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Knot theory and wild knots

Cherie Annette Reardon

KNOT THEORY

AND WILD KNOTS

A Project

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

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ABSTRACT

The primary mathematical question concerning knots remains their classification, in other words, the problem of the comparison of two arbitrary knots. From a physical point of view, consider the knots to be constructed from deformable rubber. Then the question becomes: **Suppose we are given two knots K_1 and K_2. Is it possible to manipulate K_1 by stretching and twisting, without tearing, to transform K_1 into K_2?** However, this process is uncertain and potentially tedious. A mathematical approach is required.

It has been established that if two knots are topologically equivalent, fundamental groups associated with each knot are isomorphic although the converse is not necessarily true. That is, two knots with the same group may be topologically inequivalent. The fundamental group of a knot, denoted Π_1, is considered a knot invariant.

Definition: *A knot invariant on the set of all knots $\mathcal{K}$ is a function which assigns to each knot K*

$$I: \mathcal{K} \rightarrow \theta$$

an object I(K) in θ in such a manner that knots which are of the same type are assigned to equivalent objects.[1]

Initial approaches to solutions of problems in knot theory stemmed from a combinatorial point of view. In

addition, graph theory played its part in the development of knot theory as have noncommutative algebra and algebraic topology.

The goal of this project is to study the relationship between algebraic invariants and topology which are used to determine whether two knots are distinct. Special attention will be given to the topological invariants of knots coming from algebraic topology.

ACKNOWLEDGMENTS

Special thanks to Dr. Rolland Trapp for introducing me to the world of knot theory, and for his expertise, time and endless patience in helping me to unravel knotty questions. I am also grateful to Dr. Davida Fischman for her wise counsel and valuable suggestions concerning this project. In addition, Dr. Dan Rinne's support and goodwill always kept me focused on the important issues.

Dr. Terry Hallett will never know how much I have valued her help, advice and encouragement these past twelve years. Her quietly positive attitude and never failing good humor have always cheered and inspired me.

DEDICATION

To Rahima, Patrick and Diana

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INTRODUCTION

Flying a kite, sailing a boat, tying down an airplane, a viral attack on a DNA molecule, connecting two lengths of rope together and rappelling down a rock face all make extensive use of different types of knots and links. In order to study knots from a mathematical perspective take a length of twine, tie a knot in it, then tie the loose ends of the twine together. The result is a knot with no loose ends. This physical object provides the analogue for the knots and knot theory studied by mathematicians for more than a century. A diverse range of scientists, from the organic chemist to the mathematical physicist, consider the knotty problems of knot theory of significant relevance to their own work.

CHAPTER ONE

Section 1

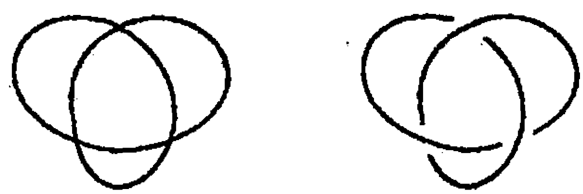
Knots, Links and Classes of Knots

The initial objective in this section is to introduce knots, links, equivalence of knots and special classes of knots. In the effort to distinguish one knot from another and identify equivalent knots, some applicable topological topics concerning mappings between spaces are first reviewed.

Given two topological spaces, X and Y, a *homeomorphism* from X to Y is a mapping $f: X \to Y$ that is bicontinuous and bijective.

Definition 1.1.1: *A knot K is the homeomorphic image of the unit circle $S^1 \subset \mathbb{R}^2$ into $\mathbb{R}^3$. A link L of n components is the homeomorphic image of n disjoint copies of S^1. Hence, a link of one component is a knot.*

