From Grothendieck Groups to Generators: The Discrete Cluster Categories of Type A_∞

by

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Abstract

In this thesis we look at two closely related families of categories: the discrete cluster categories of Dynkin type A_∞, and their completions in the sense of Paquette and Yıldırım.

We compute the triangulated Grothendieck group of the discrete cluster categories of Dynkin type A_∞, as well as their Paquette-Yıldırım completions. Further, we provide a counterexample to a theorem by Palu and provide a corrected statement of the result.

We also introduce the concept of homologically connected objects, and show that any object in the Paquette-Yıldırım completion of a discrete cluster category of Dynkin type A_∞ can be decomposed into homologically connected direct summands, and that the smallest thick subcategory containing an object is determined by its decomposition into homologically connected direct summands. This allows us to classify the classical generators of the Paquette-Yıldırım completions of the discrete cluster categories of Dynkin type A_∞, and associate an integer to each classical generator that is an upper bound on their generation time. This allows us to compute an upper bound for the Orlov spectrum, and to compute the Rouquier dimension of the Paquette-Yıldırım completions.

Further, we compute the graded endomorphism ring of a chosen classical generator as a $\mathbb{Z}$-graded, upper triangular matrix ring with polynomial rings and Laurent polynomial rings as entries.

Contents

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Author’s Declaration

I declare that, except where explicit reference is made to the contribution of others, this dissertation is the result of my own work and has not been submitted for any other degree at the University of Glasgow or any other institution.

Introduction

Discrete cluster categories of Dynkin type A_∞ were introduced by Igusa and Todorov in [39] as part of a larger family of cluster categories that are stable Frobenius categories coming from cyclic posets. They are particularly well behaved triangulated categories and provide a practical setting in which to study cluster categories with infinitely many isomorphism classes of indecomposable objects.

We may model a discrete cluster category of Dynkin type A_∞, labelled $\mathcal{C}_n$ for $n \geq 1$, using a disc with an infinite set of marked points $\mathscr{M}$ on the boundary, and a set of n accumulation points. That is, points on the boundary of the disc satisfying some limit condition of marked points from both clockwise and anticlockwise directions. The isomorphism classes of indecomposable objects of $\mathcal{C}_n$ are in bijection with the isotopy classes of arcs between marked points that are not isotopic to boundary segments of the disc, and an Ext^1-space between indecomposable objects is non-trivial if and only if the corresponding arcs *cross* on the interior of the disc.

We also consider a second family of closely related categories in this thesis, the Paquette-Yıldırım completion [58] of the discrete cluster category of Dynkin type A_∞, denoted $\overline{\mathcal{C}}_n$. The category $\overline{\mathcal{C}}_n$ is constructed by Paquette and Yıldırım as a Verdier localisation of $\mathcal{C}_{2n}$, and can be modelled in a similar fashion to $\mathcal{C}_n$, by considering the accumulation points to be contained within the set of marked points.

In the first part of this thesis we compute the triangulated Grothendieck groups of $\mathcal{C}_n$ and $\overline{\mathcal{C}}_n$, and in the second part we look at classical generators in $\overline{\mathcal{C}}_n$. Our main results in the second part includes classifying the classical generators and computing bounds for the Orlov spectrum of $\overline{\mathcal{C}}_n$, as well as computing the graded endomorphism ring of a particular classical generator of $\overline{\mathcal{C}}_n$ with some desirable properties.

Cluster Algebras

Fomin and Zelevinsky introduced the concept of cluster algebras in the early 2000's as an algebraic framework for dual canonical bases and total positivity in semisimple groups. They developed the foundational results and began the development of a rich theory surrounding cluster algebras in a series of papers [30, 31, 32], as well as [14] with Berenstein.

We may think of a cluster algebra as a commutative, unital algebra $\mathcal{A}$ that has no zero divisors and a distinguished family of generators, the *cluster variables.* These cluster variables form overlapping subsets of n elements, known as *clusters*, which follow an *exchange relation*, where a cluster variable may be removed from a cluster and uniquely replaced by another cluster variable not already in the original cluster to obtain a new cluster. The process of removing a cluster variable and replacing it with a new one is known as *mutation.* Fomin and Zelevinsky provide a number of prototypical examples of cluster algebras in the introduction to [30], including the rank 1 cluster algebra $\mathbb{C}[SL_2]$, and a cluster algebra of arbitrary rank m, the homogeneous coordinate ring $\mathbb{C}[Gr_{2,m+3}]$ of the Grassmannian of 2-dimensional subspaces of $\mathbb{C}^{m+3}$.

An important early development in cluster algebras was the classification of finite type cluster algebras, i.e. those with finitely many cluster variables. It was the main focus of [31] where the authors provided a bijection between the set of strong isomorphism classes of finite type cluster algebras and Cartan matrices of finite type.

Cluster Categories

What followed [30] was a relative explosion in interest in cluster algebras, with connections being found in topics such as: the representation theory of algebras and finite dimensional algebras, combinatorics, Poisson geometry, and Teichmüller spaces. What is particularly relevant to our interests was the attempt to *categorify* cluster algebras. This is a loosely defined concept with the aim of being able to study cluster algebras via methods in category theory, primarily achieved by constructing categories that have properties resembling those of a cluster algebra, so-called *cluster categories.* To do this, Buan-Marsh-Reineke-Reiten-Todorov [18] introduce the notion of a *cluster tilting subcategory* (which they call an Ext-configuration), which resembles a cluster in a cluster algebra, with the indecomposable objects in the cluster tilting subcategory being analogous to its cluster variables.

A subcategory $\mathcal{S}$ of a triangulated category $\mathcal{T}$ is cluster tilting if it is functorially finite and $\mathcal{S} = (\mathcal{S}[-1])^{\perp} = {}^{\perp}(\mathcal{S}[1])$, where $[1]$ denotes the suspension functor of $\mathcal{T}$, $[-1]$ is the inverse suspension functor, and $\mathcal{X}^{\perp}$ (resp. ${}^{\perp}\mathcal{X}$) denotes the right (resp. left) 0-perpendicular subcategory for some subcategory $\mathcal{X} \subseteq \mathcal{T}$, i.e.

$$\mathcal{X}^{\perp} = \{Y \in \mathcal{T} \mid \mathrm{Hom}(X, Y) = 0 \text{ for all } X \in \mathcal{X}\},$$
$${}^{\perp}\mathcal{X} = \{Y \in \mathcal{T} \mid \mathrm{Hom}(Y, X) = 0 \text{ for all } X \in \mathcal{X}\}.$$

With the motivation of categorifying cluster algebras in mind, we need to ask for more than just the existence of cluster tilting subcategories, we also wish to somehow mimic the cluster combinatorics present in a cluster algebra. To do this, we ask that a cluster category has what is called a *(weak) cluster structure*, which formalises an analogue of the exchange relations between clusters in a cluster algebra. Cluster structures were introduced by Buan, Iyama, Reiten and Scott [17], and under certain conditions cluster tilting subcategories always form a (weak) cluster structure.

The first cluster categories were introduced simultaneously by Caldero, Chapoton and Schiffler in [19], for cluster algebras of Dynkin type A_n, and by Buan, Marsh, Reineke, Reiten and Todorov in [18] for cluster categories associated to hereditary algebras more generally. The approach for hereditary algebras in general involves taking an orbit category of the bounded derived category of some finite dimensional hereditary algebra $\mathcal{H}$ over a field k,

$$\mathcal{C}_{\mathcal{H}} := \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\,\mathcal{H})/(\tau^{-1} \circ [1]),$$

where τ is the Auslander-Reiten translation and $[1]$ is the suspension functor of $\mathrm{D}^{\mathrm{b}}(\mathrm{mod}\,\mathcal{H})$. The category $\mathcal{C}_{\mathcal{H}}$ is a Hom-finite, k-linear, Krull-Schmidt, 2-Calabi-Yau triangulated category. The cluster variables of the cluster algebra associated to $\mathcal{H}$ are in a one-to-one correspondence with the indecomposable objects of the cluster category associated to $\mathcal{H}$, and the clusters correspond to the cluster-tilting subcategories [18].

Later, Amiot [1] defined a cluster category for any finite dimensional k-algebra A of global dimension ≤ 2, over a field k. This was done by taking the triangulated hull $\mathcal{C}_A$ of the orbit category $\mathrm{D}^{\mathrm{b}}(\mathrm{mod}\,A)/(\nu \circ [-2])$, where ν is a Serre functor, and showing that under certain conditions A induces a cluster tilting subcategory. This general approach also works in constructing a cluster category for Jacobi-finite quivers with potential, and in the case that A is hereditary, this is an equivalent construction to that found in [18].

An interesting phenomenon, noted already in the first work on cluster algebras by Fomin and Zelevinsky [30], is the relationship between cluster algebras/cluster categories and triangulations of marked surfaces. As noted earlier, the homogeneous coordinate ring $\mathbb{C}[Gr_{2,m+3}]$ is a cluster algebra of rank m, however the clusters are also in bijection with the triangulations of an $(m+3)$-gon, and the cluster variables are in bijection with the diagonals between non-adjacent vertices. The mutation of a cluster variable is equivalent to *flipping* the corresponding diagonal in a triangulation, that is removing a diagonal and replacing it with the unique diagonal that makes the new set of diagonals a different triangulation.

Further, these bijections can be seen in the world of cluster categories. Let A_m be a quiver with m vertices with the underlying graph of Dynkin type A, and let $\mathcal{H}$ be isomorphic to the path algebra kA_m. Then the indecomposable objects of $\mathcal{C}_{\mathcal{H}}$ are in bijection with the set of diagonals between non-adjacent vertices on a $(m+3)$-gon, and the cluster tilting subcategories are in bijection with triangulations of the $(m+3)$-gon [19]. We call this category the cluster category of Dynkin type A_m.

Much work has been done in understanding the relationship between cluster algebras/cluster categories and marked surfaces: such as in [20, 21, 28, 29, 51, 52] for cluster algebras, and in [16, 61, 69] for cluster categories.

For an arbitrary orientated Riemann surface with boundary and finitely many marked points on the boundary, an associated cluster algebra was introduced by Fomin, Shapiro and Thurston [28], and later Fomin and Thurston [29]. An explicit description of a cluster category associated to a finitely marked, orientated Riemann surface with boundary was given by Brüstle and Zhang [16] for unpunctured surfaces, whilst a generalisation for punctured surfaces was given by Qiu and Zhou [61].

Interestingly, there is another way to recover cluster categories from a marked surface, by using quivers with potential and the cluster categories introduced by Amiot in [1]. This involves associating a quiver with potential to a triangulation of a marked surface, via a process introduced by Labardini-Fragoso in [47] in an attempt to relate the mutation of quivers with potential introduced by Derksen, Weyman and Zelevinsky [23] and cluster algebras associated to triangulations of surfaces from [28]. It was shown in [47] that the flipping of a diagonal in a triangulation of a surface is equivalent to the Derksen-Weyman-Zelevinsky mutation of a quiver with potential.

The Category $\mathcal{C}_n$

A cluster category of interest to us was introduced in [38] by Holm and Jørgensen, where they consider a polynomial ring $k[T]$ as a differential graded ring, with trivial differential and T placed in homological degree 1. The category $\mathcal{D}$ is the finite derived category

$$\mathcal{D} := \mathrm{D}^{\mathrm{f}}(\mathrm{mod}_{dg}\, k[T]),$$

and $\mathcal{D}$ is a Hom-finite, k-linear, Krull-Schmidt, 2-Calabi-Yau triangulated category. They show that there is a bijection between the indecomposable objects in $\mathcal{D}$ and pairs of integers $(a, b) \in \mathbb{Z}$ such that $b > a + 1$. Further details for $\mathcal{D}$ are given in Section 2.3.

To model this bijection, a combinatorial model is associated to $\mathcal{D}$. Consider a disc with infinitely many marked points, all on the boundary, labelled by the integers. Then the isotopy classes of arcs between marked points that are non-isotopic to boundary segments are in bijection with the indecomposable objects of $\mathcal{D}$, and an Ext^1-space between indecomposable objects is non-trivial if and only if the arcs corresponding to the indecomposable objects cross.

Moreover, Holm and Jørgensen show that a subcategory $\mathcal{S} \subset \mathcal{D}$ is a maximal 1-orthogonal subcategory if and only if the indecomposable objects of $\mathcal{S}$ are in bijection with a triangulation of the ∞-gon [38]. This provides an intuitive understanding of this category as an analogue of the cluster category of Dynkin type A_m as m approaches infinity, and thus we may consider it as a cluster category associated to the quiver of Dynkin type A with infinitely many vertices, A_∞.

This interpretation is further supported by Keller and Reiten, who looked at a category equivalent to $\mathcal{D}$ in [44],

$$\mathcal{D} \simeq \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\, kQ)/(\tau \circ [-1]),$$

where Q is the linearly orientated A_∞^∞-quiver, $\cdots \to \cdot \to \cdot \to \cdot \to \cdots$. In [44], the authors find a cluster tilting subcategory of $\mathcal{D}$.

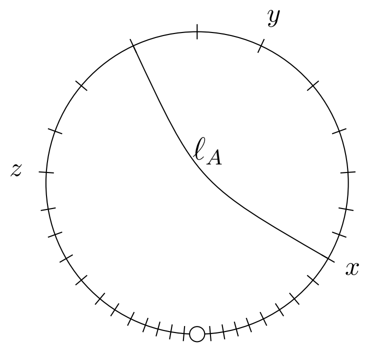


Figure 1: A combinatorial model for the category $\mathcal{D}$, with $x < y < z$ for $x, y, z \in \mathbb{Z}$. The arc ℓ_A corresponds to the indecomposable object $A \in \mathcal{D}$. The circle on the boundary is an *accumulation point*, that is a point that is a limit of marked points from both a clockwise and anti-clockwise direction.

In fact, $\mathcal{D}$ is equivalent as a triangulated category to a category in a larger family, the *discrete cluster categories of Dynkin type* A_∞, labelled here by $\mathcal{C}_n$ for all $n \geq 1$, introduced by Igusa and Todorov in [39]. Namely $\mathcal{D}$ is equivalent to $\mathcal{C}_1$. These categories are introduced as stable categories of Frobenius categories that are built from cyclic posets; in fact this process introduces a much larger class of categories than the ones discussed here, however the main topic of study in this thesis will be the discrete cluster categories of Dynkin type A_∞.

With the combinatorial model of $\mathcal{D} \simeq \mathcal{C}_1$ in mind, we can consider a combinatorial model for the category $\mathcal{C}_n$ in general as an ∞-gon having n accumulation points. Again we have a bijection between the diagonals of an ∞-gon with n accumulation points and the indecomposable objects of $\mathcal{C}_n$, with a maximal 1-orthogonal subcategory corresponding to a maximal set of non-crossing diagonals, i.e. a triangulation of the ∞-gon with n accumulation points.

These categories have been the subject of a lot of interest in recent years: Gratz-Holm-Jørgensen classified their cluster tilting subcategories and torsion pairs [36], Gratz-Zvonareva classified their thick subcategories and t-structures [37], cotorsion pairs of $\mathcal{C}_2$ were classified by Chang-Zhou-Zhu [22], and a correspondence between c-vectors of the discrete cluster categories of Dynkin type A_∞ and roots of the Borel subalgebras of $\mathfrak{sl}_\infty$ was found by Jørgensen and Yakimov [40].

We also consider a completion of the category $\mathcal{C}_n$, introduced by Paquette and Yıldırım in [58]. The Paquette-Yıldırım completion of $\mathcal{C}_n$, denoted $\overline{\mathcal{C}}_n$, is a Hom-finite, Krull-Schmidt, k-linear, triangulated category, constructed by taking a localisation of the category $\mathcal{C}_{2n}$ by a specified thick subcategory. The category $\overline{\mathcal{C}}_1$ is equivalent to the stable category of $\mathbb{Z}$-graded maximal Cohen-Macauley modules over an A_∞ curve singularity by a result of August-Cheung-Faber-Gratz-Schroll [3].

A combinatorial model for $\overline{\mathcal{C}}_n$ can be thought of as an adaptation of the combinatorial model used for $\mathcal{C}_n$, with the adaptation that the accumulation points now act as marked points as well. This means that there is an added class of *limit arcs* that have at least one of their endpoints at an accumulation point. These arcs are not present in the combinatorial model for $\mathcal{C}_n$.

Main results

Our first main result is a computation of the triangulated Grothendieck group of $\mathcal{C}_n$ for all $n \geq 1$. The Grothendieck group of a triangulated category $\mathcal{T}$, denoted $K_0(\mathcal{T})$, is the free abelian group on isomorphism classes of objects, $G_0(\mathcal{T})$, modulo the Euler relations coming from distinguished triangles in $\mathcal{T}$. The Grothendieck group of a triangulated category contains interesting information about the category, including a correspondence between subgroups of the Grothendieck group and the dense subcategories of the triangulated category, shown by Thomason [66].

To compute the Grothendieck group of $\mathcal{C}_n$, we apply a slightly altered version of a result due to Palu [57] (Theorem 3.1.5), that provides an isomorphism between the triangulated Grothendieck group of a 2-Calabi-Yau triangulated category equipped with a cluster-tilting subcategory and the split Grothendieck group of a cluster-tilting subcategory satisfying a certain property, modulo relations coming from the exchange relations of the cluster-tilting subcategory. We use this result to show the following.

Theorem 0.0.1 (3.2.5)**.** *The triangulated Grothendieck group of $\mathcal{C}_n$ is*

$$K_0(\mathcal{C}_n) \cong \mathbb{Z}^n.$$

In Remark 3.2.6, we provide a counterexample to the original statement of the theorem due to Palu, and give an updated statement of the result that is consistent with the proof given in [57]. We show that we may still apply the updated version of Palu's result in our situation.

Next we compute the triangulated Grothendieck group of $\overline{\mathcal{C}}_n$, the Paquette-Yıldırım completion of $\mathcal{C}_n$. It was shown in [58] that $\overline{\mathcal{C}}_n$ is not 2-Calabi-Yau for any $n \geq 1$, and moreover there is no Serre functor for $\overline{\mathcal{C}}_n$. Therefore the same approach to computing the Grothendieck group using Theorem 3.1.5 is not viable. Instead we consider $\overline{\mathcal{C}}_n$ as a localisation of $\mathcal{C}_{2n}$, and show that there is a short exact sequence of categories (or localisation sequence),

$$0 \to \coprod_{i=1}^{n} \mathcal{C}_1 \to \mathcal{C}_{2n} \to \overline{\mathcal{C}}_n \to 0.$$

By applying the functor $K_0(-)$ to the above sequence, we obtain the following result.

Theorem 0.0.2 (3.2.9)**.** *The triangulated Grothendieck group of $\overline{\mathcal{C}}_n$ is*

$$K_0(\overline{\mathcal{C}}_n) \cong \mathbb{Z}^n \oplus (\mathbb{Z}/2\mathbb{Z})^{n-1}.$$

The second half of this thesis is a study of the classical generators of the categories

$\overline{\mathcal{C}}_n$. In particular we obtain a classification of the classical generators of $\overline{\mathcal{C}}_n$ and provide an upper bound on their generation time.

A *classical generator* of a triangulated category $\mathcal{T}$ is an object G (or, more generally, a subcategory) such that the smallest thick subcategory of $\mathcal{T}$ that contains G is the category $\mathcal{T}$ itself. Classical generators of triangulated categories can be very useful in understanding the category as a whole, and can be used to induce equivalences of algebraic triangulated categories, as shown by Keller [41].

We classify the classical generators of $\overline{\mathcal{C}}_n$, and provide an upper bound on the generation time for each classical generator, thereby providing an upper bound for the Orlov spectrum of $\overline{\mathcal{C}}$. To do this we introduce two conditions on objects in $\overline{\mathcal{C}}_n$.

For an object $G \in \overline{\mathcal{C}}_n$, let $\langle G \rangle_1$ be the smallest full subcategory of $\overline{\mathcal{C}}_n$ containing G that is closed under direct sums, direct summands, suspension, and desuspension. We say that an object G is *homologically connected* if for any two indecomposable objects $X, Y \in \langle G \rangle_1$ there is a sequence of morphisms of degree 1 between indecomposable objects in $\langle G \rangle_1$ starting and ending at X and Y. Such a sequence is called a *zig-zag*, and we say that the number of morphisms of degree 1 in a zig-zag is the *length* of the zig-zag. A zig-zag between indecomposable objects X and Y is *minimal* if there exists no zig-zag between X and Y with a smaller length. Moreover, the supremum of lengths of minimal zig-zags in $\langle G \rangle_1$ is known as the *homological length of G*.

We show that any object in $\overline{\mathcal{C}}_n$ may be decomposed into homologically connected direct summands. Such a decomposition of an object is called a *hc (= homologically connected) decomposition of G*.

For an object $G \in \overline{\mathcal{C}}_n$, let $\mathscr{M}_G$ denoted the subset of $\mathscr{M}$, consisting of endpoints of arcs corresponding to indecomposable objects in $\langle G \rangle_1$. We say $\mathscr{M}_G$ is the *orbit of G in $\mathscr{M}$* and moreover, that G has a *complete orbit in $\mathscr{M}$* if $\mathscr{M} = \mathscr{M}_G$.

We also show that for any object G, the hc decomposition determines $\langle G \rangle$, the *thick closure* of G, i.e. the smallest thick subcategory of $\overline{\mathcal{C}}_n$ containing G. To do this, we incorporate a classification of thick subcategories of $\mathcal{C}_n$ by Gratz and Zvonareva [37], who prove that there is an isomorphism of lattices between the thick subcategories of $\mathcal{C}_n$ and non-exhaustive non-crossing partitions of $[n] = \{1, \ldots, n\}$. We prove that the orbit of objects in the hc decomposition of an object in $\mathcal{C}_n$ corresponds to a non-exhaustive non-crossing partition of $[n]$, and so determines the thick closure of the object.

Further, we show that a thick subcategory in $\overline{\mathcal{C}}_n$ is the essential image of a thick subcategory in $\mathcal{C}_{2n}$ under the localisation functor $\pi : \mathcal{C}_{2n} \to \overline{\mathcal{C}}_n$. Therefore, the thick closure of an object G in $\overline{\mathcal{C}}_n$ is determined by the orbits in $\mathscr{M}$ of the objects in the hc decomposition of G.

Subsequently, we show that a classical generator of $\overline{\mathcal{C}}_n$ must be homologically connected, and combine this with the previous result to classify the classical generators of $\overline{\mathcal{C}}_n$.

Theorem 0.0.3 (4.2.11)**.** *Let $G = \bigoplus_{i=1}^{m} G_i$ be an object in $\overline{\mathcal{C}}_n$, with G_i all indecompos-*

able. Then G is a classical generator of $\overline{\mathcal{C}}_n$ if and only if G is homologically connected and G has a complete orbit in $\mathscr{M}$.

An important thing we wish to consider with classical generators is how quickly they can generate a category, that is, how many times we need to close under cones to go from the additive hull $\operatorname{add}(G)$ of G to the triangulated category $\mathcal{T}$. We call this the *generation time of* G, and the set of generation times for all classical generators of $\mathcal{T}$ is called the *Orlov spectrum of* $\mathcal{T}$, denoted $\mathcal{O}(\mathcal{T})$ [56]. The infimum of the Orlov spectrum is known as the *Rouquier dimension of* $\mathcal{T}$ [62].

It is in general a difficult task to compute the Orlov spectrum of a triangulated category, and it is still an open question under what conditions the Orlov spectrum of a triangulated category forms an integer interval. There is also interest in finding the upper and lower bounds of the Orlov spectrum of a triangulated category, with work by Elagin-Lunts and Rouquier [24, 62].

We show that the homological length of a classical generator G is an upper limit on the generation time of G. We do this by showing that an indecomposable object M in $\overline{\mathcal{C}}_n$ is in $\langle G\rangle_{l+1}$ if there exists a minimal zig-zag of length l in $\langle G\rangle_1$ between X and Y, such that the arc corresponding to M shares an endpoint with the arc corresponding to X, and the other endpoint with the arc corresponding to Y. Therefore the homological length, the supremum of lengths of minimal zig-zags, is an upper bound on the generation time of G.

We then show that in $\overline{\mathcal{C}}_n$ there is a classical generator with homological length l for all $l \in \{1, \dots, 2n-2\}$. Moreover, we show that no classical generator has homological length $\geq 2n-1$, and so the generation time of any classical generator in $\overline{\mathcal{C}}_n$ has an upper bound of $2n-2$.

Theorem 0.0.4 (4.4.11)**.** *The Orlov spectrum of $\overline{\mathcal{C}}_n$ for $n \geq 2$ is bounded above by $2n-2$. That is*

$$\mathcal{O}(\overline{\mathcal{C}}_n) \subseteq \{1, \dots, 2n-2\}.$$

Moreover, the Rouquier dimension of $\overline{\mathcal{C}}_n$ is 1.

We also take a closer look at a particular classical generator E of $\overline{\mathcal{C}}_n$ and compute its generation time and graded endomorphism ring. In particular, E has a generation time 1, which implies the second part of Theorem 0.0.4.

We show that the graded endomorphism ring, $\operatorname{End}^*(E)$, is isomorphic to an upper triangular $(2n-1)\times(2n-1)$ matrix ring $R_n^* = \{b_{ij}\}_{1\leq i,j\leq 2n-1}$ where,

$$b_{ij} \cong \begin{cases} k[x] & \text{if } i = j \text{ and } i \text{ is odd,} \\ k[x^{\pm 1}] & \text{if } i < j, \text{ or } i = j \text{ and } i \text{ is even,} \\ 0 & \text{if } j < i, \end{cases}$$

and we have imposed a grading on R_n^* such that x is concentrated in degree -1.

Theorem 0.0.5 (4.3.10). *Let $E \in \overline{\mathcal{C}}_n$ be the classical generator in Figure 4.5. Then there is an isomorphism of graded rings*

$$\mathrm{End}^*_{\overline{\mathcal{C}}_n}(E) \cong R^*_n.$$

We construct a basis for the graded endomorphism ring $\mathrm{End}^*_{\overline{\mathcal{C}}_n}(E)$ and for the ring R^*_n, and show that there is a bijection between these bases. Further, we show that this bijection preserves the multiplication of basis elements, and so is an isomorphism between $\mathrm{End}^*_{\overline{\mathcal{C}}_n}(E)$ and R^*_n.

Chapter 1

Preliminaries

Throughout we let k be an algebraically closed field, and we consider modules as left modules.

§ 1.1 | Triangulated Categories

Triangulated categories were developed in the early 1960's independently by Puppe [59] and by Verdier in his PhD thesis, which was published much later in 1996 [67]. Puppe's approach involved primarily using homotopy categories as the main example, and defines what we today would call a *pre-triangulated category* (Definition 1.1.2). Verdier used as his main example the derived categories that had also been defined in [67], alongside his supervisor, Grothendieck.

Since then triangulated categories have become ubiquitous across many areas of mathematics. They feature prominently in areas such as representation theory, algebraic topology and algebraic geometry, but also have far reaching applications to areas like analysis and C^*-algebras.

In this section we go over some definitions, building up from candidate triangles and pre-triangulated categories, through to triangulated categories and the octahedral axiom. We then look at functors between triangulated categories and some basic properties of these functors. Throughout we follow the first two chapters of Neeman's book on the subject [55].

§ 1.1.1 | Definitions

We begin with one of the fundamental building blocks of triangulated categories, the candidate triangles.

Definition 1.1.1. [55] Let $\mathcal{S}$ be an additive category and $[1]: \mathcal{S} \to \mathcal{S}$ be an additive endofunctor of $\mathcal{S}$. We assume that $[1]$ has an inverse, denoted by $[-1]$. A *candidate triangle* in $\mathcal{S}$ (with respect to $[1]$) is a diagram of the form:

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

such that the composites $v \circ u$, $w \circ v$ and $u[1] \circ w$ are the zero morphisms. A *morphism of candidate triangles* is a set of morphisms $f = (f_1, f_2, f_3)$ such that the following diagram with candidate triangles in each row commutes:

$$\begin{CD} X @>u>> Y @>v>> Z @>w>> X[1] \\ @VV{f_1}V @VV{f_2}V @VV{f_3}V @VV{f_1[1]}V \\ X' @>u'>> Y' @>v'>> Z' @>w'>> X'[1]. \end{CD}$$

Definition 1.1.2. [55] A *pre-triangulated category* is an additive category, together with an additive automorphism $[1]$, and a class of candidate triangles (with respect to $[1]$) called *distinguished triangles*. The following definitions must hold:

TR0 Any candidate triangle which is isomorphic to a distinguished triangle is a distinguished triangle. The candidate triangle

$$X \xrightarrow{1} X \to 0 \to X[1]$$

is a distinguished triangle.

TR1 For any morphism $f : X \to Y$ there exists a distinguished triangle of the form

$$X \xrightarrow{f} Y \to Z \to X[1].$$

TR2 Consider the two candidate triangles

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

and

$$Y \xrightarrow{-v} Z \xrightarrow{-w} X[1] \xrightarrow{-u[1]} Y[1].$$

If one is a distinguished triangle, then so is the other one.

TR3 For any commutative diagram of the form

$$\begin{CD} X @>u>> Y @>v>> Z @>w>> X[1] \\ @VV{f}V @VV{g}V \\ X' @>u'>> Y' @>v'>> Z' @>w'>> X'[1] \end{CD}$$

where the rows are distinguished triangles, there is a morphism $h : Z \to Z'$, not necessarily unique, which makes the diagram

$$\begin{CD} X @>u>> Y @>v>> Z @>w>> X[1] \\ @VV{f}V @VV{g}V @VV{h}V @VV{f[1]}V \\ X' @>u'>> Y' @>v'>> Z' @>w'>> X'[1] \end{CD}$$

commutative.

When we are discussing triangles in a (pre-)triangulated category, we shall be referring to the class of distinguished triangles in said category. Not all candidate triangles are distinguished triangles, as shown by the following proposition.

Proposition 1.1.3. *[55, Proposition 1.1.20] Suppose there exists a morphism of triangles*

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f[1]} \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & X'[1] \end{array}$$

if f and g are isomorphisms, then so is h.

If we consider two candidate triangles

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1] \tag{1.1}$$

$$X \xrightarrow{u} Y \xrightarrow{v'} Z \oplus W \xrightarrow{w'} X[1] \tag{1.2}$$

where (1.1) is a distinguished triangle, then by Proposition 1.1.3 (1.2) is a distinguished triangle if and only if $W = 0$.

Given two candidate triangles and a morphism of triangles f between them we would like to be able to construct a new candidate triangle from this data. We call this new candidate triangle the mapping cone of f. A pre-triangulated category $\mathcal{T}$ is a *triangulated category* if, given a two of the three morphisms in a morphism of triangles, there is a suitable choice of the third morphism such that the mapping cone is a distinguished triangle.

Definition 1.1.4. [55] Let $\mathcal{T}$ be a pre-triangulated category. Suppose we are given a morphism of candidate triangles

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f[1]} \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & X'[1] \end{array}$$

There is a way to form a new candidate triangle out of this data

$$Y \oplus X' \xrightarrow{g'} Z \oplus Y' \xrightarrow{h'} X[1] \oplus Z' \xrightarrow{f'[1]} Y[1] \oplus X'[1]$$

where

$$g' = \begin{pmatrix} -v & 0 \\ g & u' \end{pmatrix}, \ h' = \begin{pmatrix} -w & 0 \\ h & v' \end{pmatrix}, \ f'[1] = \begin{pmatrix} -u[1] & 0 \\ f[1] & w' \end{pmatrix}.$$

This new candidate triangle is called the *mapping cone* on a map of candidate triangles.

Definition 1.1.5. [55] Let $\mathcal{T}$ be a pre-triangulated category. Then $\mathcal{T}$ is a *triangulated category* if it satisfies the following condition.

TR4' Given a diagram

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] & X[1] \arrow[d, "f[1]"] \\
X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & X'[1]
\end{tikzcd}$$

where the rows are triangles, there is, by TR3, a way to choose $h\colon Z \to Z'$ to make the diagram commutative. This h may be chosen so that the mapping cone

$$Y \oplus X' \xrightarrow{g'} Z \oplus Y' \xrightarrow{h'} X[1] \oplus Z' \xrightarrow{f'[1]} Y[1] \oplus X'[1]$$

is a triangle.

In a triangulated category, the endofunctor $[1]$ is called the *suspension functor*, and its inverse $[-1]$, is the *desuspension functor*. When we consider objects up to suspension, we mean that we consider objects up to both suspension and desuspension.

The following proposition may be regarded as a replacement for TR4' in the axioms of a triangulated category, as they have been shown to be equivalent as axioms by Proposition 1.1.6 and Theorem 1.8 of [54].

Proposition 1.1.6. *[55, Proposition 1.4.6] Let $\mathcal{T}$ be a triangulated category. Let $f : X \to Y$ and $g : Y \to Y'$ be composable morphisms, with the triangles:*

$$X \xrightarrow{f} Y \to Z \to X[1]$$
$$X \xrightarrow{gf} Y' \to Z' \to X[1]$$
$$Y \xrightarrow{g} Y' \to Y'' \to Y[1]$$

Then we can complete this to the commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\sim"] & Y \arrow[r] \arrow[d, "g"] & Z \arrow[r] \arrow[d] & X[1] \arrow[d, "\sim"] \\
X \arrow[r, "gf"] \arrow[d] & Y' \arrow[r] \arrow[d] & Z' \arrow[r] \arrow[d] & X[1] \arrow[d] \\
0 \arrow[r] \arrow[d] & Y'' \arrow[r, "\sim"] \arrow[d] & Y'' \arrow[r] \arrow[d] & 0 \arrow[d] \\
X[1] \arrow[r, "f[1]"] & Y[1] \arrow[r] & Z[1] \arrow[r] & X[2]
\end{tikzcd}$$

where the first and second rows and the second column are the given three triangles, and every row and column is a distinguished triangle.

Proposition 1.1.6 is known as TR4 or as the *Octahedral Axiom*, or occasionally as *Verdier's axiom* in homotopy theory.

There is an abundance of examples of triangulated categories throughout mathematics; for instance the stable homotopy category of spectra, the derived category of a ring, and more generally the stable category of a Frobenius category, are all triangulated categories. If there is an equivalence of triangulated categories between a triangulated category $\mathcal{T}$ and the stable category of a Frobenius category, then we say that $\mathcal{T}$ is an *algebraic* triangulated category. The derived category of a ring will be discussed later in Section 1.4.

There are often times when we may not wish to look at a triangulated category as a whole, but only look at a subcategory whilst still retaining the triangulated structure, for that we need to look to triangulated subcategories.

Definition 1.1.7. Let $\mathcal{T}$ be a triangulated category. A full additive subcategory $\mathcal{S}$ of $\mathcal{T}$ is called a *triangulated subcategory* if every object isomorphic to an object in $\mathcal{S}$ is in $\mathcal{S}$, if $\mathcal{S}[1] = \mathcal{S}$, and if for any distinguished triangle

$$X \to Y \to Z \to X[1]$$

with X and Y in $\mathcal{S}$, then Z is in $\mathcal{S}$ too.

It follows from TR2 that it is sufficient for any two of X, Y or Z to be in a triangulated subcategory $\mathcal{S}$ for the third object to also be in $\mathcal{S}$.

A particular type of triangulated subcategories that we wish to consider are thick subcategories. We say that a triangulated subcategory $\mathcal{S}$ of $\mathcal{T}$ is a *thick subcategory* if it is closed under direct summands.

§ 1.1.2 | Triangulated Functors

There is a notion of a functor between two triangulated categories such that a triangle in one category is sent to a triangle in the other category. These are known as triangulated functors.

Definition 1.1.8. Let $\mathcal{T}_1, \mathcal{T}_2$ be triangulated categories. A *triangulated functor* is an additive functor $F\colon \mathcal{T}_1 \to \mathcal{T}_2$ together with natural isomorphisms

$$\phi_X : F(X[1]) \to (F(X))[1]$$

such that for any distinguished triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

in $\mathcal{T}_1$, the candidate triangle

$$F(X) \xrightarrow{F(u)} F(Y) \xrightarrow{F(v)} F(Z) \xrightarrow{\phi_X \circ F(w)} (F(X))[1]$$

is a distinguished triangle in $\mathcal{T}_2$.

When we talk about the Grothendieck group of the Paquette-Yıldırım completion of the discrete cluster cluster category of Dynkin type A_∞, we will need to consider the kernel of a triangulated functor. This is a well defined concept that we give the definition of, as well as presenting a few elementary results concerning the kernel of a triangulated functor.

Definition 1.1.9. Let $F\colon \mathcal{T} \to \mathcal{D}$ be a triangulated functor. The *kernel* of F is defined to be the full subcategory of $\mathcal{S}$ of $\mathcal{T}$ whose objects map to objects in $\mathcal{D}$ isomorphic to 0. That is,

$$\mathcal{S} = \{X \in \mathcal{T} \mid F(X) \text{ is isomorphic to } 0\}.$$

Notably, the kernel of a triangulated functor is again triangulated and closed under direct summands, i.e. the kernel is a thick subcategory.

