ELEMENTARY AND INTEGRAL-ELEMENTARY FUNCTIONS

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ABSTRACT. By an *integral-elementary* function we mean any real function that can be obtained from the constants, $\sin x$, e^x, $\log x$, and $\arcsin x$ (defined on $(-1, 1)$) using the basic algebraic operations, composition and integration. The *rank* of an integral-elementary function f is the depth of the formula defining f. The integral-elementary functions of rank $\leq n$ are real-analytic and satisfy a common algebraic differential equation $P_n(f, f', \ldots, f^{(k)}) = 0$ with integer coefficients.

We prove that every continuous function $f: \mathbf{R} \to \mathbf{R}$ can be approximated uniformly by integral-elementary functions of bounded rank. Consequently, there exists an algebraic differential equation with integer coefficients such that its everywhere analytic solutions approximate every continuous function uniformly. This solves a problem posed by L. A. Rubel.

Using the same basic functions as above, but allowing only the basic algebraic operations and compositions, we obtain the class of *elementary* functions. We show that every differentiable function with a derivative not exceeding an iterated exponential can be uniformly approximated by elementary functions of bounded rank. If we include the function $\arcsin x$ defined on $[-1, 1]$, then the resulting class of *naive-elementary* functions will approximate every continuous function uniformly.

We also show that every sequence can be uniformly approximated by elementary functions, and that every integer sequence can be represented in the form $f(n)$, where f is naive-elementary.

1. Introduction

The investigations of this paper were motivated by the following question posed by J. Pintz: is it possible to approximate the function $\pi(x)$ (the number of primes up to x) using elementary functions in such a way that the error of the approximation is smaller than, say, $|\pi(x) - \mathrm{Li}(x)|$? Of course, the answer to this question depends on what we mean by elementary functions.

The "naive" approach is to consider a function f elementary if it can be given by a finite closed expression; that is, if f can be obtained from a given set of basic functions using a given set of operations. Our choice is then to select the admissible basic functions and operations. We shall choose one of the most restrictive possibilities, and adopt the following definition. (In the sequel, by intervals we always mean non-degenerate intervals.)

The class of *naive-elementary functions* is the smallest class NE of real functions defined on subintervals of $\mathbf{R}$ such that:

(i) NE contains the constants, the identity function x, the functions $\sin x$, e^x, $\log x$, and $\arcsin x$ (defined on $[-1, 1]$).

(ii) If f is a function defined on an interval I, and if there are functions $g, h \in NE$ such that f equals the restriction of one of the functions $g + h$, $g \cdot h$, g/h, $g \circ h$, to the interval I, then $f \in NE$. (In the case of g/h we assume that h does not vanish on I.)

Let NE_0 denote the set of functions listed in (i). If NE_n is defined, then let NE_{n+1} denote the family of all functions $f: I \to \mathbf{R}$ with the following properties: I is an interval, and there are functions $g, h \in NE_n$ such that f is the restriction of one of $g + h$, $g \cdot h$, g/h, $g \circ h$ to I. The elements of NE_n will be called the naive-elementary functions of rank n. From the minimality of the class NE it follows that $NE = \cup_{n=0}^{\infty} NE_n$. Thus every element f of NE can be given by a finite expression, and the rank of the function is given by the depth of the simplest formula defining f.

As we shall see, the class NE is surprisingly large. Namely, for every pair of continuous functions $g: \mathbf{R} \to \mathbf{R}$ and $\varepsilon: \mathbf{R} \to (0, \infty)$ there is a function $f \in NE$ such that $|f(x) - g(x)| < \varepsilon(x)$ for every $x \in \mathbf{R}$ (Theorem 6.2). Moreover, we may take $f \in NE_{19}$, and thus f can be obtained from NE_0 using only a bounded number of operations. This result provides the following answer to Pintz's question: There is a naive-elementary function f such that $|f(x) - \pi(x)| < 1$ everywhere. As for the sequence $\pi(n)$, we show that there is a naive-elementary function f such that $f(n) = \pi(n)$ for every n. In fact, this is true for any sequence of integers in place of $\pi(n)$ (Corollary 4.3).

Considering the notion of naive-elementary functions it could be objected that, although the elements of NE_0 are analytic, NE_1 already contains nondifferentiable functions (take, for example, $\arcsin(\sin x)$). Since the elementary functions are required to be analytic, it seems natural to adopt the following definition.

The class of *elementary functions* is the smallest class E of real functions defined on *open* subintervals of $\mathbf{R}$ such that:

(i) E contains the constants, the identity function x, the functions $\sin x$, e^x, $\log x$, and $\arcsin x$ (defined on $(-1, 1)$).

(ii) If f is a function defined on an open interval I, and if there are functions $g, h \in E$ such that f equals the restriction of one of the functions $g+h$, $g \cdot h$, g/h, $g \circ h$, to the interval I, then $f \in E$. (In the case of g/h we assume that h does not vanish on I.)

We define the classes E_n in the same way as above. Then, by the minimality of the class E it follows that $E = \cup_{n=0}^{\infty} E_n$. Clearly, every elementary function is analytic on its domain.

We shall prove that every differentiable function with a derivative not exceeding an iterated exponential can be uniformly approximated by elementary functions (Theorem 5.1). (This easily implies that there is an elementary function f such that $|f(x) - \pi(x)| < 1$ everywhere.) Also, if a_n is an arbitrary sequence of real numbers, and ε_n is an arbitrary sequence of positive numbers, then there is a function $f \in E_9$ such that $|f(n) - a_n| < \varepsilon_n$ for every n (Theorem 4.2). On the other hand, we do not know whether or not every sequence of integers can be represented in the form $f(n)$, where $f \in E$. We show, however, that not every real sequence is of the form $f(n)$, where $f \in NE$. This is a special case of Theorem 4.5.

Our proof showing that every continuous function can be approximated by naive-elementary functions breaks down if we are allowed to use elementary functions only. In fact, it is very likely that if a continuous function is too large or oscillates too rapidly, then it cannot be uniformly approximated by elementary functions. We shall prove, however, that if integration is also allowed, then the resulting class, defined below, approximates every continuous function.

The class of *integral-elementary functions* is the smallest class IE of real functions defined on *open* subintervals of **R** satisfying the following conditions:

(i) IE contains the constants, the identity function x, the functions $\sin x$, e^x, $\log x$, and $\arcsin x$ (defined on $(-1, 1)$).

(ii) If f is a function defined on an open interval I, and if there are functions $g, h \in IE$ such that f equals the restriction of one of the functions $g + h$, $g \cdot h$, g/h, $g \circ h$, to the interval I, then $f \in IE$. (In the case of g/h we assume that h does not vanish on I.)

(iii) If $g \in IE$ is defined on the interval I and if $a \in I$, then the function f defined by $f(x) = \int_a^x g(t)\,dt$ $(x \in I)$ also belongs to IE.

If we define the classes IE_n in the obvious way, then we have $IE = \bigcup_{n=0}^{\infty} IE_n$. Clearly, every integral-elementary function is analytic on its domain. The class IE is strictly larger than E, since it contains, for example, the nonelementary function $\int_1^x ((\sin t)/t)\,dt$ $(x > 0)$. Still, the class IE is rather small in the sense that each of its elements satisfies an algebraic differential equation, that is, an equation of the form $P(x, f, f', \ldots, f^{(k)}) = 0$, where $P(x_0, \ldots, x_{k+1})$ is a polynomial. Moreover, for every n there is a nonzero polynomial P_n with integer coefficients such that each element of IE_n satisfies the algebraic differential equation $P_n = 0$. Indeed, the elements of IE_0 satisfy one of the equations

$$f' = 0, \ f' = 1, \ f^2 + (f')^2 = 1, \ f' = f, \ x \cdot f' = 1, \ (1 - x^2) \cdot (f')^2 = 1.$$

Thus for P_0 we may take

$$x_2 \cdot (x_2 - 1) \cdot (x_1^2 + x_2^2 - 1) \cdot (x_2 - x_1) \cdot (x_0 x_2 - 1) \cdot ((1 - x_0)^2 \cdot x_2^2 - 1).$$

It is known that for every nonzero polynomial P with integer coefficients there are nonzero polynomials $Q_1, \ldots, Q_5$ with integer coefficients such that whenever the

analytic functions f and g satisfy the algebraic differential equation $P = 0$, then $f + g, f \cdot g, f/g, f \circ g, \int f$ satisfy the algebraic differential equations $Q_1 = 0, \ldots, Q_5 = 0$ respectively. (See, e.g. [3, Theorem 5.4].) This easily implies, by induction on n, that all the elements of IE_n satisfy a common algebraic differential equation $P_n = 0$, where P_n is a nonzero polynomial with integer coefficients.