A knot K is a one dimensional object, a cross section of which consists of a single point. However, a link or a knot exists in three dimensional space, with the exceptions of the unknot and unlink respectively represented by the union of one or more disjoint circles lying in the plane. Two dimensional pictures called link diagrams are used to represent and study links. To obtain a link diagram the projection map is utilized. The projection map is a function $\mathscr{P}$ which takes the ordered triple (x,y,z) in $\mathbb{R}^3$ to the ordered pair (x,y) in $\mathbb{R}^2$. Consider the knot K and the image of K, denoted K_I, shown under the projection mapping $\mathscr{P}$ in Figure 1.1. The projection of the trefoil knot on the left in Figure 1.1 fails to display important detail. It is not possible to distinguish which portions of the knot pass over other sections. This difficulty is resolved in the link diagram on the right in Figure 1.1 by leaving suitable gaps at the intersections in the projection.



a) projection b) link diagram

Figure 1.1 Trefoil knot

The point at which a knot crosses over and under itself in a link diagram is called a crossing. The curves or segments of a link diagram which are continuous are called arcs. The unbroken arc at a crossing is called an *overpass* and the two broken arcs which pass under the crossing together make the *underpass*.

Definition 1.1.2: *An <u>overpass</u> is a subarc of a knot that* *goes over at least one crossing but never goes under a crossing.*[2]

Definition 1.1.3: *An underpass is a subarc of a knot that goes under at least one crossing but never goes over a crossing.*

Sometimes an orientation is chosen to indicate direction traveled on each component of a link. Figure 1.2 depicts a four crossing link diagram of two components with these details illustrated.

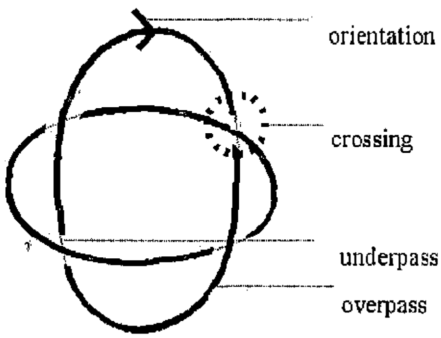


Figure 1.2 A link diagram

Usually homeomorphic space are considered equivalent in topology. Since all knots homeomorphic to the unit circle, they are all homeomorphic to one another. Therefore homeomorphism type is an uninteresting equivalence relation on knots and a rather unnatural one.

The following topological criterion is a more appropriate equivalence relation on knots.

Definition 1.1.4: *Given a topological space X, an* *isotopic deformation* *is a family of homeomorphisms h_t: X → X satisfying two properties.*

1. For all points p in X, $h_0(p)=(p)$, and,

2. the function defined by $H(t,p) = h_t(p)$ is simultaneously continuous in both variables[3] as pictured in the example of Figure 1.3.

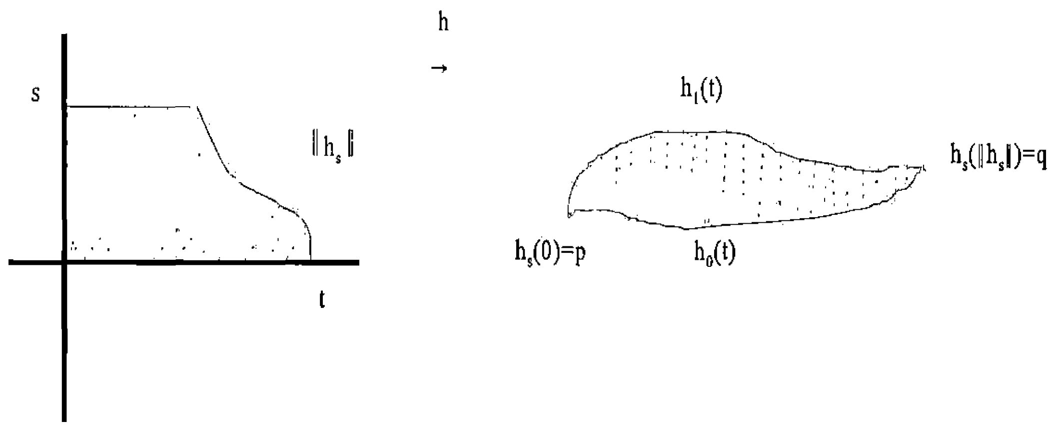


Figure 1.3 A fixed endpoint family

If two knots or links belong to the same isotopy type they are equivalent. If two links are equivalent, they are said to belong to the same link type.