Lemma 1.1.10 ([55])**.** *Let $F\colon \mathcal{T} \to \mathcal{D}$ be a triangulated functor. Then the kernel $\mathcal{S}$ of F is a triangulated subcategory of $\mathcal{T}$ and closed under direct summands.*

Proof. Let $X \in \mathcal{T}$ be in the kernel of F, then so is $X[1]$, as $F(X) \cong 0$ if and only if $F(X[1]) = (F(X))[1] \cong 0$. If

$$X \to Y \to Z \to X[1]$$

is a distinguished triangle in $\mathcal{T}$, then

$$F(X) \to F(Y) \to F(Z) \to (F(X))[1]$$

is in $\mathcal{D}$. If $F(X)$ and $F(Y)$ are isomorphic to 0, then by TR3 in Definition 1.1.2, the above triangle is isomorphic to

$$0 \to 0 \to 0 \to 0,$$

in particular $F(Z) \cong 0$. Thus, if $X, Y \in \mathcal{S}$, then so is Z, and so $\mathcal{S}$ is a triangulated subcategory.

To show that $\mathcal{S}$ is closed under direct summands, consider that F is an additive functor, then $F(X \oplus Y) = F(X) \oplus F(Y)$. But if $F(X \oplus Y)$ is isomorphic to 0, then so are $F(X)$ and $F(Y)$ since they are direct summands of 0. □

The next theorem is due essentially to Verdier (he proves it for thick subcategories) and forms the basis of Verdier localisation, which we will later see is analogous to the localisation of a category by a given subcategory. This version of the theorem is found in [55] as Theorem 2.1.8.

Theorem 1.1.11. *Let $\mathcal{T}$ be a triangulated category with $\mathcal{S}$ a triangulated subcategory. Then there is a triangulated category $\mathcal{T}/\mathcal{S}$, and a triangulated functor $F : \mathcal{T} \to \mathcal{T}/\mathcal{S}$ such that $\mathcal{S}$ is the kernel of F, and F is universal with this property. If $G : \mathcal{T} \to \mathcal{D}$ is a triangulated functor whose kernel contains $\mathcal{S}$, then it factors uniquely as*

$$\mathcal{T} \xrightarrow{F} \mathcal{T}/\mathcal{S} \to \mathcal{D}.$$

Given an autoequivalence $F\colon \mathcal{T} \to \mathcal{T}$, we may construct the *orbit category of $\mathcal{T}$ with respect to F* [42]. That is, the category $\mathcal{T}/F$ with the same objects as $\mathcal{T}$, and morphisms from X to Y are in bijection with

$$\mathrm{Hom}_{\mathcal{T}/F}(X, Y) := \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{T}}(X, F^n Y).$$

Let $\mathcal{H}$ be an hereditary, abelian category, and let $\mathcal{T} \simeq \mathrm{D^b}(\mathcal{H})$ with an autoequivalence G. Suppose that for each indecomposable object X in $\mathcal{H}$, then only finitely many objects $G^n X$, $i \in \mathbb{Z}$, are in $\mathcal{H}$. Further, suppose there exists an $N \geq 0$ such that each G-orbit of an indecomposable object in $\mathcal{T}$ contains some object $X[n]$ for $X \in \mathcal{H}$ and $0 \leq n \leq N$. Then it was shown by Keller in [42] that the orbit category $\mathcal{T}/G$ is canonically triangulated.

§ 1.1.3 | The Serre Functor

Let $\mathcal{T}$ be a k-linear, Hom-finite triangulated category with suspension functor $[1]$. A *right Serre functor* for $\mathcal{T}$ [42] is a triangulated functor $\phi\colon \mathcal{T} \to \mathcal{T}$, together with bifunctor isomorphisms

$$D\,\mathrm{Hom}_{\mathcal{T}}(X, -) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{T}}(-, \phi X),\ X \in \mathcal{T},$$

where $D = \mathrm{Hom}_k(-, k)$. The functor ϕ is unique up to isomorphism of triangles, if it exists. A *left Serre functor* for $\mathcal{T}$ is a triangulated functor $\phi'\colon \mathcal{T} \to \mathcal{T}$, and isomorphisms

$$D\,\mathrm{Hom}_{\mathcal{T}}(-, X) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{T}}(\phi' X, -),\ X \in \mathcal{T}.$$

If a triangulated category $\mathcal{T}$ has both a left and right Serre functor, we say it has *Serre duality*. Equivalently, if ϕ or ϕ' is an equivalence of categories, then $\mathcal{T}$ has Serre duality [42].

Let $\mathcal{T}$ be a k-linear, Hom-finite triangulated category with suspension functor $[1]$, and Serre duality with right Serre functor ϕ. Then we say that $\mathcal{T}$ is *Calabi-Yau of dimension d* (or, *d-Calabi-Yau*) [42], if there exists an isomorphism of triangulated functors,

$$\phi \xrightarrow{\sim} [d].$$

That is, there is an isomorphism $\mathrm{Hom}_{\mathcal{T}}(X, Y) \cong D\,\mathrm{Hom}_{\mathcal{T}}(Y, X[d])$, for all objects $X, Y \in \mathcal{T}$.

§ 1.2 | Localisation of Triangulated Categories

In this section we discuss some of the various localisations of triangulated categories and how they compare to one another. We will see an example of a localisation of a triangulated category in Section 2.4 when we talk about the Paquette-Yıldırım completion of the discrete cluster category of Dynkin type A_∞.

We primarily follow the work of Krause in [45].

Verdier Localisation

One way of localising a subcategory is by taking a set of morphisms and formally inverting them.

Let $F\colon \mathcal{D} \to \mathcal{C}$ be a functor. We say that F makes a morphism σ in $\mathcal{D}$ *invertible* if $F(\sigma)$ is invertible. The set of morphisms made invertible by F is denoted $\Omega(F)$.

Given a category $\mathcal{D}$ and a set Ω of morphisms in $\mathcal{D}$, we consider the *localisation of* $\mathcal{D}$ *with respect to* Ω, $\mathcal{D}[\Omega^{-1}]$ together with a canonical *localisation functor* $\mathcal{Q}_\Omega$,

$$\mathcal{Q}_\Omega\colon \mathcal{D} \longrightarrow \mathcal{D}[\Omega^{-1}]$$

which has the following properties;

- $\mathcal{Q}_\Omega$ makes the morphisms in Ω invertible,
- if a functor $F\colon \mathcal{D} \to \mathcal{C}$ makes the morphisms in Ω invertible, then there is a unique functor $\overline{F}\colon \mathcal{D}[\Omega^{-1}] \to \mathcal{C}$ such that $F = \overline{F} \circ \mathcal{Q}_\Omega$.

Both $\mathcal{D}[\Omega^{-1}]$ and $\mathcal{Q}_\Omega$ are essentially unique if they exist. The category $\mathcal{D}[\Omega^{-1}]$ has the same objects as $\mathcal{D}$. Morphisms in $\mathcal{D}[\Omega^{-1}]$ can be defined by taking the quiver Q with the set of vertices $\operatorname{Ob}\mathcal{D}$, and the set of arrows the disjoint union $\operatorname{Mor}\mathcal{D} \sqcup \Omega^{-1}$, where $\Omega^{-1} = \{\sigma^{-1}\colon Y \to X \mid \sigma\colon X \to Y \in \Omega\}$. Let $\mathcal{P}$ be the set of paths in Q, together with concatenation denoted by $\circ_{\mathcal{P}}$. Then $\operatorname{Mor}\mathcal{D}[\Omega^{-1}]$ is defined as the set of equivalence relations generated by the relations:

- $\alpha \circ_{\mathcal{P}} \beta = \alpha \circ \beta$ for all composable morphisms $\alpha, \beta \in \operatorname{Mor}\mathcal{D}$,
- $\mathrm{id}_{\mathcal{P}} X = \mathrm{id}_{\mathcal{D}} X$ for all $X \in \operatorname{Ob}\mathcal{D}$,
- $\sigma^{-1} \circ_{\mathcal{P}} \sigma = \mathrm{id}_{\mathcal{P}} X$ and $\sigma \circ_{\mathcal{P}} \sigma^{-1} = \mathrm{id}_{\mathcal{P}} Y$ for all $\sigma\colon X \to Y$ in Ω.

Composition of morphisms in $\mathcal{D}[\Omega^{-1}]$ is induced by composition in $\mathcal{P}$.

The functor Q_Ω is the identity on objects and on morphisms is the composition

$$\operatorname{Mor}\mathcal{D} \xrightarrow{\text{inc.}} \operatorname{Mor}\mathcal{D} \sqcup \Omega^{-1} \xrightarrow{\text{inc.}} \mathcal{P} \xrightarrow{\text{can.}} \mathcal{D}[\Omega^{-1}].$$

Let $\mathcal{C}$ be a triangulated subcategory of a triangulated category $\mathcal{D}$, and denote by $\Omega(\mathcal{C})$ the set of morphisms in $\mathcal{T}$ such that $f \in \Omega(\mathcal{C})$ if Z in the triangle

$$X \xrightarrow{f} Y \to Z \to X[1]$$

is in $\mathcal{C}$ [45, Lemma 4.6.1]. Then the *Verdier localisation of* $\mathcal{D}$ *by* $\mathcal{C}$ is defined as the localisation of $\mathcal{D}$ by $\Omega(\mathcal{C})$,

$$\mathcal{D}/\mathcal{C} := \mathcal{D}[\Omega(\mathcal{C})^{-1}].$$

Here $\mathcal{D}/\mathcal{C}$ is the same category as in Theorem 1.1.11.

Calculus of Fractions

Another way to localise a category by a set of morphisms satisfying certain properties, is to define a new category with the same objects as the original category, but where morphisms are replaced by equivalence classes of left fractions.

Definition 1.2.1. Let $\mathcal{D}$ be a category, and let Ω be a collection of maps in $\mathcal{D}$. We say Ω admits a *calculus of left fractions* if it satisfies:

- if τ, σ are composable morphisms in Ω, then $\tau \circ \sigma$ is in Ω, and the identity morphism is in Ω for all objects in $\mathcal{D}$,
- each pair of morphisms $X' \overset{\sigma}{\leftarrow} X \overset{\alpha}{\rightarrow} Y$, with σ in Ω can be completed to a commutative diagram
$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "\sigma"] & Y \arrow[d, "\sigma'"] \\ X' \arrow[r, "\alpha'"] & Y' \end{tikzcd}$$
such that σ' is in Ω,
- let $\alpha, \beta \colon X \to Y$ be morphisms in $\mathcal{D}$. If there is a morphism $\sigma \colon X' \to X$ in Ω such that $\alpha \circ \sigma = \beta \circ \sigma$, then there exists a morphism $\tau \colon Y \to Y'$ in Ω such that $\tau \circ \alpha = \tau \circ \beta$.

For Ω to admit a *calculus of right fractions* is defined dually, and we say that Ω admits a *multiplicative system* if it admits both a calculus of right fractions and a calculus of left fractions.

Given a collection of maps Ω that admits a calculus of left fraction, we follow [45] in defining a new category $\Omega^{-1}\mathcal{D}$ as follows. The objects are those of $\mathcal{D}$. Given two objects X and Y in $\Omega^{-1}\mathcal{D}$, we call a pair (τ, σ) of morphisms

$$X \overset{\tau}{\rightarrow} Y' \overset{\sigma}{\leftarrow} Y$$

in $\mathcal{D}$ with $\sigma \in \Omega$ a *left fraction*. The morphisms in $\Omega^{-1}\mathcal{D}$ are equivalence classes $[\tau, \sigma]$ of left fractions, where (τ_1, σ_1) and (τ_2, σ_2) are equivalent if there is a third left fraction (τ_3, σ_3) such that there is a commutative diagram,

$$\begin{tikzcd} & Y_1 \arrow[d] & \\ X \arrow[ur, "\tau_1"] \arrow[r, "\tau_3"] \arrow[dr, "\tau_2"'] & Y_3 & Y \arrow[ul, "\sigma_1"'] \arrow[l, "\sigma_3"'] \arrow[dl, "\sigma_2"] \\ & Y_2 \arrow[u] & \end{tikzcd}$$

with $\sigma_3 \in \Omega$.

The composition of equivalence classes $[\tau, \sigma]$ and $[\alpha, \beta]$ is by definition the equivalence class $[\alpha' \circ \tau, \sigma' \circ \beta]$, forming the following commutative diagram.

$$\begin{tikzcd}
 & & Z'' & & \\
 & Y' \arrow[ur, "\beta'"] & & Z' \arrow[ul, "\sigma'"'] & \\
X \arrow[ur, "\tau"] & & Y \arrow[ul, "\sigma"'] \arrow[ur, "\alpha"] & & Z \arrow[ul, "\beta"']
\end{tikzcd}$$

The morphisms σ' and β' are those obtained from the second condition of Definition 1.2.1. There exists a canonical functor

$$\mathcal{R}_\Omega \colon \mathcal{D} \to \Omega^{-1}\mathcal{D}$$

that is the identity on objects, and takes a morphism $\alpha\colon X \to Y$ to the equivalence class $[\alpha, \mathrm{id}_Y]$.

Both of the above constructions of a localisation of a triangulated category are in fact equivalent.

Proposition 1.2.2 ([45])**.** *The functor $F\colon \Omega^{-1}\mathcal{D} \to \mathcal{D}[\Omega^{-1}]$, which is the identity on objects and takes a morphism $[\alpha, \sigma]$ to $(Q_\Omega \sigma)^{-1} \circ Q_\Omega \alpha$ is an isomorphism.*

Now let $\mathcal{T}$ be a triangulated category and let Ω be a multiplicative system. Then Ω is *compatible with the triangulated structure of* $\mathcal{T}$ if

- given $\omega \in \Omega$, then $\omega[n] \in \Omega$ for all $n \in \mathbb{Z}$,
- given a morphism of triangles $\phi = (\phi_1, \phi_2, \phi_3)$ with ϕ_1 and ϕ_2 in Ω, then there exists a morphism $(\phi_1, \phi_2, \phi_3')$ with ϕ_3' in Ω.

The next lemma is Lemma 4.3.1 in [45].

Lemma 1.2.3. *Let $\mathcal{T}$ be a triangulated category and let Ω be a multiplicative system that is compatible with the triangulation of $\mathcal{T}$. Then the localisation $\mathcal{T}[\Omega^{-1}]$ carries a unique triangulated structure such that the localisation functor $\mathcal{T} \to \mathcal{T}[\Omega^{-1}]$ is exact.*

That is, the localisation of a triangulated category is canonically a triangulated category, and the localisation functor a triangulated functor. Moreover, let $\mathcal{S}$ be a thick subcategory of $\mathcal{T}$, and let $\Omega(\mathcal{S})$ be the collection of morphisms such that $f \in \Omega(\mathcal{S})$ if $\operatorname{cone} f \in \mathcal{S}$. Then $\Omega(\mathcal{S})$ admits a multiplicative system [67], and so the localisation $\mathcal{T}/\mathcal{S}$ is well defined for any thick subcategory $\mathcal{S}$.

§ 1.3 | Differential Graded Algebras

§ 1.3.1 | Chain Complexes

Definition 1.3.1. Let $\mathcal{A}$ be an additive category. A *chain complex* in $\mathcal{A}$ is a collection of objects $(A_i)_{i\in\mathbb{Z}} \in \mathcal{A}$, along with differentials $d_n^A\colon A_n \to A_{n-1}$ such that $d_n^A \circ d_{n+1}^A = 0$

for all $n \in \mathbb{Z}$.

Given two chain complexes, A and B, a morphism between them is a collection of maps $f_n \colon A_n \to B_n$ for all $n \in \mathbb{Z}$ such that the diagram

$$\begin{tikzcd} A_n \arrow[r, "d_n^A"] \arrow[d, "f_n"] & A_{n-1} \arrow[d, "f_{n-1}"] \\ B_n \arrow[r, "d_n^B"] & B_{n-1} \end{tikzcd}$$

commutes. We denote the category of chain complexes over $\mathcal{A}$ by $\mathrm{Ch}(\mathcal{A})$.

We say that a chain complex A is bounded below if $A_i = 0$ for all $i \leq n$ for some $n \in \mathbb{Z}$. Similarly, we say that a chain complex B is bounded above if $B_i = 0$ for all $i \geq m$ for some $m \in \mathbb{Z}$. If a chain complex is both bounded above and bounded below, then we simply say it is bounded. The subcategory of $\mathrm{Ch}(\mathcal{A})$ made up of bounded (resp. bounded above, resp. bounded below) chain complexes is denoted $\mathrm{Ch}^b(\mathcal{A})$ (resp. $\mathrm{Ch}^-(\mathcal{A})$, resp. $\mathrm{Ch}^+(\mathcal{A})$).

There exists a dual notion of a chain complex in $\mathcal{A}$, called a *cochain complex*, which is a collection of objects $(C^i)_{i\in\mathbb{Z}} \in \mathcal{A}$ with a differential $d_C^n \colon C^n \to C^{n+1}$.

Example 1.3.2. Let $\mathcal{A}$ be the category of modules over the polynomial ring in one variable, $\mathrm{mod}(k[x])$. Then we can consider the chain of modules

$$\cdots \to k[x]/(x^2) \xrightarrow{x} k[x]/(x^2) \xrightarrow{x} k[x]/(x^2) \xrightarrow{x} k[x]/(x^2) \xrightarrow{x} k[x]/(x^2) \to \cdots .$$

The map $d = x$ is a differential as $x^2 = 0$, and so this is a chain complex in $\mathcal{A}$. Additionally, we could also look at the chain complex

$$\cdots \to k[x]/(x^2) \xrightarrow{0} k[x]/(x^2) \xrightarrow{x} k[x]/(x^2) \xrightarrow{0} k[x]/(x^2) \xrightarrow{x} k[x]/(x^2) \to \cdots ,$$

which is isomorphic in each degree to the first chain complex, but has a different differential. This different differentials mean that these chain complexes are non-isomorphic as chain complexes, even if they are isomorphic in each degree.

If we endow the category of chain complexes over an abelian category $\mathcal{A}$ with a class of pairs (i, p) such that (i_n, p_n) is a split short exact sequence for all $n \in \mathbb{Z}$, then $\mathrm{Ch}(\mathcal{A})$ is an exact category. Moreover, for some $A \in \mathrm{Ch}(\mathcal{A})$, we may define the object $IA \in \mathrm{Ch}(\mathcal{A})$ such that

$$(IA)_n := A_n \oplus A_{n+1}, \;\; d_n^{IA} = \begin{pmatrix} 0 & \mathbf{1} \\ 0 & 0 \end{pmatrix}.$$

The object IA is both projective and injective for all $A \in \mathrm{Ch}(\mathcal{A})$, and $\mathrm{Ch}(\mathcal{A})$ has enough projectives and enough injectives. Such a category, one that is exact, has enough projectives, has enough injectives, and where the class of projective and injective objects coincide is called a *Frobenius category*.

§ 1.3.2 | Graded Rings and Algebras

Before talking about differential graded algebras it makes sense to discuss graded rings and algebras.

Definition 1.3.3. Let R be a ring and let $(I, \cdot)$ be a monoid. Then R is an *I-graded ring* if R has a decomposition of additive groups

$$R = \bigoplus_{i \in I} R_i,$$

such that for all $i, j \in I$

$$R_i R_j \subseteq R_l,$$

where $i \cdot j = l \in I$. We say that a non-zero element $r \in R_i$ is *homogeneous of degree i*.

Generally, graded rings are considered with an $\mathbb{N}$-grading or a $\mathbb{Z}$-grading, where we consider both $\mathbb{N}$ and $\mathbb{Z}$ under addition. It is possible to consider any ring R as a $\mathbb{N}$-graded (or $\mathbb{Z}$-graded) ring using the *trivial grading*, where we set R as R_0 and all R_i to be trivial for $i \neq 0$. We say that a ring with a trivial grading is *trivially graded*. However the theory of graded rings is much richer than that of trivially graded rings, and gradings can be placed onto many rings.

Example 1.3.4. Let $R = k[x]$ be the polynomial ring in one variable over a field k. Then R is considered to be an $\mathbb{N}$-graded ring with $R_i = \{ax^i \mid a \in k\}$, in other words an element r is of homogeneous degree i if $r = ax^i$ for some scalar $a \in k$. It is clear that this satisfies the conditions to be an $\mathbb{N}$-graded ring, as each R_i is an additive group, and any two elements $ax^i \in R_i$ and $bx^j \in R_j$ multiply to give $abx^{i+j} \in R_{i+j}$. This can be extended to polynomial rings in multiple variables, where a ring $S = k[x_1, \ldots, x_n]$ is $\mathbb{N}$-graded with a decomposition $S = \bigoplus_{i \in \mathbb{N}} S_i$ such that

$$S_i = \langle x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n} \in k[x_1, \ldots, x_n] \mid m_1 + m_2 + \ldots + m_n = i \rangle,$$

is an additive abelian group.

When we consider rings without gradings, we often look at the modules over the ring. This is no different when we consider a graded ring, when we can look at the graded modules over a graded ring.

Definition 1.3.5. Let R be an I-graded ring and let M be an R-module. Then M is an *I-graded R-module* if

- $M = \bigoplus_{i \in I} M_i$, where each M_i is an abelian group,
- $R_i M_j \subseteq M_{i \cdot j}$ for all $i, j \in I$.

Let V, W both be I-graded R-modules, then a morphism of R-modules $f : V \to W$ is a *morphism of graded R-modules* if it additionally satisfies $f(V_i) \subset W_i$ for all $i \in I$. We denote the category of graded R-modules by $\mathrm{Mod}_{\mathrm{gr}} R$.

Any graded ring can be considered as a graded module over itself, and we can consider a graded ring as a module over a trivially graded ring. For instance, let R be a commutative, trivially graded ring and let $S = R[x_1, \dots, x_n]$ be a polynomial ring in n variables over R. Then S is an $\mathbb{N}$-graded R-module with the grading given by the same grading on S when viewed as a ring.

It is also possible to define a graded Hom-space between two graded R-modules Let V, W be two graded R-modules, then the grading on the space of morphisms is;

$$Hom_{\mathrm{Mod}_{\mathrm{gr}} R}(V, W)_n = \{f \in \mathrm{Hom}_R(V, W) \mid f(V_p) \subset W_{p+n} \, \forall p \in \mathbb{Z}\}.$$

Additionally, it is possible for a module to have two distinct gradings, making them non-isomorphic in $\mathrm{gr}\, R - \mathrm{mod}$. For example, take the polynomial ring over R in n variables, $R[x_1, \dots, x_n]$. Then it can be considered with an $\mathbb{N}$-grading as in Example 1.3.4, or it could also be considered with the trivial grading.

As with graded modules, there is a notion of a graded algebra.

Definition 1.3.6. Let A be an algebra over an I-graded ring R. Then we say that A is an *I-graded R-algebra* if $A = \bigoplus_{i \in I} A_i$ is a I-graded R-module and

$$A_i A_j \subseteq A_{i \cdot j}$$

for all $i, j \in I$.

Notably, if R is trivially graded then we simply require all graded pieces A_i to be left R-modules. Examples of graded algebras include the tensor algebra over a vector space, exterior algebras and symmetric algebras. One may also look at Clifford algebras, which are an example of a $\mathbb{Z}/2\mathbb{Z}$-graded algebra, also known as a *superalgebra.*

§ 1.3.3 | The Differential

Graded rings can be generalised further to something called differential graded rings. These are graded rings that also carry the structure of a chain complex (Definition 1.3.1) and are often seen as a powerful tool in areas such as homological algebra and homotopy theory.

Definition 1.3.7. A *differential graded ring*, R, is a $\mathbb{Z}$-graded ring endowed with a degree one morphism $d : R \to R$, called the *differential*, such that $d^2 = 0$, and d satisfies the *graded Leibniz rule.* That is,

$$d(ab) = d(a)b + (-1)^{\deg(a)} a\, d(b),$$

for all $a, b \in R$.

A *differential graded R-module*, M, is a graded R-module with a differential d_M such that $d_M^2 = 0$, and

$$d_M(am) = d_M(a)m + (-1)^{\deg(a)} a\, d_M(m),$$

for all $a \in R$ and $m \in M$. A *morphism of differential graded R-modules* $f : M \to N$ is a morphism of graded rings that also satisfies $f \circ d_M = d_N \circ f$.

We denote the category of differential graded (=dg) R-modules as $\mathrm{Mod}_{\mathrm{dg}}R$. As with the trivial grading there exists a trivial differential, $d = 0$, which makes all $\mathbb{Z}$-graded rings into differential graded rings. For an example of a dg ring with a non-trivial differential, let $R = k[x]$, a polynomial ring in one variable over a field, with grading given by $x^i \in R_i$. Let the differential be

$$d(ax^i) = \begin{cases} 0 & \text{if } i \text{ is even,} \\ ax^{i+1} & \text{if } i \text{ is odd,} \end{cases}$$

for all $a \in k$. It is clear that $d^2 = 0$, and to check that d satisfies the graded Leibniz rule, we have

$$d(ax^i \cdot bx^j) = d(ax^i) \cdot bx^j + (-1)^i ax^i \cdot d(bx^j).$$

If i, j are even then both sides are zero, similarly if i, j are odd then both sides are also zero. If i is odd and j even, then $d(ax^i \cdot bx^j) = abx^{i+j+1}$ and $d(ax^i) \cdot bx^j + (-1)^i ax^i \cdot d(bx^j) = abx^{i+j+1} + 0 = abx^{i+j+1}$. Finally, if i is even and j odd, then $d(ax^i \cdot bx^j) = abx^{i+j+1}$ and $d(ax^i) \cdot bx^j + (-1)^i ax^i \cdot d(bx^j) = 0 + (-1)^i abx^{i+j+1} = abx^{i+j+1}$, and so d satisfies the graded Leibniz rule.

The grading of $Hom_{\mathrm{Mod}_{\mathrm{gr}}R}(V, W)$ naturally induces a grading on $Hom_{\mathrm{Mod}_{\mathrm{dg}}R}(V, W)$, which can be seen as a dg R-module with the differential

$$\partial f : d_W \circ f - (-1)^{\deg(f)} f \circ d_V.$$

In particular, $\mathrm{Mod}_{\mathrm{dg}}R$ is enriched over dg abelian groups. The category $\mathrm{Mod}_{\mathrm{dg}}R$ is also equipped with a family of endofunctors known as the *shift functors*;

$$\begin{aligned} [n] :& \mathrm{Mod}_{\mathrm{dg}}R \to \mathrm{Mod}_{\mathrm{dg}}R \\ & (\bigoplus_{i \in \mathbb{Z}} V_i, d_V) \mapsto (\bigoplus_{i \in \mathbb{Z}} V_{i+1}, (-1)^n d_V). \end{aligned}$$

Definition 1.3.8. Let A be a dg R-module endowed with a degree zero morphism $m : A \otimes A \to A$ such that,

$$d_A(ab) = d_A(a)b + (-1)^{\deg(a)} a d_A(b),$$

where $ab := m(a \otimes b)$, then A is a *differential graded algebra* over R. A *morphism* of differential graded algebras over R is a degree zero morphism $f : A \to A'$ such that $f \circ d_A = d_{A'} \circ f$.

Examples of dg algebras include the tensor algebra of a vector space, De-Rham algebras, and Koszul complexes.

§ 1.4 | Derived Categories

The derived category has its roots in Verdier's PhD thesis [67], taking form as the primary example of a triangulated category. Since then derived categories have become indispensable, with algebraic geometry and homological algebra making particularly heavy use of them.

In this section we build up to the seemingly abstract definition of a derived category by first looking at chain complexes and homotopic maps. We then look at homology functors and quasi-isomorphisms, all necessary ingredients in the construction of a derived category.

The Homotopy Category

Let A and B be objects in $\mathrm{Ch}(\mathcal{A})$. Then we can impose an equivalence relation on $\mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(A, B)$ by relating morphisms that are *homotopic* to each other.

Definition 1.4.1. Let $\mathrm{Ch}(\mathcal{A})$ be the category of chain complexes over an additive category $\mathcal{A}$. We say that two morphisms $f, g\colon A \to B$, for any $A, B \in \mathrm{Ch}(\mathcal{A})$, are *homotopic* (or, equivalently, $f - g$ is *null-homotopic*) if there exists a collection of maps $h_n\colon A_n \to B_{n+1}$ such that

$$f_n - g_n = d^B_{n+1} h_n - h_{n-1} d^A_n$$

for all $n \in \mathbb{Z}$. If f and g are homotopic, we write $f \sim g$.

When showing whether or not two morphisms are homotopic, we often visualise the problems using the following diagram.

$$\begin{tikzcd}
\cdots \arrow[r] & A_{n+1} \arrow[r, "d^A_{n+1}"] \arrow[d, shift right, "f_{n+1}"'] \arrow[d, shift left, "g_{n+1}"] & A_n \arrow[r, "d^A_n"] \arrow[dl, "h_n"'] \arrow[d, shift right, "f_n"'] \arrow[d, shift left, "g_n"] & A_{n-1} \arrow[r] \arrow[dl, "h_{n-1}"'] \arrow[d, shift right, "f_{n-1}"'] \arrow[d, shift left, "g_{n-1}"] & \cdots \\
\cdots \arrow[r] & B_{n+1} \arrow[r, "d^B_{n+1}"] & B_n \arrow[r, "d^B_n"] & B_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Definition 1.4.2. Let $\mathcal{A}$ be an additive category, and $\mathrm{Ch}(\mathcal{A})$ the category of chain complexes over $\mathcal{A}$. Then the *homotopy category*, $\mathrm{K}(\mathcal{A})$ has the same objects as $\mathrm{Ch}(\mathcal{A})$, and morphism spaces

$$\mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(A, B) = \mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(A, B)/\sim .$$

The homotopy category of an additive category $\mathcal{A}$, $\mathrm{K}(\mathcal{A})$, is a triangulated category [68, Proposition 10.2.4].

The homotopy category of bounded chain complexes in $\mathcal{A}$ is denoted by $\mathrm{K}^b(\mathcal{A})$, and the category of bounded below (resp. bounded above) chain complexes is denoted $\mathrm{K}^+(\mathcal{A})$ (resp. $\mathrm{K}^-(\mathcal{A})$).

§ 1.4.1 | Homology and the Derived Category

From now on, we consider the category $\mathcal{A}$ to be abelian. That is, an additive category where every morphism has a kernel and cokernel, and every epimorphism (resp. monomorphism) is a cokernel (resp. kernel) of some morphism.

Let $A, B \in \mathrm{K}(\mathcal{A})$, and suppose there is a morphism of chain complexes in $\mathrm{K}(\mathcal{A})$, $f\colon A \to B$. Then, for all $n \in \mathbb{Z}$, we may construct the commutative diagram

$$\begin{tikzcd}
 & \operatorname{im} d^A_{n+1} \arrow[r, "\phi"] & \ker d^A_n \arrow[d, "p"] & & \\
\cdots \arrow[r] & A_{n+1} \arrow[u, "\overline{d}^A_{n+1}"] \arrow[r, "d^A_{n+1}"] \arrow[d, "f_{n+1}"] & A_n \arrow[r, "d^A_n"] \arrow[d, "f_n"] & A_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & B_{n+1} \arrow[r, "d^B_{n+1}"] \arrow[d, "\overline{d}^B_{n+1}"] & B_n \arrow[r, "d^B_n"] & B_{n-1} \arrow[r] & \cdots \\
 & \operatorname{im} d^B_{n+1} \arrow[r, "\psi"] & \ker d^B_n \arrow[u, "l"] & &
\end{tikzcd}$$

where p is the kernel of d^A_n, and l is the kernel of d^B_n. By the universal property of the kernel, and the fact that $d^B_n f_n p = f_{n-1} d^A_n p = 0$, there exists a unique morphism $\alpha\colon \ker d^A_n \to \ker d^B_n$ such that $l\alpha = f_n p$.

Definition 1.4.3. Let $A, B \in \mathrm{K}(\mathcal{A})$ be two chain complexes. We define the n^{th} *homology functor*, $\mathrm{H}_n\colon \mathrm{Ch}(\mathcal{A}) \to \mathcal{A}$, to act on objects by taking $H_n(A) \cong \ker d^A_n / \operatorname{im} d^A_{n+1}$. Given a map of chain complexes, $f\colon A \to B$, then $\mathrm{H}_n(f)\colon \mathrm{H}_n(A) \to \mathrm{H}_n(B)$ is the unique morphism such that $\operatorname{coker} \psi \circ \alpha = \mathrm{H}_n(f) \circ \operatorname{coker} \phi$.

We say that a chain complex B is *exact* if $H_n(B) = 0$ for all $n \in \mathbb{Z}$.

Example 1.4.4. Let $\mathcal{A} = \mathrm{mod}(k[x])$ and let A be the chain complex as in Example 1.3.2. Then the kernel of d_i is the ideal of $k[x]/(x^2)$ generated by x, and the image of d_{i+1} is also the ideal of $k[x]/(x^2)$ generated by x. Therefore $\mathrm{H}_i(A) = \operatorname{coker} \phi \cong \ker d_i / \operatorname{im} d_{i+1} = 0$ for all $i \in \mathbb{Z}$.

Definition 1.4.5. Let $A, B \in \mathrm{K}(\mathcal{A})$ be two objects with a morphism $f\colon A \to B$. We call f a *quasi-isomorphism* if $\mathrm{H}^n(f)\colon \mathrm{H}^n(A) \to \mathrm{H}^n(B)$ is an isomorphism for all $n \in \mathbb{Z}$. The set of quasi-isomorphisms in $\mathrm{K}(\mathcal{A})$ is labelled $S^{qi}(\mathcal{A})$.

It is possible that two objects $A, B \in \mathrm{K}(\mathcal{A})$ with an isomorphism between $\mathrm{H}_n(A)$ and $\mathrm{H}_n(B)$ for all $n \in \mathbb{Z}$, are not quasi-isomorphic. This is as there is no guarantee that there exists a morphism $f\colon A \to B$ such that $\mathrm{H}_n(f)$ is an isomorphism in $\mathcal{A}$ for all $n \in \mathbb{Z}$.

Definition 1.4.6. Let $\mathcal{A}$ be an abelian category. Then the *derived category of* $\mathcal{A}$, $D(\mathcal{A})$, is the localisation of $\mathrm{K}(\mathcal{A})$ by $S^{qi}(\mathcal{A})$, i.e. we formally invert all morphisms $f \in S^{qi}(\mathcal{A})$.

The derived category of an abelian, k-linear category is always a triangulated, k-linear category. Throughout we shall make reference to the bounded below derived category, $\mathrm{D}^+(-)$, the bounded above derived category, $\mathrm{D}^-(-)$, and the bounded derived category, $\mathrm{D}^{\mathrm{b}}(-)$, which are defined as subcategories of $D(-)$ with objects being complexes that are bounded below, bounded above, and bounded, respectively.

Let $\mathcal{A}, \mathcal{B}$ be abelian categories such that there exists an equivalence between $D(\mathcal{A})$ and $D(\mathcal{B})$. Then we say that $\mathcal{A}$ and $\mathcal{B}$ are *derived equivalent.* Moreover, two rings R, S are also said to be derived equivalent if $D(\mathrm{Mod}R)$ is equivalent to $D(\mathrm{Mod}S)$.

§ 1.5 | Auslander-Reiten Theory

Auslander-Reiten theory was developed in a series of papers by Auslander and, later, Auslander and Reiten [4, 5, 6, 7], to study the representation theory of Artinian rings, and involves the use of Auslander-Reiten sequences and Auslander-Reiten quivers. Some aspects of Auslander-Reiten theory are necessary for the understanding of cluster categories, and so we go over some of the basic ideas here.

§ 1.5.1 | The Auslander-Reiten Translation

To define the Auslander-Reiten translation functor, we begin by letting A be a finite dimensional k-algebra. Let $\operatorname{proj} A$ be the category of finitely generated projective A-modules, and $\operatorname{inj} A$ the category of finitely generated injective A-modules. Then we have dualities

$$D := \mathrm{Hom}_k(-, k) : \operatorname{mod} A \leftrightarrow \operatorname{mod} A^{\mathrm{op}},$$
$$(-)^* := \mathrm{Hom}_A(-, A) : \operatorname{proj} A \leftrightarrow \operatorname{proj} A^{\mathrm{op}},$$

which induce the equivalence

$$\nu := D(-)^* : \operatorname{proj} A \to \operatorname{inj} A,$$

called the *Nakayama functor.* For an object $X \in \operatorname{mod} A$ with a minimal projective resolution

$$P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} X \to 0,$$

we define τX, the *Auslander-Reiten translation* of X, by the exact sequence

$$0 \to \tau X \to \nu P_0 \xrightarrow{\nu d_1} \nu P_1.$$

The Auslander-Reiten translation is a bijection between the isomorphism classes of non-projective objects in $\operatorname{mod} A$ and the isomorphism classes of non-injective objects, given in Theorem 1.5.1.

§ 1.5.2 | Auslander-Reiten Quivers

Let A be a finite dimensional k-algebra, and let $M, N \in \operatorname{mod} A$. Then we say a morphism $f \colon M \to N$ is *irreducible* if f is not a split epimorphism, a split monomorphism, and $f = gh$ implies either g is a split epimorphism and h is a split monomorphism. In particular, if $f \neq 0$ is an irreducible morphism, then f is either an epimorphism or a monomorphism.

A short exact sequence

$$0 \to M \xrightarrow{f} N \xrightarrow{g} L \to 0$$

is said to be an *Auslander-Reiten sequence* if f, g are both irreducible morphisms, and M, L are indecomposable objects in $\operatorname{mod} A$. Auslander-Reiten sequences are uniquely determined up to isomorphism by the objects at either end of the sequence. Also, as an Auslander-Reiten sequence does not split by definition, then M cannot be injective and L cannot be projective. In fact, the following theorem shows that all non-injective indecomposable objects sit on the left of an Auslander-Reiten sequence, and that all non-projective indecomposable objects sit on the right of an Auslander-Reiten sequence.

