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VERTEX CONTROL OF FLOWS IN NETWORKS

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Abstract. We study a transport equation in a network and control it in a single vertex. We describe all possible reachable states and prove a criterion of Kalman type for those vertices in which the problem is maximally controllable. The results are then applied to concrete networks to show the complexity of the problem.

1. **Introduction.** To motivate the problem treated in this paper we may start from the following situation. Consider a closed network of tubes (such as a system of wires or a circuit) in which material (electrons, information packets, goods) is flowing with constant speed c_j on each edge e_j with no friction or loss. In the nodes v_i of the network the material is redistributed into the tubes according to certain weights ω_{ij}^- satisfying a Kirchhoff law. Simplifying the physical laws and concentrating on the structure of the network, this situation can be described by a system of linear transport equations on the edges

$$\frac{\partial}{\partial t} x_j(t,s) = c_j \frac{\partial}{\partial s} x_j(t,s) \tag{1}$$

and conditions in the vertices saying that

$$\text{outgoing flow on edge } \mathsf{e}_j = \omega_{ij}^- \sum \text{incoming flows into vertex } \mathsf{v}_i$$

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for every edge e_j leaving from vertex v_i (see Section 3 for precise definitions). The authors of [9] proposed a semigroup approach to the study of such a system. Combining graph theoretical and functional analytic tools, they proved well-posedness and described the asymptotic behavior of the solutions by the structure of the underlying graph. Based on their results we ask the following question:

Which states (mass distributions) in the network can be approximately reached by controlling the flow (by adding or subtracting material) in a single vertex?

Let us first look at a very simple example. Consider the network described by the directed and weighted graph presented in Figure 1 with all $c_j = 1$ and the weights α and $1-\alpha$ representing the proportions of the mass leaving the vertex v_1 into the edges e_1 and e_3, respectively.

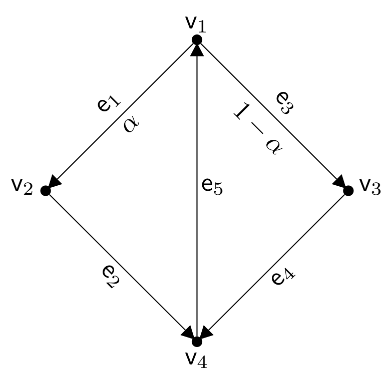

FIGURE 1. The graph G_1

This simple network already shows two typical phenomena that occur in general.

1. The mass distributions on the edges e_1 and e_3 will always satisfy the ratio $\frac{\alpha}{1-\alpha}$. Therefore not every mass distribution on the edges can be attained.
2. Taking 1 into account, all other distributions can be achieved if we control in the vertices v_2 or v_3 but not by controlling in v_1 or v_4 (see Example 5.2).

In this paper we give a complete description of the *maximal* space of reachable states for any given network (see Lemma 4.1). Moreover, we characterize by a finite dimensional Kalman-type condition those vertices from where this maximal space can be attained (see Theorem 4.4).

It seems that the researchers in graph theory have not yet investigated the properties of these vertices. The examples in Section 5 demonstrate their interesting and complex behavior and we believe they deserve a thorough treatment (see also open problems in Section 6).

Let us remark that controlling the network in more than one vertex is obviously an easier task which can be also seen from the modified Kalman condition in Corollary 4.7.

We point out that the analogous control problem for the wave instead of the transport equation on a graph has been studied intensively. We refer to the systematic treatment in [4]. However, a deeper connection between the results in the two cases remains to be unveiled.

The paper is organized in the following way. In Section 2 we start by some basic definitions from control theory (we refer to [2] and [5] for more detailed explanation). To tackle our network control problem we use the abstract results from [6]. In Section 3 we collect terminology and results for the study of flows in networks. We use the semigroup approach as developed in [9], [10], [11], and [3]. For graph theoretical notions we refer to [1] and [7], while those needed here are all defined in [9]. We introduce our network control problem in Section 4. We apply the abstract results to vertex control in networks obtaining Theorem 4.4 as the main result. In Section 5 we explain our results for concrete graphs. In Section 6 we conclude with some open problems addressing a community of researchers from different areas.

2. **Abstract boundary control.**

We start by recalling some notions from abstract control theory.

Abstract Framework 2.1. We consider

(i) three Banach spaces X, ∂X and U called the *state*, *boundary* and *control space*, respectively;
(ii) a closed, densely defined *system operator* $A_m : D(A_m) \subseteq X \to X$;
(iii) a *boundary operator* $Q \in \mathcal{L}([D(A_m)], \partial X)$;
(iv) a *control operator* $B \in \mathcal{L}(U, \partial X)$.

The *abstract boundary control system* $\Sigma_{\mathrm{BC}}(A_m, B, Q)$ associated to the *abstract Cauchy problem with boundary control* on the Banach space X with boundary space ∂X and control space U is defined as

$$\begin{cases} \dot{x}(t) = A_m x(t), & t \geq 0, \\ Qx(t) = Bu(t), & t \geq 0, \\ x(0) = x_0. \end{cases} \tag{2}$$

The function $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}_+, U)$ and a function $x(\cdot) = x(\cdot, x_0, u) \in \mathrm{C}^1(\mathbb{R}_+, X)$ with $x(t) \in D(A_m)$ for all $t \geq 0$ satisfying (2) is called a *classical solution*.