We shall prove that for every pair of continuous functions $g: \mathbf{R} \to \mathbf{R}$ and $\varepsilon: \mathbf{R} \to (0, \infty)$ there is a function $f \in IE_{19}$ such that $|f(x) - g(x)| < \varepsilon(x)$ for every $x \in \mathbf{R}$ (Theorem 6.1). By the previous remark, this implies that there is a nontrivial algebraic differential equation with integer coefficients, $P = 0$, with the following property: for every pair of continuous functions $g: \mathbf{R} \to \mathbf{R}$ and $\varepsilon: \mathbf{R} \to (0, \infty)$ there is a solution f of $P = 0$ such that f is everywhere analytic on $\mathbf{R}$, and $|f(x) - g(x)| < \varepsilon(x)$ for every $x \in \mathbf{R}$.

Whether such an algebraic differential equation exists was asked by L. A. Rubel in [5] (see also Problem 19 in [6] and [7], and Conjecture 6.2 in [3]). A partial solution was given by M. Boshernitzan in [3]. He proved that the family of functions

$$\int_0^{x+a} \frac{bd \cos(e^t)}{1 + d^2 - \cos(bt)} dt \quad (a, b, c, d \in \mathbf{R}, \ d > 0)$$

is dense in $C(I)$ for any compact interval $I \subset \mathbf{R}$. It is easy to see that these functions belong to IE_7. In [3], M. Boshernitzan also constructed an algebraic differential equation such that its *polynomial* solutions are dense in $C(I)$ for every compact interval I.

The history of Rubel's problem goes back to a false conjecture of Borel claiming that the solutions of an algebraic differential equation cannot grow faster than an iterated exponential (for details see [3],[6],[7]). The simplest counterexample to Borel's conjecture was constructed in [2]. Let

$$f_\alpha(x) = \sin^2 \pi x + \sin^2 \pi \alpha x = (2 - \cos 2\pi x - \cos 2\pi \alpha x)/2. \qquad (1.1)$$

In [2] it is shown that for every increasing function $\phi: [1, \infty) \to \mathbf{R}$ there is an irrational number α, and there is a sequence $x_n \to \infty$ such that $f_\alpha(x_n)^{-1} > \phi(x_n)$ for every n. On the other hand, it is easy to prove that the functions f_α^{-1} satisfy a common algebraic differential equation independent of α.

Our proof of Theorem 6.2 is based on the observation that for every increasing function $\phi: [1, \infty) \to \mathbf{R}$ there are irrational numbers α, β such that $f_\alpha(2^n)^{-1} + f_\beta(2^n)^{-1} > \phi(n)$ for every n (Theorem 2.1). Then we use an interpolation formula involving NE-functions (Lemma 3.5) to prove that every continuous function can be dominated by a naive-elementary function (Theorem 3.6).

It was proved in [1] that for every increasing function $\phi: [1, \infty) \to \mathbf{R}$ there is an irrational number α and a sequence $x_n \to \infty$ such that $\int_1^{x_n} f_\alpha(t)^{-1} dt > \phi(x_n)$ for every n. Our proof of Theorem 6.1, in turn, uses the fact that for every increasing function $\phi: [1, \infty) \to \mathbf{R}$ there are irrational numbers α, β such that $\int_1^x (f_\alpha(t)^{-1} + f_\beta(t)^{-1}) dt > \phi(x)$ for every $x \geq 2$ (Theorem 3.2). On the other hand,

we show that the functions $\int_1^x f_\alpha(t)^{-1} dt$ do not dominate all increasing functions: for every irrational α there is a sequence $x_i \to \infty$ such that $\int_1^{x_i} f_\alpha(t)^{-1} dt < 40 x_i \log x_i$ (Theorem 3.4).

If an integral-elementary or naive-elementary function f is defined on an interval I, then f must be analytic on a subinterval of I. Consequently, the classes IE and NE do not contain all continuous functions. We prove that even if we enlarge the set of basic functions NE_0 by an arbitrary countable set of continuous functions, the resulting class cannot contain all continuous functions (Theorem 6.4).

2. Dominating sequences with elementary functions

In this section we show that every sequence can be dominated by elementary functions of bounded rank. Recall that the function f_α was defined in (1.1). Clearly, if α is irrational, then the only real root of f_α is at $x = 0$.

THEOREM 2.1.   *For an arbitrary sequence of real numbers, A_n ($n = 1, 2, \ldots$), there is an elementary function f of rank 7 such that f is defined everywhere on $\mathbf{R}$, and $f(n) > A_n$ for every $n \in \mathbf{N}$. Namely, the function $f(x) = f_\alpha(2^x)^{-1} + f_\beta(2^x)^{-1}$ has this property, where α and β are irrational numbers depending on the sequence A_n.*

Let $\|x\|$ denote the distance of the real number x to the nearest integer. We shall frequently use the fact that

$$2\|x\| \le |\sin \pi x| \le \pi \|x\| \tag{2.1}$$

for every real number x.

LEMMA 2.2.   *Let C_n ($n = 1, 2, \ldots$) be an arbitrary sequence of real numbers. Then there are irrational numbers α, β such that*

$$\max\left(\frac{1}{\|2^n\alpha\|}, \frac{1}{\|2^n\beta\|}\right) > C_n \tag{2.2}$$

holds for every $n = 1, 2, \ldots$.

Proof.   Let $a_1 < a_2 < \ldots$ be a sequence of positive integers such that $a_1 = 1$, and

$$a_{k+1} > 2a_k + 10k + \max\{|C_n|: n \le a_k\} \quad (k = 1, 2, \ldots). \tag{2.3}$$

We put $\alpha = \sum_{i=1}^\infty 2^{-a_{2i}}$ and $\beta = \sum_{i=0}^\infty 2^{-a_{2i+1}}$. Then α, β are irrational, as their dyadic expansions are not eventually periodic. Let n be a positive integer, and suppose that $a_{k-1} \le n < a_k$. If k is even, then

$$\|2^n\beta\| = \sum_{i=k/2}^\infty 2^{n-a_{2i+1}} \le 2^{n-a_{k+1}} \cdot \left(1 + 2^{-9}\right), \tag{2.4}$$

as $a_{k+2} \geq a_{k+1} + 10$. Thus $\|2^n \beta\|^{-1} \geq 2^{a_{k+1}-a_k-1} > 2^{C_n} > C_n$ by the choice of a_{k+1}. If k is odd then we find, in the same way, that $\|2^n \alpha\|^{-1} > C_n$, and thus (2.2) holds in both cases.   □

Proof of Theorem 2.1.   Let α, β be irrational numbers satisfying (2.2) with $C_n = \pi \cdot \sqrt{|A_n|}$. Then by (2.1), we have,

$$f_\alpha(2^n)^{-1} + f_\beta(2^n)^{-1} = \sin^{-2}(\pi \alpha 2^n) + \sin^{-2}(\pi \beta 2^n) \geq (\pi \|2^n \alpha\|)^{-2} + (\pi \|2^n \beta\|)^{-2}$$

$$\geq \frac{1}{\pi^2} \max \left(\|2^n \alpha\|^{-2}, \|2^n \beta\|^{-2} \right) > \frac{1}{\pi^2} C_n^2 = |A_n|$$

for every $n \geq 1$. Since $f_\alpha(2^x)^{-1} + f_\beta(2^x)^{-1} \in E_7$, this completes the proof.   □

The next supplement to Theorem 2.1 will be needed in later applications.

LEMMA 2.3.   *Let $1 = a_1 < a_2 < \ldots$ be a sequence of integers satisfying (2.3), and let α, β be as in the proof of Lemma 2.2. Then the function $f(x) = f_\alpha(2^x)^{-1} + f_\beta(2^x)^{-1}$ has the additional property that $|f(n) - f(m)| \geq 1$ for every $1 \leq n < m$.*

Proof.   We shall prove first that if $a_{k-1} \leq n < a_k$ then

$$0.9 \cdot \pi^{-2} \cdot 2^{2(a_{k+1}-n)} < f(n) < 1.2 \cdot \pi^{-2} \cdot 2^{2(a_{k+1}-n)}. \tag{2.5}$$

Suppose that k is even (the case when k is odd can be treated similarly). Let $\|2^n \beta\| = \theta$; then

$$2^{n-a_{k+1}} < \theta < 1.01 \cdot 2^{n-a_{k+1}}$$

by (2.4). Thus we have

$$f(n) > f_\beta(2^n)^{-1} = \sin^{-2}\left(\pi \beta 2^n\right) \geq \pi^{-2} \theta^{-2} \geq \pi^{-2} \cdot 1.01^{-2} \cdot 2^{2(a_{k+1}-n)},$$

which proves the first inequality of (2.5). Since $\pi \theta < 0.1$ and $\cos x > 0.99$ for every $0 \leq x \leq 0.1$, we have $\sin(\pi \theta) > 0.99 \pi \theta$, and thus

$$f_\beta(2^n)^{-1} = \sin^{-2}\left(\pi \beta 2^n\right)$$

$$= \sin^{-2}(\pi \theta) < 1.1 \cdot \pi^{-2} \theta^{-2} < 1.1 \cdot \pi^{-2} \cdot 2^{2(a_{k+1}-n)}. \tag{2.6}$$