Theorem 1.5.1. *[6] Let A be a finite dimensional algebra and let M be an indecomposable object in* $\operatorname{mod} A$*, then*

- *if M is not projective, then there exists an Auslander-Reiten sequence of the form* $0 \to \tau M \to E \to M \to 0$.
- *if M is not injective, then there exists an Auslander-Reiten sequence of the form* $0 \to M \to E' \to \tau^{-1}M \to 0$.

Here τ is the Auslander-Reiten translation, and τ^{-1} is the inverse Auslander-Reiten translation.

Another tool from Auslander-Reiten theory that we make use of is the *Auslander-Reiten quiver* of a finite dimensional algebra A, which is a quiver with relations (Q_A, I_A). The quiver Q_A has a set of vertices in bijection with the set of isomorphism classes of indecomposable objects in $\operatorname{mod} A$. Let $M, N \in \operatorname{mod} A$ be indecomposable objects, and suppose there is an irreducible morphism from M to N, then there is an arrow in the Auslander-Reiten quiver starting at the vertex corresponding to M and ending at the vertex corresponding to N.

Incidentally, we may see the Auslander-Reiten sequences, and so the Auslander-Reiten translation, from the Auslander-Reiten quiver. Let $M \in \operatorname{mod} A$ be indecomposable, and suppose that

$$0 \to \tau M \to L_1 \oplus L_2 \oplus \cdots \oplus L_i \to M \to 0$$

is the Auslander-Reiten sequence ending at M. Then there exists a *mesh* in Q_A, a collection of paths starting at the vertex corresponding to τM and ending at the vertex corresponding to M.

$$\begin{tikzcd}
 & L_1 \arrow[ddr, "b_1"] & \\
 & L_2 \arrow[dr, "b_2"] & \\
\tau M \arrow[uur, "a_1"] \arrow[ur, "a_2"] \arrow[dr, "a_i"'] & \vdots & M \\
 & L_i \arrow[ur, "b_i"'] &
\end{tikzcd}$$

Moreover, we consider Q_A with a set of relations I_A, known as the *mesh relations*. For each mesh, using the notation in the above diagram, we have the k-linear sum of paths $\sum_{j=1}^{i} b_j a_j$, and I_A is generated by the sums of paths corresponding to the set of all meshes in Q_A. The path algebra of the Auslander-Reiten quiver is kQ_A/I_A, the path algebra of the underlying quiver modulo the mesh relations.

By an abuse of notation, for the Auslander-Reiten quiver (Q_A, I_A) of a finite dimensional algebra A, we shall drop the set of relations and denote the Auslander-Reiten quiver by Q_A.

Example 1.5.2. Let $Q = 1 \to 2 \to 3$ be a linearly orientated quiver of Dynkin type A_3, and let B be the path algebra kQ. Let P_i be the indecomposable projective module associated to the vertex i, I_i be the indecomposable injective module associated to the vertex i, and S_i be the indecomposable simple module associated to the vertex i. We have the three isomorphisms $P_3 \cong I_1$, $P_1 \cong S_1$, and $I_3 \cong S_3$. Then $\operatorname{mod} B$ has three indecomposable, non-projective modules, S_2, I_2, I_1. We know by Theorem 1.5.1 that these are the only indecomposable objects that sit on the right of an Auslander-Reiten sequence. These Auslander-Reiten sequences are

$$\begin{gathered}
0 \to P_3 \to P_2 \to S_2 \to 0, \\
0 \to P_2 \to P_1 \oplus S_2 \to I_2 \to 0, \\
0 \to S_2 \to I_2 \to I_1 \to 0.
\end{gathered}$$

These Auslander-Reiten sequences respectively give us the following meshes

$$\begin{tikzcd}[column sep=small]
 & P_2 \arrow[dr] & & & P_1 \arrow[dr] & & & I_2 \arrow[dr] & \\
P_3 \arrow[ur] & & S_2, & P_2 \arrow[ur] \arrow[dr] & & I_2, & S_2 \arrow[ur] & & I_1. \\
 & & & & S_2 \arrow[ur] & & & &
\end{tikzcd}$$

Hence the Auslander-Reiten quiver Q_B has the form

$$\begin{tikzcd}
& & P_1 \arrow[dr, "\beta_2"] & & \\
& P_2 \arrow[ur, "\alpha_2"] \arrow[dr, "\beta_1"'] & & I_2 \arrow[ll, dotted] \arrow[dr, "\beta_3"] & \\
P_3 \arrow[ur, "\alpha_1"] & & S_2 \arrow[ll, dotted] \arrow[ur, "\alpha_3"'] & & I_1, \arrow[ll, dotted]
\end{tikzcd}$$

where the dotted arrows takes an indecomposable object M to its Auslander-Reiten translation τM. The mesh relations of Q_B are $I_B = \langle \beta_1\alpha_1,\ \beta_2\alpha_2 + \alpha_3\beta_1,\ \beta_3\alpha_3 \rangle$.

§ 1.6 | Cluster Categories

Cluster categories associated to hereditary algebras were introduced shortly after the introduction of cluster algebras, and were intended to be a categorification of cluster algebras. They were introduced in [18] as an orbit category of the bounded derived category of an hereditary algebra $\mathcal{H}$, and in [19] for the A_n case as the category of representations over a quiver with relations.

Another construction of cluster categories, this time for algebras of global dimension ≤ 2, was introduced by Amiot in [1]. This cluster category has an equivalence to the cluster category found in [18] when the algebra is hereditary, and so may be seen as an extension of this construction.

§ 1.6.1 | Cluster Categories From Hereditary Algebras

We recall that, given a triangulated category $\mathcal{T}$ and an autoequivalence $F\colon \mathcal{T} \to \mathcal{T}$, we can construct the orbit category $\mathcal{T}/F$ with the same objects as $\mathcal{T}$, and

$$\operatorname{Hom}_{\mathcal{T}/F}(X, Y) \cong \bigoplus_{n\in\mathbb{Z}} \operatorname{Hom}_{\mathcal{T}}(X, F^n Y)$$

Definition 1.6.1. Let $\mathcal{H}$ be an hereditary algebra. Then the *cluster category associated to* $\mathcal{H}$ is defined to be the $\tau^{-1} \circ [1]$-orbit category of the bounded derived category of $\mathcal{H}$, i.e.

$$\mathcal{C}_{\mathcal{H}} := \mathrm{D}^{\mathrm{b}}(\mathcal{H})/(\tau^{-1} \circ [1]).$$

Let $\mathcal{T}$ be a triangulated category with suspension functor $[1]$, and let $\mathcal{S}$ be a subcategory of $\mathcal{T}$. Let $f \in \operatorname{Hom}_{\mathcal{T}}(X, Y)$, then, we call f a *right $\mathcal{S}$-approximation* of $Y \in \mathcal{T}$ if $X \in \mathcal{S}$ and

$$\operatorname{Hom}(-, X) \xrightarrow{f\cdot} \operatorname{Hom}(-, Y) \to 0$$

is exact as functors on $\mathcal{S}$. We call $\mathcal{S}$ a *contravariantly finite subcategory* of $\mathcal{T}$ if any $Y \in \mathcal{T}$ has a right $\mathcal{S}$-approximation. A *left $\mathcal{S}$-approximation* and a *covariantly finite subcategory* are dually defined. We say that a contravariantly and covariantly finite subcategory is *functorially finite.*

Contravariantly finite subcategories were first introduced in [8], with respect to finitely generated module categories of artin algebras.

Now, let $\mathcal{S}$ be a functorially finite subcategory of $\mathcal{T}$, then we say $\mathcal{S}$ is a *cluster tilting subcategory* of $\mathcal{T}$ if

$$\mathcal{S} = (\mathcal{S}[-1])^{\perp} = {}^{\perp}(\mathcal{S}[1])$$

where

$$\mathcal{X}^{\perp} = \{Y \in \mathcal{T} \,|\, \mathrm{Hom}(X, Y) = 0 \text{ for all } X \in \mathcal{X}\}$$

and

$${}^{\perp}\mathcal{X} = \{Y \in \mathcal{T} \,|\, \mathrm{Hom}(Y, X) = 0 \text{ for all } X \in \mathcal{X}\}$$

for a subcategory $\mathcal{X}$ of $\mathcal{T}$. In a 2-Calabi-Yau, Hom-finite triangulated category, ${}^{\perp}(\mathcal{X}[1]) = (\mathcal{X}[-1])^{\perp}$ for all subcategories $\mathcal{X}$, and so it suffices to only show that $\mathcal{X} = {}^{\perp}(\mathcal{X}[1])$, or $\mathcal{X} = (\mathcal{X}[-1])^{\perp}$.

Exchange pairs in a cluster category were introduced in [18] as an extension of the notion of the completion of an almost complete basic tilting module over a hereditary algebra, to the theory of cluster categories and almost cluster tilting objects. They also provide us with an analogue of the idea of mutation of clusters in a cluster algebra, and are hence useful to understand when looking at cluster categories generally. More specifically, we wish to understand them as they play a crucial role in a result in [57] which we will use in the proof of Theorem 3.2.5.

Definition 1.6.2. Let $\mathcal{T}$ be a triangulated category, and let $\mathcal{S} \subset \mathcal{T}$ be a cluster tilting subcategory, with $M \in \mathcal{S}$. Let $\mathcal{S}'$ be the full additive subcategory of $\mathcal{S}$ containing all the same indecomposables as $\mathcal{S}$ except M, i. e. $M \notin \mathcal{S}'$. Then we call $\mathcal{S}'$ an *almost cluster tilting subcategory* of $\mathcal{T}$, and M a *complement* of $\mathcal{S}'$.

It was shown in [18] that there are two complements for each almost cluster tilting subcategory $\mathcal{S}'$, which we label M and M^* respectively. We call the pair (M, M^*) an *exchange pair*.

Let $\mathcal{S}'$ be an almost cluster tilting subcategory of a cluster category $\mathcal{T}$, and let M and M' be complements of $\mathcal{S}'$. Let $\mathcal{S}$, resp. $\mathcal{R}$, be the cluster tilting subcategory containing $\mathcal{S}'$ and M, resp. M'. Then we obtain the cluster tilting subcategory $\mathcal{R}$ from $\mathcal{S}$ by removing the indecomposable object M and replacing it with the indecomposable object M'. This operation is the *mutation of $\mathcal{S}$ at M*, denoted $\mu_M(\mathcal{S})$.

For every exchange pair (M, M^*) we have two non-split triangles due to [18], called the *exchange triangles*:

$$M \to B_{M^*} \to M^* \to M[1] \quad \text{and} \quad M^* \to B_M \to M \to M^*[1]$$

where the maps $B_M \to M$ and $B_{M^*} \to M^*$ are right $\mathcal{S}$-approximations, where $\mathcal{S}$ is the cluster tilting subcategory containing both the almost cluster tilting subcategory $\mathcal{S}'$ and its complement M.

Here we show an example of a cluster category associated to an hereditary algebra, and then give a cluster tilting subcategory and its mutation at an indecomposable object.

Example 1.6.3. Let $\mathcal{H} = \operatorname{mod} A_3$, where A_3 is the linearly orientated quiver

$$1 \longrightarrow 2 \longrightarrow 3.$$

Then the bounded derived category $\mathrm{D^b}(\mathcal{H})$ has the Auslander-Reiten quiver

$$\begin{tikzcd}
& I_1[-1] \arrow[dr] & & P_1 \arrow[ll, dotted] \arrow[dr] & & P_3[1] \arrow[ll, dotted] \arrow[dr] & & S_2[1] \arrow[ll, dotted] \arrow[dr] & & I_1[1] \arrow[ll, dotted] & \\
\cdots & & P_2 \arrow[ur] \arrow[dr] & & I_2 \arrow[ll, dotted] \arrow[ur] \arrow[dr] & & P_2[1] \arrow[ll, dotted] \arrow[ur] \arrow[dr] & & I_2[1] \arrow[ll, dotted] \arrow[ur] \arrow[dr] & & \cdots \\
& P_3 \arrow[ur] & & S_2 \arrow[ll, dotted] \arrow[ur] & & I_1 \arrow[ll, dotted] \arrow[ur] & & P_1[1] \arrow[ll, dotted] \arrow[ur] & & P_3[2] \arrow[ll, dotted] &
\end{tikzcd}$$

where P_i (resp. I_i) is the projective (resp. injective) module associated to vertex i, S_i is the simple module at vertex i, and $[1]$ is the suspension functor. The Auslander-Reiten translation τ of an object is given by the dotted arrows.

The cluster category $\mathcal{C}_\mathcal{H}$ is the orbit category of $\mathrm{D^b}(\mathcal{H})$ with respect to the functor $\tau^{-1} \circ [1]$, and has the Auslander-Reiten quiver

$$\begin{tikzcd}
& & P_1 \arrow[dr] & & P_3[1] \arrow[dr] & & \color{red}{P_3} \arrow[dr, red] & & \\
& P_2 \arrow[ur] \arrow[dr] & & I_2 \arrow[ur] \arrow[dr] & & P_2[1] \arrow[ur, red] \arrow[dr] & & \color{red}{P_2} \arrow[dr, red] & \\
P_3 \arrow[ur] & & S_2 \arrow[ur] & & I_1 \arrow[ur] & & P_1[1] \arrow[ur, red] & & \color{red}{P_1}
\end{tikzcd}$$

where the objects in red are isomorphic to the objects on the left hand side of the quiver. We represent the Auslander-Reiten quiver like this to show the Möbius strip-like behaviour that the Auslander-Reiten quiver of a cluster category exhibits, as there are non-zero irreducible morphisms from the objects $P_i[1]$ to P_{i-1} for $i = 2, 3$.

There is a canonical cluster tilting subcategory of $\mathcal{C}_\mathcal{H}$ given by $\operatorname{add}(\bigoplus_{i=1}^3 P_i)$, the additive subcategory of objects corresponding to direct sums of projective modules of $\mathcal{H}$. Denote this cluster tilting subcategory $\mathcal{T}$. Suppose we want to mutate $\mathcal{T}$ at the object P_2, and say $\mathcal{T}' = \operatorname{add}(P_1 \oplus P_3)$. Then to find the exchange pair (P_2, M) we need an indecomposable object $M \ncong P_i$ such that $\operatorname{Hom}_{\mathcal{C}_\mathcal{H}}(M, P_1[1]) = \operatorname{Hom}_{\mathcal{C}_\mathcal{H}}(M, P_3[1]) = 0$. By highlighting the non-zero morphisms into $P_1[1]$ and $P_3[1]$ on the Auslander-Reiten quiver of $\mathcal{C}_\mathcal{H}$,

$$\begin{tikzcd}
& & P_1 \arrow[dr, dotted] & & P_3[1] \arrow[dr] & & \\
& P_2 \arrow[ur, dotted] \arrow[dr, dotted] & & I_2 \arrow[ur] \arrow[dr, dotted] & & P_2[1] \arrow[dr] & \\
P_3 \arrow[ur, dotted] & & S_2 \arrow[ur] & & I_1 \arrow[ur, dotted] & & P_1[1]
\end{tikzcd}$$

we can see that the only object M not isomorphic to P_i for any $i = 1, 2, 3$, such that $\operatorname{Hom}_{\mathcal{C}_\mathcal{H}}(M, P_1[1]) = \operatorname{Hom}_{\mathcal{C}_\mathcal{H}}(M, P_3[1]) = 0$, is the object I_1.

Therefore the cluster tilting subcategory $\mathcal{S} = \mu_{P_2}(\mathcal{T})$ is the additive subcategory,

$$\mathcal{S} = \mathrm{add}(P_1 \oplus P_3 \oplus I_1).$$

§ 1.6.2 | Quivers and Fomin-Zelevkinsky Mutation

The process of mutation of a quiver was introduced by Fomin and Zelevinsky in [30] as a combinatorial aspect to their definition of a cluster algebra. Here we go through the mutation of a quiver at a chosen vertex, under some mild conditions on the quiver.

Let Q be a quiver without loops and 2-cycles, and let i be a vertex of Q. The *Fomin-Zelevinksy mutation of Q at i* is constructed as follows:

1. for all subquivers $j \to i \to l$, we add a new arrow $j \to l$,
2. reverse all arrows that start or end at i,
3. chose a maximal set of 2-cycles and delete the set from the quiver.

In the case of i being a sink or source, we simply reverse all of the arrows incident to i. The operation of quiver mutation is an involution, that is $\mu_i \mu_i Q = Q$ for all $i \in Q_0$.

We label the new quiver obtained from mutation of Q at i by $\mu_i Q$, and we say that two quivers Q and Q' are *mutation equivalent* if we can mutate Q finitely many times to obtain Q'. The online applet [43] by Keller can be used to check whether two quivers are mutation equivalent.

Example 1.6.4. We look again at the quiver $Q = A_3$ with linear orientation,

$$1 \to 2 \to 3$$

and we mutate at the vertex 2.

By the first step in the construction, we add an arrow from vertex 1 to vertex 3 and get the quiver

$$\begin{tikzcd} & 2 \arrow[dr] & \\ 1 \arrow[ur] \arrow[rr] & & 3. \end{tikzcd}$$

Next we reverse all of the arrows incident with vertex 2, and get

$$\begin{tikzcd} & 2 \arrow[dl] & \\ 1 \arrow[rr] & & 3, \arrow[ul] \end{tikzcd}$$

and since there are no 2-cycles to delete, we are done. So the mutation of the quiver

Q at vertex 2 is

$$\mu_2 Q = \quad \begin{tikzcd} & 2 \arrow[dl] & \\ 1 \arrow[rr] & & 3. \arrow[ul] \end{tikzcd}$$

As an example of showing that mutation is an involution, we now mutate the quiver $\mu_2 Q$, again at vertex 2. First we add an arrow $3 \to 1$, as there is a subquiver $3 \to 2 \to 1$,

$$\begin{tikzcd} & 2 \arrow[dl] & \\ 1 \arrow[rr, shift left] & & 3. \arrow[ul] \arrow[ll, shift left] \end{tikzcd}$$

Now we reverse all arrows incident to 2,

$$\begin{tikzcd} & 2 \arrow[dr] & \\ 1 \arrow[ur] \arrow[rr, shift left] & & 3. \arrow[ll, shift left] \end{tikzcd}$$

Finally, given there is a 2-cycle between the vertices 1 and 3, we delete this to get

$$1 \to 2 \to 3,$$

and so we see that $\mu_2\mu_2 Q = Q$.

A quiver Q is said to be of *finite mutation type*, if there are only finitely many quivers in the mutation equivalence class of Q. All of the quivers of finite mutation type were classified by Felikson, Shapiro and Tumarkin across two papers [25, 26], where they show that a quiver is of finite mutation type if it satisfies some decomposition condition, or is mutation equivalent to one of the 11 exceptional cases. Namely, quivers of Dynkin type A_n and D_m are of finite mutation type.

§ 1.6.3 | Cluster Structures

Not all categories that have cluster tilting subcategories will be cluster categories as in Definition 1.6.1. The study of cluster tilting subcategories in 2-Calabi-Yau triangulated categories developed into the notion of a *cluster structure* on a category [17].

Definition 1.6.5. Let $\mathcal{T}$ be a k-linear, Hom-finite, Krull-Schmidt, triangulated category. Let $\mathcal{S}$ be a cluster tilting subcategory of $\mathcal{T}$, and let $\mathcal{S}' \subset \mathcal{S}$ be an almost cluster tilting subcategory. Then we say $\mathcal{T}$ has a *weak cluster structure* if the following hold:

1. Let $M \in \mathcal{S}$ be an indecomposable object, then M can be replaced by another indecomposable object $M' \in \mathcal{C}$ such that the new subcategory $\mathcal{R}$ is again a cluster tilting subcategory.

2. For each $M \in \mathcal{S}$, there are triangles $M' \xrightarrow{f} B \xrightarrow{g} M \to M'[1]$ and $M \xrightarrow{s} B' \xrightarrow{t} M' \to M[1]$, where g and t are minimal right $\mathcal{S}'$-approximations and f and s are minimal left $\mathcal{S}'$-approximations.

We say $\mathcal{T}$ has a *cluster structure* if the following hold:

3. The Gabriel quiver of $\mathrm{End}_{\mathcal{T}}(\mathcal{S})$ has no loops or 2-cycles.

4. The Gabriel quiver of $\mathrm{End}_{\mathcal{T}}(\mathcal{R})$ is the Gabriel quiver of $\mathrm{End}_{\mathcal{T}}(\mathcal{S})$ under Fomin-Zelevinsky mutation at a vertex corresponding to M.

The discrete cluster categories of Dynkin type A_∞ discussed in the rest of this thesis are not cluster categories in the sense of [18]. However they do have a cluster structure, which was shown in [38] for the one accumulation point case, and shown directly for n accumulation points in [35]. More generally, it was shown that any algebraic, 2-Calabi-Yau triangulated categories that admits a directed cluster tilting subcategory has a cluster structure by Šťovíček and van Roosmalen [65].

The completion of the discrete cluster categories of Dynkin type A_∞ used in this thesis, due to Paquette and Yıldırım [58], do not have a cluster structure however, or even a weak cluster structure. This is due to Theorem 4.4 in [58], which states a cluster tilting subcategory must have an object corresponding to a *limit arc* (Subsection 2.4), which cannot be mutated. Thus the completion does not satisfy the first axiom in Definition 1.6.5.

Chapter 2

Cluster Categories and Surfaces

In this chapter we take a closer look at a class of cluster categories, those which can be defined via a marked surface. The connections between cluster combinatorics and surfaces has be known since the introduction of cluster algebras by Fomin and Zelevinsky in the early part of the century [30], and has been studied in great detail from the perspective of both surfaces, cluster algebras and cluster categories.

We begin this chapter by looking at one of the earliest connections between cluster combinatorics and marked surfaces, that of the bijection between the clusters of the cluster algebra associated to A_n and the triangulations of an $(n+3)$-gon [33]. From there, we define one of the main topics of interest to this thesis, the discrete cluster category of Dynkin type A_∞, first introduced for a single accumulation point in [38] as a subcategory of a derived category of a polynomial ring seen as a differential graded ring, and later generalised for finitely many accumulation points in [39] as a stable Frobenius category constructed via cyclic posets. The construction of the discrete cluster category of Dynkin type A_∞ used throughout will be the one in [35], which is more combinatoric in nature. In the interest of completeness, and as a more algebraic view of these categories, we also provide the construction due to Holm and Jørgensen in [38].

We then look at the other main interest in this thesis, a completion of the discrete cluster category of Dynkin type A_∞ due to Paquette and Yıldırım [58]. Although they may not be a cluster category in the sense of Definition 1.6.1, nor do they have a cluster structure, they are still a powerful tool in which to study cluster combinatorics, and have even been shown to arise in other settings, being equivalent to a category of graded maximal Cohen-Macaulay modules over an A_∞ singularity in the single accumulation point case [3]. As with the non-completed case, there are alternative constructions of the completed version. We go through some of these other constructions after describing the Paquette-Yıldırım completion.

Finally, we conclude the chapter by looking at some more marked surfaces and their associated cluster categories. For instance, we look at the marked surface associated to the cluster category of Dynkin type D_n [63], and the cluster category associated to a marked surface satisfying some elementary properties [16, 61].

§ 2.1 | The Cluster Category of Dynkin Type A_n

The cluster algebra of Dynkin type A_n is one of the earliest examples of a cluster algebra, studied in [33, 31] shortly after the introduction of cluster algebras. In [33] it was shown that there is a bijection between the diagonals of an $(n+3)$-gon and the cluster variables of the cluster algebra of type A_n, and further that there was a bijection between the triangulations of the $(n+3)$-gon and the clusters. This representation carried over to the categorification of these cluster algebras into cluster categories, where the cluster variables correspond to indecomposable objects, and set of cluster variables in a cluster corresponds to the set of indecomposable objects in a cluster-tilting subcategory.

The mutation of an indecomposable object in a cluster tilting subcategory of $\mathcal{C}_{A_n}$ can be viewed as replacing the corresponding arc in a triangulation of an $(n+3)$-gon with the unique arc that forms a different triangulation such that all other arcs in the triangulation remain.

Cluster categories constructed from hereditary algebras inherit a natural triangulated structure by a result of Keller [42] and so have an autoequivalence, known as the *suspension* (or *shift*) functor and denoted by $[1]$. In the case of cluster categories arising from non-punctured surfaces, there is an appropriate choice of bijection between indecomposable objects and arcs such that the suspension functor acts on arcs by a single rotation of both endpoints in a direction opposite to the orientation of the surface, as shown in the Figure 2.1. For the cluster category of Dynkin type A_n, this bijection is given in [19], where they associate a category to the set of arcs in the $(n+3)$-gon where the Auslander-Reiten translation acts on an arc by rotating the endpoints in the opposite orientation, and showing that the category is equivalent to the cluster category of A_n defined in [18].

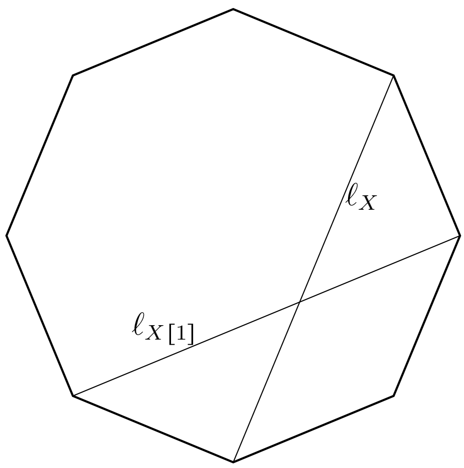


Figure 2.1: The suspension functor [1] acting upon the arc ℓ_X, corresponding to the object X in the cluster category $\mathcal{C}_{A_5}$.

We use Example 1.6.3 to give an example of how the bijection between indecomposable objects in $\mathcal{C}_{A_n}$ and the arcs of the $(n+3)$-gon can be constructed, using an adaptation of the coordinate system used in [38].

Example 2.1.1. The Auslander-Reiten quiver for $\mathcal{C}_{A_3}$ is given in Example 1.6.3.

$$\begin{tikzcd}[column sep=small]
 & & P_1 \arrow[dr] & & P_3[1] \arrow[dr] & & \\
 & P_2 \arrow[ur] \arrow[dr] & & I_2 \arrow[ur] \arrow[dr] & & P_2[1] \arrow[dr] & \\
P_3 \arrow[ur] & & S_2 \arrow[ur] & & I_1 \arrow[ur] & & P_1[1]
\end{tikzcd}$$

From here, we may associate a coordinate system to the indecomposable objects, with coordinates being ordered pairs in $\mathbb{Z}/6\mathbb{Z}$. Let $P_3 = (0,2)$, then we define the coordinates for each indecomposable object in the Auslander-Reiten quiver by using the following rule for each mesh relation.

$$\begin{tikzcd}
 & (i, j+1) \arrow[dr] & \\
(i,j) \arrow[ur] \arrow[dr] & & (i+1, j+1) \\
 & (i+1, j) \arrow[ur] &
\end{tikzcd}$$

We can now relabel the Auslander-Reiten quiver with the coordinate systems to get;

$$\begin{tikzcd}[column sep=small]
 & & (0,4) \arrow[dr] & & (1,5) \arrow[dr] & & \\
 & (0,3) \arrow[ur] \arrow[dr] & & (1,4) \arrow[ur] \arrow[dr] & & (2,5) \arrow[dr] & \\
(0,2) \arrow[ur] & & (1,3) \arrow[ur] & & (2,4) \arrow[ur] & & (3,5)
\end{tikzcd}$$

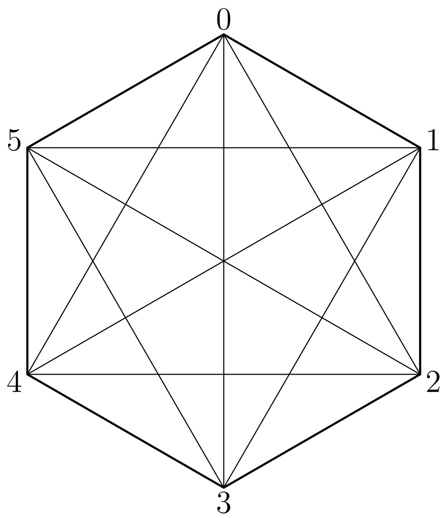


Figure 2.2: A representation of the set of indecomposable objects of $\mathcal{C}_{A_3}$ seen as arcs on the hexagon.

The indecomposable objects of a cluster tilting subcategories of $\mathcal{C}_{A_n}$ are in bijection with triangulations of a hexagon, and mutation of an object in a cluster tilting subcategory is shown below.

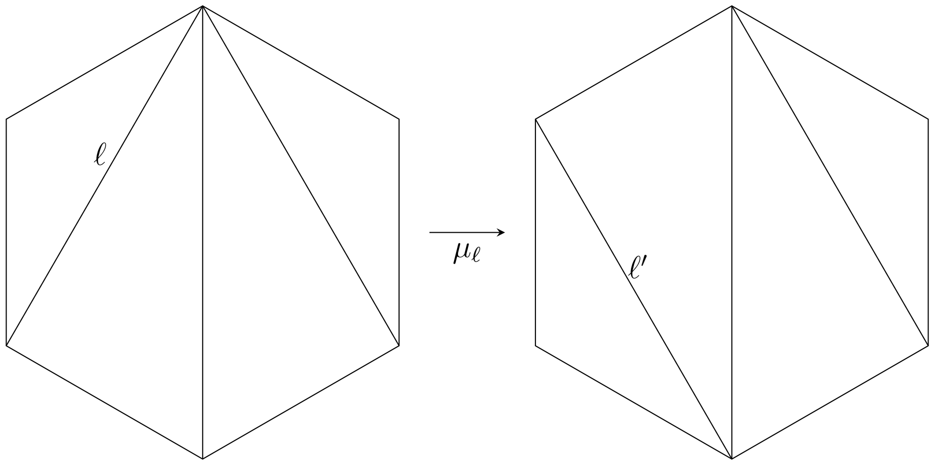


Figure 2.3: A representation of the mutation of a cluster tilting subcategory by the indecomposable object corresponding to the arc ℓ.

§ 2.2 | Discrete Cluster Categories of Type A_∞

Our main object of interest is the family of categories known as the discrete cluster categories of type A_∞, as well as their completions in the sense of Paquette-Yıldırım. Although discrete cluster categories of Dynkin type A_∞, which we label $\mathcal{C}_n$ where n represents the number of accumulation points, precede the work by Gratz, Holm, and Jørgensen [35] by a few years (particularly in [38], [39]), we follow their construction here. Later on, we pass to a completion of the discrete cluster categories of Dynkin type A_∞, labelled $\overline{\mathcal{C}}_n$, following the Paquette-Yıldırım construction [58].

We start by defining the marked surface we use to construct the category $\mathcal{C}_n$, where we always consider the circle S^1 to have an anti-clockwise orientation.

Definition 2.2.1. A subset $\mathscr{M}$ of the circle S^1 is called *admissible* if it satisfies the following conditions:

1. $\mathscr{M}$ has infinitely many elements,
2. $\mathscr{M} \subset S^1$ is a discrete subset,
3. $\mathscr{M}$ satisfies the *two-sided limit condition*, i.e. each $x \in S^1$ which is the limit of a sequence in $\mathscr{M}$ is a limit of both an increasing and decreasing sequence from $\mathscr{M}$ with respect to the cyclic order.

We call the points at which the two-sided limits in $\mathscr{M}$ converge *accumulation points*. Note that these accumulation points are not in $\mathscr{M}$, as they do not have defined successors and predecessors, meaning $\mathscr{M}$ would not be discrete if they were to be included. As in [35], we may think of $\mathscr{M}$ as the vertices of the ∞-gon. For an admissible subset $\mathscr{M} \in S^1$, we label the set of accumulation points $L(\mathscr{M})$, and give them a cyclic ordering induced by the orientation of S^1.

Let $a \in L(\mathscr{M})$ be an accumulation point, we define the *successor* $a^+ \in L(\mathscr{M})$ of a (resp. *predecessor* $a^- \in L(\mathscr{M})$ of a) to be the accumulation point such that $a \leq b \leq a^+$ with $b \in L(\mathscr{M})$ ($a^- \leq c \leq a$ with $c \in L(\mathscr{M})$) implies $b = a$ or $b = a^+$ (resp. $c = a^-$ or $c = a$). We define the *segment* between a and a^+ to be the set of marked points $\{x \in \mathscr{M} \mid a < x < a^+\}$, where the ordering is given by the orientation of S^1. If $|L(\mathscr{M})| = n$, then there exists n segments in $\mathscr{M}$, and notably all of $\mathscr{M}$ is one segment in the one accumulation point case.

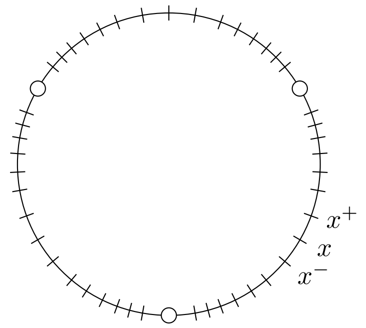


Figure 2.4: An admissible subset $\mathscr{M}$ of S^1. The marked points in $\mathscr{M}$ converge to the accumulation points represented as small circles, and each marked point x has both a predecessor and a successor, labelled x^- and x^+ respectively.

Definition 2.2.2. [35, Definition 0.2] A *arc of* $\mathscr{M}$ is a subset $\ell_X = \{x_0, x_1\} \subset \mathscr{M}$ where $x_1 \notin \{x_0^-, x_0, x_0^+\}$, where x^+ and x^- are the successor and predecessor respectively to $x \in \mathscr{M}$. If $\ell_Y = \{y_0, y_1\}$ is another arc, then ℓ_X and ℓ_Y *cross* if $x_0 < y_0 < x_1 < y_1 < x_0$ or $x_0 < y_1 < x_1 < y_0 < x_0$.

If we consider the disc D bounded by S^1, then we may think of an arc $\ell_X = \{x_0, x_1\}$ as being a representative of the isotopy class of non-self-intersecting curves in D between the marked points x_0 and x_1. Two arcs cross if any of their representative curves cross in the interior of D.

There are two types of arcs for us to consider; *short arcs* are the arcs that have both endpoints in the same segment, and *long arcs* are the arcs with endpoints in different segments.

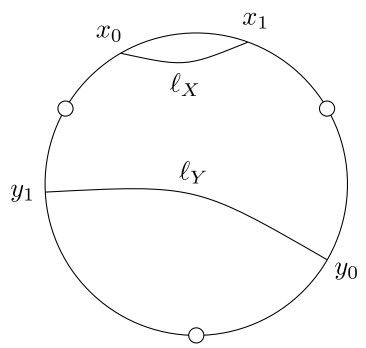


Figure 2.5: Two arcs of $\mathscr{M}$, where ℓ_X is a short arc and ℓ_Y is a long arc.

By considering this combinatorial description of arcs and crossings, in [35] the authors associate a k-linear, Hom-finite, Krull-Schmidt, 2-Calabi-Yau, triangulated category to $\mathscr{M}$, labelled $\mathcal{C}(S^1, \mathscr{M})$ and called a *discrete cluster category of Dynkin type* A_∞. The arcs of the ∞-gon are in bijection with the indecomposable objects of $\mathcal{C}(S^1, \mathscr{M})$, where ℓ_X is the arc corresponding to the indecomposable object $X \in \mathcal{C}(S^1, \mathscr{M})$. Given two indecomposable objects $X, Y \in \mathcal{C}(S^1, \mathscr{M})$, then $\mathrm{Hom}_{\mathcal{C}(S^1, \mathscr{M})}(X, Y) \cong k$ if the corresponding arcs ℓ_X and $\ell_{Y[-1]}$ cross, otherwise there are no morphisms from X to Y.

Let $\mathscr{M}$ and $\mathscr{M}'$ be admissible subsets of S^1 such that $|L(\mathscr{M})| = |L(\mathscr{M}')| = n$, then there is an equivalence of categories between $\mathcal{C}(S^1, \mathscr{M})$ and $\mathcal{C}(S^1, \mathscr{M}')$. Therefore for all $n \in \mathbb{Z}_{\geq 1}$, we will consider an admissible subset of S^1 with n accumulation points, $\mathscr{M}^n$, and so we consider the category $\mathcal{C}_n := \mathcal{C}(S^1, \mathscr{M}^n)$ as the representative of the equivalence class of discrete cluster categories of Dynkin type A_∞ with n accumulation points.

We have the following in $\mathcal{C}_n$;

1. $\mathcal{C}_n$ is 2-Calabi-Yau, that is there are natural isomorphisms
$$\mathrm{Hom}_{\mathcal{C}_n}(X, Y) \cong D\,\mathrm{Hom}_{\mathcal{C}_n}(Y, X[2]),$$
where $D(-) = \mathrm{Hom}_k(-, k)$ and $[1] : \mathcal{C}_n \to \mathcal{C}_n$ is the suspension functor.

2. The suspension functor acts on arcs by taking $\ell_X = \{x_0, x_1\}$ to $\ell_{X[1]} = \{x_0^-, x_1^-\}$.

3. We have
$$\mathrm{Ext}^1(X, Y) \cong \begin{cases} k & \text{if } \ell_X \text{ and } \ell_Y \text{ cross,} \\ 0 & \text{else,} \end{cases}$$
for indecomposable objects $X, Y \in \mathcal{C}_n$.

