are built up from m-multicherries in the following way. We start with m vertices that are contained in a hyperedge and form a base. At each step a new vertex is added to the graph. We choose one of the existing bases uniformly at random. The new vertex is connected to all of the vertices of the chosen base with single edges. Moreover, m different new bases are added; each of them contains the new vertex and $m-1$ of the vertices of the chosen base.

Remark 3. One can say that the new bases are the children of the chosen old base. In this sense we get a model of m-ary trees that is similar to the previous case, but vertices are uniformly chosen at each step. In this case every child behaves like its parent; for symmetry reasons their degrees have the same limit distributions.

This graph is not a tree, however, the jth child of a given vertex is uniquely determined. We distinguish one vertex of the initial configuration, this is the root, denoted by $\emptyset$. Now we can find vertex x for all $x \in \mathcal{N}$, keeping in mind that they are not all different. The initial configuration is symmetric, hence the results hold for all choices of the root.
In [9] (Theorem 3.1 with $k=1$) it is proved that the number vertices that are attached to some of the starting vertices divided by $n^{(m-1)/m}$ converges almost surely to a positive random variable ζ. The following proposition can be proved using this result and the method of Theorem 1.

Theorem 2. *Let $k \in \mathbb{Z}_+$ and $x = (x_1, \ldots, x_k) \in \mathcal{N}$ be fixed. The following hold in a random m-multitree.*

$$\frac{\deg(x, G_n)}{n^{(m-1)/m}} \to \zeta_x$$

with probability 1, with some positive random variable ζ_x. The distribution of ζ_x is the same as the distribution of $\zeta_\emptyset \cdot \xi_1 \cdot \ldots \cdot \xi_k$, where

- *$\zeta_\emptyset, \xi_1, \ldots, \xi_k$ are independent random variables;*
- *$\zeta_\emptyset$ is positive;*
- *ξ_1 has distribution Beta $\left(1 + \frac{1}{m-1},\ x_1 - 1\right)$ if $x_1 > 1$; $\xi_1 \equiv 1$ if $x_1 = 1$;*
- *ξ_s has distribution Beta $\left(1 + \frac{1}{m-1},\ x_s\right)$ for $2 \le s \le k$.*

Sketch of the proof. For $x = \emptyset$ the statement easily follows from Theorem 3.1. of [9].
For $k=1$ we can use the method of Theorem 1 for the bases. Every vertex, except the first m, is contained in m bases when it is born, and this number increases by $m-1$ at every step it gets a new edge. A vertex of the initial configuration is contained in $1 + j(m-1)$ bases when its jth child is born. Other vertices are contained in $m + j(m-1)$ bases when their jth child is born.
We choose bases uniformly. We use the same argument with the urn models for the bases; the bases containing the jth child and some of the bases containing $\emptyset$ are white, others are black. Now we have that $c = m-1$. Since there is a deterministic linear connection between the number of bases containing a given vertex and its degree, similar results hold for the degrees.
Finally we can complete the proof by induction on k. □

5 Functional central limit theorem

In this section we give a functional version of the main results in the random tree model with linear weight function. Similar results can be proved for the models of Section 4, we omit the details. These are based on the results of Gouet [6]. Weak convergence of processes is the weak convergence of the induced probabilities on the Skorohod space $D[0,\infty)$. It is denoted by $\Rightarrow$.

Theorem 3. *Let $k \in \mathbb{Z}_+$ and $x = (x_1, \ldots, x_k) \in \mathcal{N}$ be fixed. Then the following holds in the random tree model of Section 2.2.*

$$n^{-1/(2(2+\beta))}\left(\deg\left(x, \lfloor nt^{(2+\beta)}\rfloor\right) - n^{1/(2+\beta)} t\zeta_x\right) \Rightarrow B\left(\zeta_x t\right),$$

for $t > 0$ as $n \to \infty$, where B is a Brownian motion, independent of ζ_x.

Proof. For the root, i.e., $x = \emptyset$, the statement follows directly from the results of Gouet [6] by using the following Pólya urn scheme. Colour the total weight of the root white, and all the other weights black. At each step, we choose a colour with probabilities proportional to the weights. We will use the notations of Gouet [6]. If white is chosen, white is increased by $a = 1$, and black is increased by $b = 1 + \beta$, due to the new vertex. If the black is chosen, white does not change, we have $c = 0$, and black is increased by $d = 2 + \beta$. At the beginning we have $1 + \beta$ of white weight and none of black. It is easy to check that this model is tenable, i.e., the urn is not initially empty, the total weight increases by the same amount $s = 2 + \beta \geq 1$ at every step, $a \neq c$, finally, the process does not stop because of impossible removals, since there are no removals in our case.
The conditions of Proposition 2.2. of [6] are also satisfied; $s > a > 0$, $bc = 0$, $\max(b, c) > 0$. This immediately gives the result.
Now let $x \in \mathcal{N}$, $x \neq \emptyset$ be fixed. We will condition again on the birth time of x. Let $E_{x,j}$ denote the event that x is born in the jth step $(j = 1, 2, \ldots)$. Given $E_{x,j}$, colour the total weight of vertex x white, and all the other weights black. Like in the case of the root, we get an urn model examined by Gouet with replacement matrix

$$\begin{pmatrix} a & b \\ 0 & s \end{pmatrix} = \begin{pmatrix} 1 & 1+\beta \\ 0 & 2+\beta \end{pmatrix}.$$

Proposition 2.2. of [6] can be applied to get that the weak convergence holds on the conditional probability space given by $E_{x,j}$. Formally, let

$$S(n,t) = n^{-1/(2(2+\beta))}\left(\deg\left(x, \lfloor nt^{(2+\beta)}\rfloor\right) - n^{1/(2+\beta)} t\zeta_x\right) \quad (n \geq 1,\ t > 0);$$

$B^*(t) = B\left(\zeta_x t\right)$ for $t > 0$. Let C be a fixed open Borel set of the Skorohod space D. Thus we got that

$$\liminf_{n\to\infty} \mathbb{P}\left(S(n,\cdot) \in C \middle| E_{x,j}\right) \geq \mathbb{P}\left(B^*(\cdot) \in C \middle| E_{x,j}\right).$$

Since $E_{x,j}$ are disjoint events for different values of j, this implies that

$$\liminf_{n\to\infty} \mathbb{P}\left(S(n,\cdot) \in C \middle| \bigcup_{j=1}^{J} E_{x,j}\right) \geq \mathbb{P}\left(B^*(\cdot) \in C \middle| \bigcup_{j=1}^{J} E_{x,j}\right)$$

for all $J \in \mathbb{N}$. On the other hand, it is clear that

$$\lim_{J\to\infty} \sum_{j=J+1}^{\infty} \mathbb{P}\left(E_{x,j}\right) = 0.$$

Now it is easy to see that

$$\liminf_{n\to\infty} \mathbb{P}(S(n,\cdot)\in C) \geq \mathbb{P}\left(B^*(\cdot)\in C\right).$$

Since this holds for all open Borel sets C, the weak convergence is proved.

□

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