Definition 1.1.5: *For two unoriented knots, K_1 and K_2, to be considered of the same* *isotopy type*, *there must exist an isotopic deformation $\{h_t\}$ of R^3 such that $h_1(K_1) = K_2$.*

Given a fixed point $p \subset K_1 \subset R^3$, $h_t(p)$ traces the path from the original position at K_1 through the deformation to K_2. The unknot, trefoil, figure eight and Hopf link pictured in Figure 1.4 all belong to distinct knot or link types.

a) unknot b) trefoil c) figure eight d) Hopf link

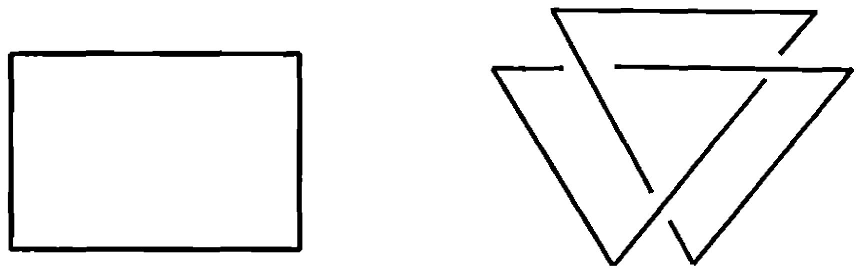
Figure 1.4 Distinct knot types

Knots which are of the same isotopy class are represented in Figure 1.5 by two diagrams of the unknot.

Figure 1.5 Two link diagrams of the unknot

<u>**Definition 1.1.6:**</u> *A <u>polygonal, or piecewise, linear knot</u> consists of the union of a finite number of closed segments called edges, the end points of which are the vertices of the knot.*

A knot which is equivalent to a polygonal knot is considered *tame*. The unknot and trefoil knots are depicted as polygonal knots in Figure 1.6. Knots and links are primarily pictured as in Figure 1.5, with curves rather than straight segments.



a) unknot b) trefoil

Figure 1.6 Polygonal knots

A knot which is inequivalent to a polygonal knot is considered *wild*. Tame knots are most often studied but a discussion of a proof by Ralph H. Fox that wild knots do exist[4] appears in Chapter 3 of this paper. See Figure 1.7. The remarkable property of this wild knot is that the number of loops outside a ball of radius ϵ, centered at the wild point p increases without bound as the radius $\epsilon \to 0$.

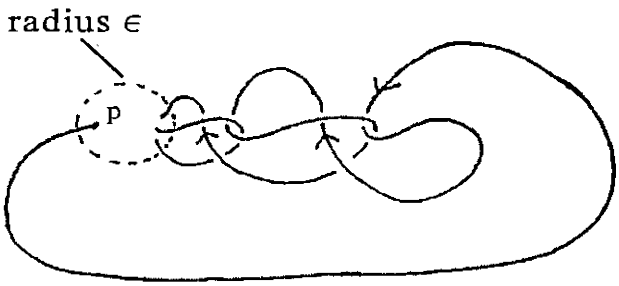


Figure 1.7 A wild knot

Theorem 1.1.7: *Every homeomorphism of $\mathbb{R}^3$ onto itself is either orientation preserving or orientation reversing.*[5]

The mirror image of a knot K is the image of the knot under the reflection $\mathscr{R}$ defined by $(x,y,z) \to (x,y,-z)$. Figure 1.8 shows a trefoil and its mirror image under the reflection $\mathscr{R}$.