4. If $\ell_X = \{x_0, x_1\}$ and $\ell_Y = \{y_0, y_1\}$ with $x_0 \leq y_0 \leq x_1^{--} < x_1 \leq y_1 \leq x_0^{--}$, then a morphism $X \to Y$ factors through S if and only if $\ell_S = \{s_0, s_1\}$ with $x_0 \leq s_0 \leq y_0$ and $x_1 \leq s_1 \leq y_1$.

5. Let $\ell_X = \{x_0, x_1\}$ and $\ell_Y = \{y_0, y_1\}$ such that $\mathrm{Ext}^1(X, Y) \cong k$, then we have the following distinguished triangles in $\mathcal{C}_n$
$$X \to A \oplus B \to Y \to X[1]$$
$$Y \to C \oplus D \to X \to Y[1],$$
where $\ell_A = \{x_0, y_1\}, \ell_B = \{y_0, x_1\}, \ell_C = \{x_0, y_0\}$ and $\ell_D = \{x_1, y_1\}$.

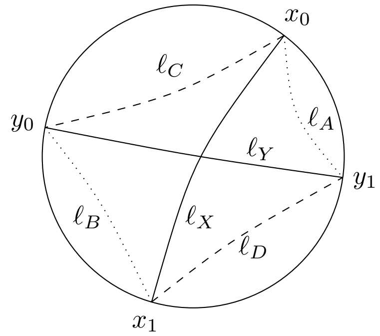


Figure 2.6: Arcs corresponding to the distinguished triangles induced by non-trivial $\mathrm{Ext}^1(X, Y) \cong \mathrm{Ext}^1(Y, X)$ groups.

Notice that Ext^1-spaces are symmetric in the two arguments, which is a consequence of $\mathcal{C}_n$ having the 2-Calabi-Yau property.

Definition 2.2.3. Let X be an indecomposable object in $\mathcal{C}_n$. Let $\mathscr{M}_X \subseteq \mathscr{M}$ be the set of marked points that are the endpoints of the arcs corresponding to suspensions and desuspensions of X.

If $A \cong \bigoplus_{i=1}^{l} X_i$, with all X_i indecomposable, then $\mathscr{M}_A = \bigcup_{i=1}^{l} \mathscr{M}_{X_i}$. We call $\mathscr{M}_A$ the *orbit of A in $\mathscr{M}$*. If we have $\mathscr{M}_A = \mathscr{M}$, then we say ℓ_A has a *complete orbit in $\mathscr{M}$*.

In particular, $\mathscr{M}_X$ is equal to the union of the segments containing an endpoint of ℓ_X, and so any object in $\mathcal{C}_1$ has a complete orbit in $\mathscr{M}$.

The cluster-tilting subcategories of $\mathcal{C}_n$ were classified by Gratz, Holm and Jørgensen in [35] with combinatorial descriptions. For this, they defined what it means for a sequence of marked points in $\mathscr{M}$ to converge to a point.

Definition 2.2.4. [35, Definition 1.4] Let $\{x_i\}_{i\in\mathbb{Z}_{i\geq 0}}$ be a convergent sequence in $\mathscr{M}$. If $\{x_i\}_{i\in\mathbb{Z}_{i\geq 0}}$ converges to $p \in L(\mathscr{M})$, then we write $x_i \to p$.

- We say that $x_i \to p$ *from below* if there is a $\mu \in S^1\backslash\{p\}$ such that $x_i \in [\mu, p]$ from some step.
- We say that $x_i \to p$ *from above* if there is a $\nu \in S^1\backslash\{p\}$ such that $x_i \in [p, \nu]$ from some step.

Furthermore, they introduced two different combinatorially defined sets of arcs based on convergence of arcs to a point, known as a leapfrog and a fountain, that can either form part of a triangulation, or a whole triangulation of $(S^1, \mathscr{M})$. These collections are important in the classification of the cluster-tilting subcategories of $\mathcal{C}_n$.

Definition 2.2.5. [35, Definition 0.4] Let $\mathscr{X}$ be a set of arcs of $\mathscr{M}$.

- Let $a \in L(\mathscr{M})$, then we say that *$\mathscr{X}$ has a leapfrog converging to $a \in L(\mathscr{M})$* if there is a sequence $\{x_i, y_i\}_{i\in\mathbb{Z}_{\geq 0}}$ of arcs from $\mathscr{X}$ with $x_i \to a$ from below and $y_i \to a$ from above. We call the set of arcs $\{x_i, y_i\}_{i\in\mathbb{Z}_{\geq 0}}$ a *leapfrog*.

- Let $a \in L(\mathscr{M})$ and $z \in \mathscr{M}$, then we say that $\mathscr{X}$ *has a right fountain at* z *converging to* a if there is a sequence $\{z, x_i\}_{i\in\mathbb{Z}_{\geq 0}}$ from $\mathscr{X}$ with $x_i \to a$ from below. We say that $\mathscr{X}$ *has a left fountain at* z *converging to* a if there is a sequence $\{z, y_i\}_{i\in\mathbb{Z}_{\geq 0}}$ from $\mathscr{X}$ with $y_i \to a$ from above.

- We say that $\mathscr{X}$ *has a fountain at* z *converging to* a if it has a right fountain and a left fountain at z converging to a. We call the set of arcs $\{z, x_i\}_{i\in\mathbb{Z}_{\geq 0}} \cup \{z, y_i\}_{i\in\mathbb{Z}_{\geq 0}}$ a *fountain.*

This definition does not stipulate that either a fountain or a leapfrog have to be maximal in any sense, the endpoints of the arcs simply have to form a subset of $\mathscr{M}$ that converges to an accumulation point, from either above, below, or both.

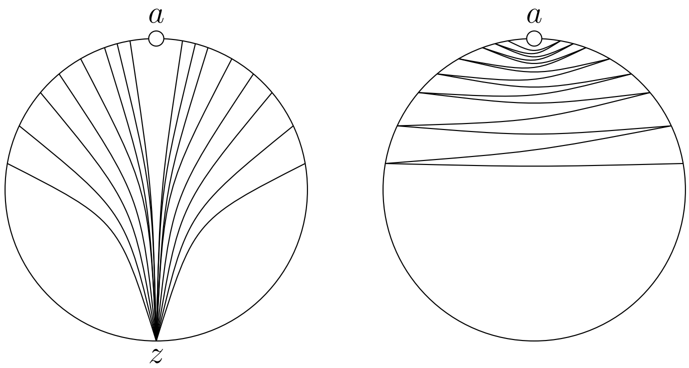


Figure 2.7: An example of a fountain at z converging to a on the left, and an example of a leapfrog converging to a on the right.

§ 2.2.1 | Cluster Tilting subcategories of $\mathcal{C}_n$

When we are working in a 2-Calabi-Yau category, it is clear that $(\mathcal{S}[-1])^{\perp} = {}^{\perp}(\mathcal{S}[1])$, and so we only need to show that $\mathcal{S} = {}^{\perp}(\mathcal{S}[1])$ for $\mathcal{S}$ to be a cluster tilting subcategory. The cluster tilting subcategories of $\mathcal{C}_n$ were classified by Gratz-Holm-Jørgensen in [35] for any $n \geq 1$.

Let $\operatorname{add}(\mathcal{X})$ denote the full additive subcategory of $\mathcal{C}_n$, for some collection of objects $\mathcal{X} \subset \operatorname{Ob}(\mathcal{C}_n)$.

Theorem 2.2.6. *[35, Theorem 5.7] Let $\mathscr{X}$ be a set of arcs in $(S^1, \mathscr{M})$, and let $\mathcal{X}$ be the set of objects in $\mathcal{C}_n = \mathcal{C}(S^1, \mathscr{M})$ corresponding to the set $\mathscr{X}$. Then $\operatorname{add}(\mathcal{X})$ is a cluster tilting subcategory of $\mathcal{C}_n$ if and only if $\mathscr{X}$ is a maximal set of pairwise non-long arcs, such that for each $a \in L(\mathscr{M})$, the set $\mathscr{X}$ has a fountain or leapfrog converging to a.*

In Section 3.2.2, we will need to chose a cluster tilting subcategory of $\mathcal{C}_n$ to work with, for each $n \geq 1$. For some technical reasons, it makes more sense for us to consider a cluster tilting subcategory with a leapfrog converging to each accumulation point rather than a fountain. With this in mind, from now on we shall fix a cluster tilting subcategory of $\mathcal{C}_n$ with which we want to work.

Construction 2.2.7. Given we have n accumulation points induced by $\mathscr{M}$ then we have $L(\mathscr{M}) = \{a_1, \dots, a_n\}$, which has a cyclic ordering. Let $\{z_1, \dots, z_n\} \subset \mathscr{M}$ be a set of marked points such that $z_i \in (a_i, a_{i+1})$ for all $i = 1, \dots, n$. By taking the arcs $\ell_{Z_i} = \{z_i, z_{i+1}\}$ for all $i = 1, \dots, n$ we inscribe an n-gon inside the circle; the arc ℓ_{Z_i} shall correspond to the object $Z_i \in \mathcal{C}_n$ for all $i = 1, \dots, n$. For all $a_i \in L(\mathscr{M})$, let $\{z_i, z_{i+1}\}$ be the first arc in a leapfrog converging to a_{i+1}, such that if the arc $\{x, y\}$ is in the leapfrog, then so are either the arcs $\{x^+, y\}$ and $\{x, y^+\}$, or the arcs $\{x^-, y\}$ and $\{x, y^-\}$. We label the leapfrog converging to a_i by $\mathscr{L}_i$. Without loss of generality, we may let $\ell_{Y_i} = \{z_i, z_{i+1}^-\}$ be in $\mathscr{L}_{i+1}$. Notice that this implies that for every arc $\ell_M \in \mathscr{L}_{i+1}$ that is not ℓ_{Z_i}, there are exactly two other arcs, $\ell_{M^-}, \ell_{M^+} \in \mathscr{L}_{i+1}$ that share an endpoint with ℓ_M.

For $n \geq 4$, we give the inscribed n-gon a fan triangulation; that is, we add the arcs $\ell_{X_i} = \{z_1, z_i\}$ for all $i = 2, \dots, n$. Notice that this means that we have $X_2 \cong Z_1$ and $X_n \cong Z_n$. This collection of arcs will look like Figure 2.8, along with a leapfrog $\mathscr{L}_i$ converging to each accumulation point.

Theorem 2.2.6 ensures that this construction corresponds to a cluster tilting subcategory of $\mathcal{C}_n$, as the collection of arcs is a maximal set of non-long arcs, and each accumulation point has a leapfrog converging to it.

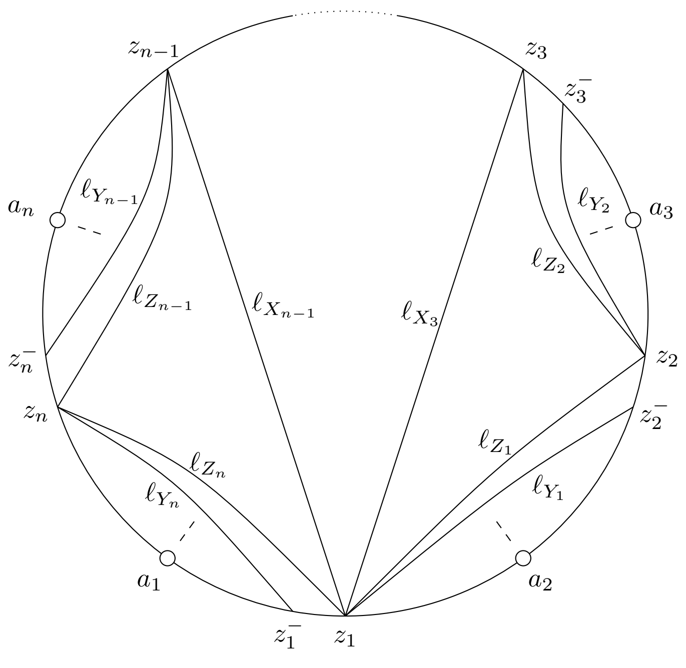


Figure 2.8: The collection of arcs corresponding to the cluster tilting subcategory we wish to consider for n two-sided accumulation points, where leapfrogs are represented by the dashed lines. By Theorem 2.2.6, this collection of leapfrogs together with the fan triangulation of the inscribed n-gon corresponds a cluster tilting subcategory of $\mathcal{C}_n$.

Given some cluster tilting subcategory $\mathcal{T}$, with an exchange pair (M, M^*) and $M \in \mathcal{T}$, then by [35] the arc ℓ_{M^*} must cross an arc ℓ_X for some indecomposable object

$X \in \mathcal{T}$, as $\mathcal{T}$ corresponds to a maximal set of pairwise non-long arcs. However, there exists a cluster tilting subcategory $\mathcal{S}$ with $M^* \in \mathcal{S}$ such that $\mathcal{S}$ and $\mathcal{T}$ share an almost cluster tilting subcategory $\mathcal{T}'$, which has complements M and M^*. This implies that the arcs ℓ_M and ℓ_{M^*} must cross. Therefore we have $\mathrm{Ext}^1(M, M^*) \cong k$, and therefore $\dim_k \mathrm{Hom}_{\mathcal{C}_n}(M, M^*[1]) = 1$. By the symmetry of $\mathrm{Ext}^1(-,-)$ inherent in 2-Calabi-Yau triangulated categories, we also have $\dim_k \mathrm{Hom}_{\mathcal{C}_n}(M^*, M[1]) = 1$.

It is important to note that by definition $B_M, B_{M^*} \in \mathcal{T}$ (Section 1.6), and the pair of arcs ℓ_{B_M} and $\ell_{B_{M^*}}$ form a quadrilateral with the arcs ℓ_M and ℓ_{M^*}.

§ 2.3 | The Holm-Jørgensen Construction

Holm and Jørgensen were the first to look at the ∞-gon through the lens of a cluster category in [38]. They construct a subcategory of a derived category of a differential graded algebra, that can be represented as the arcs on an ∞-gon (or, equivalently, as curves between non-consecutive integers on the number line). This construction is shown to be equivalent to $\mathcal{C}_1$ in [39].

Let $R = k[T]$ be the polynomial ring over a field k, seen as a differential graded algebra with zero differential and T placed in homological degree 1. Then $\mathcal{D} = \mathrm{D}^f(R)$ is the subcategory of the derived category of R with objects having finite dimensional homology over k. This is a Hom-finite, k-linear, Krull-Schmidt, 2-Calabi-Yau, triangulated category, where the indecomposable objects are isomorphic to the object

$$X_r = R/(T^{r+1})$$

for each integer $r \geq 0$, up to suspension.

The Auslander-Reiten quiver of $\mathcal{D}$ is

$$\begin{tikzcd}[column sep=small]
 & X_3[0] \arrow[dr] & & X_3[-1] \arrow[dr] & & X_3[-2] \arrow[dr] & & X_3[-3] \\
X_2[1] \arrow[ur] \arrow[dr] & & X_2[0] \arrow[ur] \arrow[dr] & & X_2[-1] \arrow[ur] \arrow[dr] & & X_2[-2] \arrow[ur] \arrow[dr] & \\
 & X_1[1] \arrow[ur] \arrow[dr] & & X_1[0] \arrow[ur] \arrow[dr] & & X_1[-1] \arrow[ur] \arrow[dr] & & X_1[-2] \\
X_0[2] \arrow[ur] & & X_0[1] \arrow[ur] & & X_0[0] \arrow[ur] & & X_0[-1] \arrow[ur] &
\end{tikzcd}$$

Figure 2.9: The Auslander-Reiten quiver of $\mathcal{D}$, $Q_{\mathcal{D}}$.

In [38] the authors associate a coordinate system to the indecomposable objects of $\mathcal{D}$ according to the rule

$$(m, n) = X_{n-m-2}[-n].$$

This allows them to associate the indecomposable objects to curves between integers on the number line, which are non-consecutive because we require $n - m - 2 \geq 0$, therefore $m < n$ and $n - m \geq 2$. It is also clear from the Auslander-Reiten quiver that $(m, n)[1] = (m-1, n-1)$.

The morphism space between two indecomposable objects may also be seen on the Auslander-Reiten quiver $Q_{\mathcal{D}}$, in Figure 2.9, however we use the following definition and proposition found in [38].

Definition 2.3.1. Let $X = (i, j)$ be an indecomposable object in $\mathcal{D}$. The unbounded subsets $H^-(X)$ and $H^+(X)$ of $Q_{\mathcal{D}}$ are given by

$$\begin{aligned} H^-(X) =& \{(m,n) \,|\, m \leq i-1,\ i+1 \leq n \leq j-1\}, \\ H^+(X) =& \{(m,n) \,|\, j+1 \leq n,\ i+1 \leq m \leq j-1\}. \end{aligned}$$

We write $H(X) = H^-(X) \cup H^+(X)$.

Using this definition, Holm and Jørgensen prove the following proposition.

Proposition 2.3.2 ([38]). *Let X and Y be indecomposable objects in $\mathcal{D}$. Then*

$$\mathrm{Hom}_{\mathcal{D}}(X,Y) \cong \begin{cases} k, & \textit{for } Y \in H(X[1]), \\ 0, & \textit{else.} \end{cases}$$

They later prove that the two curves on the number line associated to X and Y cross if and only if $Y \in H(X[1])$ if and only if $X \in H(Y[-1])$. By considering the indecomposable objects in $\mathcal{D}$ as curves between non-consecutive integers on the number line, and that morphism spaces between objects are induced by the crossing of these curves, it is relatively straightforward to see that $\mathcal{D}$ and $\mathcal{C}_1$ are equivalent as categories.

§ 2.4 | A Completion of $\mathcal{C}_n$

We now give a completion of discrete cluster categories of Dynkin type A_∞, which was first presented by Paquette and Yıldırım in [58]. In the case of a single accumulation point, there is an alternative yet equivalent completion due to Fisher [27], which involves formally adding a certain class of homotopy colimits into $\mathcal{C}_1$. However we stay with the Paquette-Yıldırım construction, which is much more combinatoric in nature, and which also provides us with a useful inbuilt functor between $\mathcal{C}_{2n}$ and a completion of a discrete cluster category.

§ 2.4.1 | The Paquette Yıldırım Construction

We follow the construction given in [58], and state some results about the construction that will be of use to us.

Construction 2.4.1. [58] Let $a_1, \ldots, a_n \in L(\mathscr{M})$ be the accumulation points of an admissible subset $\mathscr{M}$ of S^1, which corresponds to the category $\mathcal{C}(S^1, \mathscr{M})$. Then we replace each $a_i \in L(\mathscr{M})$ with an interval $[a_i^+, a_i^-]$ containing the points $a_{ij} \in (a_i^+, a_i^-)$ for all $j \in \mathbb{Z}$, such that $a_{ij} < a_{ij'}$ if and only if $j < j'$, and we have $\lim_{j \to \infty} a_{ij} = a_i^+$ and $\lim_{j \to -\infty} a_{ij} = a_i^-$. We denote this new admissible subset of S^1 as $\mathscr{M}'$. Let $\mathcal{D}_n$ be the full subcategory of $\mathcal{C}(S^1, \mathscr{M}')$ consisting of objects that correspond to short arcs with endpoints in one of the segments (a_i^-, a_i^+).

Let Ω to be the set of morphisms f in $\mathrm{Mor}(\mathcal{C}_n)$ such that the cone of f corresponds to an arc with both endpoints in an interval (a_i^-, a_i^+) for some $i = 1, \dots, n$, i.e.

$$\Omega = \{f \in \mathrm{Mor}(\mathcal{C}_n) \mid \mathrm{cone}\, f \in \mathcal{D}_n\}.$$

Then the completion of $\mathcal{C}(S^1, \mathscr{M}) = \mathcal{C}_n$, in the sense of Paquette-Yıldırım, is defined to be the localisation $\overline{\mathcal{C}}(S^1, \mathscr{M}) := \mathcal{C}(S^1, \mathscr{M}')[\Omega^{-1}]$. As before, we shall denote the completion by $\overline{\mathcal{C}}_n$ when the admissible subset $\mathscr{M}$ has n accumulation points and need not be specified.

As well as the short arcs and long arcs that we consider in $\mathcal{C}_n$, there are two more types of arcs to think about in the Paquette-Yıldırım completion; *limit arcs* are the arcs where a single endpoint is at an accumulation point, and *double limit arcs* are those arcs with both endpoints at accumulation points.

The following lemma shows how the Ext^1-spaces are given in $\overline{\mathcal{C}}_n$.

Lemma 2.4.2. *[58, Prop. 3.14] Let $X, Y \in \overline{\mathcal{C}}_n$ be indecomposable objects. Then* $\mathrm{Hom}_{\overline{\mathcal{C}}_n}(X, Y[1])$ *is at most one dimensional. It is one dimensional if and only if one of the following conditions are met for the arcs ℓ_X and ℓ_Y:*

- *ℓ_X, ℓ_Y cross,*
- *$\ell_x \neq \ell_Y$ share exactly one accumulation point, and we can go from ℓ_X to ℓ_Y by rotating ℓ_X about their common endpoint following the orientation of S^1,*
- *$\ell_X = \ell_Y$ are double limit arcs.*

Unfortunately, Lemma 2.4.2 shows that $\overline{\mathcal{C}}_n$ is no longer 2-Calabi-Yau, as any two limit arcs that share an endpoint at an accumulation point have a non-trivial Ext^1-space in only one direction.

It is useful for us to note that Construction 2.4.1 is equivalent to the Verdier localisation of $\mathcal{C}_{2n}$ by the full subcategory $\mathcal{D}_n$, which we will see is equivalent to the disjoint union of n copies of $\mathcal{C}_1$.

Proposition 2.4.3. *There is an equivalence of categories:*

$$\mathcal{D}_n \simeq \bigsqcup_{i=1}^{n} \mathcal{C}_1,$$

where $\bigsqcup_{i=1}^{n} \mathcal{C}_1$ denotes the disjoint union of n copies of $\mathcal{C}_1$.

Proof. We note that $\mathcal{D}_n$ only has objects corresponding to short arcs on n segments, and no two short arcs on different segments cross, so there are no morphisms between objects corresponding to short arcs on different segments. In other words, any two subcategories that correspond to arcs on different segments, are orthogonal. We label the segments $s_1, \ldots, s_n$.

There exists a canonical functor ψ from $\mathcal{D}_1$, a subcategory of $\mathcal{C}_2 = \mathcal{C}(S^1, \mathscr{M}')$, to $\mathcal{C}_1 = \mathcal{C}(S^1, \mathscr{M})$, that takes an arc $\ell_X = \{x', y'\}$ in $\mathcal{D}_1$ to an arc $\ell_{\psi X} = \{x, y\}$ in $\mathcal{C}_1$, such that for every marked point $x' < z' < y'$ with $x', y', z' \in \mathscr{M}'$, there exists a marked point $x < z < y$ with $x, y, z \in \mathscr{M}$, and every arc with $x' \in \mathscr{M}'$ as an endpoint is sent to an arc with $x \in \mathscr{M}$ as an endpoint. The functor ψ is fully faithful on Ext^1-spaces as two arcs that cross in $\mathcal{D}_1$ are sent to two arcs that also cross in $\mathcal{C}_1$. This is an equivalence of categories between $\mathcal{D}_1$ and $\mathcal{C}_1$.

There also exists a fully faithful functor ρ_i from $\mathcal{D}_n$ to $\mathcal{D}_1$ that is the functor that canonically maps objects corresponding to arcs in the segment s_i to $\mathcal{D}_1$, and is the zero functor on all other objects. This is an equivalence of categories between $\mathcal{D}_1$ and the full subcategory of $\mathcal{D}_n$ corresponding to arcs in the segment s_i. The category $\mathcal{D}_n$ has n mutually orthogonal subcategories equivalent to $\mathcal{D}_1$ by construction, and so the canonical functor

$$\rho = \sum_{i=1}^{n} (\rho_i) \colon \mathcal{D}_n \to \bigsqcup_{i=1}^{n} \mathcal{D}_1$$

is an equivalence of categories.

Hence there is an equivalence of categories:

$$\iota := \rho^{-1} \circ \left(\bigsqcup_{i=1}^{n} \psi \right) : \bigsqcup_{i=1}^{n} \mathcal{C}_1 \xrightarrow{\sim} \mathcal{D}_n.$$

□

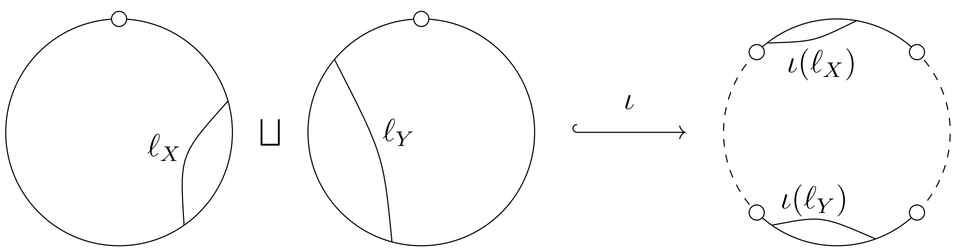


Figure 2.10: How the equivalence ι from $\mathcal{C}_1 \sqcup \mathcal{C}_1$ to $\mathcal{D}_2$ acts on corresponding arcs.

§ 2.4.2 | Alternative Completions of $\mathcal{C}_1$

The completion of $\mathcal{C}_1$ in the Paquetter-Yıldırım sense is not the only completion that has been discussed in the literature. In this section we briefly discuss some other completions of $\mathcal{C}_1$, found in [27] and [12] respectively.

The Fisher Completion

In [27], Fisher constructs a completion of $\mathcal{C}_1$ by formally including a certain class of homotopy colimits into $\mathcal{C}_1$. To do this they define a *slice* in the Auslander-Reiten quiver $Q_{\mathcal{D}}$, Figure 2.9, as a collection of vertices and arrows associated to a direct system of irreducible morphisms

$$X_0[n] \to X_1[n-1] \to \cdots \to X_i[n-i] \to \cdots$$

and then take the homotopy colimit of this collection, which is defined as follows.

Definition 2.4.4 ([60])**.** Let $\{f_n : X_n \to X_{n+1} \mid n \in \mathbb{N}\}$ be a sequence of morphisms in $\mathcal{T}$, a triangulated category. The *homotopy colimit* of this sequence is an object $\operatorname{hocolim}_i X_i = X \in \mathcal{T}$ that fits into the distinguished triangle

$$\bigoplus_{n\in\mathbb{N}} X_n \xrightarrow{\psi} \bigoplus_{n\in\mathbb{N}} X_n \to X \to \bigoplus_{n\in\mathbb{N}} X_n[1].$$

Where ψ is the morphism given by the infinite matrix

$$\psi = \begin{pmatrix} e_{X_1} & f_1 & 0 & 0 & \cdots \\ 0 & e_{X_2} & f_2 & 0 & \cdots \\ 0 & 0 & e_{X_3} & f_3 & \cdots \\ 0 & 0 & 0 & e_{X_4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with e_{X_n} denoting the identity morphism on X_n.

Let $\mathcal{D} = \mathrm{D}(k[T])$, where $k[T]$ is seen as a differential graded ring with T placed in homological degree 1. That is, the ambient derived category of the category considered in [38] by Holm and Jørgensen (Section 2.3), and so $\mathcal{C}_1 \subset \mathcal{D}$. Let

$$X_0[n] \to X_1[n-1] \to \cdots \to X_i[n-i] \to \cdots$$

be a slice in the Auslander-Reiten quiver of $\mathcal{C}_1$. We label the homotopy colimit of this slice $E_n \in \mathcal{D}$, and there is a natural identification $E_n[m] = E_{n+m}$ in $\mathcal{D}$ for all $m \in \mathbb{Z}$.

Then Fisher defines the completion of $\mathcal{C}_1$ to be the full subcategory

$$\widehat{\mathcal{C}}_1 = \operatorname{add}\{E_n, X_m[i]\} \subset \mathcal{D}$$

for all $n, i \in \mathbb{Z}$ and $m \in \mathbb{N}$. We do not consider arbitrary coproducts inside of $\widehat{\mathcal{C}}_1$.

It was shown by August, Cheung, Faber, Gratz and Schroll in [3] that there is an equivalence of triangulated categories between $\widehat{\mathcal{C}}_1$ and $\overline{\mathcal{C}}_1$. In fact, it was shown that these completions occur naturally as the category of $\mathbb{Z}$-graded, maximal Cohen-Macaulay modules over $\mathbb{C}[x, y]/(x^2)$, with x in degree 1 and y in degree -1.

§ 2.5 | Marked Surfaces and Cluster Categories

From the introduction of cluster algebras by Fomin and Zelevinsky in [31] it was known that there was a connection between cluster algebras and triangulations of marked surfaces. They have a brief discussion on an example of a cluster algebra, the homogeneous coordinate ring $\mathbb{C}[Gr_{2,n+3}]$ of the Grassmannian of 2 dimensional subspaces of $\mathbb{C}^{n+3}$, and how its cluster variables are in bijection with the diagonals of an $(n+3)$-gon, and clusters in bijection with triangulations of the $(n+3)$-gon. In particular, the cluster categories considered throughout this thesis can be loosely seen as a limit of these categories as n tends to infinity. Since then there has been a lot of progress on the relationship between marked surfaces and cluster algebras/categories, and in this section we briefly go over the cluster categories of Dynkin type D_n, as well as looking at marked surfaces more generally.

Although not talked about explicitly here, a combinatorial model for the cluster categories of Dynkin type E_6, E_7 and E_8, along with the combinatorial model for a family of related cluster categories, was found by Lamberti in [48].

§ 2.5.1 | Cluster Categories of Dynkin Type D_n

After the cluster categories of Dynkin type A_n, the next combinatorial model to follow was introduced by Schiffler in [63] for the cluster category of Dynkin type D_n. Schiffler constructs a cluster category from an n-gon with a single puncture, where some objects are in correspondence to ordered pairs of vertices (a, b) of the n-gon, where b is not the anti-clockwise successor of a. They also introduce a collection of *tagged arcs*, such that every vertex x induces two tagged arcs, $\ell_{M_x^-}$ and $\ell_{M_x^+}$, with both endpoints at the same vertex, and represented by a (potentially tagged) arc between the vertex and the puncture.

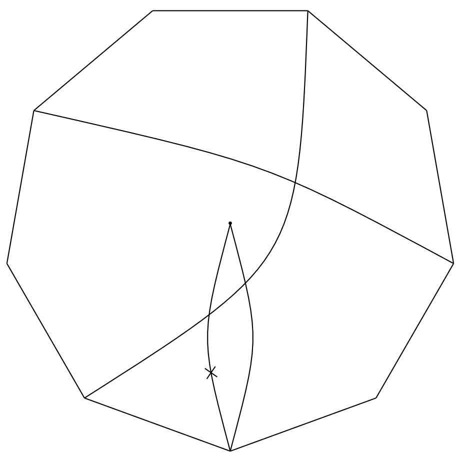


Figure 2.11: Some arcs on the punctured n-gon, where the cross denotes the tagged arc $\ell_{M_x^-}$ and the other arc with the same endpoints is $\ell_{M_x^+}$.

Schiffler then builds a category $\mathcal{D}$ by defining the tagged arcs in the punctured n-gon to be the indecomposable objects, and showing that the dimension of $\mathrm{Ext}^1(M, N)$ over a field k is equal to the *crossing number* of the two indecomposable objects M and N. The crossing number is defined to be the minimal number of intersections of two arcs ℓ_M and ℓ_N on the interior of the n-gon.

The suspension functor (denoted by the Auslander-Reiten translation, τ, in [63]) acts on tagged arcs with distinct endpoints in the same way as the non-punctured n-gon, by rotating each endpoint anti-clockwise to its successor, and acts on a tagged arc with a single endpoint, say M_x^-, by rotating the endpoint anti-clockwise to its successor, and by changing the sign of the tagged arc, i.e. $\tau M_x^- \cong M_{x'}^+$ where x' is the anti-clockwise successor to x.

§ 2.5.2 | Cluster Categories Via Marked Surfaces

There has been an abundance of work dedicated to understanding the cluster combinatorics and cluster structures inherent to marked surfaces in the years since the introduction of cluster algebras; such as in [28, 29, 20, 21, 52, 51] for cluster algebras, and in [16, 61, 69] for cluster categories.

In a series of papers [28, 29], Fomin and Thurston (as well as Shapiro in [28]) construct a cluster algebra from an arbitrary Riemann surface with boundary and finitely many marked points on each boundary segment. They go on to prove a series of results for these cluster algebras by using techniques found in combinatorial topology and hyperbolic geometry, exploiting a connection that until then had been commonly used in the reverse direction, using cluster algebra theory in certain topological and geometric settings.

Moving into the world of categories, Brüstle and Zhang [16] explicitly described a cluster category from a marked surface with finitely many marked points and without punctures. It was already known that a cluster category may be associated to a marked surface without punctures by combining two ideas; cluster categories from quivers with potential [1], and quivers with potential from triangulated surfaces [2, 47]. However, the aim of [16] is to combinatorically describe the objects and irreducible morphisms in the cluster category associated to a finitely marked surface without punctures. To do this, they classify the indecomposable objects as string objects and band objects, named for their connection to the objects in the module category of a string algebra. The string objects, similar to the objects in $\mathcal{C}_n$, are given by the homotopy classes of arcs between marked points, whilst the band objects are given by scalar products (by elements of $k^* = k\backslash\{0\}$) of elements in the fundamental group of the surface modulo certain relations.

This explicit description was generalised to the case of a punctured surface by Qiu and Zhou in [61]. To achieve this they combine the ideas presented in [16] for the non-punctured case, and the tagged arcs used in the combinatorial model for the cluster category associated to D_n [63].

§ 2.5.3 | Quivers With Potential Arising From Triangulations

For a surface S with a collection of marked points $\mathbb{M} \subset S$ we may construct a triangulation of the surface under some mild assumptions on $\mathbb{M}$. It was noticed in [28, 34] that for a triangulation of a marked surface $(S, \mathbb{M})$ we may associate a quiver with potential.

In this section we describe the construction of a quiver with potential arising from a triangulation of a surface, using [46] as a reference.

Definition 2.5.1. Let Q be a quiver, and let P be a possibly infinite linear combination of cycles in Q, with coefficients in some field k. Then P is a *potential* if no two cycles in P with non-zero coefficients can be obtained from each other by rotation. We say that the pair (Q, P) is a *quiver with potential.*

Example 2.5.2. Let Q be the quiver

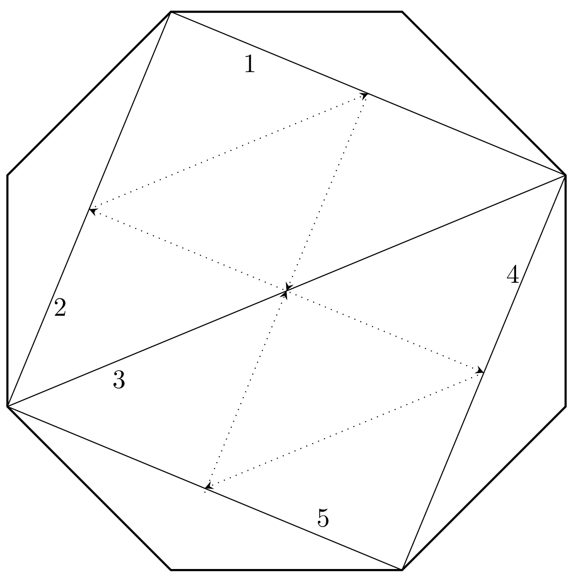
$$\begin{tikzcd}
1 \arrow[dr, "\alpha_1"] & & 4 \arrow[dd, "\alpha_4"] \\
& 3 \arrow[dl, "\alpha_3"] \arrow[ur, "\alpha_6"] & \\
2 \arrow[uu, "\alpha_2"] & & 5 \arrow[ul, "\alpha_5"]
\end{tikzcd}$$

and let $P = \alpha_1\alpha_2\alpha_3 + \alpha_6\alpha_5\alpha_4$. Then P is a potential, and the pair (Q, P) is a quiver with potential.

Construction 2.5.3. Let $(S, \mathbb{M})$ be a marked surface with all marked points on the boundary of S, and let τ denote a triangulation of $(S, \mathbb{M})$, with an arc in τ if it is non-isotopic to the boundary of S. The vertices of Q are in bijection with the arcs in τ, and an arrow α between two vertices i and j only exists if the arcs ℓ_i and ℓ_j are in the same triangle of τ. The arrow α goes from i to j if ℓ_j is anti-clockwise of ℓ_i about their shared endpoint.

The potential P is the linear combination of cycles coming from a triangle where all three edges are non-isotopic to the boundary of S, and cycles coming from linear paths around a puncture in S.

Example 2.5.4. Let S be an circle and $|\mathbb{M}| = 8$, with $\mathbb{M} \subset \partial S$, i.e. there are no punctures in $(S, \mathbb{M})$. Let τ be the following triangulation of $(S, \mathbb{M})$.