We are mainly concerned to describe all states a given system can possibly attain. Therefore we define

Definition 2.2. The *approximate reachability space* associated to (2) is

$$\begin{aligned} \mathcal{R}^{\mathrm{BC}} := \operatorname{cl} \big\{ & y \in X \mid \exists t > 0 \text{ and } u(\cdot) \in \mathrm{L}^1([0,t], U) \text{ such that } y = x(t), \\ & \text{where } x(\cdot) \text{ is the solution of (2) with } x(0) = 0 \big\}. \end{aligned} \tag{3}$$

The boundary control system $\Sigma_{\mathrm{BC}}(A_m, B, Q)$ is called *approximately boundary controllable*, if $\mathcal{R}^{\mathrm{BC}} = X$.

In the following Lemma we collect some results due to Greiner [8, Lem. 1.2, Lem. 1.3] we shall need to describe the space $\mathcal{R}^{\mathrm{BC}}$.

Lemma 2.3. *Assume the following properties to hold.*

(i) The restricted operator $A \subset A_m$ with domain $D(A) := \ker Q$ generates a strongly continuous semigroup $(T(t))_{t \geq 0}$ on X;
(ii) the boundary operator $Q : D(A_m) \to \partial X$ is surjective.

Then for each $\lambda \in \rho(A)$, $D(A_m) = D(A) \oplus \ker(\lambda - A_m)$, the operator $Q|_{\ker(\lambda - A_m)}$ is invertible and the inverse $Q_\lambda := (Q|_{\ker(\lambda - A_m)})^{-1} : \partial X \to \ker(\lambda - A_m) \subseteq X$ is bounded.

For $\lambda \in \rho(A)$ we call the operator Q_λ, introduced in Lemma 2.3, the *Dirichlet operator* and define

$$B_\lambda := Q_\lambda B \in \mathcal{L}(U, \ker(\lambda - A_m)).$$

Recall that the *spectral bound* of A is defined as $\omega_0(A) = \sup\{\mathrm{Re}\lambda : \lambda \in \sigma(A)\}$. By $\mathrm{rg}(C)$ we denote the *range* of an operator C. The following has been shown in [6].

Theorem 2.4. *The approximate reachability space $\mathcal{R}^{\mathrm{BC}}$ of $\Sigma_{BC}(A_m, B, Q)$ coincides with*

(i) the smallest closed, $(T(t))_{t\geq 0}$-invariant subspace of X containing $\mathrm{rg}(B_\mu)$ for all μ sufficiently large,
(ii) the smallest closed, $R(\mu, A)$-invariant, $\mu > \omega > \omega_0(A)$, subspace of X containing $\mathrm{rg}(B_\mu)$ for all μ sufficiently large, and
(iii) $\overline{\mathrm{span}}\big(\bigcup_{\lambda>\omega} \mathrm{rg}(B_\lambda)\big)$ for some $\omega > \omega_0(A)$.

The approximate reachability space is related to a subspace, which is independent of the specific control operator.

Definition 2.5. The *maximal reachability space* associated to (2) is

$$\mathcal{R}^{\mathrm{BC}}_{\max} := \overline{\mathrm{span}} \bigcup_{\lambda > \omega_0(A)} \ker(\lambda - A_m). \tag{4}$$

The system $\Sigma_{\mathrm{BC}}(A_m, B, Q)$ is called *maximally controllable* if $\mathcal{R}^{\mathrm{BC}} = \mathcal{R}^{\mathrm{BC}}_{\max}$.

It has been shown in [6] that $\mathcal{R}^{\mathrm{BC}} \subset \mathcal{R}^{\mathrm{BC}}_{\max}$, so $\mathcal{R}^{\mathrm{BC}}_{\max}$ is indeed the largest possible space of states that can be approximately reached by applying some boundary control B. We will see that the relevant question for controllability of our network problem is when are these two spaces equal.

3. **Flows in networks.** We consider a finite network modeled by a simple, directed graph. We denote by $\mathsf{V} = \{\mathsf{v}_1, \dots, \mathsf{v}_n\}$ the set of vertices and by $\mathsf{E} = \{\mathsf{e}_1, \dots, \mathsf{e}_m\}$ the set of (directed) edges of the graph. The edges are parameterized on the interval $[0, 1]$, to the contrary of their directions. The vertex $\mathsf{e}_j(0)$ is thus called the *head* and the vertex $\mathsf{e}_j(1)$ the *tail* of the edge $\mathsf{e}_j \in \mathsf{E}$. The edge e_j is an *incoming edge* for the vertex v_i if $\mathsf{v}_i = \mathsf{e}_j(0)$ holds, and it is called an *outgoing edge* for v_i if $\mathsf{v}_i = \mathsf{e}_j(1)$ holds. We assume that in every vertex there is at least one incoming as well as at least one outgoing edge.

We will use the following *graph matrices* (see [9, Sect. 1]) to describe the structure of the network.