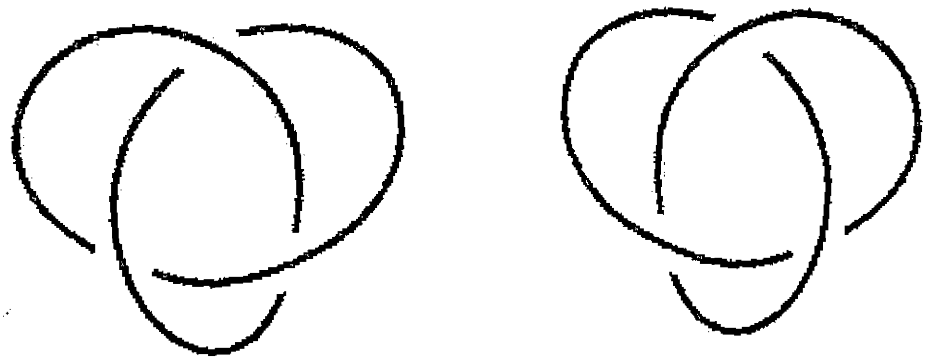


Figure 1.8 Trefoil and its mirror image

An amphicheiral knot K is one which is equivalent to its mirror image K_m. Thus K and K_m are isotopic. One example of an amphicheiral knot is the figure eight knot. The left hand trefoil and the right hand trefoil are examples of non-amphicheiral knots.[6]

Another interesting class of knots are invertible knots. An oriented knot is called invertible if it is possible to deform the knot back to itself with its original orientation sent to an opposite orientation. The right and left handed trefoil are invertible; simply rotate them about a vertical axis as indicated in Figure 1.9.[7]

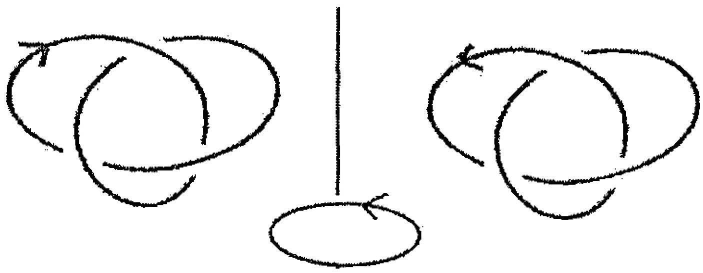


Figure 1.9 The trefoil rotated about a vertical axis

Figure 1.10 A non-invertible knot

The knot 8_{17} in Figure 1.10 is not invertible, however, this is difficult to prove and outside the scope of this paper.[8]

Section Two

The Fundamental Group

Knot theory concerns itself with the group of the complementary space, R^3-K, of a knot K. Algebraic topology is the field which uses algebraic structures such as groups and group homomorphisms to study topological spaces. The study of the fundamental group, denoted π_1, of an arbitrary topological space can be applied to knot theory to assist in distinguishing between knots.

The fundamental group provides a method of associating groups with topological spaces and homomorphisms between associated groups with continuous functions which map one space to another. For example, if two topological spaces are homeomorphic then their fundamental groups are isomorphic. Consequently, this algebraic approach provides some very useful information. However, the algebra reflects only a portion of the complete topological picture. This tool is limited in that if the fundamental groups are isomorphic this is not sufficient information to conclude that the associated topological spaces are equivalent.

Before we can define π_1 we must examine the set of paths in a topological space X. Once we have examined the set of paths in X we will then focus on loops in X which begin and end at the same point. Consider a particle moving through space. The particle travels over a particular path during a given period of time. Assume motion begins at time $t = 0$ and continues until a designated stopping time, $t \geq 0$. The *path a* in the topological space X is a continuous mapping $a\colon [0,\|a\|] \rightarrow X$. A *path b* is similarly a continuous mapping $b\colon [0,\|b\|] \rightarrow X$. The stopping time of *path a* is $\|a\| \geq 0$, $a(0)$ is the initial point of *path a*, and the terminal point is $a(\|a\|)$ with similar conditions for *path b*. We denote by *path* a^{-1} the path obtained in which *path a* is traversed in the opposite direction. Since different paths may share the same image points it is important to distinguish between the actual path traversed during the interval $[0,\|a\|]$ and the set of image points. For *path a* and *path b* to be equal they must share the same domain of definition, i.e., $\|a\| = \|b\|$, and for all t in the domain, $a(t) = b(t)$. As an example, let a topological space X consist of $S^1 \subset \mathbb{R}^2$. Using all possible pairs of polar coordinates $\langle 1, \theta \rangle$, let *path a* and *path b* range as follows.

$$a(t) = (1,t), \qquad 0 \leq t \leq 2\pi$$

$$b(t) = (1,t), \qquad 0 \leq t \leq 2\pi$$

Although *path a* and *path b* are equal, a third path, *path c*, $c\colon [0,\|c\|] \rightarrow X$, is not equal to either *path a* or *path b*, given the following domain of definition.

$$c(t) = (1,2t), \qquad 0 \le t \le 2\pi$$

Even though *path c* has the same starting time, stopping time and set of image points as the other two paths, *path c* does not share the same domain of definition[9].