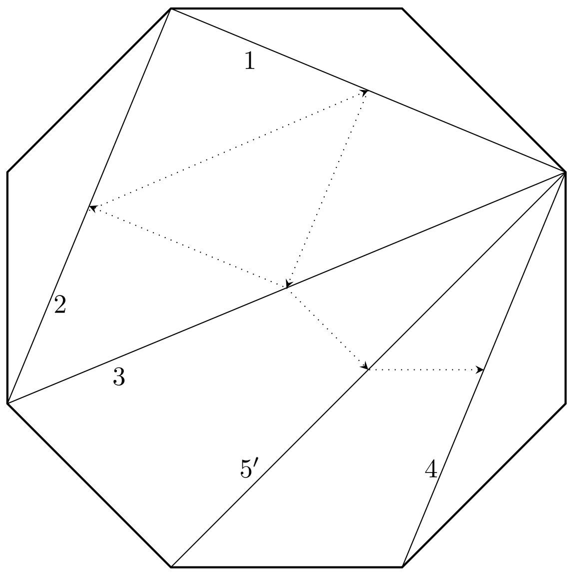


The dotted arrows form the quiver Q_τ associated to the triangulation τ. Therefore, we get the quiver with potential (Q_τ, P_τ) that has the following quiver,

$$\begin{tikzcd}
1 \arrow[dr, "\alpha_1"] & & 4 \arrow[dd, "\alpha_4"] \\
& 3 \arrow[dl, "\alpha_3"] \arrow[ur, "\alpha_6"] & \\
2 \arrow[uu, "\alpha_2"] & & 5 \arrow[ul, "\alpha_5"]
\end{tikzcd}$$

with potential $P_\tau = \alpha_1\alpha_2\alpha_3 + \alpha_6\alpha_5\alpha_4$.

Moreover, if we flip τ at the arc labelled 5, that is, remove the arc 5 from τ and replace it with the unique new arc $5'$ that forms a triangulation τ', then we get a new quiver with potential from τ'.



From τ' we get the quiver with potential $(Q_{\tau'}, P_{\tau'})$ given by the quiver

$$Q_{\tau'} = \quad \begin{tikzcd}
1 \arrow[dr, "\beta_1"] & & & \\
& 3 \arrow[dl, "\beta_3"] \arrow[r, "\beta_{5'}"] & 5' \arrow[r, "\beta_4"] & 4, \\
2 \arrow[uu, "\beta_2"] & & &
\end{tikzcd}$$

and the potential $P_{\tau'} = \beta_1\beta_2\beta_3$.

Note that the quiver $Q_{\tau'}$ is in fact the quiver Q_τ under Fomin-Zelevinsky mutation at the vertex 5.

The phenomenon exhibited in Example 2.5.4, that flipping an arc is equivalent to Fomin-Zelevinsky mutation at the corresponding vertex of the associated quiver, is no accident. In fact, it was shown by Fomin, Shapiro and Thurston in [28] that this is true for any triangulation of a marked surface that can be triangulated.

Proposition 2.5.5. *Let τ be an ideal triangulation of a marked surface. Suppose τ' is* *obtained by flipping an arc labelled i in τ. Then $Q_{\tau'} = \mu_i(Q_{\tau'})$.*

The language used in [28] is that of signed adjacency matrices, however there is a natural way of obtaining a signed adjacency matrix from a quiver without loops and 2-cycles, and these statements are equivalent for our purposes.

Furthermore, using Proposition 2.5.5, it is also shown in [28] that the cluster algebra associated to a surface is uniquely determined by the surface, and does not depend on a choice of initial triangulation.

Chapter 3

Grothendieck Groups

In this chapter we look at Grothendieck groups and some of their basic properties. We then compute the triangulated Grothendieck group of discrete cluster categories of Dynkin type A_∞. Subsequently, we compute the triangulated Grothendieck group of the Paquette-Yıldırım completion of the discrete cluster categories of Dynkin type A_∞.

§ 3.1 | Grothendieck Groups

The Grothendieck group of a category is a way of constructing an abelian group on the set of objects in the category in a universal way. This is a generalisation of the Grothendieck group over a commutative monoid, and obeys the same universal property as found in the commutative monoid setting.

Given an commutative monoid M with Grothendieck group K, there exists a monoid homomorphism $i : M \to K$ such that for any monoid homomorphism $f : M \to A$ for an abelian group A, there is a unique group homomorphism $g : K \to A$ such that $f = g \circ i$. That is, the following diagram commutes

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[dr, "i"'] & A. \\ & K. \arrow[u, dashed, "g"'] \end{tikzcd}$$

The general idea of the Grothendieck group of a category is to take the free abelian group generated by the isomorphism classes of objects of the category, then taking the quotient by a set of relations $[Y] = [X] + [Z]$ for all triples (X, Y, Z) in a distinguished class of triples. Here we discuss three different Grothendieck groups a category may have, depending upon the class of distinguished triples that we wish to consider. Throughout, for some category $\mathcal{C}$, we shall use the free abelian group, $G_0(\mathcal{C})$, with basis $\{[X] \mid X \in \mathcal{C}\}$, with $[X]$ denoting the isomorphism class of $X \in \mathcal{C}$.

In an additive category, that is, a category with a zero object, finite products and finite coproducts, such that there is an isomorphism between the product and coproduct of any two objects, we consider (X, Y, Z) as a distinguished triple if and only if $Y \cong X \oplus Z$.

Definition 3.1.1. Let $\mathcal{S}$ be an additive category. The *split Grothendieck group of* $\mathcal{S}$, $K_0^{\mathrm{add}}(\mathcal{S})$, is the group

$$K_0^{\mathrm{add}}(\mathcal{S}) = G_0(\mathcal{S})/\langle [X \oplus Y] - [X] - [Y] \,|\, X,\ Y \in \mathcal{S}\rangle.$$

When working with the split Grothendieck group, we will want to use the following well-known result:

Proposition 3.1.2. *[53, Proposition 2.1] Let $\mathcal{T}$ be a Krull-Schmidt category with a collection of non-isomorphic indecomposable objects $\{X_i\}_{i\in I}$. The split Grothendieck group of $\mathcal{T}$ is the free abelian group*

$$K_0^{\mathrm{add}}(\mathcal{T}) \cong \bigoplus_{i\in I} \mathbb{Z}\cdot [X_i]$$

with basis $\{[X_i]\}_{i\in I}$.

In a small abelian category, we choose a class of distinguished triples such that if

$$0 \to X \to Y \to Z \to 0$$

is a short exact sequence, then (X, Y, Z) is a distinguished triple.

Definition 3.1.3. Let $\mathcal{A}$ be a small abelian category. The *abelian Grothendieck group of* $\mathcal{A}$, $K_0^{\mathrm{ab}}(\mathcal{A})$, is the group

$$K_0^{\mathrm{ab}}(\mathcal{A}) = G_0(\mathcal{A})/\langle [Y] - [X] - [Z] \,|\, 0 \to X \to Y \to Z \to 0 \text{ a short exact sequence}\rangle.$$

For a small triangulated category, a triple (X, Y, Z) is a distinguished triple if the triangle

$$X \to Y \to Z \to X[1]$$

is a distinguished triangle. This is the Grothendieck group that we will focus on from now on, and the one that we will compute for $\mathcal{C}_n$ and $\overline{\mathcal{C}}_n$.

Definition 3.1.4. Let $\mathcal{T}$ be a small triangulated category. Then the *triangulated Grothendieck group of* $\mathcal{T}$, $K_0(\mathcal{T})$, is the group

$$K_0(\mathcal{T}) = G_0(\mathcal{T})/\langle [Y] - [X] - [Z] \,|\, X \to Y \to Z \to X[1] \text{ a triangle in } \mathcal{T}\rangle.$$

From now on, whenever we refer to the Grothendieck group, this will be used to mean the triangulated Grothendieck group.

Generally when we talk about Grothendieck groups of categories, we wish to avoid categories with arbitrary coproducts, as these categories are guaranteed to have trivial Grothendieck groups due to something called the *Eilenberg swindle* [11], sometimes known as the Mazur swindle in the context of geometric topology [49, 50].

This is where we take any object $X \in \mathcal{C}$ and show that it must be zero in the Grothendieck group via the isomorphism $\bigoplus_{i=0}^{\infty} X \oplus X \cong \bigoplus_{i=0}^{\infty} X$, and the relation

$$[\bigoplus_{i=0}^{\infty} X] \oplus [X] = [\bigoplus_{i=0}^{\infty} X],$$

and so $[X] = 0$ in the Grothendieck group. This holds for any of the categories that we have defined the Grothendieck group for, and so we limit ourselves to essentially small categories.

All of the different types of Grothendieck groups have analogous definitions, so it would be natural to ask whether there is any relationship between them. For example, let $\mathcal{A}$ be an abelian category, and its bounded derived category $\mathrm{D^b}(\mathcal{A})$, then if we know anything about $K_0(\mathrm{D^b}(\mathcal{A}))$, do we have any information about $K_0^{\mathrm{ab}}(\mathcal{A})$?

We do in fact get a positive answer to this question, and there exists an isomorphism $K_0(\mathrm{D^b}(\mathcal{A})) \cong K_0^{\mathrm{ab}}(\mathcal{A})$. Similarly, we have Proposition 3.1.2 that gives us a way of computing the split Grothendieck group of a category without having to look at every possible relation, so is there a way that we can do this for an abelian or triangulated Grothendieck group? Unfortunately, there is no way of doing this in general, however there is a theorem due to Palu [57] that we can use in our situation, that relates the triangulated Grothendieck group of a 2-Calabi-Yau triangulated category to a quotient of the split Grothendieck group of a cluster tilting subcategory of the triangulated category.

Theorem 3.1.5. *[57, Theorem 10] Let $\mathcal{C}$ be a* Hom*-finite, 2-Calabi-Yau triangulated category with a cluster tilting subcategory $\mathcal{T}$, and let $M \in \mathrm{ind}(\mathcal{T})$. Let B_M and B_{M^*} be the central objects in the exchange triangles for the exchange pair (M, M^*). Then, if $K_0^{\mathrm{ab}}(\mathrm{mod}\,\mathcal{T})$ is generated by all classes $[S_N]$ of simple $\mathcal{T}$-modules, the triangulated Grothendieck group of $\mathcal{C}$ is the quotient of the split Grothendieck group of the cluster tilting subcategory $\mathcal{T}$ by all relations $[B_{M^*}] - [B_M]$:*

$$K_0(\mathcal{C}) \cong K_0^{\mathrm{add}}(\mathcal{T}) / \langle [B_{M^*}] - [B_M] \,|\, M \in \mathrm{ind}(\mathcal{T}) \rangle$$

It should be noted that this is not the original statement in [57], as the condition that $K_0^{\mathrm{ab}}(\mathrm{mod}\,\mathcal{T})$ be generated by the simple modules in $\mathrm{mod}\,\mathcal{T}$ is implicitly used in the proof, but not explicitly stated as a requirement. Here $\mathrm{mod}\,\mathcal{T}$ is used to denote the category of $\mathcal{T}$-modules, i.e. the category of k-linear, contravariant functors from $\mathcal{T}$ to the category of k-vector spaces, and $K_0^{\mathrm{ab}}(-)$ is used to mean the abelian Grothendieck group, where triples arise from short exact sequences, see Remark 3.2.6 for further details.

§ 3.2 | The Grothendieck Groups of Discrete Cluster Categories

In this section we look at some previous results on the Grothendieck groups of discrete cluster categories, before going on to compute the Grothendieck groups of $\mathcal{C}_n$ and $\overline{\mathcal{C}}_n$.

§ 3.2.1 | Discrete Cluster Categories of Dynkin Type

The Grothendieck group of the cluster category of a hereditary algebra was studied in [10] by Barot, Kussin and Lenzing. They compute Grothendieck groups for a number of cluster categories, including the cluster categories of finite dimensional hereditary algebras, and the cluster categories of canonical algebras, equivalently, the cluster category of the hereditary category of coherent sheaves on a weighted projective line.

For our purposes, we care about the Grothendieck groups of the cluster categories of a connected, hereditary, representation-finite algebra, for which they give the following classification.

Proposition 3.2.1. *[10] Let Δ be a Dynkin diagram, and let $\mathcal{H}_\Delta$ be the hereditary, representation-finite algebra associated to Δ. Then we have the following:*

$$K_0(\mathcal{C}_\mathcal{H}) \cong \begin{cases} 0 & \text{if } \Delta = A_m, E_m \text{ with } m \text{ even,} \\ \mathbb{Z} & \text{if } \Delta = A_m, D_m, E_7 \text{ with } m \text{ odd,} \\ \mathbb{Z}^2 & \text{if } \Delta = D_m \text{ with } m \text{ even.} \end{cases}$$

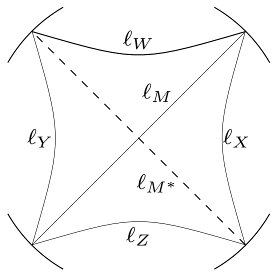
Here $\mathcal{C}_\mathcal{H}$ is the cluster category associated to $\mathcal{H}$ in the sense of Buan, Marsh, Reineke, Reiten, and Todorov [18].

What is interesting to note is the periodic behaviour of the Grothendieck groups of Dynkin type A_m and D_m as m increases. This may lead us to naïvely assume that the Grothendieck group of $\mathcal{C}_n$ may also exhibit a similar periodic behaviour as n increases, or that it may stabilise. However, we show in Theorem 3.2.5 that this does not happen, and the dimension of $K_0(\mathcal{C}_n)$ increases as n does.

§ 3.2.2 | The Grothendieck Group of $\mathcal{C}_n$

Let $\mathcal{T}$ be a cluster tilting subcategory of $\mathcal{C}_n$. As $\mathcal{C}_n$ is a Krull-Schmidt category, this implies by Proposition 3.1.2 that $K_0^{\mathrm{add}}(\mathcal{T})$ is exactly the free abelian group generated by the indecomposable objects $X \in \mathcal{T}$. This leaves us with only the exchange pairs of $\mathcal{T}$ to find, and thus we can compute the Grothendieck group of $\mathcal{C}_n$. The exchange pairs are found with the following lemma.

Lemma 3.2.2. *Let $\mathcal{T}$ be a cluster tilting subcategory of $\mathcal{C}_n$, and let $\mathscr{T}$ be the corresponding collection of arcs. Let $X, Y, W, Z \in \mathcal{T}$ be indecomposable or zero objects, such that the corresponding arcs in $\mathscr{T}$ form a quadrilateral. Let $M \in \operatorname{ind}(\mathcal{T})$ correspond to an arc ℓ_M as in the following diagram.*



Let $M^* \in \mathcal{C}_n \backslash \mathcal{T}$ be the unique indecomposable object such that (M, M^*) is an exchange pair. Then

$$B_M \cong W \oplus Z, \; B_{M^*} \cong X \oplus Y.$$

Proof. By [39], there exists two triangles for the pair of long arcs ℓ_M and $\ell_{M'}$,

$$M \to X \oplus Y \to M^* \to M[1]$$
$$M^* \to W \oplus Z \to M \to M^*[1].$$

However we also have the exchange triangles [18],

$$M \to B_{M^*} \to M^* \to M[1] \quad \text{and} \quad M^* \to B_M \to M \to M^*[1].$$

By using the fact that $\operatorname{Ext}^1(M, M^*) \cong \operatorname{Ext}^1(M^*, M) \cong k$, we now only need to show that none of these triangles split.

To split, the triangle

$$M \xrightarrow{u} X \oplus Y \xrightarrow{v} M^* \xrightarrow{w} M[1]$$

must have a retraction v, however we have

$$\operatorname{Hom}(M^*, X \oplus Y) \cong \operatorname{Ext}^1(M^*, X[-1] \oplus Y[-1]) = 0$$

where the final equality holds as neither $\ell_{X[-1]}$ nor $\ell_{Y[-1]}$ cross ℓ_{M^*}, and so have trivial Ext^1-spaces. Hence there are no maps $v' : M^* \to X \oplus Y$, and so v cannot be a retract. Thus the triangle doesn't split.

A similar argument works for the triangle

$$M^* \to W \oplus Z \to M \to M^*[1]$$

The two exchange triangles are non-split by definition. Hence we have $B_M \cong W \oplus Z$ and $B_{M^*} \cong X \oplus Y$. □

Before we move on, we recall the cluster tilting subcategory from Construction 2.2.7, shown in Figure 3.1.

Construction 3.2.3. We construct a cluster tilting subcategory $\mathcal{T}_n$ of $\mathcal{C}_n$. Given there are n accumulation points, we have $L(\mathscr{M}) = \{a_1, \ldots, a_n\}$ with a cyclic order.

Let $\{z_1, \ldots, z_n\} \subset \mathscr{M}$ be a set of marked points such that $a_{i-1} < z_i < a_i$ for all $i = 1, \ldots, n$. By taking the arcs $\ell_{Z_i} = \{z_i, z_{i+1}\}$ for all $i = 1, \ldots, n$ we inscribe an n-gon inside the circle; the arc ℓ_{Z_i} shall correspond to the object $Z_i \in \mathcal{C}_n$ for all $i = 1, \ldots, n$. For all $a_i \in L(\mathscr{M})$, let $\{z_i, z_{i+1}\}$ be the first arc in a leapfrog converging to a_i, such that if the arc $\{x, y\} \neq \{z_i, z_{i+1}\}$ is in the leapfrog, then so are either the arcs $\{x^+, y\}$ and $\{x, y^+\}$, or the arcs $\{x^-, y\}$ and $\{x, y^-\}$. We label the collection of arcs in the leapfrog converging to a_i by $\mathscr{L}_i$. Without loss of generality, we may let $\ell_{Y_i} = \{z_i, z_{i+1}^-\}$ be in $\mathscr{L}_i$. Notice that this implies that for every arc $\ell_M \in \mathscr{L}_i$ that is not ℓ_{Z_i}, there are exactly two other arcs, $\ell_{M^-}, \ell_{M^+} \in \mathscr{L}_i$ that share an endpoint with ℓ_M.

For $n \geq 4$, we give the inscribed n-gon a fan triangulation; that is, we add the arcs $\ell_{X_i} = \{z_1, z_i\}$ for all $i = 2, \ldots, n$. Notice that this means that we have $X_2 \cong Z_1$ and $X_n \cong Z_n$. This collection of arcs will look like Figure 2.8, along with a leapfrog $\mathscr{L}_i$ converging to each accumulation point.

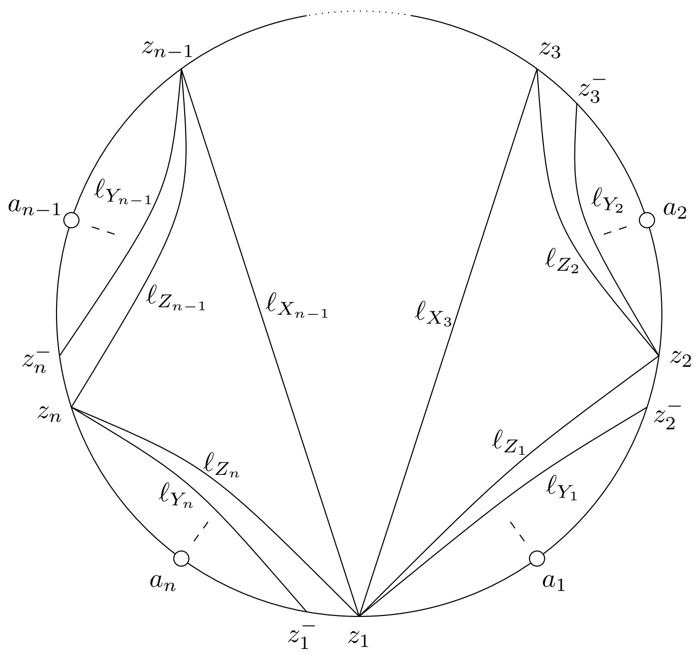


Figure 3.1: The collection of arcs corresponding to the cluster tilting subcategory we wish to consider for n two-sided accumulation points, where leapfrogs are represented by the dashed lines. By Theorem 2.2.6, this collection of leapfrogs together with the fan triangulation of the inscribed n-gon corresponds a cluster tilting subcategory of $\mathcal{C}_n$.

As $\mathcal{T}_n$ corresponds to a triangulation of a marked surface, we may associate a quiver with potential $Q(\mathcal{T}_n)$ to $\mathcal{T}_n$ following Construction 2.5.3.

Proposition 3.2.4. *Let $\mathcal{T}_n$ be the cluster tilting subcategory in Construction 2.2.7, and let $\mathscr{L}_i$ be the collection of arcs in the leapfrog converging to a_i. Then there is a subquiver of $Q(\mathcal{T}_n)$ with vertices corresponding to arcs in $\mathscr{L}_i$ of the following form;*

$$\cdot \longleftarrow \cdot \longrightarrow \cdot \longleftarrow \cdot \longrightarrow \cdot \longleftarrow \cdot \longrightarrow \cdot \longleftarrow \cdots.$$

Proof. Let $\ell_M = \{m_1, m_2\}$ be an arc in the leapfrog $\mathscr{L}_i$, such that $z_i < m_1 < a_i < m_2 < z_{i+1}$. Then we have two cases, either $\ell_{M'} = \{m_1, m_2^-\}$ and $\ell_{M''} = \{m_1^-, m_2\}$ are both arcs in $\mathscr{L}_i$, or $\ell_{N'} = \{m_1, m_2^+\}$ and $\ell_{N''} = \{m_1^+, m_2\}$ are both arcs in $\mathscr{L}_i$.

In the first case, $\ell_{M'}$ and $\ell_{M''}$ are both clockwise rotations of ℓ_M about their respective shared endpoints. Moreover, ℓ_M and $\ell_{M'}$ form a triangle in the triangulation associated to $\mathcal{T}_n$, along with an arc $\{m_2, m_2^-\}$, which is isotopic to the boundary. Hence there is a arrow in $Q(\mathcal{T}_n)$ from the vertex corresponding to $\ell_{M'}$ to the vertex corresponding to ℓ_M, and a similar argument finds an arrow from the vertex corresponding to $\ell_{M''}$ to the vertex corresponding to ℓ_M. Thus the following quiver is a subquiver of $Q(\mathcal{T}_n)$,

$$1 \longrightarrow 2 \longleftarrow 3.$$

The second case is when $\ell_{N'}$ and $\ell_{N''}$ are both anti-clockwise rotations of ℓ_M about their respective shared endpoints. From these we get a subquiver of $Q(\mathcal{T}_n)$,

$$1' \longleftarrow 2' \longrightarrow 3'.$$

We note that if ℓ_M is of the type in the first case, then the arcs $\ell_{M'}$ and $\ell_{M''}$ will be of the type in the second case, and vice versa with the arcs $\ell_{N'}$ and $\ell_{N''}$ of the type in the first case.

Thus we get a subquiver of $Q(\mathcal{T}_n)$,

$$\cdot \longleftarrow \cdot \longrightarrow \cdot \longleftarrow \cdot \longrightarrow \cdot \longleftarrow \cdot \longrightarrow \cdot \longleftarrow \cdots,$$

where the terminal vertex corresponds to the arc ℓ_{Z_i}. □

The subquiver in Proposition 3.2.4 is the full quiver (with trivial potential) associated to the cluster tilting subcategory $\mathcal{T}_1 \subset \mathcal{C}_1$.

In the case of $n = 2$, then the quiver associated to $\mathcal{T}_2 \subset \mathcal{C}_2$ is given by two copies of the subquiver in Proposition 3.2.4 with the terminal vertices identified, i.e. the quiver,

$$\cdots \longrightarrow \cdot \longleftarrow \cdot \longrightarrow \cdot \longleftarrow \cdot \longrightarrow \cdot \longleftarrow \cdot \longrightarrow \cdot \longleftarrow \cdots.$$

Furthermore, we consider Proposition 3.2.4 to find the quiver with potential associated to $\mathcal{T}_n$ for $n \geq 3$ (Figure 3.2).

The path algebra $\mathcal{A} \cong kQ(\mathcal{T}_n)/P$ is generated by paths of finite length, with the longest path $\beta_{n-2}\alpha_{n-2}\ldots\beta_2\alpha_2\beta_1\alpha_1\delta$, with length $2n-3$. Hence, any object in $\operatorname{mod}\mathcal{A}$ can be generated by a finite sum of simple objects in $K_0^{\mathrm{ab}}(\operatorname{mod}\mathcal{A}) \cong K_0^{\mathrm{ab}}(\operatorname{mod}\mathcal{T}_n)$. Therefore we may use Theorem 3.1.5, combined with Lemma 3.2.2, to obtain the following result.

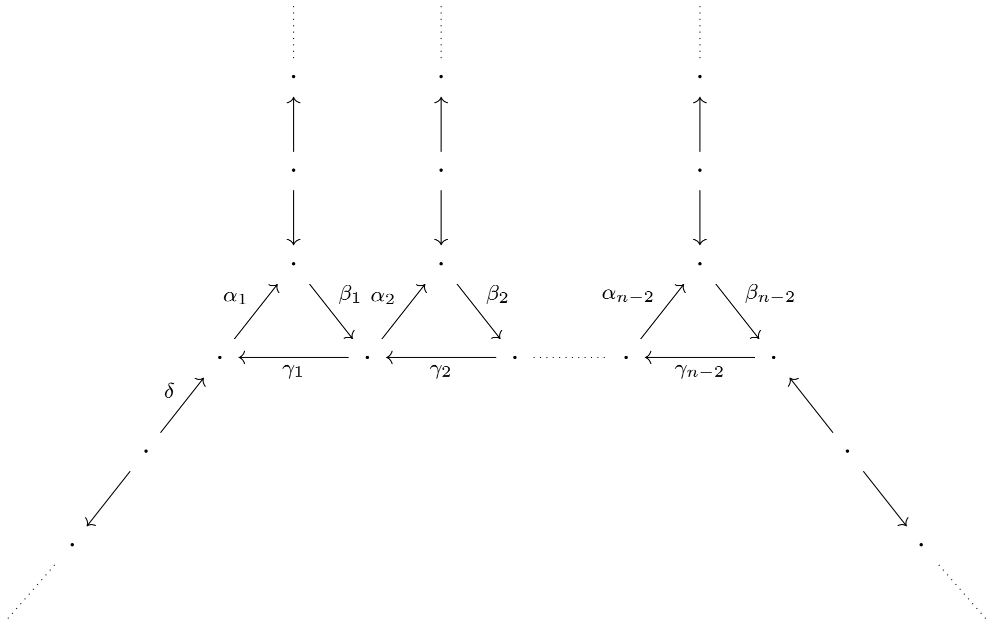


Figure 3.2: The quiver with potential associated to the cluster tilting subcategory $\mathcal{T}_n \subset \mathcal{C}_n$ for $n \geq 3$, where each arm is a subquiver of the type in Proposition 3.2.4. The potential is $P = \langle \sum_{i=1}^{n-2} \gamma_i \beta_i \alpha_i \rangle$.

Theorem 3.2.5. *Then the discrete cluster category of Dynkin type A_∞ with n accumulation points, $\mathcal{C}_n$, has the triangulated Grothendieck group,*

$$K_0(\mathcal{C}_n) \cong \mathbb{Z}^n.$$

Moreover, for $n \geq 2$, the elements $\{[Y_1], [X_2], \ldots, [X_n]\}$ forms a basis of $K_0(\mathcal{C}_n)$.

Proof. One may see by Lemma 3.2.2 that each leapfrog $\mathscr{L}_i$ induces two basis elements $[Z_i], [Y_i]$ in $K_0^{\mathrm{add}}(\mathcal{T}_n)/\langle [B_M] - [B_{M^*}] \rangle$. To see this, take some arc $\ell_M \in \mathscr{L}_{i+1}$, with arcs $\ell_{M^-}, \ell_{M^+} \in \mathscr{L}_{i+1}$ sharing an endpoint with ℓ_M. Then one of the pair of objects B_M and B_{M^*} coming from the exchange triangles must be trivial, and the other have two indecomposable direct summands, due to the construction of the leapfrog. From this, and by the relation $[B_M] - [B_{M^*}] = 0$, one may see that we get that we get $[M^-] = [M^+]$ using Lemma 3.2.2. Then, via induction on the arcs in the leapfrog, one may see that every arc $\ell_M \in \mathscr{L}_{i+1}$ corresponds to either $[Z_i]$ or $[Y_i]$ in $K_0^{\mathrm{add}}(\mathcal{T}_n)/\langle [B_M] - [B_{M^*}] \rangle$.

To prove the claim for all $n > 0$, we shall prove it for $n = 1, 2, 3$ respectively, and then for $n \geq 4$.

In the case $n = 1$, we only have $a_1 \in L(\mathscr{M})$. Thus the first non-trivial arc in $\mathscr{L}_1$ is the arc $\ell_{Z_1} = \{z_1^-, z_1^+\}$, which we label Z_1 in lieu of the arc $\{z_1, z_2\}$. Let us take the arc $\ell_{Y_1} = \{z_1^{--}, z_1^+\}$. From this, we may see that the arc $\ell_{Z_1^*}$ has endpoints $\{z_1^{--}, z_1\}$. Using Lemma 3.2.2 we see that $B_{Z_1} \cong Y_1$ and $B_{Z_1^*} = 0$, so we have $[B_{Z_1}] = [Y_1]$ and $[B_{Z_1^*}] = 0$, meaning $[B_{Z_1}] - [B_{Z_1^*}] = [Y_1]$. There are no more relations in $K_0^{\mathrm{add}}(\mathcal{T}_1)$ left to consider, and so by Theorem 3.1.5 [57],

$$K_0(\mathcal{C}_1) \cong \mathbb{Z}.$$

In the case of $n = 2$, the arc $\ell_{Z_1} = \{z_1, z_2\}$ is contained in both $\mathscr{L}_1$ and $\mathscr{L}_2$, and thus we have $Z_1 \cong Z_2$. One can see from Lemma 3.2.2 that $B_{Z_1} \cong Y_1 \oplus Y_2$ and $B_{Z_1^*} \cong 0$, and thus we have $[B_{Z_1}] - [B_{Z_1^*}] = [Y_1] + [Y_2]$, which means that when we quotient by $[B_{Z_1}] - [B_{Z_1^*}]$ we get $[Y_2] = -[Y_1]$. Meaning we have only two basis elements in $K_0^{\text{add}}(\mathcal{T}_2)/\langle [B_M] - [B_{M^*}]\rangle$, and there are no more relations to consider. Hence, by Theorem 3.1.5 [57],

$$K_0(\mathcal{C}_2) \cong \mathbb{Z}^2.$$

When we have $n = 3$, we must consider the triangle consisting of sides being the arcs $\ell_{Z_1} = \{z_i, z_{i+1}\}$, with $i = 1, 2, 3$. The exchange pair of Z_1 consists of $B_{Z_1} \cong Z_2 \oplus Y_1$ and $B_{Z_1^*} \cong Z_3$, giving us the relation $[Z_2] + [Y_1] = [Z_3]$. Similarly, via the exchange pairs at Z_2 and Z_3 respectively, we have the relations,

$$[Z_3] + [Y_2] = [Z_1]$$
$$[Z_1] + [Y_3] = [Z_2].$$

Which means that we have three basis elements of $K_0^{\text{add}}(\mathcal{T}_3)/\langle [B_M] - [B_{M^*}]\rangle$, which are the elements $\{[Z_1], [Z_2], [Z_3]\}$, with no more relations to consider, and so by Theorem 3.1.5,

$$K_0(\mathcal{C}_3) \cong \mathbb{Z}^3.$$

For $n \geq 4$, we have the additional arcs forming the fan triangulation of the inscribed n-gon to consider. The corresponding objects to these arcs induce the elements $[X_i]$ in $K_0^{\text{add}}(\mathcal{T}_n)$, for all $i = 3, \ldots, n-1$. In these cases we wish to consider the exchange pairs on two different families of objects, $\{Z_i\}_{i=1,\ldots,n}$ and $\{X_j\}_{j=3,\ldots,n-1}$.

For all X_i, $i = 3, \ldots, n-1$ using Lemma 3.2.2, we have the exchange pairs $B_{X_i} \cong Z_i \oplus X_{i-1}$ and $B_{X_i^*} \cong X_{i+1} \oplus Z_{i-1}$. These exchange pairs give us the relations,

$$[Z_i] + [X_{i-1}] = [Z_{i-1}] + [X_{i+1}], \qquad \text{for all } i = 3, \ldots, n-1,$$

where we use the fact that by definition $X_2 \cong Z_1$ and $X_n \cong Z_n$. By inspection at the arcs ℓ_{Z_i} for $i = 2, \ldots, n-1$, one can see that by Lemma 3.2.2 we have the exchange pairs $B_{Z_i} \cong X_{i+1} \oplus Y_i$ and $B_{Z_i^*} \cong X_i$. This means we have the set of relations

$$[X_i] = [X_{i+1}] + [Y_i],$$

for $i = 2, \ldots, n-1$.

It only remains to check the relations induced by the exchange pairs for Z_1 and Z_n.

One may easily check that we have the exchange pairs by applying Lemma 3.2.2,

$$B_{Z_1} \cong Z_2 \oplus Y_1 \qquad B_{Z_1^*} \cong X_3$$
$$B_{Z_n} \cong Y_n \oplus X_{n-1} \qquad B_{Z_n^*} \cong Z_{n-1}.$$

These exchange pairs then induce the relations,

$$[X_3] = [Z_2] + [Y_1]$$
$$[Z_{n-1}] = [Y_n] + [X_{n-1}].$$

This means that all of the relations $\langle B_M - B_{M^*} \rangle$, for M an indecomposable summand of the cluster tilting object T, have been found. These are

$$[Z_i] = -[X_{i-1}] + [X_{i+1}] + [Z_{i-1}] \qquad \text{for all } i = 3, \ldots, n-1 \tag{3.1}$$
$$[Y_i] = -[X_{i+1}] + [X_i] \qquad \text{for all } i = 2, \ldots, n-1 \tag{3.2}$$
$$[Z_2] = [Y_1] + [X_3] \tag{3.3}$$
$$[Y_n] = [Z_{n-1}] - [X_{n-1}]. \tag{3.4}$$

By considering (3) and then (1) inductively, we find each $[Z_i]$ for $i = 2, \ldots, n-1$ in terms of the elements $\{[Y_1], [X_2], \ldots, [X_n]\}$. We may also find all $[Y_i]$ for $i = 2, \ldots, n$ in terms of the same set of elements, by considering (2) and (4). Note, given we also have $X_2 \cong Z_1$ and $X_n \cong Z_n$, so therefore $[X_2] = [Z_1]$ and $[X_n] = [Z_n]$, meaning we have found all elements in terms of linear combinations of the set of elements $\{[Y_1], [X_2], \ldots, [X_n]\}$.

Given there are no more relations to consider, this means we have a basis for the group $K_0^{\text{add}}(\mathcal{T}_n)/\langle B_M - B_{M^*} \rangle$;

$$\{[Y_1], [X_2], \ldots, [X_n]\}.$$

All of these elements in the basis have infinite order, as there exists no relation such that $m[A] = 0$ for any $m \in \mathbb{Z}$ and $A \in \operatorname{ind}(\mathcal{T}_n)$; and so, by Theorem 3.1.5,

$$K_0(\mathcal{C}_n) \cong \mathbb{Z}^n.$$

□

Remark 3.2.6. A natural question to ask is why this choice of cluster tilting subcategory is being used in particular. We choose this cluster tilting subcategory primarily because it satisfies an implicit condition required to apply Theorem 3.1.5, which other cluster tilting subcategories do not necessarily satisfy. This is that, given a cluster tilting subcategory $\mathcal{T}$, then $K_0^{\text{ab}}(\operatorname{mod} \mathcal{T})$ must be generated by the simple $\mathcal{T}$-modules, S_M associated to the indecomposable object $M \in \mathcal{T}$.

For example, take the additive subcategory $\mathcal{S} \subset \mathcal{C}_1$ with indecomposable objects that correspond to a fountain at $z \in \mathscr{M}$ converging to $a \in L(\mathscr{M})$. This is a cluster tilting subcategory by [35] when the corresponding arcs are maximal as a set of pairwise non-crossing arcs. When we apply Theorem 3.1.5 to $\mathcal{S}$ we get the result that $K_0(\mathcal{C}_1) \cong \mathbb{Z}^2$. We know this cannot be true by considering only the triangles in $\mathcal{C}_1$ that correspond to the crossings of arcs, which shows that $K_0(\mathcal{C}_1)$ must be a quotient of $\mathbb{Z}$.

We also see that $\mathcal{S}$ does not satisfy the condition of $K_0^{\mathrm{ab}}(\operatorname{mod}\mathcal{S})$ is generated by simple $\mathcal{S}$-modules. To do this, we can associate a quiver to a triangulation (see Subsection 2.5.3), and for $\mathcal{S}$ this quiver with potential has infinite length paths between vertices, and so a projective $\mathcal{S}$-module P may not be the sum of finitely many simple $\mathcal{S}$-modules in $K_0^{\mathrm{ab}}(\operatorname{mod}\mathcal{S})$. Thus $K_0^{\mathrm{ab}}(\operatorname{mod}\mathcal{S})$ is not generated by the simple $\mathcal{S}$-modules.

This is a condition that is mentioned within the proof of Theorem 3.1.5 in [57], however is not mentioned within the statement of the theorem. Whilst this does serve as a counterexample to the theorem as it is originally stated, the author still believes that a form of the theorem could still be used on cluster tilting subcategories without this condition, however a more explicit computation may be necessary, and it would be preferable to find a cluster tilting subcategory satisfying this condition if possible.

The author does not currently know of any 2-Calabi-Yau, triangulated category with at least one cluster tilting subcategory, such that there does not exist a cluster tilting subcategory that satisfies the conditions of Theorem 3.1.5.