Definition 3.1. (i) The *outgoing incidence matrix* $\Phi^- = \big(\phi^-_{ij}\big)_{n\times m}$ has entries

$$\phi^-_{ij} := \begin{cases} 1, & \mathsf{v}_i \text{ tail of } \mathsf{e}_j, \\ 0, & \text{else;} \end{cases}$$

(ii) The *incoming incidence matrix* $\Phi^+ = \big(\phi^+_{ij}\big)_{n\times m}$ has entries

$$\phi^+_{ij} := \begin{cases} 1, & \mathsf{v}_i \text{ head of } \mathsf{e}_j, \\ 0, & \text{else;} \end{cases}$$

(iii) The *weighted outgoing incidence matrix* is $\Phi^-_w = \big(\omega^-_{ij}\big)_{n\times m}$, where $0 \leq \omega^-_{ij} \leq 1$ satisfy $\omega^-_{ij} = 0 \Leftrightarrow \phi^-_{ij} = 0$ and $\sum_{j=1}^m \omega^-_{ij} = 1$ for all $i = 1, \dots, n$;

(iv) The *weighted adjacency matrix* is defined by $\mathbb{A} = (a_{ik})_{n\times n} := \Phi^+(\Phi_w^-)^\top$;
(v) The *weighted adjacency matrix of the line graph* is defined as $\mathbb{B} = (b_{lj})_{m\times m} := (\Phi_w^-)^\top \Phi^+$.

As examples for the graph matrices Φ_w^- and $\mathbb{A}$ we refer to Example 4.2 in Section 5.

Remark 3.2. Simple computations show that

$$\Phi^-(\Phi_w^-)^\top = I_{\mathbb{C}^n} \tag{5}$$

and that both adjacency matrices $\mathbb{A}$ and $\mathbb{B}$ are column stochastic hence

$$\|\mathbb{A}\|_1 = 1 \quad \text{and} \quad \|\mathbb{B}\|_1 = 1.$$

Furthermore, the relation

$$(\Phi_w^-)^\top \mathbb{A} = \mathbb{B}(\Phi_w^-)^\top \tag{6}$$

holds.

The mathematical model for flows in networks is as follows (see [9, Sect. 1]).

- We consider transport equations on the m edges of the graph:
$$\frac{\partial}{\partial t} x_j(t,s) = c_j \frac{\partial}{\partial s} x_j(t,s), \tag{7}$$
where $c_j > 0$ is the velocity of the flow on the edge e_j.
- The boundary conditions say that in each vertex v_i the incoming flow is distributed on the outgoing edges by fixed proportions ω_{ij}^-, called the *weights* of the edges e_j in vertex v_i:
$$\phi_{ij}^- x_j(t,1) = \omega_{ij}^- \sum_{k=1}^m \phi_{ik}^+ x_k(t,0). \tag{8}$$
Observe that by the assumption in Definition 3.1(iii) this implies that the Kirchhoff law is satisfied, i.e., the total incoming flow mass equals the total outgoing flow mass.
- We further need initial conditions on the edges:
$$x_j(0,s) = f_j(s). \tag{9}$$

In the above formulas

$t \geq 0$ is the time variable,
$s \in [0,1]$ is the space variable on the edges,
$j = 1,\ldots,m$ are the indices of edges, and
$i = 1,\ldots,n$ are the indices of vertices.

We now rewrite this in the form of an abstract Cauchy problem

$$\begin{cases} \dot{x}(t) = A\,x(t), \ t \geq 0, \\ x(0) = f, \end{cases} \tag{10}$$

on X, where

- $X := \mathrm{L}^1([0,1],\mathbb{C}^m)$,
- $A := \operatorname{diag}\left(c_j \frac{d}{ds}\right)_{j=1,\ldots,m}$ with the domain (see [9, Sect. 2])

$$D(A) := \{g \in \mathrm{W}^{1,1}([0,1],\mathbb{C}^m) \mid g(1) \in \operatorname{rg}(\Phi_w^-)^\top \text{ and } \Phi^- g(1) = \Phi^+ g(0)\}, \tag{11}$$

- $x(t) = x(t,\cdot),\ f = (f_1,\ldots,f_m)^\top$.

In the domain of A, the first condition

$$g(1) \in \operatorname{rg}(\Phi_w^-)^\top$$

means that in every vertex the total incoming flow is distributed in (given) weighted proportions to the outgoing edges. The second condition

$$\Phi^- g(1) = \Phi^+ g(0)$$

is the Kirchhoff's law in each vertex.

By [9, Cor. 2.7] this problem is well-posed. Furthermore, it was shown in [3] that in case when all the velocities c_j coincide, we can explicitly describe the semigroup governing the problem. For this reason and in order to simplify our control problem, we assume in the following that

$$c_j = 1,\ j = 1, \ldots, m. \tag{12}$$

Proposition 3.3. *Let (12) holds. Then the domain (11) of the operator A can be written as*

$$D(A) = \left\{ g \in \mathrm{W}^{1,1}([0,1], \mathbb{C}^m) \mid g(1) = \mathbb{B}\, g(0) \right\}$$

and $(A, D(A))$ generates the strongly continuous semigroup

$$[T(t)g](s) = \mathbb{B}^n\, g(t+s-n) \quad \textit{if}\ \ t+s \in [n, n+1)\ \textit{for}\ n \in \mathbb{N},$$

where $\mathbb{B}^0 := I$. Moreover, the spectral bound $\omega_0(A) = 0$.