If the terminal point of one path is the initial point of another it is possible to define the product of these two paths. Let $a(\|a\|) = b(0)$. Then $a*b$, the product of *path a* and *path b*, may be defined $(a*b)(t) = a(t)$ for $0 \le t \le \|a\|$ and as $b(t - \|a\|)$ for $\|a\| \le t \le \|a\| + \|b\|$. Since both *path a* and *path b* are defined as continuous paths in X, $a*b$ also describes a continuous path consisting of the concatenation of *path a* with *path b* as illustrated in Figure 1.11.

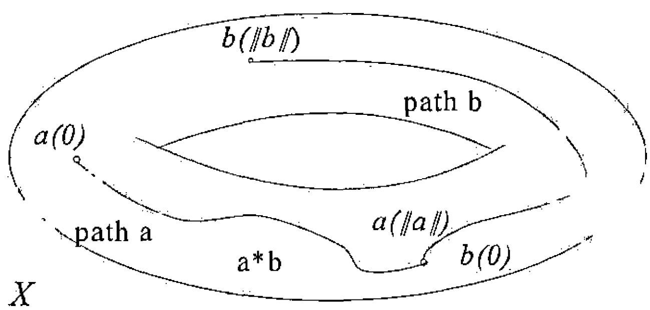


Figure 1.11 Path a*b in a torus

Similarly, we can define the product of three *paths a, b* and *c;* then

1) $a*b$ and $b*c$ are defined,
2) $a*(b*c)$ is defined,
3) $(a*b)*c$ is defined,

and it is not hard to verify that $a*(b*c) = (a*b)*c$ and multiplication of paths is associative when defined. So far we have discussed a set of paths with the binary operation of path multiplication which is associative. In order to obtain a group structure, we must impose the restriction that all paths begin and end at a given point, say p in X. Our new focus becomes the study of *p-based loops*. Certainly, any two *p-based loops* can be multiplied and the product of any two *p-based loops* will also be a *p-based loop*, so this set is closed under * multiplication. In addition, the identity path at *point* p is a multiplicative identity.

In order to complete the construction of inverse elements, and hence a group structure, we must define an equivalence relation on paths. A set whose elements consist of equivalence classes of paths restricted to *p-based* loops will provide us with the necessary structure for the fundamental group. The appropriate equivalence relation is homotopy, which is similar to isotopy.

To define a homotopy, let us examine a collection of paths $h_s \subset X,\ 0 \le s \le 1$. This collection $\{h_s\}$ will be called a continuous family of paths provided the stopping time $\|h_s\|$ is continuously dependent on s, and the function h defined by $h(s,t) = h_s(t)$ maps the closed region $\{s,t \mid 0 \le s \le 1,\ 0 \le t \le \|h_s\|\}$ continuously into X. It is also required that $h(s,t)$ must be simultaneously continuous with respect to both s and t. Further structure is imposed on $\{h_s\}$ by requiring it to be a fixed endpoint family of paths. That is, points p and q exist in X such that $h_s(0) = p$ and $h_s(\|h_s\|) = q$ for all s in the interval $0 \le s \le 1$. Two *paths* a and b in a topological space X are homotopic, denoted *path* $a \sim$ *path* b, if a fixed endpoint family, $\{h_s\}$, of paths exists such that *path* $a = h_o$ and *path* $b = h_1$. As a result, the set of all such fixed endpoint families of paths partition the set of all paths in X into equivalence classes. The equivalence class of paths containing *path* a will be denoted by $[a]$.

Definition 1.2.1: *The fundamental groupoid of X, $\Gamma(X)$, is defined to be the set of all equivalence classes of paths in X.*

A geometric interpretation of equivalence of paths is that *path* a and *path* b are considered equivalent if and only if one can be deformed into the other in X without moving the endpoints.