The fractional part of $2^n \alpha$ equals

$$\sum_{i=k/2}^{\infty} 2^{n-a_{2i}}.$$

If $n = a_k - 1$ then this gives $\|2^n \alpha\| > 1/4$ and

$$f_\alpha(2^n)^{-1} = \sin^{-2}\left(\pi \alpha 2^n\right) \leq (2\|2^n \alpha\|)^{-2} < 4 < 0.1 \cdot \pi^{-2} \cdot 2^{2(a_{k+1}-n)}.$$

On the other hand, if $n < a_k - 1$, then $\|2^n \alpha\| > 2^{n-a_k}$, and thus

$$f_\alpha(2^n)^{-1} \leq (2\|2^n \alpha\|)^{-2} < 2^{2(a_k - n)} < 0.1 \cdot \pi^{-2} \cdot 2^{2(a_{k+1} - n)},$$

that is,

$$f_\alpha(2^n)^{-1} < 0.1 \cdot \pi^{-2} \cdot 2^{2(a_{k+1} - n)}$$

holds in both cases. Adding this inequality to (2.6), we obtain the second inequality of (2.5).

Now let $1 \leq n < m$ be arbitrary integers, and suppose $a_{k-1} \leq n < a_k$ and $a_{j-1} \leq m < a_j$. Then the numbers $2(a_{k+1} - n) = N$ and $2(a_{j+1} - m) = M$ are different. Indeed, if $k = j$ then $N > M$, and if $k < j$ then, by $a_{j+1} > 2a_j$ we have

$$a_{j+1} - m > a_{j+1} - a_j > a_j \geq a_{k+1} \geq a_{k+1} - n$$

and thus $M > N$. For example, if $N > M$ then by (2.5) we have

$$\begin{aligned}
f(n) - f(m) &> 0.9 \cdot \pi^{-2} \cdot 2^N - 1.2 \cdot \pi^{-2} \cdot 2^M \\
&\geq 1.8 \cdot \pi^{-2} \cdot 2^M - 1.2 \cdot \pi^{-2} \cdot 2^M \\
&= 0.6 \cdot \pi^{-2} \cdot 2^M > 1,
\end{aligned}$$

as $M \geq 5$. $\quad \Box$

3. Dominating continuous functions with integral-elementary and naive-elementary functions

In this section our aim is to show that every continuous function on $\mathbf{R}$ can be dominated by integral-elementary and naive-elementary functions of bounded rank.

LEMMA 3.1. *We have*

$$\int_{n-1}^{n} \frac{dt}{f_\alpha(t)} \geq \frac{1}{50\|n\alpha\|}$$

whenever α is irrational, $|\alpha| < 1$ and $n \in \mathbf{N}$.

Proof. By (2.1) we have

$$f_\alpha(n) = \sin^2 \pi \alpha n \leq \pi^2 \|n\alpha\|^2,$$

and

$$|f_\alpha'(n)| = |\pi \alpha \sin 2\pi \alpha n| \leq |2\pi \alpha \sin \pi \alpha n| \leq 2\pi^2 \|n\alpha\|. \tag{3.1}$$

Since $|f''_\alpha| \le 4\pi^2$ everywhere, Taylor's formula gives

$$|f_\alpha(n+t)| = \left| f_\alpha(n) + f'_\alpha(n)t + \frac{f''_\alpha(c)}{2}t^2 \right| \le \pi^2 \left(\|n\alpha\|^2 + 2\|n\alpha\||t| + 2t^2 \right)$$

for every t. Consequently, for $0 \le t \le \|n\alpha\|$ we obtain $|f_\alpha(n-t)| \le 5\pi^2\|n\alpha\|^2$. Thus

$$\int_{n-1}^n \frac{dt}{f_\alpha(t)} \ge \int_{n-\|n\alpha\|}^n \frac{dt}{f_\alpha(t)} \ge \frac{\|n\alpha\|}{5\pi^2\|n\alpha\|^2} \ge \frac{1}{50\|n\alpha\|}. \qquad \square$$

THEOREM 3.2. *For every continuous function g: $[1, \infty) \to \mathbf{R}$ there are irrational numbers α, β such that*

$$\int_1^x \left(\frac{1}{f_\alpha(t)} + \frac{1}{f_\beta(t)} \right) dt > g(x) \quad (x \ge 2).$$

Proof. Let α, β be irrationals satisfying (2.2) with

$$C_n = 50 \cdot \max\{|g(x)|: x \in [1, 2^{n+1}]\}.$$

We may assume $\alpha, \beta \in (0, 1)$. Let $x \ge 2$ and $n = [\log x / \log 2]$; then $n \in \mathbf{N}$, and $2^n \le x \le 2^{n+1}$. Putting $F(x) = f_\alpha(x)^{-1} + f_\beta(x)^{-1}$ by Lemma 3.1 we have

$$\int_1^x F(t)\,dt \ge \int_{2^n-1}^{2^n} F(t)\,dt \ge \frac{1}{50\|2^n\alpha\|} + \frac{1}{50\|2^n\beta\|} > \frac{1}{50}C_n \ge g(x). \qquad \square$$

THEOREM 3.3. *For every continuous function h: $\mathbf{R} \to \mathbf{R}$ there is an integral-elementary function f of rank 8 such that f is defined everywhere on $\mathbf{R}$, and $f(x) > h(x)$ for every $x \in \mathbf{R}$.*

Proof. Let h: $\mathbf{R} \to \mathbf{R}$ be continuous, and put

$$g(x) = \max\{|h(t)|: |t| \le |x|\} \quad (x \in \mathbf{R}).$$

Then g is also continuous and hence, by Theorem 3.2, there are irrational numbers, α and β such that $\int_1^x (f_\alpha(t)^{-1} + f_\beta(t)^{-1})\,dt > g(x)$ for every $x \ge 2$. Let

$$f(x) = \int_1^{x^2+2} (f_\alpha(t)^{-1} + f_\beta(t)^{-1})\,dt.$$

Then $f \in IE_8$, and $f(x) > g(x^2 + 2) \ge h(x)$ holds for every x. $\quad \square$

For the sake of completeness we show that the functions $\int_1^x f_\alpha(t)^{-1}dt$ do not dominate every continuous function.

THEOREM 3.4.   *For every irrational α there is a sequence $x_i \to \infty$ such that*

$$\int_1^{x_i} \frac{dt}{f_\alpha(t)} < 40 \cdot x_i \log x_i$$

for every i.

Since this result is independent of the rest of the paper, we shall give the proof in the appendix. Our next aim is to dominate continuous functions with naive-elementary functions. To this end we shall need two auxiliary functions, p and q, defined as follows. Let $p(0) = 0$, $p(1/4) = p(1/2) = 1$, $p(3/4) = p(1) = 0$, let p be linear on the intervals $[(i - 1)/4, i/4]$ $(i = 1, 2, 3, 4)$, and let p be periodic mod 1 on $\mathbf{R}$. Let $q(x) = n$ if $x \in [n, n + (3/4)]$ $(n \in \mathbf{Z})$, and let q be linear on the intervals $[n + (3/4), n + 1]$ $(n \in \mathbf{Z})$. We show that $p, q \in NE$. Since

$$\|x\| = \frac{1}{2\pi} \arcsin(\sin(2\pi x - (\pi/2))) + \frac{1}{4}, \qquad (3.2)$$

the function $\|.\|$ belongs to NE_6. It is easy to check that

$$p(x) = 2\|x\| + 2\|x + (1/4)\| - (1/2) \qquad (3.3)$$

and

$$q(x) = x - \|x\| + \|x + (1/4)\| - (1/2) \cdot \|2x\| - (1/4). \qquad (3.4)$$

Since the functions $2\|x + (1/4)\| - (1/2)$ and $(1/2) \cdot \|2x\| + (1/4)$ are still of rank 6, we have $p \in NE_7$ and $q \in NE_8$.

LEMMA 3.5.   *Let $F: \mathbf{R} \to \mathbf{R}$ be an arbitrary function, and put*

$$F_1(x) = p(x) \cdot F(q(x)) + (1 - p(x)) \cdot F(q(x + (1/2))) \quad (x \in \mathbf{R}). \qquad (3.5)$$

Then $F_1(n) = F(n)$ for every $n \in \mathbf{Z}$, and F_1 is piecewise linear and monotone in each interval $[n, n + 1]$ $(n \in \mathbf{Z})$.