Proof. See [3, Propositions 3.1 and 3.4] and [9, Corollary 3.5]. □

Remark 3.4. If needed one may work in the (reflexive) space $X_p := L^p([0,1], \mathbb{C}^m)$, $1 < p < \infty$, where the same proposition holds for the bounded (but not necessarily contractive) flow semigroup.

4. Vertex control in networks.

Now we focus on maximal controllability of flows in networks by controls acting in one of the vertices only. Throughout the section we will assume that the condition (12) holds.

We start by (10) and add appropriate boundary and control operators to obtain an abstract Cauchy problem with boundary control. In our setting $X = \mathrm{L}^1([0,1], \mathbb{C}^m)$ is the *state space* for our problem while the *boundary space* is $\partial X := \mathbb{C}^n$ corresponding to the vertices of the graph. We further need the following notations and results from [9].

The *outgoing boundary operator* $L : X \to \partial X$ is

$$Lg := \Phi^- g(1), \quad D(L) := W^{1,1}\left([0,1], \mathbb{C}^m\right),$$

while the *incoming boundary operator* $M : X \to \partial X$ is

$$Mg := \Phi^+ g(0), \quad D(M) := W^{1,1}\left([0,1], \mathbb{C}^m\right).$$

Both operators are bounded and map from the Banach space $W^{1,1}\left([0,1], \mathbb{C}^m\right)$ to $\mathbb{C}^n$. Then the domain of A defined in (11) can be written as

$$D(A) = \left\{ g \in \mathrm{W}^{1,1}([0,1], \mathbb{C}^m) \mid g(1) \in \operatorname{rg}(\Phi_w^-)^\top \text{ and } (L-M)g = 0 \right\}.$$

Hence, defining $A_m = \frac{d}{ds}$ with domain

$$D(A_m) = \left\{ g \in \mathrm{W}^{1,1}([0,1], \mathbb{C}^m) \mid g(1) \in \operatorname{rg}(\Phi_w^-)^\top \right\} \tag{13}$$

and the *boundary operator*

$$Q := L - M \in \mathcal{L}([D(A_m)], \partial X), \tag{14}$$

the abstract Cauchy problem (10) obtains the form

$$\begin{cases} \dot{x}(t) = A_m x(t), & t \geq 0, \\ Qx(t) = 0, & t \geq 0, \\ x(0) = x_0. \end{cases}$$

Finally, we impose control in the vertex $\mathsf{v} = \mathsf{v}_i \in \mathsf{V}$ for some fixed $i \in \{1, \dots, n\}$. In the following we identify this vertex with a vector $\mathsf{v} \in \mathbb{C}^n$

$$\mathsf{v} = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i^{\text{th}}.$$

As (one dimensional) *control space* U and *control operator* B we choose

$$U := \mathbb{C}, \quad B : U \to \operatorname{span}\{\mathsf{v}\} \subset \partial X = \mathbb{C}^n,$$

where B is any (bounded) linear operator acting between the given vector spaces. With these notations we arrive at an abstract Cauchy problem with boundary control of the form (2).

$$\begin{cases} \dot{x}(t) = A_m x(t), & t \geq 0, \\ Qx(t) = Bu(t), & t \geq 0, \\ x(0) = x_0. \end{cases} \tag{15}$$

Now we can apply the abstract results from Section 2 to our problem. Since the eigenvectors of A_m have to satisfy the boundary conditions in the vertices (c.f. (13)), it follows that in general $\mathcal{R}^{\mathrm{BC}}_{\max}$ from (4) cannot be equal to the state space $X = \mathrm{L}^1([0,1], \mathbb{C}^m)$ (see section 5 for some concrete examples). Our aim is to find out when $\mathcal{R}^{\mathrm{BC}} = \mathcal{R}^{\mathrm{BC}}_{\max}$ can be achieved, i.e., when the system is maximally controllable. For this purpose we will give explicit descriptions of $\mathcal{R}^{\mathrm{BC}}_{\max}$ and $\mathcal{R}^{\mathrm{BC}}$ in terms of the graph matrices. In the following ε_λ denotes the exponential function

$$\varepsilon_\lambda(s) := \mathrm{e}^{\lambda s} \text{ for } s \in [0,1] \text{ and some } \lambda \in \mathbb{C}.$$

Lemma 4.1. *The maximal reachability space* $\mathcal{R}^{\mathrm{BC}}_{\max}$ *is equal to*

$$\mathcal{R}^{\mathrm{BC}}_{\max} = \operatorname{span} \left\{ \begin{pmatrix} a_1 g \\ \vdots \\ a_m g \end{pmatrix} \middle| \, g \in L^1([0,1], \mathbb{C}) \text{ and } \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} \in \operatorname{rg}(\Phi_w^-)^\top \right\}.$$

Briefly,

$$\mathcal{R}^{\mathrm{BC}}_{\max} = L^1([0,1], \mathbb{C}) \otimes \operatorname{rg}(\Phi_w^-)^\top = L^1([0,1], \mathbb{C}) \otimes (\Phi_w^-)^\top \mathbb{C}^n. \tag{16}$$

Proof. Using [9, p. 147] we have that

$$\ker(\lambda - A_m) = \left\{ \begin{pmatrix} a_1 \varepsilon_\lambda \\ \vdots \\ a_m \varepsilon_\lambda \end{pmatrix} \middle| \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} \in \operatorname{rg}(\Phi_w^-)^\top \right\}. \tag{17}$$

Observe that by the Stone-Weierstrass theorem

$$\overline{\operatorname{span}} \bigcup_{\lambda > \omega_0(A)} \{\varepsilon_\lambda\} = \mathrm{L}^1([0,1], \mathbb{C}),$$

hence we are done. □

In order to describe $\mathcal{R}^{\mathrm{BC}}$ using Theorem 2.4(iii) we will need the form of the operators $B_\lambda = Q_\lambda B$ for λ large enough. We start with

$$Q_\lambda = (Q|_{\ker(\lambda - A_m)})^{-1}$$

from Lemma 2.3 and compute it for the boundary operator Q defined in (14).