§ 3.2.3 | The Grothendieck Groups of $\overline{\mathcal{C}}_n$

Here we compute the triangulated Grothendieck group of the completed discrete cluster category of Dynkin type A_∞ with n two-sided accumulation points, $\overline{\mathcal{C}}_n$. A similar approach to the main result used in the previous section would not be feasible, as the results in [57] require the category in question to be 2-Calabi-Yau, a property that $\overline{\mathcal{C}}_n$ does not have due to the lack of Serre functor. Therefore a new approach must be made, one for which the 2-Calabi-Yau property is not necessary.

This new approach involves using the localisation functor used in [58] (Construction 2.4.1) to construct the category $\overline{\mathcal{C}}_n$ from the category $\mathcal{C}_{2n}$. As shown in Proposition 2.4.3 this essentially surjective functor π has a kernel of n copies of the category $\mathcal{C}_1$, and so fits into the short exact sequence:

$$0 \to \coprod_{i=1}^{n} \mathcal{C}_1 \xrightarrow{\rho} \mathcal{C}_{2n} \xrightarrow{\pi} \overline{\mathcal{C}}_n \to 0.$$

Before we get to the main result and its proof, we must provide some useful statements that hold for both $\mathcal{C}_n$ and $\overline{\mathcal{C}}_n$. We state these lemmas for $\mathcal{C}_n$, however note that the statements and proofs are analogous for $\overline{\mathcal{C}}_n$.

Lemma 3.2.7. *Let $W \in \mathcal{C}_n$ be any indecomposable object such that the arc ℓ_W is a short arc. Then ℓ_W has an odd number of marked points between its two endpoints if and only if $[W] \neq 0$.*

Proof. We will show this inductively.

Let ℓ_{W_i} be an arc with $i \in \mathbb{Z}_{>0}$ marked points between its two endpoints, with all ℓ_{W_i} sharing one endpoint, i.e. $\{\ell_{W_i}\}_{i>0}$ is a left sided fountain. It is clear to see that ℓ_{W_0} is isotopic to a boundary segment between two adjacent marked points, and therefore a zero object, so $[W_0] = 0$.

Next, we have W_1, to which we will assign $[W_1] \in K_0(\mathcal{C}_n)$. We can then use the following triangle to find all subsequent $[W_i] \in K_0(\mathcal{C}_n)$;

$$W_i \to W_{i+1} \to W_1[i] \to W_i[1].$$

This is verified as a distinguished triangle by construction of a quadrilateral of arcs, with the arcs ℓ_{W_i} and $\ell_{W_1[i]}$ crossing.

From this, we can see that in $K_0(\mathcal{C}_n)$ we have;

$$[W_{i+1}] = [W_i] + (-1)^i [W_1],$$

and hence, using induction on i, we find

$$[W_i] = \begin{cases} 0 & \text{if } i \text{ is even} \\ [W_1] & \text{if } i \text{ is odd} \end{cases}$$

□

Recall from Figure 2.8 and the proof of Theorem 3.2.5 that we have the set of arcs $\{\ell_{Y_1}, \ell_{X_2}, \ldots, \ell_{X_n}\}$ that correspond to a basis in $K_0(\mathcal{C}_n)$. For reference, we give a figure with these arcs on it.

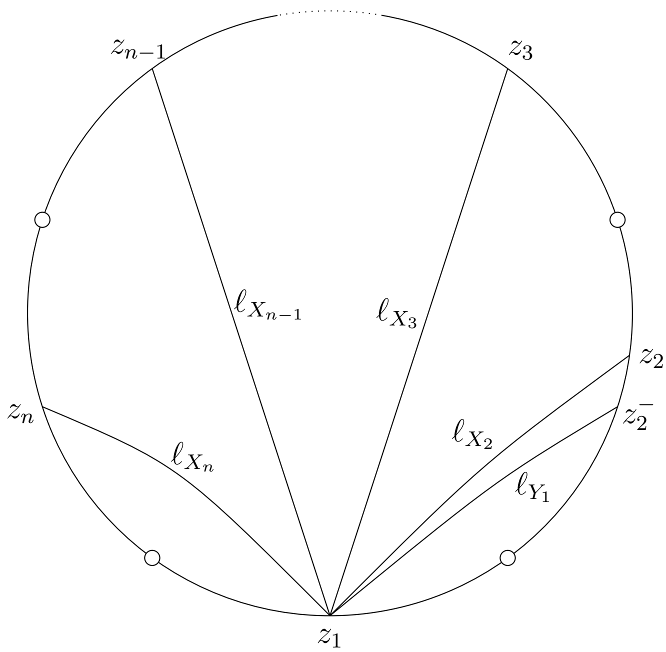


Figure 3.3: The set of arcs corresponding to the chosen representatives of the basis of $K_0(\mathcal{C}_n)$.

Lemma 3.2.8. *Let $W \in \mathcal{C}_{2n}$ be indecomposable with ℓ_W a short arc such that $[W] \neq 0$ in $K_0(\mathcal{C}_{2n})$, then there exists $j \in \mathbb{Z}$ such that there exists a triangle of the form*

$$X_i \to A \oplus W[j] \to X_i[l] \to X_i[1],$$

for some $i \in \{2, \dots, n\}$ with l even, and where ℓ_A is a short arc. Moreover, one of the following is true,

- *z_1 is an endpoint of ℓ_A, and $[A] = [X_2] + [Y_1]$,*
- *or z_1 is an endpoint of $\ell_{W[j]}$, and $(-1)^j[W] = [X_2] + [Y_1]$.*

Proof. There exists some $j \in \mathbb{Z}$ such that $\ell_{W[j]} = \{z_i, x\}$, where $a_i < x < z_i < a_{i+1}$. By Lemma 3.2.7, there are an odd number of marked points between z_i and x.

We consider two cases, where ℓ_W has endpoints on the same segment as z_1, and where ℓ_W has endpoints on the same segment as z_i for $i \neq 1$.

Suppose $i \neq 1$, then $\ell_{W[j]}$ shares the endpoint z_i with ℓ_{X_i}, and the endpoint x with $\ell_{X_i[l]}$, where $l > 0$ is even. We see that l is even as there is an odd number of marked points between z_i and x, and the endpoints of $\ell_{M[1]}$ are the predecessor of the endpoints of ℓ_M with respect to the anti-clockwise orientation, for any indecomposable object $M \in \mathcal{C}_n$. Therefore, if there are m marked points between the marked points $a > b \in \mathscr{M}$, and ℓ_M has an endpoint at a, then $\ell_{M[m+1]}$ has an endpoint at b.

Also, the arcs ℓ_{X_i} and $\ell_{X_i[l]}$ cross, meaning there is a triangle

$$X_i \to A \oplus B \to X_i[l] \to X_i[1],$$

where $\ell_B = \{z_i, z_i - l\}$ and $\ell_A = \{z_1, z_1 - l\}$. Therefore $B \cong W[j]$ and ℓ_A is a short arc with z_1 as an endpoint.

To show that $[A] = [X_2] + [Y_1]$, consider the objects X_2 and $Y_1[l]$. Then the corresponding arcs are $\ell_{X_2} = \{z_1, z_2\}$ and $\ell_{Y_1[l]} = \{z_1 - l, z_2^- - l\}$ respectively, and ℓ_{X_2} crosses $\ell_{Y_1[l]}$. Hence we have a triangle

$$X_2 \to A' \oplus B' \to Y_1[l] \to X_2[1],$$

with $\ell_{A'} = \{z_1, z_1 - l\}$ and $\ell_{B'} = \{z_2, z_2^- - l\}$, so there is an even number of marked points between the endpoints of the short arc $\ell_{B'}$ and $A' \cong A$. In the Grothendieck group, this means we have the relation

$$[A] + [B] = [X_2] + [Y_1[l]].$$

However, $[B] = 0$ by Lemma 3.2.7 and l is even so $[Y_1[l]] = [Y_1]$, therefore we get $[A] = [X_2] + [Y_1]$.

Now suppose that ℓ_W has endpoints on the same segment as z_1. Then there exists some $j \in \mathbb{Z}$ such that $\ell_{W[j]} = \{z_1, z_1 - l\}$, still with $l > 0$ even. For any $i = 2, \dots, n$, there exists a triangle

$$X_i \to A \oplus W[j] \to X_i[l] \to X_i[1]$$

with $\ell_A = \{z_i, z_i - l\}$, as ℓ_{X_i} crosses $\ell_{X_i[l]}$ and both share an endpoint with $\ell_{W[j]}$.

To show that $(-1)^j[W] = [X_2] + [Y_1]$, we use the same argument as when ℓ_W has endpoints in the same segment as z_i, to show that $[W[j]] = [X_2] + [Y_1]$, and note that $j \in \mathbb{Z}$ could be either odd or even, and so we use $(-1)^j[W] = [W[j]] = [X_2] + [Y_1]$ to prove our statement. $\square$

With these results, we can now compute the Grothendieck group for $\overline{\mathcal{C}}_n$.

Theorem 3.2.9. *Let $\overline{\mathcal{C}}_n$ be the completion of a discrete cluster category of Dynkin type A_∞ with n two-sided accumulation points. Then $\overline{\mathcal{C}}_n$ has the triangulated Grothendieck group:*

$$K_0(\overline{\mathcal{C}}_n) \cong \mathbb{Z}^n \oplus (\mathbb{Z}/2\mathbb{Z})^{n-1}$$

Proof. We have the localisation functor $\pi\colon \mathcal{C}_{2n} \to \overline{\mathcal{C}}_n$, which has kernel n copies of $\mathcal{C}_1$ by Proposition 2.4.3. This means that we have a short exact sequence of categories (in the language of [64]):

$$0 \to \bigsqcup_{i=1}^{n} \mathcal{C}_1 \xrightarrow{\rho} \mathcal{C}_{2n} \xrightarrow{\pi} \overline{\mathcal{C}}_n \to 0$$

We also use the fact that the Grothendieck group functor, $K_0(-)$, is right exact [64, Fact 1.2], and so that means we have the commutative diagram with exact rows:

$$\begin{tikzcd}
K_0(\bigsqcup_{i=1}^{n} \mathcal{C}_1) \arrow[r, "f"] \arrow[d, "\cong"] & K_0(\mathcal{C}_{2n}) \arrow[r, "g"] \arrow[d, "\cong"] & K_0(\overline{\mathcal{C}}_n) \arrow[r] \arrow[d, "\cong"] & 0 \\
\mathbb{Z}^n \arrow[r, "f"] & \mathbb{Z}^{2n} \arrow[r, "g"] & K_0(\overline{\mathcal{C}}_n) \arrow[r] & 0
\end{tikzcd}$$

where the first and second isomorphisms come from Theorem 3.2.5.

Therefore, given g is a surjection, it is sufficient to describe the map f and find its cokernel.

We know that the functor $\rho : \bigsqcup_{i=1}^{n} \mathcal{C}_1 \to \mathcal{C}_{2n}$ is injective on objects, with each copy of $\mathcal{C}_1$ mapping into alternating segments in $\mathcal{C}_{2n}$. This is due to the choice of $\mathcal{D}$ by Paquette and Yıldırım in the construction of $\overline{\mathcal{C}}_n$ [58]. We will label the basis elements of $K_0(\bigsqcup_{i=1}^{n} \mathcal{C}_1)$ as $\{[M_1], \dots, [M_n]\}$, with corresponding arcs $\ell_{M_1}, \dots, \ell_{M_n} \in \bigsqcup_{i=1}^{n} \mathcal{C}_1$. Further, we shall choose the labelling such that $\rho(M_1) \cong X$, where X is the object corresponding to the arc $\ell_X = \{z_1^{--}, z_1\}$, and so we have $f([M_1]) = [\rho(M_1)] = [X]$. Generally we say that the arc $\ell_{\rho(M_i)} = \{z_{2i-1}^{--}, z_{2i-1}\}$ corresponds to the object $\rho(M_i)$, and so $[\rho(M_i)] \neq 0$ by Lemma 3.2.7.

Notice that if we choose an object, W, that corresponds to the arc $\ell_W = \{z_i^{--}, z_i\}$ and so $[W] \neq 0$ by Lemma 3.2.7, then X is the unique object, corresponding to the arc

ℓ_X, fitting into the distinguished triangle

$$X_i \to X \oplus W \to X_i[2] \to X_i[1]$$

and so $[X] = [X_2] + [Y_1]$, by Lemma 3.2.8.

Given $[\rho(M_i)] \neq 0$ we may now apply Lemma 3.2.8, giving us the triangle

$$X_{2i-1} \to X \oplus \rho(M_i) \to X_{2i-1}[2] \to X_{2i-1}[1]$$

with $[X] = [X_2] + [Y_1]$. This in turn means we have

$$[\rho(M_i)] = \begin{cases} 2[X_{2i-1}] - [X_2] - [Y_1] & \text{if } i = 2, \dots, n \\ [X_2] + [Y_1] & \text{if } i = 1 \end{cases}$$

By summing $[\rho(M_i)]$ and $[\rho(M_1)]$, we get $[\rho(M_i)] + [\rho(M_1)] = 2[X_{2i-1}]$, and so we get the image of f being the span of the elements $\{[X_2]+[Y_1], 2[X_{2i-1}]\}$ for all $i = 2, \dots, n$. This gives us $\operatorname{im}(f) \cong (2\mathbb{Z})^{n-1} \oplus \mathbb{Z}$, and so by the exact sequence

$$\mathbb{Z}^n \xrightarrow{f} \mathbb{Z}^{2n} \xrightarrow{g} K_0(\overline{\mathcal{C}_n}) \to 0,$$

we get $K_0(\overline{\mathcal{C}_n}) \cong \mathbb{Z}^{2n}/\ker(g) \cong \mathbb{Z}^{2n}/\operatorname{im}(f)$, where the last equivalence comes from the above sequence being exact. Hence we get

$$K_0(\overline{\mathcal{C}}_n) \cong \mathbb{Z}^n \oplus (\mathbb{Z}/2\mathbb{Z})^{n-1}.$$

$\square$

Chapter 4

Generators of $\overline{\mathcal{C}}_n$

In this chapter we start by looking at what it means for an object to generate a triangulated category, and how we can use the set of generating objects of a triangulated category to define a dimension on the category. We follow this up by classifying the generators of $\overline{\mathcal{C}}_n$, the completed discrete cluster categories of Dynkin type A_∞.

Then we compute the graded endomorphism ring of a particularly nice generator of $\overline{\mathcal{C}}_n$ in an attempt to obtain a homological description of $\overline{\mathcal{C}}_n$ for all $n \geq 1$, similar to that of $\mathcal{C}_1$ discussed by Holm and Jørgensen in [38].

Finally, we introduce a homological length for an object in a triangulated category, and show that the homological length of a generator in $\overline{\mathcal{C}}_n$ is an upper bound for the time that generator takes to generate $\overline{\mathcal{C}}_n$. Subsequently, we use this to provide an upper bound for the Orlov spectrum of $\overline{\mathcal{C}}_n$.

With the exception of the section on the graded endomorphism ring of a given generator, the results in this chapter can be extended to the non-completed categories $\mathcal{C}_n$, with only a few subtleties to contend with.

§ 4.1 | Generators of a Triangulated Category

Objects that generate a triangulated category have been the subject of study for many years, notably being used to construct equivalences of categories by Keller [41] for algebraic triangulated categories. There are a few different definitions of what it means for an object to be a generator, with various implications between the definitions. Throughout when we say generator, we shall mean what may be referred to as a classical generator by some authors. Many of the definitions in the section are adapted versions of the definitions given in [15] by Bondal and Van den Bergh.

Recall the definition of a thick subcategory.

Definition 4.1.1. Let $\mathcal{C}$ be a triangulated category with a triangulated subcategory $\mathcal{B}$, then we say $\mathcal{B}$ is a *thick subcategory* of $\mathcal{C}$ if it is closed under direct summands.

It is important to understand thick subcategories, as they play a crucial role in the definition of classical generators of a triangulated category.

Definition 4.1.2. Let $\mathscr{E} = \{E_i\}_{i \in I}$ be a collection of objects in $\mathcal{C}$. We say that $\mathscr{E}$ *classically generates* $\mathcal{C}$ if the smallest thick subcategory of $\mathcal{C}$ containing $\mathscr{E}$ is $\mathcal{C}$ itself.

Some authors use the term *classical generator* to distinguish them from other notions of generators, for instance a *(weak) generator* E in a triangulated category is an object such that for all objects K, there exists a non-zero morphism $E \to K[n]$ for some $n \in \mathbb{Z}$. It should be noted that all classical generators are weak generators, however the converse is not always true. Whenever we refer to generators, we shall always be referring to classical generators.

Following [13], we define an operation on two subcategories of a triangulated category to obtain another subcategory containing both of the original subcategories.

Definition 4.1.3. Let $\mathcal{R}, \mathcal{S} \subset \mathcal{C}$ be two subcategories, then $\mathcal{R} \star \mathcal{S}$ is the full subcategory of direct summands of all objects Y such that there exists a triangle

$$X \to Y \to Z \to X[1]$$

with $X \in \mathcal{R}$ and $Z \in \mathcal{S}$.

The following proposition shows us that given three subcategories $\mathcal{R}, \mathcal{S}, \mathcal{T} \subseteq \mathcal{C}$, then the subcategory $\mathcal{R} \star \mathcal{S} \star \mathcal{T}$ is well defined.

Proposition 4.1.4. *The operation $\star$ is associative.*

Proof. Let $\mathcal{R}, \mathcal{S}, \mathcal{T}$ be subcategories of a triangulated category $\mathcal{C}$. Suppose that $Z \in (\mathcal{R} \star \mathcal{S}) \star \mathcal{T}$, we show that $Z \in \mathcal{R} \star (\mathcal{S} \star \mathcal{T})$.

As $Z \in (\mathcal{R} \star \mathcal{S}) \star \mathcal{T}$, then we can choose some object $X \in \mathcal{R} \star \mathcal{S}$ such that we have two triangles,

$$A \xrightarrow{f} X \to B \to A[1]$$

and

$$X \xrightarrow{g} Z' \to C \to X[1]$$

with $A \in \mathcal{R}$, $B \in \mathcal{S}$ and $C \in \mathcal{T}$, with Z a direct summand of $Z' \in (\mathcal{R} \star \mathcal{S}) \star \mathcal{T}$. Given the two composable morphisms f and g, we can construct the triangle

$$A \xrightarrow{gf} Z' \to D \to A[1].$$

With these three triangles, we can apply the octahedral axiom to get

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\sim"] & X \arrow[r] \arrow[d, "g"] & B \arrow[r] \arrow[d] & A[1] \arrow[d] \\
A \arrow[r, "gf"] \arrow[d] & Z' \arrow[r] \arrow[d] & D \arrow[r] \arrow[d] & A[1] \arrow[d] \\
0 \arrow[r] \arrow[d] & C \arrow[r, "\sim"] \arrow[d] & C \arrow[r] \arrow[d] & 0 \arrow[d] \\
A[1] \arrow[r] & X[1] \arrow[r] & B[1] \arrow[r] & A[2].
\end{tikzcd}$$

From the commutative diagram, we see that $D \in \mathcal{S} \star \mathcal{T}$, and so $Z' \in \mathcal{R} \star (\mathcal{S} \star \mathcal{T})$, hence $Z \in \mathcal{R} \star (\mathcal{S} \star \mathcal{T})$ as it is closed under direct summands.

Therefore $(\mathcal{R} \star \mathcal{S}) \star \mathcal{T}$ is a full subcategory of $\mathcal{R} \star (\mathcal{S} \star \mathcal{T})$, and to show the reverse inclusion is a dual argument, so we see that $\star$ is associative. □

Given some subcategory $\mathcal{R} \subseteq \mathcal{C}$, it is possible to build an iterative series of subcategories from $\mathcal{R}$ using this operation. These subcategories prove to be a useful tool in determining whether or not a collection of objects classically generate $\mathcal{C}$ or not.

Definition 4.1.5. Let $\mathscr{G} = \{G_j\}_{j \in J}$ be a collection of objects in $\mathcal{C}$. We denote by $\langle \mathscr{G} \rangle_1 \subset \mathcal{C}$ the full subcategory consisting of all direct summands of finite coproducts of suspensions of objects in $\mathscr{G}$. Further, we define the full subcategory $\langle \mathscr{G} \rangle_{n+1}$ to be

$$\langle \mathscr{G} \rangle_{n+1} := \langle \mathscr{G} \rangle_n \star \langle \mathscr{G} \rangle_1.$$

Moreover, we define $\langle \mathscr{G} \rangle$ to be the union of all $\langle \mathscr{G} \rangle_n$, i.e.

$$\langle \mathscr{G} \rangle := \bigcup_n \langle \mathscr{G} \rangle_n.$$

We assume all $\langle G \rangle_i$ to be closed under isomorphism. It is clear from the definition that $\langle \mathscr{G} \rangle_n \subset \langle \mathscr{G} \rangle_{n+1}$ for all $n \geq 1$. This is due to having the following triangle

$$X \to X \to 0 \to X[1]$$

for any $X \in \langle \mathscr{G} \rangle_n$, and so $X \in \langle \mathscr{G} \rangle_{n+1}$.

Notice also that by Proposition 4.1.4 we may see that $\langle \mathscr{G} \rangle_n$ may be obtained via the operation of any two $\langle \mathscr{G} \rangle_a$ and $\langle \mathscr{G} \rangle_b$, as long as $a + b = n$. For instance, we define $\langle \mathscr{G} \rangle_n$ as $\langle \mathscr{G} \rangle_{n-1} \star \langle \mathscr{G} \rangle_1$, but we also have

$$\begin{aligned} \langle \mathscr{G} \rangle_n &= \langle \mathscr{G} \rangle_{n-1} \star \langle \mathscr{G} \rangle_1 \\ &= (\langle \mathscr{G} \rangle_{n-2} \star \langle \mathscr{G} \rangle_1) \star \langle \mathscr{G} \rangle_1 \\ &= \langle \mathscr{G} \rangle_{n-2} \star (\langle \mathscr{G} \rangle_1 \star \langle \mathscr{G} \rangle_1) \\ &= \langle \mathscr{G} \rangle_{n-2} \star \langle \mathscr{G} \rangle_2, \end{aligned}$$

where the third equality comes from Proposition 4.1.4.

Throughout the rest of this thesis, we need only consider finite collections of objects, and so can express things in terms of objects in $\overline{\mathcal{C}}_n$. In other words, we consider objects that classically generate $\mathcal{C}$ and we consider the subcategory $\langle \mathrm{add}(G[i] \mid i \in \mathbb{Z}) \rangle \subseteq \mathcal{C}$ for an object $G \in \mathcal{C}$, which we denote as $\langle G \rangle$. If an object E classically generates a triangulated category $\mathcal{C}$, then we say E is a *classical generator*.

Next, we show that there is another way to characterise the subcategory $\langle E \rangle$ for some object $E \in \mathcal{C}$, which will be useful when we look for generators of a triangulated category.

Proposition 4.1.6. *Let $\mathcal{C}$ be a triangulated category, and let $E \in \mathcal{C}$. Then $\langle E \rangle \subseteq \mathcal{C}$ is the smallest thick, triangulated subcategory containing E.*

Proof. We must show that $\langle E \rangle$ is closed under direct summands, suspension and extensions, and is the smallest such subcategory containing E. By definition, $\langle E \rangle$ is closed under direct summands.

Let $X, Y \in \langle E \rangle$ such that there exists a triangle

$$X \to Z \to Y \to X[1].$$

Further, there exists some $i, j \in \mathbb{Z}$ such that $X \in \langle E \rangle_i$ and $Y \in \langle E \rangle_j$. Thus $Z \in \langle E \rangle_{i+j} \subseteq \langle E \rangle$, and so $\langle E \rangle$ is closed under extension.

For closure under suspension, we know that $\langle E \rangle_1$ is closed under suspension by definition. Now assume that $\langle E \rangle_n$ is closed under suspension, and that some object $X \in \langle E \rangle_{n+1}$ is the central object in a triangle

$$Z \to X \to Y \to Z[1]$$

where $Z \in \langle E \rangle_n$ and $Y \in \langle E \rangle_1$. Then, by TR2 in Defnition 1.1.5, there exists a triangle

$$Z[1] \to X[1] \to Y[1] \to Z[2],$$

which implies that $X[1] \in \langle E \rangle_{n+1}$ as $\langle E \rangle_n$ and $\langle E \rangle_1$ are closed under suspension, and so $\langle E \rangle_{n+1}$ is also closed under suspension. Hence it is only left to show that $\langle E \rangle$ is the smallest such subcategory of $\mathcal{C}$ containing E.

Suppose there exists a thick subcategory $\mathcal{B} \subset \langle E \rangle \subseteq \mathcal{C}$, and an object $A \in \langle E \rangle$ but not in $\mathcal{B}$. Further, suppose that $A \in \langle E \rangle_{n+1}$, then we have a triangle

$$X_n \to A \to Y_1 \to X_n[1]$$

with $X_n \in \langle E \rangle_n$ and $Y_1 \in \langle E \rangle_1$. If Y_1 is not in $\mathcal{B}$, then $E \notin \mathcal{B}$ as Y_1 is a direct sum of suspensions of direct summands of E, and $\mathcal{B}$ is closed under direct sums, direct summands, and suspension.

Therefore we suppose that $Y_1 \in \mathcal{B}$. Then $X_n \notin \mathcal{B}$, as $\mathcal{B}$ is thick and $A \notin \mathcal{B}$. Repeating this, we obtain a series of triangles

$$\begin{gathered} X_{n-1} \to X_n \to Y_2 \to X_{n-1}[1], \\ \vdots \\ X_{i-1} \to X_i \to Y_{n-i+2} \to X_{i-1}[1], \\ \vdots \\ X_1 \to X_2 \to Y_n \to X_1[1], \end{gathered}$$

where $X_i \in \langle E \rangle_i$, $Y_j \in \langle E \rangle_1$ and $X_i \notin \mathcal{B}$ for all $i = 1, \ldots, n$. However $X_1 \in \langle E \rangle_1$ and so is a direct sum of objects isomorphic to some suspension of a direct summand of E, and so E cannot be in $\mathcal{B}$, as $\mathcal{B}$ is closed under suspension and direct summands. Hence $\langle E\rangle$ is the smallest thick, triangulated subcategory of $\mathcal{C}$ containing E. □

As an immediate consequence of this proposition, we see that classical generators have the following property.

Corollary 4.1.7. *An object $E \in \mathcal{C}$ is a classical generator if and only if $\langle E\rangle = \mathcal{C}$.*

If, for some $n \in \mathbb{Z}$, we have $\langle E\rangle_n = \mathcal{C}$, then we call E a *strong generator of* $\mathcal{C}$ and say that E generates $\mathcal{C}$ in n steps. Note that if one classical generator of a category $\mathcal{C}$ is a strong generator, then all other classical generators are also strong generators. To show this, let $E \in \mathcal{C}$ be a strong generator and consider another classical generator $G \in \mathcal{C}$. By definition, E must be contained in some subcategory $\langle G\rangle_n$ of $\mathcal{C}$, and given E generates $\mathcal{C}$ in finite steps, say m steps, then we must have $\langle G\rangle_{nm} = \mathcal{C}$. Hence G is a strong generator.

We shall use the following definition throughout the next few sections, as it is an important requirement used in the proofs of some of the results stated.

Definition 4.1.8. Let $G = \bigoplus_{j\in J} G_j$ be a classical (resp. strong) generator of some triangulated category $\mathcal{C}$. We say that G is a *minimal classical (resp. strong) generator* of $\mathcal{C}$ if there exists no classical (resp. strong) generator, G', such that $G \cong G' \oplus G_j$ for some $j \in J$.

If a minimal strong generator $E \in \mathcal{C}$ generates $\mathcal{C}$ in n steps, then a strong generator $E' \cong E \oplus F$, for some $F \in \mathcal{C}$, must generate $\mathcal{C}$ in $\leq n$ steps, as $\langle E\rangle_i \subseteq \langle E'\rangle_i$ for all $i \geq 0$, and $\langle E\rangle_n = \mathcal{C}$.

§ 4.2 | Generators of $\overline{\mathcal{C}}_n$

In this section we introduce homologically connected objects, and show that under certain conditions, any object in a triangulated category may be decomposed into homologically connected direct summands. We use the notion of homologically connected to classify the classical generators of $\overline{\mathcal{C}}_n$.

§ 4.2.1 | Homologically Connected Objects

We say that a morphism $f \in \operatorname{Ext}^i(X,Y)$ is a *morphism of degree i*.

Definition 4.2.1. Let $\mathcal{T}$ be a Hom-finite, Krull-Schmidt triangulated category. Let $G = \bigoplus_{i=1}^m G_i$ be an object in $\mathcal{T}$. Then we say G is *homologically connected* if for any two indecomposable objects F_1 and F_{l+1} in $\langle G\rangle_1$, then there is some finite set of non-zero morphisms of degree 1, $f_1, \ldots, f_l$, between indecomposable objects in $\langle G\rangle_1$ that form a sequence between F_1 and F_{l+1};

$$F_1 \overset{f_1}{\text{———}} F_2 \text{-------} F_l \overset{f_l}{\text{———}} F_{l+1}.$$

We call these sequences a *zig-zag from F_1 to F_{l+1}*.

The direction of the morphisms and their composition is not required, it is sufficient to know that there is a zig-zag between all indecomposable direct summands of G. We say that a zig-zag between two indecomposable objects has *length* l if there are l morphisms of degree 1 in the zig-zag, and a zig-zag in *minimal* if it has the smallest length of all zig-zags between the same objects. We fix it so that a minimal zig-zag between two isomorphic objects has length zero. The *homological length* of a homologically connected object G is the supremum of the length of all minimal zig-zags between any two indecomposable direct summands of G. If no supremum exists, then we say that the homological length is ∞.

Let $f \in \mathrm{Ext}^1(X, Y)$ be a morphism of degree 1, then we say that an object Z in a distinguished triangle

$$Y \to Z \to X \to Y[1],$$

is an *extension of* X by Y.

The following proposition shows us how to reduce the length of a zig-zag by using the cones of morphisms in the zig-zag.

Lemma 4.2.2. *Let $X, Y, Z \in \overline{\mathcal{C}}_n$ be indecomposable objects such that*

$$X \longrightarrow Y \longrightarrow Z$$

is a minimal zig-zag between $X, Z \in \langle G\rangle$, for a homologically connected object $G \in \overline{\mathcal{C}}_n$. Let $X \in \langle G\rangle_a$ and $Y \in \langle G\rangle_b$, such that $Y \ncong X[\pm 1]$ if ℓ_X is a limit arc. Then there exists an indecomposable object $A \in \langle G\rangle_{a+b}$ that is a direct summand of an extension of X by Y, or a direct summand of an extension of Y by X, such that there exists a minimal zig-zag

$$A \longrightarrow Z.$$

If ℓ_X is a limit arc and $Y \cong X[1]$ (resp. $Y \cong X[-1]$), then such an A exists as a direct summand of an extension of $Y[1]$ by X (resp. A is a direct summand of an extension of X by $Y[-1]$).

Proof. We have two cases to consider, where ℓ_X and ℓ_Y cross, and where they share an endpoint. First, we consider when ℓ_X and ℓ_Y cross.

Let $\ell_X = \{x_1, x_2\}, \ell_Y = \{y_1, y_2\}$ and $\ell_Z = \{z_1, z_2\}$, where $x_1 < y_1 < x_2 < y_2 < x_1$. Let $f \in \mathrm{Ext}^1(X, Y)$, then there exists a triangle

$$Y \longrightarrow A \oplus A' \longrightarrow X \longrightarrow Y[1]$$

where $\ell_{A\oplus A'} = \ell_A \oplus \ell_{A'} = \{x_1, y_1\} \oplus \{x_2, y_2\}$, and $A, A' \in \langle G\rangle_{a+b}$ as $X \in \langle G\rangle_a$ and $Y \in \langle G\rangle_b$. As there is a non-zero morphism of degree 1 between Y and Z in at least one direction, then either ℓ_Y and ℓ_Z cross, or they share an endpoint. If ℓ_Z and ℓ_Y share an endpoint, then ℓ_Z and ℓ_A share an endpoint (or ℓ_Z and $\ell_{A'}$ share an endpoint), and therefore there is a morphism of degree 1 in some direction between Z and A (or between Z and A').

Suppose ℓ_Z and ℓ_Y cross, and further suppose $y_1 < z_1 < y_2 < z_2 < y_1$. However, we know ℓ_X and ℓ_Z cannot cross or share an endpoint as there are no morphisms of degree 1 between X and Z by minimality of the zig-zag. Therefore one of the following holds

$$y_1 < x_2 \leq z_1 < y_2 < z_2 \leq x_1 < y_1,$$
$$y_1 < z_2 \leq x_2 < y_2 < x_1 \leq z_1 < y_1,$$

and so $\ell_{A'}$ and ℓ_Z cross and thus there is a morphism of degree 1 between A' and Z.

Now suppose $\ell_X = \{x, x_1\}$ and $\ell_Y = \{x, y_1\}$ share an endpoint at an accumulation point. We consider two cases, one where $Y \ncong X[\pm 1]$, and the second case where $Y \cong X[1]$ (or, $Y \cong X[-1]$). For the first case, let $x < x_1 < y_1^- < x$, so we get the triangle

$$X \to A \to Y \to X[1]$$

where $\ell_A = \{x_1, y_1\}$. Let $\ell_Z = \{z_1, z_2\}$, as there is a morphism of degree 1 between Y and Z, but no morphisms of degree 1 between X and Z, then either $x_1 < z_1 < y_1 < z_2 < x < x_1$, or one of the following holds and y_1 is an accumulation point,

$$x_1 < z_1 < y_1 = z_2 < x < x_1,$$
$$x_1 < y_1 = z_1 < z_2 < x < x_1.$$

Therefore ℓ_A either shares an endpoint with ℓ_Z at an accumulation point, or they cross, and so there is a morphism of degree 1 between A and Z. Further, if $X \in \langle G\rangle_a$ and $Y \in \langle G\rangle_b$, then $A \in \langle G\rangle_{a+b}$. A similar argument holds when we consider $x < y_1 < x_1^- < x$.

Next we focus on the second case where $Y \cong X[1]$ (resp. $Y \cong X[-1]$). There exists a non-zero morphism of degree 1 between X and $Y[1]$ (resp. between X and $Y[-1]$) by [58]. Any ℓ_Z that crosses ℓ_Y but not ℓ_X either crosses $\ell_{Y[1]}$ (resp. crosses $\ell_{Y[-1]}$), or $\ell_Z = \{x_1, x_1^{--}\}$ (resp. $\ell_Z = \{x_1, x_1^{++}\}$), and in this case Z is isomorphic to A.

If ℓ_Z crosses $\ell_{Y[1]}$ (resp. crosses $\ell_{Y[-1]}$) we are back in the previous case, and so there exists some object $A \in \langle G\rangle_{a+b}$ such that there exists a zig-zag

$$A \text{ ——— } Z.$$

$\square$

The new zig-zags that are formed using Lemma 4.2.2 are minimal zig-zags, which we show with the next lemma.

Lemma 4.2.3. *Let $G \in \overline{\mathcal{C}}_n$ be homologically connected, and let*

$$G_1 \text{ ——— } G_2 \text{ ——— } \cdots \text{ ——— } G_{d+1},$$

be a minimal zig-zag of objects in $\langle G\rangle_1$, with length d. Then there exists a minimal *zig-zag*

$$M_i \text{ ——— } G_{i+1} \text{ ——— } \cdots \text{ ——— } G_{d+1},$$

of length $d-i+1$, with $M_i \in \langle G \rangle_i$, and $2 \leq i \leq d$.

Proof. Let $\ell_{G_i} = \{y_i, z_i\}$ for all $i = 1, \ldots, d+1$. By Lemma 4.2.2, there exists an object $M_2 \in \langle G \rangle_2$ such that

$$M_2 \text{ ——— } G_3 \text{ ——— } \cdots \text{ ——— } G_d \text{ ——— } G_{d+1}$$

is a zig-zag. We repeat this process $d-1$ times using Lemma 4.2.2, producing a series of zig-zags for $i = 2, \ldots, d$,

$$M_i \text{ ——— } G_{i+1} \text{ ——— } \cdots \text{ ——— } G_{d+1},$$

where $M_i \in \langle G \rangle_{i-1}$. We show that this zig-zag is minimal.

If $G_{i+1} \ncong M_i[\pm 1]$, or if M_i is a long arc, then we may have $\ell_{M_{i+1}} = \{y_1, z_{i+1}\}$. This is because G_{i+1} and M_i are indecomposable objects with a morphism of degree 1 between them, and so the cone of the morphism $G_{i+1}[-1] \to M_i$ (or $M_i[-1] \to G_{i+1}$) is a direct sum of objects corresponding to arcs sharing endpoints with $\ell_{G_{i+1}}$ and ℓ_{M_i}.

If M_i is a limit arc and $G_{i+1} \cong M_i[1]$ (resp. $G_{i+1} \cong M_i[-1]$), then we may replace G_{i+1} in the zig-zag with $G_{i+1}[1] \cong M_i[2]$ (resp. $G_{i+1}[-1] \cong M_i[-2]$). This is possible as G_{i+2} cannot be a short arc by Corollary 4.2.12, and $\ell_{G_{i+2}}$ crosses $\ell_{G_{i+1}}$ but not ℓ_{M_i}, so $\ell_{G_{i+2}}$ must cross $\ell_{G_{i+1}[1]}$ (resp. $\ell_{G_{i+1}[-1]}$). In this case we have $\ell_{M_{i+1}} = \{x_1, z_i^-\}$ (resp. $\ell_{M_{i+1}} = \{y_1, z_i^+\}$).