Proof.   Let $\{x\}$ denote the fractional part of the real number x. It is easy to check that

$$F_1(x) = \begin{cases} F([x]) & \text{if } \{x\} \le 1/2; \\ p(x) \cdot F([x]) + (1 - p(x)) \cdot F([x] + 1) & \text{if } 1/2 \le \{x\} \le 3/4; \\ F([x] + 1) & \text{if } \{x\} \ge 3/4. \end{cases} \qquad (3.6)$$

Then the statement of the lemma follows from the fact that p is linear in $[1/2, 3/4]$. $\square$

THEOREM 3.6.   *For every continuous function $h: \mathbf{R} \to \mathbf{R}$ there is a naive-elementary function f of rank 11 such that f is defined everywhere on $\mathbf{R}$, and $f(x) > h(x)$ for every $x \in \mathbf{R}$.*

Proof. Let $A_n = \max\{|h(x)|: |x| \le n + 1\}$ $(n = 1, 2, \ldots)$. By Theorem 2.1, there is a function $f_1 \in E_7$ such that $f_1(n) > A_n$ for every $n = 1, 2, \ldots$. Let $f_2(x) = f_1(x^2 + 1)$; then $f_2 \in E_8$, and $f_2(n) > \max\{|h(x)|: x \in [n - 1, n + 1]\}$ for every $n \in \mathbf{Z}$. Finally, if we put

$$f(x) = p(x) \cdot f_2(q(x)) + (1 - p(x)) \cdot f_2(q(x + (1/2))) \quad (x \in \mathbf{R}),$$

then $f \in NE_{11}$, and it follows from Lemma 3.5 that $f > h$ everywhere on $\mathbf{R}$. $\quad\square$

4. Approximating sequences with elementary functions

In this section we shall prove that every sequence can be approximated with arbitrary precision by an elementary function of bounded rank.

LEMMA 4.1. *Let x_n and c_n $(n = 1, 2, \ldots)$ be two sequences of real numbers such that $\inf_{n \ne m} ||x_n| - |x_m|| > 0$ and $|c_n| \le |x_n|$ for every n. Then there are real numbers γ, M such that the function $f(x) = x \cdot \sin(\gamma \cdot e^{M \cdot x^2})$ satisfies $|f(x_n) - c_n| \le 1$ for every $n = 1, 2, \ldots$.*

Proof. We may assume $x_n \ne 0$ for every n. After rearranging the sequences we may also suppose that $|x_1| < |x_2| < \ldots$. Let $\inf_n(|x_{n+1}| - |x_n|) = \delta > 0$, and put $M = 10/\delta$. Let

$$I_n = \left[\frac{c_n}{x_n} - \frac{1}{|x_n|}, \frac{c_n}{x_n} + \frac{1}{|x_n|} \right] \cap [-1, 1];$$

then $|I_n| \ge 1/|x_n|$ for every n. We have to prove that the intersection of the sets

$$E_n = \left\{ \gamma \in \mathbf{R}: \sin\left(\gamma \cdot e^{M \cdot x_n^2} \right) \in I_n \right\} \quad (n = 1, 2, \ldots)$$

is nonempty. Since $|\sin x - \sin y| \le |x - y|$ for every x, y, there is a closed interval $J_n \subset [-\pi/2, \pi/2]$ for every n, such that $|J_n| \ge 1/|x_n|$ and $\sin(J_n) \subset I_n$. The set E_n is periodic mod $p_n = 2\pi e^{-M \cdot x_n^2}$, and contains the interval $K_n = \{y \cdot e^{-M \cdot x_n^2}: y \in J_n\}$. Therefore, $K_n + p_n k \subset E_n$ for every $k \in \mathbf{Z}$. To complete the proof, it is enough to find integers $k_1, k_2, \ldots$ such that

$$K_n + p_n k_n \supset K_{n+1} + p_{n+1} k_{n+1} \tag{4.1}$$

for every n. Indeed, in this case the intersection $\bigcap_{n=1}^{\infty}(K_n + p_n k_n)$ is nonempty, and then so is $\bigcap_{n=1}^{\infty} E_n$. Let $k_1 = 0$ and suppose that k_n has been selected. Now

$$p_{n+1} = 2\pi \cdot e^{-M \cdot x_{n+1}^2} \le 2\pi \cdot |J_n| \cdot |x_n| \cdot e^{-M \cdot (|x_n| + \delta)^2}$$

$$< 2\pi \cdot |J_n| \cdot e^{-M x_n^2} \cdot |x_n| \cdot e^{-2M\delta \cdot |x_n|} = 2\pi \cdot |K_n| \cdot |x_n| \cdot e^{-20|x_n|} < |K_n|/2,$$

and thus the interval $K_n + p_n k_n$ is longer than $2p_{n+1}$, twice the period of E_{n+1}. Since $|K_{n+1}| \le p_{n+1}$, it follows that K_{n+1} can be translated by an integer multiple of p_{n+1} such that the translated copy is covered by $K_n + p_n k_n$; that is, (4.1) holds. $\quad\square$

THEOREM 4.2.   *If a_n $(n = 1, 2, \ldots)$ are arbitrary real numbers and ε_n $(n = 1, 2, \ldots)$ are arbitrary positive numbers, then there is an elementary function f of rank 9 such that f is defined everywhere on $\mathbf{R}$ and $|f(n) - a_n| < \varepsilon_n$ for every $n \in \mathbf{N}$.*

Proof.   By Theorem 2.1 we can choose a function $w \in E_7$ such that $w > 0$ everywhere and $w(n) > 1/\varepsilon_n$ for every n. Applying Theorem 2.1 and Lemma 2.3, we find a function $v \in E_7$ with the following properties: v is defined everywhere, $v(n) > |a_n| \cdot w(n)$ for every $n \in \mathbf{N}$, and $|v(n) - v(m)| \geq 1$ for every $n \neq m$. Applying Lemma 4.1 with $x_n = v(n)$ and $c_n = a_n \cdot w(n)$ we obtain a function $g \in E_6$ such that $|g(v(n)) - a_n \cdot w(n)| \leq 1$ for every $n = 1, 2, \ldots$. Then the function $f = (g \circ v)/w$ belongs to E_9, and satisfies

$$|f(n) - a_n| = |g(v(n)) - a_n \cdot w(n)|/w(n) \leq 1/w(n) < \varepsilon_n$$

for every n.   □

COROLLARY 4.3.   *Let a_n $(n \in \mathbf{N})$ be an arbitrary sequence of integers. Then there is an $f \in NE_{10}$ such that $f(n) = a_n$ for every $n \in \mathbf{N}$.*

Proof.   Let $g \in E_9$ be such that $a_n < g(n) < a_n + (3/4)$ for every n. Then the function $f = q \circ g$ satisfies the requirements (compare (3.4)).   □

Since the function q does not belong to IE, the following question remains open.

PROBLEM 4.4.   *Let a_n $(n \in \mathbf{N})$ be an arbitrary sequence of integers. Does there exist an integral-elementary (or even an elementary) function f such that $f(n) = a_n$ for every n?*

In light of the statements of Theorem 4.2 and Corollary 4.3 it is natural to ask whether or not every sequence is actually equal to the sequence $f(n)$ with a suitable naive-elementary function f. In the next theorem we show that the answer is negative. Let

$$[0, 1]^{\mathbf{N}} = \{(x_1, x_2, \ldots) \colon x_i \in [0, 1] \ (i = 1, 2, \ldots)\}$$

denote the set of sequences in $[0, 1]$, and let μ denote the product measure on $[0, 1]^{\mathbf{N}}$, where each component is endowed with the Lebesgue measure λ. We shall prove that the set

$$\{(f(n))_{n=1}^{\infty} \in \mathbf{R}^{\mathbf{N}} \colon f \in NE \cup IE\} \cap [0, 1]^{\mathbf{N}}$$

is of μ-measure zero.

Let f be a real valued function defined on a set $H \subset \mathbf{R}^n$. We say that f is Lipschitz α, if there is a constant K such that $|f(y) - f(x)| \leq K|y - x|^{\alpha}$ for every $x, y \in H$. We say that f is locally Lipschitz α, if every $x \in H$ has a neighbourhood U such that

the restriction of f to $H \cap U$ is Lipschitz α. Finally, we say that f is locally Lipschitz if there is $\alpha > 0$ such that f is locally Lipschitz α. Clearly, every C^1 function is locally Lipschitz. Since

$$| \arcsin x - \arcsin y | \leq \frac{\pi}{\sqrt{2}} |x - y|^{1/2} \tag{4.2}$$

for every $x, y \in [-1, 1]$, the function $\arcsin x$ is Lipschitz $1/2$ on $[-1, 1]$.