Lemma 4.2. *For $\lambda > 0 = \omega_0(A)$ we have*

$$Q_\lambda = \varepsilon_\lambda (\Phi_w^-)^\top R\left(\mathrm{e}^\lambda, \mathbb{A}\right),$$

where $R\left(\mathrm{e}^\lambda, \mathbb{A}\right) := \left(\mathrm{e}^\lambda - \mathbb{A}\right)^{-1}$ denotes the resolvent of $\mathbb{A}$ in e^λ.

Proof. First compute

$$\begin{aligned}(L-M)\varepsilon_\lambda(\Phi_w^-)^\top R\left(\mathrm{e}^\lambda, \mathbb{A}\right) &= \mathrm{e}^\lambda \Phi^- (\Phi_w^-)^\top R\left(\mathrm{e}^\lambda, \mathbb{A}\right) - \Phi^+ (\Phi_w^-)^\top R\left(\mathrm{e}^\lambda, \mathbb{A}\right)\\ &= \mathrm{e}^\lambda R\left(\mathrm{e}^\lambda, \mathbb{A}\right) - \mathbb{A} R\left(\mathrm{e}^\lambda, \mathbb{A}\right) = I_{\mathbb{C}^n},\end{aligned}$$

where we used (5).
By (17), taking any $f \in \ker(\lambda - A_m)$ we have

$$f = \begin{pmatrix} a_1 \varepsilon_\lambda \\ \vdots \\ a_m \varepsilon_\lambda \end{pmatrix} \text{ for some } \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} = (\Phi_w^-)^\top d,\ d \in \mathbb{C}^n.$$

Hence, using $\|\mathrm{e}^{-\lambda}\mathbb{A}\|_1 < 1$ for $\lambda > 0$, Definition 3.1, (5) and (6) we have

$$\begin{aligned}&\varepsilon_\lambda(\Phi_w^-)^\top R\left(\mathrm{e}^\lambda, \mathbb{A}\right)(L-M)f =\\ &\quad= \varepsilon_\lambda(\Phi_w^-)^\top R\left(\mathrm{e}^\lambda, \mathbb{A}\right)\left(\Phi^- f(1) - \Phi^+ f(0)\right)\\ &\quad= \varepsilon_\lambda \mathrm{e}^{-\lambda}(\Phi_w^-)^\top \sum_{k=0}^\infty \mathrm{e}^{-\lambda k}\mathbb{A}^k \left(\mathrm{e}^\lambda \Phi^- \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} - \Phi^+ f(0)\right)\\ &\quad= \varepsilon_\lambda \mathrm{e}^{-\lambda}\left(\mathrm{e}^\lambda \sum_{k=0}^\infty \mathrm{e}^{-\lambda k}\mathbb{B}^k (\Phi_w^-)^\top d - \sum_{k=0}^\infty \mathrm{e}^{-\lambda k}\mathbb{B}^{k+1} f(0)\right)\\ &\quad= \varepsilon_\lambda \left(\sum_{k=0}^\infty \mathrm{e}^{-\lambda k}\mathbb{B}^k \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} - \sum_{k=0}^\infty \mathrm{e}^{-\lambda(k+1)}\mathbb{B}^{k+1}\begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix}\right)\\ &\quad= \begin{pmatrix} a_1 \varepsilon_\lambda \\ \vdots \\ a_m \varepsilon_\lambda \end{pmatrix} = f.\end{aligned}$$

We have thus shown that

$$\varepsilon_\lambda(\Phi_w^-)^\top R\left(\mathrm{e}^\lambda, \mathbb{A}\right)(L-M) = I_{\ker(\lambda - A_m)},$$

hence we are done. □

Using Theorem 2.4(iii) we obtain the following.

Corollary 4.3. *There exists $\omega > 0$ such that*

$$\mathcal{R}^{\mathrm{BC}} = \overline{\operatorname{span}} \bigcup_{\lambda > \omega} \left\{ \varepsilon_\lambda (\Phi_w^-)^\top R\left(\mathrm{e}^\lambda, \mathbb{A}\right) \mathsf{v} \right\} \tag{18}$$

$$= L^1\left([0,1], \mathbb{C}\right) \otimes (\Phi_w^-)^\top \left(\operatorname{span}\left\{\mathsf{v}, \mathbb{A}\mathsf{v}, \dots, \mathbb{A}^{n-1}\mathsf{v}\right\}\right). \tag{19}$$