As the zig-zag

$$G_1 \text{ ——— } G_2 \text{ ——— } \cdots \text{ ——— } G_d \text{ ——— } G_{d+1}$$

is minimal, then ℓ_{G_a} does not cross any arc ℓ_{G_j} for $j = a+2, \ldots, d+1$, or share an accumulation as an endpoint. However, as ℓ_{M_i} shares one endpoint with ℓ_{G_1} and another with ℓ_{G_i}, then ℓ_{G_j}, $j \geq i+2$ can only cross ℓ_{M_i} if it also cross some ℓ_{G_a}, or shares an endpoint at an accumulation point with some ℓ_{G_a}, for $a \leq i$. This cannot happen, and so there exists no morphisms of degree 1 between M_i and G_j for $j \geq i+2$, and so the zig-zag

$$M_i \text{ ——— } G_{i+1} \text{ ——— } \cdots \text{ ——— } G_{d+1},$$

is minimal with $M_i \in \langle G \rangle_i$. □

Next we show when we can expect an indecomposable object to be homologically connected.

Proposition 4.2.4. *Let $X \in \overline{\mathcal{C}}_n$ be an indecomposable object. Then X is homologically connected.*

Proof. We must check the four different types of arc; double limit arcs, limit arcs, long arcs, and short arcs.

If ℓ_X is a double limit arc, then $X \cong X[i]$ for all $i \in \mathbb{Z}$, and so X is homologically connected. Similarly, if ℓ_X is a limit arc, then by [58] there exists a morphism of degree 1 in $\mathrm{Ext}^1(X, X[i])$ for all $i < 0$, so X is homologically connected.

Now suppose that ℓ_X is a long arc, then there exists a morphism of degree 1 in $\mathrm{Ext}^1(X, X[i])$ for all $i \in \mathbb{Z}\backslash\{0\}$, as ℓ_X and $\ell_{X[i]}$ cross for all $i \neq 0$. So X is homologically connected.

Now let ℓ_X be a short arc. Then ℓ_X and $\ell_{X[1]}$ cross, and so

$$\mathrm{Ext}^1(X, X[1]) \cong k.$$

Therefore there exists a non-zero morphism of degree 1 in $\mathrm{Ext}^1(X, X[1])$, and so there exists a sequence of morphisms of degree 1 from $X[i]$ to X for all $i \geq 0$. Hence there is a zig-zag between any two suspensions of X, and so X is homologically connected. $\square$

Next, we show that any object in $\overline{\mathcal{C}}_n$, and more generally any object in a Hom-finite, Krull-Schmidt triangulated category, can be decomposed into a direct sum of homologically connected summands.

Lemma 4.2.5. *Let $\mathcal{T}$ be a* Hom*-finite, Krull Schmidt triangulated category such that each indecomposable object is homologically connected, and let G be an object in $\mathcal{T}$. Then there exists a decomposition of $G \cong \oplus_{i=1}^{l} X_i$, such that X_a is homologically connected for all $a = 1, \ldots, l$, and $X_a \oplus X_b$ is not homologically connected for $a \neq b$.*

Proof. Suppose that $G \cong \bigoplus_{i=1}^{m} G_i$ where G_i are all indecomposable. We construct a graph Q with m vertices, with an edge between i and j if $G_i \oplus G_j$ is homologically connected. Then we either have a set of disjoint graphs, or the graph is connected. We label the connected subgraphs $Q_1, \ldots, Q_l$.

For a connected subgraph Q_a with vertices $\{a_1, \ldots, a_p\}$, let $X_a \cong \bigoplus_{i=1}^{p} G_{a_i}$. Then X_a is homologically connected as each indecomposable direct summand is homologically connected by assumption, and an edge between a_i and a_j represents a zig-zag between G_{a_i} and G_{a_j} in $\langle X_a \rangle_1$. Hence there exists a zig-zag between any two indecomposable objects in $\langle X_a \rangle_1$, meaning X_a is homologically connected.

As there are no morphisms of degree 1 between the indecomposable objects corresponding to vertices in disjoint subgraphs, then each X_a and X_b, for $a \neq b$, have no morphisms of degree 1 between them, and so $X_a \oplus X_b$ is not homologically connected. Therefore we have

$$G \cong X_1 \oplus \cdots \oplus X_l,$$

such that each X_a is homologically connected and $X_a \oplus X_b$ is not homologically connected for any $a \neq b$. $\square$

We call such a decomposition of an object X a *hc (=homologically connected) decomposition of X*.

§ 4.2.2 | Thick Subcategories

The thick subcategories of $\mathcal{C}_n$ were classified by Gratz and Zvonareva in [37]. In fact, they prove that there exists an isomorphism of lattices between the thick subcategories of $\mathcal{C}_n$, thick$(\mathcal{C}_n)$, and *non-exhaustive non-crossing partitions of* $[n] = \{1, \dots, n\}$.

Let $\mathcal{P} = \{B_m \subseteq [n] \mid m \in I\}$ be a collection of non-empty subsets of $[n]$ for some indexing set I. Then Gratz and Zvonareva [37] define $\mathcal{P}$ to be a *non-exhaustive non-crossing partitions of* $[n]$ if $B_{m_1} \cap B_{m_2} = 0$ when $m_1 \neq m_2 \in I$, and whenever

$$1 \leq i < k < j < l \leq n$$

for $i, j, k, l \in [n]$ with $i, j \in B_{m_1}$ and $k, l \in B_{m_2}$ for $m_1, m_2 \in I$, then $m_1 = m_2$. The set of non-exhaustive non-crossing partitions of $[n]$ is denoted NNC_n.

Let $\mathcal{P} = \{B_m \subseteq [n] \mid m \in I\}$ be a non-exhaustive non-crossing partition of $[n]$. The authors of [37] consider the full subcategory $\langle \mathcal{P} \rangle \subseteq \mathcal{C}_n$ that is closed under direct sums and direct summands, and contains the zero object,

$$\langle \mathcal{P} \rangle := \mathrm{add}\{X \in \mathcal{C}_n \mid \ell_X = \{x, y\},\ x, y \in \bigcup_{i \in B_m} (a_i, a_{i+1}), \text{for some}\, m \in I\}.$$

Recall that $a_1, \dots, a_n$ are the accumulation points in the combinatorial model for $\mathcal{C}_n$.

Theorem 4.2.6. *[37, Theorem 3.7] There is an isomorphism of lattices*

$$NNC_n \cong \mathrm{thick}(\mathcal{C}_n).$$

Under this isomorphism a non-exhaustive non-crossing partition $\mathcal{P}$ corresponds to the thick subcategory $\langle \mathcal{P} \rangle$.

It is noted by Gratz and Zvonareva [37] that in general for $\mathcal{P} = \{B_m \subseteq [n] \mid m \in I\}$, the subcategory $\langle \mathcal{P} \rangle$ is equivalent to the union of mutually orthogonal thick subcategories of the form $\langle \{B_m\} \rangle$. Here, we show how to construct $\langle G \rangle$ for any object $G \in \mathcal{C}_n$ in terms of non-exhaustive non-crossing partitions of $[n]$.

Lemma 4.2.7. *Let $F \in \mathcal{C}_n$ be a object, and let $F \cong \bigoplus_{i \in I} F_i$ be hc decomposition of F. Let $\{B_{m_i} \mid i \in I\}$ be a collection of subsets of $[n]$ such that $\mathscr{M}_{F_i} = \bigcup_{p \in B_{m_i}} (a_p, a_{p+1})$ for all $i \in I$.*

Then $\mathcal{P} = \{B_{m_i} \mid i \in I\}$ is a non-exhaustive non-crossing partition of $[n]$. Moreover, $\langle F \rangle$ is equivalent to $\langle \mathcal{P} \rangle$.

Proof. First, we show that the collection $\mathcal{P}$ is a non-exhaustive non-crossing partition of $[n]$.

Let B_{m_i}, B_{m_j} be two subsets of $[n]$, corresponding to the homologically connected objects F_i and F_j respectively. As F_i and F_j are non-isomorphic objects in a hc decomposition of F, there are no morphism of degree 1 between indecomposable objects in $\langle F_i \rangle_1$ and indecomposable objects in $\langle F_j \rangle_1$. That is, $\mathscr{M}_{F_i} \cap \mathscr{M}_{F_j} = 0$ as arcs corresponding to an indecomposable object $A \in \langle F_i \rangle_1$ and an indecomposable object

$B \in \langle F_j \rangle_1$ cannot cross, and in particular ℓ_A and $\ell_{B[1]}$ cannot share an endpoint. Hence $B_{m_i} \cap B_{m_j} = 0$ if $m_i \neq m_j$.

Let $e, f, g, h, \in [n]$ such that

$$1 \leq e < g < f < h \leq n,$$

with $e, f \in B_{m_i}$ and $g, h \in B_{m_j}$. As F_i and F_j are homologically connected, there exists zig-zags

$$X_1 \;\text{———}\; X_2 \;\text{———}\; X_3 \;\text{-------}\; X_{s-1} \;\text{———}\; X_s,$$

$$Y_1 \;\text{———}\; Y_2 \;\text{———}\; Y_3 \;\text{-------}\; Y_{t-1} \;\text{———}\; Y_t,$$

with $X_1, \ldots, X_s \in \langle F_i \rangle_1$ and $Y_1, \ldots, Y_t \in \langle F_j \rangle_1$, such that ℓ_{X_1} has an endpoint $x \in (a_e, a_{e+1})$, ℓ_{X_s} has an endpoint in $x' \in (a_f, a_{f+1})$, ℓ_{Y_1} has an endpoint in $y \in (a_g, a_{g+1})$, and ℓ_{Y_t} has an endpoint in $y' \in (a_h, a_{h+1})$. As there is a morphism of degree 1 between $X_{s'}$ and $X_{s'+1}$, then $\ell_{X_{s'}}$ and $\ell_{X_{s'+1}}$ must cross, similarly $\ell_{Y_{t'}}$ and $\ell_{Y_{t'+1}}$ must cross.

Let ℓ_X be an arc with endpoints $\{x, x'\}$, such that ℓ_X traces $\ell_{X_{s'}}$ until $\ell_{X_{s'}}$ and $\ell_{X_{s'+1}}$ cross, then ℓ_X traces $\ell_{X_{s'+1}}$ for all $s' = 1, \ldots, s-1$. We define $\ell_Y = \{y, y'\}$ similarly for $t' = 1, \ldots, t-1$. As $x < y < x' < y' < x$, then ℓ_X and ℓ_Y must cross, therefore $\ell_{X_{s'}}$ and $\ell_{Y_{t'}}$ must cross for some $s' \in \{1, \ldots, s\}$ and some $t' \in \{1, \ldots, t\}$. Hence there is a morphism of degree 1 between $X_{s'} \in \langle F_i \rangle_1$ and $Y_{t'} \in \langle F_j \rangle_1$, and so $F_i \oplus F_j$ must be homologically connected. However, F_i and F_j are both part of a hc decomposition of F, and so by Lemma 4.2.5, $F_i \oplus F_j$ is only homologically connected if $i = j$. Therefore $m_i = m_j$, and so $\mathcal{P}$ is a non-exhaustive non-crossing partition of $[n]$.

To show that $\langle F \rangle$ is equivalent to $\langle \mathcal{P} \rangle$, we note that $\langle F \rangle$ is a thick subcategory and so is equivalent to $\langle \mathcal{P}' \rangle$ for some non-exhaustive non-crossing partition $\mathcal{P}'$. Suppose $\langle \mathcal{P}' \rangle \subset \langle \mathcal{P} \rangle$, then there exists some $p \in [n]$ such that indecomposable objects corresponding to arcs with an endpoint in (a_p, a_{p+1}) are in $\langle \mathcal{P} \rangle$ but not $\langle \mathcal{P}' \rangle$. However, there exists an indecomposable direct summand of F that corresponds to an arc with an endpoint in (a_p, a_{p+1}), as $\mathscr{M}_{F_i} = \bigcup_{p \in B_{m_i}} (a_p, a_{p+1})$ for all $i \in I$ and $\mathcal{P} = \{B_{m_i} \mid i \in I\}$. Therefore $F \notin \langle \mathcal{P}' \rangle$, and so $\langle F \rangle$ is equivalent to $\langle \mathcal{P} \rangle$. $\square$

Note that $\mathcal{C}_n$ satisfies the axioms of Lemma 4.2.5, as $\mathcal{C}_n$ is a full triangulated subcategory of $\overline{\mathcal{C}}_n$ [58], and so every indecomposable object is homologically connected by Lemma 4.2.4.

We provide two examples of thick subcategories of $\mathcal{C}_6$ as illustration.

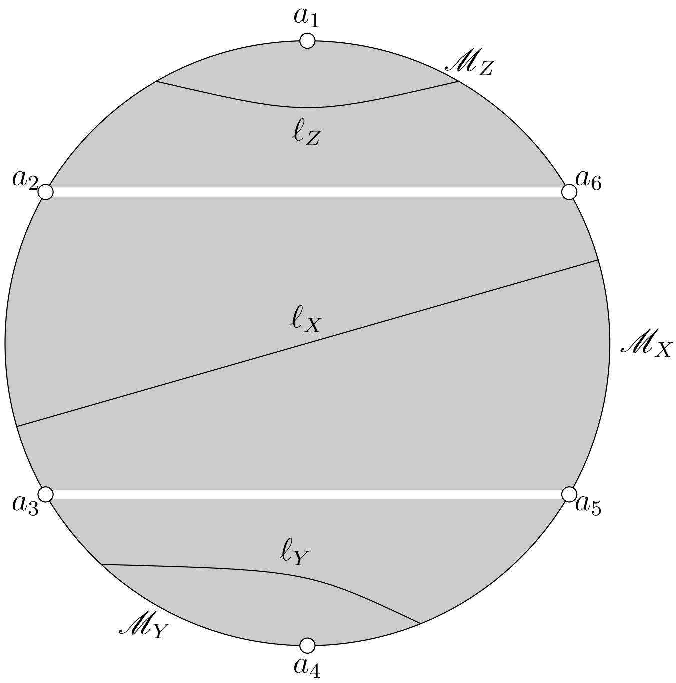


Figure 4.1: A representation of the thick subcategory $\mathcal{T}_1$ of $\mathcal{C}_6$ containing the object $W \cong X \oplus Y \oplus Z$. The indecomposable objects correspond to the arcs that are entirely contained in the shaded area, with non-exhaustive non-crossing partition $\mathcal{P} = \{\{1,6\},\{2,5\},\{3,4\}\}$.

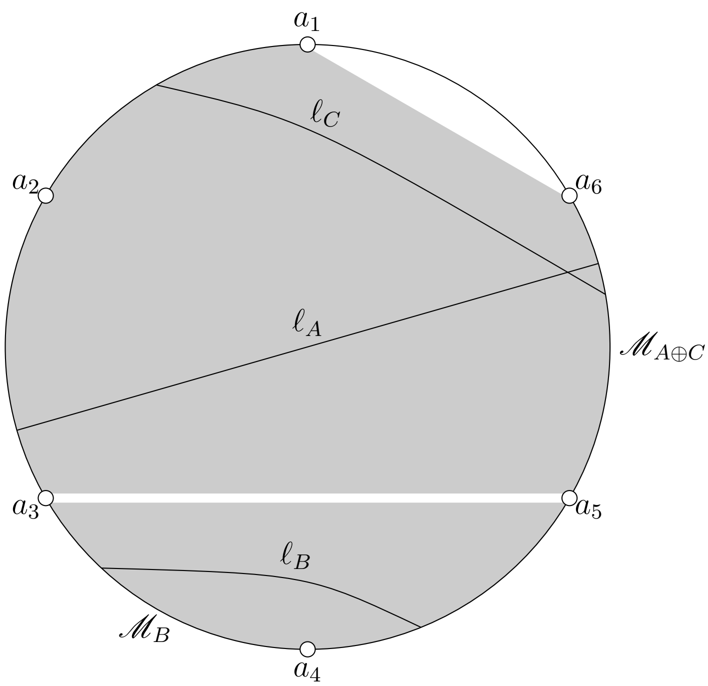


Figure 4.2: A representation of the thick subcategory $\mathcal{T}_2$ of $\mathcal{C}_6$ containing the objects $D \cong A \oplus B \oplus C$. The indecomposable objects correspond to the arcs that are entirely contained in the shaded area, with non-exhaustive non-crossing partition $\mathcal{P}' = \{\{1,2,5\},\{3,4\}\}$.

We may use the classification of thick subcategories in $\mathcal{C}_{2n}$ to classify the thick subcategories in $\overline{\mathcal{C}}_n$ by using the localisation functor $\pi : \mathcal{C}_{2n} \to \overline{\mathcal{C}}_n$ from [58].

Proposition 4.2.8. *Let $\varphi : \mathcal{C} \to \overline{\mathcal{C}}$ be a localisation functor, and let $\mathcal{T}' \subseteq \overline{\mathcal{C}}$ be a thick subcategory. Then $\mathcal{T}'$ is equivalent to the essential image of some thick subcategory $\mathcal{T} \subseteq \mathcal{C}$.*

Proof. Let $\mathcal{D} \subseteq \mathcal{C}$ be the subcategory defined as follows

$$\mathcal{D} := \{X \in \mathcal{C} \,|\, 0 \ncong \varphi(X) \in \mathcal{T}'\},$$

and let $\mathcal{T}$ be the thick closure of $\mathcal{D}$. We show that $\varphi(\mathcal{T}) \simeq \mathcal{T}'$.

As φ is an identity on objects, and $\mathcal{T}'$ is a thick subcategory then $\mathcal{D}$ is closed under direct summands. Now let $X, Y \in \mathcal{D}$ such that there exists a triangle

$$X \to Z \oplus Z' \to Y \to Z[1]$$

with $Z \in \mathcal{T}$ but not in $\mathcal{D}$, and $Z' \in \mathcal{D}$. The localisation functor φ is triangulated and therefore induces a triangle in $\overline{\mathcal{C}}_n$,

$$\varphi(X) \to \varphi(Z) \oplus \varphi(Z') \to \varphi(Y) \to \varphi(X[1]).$$

However, as $\varphi(X), \varphi(Y), \varphi(Z') \in \mathcal{T}'$ but $\varphi(Z) \notin \mathcal{T}' \backslash \{0\}$, then $\varphi(Z) \cong 0$ as $\mathcal{T}'$ is thick.

Therefore any object in $\mathcal{T}$ but not in $\mathcal{D}$ is isomorphic to a zero object in $\overline{\mathcal{C}}$, and so $\varphi(\mathcal{T}) \simeq \mathcal{T}'$. □

The thick subcategories of $\mathcal{C}_6$ from Figures 4.1 and 4.2 respectively induce the following thick subcategories in $\overline{\mathcal{C}}_3$.

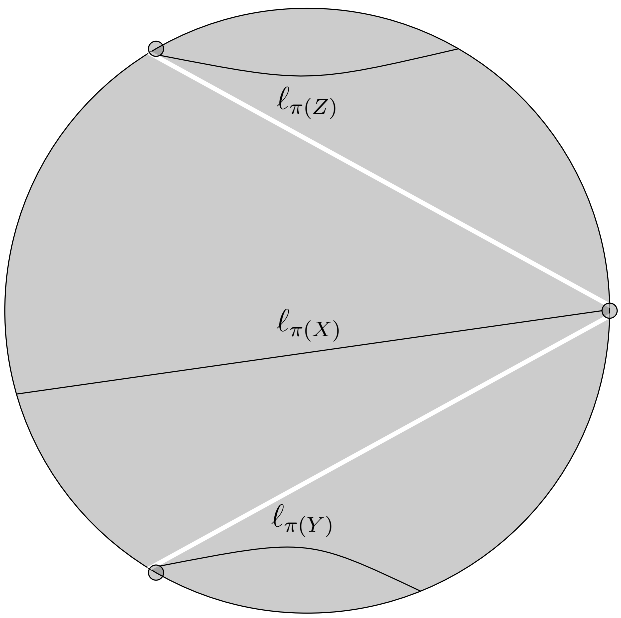


Figure 4.3: The thick subcategory $\mathcal{T}_1'$ of $\overline{\mathcal{C}}_3$ that is equivalent to $\pi(\mathcal{T}_1)$. Again, indecomposable objects in $\mathcal{T}_1'$ are in correspondence with the arcs entirely contained in the shaded area.

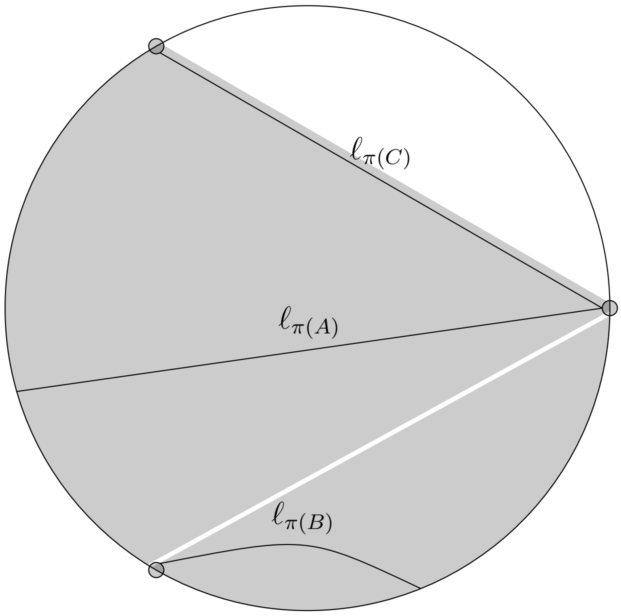


Figure 4.4: The thick subcategory $\mathcal{T}_2'$ of $\overline{\mathcal{C}}_3$ that is equivalent to $\pi(\mathcal{T}_2)$.

Recall from Definition 2.2.3 that an object $X \in \overline{\mathcal{C}}_n$ has an orbit in $\mathscr{M}$ denoted by $\mathscr{M}_X$ (more specifically, ℓ_X has an orbit in $\mathscr{M}$), which corresponds to the union of segments and accumulation points containing an endpoint of an arc corresponding to a direct summand of X. Also recall that X (again, specifically ℓ_X) has a complete orbit in $\mathscr{M}$ if $\mathscr{M}_X = \mathscr{M}$. Here, we classify the thick subcategories of $\overline{\mathcal{C}}_n$ in terms of orbits of homologically connected objects.

Lemma 4.2.9. *Let G be an object in $\overline{\mathcal{C}}_n$, with hc decomposition $G \cong \bigoplus_{i\in I} G_i$. Then an indecomposable object $X \in \overline{\mathcal{C}}_n$ is in $\langle G\rangle$ if and only if $\mathscr{M}_X \subseteq \mathscr{M}_{G_i}$ for some $i \in I$. That is, $\langle G\rangle$ is completely determined by the disjoint union $\bigsqcup_{i\in I} \mathscr{M}_{G_i}$.*

Proof. The subcategory $\langle G\rangle$ is a thick subcategory of $\overline{\mathcal{C}}_n$, and so by Proposition 4.2.8 there exists a thick subcategory $\mathcal{T}$ of $\mathcal{C}_{2n}$ such that $\pi(\mathcal{T}) \simeq \langle G\rangle$. Moreover, $\mathcal{T} \simeq \langle F\rangle$ for some object $F \in \mathcal{C}_{2n}$ by Lemma 4.2.7, where $\pi(F) \cong G$. Therefore an indecomposable object $U \in \mathcal{C}_{2n}$ is in $\mathcal{T}$ if and only if ℓ_U has both endpoints in $\mathscr{M}_{F_j}$, for some F_j in the hc decomposition of F, by Theorem 4.2.6 and Lemma 4.2.7. Hence $\ell_{\pi(U)}$ has both endpoints in $\mathscr{M}_{\pi(F_j)} \subseteq \mathscr{M}_{G_i}$ for some $i \in I$ if and only if $\pi(U) \in \langle G\rangle$. □

§ 4.2.3 | Generators

This next proposition shows that being homologically connected is a necessary condition for an object to be a generator of $\overline{\mathcal{C}}_n$.

Proposition 4.2.10. *Let G be a generator of $\overline{\mathcal{C}}_n$. Then G is homologically connected.*

Proof. Suppose that $G \in \overline{\mathcal{C}}_n$ is not homologically connected. We show that G cannot be a generator of $\overline{\mathcal{C}}_n$.

By Lemma 4.2.5, there exists a hc decomposition of G,

$$G \cong \bigoplus_{i\in I} G_i.$$

Further, Lemma 4.2.9 tells us that an object $X \in \overline{\mathcal{C}}_n$ is in $\langle G \rangle$ if and only if $\mathscr{M}_X \subseteq \mathscr{M}_{G_i}$ for some $i \in I$. Now let $Y \in \overline{\mathcal{C}}_n$ correspond to the arc $\ell_Y = \{y_1, y_2\}$, such that $y_1 \in \mathscr{M}_{G_j}$ and $y_2 \in \mathscr{M}_{G_{j'}}$ with $j \neq j' \in I$. Then $\mathscr{M}_Y \not\subset \mathscr{M}_{G_i}$ for any $i \in I$, hence $Y \notin \langle G \rangle$, so G is not a classical generator of $\overline{\mathcal{C}}_n$. □

Finally, we can combine Lemma 4.2.9 and Proposition 4.2.10 to classify all of the generators of $\overline{\mathcal{C}}_n$, and moreover, show that they all must be strong generators too.

Theorem 4.2.11. *Let G be an object in $\overline{\mathcal{C}}_n$, then G is a generator of $\overline{\mathcal{C}}_n$ if and only if G is homologically connected and G has a complete orbit of $\mathscr{M}$.*

Proof. Let G be a generator, then by Proposition 4.2.10 G is homologically connected, and by Proposition 4.2.9 $\mathscr{M}_X \subseteq \mathscr{M}_G$ for all $X \in \overline{\mathcal{C}}_n$, and so G has a complete orbit in $\mathscr{M}$.

Let G be homologically connected and have a complete orbit in $\mathscr{M}$, then by Proposition 4.2.9 all indecomposable objects are in $\langle G \rangle$, and so G is a generator of $\overline{\mathcal{C}}_n$. □

It follows from Theorem 4.2.11 that no short arcs may be direct summands of a minimal strong generator of $\overline{\mathcal{C}}_n$ for all n.

Corollary 4.2.12. *Let ℓ_X be a short arc. Then X cannot be a direct summand of a minimal strong generator of $\overline{\mathcal{C}}_n$.*

Proof. Let ℓ_X be a short arc, and G be a generator with X as a direct summand, and let F be an object such that $G \cong F \oplus X$.

Let G have no other direct summands with endpoints in the same segment as ℓ_X. Then $\mathrm{Ext}^i(X, F) = 0$ for all $i \in \mathbb{Z}$, and so G is not homologically connected.

Now let Y be an indecomposable direct summand of F such that $\mathscr{M}_X \subsetneq \mathscr{M}_Y$. If ℓ_Y has only one endpoint on the segment shared by ℓ_X, then ℓ_Y is a long arc or limit arc, and thus there exists a triangle

$$Y[j] \to X \oplus Z \to Y[l] \to Y[j+1]$$

and so $X \in \langle Y \rangle$ and so F is also a generator of $\overline{\mathcal{C}}_n$.

If $\mathscr{M}_X = \mathscr{M}_Y$, then Proposition 4.2.9 implies that $X \in \langle Y \rangle$, as Proposition 4.2.4 means that Y must be homologically connected.

Therefore, G cannot be a minimal generator if it has a short arc as an indecomposable direct summand. □

Lemma 4.2.13. *Let G be a generator of $\overline{\mathcal{C}}_n$, and suppose there exists a zig-zag in $\langle G \rangle_1$,*

$$M_1 \text{ ——— } M_2 \text{ ------- } M_d \text{ ——— } M_{d+1} \text{ ——— } M_1,$$

such that $M_1 \ncong M_i[j]$ for all $i = 2, \ldots, d+1$ and $j \in \mathbb{Z}$, and ℓ_{M_1} shares an endpoint each with $\ell_{M_2[a_2]}$ and $\ell_{M_{d+1}[a_{d+1}]}$ for some $a_2, a_{d+1} \in \mathbb{Z}$. Then the object F such that $G \cong F \oplus M_1$ is a generator of $\overline{\mathcal{C}}_n$.

Proof. As ℓ_{M_1} shares an endpoint each with $\ell_{M_2[a_2]}$ and $\ell_{M_{d+1}[a_{d+1}]}$, then F has a complete orbit in $\mathscr{M}$. We now need to show that F is homologically connected.

Let $\ell_{M_i[a_i]} = \{x_i, y_i\}$, where $x_i < x_{i+1} \leq y_i < y_{i+1} < x_i$ and $x_{i+1} = y_i$ if and only if y_i is an accumulation point. That is, for some $m \in \mathscr{M}$ such that $x_1 < m < y_1 < x_1$, then $x_i \leq m \leq y_i < x_i$ for some $i = 2, \ldots, d+1$, again with equality if and only if m is an accumulation point.

Suppose an arc $\ell_N = \{n_1, n_2\}$ crosses ℓ_{M_1}, and N is in $\langle G \rangle_1$. Then ℓ_N must also cross some arc ℓ_{M_i} for $i = 2, \ldots, d+1$, or share an endpoint with ℓ_{M_i} at an accumulation point. In either case, there exists a zig-zag of length 1 between N and M_i, and so F is homologically connected.

Therefore, Theorem 4.2.11 tells us that F is a generator of $\overline{\mathcal{C}}_n$. □

§ 4.3 | Graded Endomorphism Ring of E

In this section we look at a particular generator of $\overline{\mathcal{C}}_n$, which we label E, and compute its graded endomorphism ring as a matrix ring over the field k. This generator is chosen because it has some particular properties that we want when considering its graded endomorphism ring, for instance that the algebra is upper triangular.

§ 4.3.1 | The Generator E

The generator E can be thought of as similar to a fan triangulation of a polygon, with all arcs sharing an endpoint at a single accumulation point, and with an endpoint contained in one of the segments or at an accumulation point.

More explicitly, let $X_i \in \overline{\mathcal{C}}_n$ correspond to the arc $\ell_{X_i} = \{a_1, z_i\}$ for all $i = 1, \ldots, n$, and $Y_j \in \overline{\mathcal{C}}_n$ correspond to the arc $\ell_{Y_j} = \{a_1, a_{j+1}\}$ for $j = 1, \ldots, n-1$. Then we let E be the direct sums of all X_i's and Y_j's, i.e.

$$E = \Big(\bigoplus_{i=1}^{n} X_i\Big) \oplus \Big(\bigoplus_{j=1}^{n-1} Y_j\Big).$$

Proposition 4.3.1. *An indecomposable object is in $\langle E \rangle_1$ if and only if it corresponds to an arc of the form (a_1, y), for $a_1 \neq y \in \mathscr{M}$. Thus E has a complete orbit in $\mathscr{M}$.*

Proof. Let $M \in \langle E \rangle_1$ be indecomposable, then by definition M is a direct summand of $E[p]$ for some $p \in \mathbb{Z}$. This means that either $M \cong X_i[a]$ for some $i \in \{1, \ldots, n\},\ a \in \mathbb{Z}$, or $M \cong Y_j[b]$ for some $j \in \{1, \ldots, n-1\},\ b \in \mathbb{Z}$.

If $M \cong X_i[a]$, then it corresponds to the arc $\{a_1, z_i - a\}$ by [58]. If $M \cong Y_j[b]$, then it corresponds to the arc $\{a_1, a_{j+1}\}$, also by [58]. Therefore, when $M \in \langle E \rangle_1$ is indecomposable, ℓ_M is the form $\{a_1, y\}$ for $a_1 \neq y \in \mathscr{M} \cup L(\mathscr{M})$.

Now let $\ell_N = \{a_1, y\}$ for some $a_1 \neq y \in \mathscr{M} \cup L(\mathscr{M})$. If $y \in L(\mathscr{M})$, then $N \cong Y_j$ for some $1 \leq j \leq n-1$. Similarly, if $y \in \mathscr{M}$, then $N \cong X_i[a]$ for some $1 \leq i \leq n$ and $a \in \mathbb{Z}$. In both cases, N is isomorphic to an indecomposable summand of a suspension of E, and therefore $N \in \langle E \rangle_1$. □

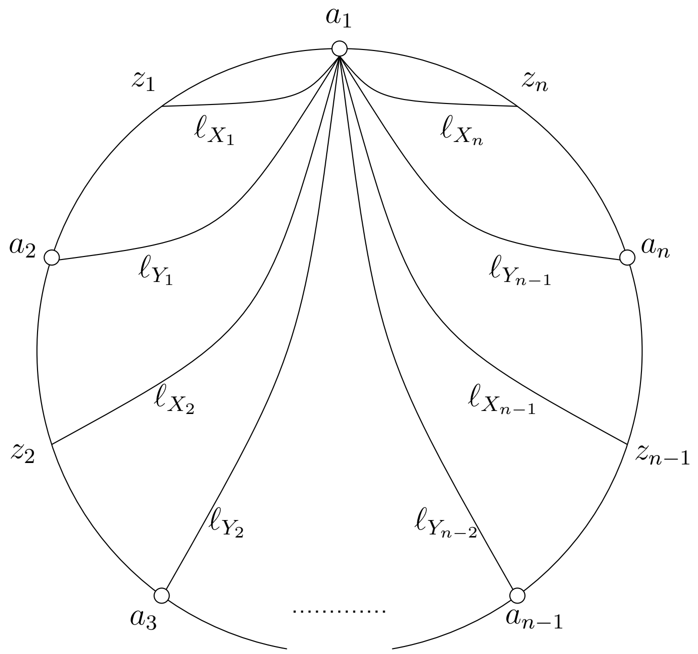


Figure 4.5: The arcs corresponding to X_i's and Y_j's.

Lemma 4.3.2. *The object E is a generator of $\overline{\mathcal{C}}_n$.*

Proof. By Proposition 4.3.1, E has a complete orbit in $\mathscr{M}$. Also, by [58], for any two arcs ℓ_U and ℓ_V sharing an endpoint at an accumulation point, then $\operatorname{Ext}^1(U, V) \cong k$ if and only if ℓ_U is an anti-clockwise rotation of ℓ_V about the shared endpoint. Therefore there exists a zig-zag of length 1 between all indecomposable summands of E, and so E is homologically connected. The result then follows by Theorem 4.2.11. $\square$

§ 4.3.2 | $\operatorname{End}^*(E)$

In this subsection, we state what that graded endomorphism ring of an object in a triangulated category is, and then compute the graded endomorphism ring for E. We do this in an attempt to obtain a homological description of the category $\overline{\mathcal{C}}_n$, similar to that found in [38] and [3], by using a result due to Keller [41] stating that an algebraic triangulated category $\mathcal{T}$ with a classical generator G is equivalent to $\mathrm{D}^{\mathrm{perf}}(\operatorname{RHom}(G, G))$. Therefore if it can be shown that there exists a quasi-isomorphism between $\operatorname{RHom}(E, E)$ and $\operatorname{End}^*(E)$, then there is a triangulated equivalence between $\overline{\mathcal{C}}_n$ and $\mathrm{D}^{\mathrm{perf}}(\operatorname{End}^*(E))$.

Definition 4.3.3. Let M be an object of a triangulated category $\mathcal{T}$, then the *graded endomorphism ring* of M, denoted $\operatorname{End}^*(M)$, is the graded ring with i^{th} degree

$$\operatorname{End}^i_{\mathcal{T}}(M) = \operatorname{Ext}^i_{\mathcal{T}}(M, M) \cong \operatorname{Hom}_{\mathcal{T}}(M, M[i]).$$

Let $f \in \operatorname{End}^i_{\mathcal{T}}(M)$ and $g \in \operatorname{End}^j_{\mathcal{T}}(M)$. Then we define multiplication as

$$gf := g[i] \circ f \in \operatorname{End}^{i+j}_{\mathcal{T}}(M).$$

To compute the graded endomorphism ring of E, it is worth first looking at the graded endomorphism rings of the indecomposable direct summands of E. We follow the concept of a similar proof found in [3].

Lemma 4.3.4. *Let $X_i \in \overline{\mathcal{C}}_n$ be the indecomposable object that corresponds to the arc $\{a_1, z_i\}$. Then*

$$\mathrm{End}^*_{\overline{\mathcal{C}}_n}(X_i) \cong k[x]$$

as graded rings, with x concentrated in degree -1.

Proof. We follow part of the proof of [3, Prop. 3.6].

By [58, Prop. 3.14], we have

$$\dim_k(\mathrm{End}^l_{\overline{\mathcal{C}}_n}(X_i)) = \begin{cases} 1 & \text{if } l \leq 0, \\ 0 & \text{if } l > 0, \end{cases}$$

which agrees on dimensions with $k[x]$ considered as a graded ring with x concentrated in degree -1. All that is left to show is that the ring structures on $\mathrm{End}^*_{\overline{\mathcal{C}}_n}(X_i)$ and $k[x]$ agree.