Let $\mathcal{F}$ be a family of functions defined on subintervals of $\mathbf{R}$. We shall denote by $IE(\mathcal{F})$ the smallest class satisfying the following conditions: (i) $IE(\mathcal{F})$ contains $\mathcal{F}$ and also the constant functions; (ii) if f is a function defined on an interval I, and if there are functions $g, h \in IE(\mathcal{F})$ such that f equals the restriction of one of the functions $g + h$, $g \cdot h$, g/h, $g \circ h$, to the interval I, then $f \in IE(\mathcal{F})$; and finally, (iii) if $g \in IE(\mathcal{F})$ is defined on the interval I and if $a \in I$, then the function f defined by $f(x) = \int_a^x g(t)\, dt$ $(x \in I)$ also belongs to $IE(\mathcal{F})$.

THEOREM 4.5. *Let $\mathcal{F}$ be a countable family of locally Lipschitz functions defined on subintervals of $\mathbf{R}$. Then the set*

$$S_{\mathcal{F}} = \{(f(n))_{n=1}^{\infty} : f \in IE(\mathcal{F})\} \cap [0, 1]^{\mathbf{N}}$$

is of μ-measure zero.

The proof is given in the appendix. The local Lipschitz property cannot be replaced by continuity in the previous theorem. Moreover, there is a single continuous function $F \colon \mathbf{R} \to \mathbf{R}$ such that

$$[0, 1]^{\mathbf{N}} \subset \{(f(n))_{n=1}^{\infty} : f \in IE(\{F\})\}.$$

Indeed, since the Hilbert cube $[0, 1]^{\mathbf{N}}$ is the continuous image of $[0, 1]$, there are continuous functions $\phi_n \colon [0, 1] \to [0, 1]$ such that

$$[0, 1]^{\mathbf{N}} = \{(\phi_n(t))_{n=1}^{\infty} : t \in [0, 1]\}.$$

Let $F(2n + x) = \phi_n(x)$ for every $n \in \mathbf{N}$ and $x \in [0, 1]$, and let F be extended to $\mathbf{R}$ as a continuous function. If $a_n \in [0, 1]^{\mathbf{N}}$ and α is such that $a_n = \phi_n(\alpha)$ for every n, then we have $f(n) = a_n$ $(n \in \mathbf{N})$, where $f(x) = F(2x + \alpha) \in IE(\{F\})$.

In this example the set $\{(f(n))_{n=1}^{\infty} : f \in IE(\{F\})\}$ will actually contain every sequence. In fact, every sequence (a_n) can be written in the form $(b_n^{-1} - c_n^{-1})$, where $b_n, c_n \in (0, 1)$, and thus $a_n = f(n)$, where

$$f(x) = \frac{1}{F(2x + \beta)} - \frac{1}{F(2x + \gamma)}$$

with suitable $\beta, \gamma \in [0, 1]$.

We also remark that analogous results concerning the set of sequences represented by differentially algebraic functions were proved in [4].

5. Approximation of functions by elementary functions

In this section our aim is to prove that differentiable functions with not too large derivatives can be uniformly approximated by elementary functions. Recall that $e_k(x)$ denotes the iterated exponential function; that is $e_1(x) = e^x$ and $e_{k+1}(x) = e^{e_k(x)}$ ($k = 1, 2, \ldots$).

THEOREM 5.1. *Let $g: \mathbf{R} \to \mathbf{R}$ be a differentiable function, and suppose that $|g'(x)| \le e_k(|x|)$ for every $x \in \mathbf{R}$ with a suitable positive integer k. Then for every $n \in \mathbf{N}$ there is an elementary function $f: \mathbf{R} \to \mathbf{R}$ such that $|f(x) - g(x)| < 1/e_n(|x|)$ everywhere on $\mathbf{R}$.*

LEMMA 5.2. *Let $g: \mathbf{R} \to \mathbf{R}$ be a continuous function such that $|g(y) - g(x)| \le 1$ whenever $|y - x| \le 1$. Then there is $f \in E_{17}$ such that $|f(x) - g(x)| < 7$ for every $x \in \mathbf{R}$.*

Proof. First we suppose that $g(0) = 0$, and construct a function $f \in E_{16}$ such that $|f - g| < 7$. Then the statement of the general case will follow by considering the function $g - g(0)$ instead of g, and by adding the constant $g(0)$ to f.

Since $g(0) = 0$, the assumption on g implies that $|g(n)| \le |n|$ for every $n \in \mathbf{Z}$. Then we can choose real numbers c_n such that $|c_n - g(n)| \le 1/3$, and $|c_n| \le |n + (1/3)|$ ($n \in \mathbf{Z}$). Since $||n + (1/3)| - |m + (1/3)|| \ge 1/3$ for every $n, m \in \mathbf{Z}$, $n \ne m$, we may apply Lemma 4.1, and find real numbers γ and M such that the function $f_1(x) = x \cdot \sin(\gamma \cdot e^{M \cdot x^2})$ satisfies $|f_1(n + (1/3)) - c_n| \le 1$ for every $n \in \mathbf{Z}$. Let $F(x) = f_1(x + (1/3))$, then $F \in E_7$ and

$$|F(n) - g(n)| \le |f_1(n + (1/3)) - c_n| + |c_n - g(n)| < 2$$

for every $n \in \mathbf{Z}$. Let F_1 be defined by (3.5). If $n \le x < n + 1$, then we have $|F(n) - g(n)| < 2$, $|F(n+1) - g(n+1)| < 2$, $|g(x) - g(n)| \le 1$ and $|g(x) - g(n+1)| \le 1$, and hence $|F(n) - g(x)| < 3$ and $|F(n+1) - g(x)| < 3$. Thus, by Lemma 3.5, we have $|F_1(x) - g(x)| < 3$.

In the sequel we shall write $\exp(x)$ for e^x. We put $K = 8(|M\gamma| + |M| + 1)$, $\varepsilon = \exp(-5K)$ and $\delta(x) = \varepsilon \cdot \exp(-2Kx^2)$. Let

$$R(x) = \frac{1}{2\pi} \arcsin\left((1 - \delta(x)^2) \sin(2\pi x - (\pi/2))\right) + \frac{1}{4};$$

then it follows from (3.2) and (4.2) that

$$|R(x) - \|x\|| < \delta(x) \tag{5.1}$$

for every $x \in \mathbf{R}$. Also, $R \in E_9$, and hence the functions

$$p_1(x) = 2 \cdot R(x) + 2 \cdot R(x + (1/4)) - (1/2)$$

and

$$q_1(x) = x - R(x) + R(x + (1/4)) - (1/2) \cdot R(2x) - (1/4)$$

belong to E_{13}. Now we define

$$f(x) = p_1(x) \cdot F(q_1(x)) + (1 - p_1(x)) \cdot F(q_1(x + (1/2)));$$

then $f \in E_{16}$. We shall prove that $|f - F_1| < 4$. Since $|F_1 - g| < 3$, this will finish the proof. Let $Q = q(x + (1/2))$ and $Q_1 = q_1(x + (1/2))$. Then, by (3.5) we have

$$
\begin{aligned}
|f - F_1| &\leq |p \cdot F(q) - p_1 \cdot F(q_1)| + |(1 - p) \cdot F(Q) - (1 - p_1) \cdot F(Q_1)| \\
&\leq p \cdot |F(q) - F(q_1)| + |p_1 - p| \cdot |F(q_1)| + (1 - p) \cdot |F(Q) - F(Q_1)| \\
&\quad + |p_1 - p| \cdot |F(Q_1)| \\
&\overset{\text{def}}{=} A + B + C + D.
\end{aligned}
$$

We shall prove that each of A, B, C, D is less than 1. Since $\exp(K|x|) \leq \exp(K) \cdot \exp(Kx^2)$ for every x, we have

$$
\begin{aligned}
\delta(x + (1/4)) &= \varepsilon \cdot \exp(-2K(x + (1/4))^2) < \varepsilon \cdot \exp(-2Kx^2) \cdot \exp(K|x|) \\
&< \varepsilon \cdot \exp(-2Kx^2) \cdot \exp(K) \cdot \exp(Kx^2) = \exp(-4K) \cdot \exp(-Kx^2).
\end{aligned}
$$

By (3.3), (3.4) and (5.1) this implies

$$
\begin{aligned}
|p(x) - p_1(x)| &\leq 2 \cdot \delta(x) + 2 \cdot \delta(x + (1/4)) < 4 \cdot \exp(-4K) \cdot \exp(-Kx^2) \\
&< \exp(-3) \cdot \exp(-x^2) < \frac{1}{|x| + 3},
\end{aligned} \tag{5.2}
$$

and

$$
\begin{aligned}
|q(x) - q_1(x)| &\leq \delta(x) + \delta(x + (1/4)) + \delta(2x) < 2 \cdot \delta(x) + \delta(x + (1/4)) \\
&< 4 \cdot \exp(-4K) \cdot \exp(-Kx^2) \\
&< \frac{1}{2} \cdot \exp(-K) \cdot \exp(-Kx^2) < 1.
\end{aligned} \tag{5.3}
$$