Proof. We only have to prove the second equality. By Proposition 3.3 together with Theorem 2.4 we have

$$T(1)\left(\varepsilon_\lambda (\Phi_w^-)^\top R\left(\mathrm{e}^\lambda, \mathbb{A}\right) \mathsf{v}\right) = \varepsilon_\lambda \mathbb{B} (\Phi_w^-)^\top R\left(\mathrm{e}^\lambda, \mathbb{A}\right) \mathsf{v} \in \mathcal{R}^{\mathrm{BC}}.$$

Using (6) we obtain

$$\varepsilon_\lambda (\Phi_w^-)^\top R\left(\mathrm{e}^\lambda, \mathbb{A}\right) \mathbb{A}\mathsf{v} \in \mathcal{R}^{\mathrm{BC}}.$$

Applying $T(1)$ to this vector again yields

$$T(1)\left(\varepsilon_\lambda (\Phi_w^-)^\top R\left(\mathrm{e}^\lambda, \mathbb{A}\right) \mathbb{A}\mathsf{v}\right) = \varepsilon_\lambda (\Phi_w^-)^\top R\left(\mathrm{e}^\lambda, \mathbb{A}\right) \mathbb{A}^2\mathsf{v} \in \mathcal{R}^{\mathrm{BC}}.$$

Continuing this procedure we obtain that

$$\varepsilon_\lambda (\Phi_w^-)^\top R\left(\mathrm{e}^\lambda, \mathbb{A}\right) \mathbb{A}^k\mathsf{v} \in \mathcal{R}^{\mathrm{BC}},\ k = 0, 1, \dots$$

Since $\mathcal{R}^{\mathrm{BC}}$ is a linear subspace, we also have that

$$\begin{aligned} \mathrm{e}^\lambda \cdot \varepsilon_\lambda (\Phi_w^-)^\top R\left(\mathrm{e}^\lambda, \mathbb{A}\right) \mathbb{A}^k\mathsf{v} - \varepsilon_\lambda (\Phi_w^-)^\top R\left(\mathrm{e}^\lambda, \mathbb{A}\right) \mathbb{A}^{k+1}\mathsf{v} \\ = \varepsilon_\lambda (\Phi_w^-)^\top R\left(\mathrm{e}^\lambda, \mathbb{A}\right) \left(\mathrm{e}^\lambda - \mathbb{A}\right) \mathbb{A}^k\mathsf{v} \\ = \varepsilon_\lambda (\Phi_w^-)^\top \mathbb{A}^k\mathsf{v} \in \mathcal{R}^{\mathrm{BC}},\ k = 0, 1, \dots, n-1. \end{aligned}$$

Using the Stone-Weierstrass theorem and the Neumann series expansion of $R\left(\mathrm{e}^\lambda, \mathbb{A}\right)$, we finally obtain the result. ☐

Let us emphasize that the space $\mathcal{R}^{\mathrm{BC}}_{\max}$ consists of all possible states in the network and is independent of the control. The space $\mathcal{R}^{\mathrm{BC}} \subseteq \mathcal{R}^{\mathrm{BC}}_{\max}$ however depends on the specific control operator, in our case on the vertex in which the control takes place. We are now able to characterize the vertices in which these two spaces coincide.

Theorem 4.4. *The following assertions are equivalent for a vertex* v.

(a) $\mathcal{R}^{\mathrm{BC}} = \mathcal{R}^{\mathrm{BC}}_{\max}$, *i.e., the flow is* maximally controllable *in the vertex* v.
(b) $\operatorname{span}\left\{\mathsf{v}, \mathbb{A}\mathsf{v}, \dots, \mathbb{A}^{n-1}\mathsf{v}\right\} = \mathbb{C}^n$.

Proof. Using (16) and (19), (a) is equivalent to

$$\begin{aligned} \mathcal{R}^{\mathrm{BC}}_{\max} &= L^1\left([0,1], \mathbb{C}\right) \otimes (\Phi_w^-)^\top \mathbb{C}^n \\ &= L^1\left([0,1], \mathbb{C}\right) \otimes (\Phi_w^-)^\top \left(\operatorname{span}\left\{\mathsf{v}, \mathbb{A}\mathsf{v}, \dots, \mathbb{A}^{n-1}\mathsf{v}\right\}\right) \\ &= \mathcal{R}^{\mathrm{BC}}. \end{aligned}$$

Since $(\Phi_w^-)^\top$ is injective hence left invertible, this holds if and only if

$$\operatorname{span}\left\{\mathsf{v}, \mathbb{A}\mathsf{v}, \dots, \mathbb{A}^{n-1}\mathsf{v}\right\} = \mathbb{C}^n.$$

☐

Remark 4.5. Assertion (b) in Theorem 4.4 is a *Kalman-type condition*, well-known in control theory. In our situation, it guarantees that by controlling in the vertex v the largest possible space of mass distributions in the network can be (approximately) reached.

In concrete situations (and for large graphs) it may be quite difficult to verify this Kalman-type criterion. In particular, it depends on the structure of the graph *and* on the distribution of the weights ω_{ij}^- (see Example 5.6 below). Therefore it is desirable to have a sufficient condition for maximal controllability which can be seen directly (and only) from the graph.

Remark 4.6. If there exists a vertex v_j in G such that the shortest (directed) path between v_i and v_j has length $n-1$, then the condition (b) in Theorem 4.4 is satisfied for the vertex v_i (see e.g. [7, Lemma 2.5]).

This condition of the graph can be tested in linear time if the vertex is given. Unfortunately it is far from being also necessary (see Example 5.3 or 5.6).