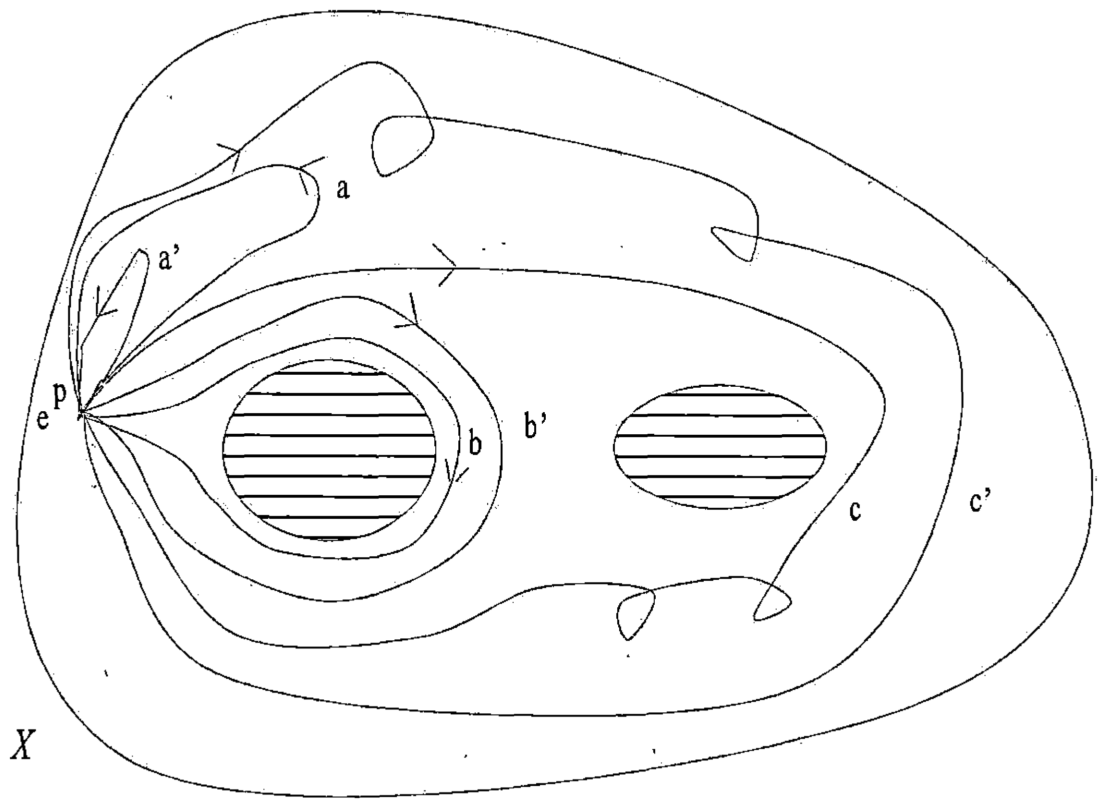
Multiplication of paths induces a multiplication in the fundamental groupoid $\Gamma(X)$. To illustrate this let us consider four paths in X, path a, *path* a_1, *path* b and path b_1 with $a*b$ defined, and with *path* a ~ *path* a_1 and path b ~ *path* b_1. Let *path* a and *path* b be contained in the fixed endpoint family $\{h_s\}$ and *path* a_1 and *path* b_1 be contained in the fixed endpoint family $\{k_s\}$. The collection of paths $\{h_s*k_s\}$ will again be a fixed endpoint family with a_1*b_1 defined and with $a*b \sim a_1*b_1$. Since $a*b$ is defined, the product of the equivalence classes may be defined as $[a]*[b] = [a*b]$. Thus multiplication in $\Gamma(X)$ is well defined. The inverse of an element $[a]$ is defined as $[a]^{-1} = [a^{-1}]$.