Since $x - 1 \leq q(x) \leq x$, we have $x - 2 \leq q_1(x) \leq x + 1$. Thus

$$|F(q_1(x))| = |f_1(q_1(x) + (1/3))| \leq |q_1(x) + (1/3)| \leq |x| + 2,$$

and $|F(Q_1)| \leq |x + (1/2)| + 2 < |x| + 3$. By (5.2), this gives $B < 1$ and $D < 1$. Since $0 \leq p \leq 1$, in order to prove $A < 1$ and $C < 1$, it is enough to show that $|F(q_1) - F(q)| < 1$. We have

$$F(q_1(x)) - F(q(x)) = F'(c)(q_1(x) - q(x)) = f_1'(c + (1/3))(q_1(x) - q(x)), \tag{5.4}$$

where $c \in (x - 2, x + 1)$. If $d = c + (1/3)$ then $d \in (x - 2, x + 2)$, and thus

$$|f_1'(d)| \leq 1 + |d| \cdot |\gamma| \cdot \exp(Md^2) \cdot 2|Md|$$
$$< 1 + 2|M\gamma|(|x| + 2)^2 \cdot \exp(M(|x| + 2)^2)$$
$$< 2 \cdot \exp(K) \cdot \exp(Kx^2).$$

Thus (5.3) and (5.4) yield $|F(q_1) - F(q)| < 1$, which completes the proof. $\square$

Proof of Theorem 5.1. Let D_i denote the family of all differentiable functions $h: \mathbf{R} \to \mathbf{R}$ satisfying $|h'(x)| \leq e_i(|x|)$ for every $x \in \mathbf{R}$, and put $D = \cup_{i=1}^{\infty} D_i$. If $h \in D_i$ then $|h(x)| \leq |h(0)| + e_i(|x|)$ for every x, as $e_i' \geq e_i$ on $[0, \infty)$. Since $e_i(|x|)^2 \leq e_{i+1}(|x|)$, this implies that $h_1 \cdot h_2 \in D$ and $h_1 \circ h_2 \in D$ for every $h_1, h_2 \in D$.

Let $c_1(x) = e^x + e^{-x}$ and $c_{i+1}(x) = c_1(c_i(x))$ for every $x \in \mathbf{R}$ and $i = 1, 2, \ldots$. It is easy to prove by induction that $c_i \in D$ and $c_i(x) > e_i(|x|)$ ($x \in \mathbf{R}$) for every i. Also, the functions c_i are even and satisfy $x \cdot c_i'(x) \geq 0$ for all $x \in \mathbf{R}$. Let $v_i(x) = x \cdot c_i(x)$. Then $v_i \in E \cap D$ and $v_i'(x) \geq c_i(x)$ everywhere.

Let $k, n \in \mathbf{N}$ and $g: \mathbf{R} \to \mathbf{R}$ be as in the theorem, and put $g_1 = 7 \cdot g \cdot c_n$. Since $g \in D$ by assumption, it follows that $g_1 \in D_i$ for a suitable i. Then $v_i'(x) > e_i(|x|) \geq |g_1'(x)|$ everywhere, and thus $|v_i(x) - v_i(y)| \geq |g_1(x) - g_1(y)|$ for every x, y. Let $g_2 = g_1 \circ v_i^{-1}$, then $|g_2(x) - g_2(y)| \leq |x - y|$ for every x, y. By Lemma 5.2, there is $u \in E$ such that $|u - g_2| < 7$. Let $f = (u \circ v_i)/(7c_n)$. Then $f \in E$, and

$$|f - g| = \left| \frac{u \circ v_i}{7c_n} - \frac{g_1}{7c_n} \right| = \frac{|u \circ v_i - g_2 \circ v_i|}{7c_n} < \frac{1}{c_n} < \frac{1}{e_n(|x|)}. \qquad \square$$

6. Approximation of functions by integral-elementary and naive-elementary functions

In this section we shall prove that every continuous function can be approximated with an arbitrarily small error by integral-elementary and naive-elementary functions of bounded rank.

THEOREM 6.1. *For every pair of continuous functions $g: \mathbf{R} \to \mathbf{R}$ and $\varepsilon: \mathbf{R} \to (0, \infty)$ there is a function $f \in IE_{19}$ defined everywhere on $\mathbf{R}$ such that $|f(x) - g(x)| < \varepsilon(x)$ for every $x \in \mathbf{R}$.*

Proof. By Theorem 3.3, there is a function $w \in IE_8$ such that $w > 7/\varepsilon$ everywhere. Let $g_1 = g \cdot w$. Since g_1 is continuous on $\mathbf{R}$, there is a positive number $\delta_n < 1$ for every $n \in \mathbf{Z}$ such that if $x, y \in [n - 2, n]$ and $|y - x| \leq \delta_n$, then $|g_1(y) - g_1(x)| < 1$. Let h be a continuous function such that $h(x) > 1/\delta_n$ for every $x \in [n - 2, n]$ and $n \in \mathbf{Z}$. Applying Theorem 3.3 again we obtain a function $v_1 \in IE_8$ such that $v_1 > h$. Let $v(x) = \int_0^x v_1(t) \, dt$. Then $v \in IE_9$, v is a strictly increasing homeomorphism of $\mathbf{R}$ onto itself, and $v'(x) > h(x) > 1$ everywhere.

We show that if $x, y \in \mathbf{R}$ and $|v(y) - v(x)| \le 1$ then $|g_1(y) - g_1(x)| < 1$. Let $x, y \in \mathbf{R}$ be fixed such that $|v(y) - v(x)| \le 1$. Then we have $|y - x| \le |v(y) - v(x)| \le 1$, and thus there is $n \in \mathbf{Z}$ such that $x, y \in [n - 2, n]$. Then

$$|y - x|/\delta_n \le \left| \int_x^y h(t)\, dt \right| \le |v(y) - v(x)| \le 1,$$

from which we obtain $|y - x| \le \delta_n$ and $|g_1(y) - g_1(x)| < 1$.

Let $g_2 = g_1 \circ v^{-1}$. Then g_2 is continuous on $\mathbf{R}$ and has the property that $|g_2(y) - g_2(x)| < 1$ holds whenever $|y - x| \le 1$. By Lemma 5.2, there is $u \in E_{17}$ such that $|u - g_2| < 7$ everywhere on $\mathbf{R}$. Let $f = (u \circ v)/w$. Then we have $f \in IE_{19}$, and

$$|f - g| = \left| \frac{u \circ v}{w} - \frac{g_1}{w} \right| = \frac{|u \circ v - g_2 \circ v|}{w} < \frac{7}{w} < \varepsilon. \qquad \square$$

THEOREM 6.2.   *For every pair of continuous functions* $g \colon \mathbf{R} \to \mathbf{R}$ *and* $\varepsilon \colon \mathbf{R} \to (0, \infty)$ *there is a function* $f \in NE_{19}$ *defined everywhere on* $\mathbf{R}$ *such that* $|f(x) - g(x)| < \varepsilon(x)$ *for every* $x \in \mathbf{R}$.

Proof.   We shall repeat the previous proof with minor modifications. By Theorem 3.6, there is a function $w \in NE_{11}$ such that $w > 7/\varepsilon$ everywhere. Then let g_1, δ_n, and h be as in the previous proof. We shall construct a function $v \in NE_{15}$ such that v is a strictly increasing homeomorphism of $\mathbf{R}$ onto itself, v is piecewise linear, and $v'(x) > h(x) > 1$ at every point x where v' exists. This will conclude the proof, since defining $g_2 = g_1 \circ v^{-1}$, u and $f = (u \circ v)/w$ in the same way as in the previous proof, it follows that $f \in NE_{19}$ and $|f - g| < \varepsilon$.

We can choose real numbers a_k ($k \in \mathbf{Z}$) such that

$$a_{k+1} - a_k > 3 + \max\{h(x) \colon x \in [k, k+1]\}$$

for every $k \in \mathbf{Z}$. Let $b_n = a_k$ if $k \in \mathbf{Z}$ and $n = 2k^2 + k + 1$, and let $b_n = 0$ if there is no $k \in \mathbf{Z}$ such that $n = 2k^2 + k + 1$. Since the function $2x^2 + x + 1$ maps $\mathbf{Z}$ injectively into $\mathbf{N}$, the definition of the sequence b_n makes sense. By Theorem 4.2, there exists a function $\phi \in E_9$ such that ϕ is defined everywhere on $\mathbf{R}$ and $|\phi(n) - b_n| < 1$ for every n. Putting $F(x) = \phi(2x^2 + x + 1)$, we have $F \in E_{10}$, and $|F(k) - a_k| < 1$ for every $k \in \mathbf{Z}$. Let F_1 be defined by (3.5), then $F_1 \in NE_{13}$ and, by Lemma 3.5, F_1 is increasing and is piecewise linear. Since $F_1(k + 1) - F_1(k) = F(k + 1) - F(k) > a_{k+1} - a_k - 2 > 1$, we have

$$\lim_{x \to \infty} F_1(x) = \infty \quad \text{and} \quad \lim_{x \to -\infty} F_1(x) = -\infty.$$

Also, it follows from (3.6) that F_1 is linear on $[k + (1/2), k + (3/4)]$, and

$$F_1'(x) > \frac{a_{k+1} - a_k - 2}{(1/4)}$$

in the interior of this interval. By the choice of the numbers a_k this implies that

$$F'_1(x) > \max\{h(x): x \in [k, k+1]\}$$

for every $x \in (k + (1/2), k + (3/4))$. This easily implies that the function

$$v(x) = F_1(x + (1/2)) + F_1(x + (1/4)) + F_1(x) + F_1(x - (1/4))$$

satisfies the requirements. $\quad\square$

As we remarked in the introduction, Theorem 6.1 implies the following.