Finally, let us mention that the proof of Theorem 4.4 can be easily adopted to the case when the control takes place in more than one vertex, obtaining the following obvious modification of the Kalman condition.

Corollary 4.7. *Assume that we control in the vertices* $\mathsf{v}_{i_1}, \dots, \mathsf{v}_{i_k}$*, and for the velocities (12) holds. Then the following assertions are equivalent.*

(a) $\mathcal{R}^{\mathrm{BC}} = \mathcal{R}^{\mathrm{BC}}_{\max}$,

(b) $\operatorname{span}\left\{\mathsf{v}_{i_1}, \mathbb{A}\mathsf{v}_{i_1}, \dots, \mathbb{A}^{n-1}\mathsf{v}_{i_1}, \dots, \mathsf{v}_{i_k}, \mathbb{A}\mathsf{v}_{i_k}, \dots, \mathbb{A}^{n-1}\mathsf{v}_{i_k}\right\} = \mathbb{C}^n$.

5. **Examples.** We conclude with some examples of networks showing the complexity of our problem already for small graphs.

Example 5.1. Starting with the basic graph C_4, the undirected cycle on 4 vertices, we first note that there is only one possible orientation of the edges yielding a strongly connected directed graph G_0, see Figure 2.

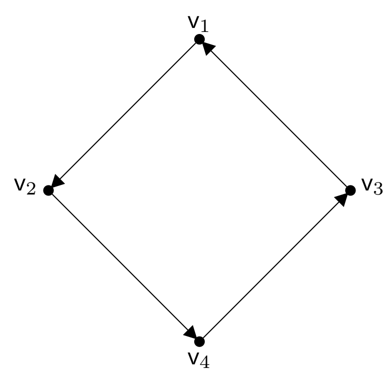

Figure 2. The graph G_0

By checking condition (b) of Theorem 4.4 it follows easily that our problem is maximally controllable in all the vertices of G_0. It is also obvious that the condition in the Remark 4.6 holds for every vertex of G_0.

Example 5.2. Now let us orient the edges of C_4 in a different way and add an extra edge from v_4 to v_1. We take the weights on the edges as

$$\omega_{11}^- = \alpha,\ \omega_{22}^- = 1,\ \omega_{13}^- = 1-\alpha,\ \omega_{34}^- = \omega_{45}^- = 1 \quad \text{for some} \quad 0 < \alpha < 1,$$

and denote the network thus obtained by G_1, see Figure 1 (in the Introduction).

Note that the network G_1 is strongly connected and its incidence and adjacency matrices are

$$(\Phi_w^-)^\top = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1-\alpha & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbb{A} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ \alpha & 0 & 0 & 0 \\ 1-\alpha & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

By Lemma 4.1,

$$\mathcal{R}_{\max}^{\mathrm{BC}} = \operatorname{span}\left\{ \begin{pmatrix} a_1 g \\ \vdots \\ a_5 g \end{pmatrix} \middle| \, g \in L^1([0,1],\mathbb{C}),\ a_i \in \mathbb{C},\ \frac{a_3}{a_1} = \frac{1-\alpha}{\alpha} \right\}.$$

Verifying the Kalman-type condition in Theorem 4.4 we obtain that

$$\mathcal{R}_{\max}^{\mathrm{BC}} = \mathcal{R}^{\mathrm{BC}} \iff \mathsf{v} = \mathsf{v}_2 \text{ or } \mathsf{v} = \mathsf{v}_3.$$

So, we can control the flow in the network G_1 only in the vertices v_2 or v_3. Also in this case, the condition from Remark 4.6 is satisfied for the vertices v_2 and v_3 – the shortest directed path between them in both directions has length 3.

Example 5.3. Let G_2 be the network obtained from G_1 by inserting an edge from v_2 to v_3 and taking the weights

$$\omega_{11}^- = \alpha,\ \omega_{22}^- = 1-\beta,\ \omega_{13}^- = 1-\alpha,\ \omega_{34}^- = \omega_{45}^- = 1,\ \omega_{26}^- = \beta \quad \text{for some} \quad 0 < \alpha, \beta < 1,$$

see Figure 3.

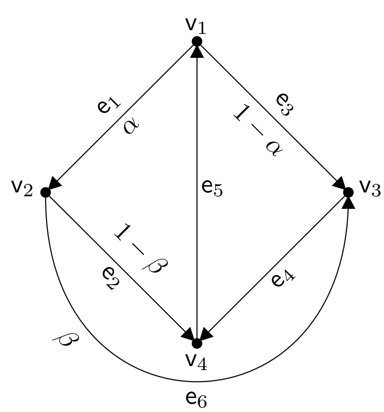

FIGURE 3. The graph G_2

Note that this is a directed version of K_4, the complete graph with 4 vertices or the tetrahedron graph. The appropriate adjacency matrix is

$$\mathbb{A} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ \alpha & 0 & 0 & 0 \\ 1-\alpha & \beta & 0 & 0 \\ 0 & 1-\beta & 1 & 0 \end{pmatrix}.$$

The maximal reachability space in this case is

$$\mathcal{R}^{\mathrm{BC}}_{\max} = \operatorname{span}\left\{ \left(\begin{array}{c} a_1 g \\ \vdots \\ a_6 g \end{array} \right) \middle| \, g \in L^1([0,1],\mathbb{C}),\ a_i \in \mathbb{C},\ \frac{a_3}{a_1} = \frac{1-\alpha}{\alpha},\ \frac{a_6}{a_2} = \frac{\beta}{1-\beta} \right\}.$$

The condition (b) in Theorem 4.4 holds for every vertex hence the problem is maximally controllable in each of the vertices. Here only the vertex v_3 has the property from the Remark 4.6.