Let $\alpha = [a]$, $\beta = [b]$, and $\gamma = [c] \in \Gamma(X)$. Because all paths belonging to an equivalence class will have the same initial and terminal point, we may choose the terminal point of a representative path in α to be the initial point of a representative path in β and the terminal point of β the initial point of a representative path in γ. From this, and the fact that path multiplication is associative, we have that multiplication in $\Gamma(X)$ is associative as well and that $\alpha*(\beta*\gamma) = (\alpha*\beta)*\gamma$.

For an element $\Gamma(X)$ to be an identity it must contain the identity path. Let ϵ be the constant path in $\Gamma(X)$. Let both $\epsilon * \alpha$ and $\beta * \epsilon$ be defined for all $\alpha, \beta \in \Gamma(X)$. Then $[\epsilon]$ is an identity if $\epsilon * \alpha = \alpha$ and $\beta * \epsilon = \beta$. It is clear that $\Gamma(X)$ inherits the semigroup structure previously defined for the set of paths in X. We define the inverse of an arbitrary element α in $\Gamma(X)$ by $[a^{-1}] = (\alpha)^{-1} = \alpha^{-1}$. Hence α^{-1} depends only upon α and remains independent of any particular representative path.

Let p be a point of X and let $\pi_1(X,p)$ be the subset of $\Gamma(X)$ in which all elements have p as both the initial and terminal point. The product of two such *p-based loops* is again a *p-based loop*. By limiting our attention to such closed paths, we now have sufficient structure established to state that $\pi_1(X,p)$, with the binary operation of path multiplication, is a group.

Theorem 1.2.2: *The group $\underline{\pi_1(X,p)}$ is called the __fundamental group__ of the topological space X relative to the basepoint p.*



$$a' \sim a,\ b' \sim b,\ c' \sim c$$
$$a \nsim b,\ b \nsim c,\ a \nsim c$$

Figure 1.12 P-based loops

The example in Figure 1.12 displays various p-based loops in a two holed annular region of the plane as well as path equivalences and orientations. It is interesting to note that the various paths reflect basic structural characteristics of X. Given a pathwise connected topological space X, the fundamental groups of X which are defined for different basepoints in X are isomorphic.

Theorem 1.2.3: *Let α be any element of $\Gamma(X)$ with initial point p and terminal point p′. Then the assignment $\beta \rightarrow \alpha^{-1}\beta\alpha$ is an isomorphism of $\pi_1(X,p)$ onto $\pi_1(X,p')$.*[10]

To show the fundamental group is a topological invariant we will demonstrate that homeomorphic spaces have isomorphic fundamental groups. To do so we must consider how continuous maps of topological spaces induce homomorphisms between fundamental groups. In order to obtain induced homomorphisms of fundamental groups given two topological spaces X and Y, let us examine a continuous mapping $f: X \rightarrow Y$. *Path a* in X induces a *path fa* in Y given by the composition $fa(t) = f(a(t))$. The stopping times $\|a\|$ and $\|fa\|$ are the same. The following theorems follow readily from the definitions.

Theorem 1.2.4: *If the product $a*b$ is defined, so is $fa*fb$, and $f(a*b) = fa*fb$.*

Theorem 1.2.5: *If path e is an identity, so is path fe.*

Theorem 1.2.6: $fa^{-1} = (fa)^{-1}$

Theorem 1.2.7: *If path a ~ path b, then path fa ~ path fb.*[11]

The consequence of these theorems is that the function f induces a mapping f^* from the fundamental groupoid $\Gamma(X)$ into the fundamental groupoid $\Gamma(Y)$ given by $f^*([a]) = [fa]$.

The characteristics of the mapping f^* are outlined in the following propositions.