THEOREM 6.3. *There is a nontrivial algebraic differential equation with integer coefficients, $P = 0$, with the following property: for every pair of continuous functions $g: \mathbf{R} \to \mathbf{R}$ and $\varepsilon: \mathbf{R} \to (0, \infty)$ there is a solution f of $P = 0$ such that f is everywhere analytic on $\mathbf{R}$, and $|f(x) - g(x)| < \varepsilon(x)$ for every $x \in \mathbf{R}$.*

If an integral-elementary or naive-elementary function f is defined on an interval I, then f must be analytic on a subinterval of I. Consequently, the classes IE and NE do not contain all continuous functions. Thus, we may ask whether or not all continuous functions can be obtained by starting from a suitable finite or countable collection of continuous functions. We close this section by showing that the answer is negative.

THEOREM 6.4. *Let $\mathcal{F}$ be a countable family of continuous functions defined on subintervals of $\mathbf{R}$. Then for every compact and infinite set $K \subset \mathbf{R}$ there is a continuous function $f: K \to \mathbf{R}$ such that $f \notin \{g|K: g \in IE(\mathcal{F})\}$.*

Proof. Let f be a continuous function defined on the compact set $A \subset \mathbf{R}$. By the modulus of continuity of f we mean the function $\omega_f: \mathbf{R}^+ \to \mathbf{R}^+$ defined by

$$\omega_f(\delta) = \max\{|f(y) - f(x)|: x, y \in A, \ |y - x| \le \delta\}.$$

Clearly, we have $\lim_{\delta \to 0+} \omega_f(\delta) = 0$. We shall prove that for every $k = 0, 1, \ldots$ there is a countable family Ω_k of functions from $\mathbf{R}^+$ to $\mathbf{R}^+$ such that

(i) $\lim_{\delta \to 0+} \tau(\delta) = 0$ for every $\tau \in \Omega_k$; and
(ii) if $f \in IE(\mathcal{F})$ is of rank k, $A \subset \mathbf{R}$ is compact, and f is defined on A, then there is a function $\tau \in \Omega_k$ such that

$$\omega_{f|A}(\delta) \le \tau(\delta) \quad (\delta > 0).$$

We shall prove this statement by induction on k. Let $f \in \mathcal{F}$ be fixed, and suppose that f is defined on the interval I. Then there are compact sets $A_i \subset I$ such that

$I = \bigcup_{i=1}^{\infty} A_i$, and whenever $A \subset I$ is compact, then $A \subset A_i$ holds for some i. Let

$$\Omega^f = \{\omega_{f|A_i} : i = 1, 2, \ldots\}.$$

Clearly, if f is defined on a compact set A, then $\omega_{f|A}(\delta) \leq \tau(\delta)$ holds for at least one $\tau \in \Omega^f$. Let

$$\Omega_0 = \bigcup_{f \in \mathcal{F}} \Omega^f.$$

Then Ω_0 satisfies (i) and (ii) for $k = 0$, since the functions of rank zero are the elements of $\mathcal{F}$ and the constants, and the modulus of continuity of a constant function is identically zero.

Let $k \geq 0$, and suppose that the countable family Ω_k has been constructed so that it satisfies (i) and (ii). We put

$$\Omega_{k+1} = \{N \cdot (\tau_1 + \tau_2) : \tau_1, \tau_2 \in \Omega_k, \ N \in \mathbf{N}\} \bigcup \{\tau_1 \circ \tau_2 : \tau_1, \tau_2 \in \Omega_k\} \bigcup \{N \cdot x : N \in \mathbf{N}\}.$$

Clearly, Ω_{k+1} is countable and satisfies (i) (with $k + 1$ instead of k). To prove (ii), let $f \in IE(\mathcal{F})$ be an arbitrary function of rank $k + 1$, and let A be a compact subset of the domain of f. Then there are functions g, h of rank k such that one of the following statements is true:

- $f = g + h$. In this case g, h must be defined on A. Let $\omega_{g|A} \leq \tau_1$ and $\tau_{h|A} \leq \tau_2$, where $\tau_1, \tau_2 \in \Omega_k$. Then $\omega_{f|A} \leq \tau_1 + \tau_2 \in \Omega_{k+1}$.
- $f = g \cdot h$. Again, g, h must be defined on A. Let $\omega_{g|A} \leq \tau_1$ and $\omega_{h|A} \leq \tau_2$, where $\tau_1, \tau_2 \in \Omega_k$. If $|g(x)|, |h(x)| \leq N$ for every $x \in A$ then $\omega_{f|A} \leq N(\tau_1 + \tau_2) \in \Omega_{k+1}$.
- $f = g/h$. This is similar to the previous case, taking into consideration that h must be nonzero on A and thus $|h(x)| > 1/N$ for every $x \in A$ with a suitable $N \in \mathbf{N}$.
- $f(x) = \int_a^x g(t)\,dt$ with an $a \in \mathbf{R}$. In this case g must be defined on a closed interval J containing A. If $|g| \leq N$ on J, then we have

$$\omega_{f|A}(\delta) \leq \omega_{f|J}(\delta) \leq N\delta.$$

Since $N \cdot x \in \Omega_{k+1}$, this completes the proof of (ii).

Let $\tau_1, \tau_2, \ldots$ be an enumeration of the elements of $\cup_{k=0}^{\infty} \Omega_k$. Then for every $f \in IE(\mathcal{F})$ and for every compact subset A of the domain of f there is an i such that $\omega_{f|A} \leq \tau_i$. Let $\delta_i > 0$ be such that $\tau_i(\delta_i) < 1/i$ ($i = 1, 2, \ldots$).

Let K be a compact infinite subset of $\mathbf{R}$, and let x be a point of accumulation of K. Then we can select a sequence x_i of distinct points of K such that $\lim_{i \to \infty} x_i = x$ and $0 < |x - x_i| < \delta_i$ for every i. Let $f(x_i) = 1/i$ ($i = 1, 2, \ldots$), and extend f continuously to K. Then $\omega_f(\delta_i) \geq 1/i > \tau_i(\delta_i)$ for every i and, consequently, f cannot be the restriction of any of the elements of $IE(\mathcal{F})$ to K. $\quad\square$

7. Appendix

Proof of Theorem 3.4.   We shall prove that for every irrational number $0 < \alpha < 1$ there is a sequence $x_i \to \infty$ such that

$$\int_1^{x_i} \frac{dt}{f_\alpha(t)} < 36 \cdot x_i \log x_i$$

for every i. Then the case of $\alpha > 1$ follows easily by using $f_{1/\alpha}(x) = f_\alpha(x/\alpha)$, and by making the substitution $u = t/\alpha$.

First we show that

$$\int_{n-(1/2)}^{n+(1/2)} \frac{dt}{f_\alpha(t)} \leq \frac{6}{\|n\alpha\|} \tag{7.1}$$

for every $n \in \mathbf{N}$. By (2.1) we have $f_\alpha(n) = \sin^2 \pi \alpha n \geq 4\|n\alpha\|^2$. Then, by (3.1) and $|f''_\alpha| \leq 4\pi^2$ we obtain

$$|f_\alpha(n+t)| = \left| f_\alpha(n) + f'_\alpha(n)t + \frac{f''_\alpha(c)}{2}t^2 \right| \geq 4\|n\alpha\|^2 - 2\pi^2\|n\alpha\||t| - 2\pi^2 t^2$$

for every t. For $|t| \leq \|n\alpha\|/10$ this gives

$$|f_\alpha(n+t)| \geq 4\|n\alpha\|^2 - 2\|n\alpha\|^2 - \|n\alpha\|^2 = \|n\alpha\|^2,$$

and thus, putting $I_1 = [n - \|n\alpha\|/10, n + \|n\alpha\|/10]$, we have

$$\int_{I_1} \frac{dt}{f_\alpha(t)} \leq \frac{|I_1|}{\|n\alpha\|^2} < \frac{1}{\|n\alpha\|}.$$

If $\|n\alpha\|/10 \leq t \leq 1/2$ then

$$f_\alpha(n+t) \geq \sin^2 \pi(n+t) = \sin^2 \pi t \geq 4t^2.$$

Therefore, denoting $I_2 = [n + \|n\alpha\|/10, n + (1/2)]$ we obtain

$$\int_{I_2} \frac{dt}{f_\alpha(t)} \leq \int_{\|n\alpha\|/10}^{1/2} \frac{dt}{4t^2} < \frac{2.5}{\|n\alpha\|}.$$

Similarly, $\int_{I_3} f_\alpha(t)^{-1} dt < 3/\|n\alpha\|$, where $I_3 = [n - (1/2), n - \|n\alpha\|/10]$. Since $[n - (1/2), n + (1/2)] = I_1 \cup I_2 \cup I_3$, (7.1) follows. If x is a positive integer then (7.1) gives

$$\int_1^x \frac{dt}{f_\alpha(t)} < 6 \sum_{n=1}^x \frac{1}{\|n\alpha\|}. \tag{7.2}$$

Let p_i/q_i denote the convergents of the continued fraction expansion of α. It is well known that $q_i \geq F_i$ holds for every i, where F_i is the sequence of Fibonacci-numbers.