Let $f \in \mathrm{End}^{-l}(X_i)$ be a morphism with $l > 0$, then by the construction of $\overline{\mathcal{C}}_n$ in [58] and using [39, Lemma 2.4.2], we see f factors through an l-fold product $\mathrm{End}^{-1}(X_i) \times \ldots \times \mathrm{End}^{-1}(X_i)$. This shows that the graded endomorphism ring of X_i is isomorphic to $k[x]$ with x in degree -1. □

Lemma 4.3.5. *Let $Y_j \in \overline{\mathcal{C}}_n$ be the indecomposable object that corresponds to the arc $\{a_1, a_{j+1}\}$. Then*

$$\mathrm{End}^*_{\overline{\mathcal{C}}_n}(Y_j) \cong k[x^{\pm 1}]$$

as graded rings, with x concentrated in degree -1.

Proof. By [58, Prop. 3.14] we have

$$\dim_k(\mathrm{End}^l_{\overline{\mathcal{C}}_n}(Y_j)) = 1$$

for all $l \in \mathbb{Z}$. We need to show that multiplication agrees.

Given $g \in \mathrm{End}^{-l}_{\overline{\mathcal{C}}_n}(Y_j)$ for $l > 0$, we may use the same approach as in Lemma 4.3.4 to show that g factors through an l-fold product $\mathrm{End}^{-1}_{\overline{\mathcal{C}}_n}(Y_i) \times \ldots \times \mathrm{End}^{-1}_{\overline{\mathcal{C}}_n}(Y_i)$. Hence

$$\bigoplus_{l \leq 0} \mathrm{End}^l_{\overline{\mathcal{C}}_n}(Y_j) \cong k[x]$$

such that x is placed in degree -1.

Let $f \in \mathrm{End}^{l'}(Y_j)$ for $l' > 0$, we show that there exists an l'-fold product $\mathrm{End}^1_{\overline{\mathcal{C}}_n}(Y_i) \times \ldots \times \mathrm{End}^1_{\overline{\mathcal{C}}_n}(Y_i)$. Let $A \in \mathcal{C}_{2n}$ such that $\pi(A) \cong Y_j$ under the localisation functor π. There exists an l-fold product $h = h_1 \cdots h_{l'} \in \mathrm{End}^{-1}_{\mathcal{C}_{2n}}(A) \times \ldots \times \mathrm{End}^{-1}_{\mathcal{C}_{2n}}(A)$ in $\mathcal{C}_{2n}$ by [39], where each h_i is in the multiplicative system Ω. Therefore in $\overline{\mathcal{C}}_n$, there exists a morphism $\pi(h_i)^{-1} \in \mathrm{End}^1(\pi(A)) \cong \mathrm{End}^1(Y_j)$, and so $f = \pi(h)^{-1} \in \mathrm{End}^{l'}(Y_j)$ factors through an l'-fold product $\mathrm{End}^1_{\overline{\mathcal{C}}_n}(Y_i) \times \ldots \times \mathrm{End}^1_{\overline{\mathcal{C}}_n}(Y_i)$. Thus

$$\bigoplus_{l' \geq 0} \mathrm{End}^{l'}_{\overline{\mathcal{C}}_n}(Y_j) \cong k[y]$$

such that y is placed in degree 1.

Let $s \in \mathrm{End}^1_{\overline{\mathcal{C}}_n}(Y_j)$, then $s = \pi(t)^{-1}$ where $t \in \Omega$, as $\mathrm{Hom}_{\mathcal{C}_{2n}}(A, A[1]) = 0$ where $\pi(A) \cong Y_j$. Then there exists $s' \in \mathrm{End}^{-1}_{\overline{\mathcal{C}}_n}(Y_j)$ such that $s' = \pi(t)$, and so $ss' = id_{Y_j}$. Therefore

$$\bigoplus_{l \in \mathbb{Z}} \mathrm{End}^{l}_{\overline{\mathcal{C}}_n}(Y_j) \cong k[x^{\pm 1}]$$

with x placed in degree -1. □

When finding a graded ring with an isomorphism to the graded endomorphism ring of E, it is necessary to find a ring with matching dimensions in each degree. Hence it is important to know the dimension of $\mathrm{End}^i(E)$ over k for all $i \in \mathbb{Z}$, which we compute in the following lemma.

Lemma 4.3.6. *The dimension of the l^{th}* Ext*-space of E is*

$$\dim_k(\mathrm{End}^l_{\overline{\mathcal{C}}_n}(E)) = \begin{cases} 2n^2 - n & \text{if } l \leq 0 \\ 2n^2 - 2n & \text{if } l > 0. \end{cases}$$

Proof. We may consider $\mathrm{End}^l(E)$ as the direct sum of all $\mathrm{Ext}^l(X_i, E)$ and $\mathrm{Ext}^l(Y_j, E)$, for $1 \leq i \leq n$ and $1 \leq j \leq n-1$ respectively. Then we may use the sum

$$\dim_k(\mathrm{End}^l(E)) = \sum_{i=1}^{n} \dim_k(\mathrm{Ext}^l(X_i, E)) + \sum_{j=1}^{n-1} \dim_k(\mathrm{Ext}^l(Y_j, E)). \tag{4.1}$$

We now need only to compute $\dim_k(\mathrm{Ext}^l(X_i, E))$ and $\dim_k(\mathrm{Ext}^l(Y_j, E))$ in each degree $l \in \mathbb{Z}$.

To compute $\dim_k(\mathrm{Ext}^l(Y_j, E))$, we may use [58, Prop. 3.14], which tells us that for some object $M \in \overline{\mathcal{C}}_n$ that corresponds to the arc $\{a_1, z\}$ for some $z \in \mathscr{M}$, then $\mathrm{Hom}(Y_j, M[1]) \cong k$ if and only if $a_{j+1} \leq z < a_1$, else $\mathrm{Hom}(Y_j, M[1])$ is trivial. Hence we have $\mathrm{Hom}(Y_j, X_i[l]) \cong \mathrm{Ext}^l(Y_j, X_i) \cong k$ for all $l \in \mathbb{Z}$ if and only if $i > j$, and $\mathrm{Hom}(Y_j, X_i[l])$ is trivial otherwise. Similarly, we have $\mathrm{Hom}(Y_j, Y_{j'}[l]) \cong \mathrm{Ext}^l(Y_j, Y_{j'}) \cong k$ if and only if $j' \geq j$. This means we have

$$\begin{aligned} \dim_k(\mathrm{Ext}^l(Y_j, E)) &= \sum_{i=1}^{n} \dim_k(\mathrm{Ext}^l(Y_j, X_i)) + \sum_{j'=1}^{n-1} \dim_k(\mathrm{Ext}(Y_j, Y_{j'})) \\ &= (n-j) + (n-j) \\ &= 2n - 2j \end{aligned}$$

For $\dim_k(\mathrm{Ext}^l(X_i, E))$, we may also use [58, Prop. 3.14], which tells us that $\mathrm{Ext}^l(X_i, Y_j) \cong k$ when $j \geq i$ and is trivial otherwise. Further, $\mathrm{Ext}^l(X_i, X_{i'}) \cong k$

if and only if $i' > i$, or if $i = i'$ and $l \leq 0$, which means that we must consider $\dim_k(\mathrm{Ext}^l(X_i, E))$ in two different cases, one when $l > 0$ and the other when $l \leq 0$. When we consider $l > 0$, we have

$$\begin{aligned}\dim_k(\mathrm{Ext}^l(X_i, E)) &= \sum_{i'=1}^{n} \dim_k(\mathrm{Ext}^l(X_i, X_{i'})) + \sum_{j=1}^{n-1} \dim_k(\mathrm{Ext}(X_i, Y_j)) \\ &= (n-i) + (n-i) \\ &= 2n - 2i,\end{aligned}$$

and when we consider $l \leq 0$,

$$\begin{aligned}\dim_k(\mathrm{Ext}^l(X_i, E)) &= \sum_{i'=1}^{n} \dim_k(\mathrm{Ext}^l(X_i, X_{i'})) + \sum_{j=1}^{n-1} \dim_k(\mathrm{Ext}(X_i, Y_j)) \\ &= (n-i+1) + (n-i) \\ &= 2n - 2i + 1.\end{aligned}$$

Finally, we apply this to equation (4.1) and get

$$\begin{aligned}\dim_k(\mathrm{End}^l(E)) &= \sum_{i=1}^{n} \dim_k(\mathrm{Ext}^l(X_i, E)) + \sum_{j=1}^{n-1} \dim_k(\mathrm{Ext}^l(Y_j, E)) \\ &= \sum_{i=1}^{n}(2n-2i+1) + \sum_{j=1}^{n-1}(2n-2j) \\ &= 2n^2 - n\end{aligned}$$

when $l \leq 0$, and

$$\begin{aligned}\dim_k(\mathrm{End}^l(E)) &= \sum_{i=1}^{n} \dim_k(\mathrm{Ext}^l(X_i, E)) + \sum_{j=1}^{n-1} \dim_k(\mathrm{Ext}^l(Y_j, E)) \\ &= \sum_{i=1}^{n}(2n-2i) + \sum_{j=1}^{n-1}(2n-2j) \\ &= 2n^2 - 2n\end{aligned}$$

when $l > 0$. $\square$

We relabel the indecomposable direct summands of E by $E_1, \ldots, E_{2n-1}$, where $X_i \cong E_{2i-1}$ and $Y_j \cong E_{2j}$. From Figure 4.5, this corresponds to $\ell_{E_{i+1}}$ being an anti-clockwise rotation of ℓ_{E_i} about their common endpoint.

Let $e^l_{i,j}$ be a non-zero vector in $k \cong \mathrm{Ext}^l(E_i, E_j)$. It is clear from Lemma 2.4.2 that $e^l_{i,j}$ only exists when $j > i$, or $j = i$ and j is odd, for all $l \in \mathbb{Z}$, and when $i = j$ for all $l \leq 0$ when j is even, and does not exist otherwise, given that $\mathrm{Ext}^l(E_i, E_j)$ is otherwise trivial.

Proposition 4.3.7. *The collection of morphisms* $\{e^l_{i,j} \mid e^l_{i,j} \neq 0\}$ *forms a basis of* $\mathrm{End}^*(E)$.

Proof. Let $f \in \mathrm{End}^*(E)$ be any non-zero morphism. Then f decomposes as a sum of non-zero morphisms $f_{i,j}^l \in \mathrm{Ext}^l(E_i, E_j)$, this is a finite sum as f is a morphism between finite coproducts. Further, given that $\mathrm{Ext}^l(E_i, E_j) \cong k$ when $1 \leq i < j \leq 2n-1$, or $j = i$ and j is odd, and $l \in \mathbb{Z}$, and also when $i = j$ and $l \leq 0$ for j even, and is trivial otherwise, then any $f_{i,j}^l$ corresponds to some element of k, and thus we can say that $f_{i,j}^l = \alpha e_{i,j}^l$ for some non-zero $\alpha \in k$. Hence we have f is equal to a finite sum of $\alpha_{i,j,l} \cdot e_{i,j}^l$ for some scalars $\alpha_{i,j,l} \in k$.

To show that $\{e_{i,j}^l\}$ is a basis, we must show that it is linearly independent. To do this, consider the morphism $e_{i,j}^l$ and say that

$$e_{i,j}^l = \alpha_{i_1,j_1,l_1} e_{i_1,j_1}^{l_1} + \ldots + \alpha_{i_p,j_p,l_p} e_{i_p,j_p}^{l_p}.$$

As each morphism $e_{i',j'}^{l'}$ does not have the domain and codomain as $e_{i,j}^l$, then this sum cannot be equal to $e_{i,j}^l$ unless $\alpha_{i_q,j_q,l_q} = 0$ for all $q = 1, \ldots, p$ thus the collection $\{e_{i,j}^l\}$ is a basis of $\mathrm{End}^*(E)$. □

We may view $\mathrm{End}^*(E)$ as a graded k-algebra, where multiplication is defined via composition of the basis morphisms, i.e. we have

$$(\alpha \cdot e_{i,j}^l) \cdot (\beta \cdot e_{i',j'}^{l'}) = \begin{cases} \alpha\beta \cdot e_{i,j'}^{l+l'} & \text{if } j = i', \\ 0 & \text{if } j \neq i', \end{cases}$$

for non-zero morphisms $e_{i,j}^l$ and $e_{i',j'}^{l'}$ and scalars $\alpha, \beta \in k$. This is as a result of [39, Lemma 2.4.2], and [58, Proposition 3.14] (see Lemma 2.4.2) for limit arcs, which state that any morphism between objects corresponding to arcs $\ell_X = \{x_1, x_2\}$ and $\ell_Y = \{y_1, y_2\}$ factors through the object corresponding to $\ell_Z = \{z_1, z_2\}$ such that $x_1 \leq z_1 \leq y_1$ and $x_2 \leq z_2 \leq y_2$.

As a result, the identity morphism $e \in \mathrm{Hom}(E, E)$ is given by

$$e = \sum_{i=1}^{2n-1} e_{i,i}^0.$$

We now construct a graded matrix ring that we later show is isomorphic to the graded endomorphism ring of E, by forming a bijective morphism on the basis elements $e_{i,j}^l$.

Definition 4.3.8. Let R_n^* be an $(2n-1) \times (2n-1)$ matrix ring such that

$$b_{ij} \cong \begin{cases} k[x] & \text{if } i = j \text{ and } i \text{ is odd}, \\ k[x^{\pm 1}] & \text{if } i < j, \text{ or } i = j \text{ and } i \text{ is even}, \\ 0 & \text{if } j < i. \end{cases}$$

We impose a grading on R_n^* such that x is concentrated in degree -1.

Example 4.3.9. Let $n = 3$, then the matrix ring R_3^* would be

$$R_3^* = \begin{pmatrix} k[x] & k[x^{\pm 1}] & k[x^{\pm 1}] & k[x^{\pm 1}] & k[x^{\pm 1}] \\ 0 & k[x^{\pm 1}] & k[x^{\pm 1}] & k[x^{\pm 1}] & k[x^{\pm 1}] \\ 0 & 0 & k[x] & k[x^{\pm 1}] & k[x^{\pm 1}] \\ 0 & 0 & 0 & k[x^{\pm 1}] & k[x^{\pm 1}] \\ 0 & 0 & 0 & 0 & k[x] \end{pmatrix}$$

and the additive group R_3^i, for $i \leq 0$, would be

$$R_3^i = \begin{pmatrix} k \cdot x^{-i} & k \cdot x^{-i} & k \cdot x^{-i} & k \cdot x^{-i} & k \cdot x^{-i} \\ 0 & k \cdot x^{-i} & k \cdot x^{-i} & k \cdot x^{-i} & k \cdot x^{-i} \\ 0 & 0 & k \cdot x^{-i} & k \cdot x^{-i} & k \cdot x^{-i} \\ 0 & 0 & 0 & k \cdot x^{-i} & k \cdot x^{-i} \\ 0 & 0 & 0 & 0 & k \cdot x^{-i} \end{pmatrix}$$

and for $i > 0$ would be

$$\begin{pmatrix} 0 & k \cdot x^{-i} & k \cdot x^{-i} & k \cdot x^{-i} & k \cdot x^{-i} \\ 0 & k \cdot x^{-i} & k \cdot x^{-i} & k \cdot x^{-i} & k \cdot x^{-i} \\ 0 & 0 & 0 & k \cdot x^{-i} & k \cdot x^{-i} \\ 0 & 0 & 0 & k \cdot x^{-i} & k \cdot x^{-i} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We denote by $I_{i,j}^l$ the element of R_n^* concentrated in degree l such that every entry is trivial, except the entry in the i^{th} row and j^{th} column which is $\varepsilon \in k$, the multiplicative identity. It is easy to check that the collection

$$\{I_{i,j}^l \,|\, 1 \leq i \leq j \leq 2n-1, l \in \mathbb{Z}, i \neq j \text{ if } l > 0\} \tag{4.2}$$

is a basis for the graded k-algebra R_n^*, where multiplication is given by

$$(\alpha I_{i,j}^l) \cdot (\beta I_{i',j'}^{l'}) = \begin{cases} \alpha\beta I_{i,j'}^{l+l'} & \text{if } j = i', \\ 0 & \text{if } j \neq i', \end{cases}$$

for some non-zero $\alpha, \beta \in k$. The multiplicative identity is thus given by

$$I = \sum_{i=1}^{2n-1} I_{i,i}^0.$$

Theorem 4.3.10. *Let E be the strong generator of $\overline{\mathcal{C}}_n$, as in Proposition 4.4.7. Then we have an isomorphism of graded k-algebras,*

$$\operatorname{End}^*_{\overline{\mathcal{C}}_n}(E) \cong R_n^*.$$

Proof. To prove this, we define a map φ from $\operatorname{End}^*(E)$ to R_n^*, and show that this is an isomorphism of graded k-algebras. Let φ act in the basis elements via,

$$\begin{aligned} \varphi\colon \operatorname{End}^*(E) &\longrightarrow R_n^* \\ e_{i,j}^l &\longmapsto I_{i,j}^l, \end{aligned}$$

and let φ respect addition and scalar multiplication. This is a well-defined map, as there is a bijective correspondence between the basis elements $e_{i,j}^l \in \operatorname{End}^*(E)$ and the basis elements $I_{i,j}^l \in R_n^*$ given by (4.2) and Proposition 4.3.7. We next show that φ is an homomorphism of graded k-algebras.

It is clear that the identity $e \in \operatorname{End}^*(E)$ maps through φ to the identity $I \in R_n^*$. Next we show that φ respects multiplication, let $e_{i,j}^l$ and $e_{i',j'}^{l'}$ be two basis morphisms in $\operatorname{End}^*(E)$, and suppose further that $i' = j$, then we have

$$\begin{aligned} \varphi(e_{i,j}^l)\varphi(e_{i',j'}^{l'}) &= I_{i,j}^l \cdot I_{i',j'}^{l'} \\ &= I_{i,j'}^{l+l'} \\ &= \varphi(e_{i,j'}^{l+l'}) \\ &= \varphi(e_{i,j}^l \cdot e_{i',j'}^{l'}). \end{aligned}$$

Now suppose that $i' \neq j$, then we get

$$\begin{aligned} \varphi(e_{i,j}^l)\varphi(e_{i',j'}^{l'}) &= I_{i,j}^l \cdot I_{i',j'}^{l'} \\ &= 0 \\ &= \varphi(0) \\ &= \varphi(e_{i,j}^l \cdot e_{i',j'}^{l'}), \end{aligned}$$

and so we see that φ respects multiplication. Finally, we must show that φ respects the grading of $\operatorname{End}^*(E)$ and R_n^*, but this comes naturally from multiplication via

$$\varphi(e_{i,j}^l)\varphi(e_{i',j'}^{l'}) \in R_n^l R_n^{l'} \subseteq R_n^{l+l'}.$$

Hence φ is a homomorphism of graded k-algebras.

Next, we see that φ is an isomorphism because it is a bijection on the basis of $\operatorname{End}^*(E)$ and R_n^*, and there exists an homomorphism of k-algebras, ρ, such that

$$\begin{aligned} \rho\colon R_n^* &\longrightarrow \operatorname{End}^*(E) \\ I_{i,j}^l &\longmapsto e_{i,j}^l, \end{aligned}$$

which is an inverse of φ. Therefore,

$$\operatorname{End}^*_{\overline{\mathcal{C}}_n}(E) \cong R_n^*.$$

□

§ 4.4 | Generation Time

The notion of a dimension on a triangulated category was introduced by Rouquier in [62] as a tool to help study the representation dimension of a finite dimensional algebra. Given an algebra A, Rouquier provides a series of lower bounds for various dimensions of A, whenever A satisfies a set of given properties. Notably, they provide a lower bound on the representation dimension of a finite dimensional algebra over a field, which is the dimension of the bounded derived category of said algebra. This allows them to provide the first known examples of algebras with representation dimension > 3, a long standing question at the time.

Throughout the rest of this thesis, we shall use Rouquier dimension to refer to the dimension of a triangulated category, with the latter term being preferred in [62].

Definition 4.4.1. Let $\mathcal{C}$ be a triangulated category. If $G \in \mathcal{C}$ is a generator, then we define the *generation time of* G to be the minimal integer m such that $\langle G\rangle_{m+1} = \mathcal{C}$. The set of generation times of generators of $\mathcal{C}$ is called the *Orlov spectrum*, denoted $\mathcal{O}(\mathcal{C})$, and the infimum of $\mathcal{O}(\mathcal{C})$ is called the *Rouquier dimension* of $\mathcal{C}$, denoted $\dim \mathcal{C}$.

If there exists no such G, then we say $\dim \mathcal{C}$ is ∞.

The Orlov spectrum, then known as the dimension spectrum, of a triangulated category was first introduced by Orlov in [56]. Orlov looks at the dimension spectra of bounded derived categories of various geometric categories, and asks the question of whether the Orlov spectrum of the bounded derived category of coherent sheaves on a smooth, quasi-projective scheme form an integer interval? It proves to be a difficult task to compute the Orlov spectrum for triangulated categories in general, and when it may not be possible to directly compute the Orlov spectrum, it may be natural to ask whether or not there are upper and lower bounds to the Orlov spectrum. We provide an interval bound for the Orlov spectrum of $\overline{\mathcal{C}}_n$ in Theorem 4.4.11, and compute the Orlov spectrum for $\overline{\mathcal{C}}_1$.

We present some previously known results about Rouquier dimensions that are of interest. Theorem 4.4.2 is a collection of theorems from [62], compiled in a manner as in [24, Theorem 3.2] which also includes results of Rouquier concerning algebraic geometry.

Theorem 4.4.2. *[62]*

- *Let A be an Artinian ring. Then $\dim \mathrm{D}^{\mathrm{b}}(A)$ is bounded above by the Loewy length of A; i.e. the minimal d such that $(\mathrm{rad}(A))^{d+1} = 0$.*
- *Let A be a Noetherian ring. Then $\dim \mathrm{Perf}(A) < \infty$ if and only if the global dimension of A is finite.*

The following two results may be considered more folklore, however both have proofs found in [24] by Elagin and Lunts.

Lemma 4.4.3. *[24] Assume $\mathcal{C}$ is a Krull-Schmidt, triangulated category. Then* $\dim \mathcal{C} = 0$ *if and only if $\mathcal{C}$ contains only finitely many indecomposables up to isomorphism and shifts.*

Proposition 4.4.4. *[24] Let A be a Noetherian ring of global dimension n. Then*

$$\dim \operatorname{Perf}(A) \leq n.$$

Ballard, Favero and Katzarkov study the Orlov spectra of triangulated categories arising from mirror symmetry in [9]. They develop techniques to associate a generator to any given object in the bounded derived category of coherent sheaves on a smooth Calabi-Yau hypersurface, and show that these generators are uniformly bounded in their generation time. More relevantly, they also compute the Orlov spectrum of the bounded derived category of the category of finitely generated modules of the path algebra of a quiver of Dynkin type A_n.

Theorem 4.4.5. *[9] Let Q be a quiver of Dynkin type A_n. Then the Orlov spectrum of* $\mathrm{D^b}(\operatorname{mod} kQ)$ *is equal to the integer interval* $\{0, \ldots, n-1\}$.

Our results begin by looking at a minimal strong generator of $\overline{\mathcal{C}}_n$.

Lemma 4.4.6. *Let G be a minimal strong generator of $\overline{\mathcal{C}}_n$. Then the homological length of G is an upper bound for the generation time of G.*

Proof. Let G have homological length l. We show that any object in $\overline{\mathcal{C}}_n$ has a generation time at most l.

Let $X \in \overline{\mathcal{C}}_n$ such that $\ell_X = \{x_1, x_2\}$. As G is a generator, then $x_1, x_2 \in \mathscr{M}_G$ by Theorem 4.2.11, and so there exists $G_1, G_{d+1} \in \langle G \rangle_1$ with corresponding arcs $\ell_{G_1} = \{x_1, z_1\}$ and $\ell_{G_{d+1}} = \{y_{d+1}, x_2\}$. Moreover, G is homologically connected, so there exists a minimal zig-zag

$$G_1 \text{ ------ } G_2 \text{ ------ } \cdots \text{ ------ } G_d \text{ ------ } G_{d+1}$$

with length d, and all $G_i \in \langle G \rangle_1$. Let $\ell_{G_i} = \{y_i, z_i\}$ for all $i = 2, \ldots, d$. We claim that $X \in \langle G \rangle_{d+1}$.

By Lemma 4.2.3, there exists a series of zig-zags of the form

$$M_i \text{ ------ } G_{i+1} \text{ ------ } \cdots \text{ ------ } G_d \text{ ------ } G_{d+1}$$

with $M_i \in \langle G \rangle_i$., and $\ell_{M_i} = \{x_1, z_i\}$. Importantly, there exists a zig-zag

$$M_d \text{ ------ } G_{d+1}$$

with $M_d \in \langle G \rangle_d$.

The above zig-zag means that there is a morphism of degree 1 between M_d and G_{d+1}, and so at least one of the following triangles exists

$$M_d \longrightarrow A \longrightarrow G_{d+1} \longrightarrow M_d[1],$$

$$G_{d+1} \longrightarrow B \longrightarrow M_d \longrightarrow G_{d+1}[1].$$

The object X must be a direct summand of either A or B, as $\ell_{M_d} = \{x_1, z_d\}$, $\ell_{G_{d+1}} = \{y_{d+1}, x_2\}$ and $\ell_X = \{x_1, x_2\}$. Therefore $X \in \langle G \rangle_{d+1}$, and so any object of $\overline{\mathcal{C}}_n$ is generated in at most l steps, therefore G has generation time at most l. □

Recall the object $E = \bigoplus_{i=1}^{n} X_i \oplus \bigoplus_{j=1}^{n-1} \in \overline{\mathcal{C}}_n$ from Figure 4.5, where X_i corresponds to the arc $\ell_{X_i} = \{a_1, z_i\}$ and Y_j corresponds to the arc $\ell_{Y_j} = \{a_1, a_{j+1}\}$.

Proposition 4.4.7. *The object $E \in \overline{\mathcal{C}}_n$ has a generation time 1.*

Proof. The object E is a generator by Lemma 4.3.2.

By Lemma 4.4.6 we know that the generation time of G is at most to the homological length of E. Given any two indecomposable objects in $\langle E \rangle_1$, we know by [58] that there exists a non-zero morphism space between them in one direction, hence the minimal zig-zag between any two indecomposable objects in $\langle E \rangle_1$ has length 1, and so the homological length of E is 1. Moreover, the arc $\ell_A = \{a_2, a_3\}$ corresponds to an indecomposable object $A \in \overline{\mathcal{C}}_n$ such that $A \notin \langle G \rangle_1$, and so the generation time of E is at least 1. Therefore E has a generation time 1. □

The following is a consequence of Proposition 4.4.7 and Lemma 4.4.3.

Corollary 4.4.8. *The Rouquier dimension of $\overline{\mathcal{C}}_n$ is 1.*

Consequently, we can compute the Orlov spectrum of $\overline{\mathcal{C}}_1$.

Corollary 4.4.9. *The Orlov spectrum of $\overline{\mathcal{C}}_1$ is*

$$\mathcal{O}(\overline{\mathcal{C}}_1) = \{1\}.$$

Proof. We show that there is no minimal generator G with homological length $l \geq 2$, and so the generation time of G has an upper bound of 1. Let

$$G_1 \longrightarrow G_2 \dashrightarrow G_l \longrightarrow G_{l+1}$$

be a zig-zag with objects in $\langle G \rangle_1$. Corollary 4.2.12 tells us that all ℓ_{G_i} must be limit arcs as G is minimal. However, Lemma 2.4.2 states that there is a non-trivial Ext^1-space between indecomposable objects corresponding to limit arcs that share an endpoint at an accumulation point, which all ℓ_{G_i} do as there is only a single accumulation point for $\overline{\mathcal{C}}_1$. Hence there is a minimal zig-zag

$$G_1 \longrightarrow G_{l+1},$$

and so the homological length of any minimal generator G is 1, therefore the upper bound on the generation time of G is also 1. $\square$

We now look at the homological length of minimal generators in $\overline{\mathcal{C}}_n$ in an effort to find a bound for the Orlov spectrum of $\overline{\mathcal{C}}_n$. To do this, we show that for any integer l up to a given value, there exists a minimal generator with homological length equal to l.

Proposition 4.4.10. *Let $1 \leq d \leq 2n-2$ be an integer. Then there exists a generator M of $\overline{\mathcal{C}}_n$ such that M has homological length d.*

Proof. We construct a generator M_d for each $1 \leq d \leq 2n-2$ using induction on the generator E.

We know that E has homological length 1 by Corollary 4.4.7, so we construct a new object from E, called M_2 by replacing Y_1 with the object Z_1, corresponding to the arc $\ell_{Z_1} = \{z_1, a_2\}$. It is clear that M_2 is homologically connected and has a complete orbit in $\mathscr{M}$, and so by Theorem 4.2.11 we know M_2 is a generator of $\overline{\mathcal{C}}_n$. We construct M_3 from M_2 by replacing X_2 with the object Z_2, corresponding to the arc $\ell_{Z_2} = \{a_2, z_2\}$, and again we see that M_3 is a generator by Theorem 4.2.11. We repeat this construction for all M_d, $1 \leq d \leq 2n-2$.

Assume that M_d has homological length d, we show that M_{d+1} has homological length $d+1$. By Lemma 4.4.6 we know that there exists at least one minimal zig-zag between indecomposable summands of M_d of length d. We have a minimal zig-zag of the form

$$H \xrightarrow{f_1} X_1 \text{-------} Z_1 \xrightarrow{f_{d-1}} Z_{d-1} \xrightarrow{f_d} Z_d,$$

where $H \ncong X_1$ is in both $\langle E \rangle_1$ and $\langle M_d \rangle_1$. However, when we replace an indecomposable summand, say N, in M_d with Z_{d+1}, we only have length 1 minimal zig-zags between Z_{d+1} and Z_d and all other minimal zig-zags between Z_{d+1} and another indecomposable summand of M_{d+1} contain Z_d up to suspension. Hence we get a minimal zig-zag

$$H \xrightarrow{f_1} X_1 \text{-------} Z_1 \xrightarrow{f_d} Z_d \xrightarrow{f_{d+1}} Z_{d+1},$$

which has length $d+1$.

To see that we get no other new minimal zig-zags of length $l > d+1$ between indecomposable summands of M_{d+1}, consider that any minimal zig-zag containing N in the middle will not be minimal as any sequence in a zig-zag of the form

$$X_1[j] \text{ —— } N \text{ —— } H$$

can be reduced to a sequence of the form

$$X_1[j] \text{ —— } H$$

by [58]. Hence a minimal zig-zag containing N may only contain N either at the start or end of the zig-zag, and so by removing N as a summand of M_{d+1} means that no minimal zig-zag of this form has an increased length.

Therefore we see that the homological length of M_{d+1} is $d+1$. Thus, by induction, we see that there exists some object with homological length d for all $d = 1, \ldots, 2n-2$. □

Here we see an example of one of the minimal strong generators constructed in Proposition 4.4.10.

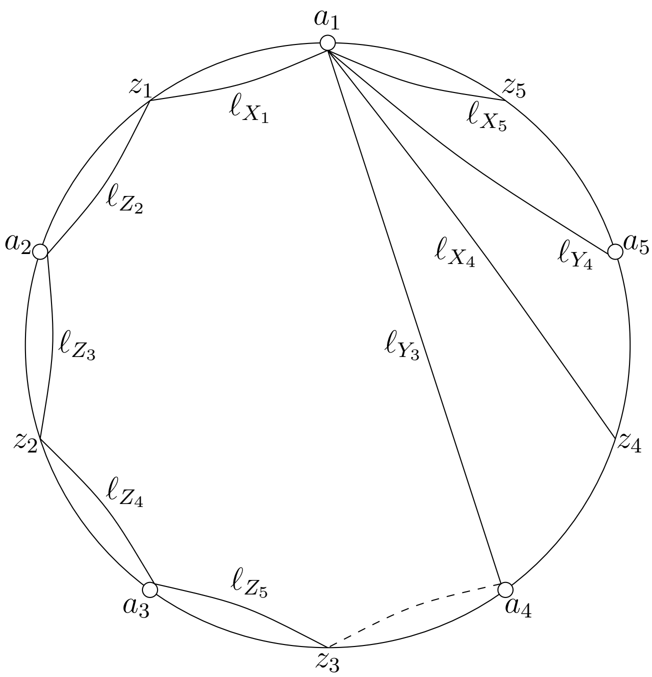


Figure 4.6: The minimal strong generator M_5 in $\overline{\mathcal{C}}_5$.

Finally, we show that there exists no minimal strong generator of $\overline{\mathcal{C}}_n$ with a homological length greater than $2n-2$, and so compute an upper bound for the Orlov spectrum of $\overline{\mathcal{C}}_n$.

Theorem 4.4.11. *The Orlov spectrum of $\overline{\mathcal{C}}_n$ for $n \geq 2$ is bounded above by $2n-2$. That is*

$$\mathcal{O}(\overline{\mathcal{C}}_n) \subseteq \{1, \ldots, 2n-2\}.$$

Proof. By Proposition 4.4.10 we know that there is a generator with homological length $2n-2$ and so has generation time at most $2n-2$ by Lemma 4.4.6, so we only need to show that there exists no minimal strong generator with homological length greater than $2n-2$. To show this, we consider two situations, one where some generator M has $\geq 2n$ non-isomorphic indecomposable direct summands, and one where M has $< 2n$ non-isomorphic indecomposable direct summands.

Suppose M has $\geq 2n$ non-isomorphic indecomposable direct summands. Then if we consider each segment and accumulation point as a vertex, and each arc as an edge, then may construct each generator as a graph with $2n$ vertices. Basic results from graph theory tell us that if we have $\geq 2n$ edges on $2n$ vertices, then we must have a loop somewhere in the graph, and this loop then corresponds to a zig-zag of the form

$$M_1[a_1] \text{ ——— } M_2[a_2] \text{ -------- } M_d[a_d] \text{ ——— } M_{d+1}[a_{d+1}] \text{ ——— } M_1[a_1],$$

such that $\ell_{M_i[a_i]}$ either crosses $\ell_{M_{i-1}[a_{i-1}]}$ and $\ell_{M_{i+1}[a_{i+1}]}$, or shares an endpoint at an accumulation point, and $\ell_{M_{d+1}[a_{d+1}]}$ crosses $\ell_{M_1[a_1]}$ or shares an endpoint at an accumulation point. Lemma 4.2.13 tells us then that the object M' such that $M \cong M' \oplus M_1$ is also a generator, and so M is not a minimal generator.

Now suppose that M has $2n-1$ non-isomorphic indecomposable direct summands. If M has homological length $2n-1$, then there must exist a minimal zig-zag of length $2n-1$, let this zig-zag be

$$M_1 \text{ ——— } M_2[m_2] \text{ -------- } M_{2n-1}[m_{2n-1}] \text{ ——— } M_i[j].$$

There exists some subsequence of this zig-zag

$$M_1 \text{ ——— } M_2[m_2] \text{ -------- } M_{i-1}[m_{i-1}] \text{ ——— } M_i[m_i],$$

of length $i-1 < 2n-1$. Suppose that there exists no morphisms of degree 1 in either direction between $M_i[m_i]$ and $M_i[j]$, then ℓ_{M_i} is a short arc, and so M is not a minimal generator by Corollary 4.2.12.

Now suppose that there does exist a morphism of degree 1 between $M_i[m_i]$ and $M_i[j]$, then we have a zig-zag

$$M_1 \text{ ——— } M_2[m_2] \text{ -------- } M_{i-1}[m_{i-1}] \text{ ——— } M_i[m_i] \text{ ——— } M_i[j],$$

of length $i \leq 2n-1$, where the length is equal when $M_i \cong M_{2n-1}$. As we only need to consider zig-zags of length $\geq 2n-1$ we may assume that $M_i \cong M_{2n-1}$, and we need to show that the zig-zag

$$M_1 \text{ ——— } M_2[m_2] \text{ -------- } M_{2n-2}[m_{2n-2}] \text{ ——— } M_{2n-1}[m_{2n-1}] \text{ ——— } M_{2n-1}[j], \tag{4.3}$$

is not minimal.

Suppose there are two arcs ℓ_{M_i} and $\ell_{M_{i'}}$ that share $a \in L(\mathscr{M})$ as an endpoint. If $i' \neq i \pm 1$, then (4.3) is not a minimal zig-zag. Therefore, suppose that $i' = i+1$, then there exists a zig-zag

$$M_i[m_i] \text{ ——— } M_{i+1}[m'_{i+1}]$$

such that we have the zig-zag

$$M_1 \text{ ——— } M_2[m_2] \text{ -------- } M_{2n-2}[m'_{2n-2}] \text{ ——— } M_{2n-1}[m'_{2n-1}] \cong M_{2n-1}[j],$$

which has length $2n-2$, and so the minimal zig-zag between M_1 and $M_{2n-1}[j]$ is at most length $2n-2$.

Now suppose that every $a_l \in L(\mathscr{M})$ is the endpoint of exactly one arc ℓ_{M_l}, for M_l indecomposable direct summands of M. Then between the n segments of $\mathscr{M}$, there are at least n long arcs corresponding to direct summands of M. However, again via seeing the segments as vertices and the long arcs as edges, there must be a cycle in the induced graph, and so M is not minimal as we could remove any of the arcs in the cycle, and the resulting object would still have a complete orbit and be homologically connected, and so M is not a minimal generator. Hence there are no minimal strong generators of $\overline{\mathcal{C}}_n$ with a homological length greater than $2n-2$. Thus $2n-2$ is an upper bound of the Orlov spectrum of $\overline{\mathcal{C}}_n$. $\square$

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