Example 5.4. Let us see what happens by inserting more vertices. Take G_1 and insert a new vertex v_5 on the edge e_5, thus obtaining the network G_3 shown in Figure 4.

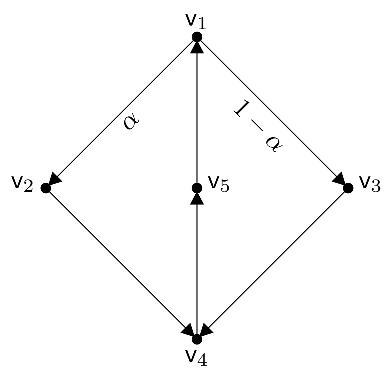

FIGURE 4. The graph G_3

We leave it to the reader to write down the appropriate matrices and see that the problem remains maximally controllable only in the vertices v_2 or v_3. Again, the vertices v_2 and v_3 satisfy the condition from Remark 4.6.

Example 5.5. The situation becomes completely different by adding one more vertex to G_3. Let G_4 be the network presented in Figure 5, for some $0 < \alpha, \beta < 1$.

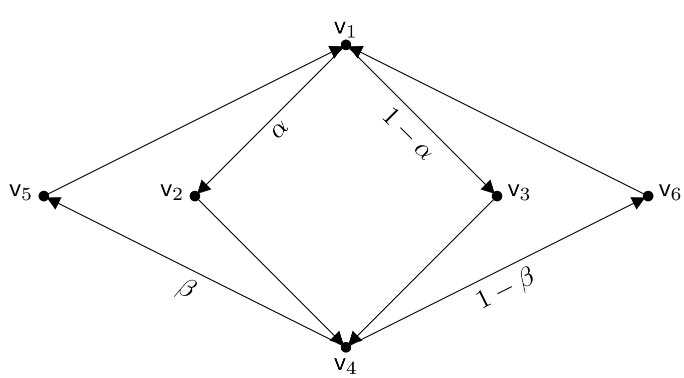

FIGURE 5. The graph G_4

Checking the Kalman-type condition for this graph we obtain

$$\mathcal{R}^{\mathrm{BC}}_{\max} \neq \mathcal{R}^{\mathrm{BC}} \qquad \text{for all vertices } \mathsf{v}_1, \ldots, \mathsf{v}_6.$$

Thus we do not have control in any of the vertices! Observe that also none of the vertices has the property described in Remark 4.6.

Example 5.6. At the end we give an example on the impact of the weights on the edges to our problem. We add two more edges to G_3 leaving the vertex v_5 and obtain an oriented version of the graph W_4, known as the *wheel on 4 vertices*. Let $0 < \alpha, \beta, \gamma < 1$ be arbitrary numbers such that $\beta + \gamma < 1$.

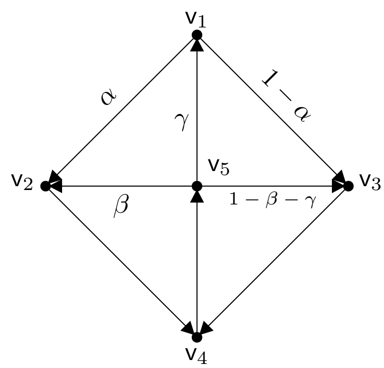

FIGURE 6. The graph G_5

The graph G_5 presented in Figure 6 admits the following adjacency matrix.

$$\mathbb{A} = \begin{pmatrix} 0 & 0 & 0 & 0 & \gamma \\ \alpha & 0 & 0 & 0 & \beta \\ 1-\alpha & 0 & 0 & 0 & 1-\beta-\gamma \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

One can easily compute that, according to Theorem 4.4, this network is maximally controllable in the vertices v_2 and v_3, independently of the particular choice of the weights. It is not controllable in v_4 and v_5, also independently of the particular choice of the weights. However, G_5 is controllable in v_1 if and only if

$$\alpha - \beta - \alpha \cdot \gamma \neq 0.$$

Hence, controllability in v_1 depends on the weights of the edges. Note that the condition from Remark 4.6 is independent of the weights and is not fulfilled for any of the vertices of G_5.

6. **Open problems.** The general results (Theorem 4.4 and Remark 4.6) and the above examples lead to the following open problems.

(i) A systematic investigation from the perspective of graph theory of the vertices having property Theorem 4.4(b) remains an interesting task.

(ii) The problem of vertex control in case when the velocities on the edges are different but rationally dependent (see [9]) can be reduced to the situation treated in Section 4. However, the case of rationally independent velocities is completely open.

(iii) More general transport processes in networks allowing space dependent velocities and absorption on the edges have been studied in [10]. The analogous control problem in this more realistic situation remains to be investigated.

(iv) It seems natural to ask what can be said about *exact* instead of approximate control (i.e. about states reachable in some fixed final time).

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