Since $F_{i+1}/F_i \rightarrow (\sqrt{5}+1)/2$, this implies that $q_{i+1}/q_i > 1.6$ for infinitely many indices i. We shall prove that if $x = q_i - 1$, where $q_{i+1}/q_i > 1.6$ and i is large enough, then $\sum_{n=1}^{x} \|n\alpha\|^{-1} < 6x \log x$. By (7.2), this will finish the proof.

Since $|\alpha - (p_i/q_i)| < 1/(q_i q_{i+1})$, we have $|n\alpha - (np_i/q_i)| < n/(q_i q_{i+1}) < 1/q_{i+1}$ for every $n < q_i$. If $np_i \equiv j \pmod{q_i}$ where $0 < j < q_i$, then this implies

$$\|n\alpha\| \geq \min\left(\frac{j}{q_i}, 1 - \frac{j}{q_i}\right) - \frac{1}{q_{i+1}} > \min\left(\frac{j}{q_i}, 1 - \frac{j}{q_i}\right) - \frac{1}{1.6 \cdot q_i}$$

$$\geq \frac{3}{8} \cdot \min\left(\frac{j}{q_i}, 1 - \frac{j}{q_i}\right).$$

If n runs through the numbers $1, \ldots, q_i - 1$ then so does $np_i \pmod{q_i}$ and hence

$$\sum_{n=1}^{x} \frac{1}{\|n\alpha\|} < \frac{8}{3} \cdot 2 \cdot \sum_{j=1}^{[x/2]} \frac{x+1}{j} < 6x \log x$$

if i is large enough.   □

Proof of Theorem 4.5.   Let $\mathcal{F}$ be a countable family of locally Lipschitz functions defined on subintervals of **R**. Let $\mathcal{P}$ denote the smallest family of functions (of arbitrary many variables) satisfying the following conditions.

(i) $\mathcal{P} \supset \mathcal{F}$.

(ii) $\mathcal{P}$ contains the function $(y, x) \mapsto y$ $(y, x \in \mathbf{R})$.

(iii) If the functions $f: H \rightarrow \mathbf{R}$ $(H \subset \mathbf{R}^{n+1})$, $g: K \rightarrow \mathbf{R}$ $(K \subset \mathbf{R}^{k+1})$ are in $\mathcal{P}$ then so are the following functions (defined on the largest set where their definition makes sense):
  $(y, z, x) \mapsto f(y, x) + g(z, x)$ $(y \in \mathbf{R}^n, z \in \mathbf{R}^k, x \in \mathbf{R})$,
  $(y, z, x) \mapsto f(y, x) \cdot g(z, x)$ $(y \in \mathbf{R}^n, z \in \mathbf{R}^k, x \in \mathbf{R})$,
  $(y, z, x) \mapsto f(y, x)/g(z, x)$ $(y \in \mathbf{R}^n, z \in \mathbf{R}^k, x \in \mathbf{R})$,
  $(y, z, x) \mapsto f(y, g(z, x))$ $(y \in \mathbf{R}^n, z \in \mathbf{R}^k, x \in \mathbf{R})$.

(iv) If the function $f: H \rightarrow \mathbf{R}$ $(H \subset \mathbf{R}^{n+1})$ belongs to $\mathcal{P}$ then so does the function $h: K \rightarrow \mathbf{R}$, where $K = \{(y, z, x): y \in \mathbf{R}^n, z \in \mathbf{R}, x \in \mathbf{R}, \{y\} \times [z, x] \subset H\}$ and $h(y, z, x) = \int_z^x f(y, t) \, dt$.

Clearly, $\mathcal{P}$ is countable. It is easy to check that if f and g are locally Lipschitz then so are the functions defined in (iii) and (iv). Since $\mathcal{F}$ consists of locally Lipschitz functions, this implies that the locally Lipschitz elements of $\mathcal{P}$ also satisfy (i)–(iv). Therefore, by the minimality of $\mathcal{P}$, it follows that each element of $\mathcal{P}$ is locally Lipschitz.

We claim that for every $f \in IE(\mathcal{F})$ there is a function $g: H \rightarrow \mathbf{R}$ $(H \subset \mathbf{R}^{n+1})$, and there is a point $y \in \mathbf{R}^n$ such that $g \in \mathcal{P}$ and $g(y, x) = f(x)$ for every $x \in \mathrm{Dom}(f)$.

This is obviously true (with $n = 0$) for the elements of $\mathcal{F}$. By (ii), this holds true for the constant functions as well. Then the general statement easily follows by induction on the rank of the function $f \in IE(\mathcal{F})$.

Let $g \colon K \to \mathbf{R}$ be given, where $K \subset \mathbf{R}^{k+1}$. Let K_n^g denote the set of points $y \in \mathbf{R}^k$ for which g is defined at (y, n). If

$$s_g = \left\{ (g(y, n))_{n=1}^{\infty} \colon y \in \bigcap_{n=1}^{\infty} K_n^g \right\},$$

then it follows from our last remark that

$$S_{\mathcal{F}} \subset \bigcup_{g \in \mathcal{P}} s_g \cap [0, 1]^{\mathbf{N}}.$$

Since $\mathcal{P}$ is countable, $\mu(S_{\mathcal{F}}) = 0$ will follow, if we can show that $\mu(s_g \cap [0, 1]^{\mathbf{N}}) = 0$ for every $g \in \mathcal{P}$.

Let $g \in \mathcal{P}$ be given, where $g \colon K \to \mathbf{R}$, $K \subset \mathbf{R}^{k+1}$, and suppose that g is locally Lipschitz with exponent α. Let $N > k/\alpha$ be fixed, and let pr_N denote the projection of $[0, 1]^{\mathbf{N}}$ to $\mathbf{R}^N$; that is, let

$$pr_N((x_1, x_2, \ldots)) = (x_1, \ldots, x_N) \quad ((x_1, x_2, \ldots) \in [0, 1]^{\mathbf{N}}).$$

We shall prove that $\lambda_N(pr_N(s_g \cap [0, 1]^{\mathbf{N}})) = 0$. This will finish the proof, as $\mu(A) \leq \lambda_N(pr_N(A))$ for every $A \subset [0, 1]^{\mathbf{N}}$.

Let the function g_n be defined by $g_n(y) = g(y, n)$ $(y \in K_n^g)$. By the local Lipschitz property of g, for every $n \in \mathbf{N}$ and $y \in K_n^g$ there is a neighbourhood U_y of y such that g_n is Lipschitz α in $K_n^g \cap U_y$. This implies, by Lindelöf's theorem, that K_n^g can be covered by a sequence $U_1^n, U_2^n, \ldots$ of sets such that g_n is Lipschitz α restricted to each U_i^n. Let $V_1, V_2, \ldots$ be an enumeration of the sets

$$\bigcap_{n=1}^{N} U_{i_n}^n \quad (i_1, i_2, \ldots, i_N \in \mathbf{N}).$$

It is clear that the sets $s_g^j = \{ (g(y, n))_{n=1}^{N} \colon y \in V_j \}$ $(j = 1, 2, \ldots)$ cover $pr_N(s_g \cap [0, 1]^{\mathbf{N}})$, and thus it is enough to show that $\lambda_N(s_g^j) = 0$ for every j. From the construction of the sets V_j it follows that the function $G_j \colon V_j \to s_g^j$ defined by

$$G_j(y) = (g(y, 1), \ldots, g(y, N)) \quad (y \in V_j)$$

is Lipschitz α for every j. Since $V_j \subset \mathbf{R}^k$, this implies that the k/α-dimensional Hausdorff-measure of $s_g^j = G_j(V_j)$ is σ-finite. Since $N > k/\alpha$, we have $\lambda_N(s_g^j) = 0$ $\square$

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