Propositions 1.2.8, (i) - (iv):

(i) *Given identity ϵ, $f^*\epsilon$ is also an identity.*

(ii) *If $\alpha * \beta$ is defined, then $(f^*\alpha) * (f^*\beta)$ is defined.*

(iii) *If $f: X \to X$ is the identity function such that $f(x) = x$, then f^* is the identity function such that $f^*\alpha = \alpha$.*

(iv) *If X under the function f maps to Y, and if Y under the function g maps to Z are both continuous mappings and $gf: X \to Z$ is the composition, then $(gf)^* = g^*f^*$.*

Given any choice of basepoint p in the topological space X it follows that f^* determines a homomorphism $f^*: (\pi(X,p) \to \pi(Y,fp)$.[12]

If X is restricted to a pathwise connected space the properties of f^* remain independent of the choice of basepoint. Given points p, $q \in X$ with p the initial point and q the terminal point of $\alpha \subset \Gamma(X)$, the following diagram commutes with the vertical mappings being isomorphisms.

$$\begin{array}{ccc} \pi(X,p) & \xrightarrow{f^*} & \pi(Y,fp) \\ \downarrow & & \downarrow \\ \beta \to \alpha^{-1}\beta\,\alpha & & \gamma \to (f^*\alpha)^{-1}\,\gamma\,(f^*\alpha) \\ \pi(X,q) & \xrightarrow{f^*} & \pi(Y,fq) \end{array}$$

Theorem 1.2.9: *If $f\colon X \to Y$ is a homeomorphism of X onto Y, the induced homomorphism $f^*\colon \pi(X,p) \to \pi(Y,fp)$ is an isomorphism onto for any basepoint p in X.*[13]

This homomorphism f^* of the fundamental group $\pi(X,p)$ induced by the continuous mapping f is the desired connection between the topological and algebraic properties of the space X. Diagrammatically, the homeomorphisms f and f^{-1} induce homomorphisms f^* and $(f^{-1})^*$ as indicated.

$$X \xrightarrow{f} Y \xrightarrow{f^{-1}} X$$

$$\pi(X,p) \xrightarrow{f^*} \pi(Y,fp) \xrightarrow{(f^{-1})^*} \pi(X,p)$$ [14]

In summary, given pathwise connected, homeomorphic topological spaces X and Y, the fundamental groups associated with spaces X and Y are isomorphic. Specifically given two knots K_1 and K_2, with respective fundamental groups $\pi(\mathbb{R}^3 - K_1)$ and $\pi(\mathbb{R}^3 - K_2)$, if it can be proven that $\pi(\mathbb{R}^3 - K_1)$ is not isomorphic to $\pi(\mathbb{R}^3 - K_2)$, then K_1 and K_2 are not equivalent knots. If K_1 and K_2 are equivalent knots their fundamental groups will be isomorphic. Although this tool has some limitations, it is extremely useful in distinguishing knots.

CHAPTER TWO

Section One

The Free Group

In order to calculate the fundamental group of a particular knot K it is essential to define and describe a method of presenting the group. This is accomplished by determining generators of the group along with specific relationships among those generators which are called defining relations. The free group, denoted $F[\mathcal{A}]$, is an important component of this development and is our starting point. The development of the presentation of a group will be made first in a general sense and then as applied to fundamental groups of knots, both tame and wild.

Consider a set $\mathcal{A}$ with elements of the form $a, b, c...,$ and cardinality α. Any element raised to an integral power is called a syllable, for example, a^n. A *word* is constructed by concatenation of syllables. The *empty word* is denoted by *1*, and any element to the zero power also denotes the identity. A *product* of two words consists of concatenation. It is clear this operation is associative and the *empty word* is a left and a right identity. The familiar rules of powers apply to the exponents.

Given words u, v, w_1, and w_2 such that $u = w_1a^0w_2$, and

$v = w_1 w_2$, we say that v is obtained from u by an *elementary contraction of type I* or that u is obtained from v by an *elementary expansion of type I.* If $u = w_1 a^m a^n w_2$ and $v = w_1 a^{m+n} w_2$, we say that v is obtained from u by an *elementary contraction of type II* or that u is obtained from v by an *elementary expansion of type I.*

Definition 2.1.1: *A word u is considered reduced, denoted u_r, if it is not possible to perform any contractions of either type I or type II.*

Definition 2.1.2: *Words u and v are considered equivalent, denoted $u \sim v$, if through a succession of elementary contractions and elementary expansions one may be rewritten as the other.*

The collection of all words formed from the set $\mathcal{A}$ will consequently be partitioned into equivalence classes. Let us denote by $[u]$ the equivalence class represented by the word u. The set of equivalence classes of words is denoted $F[\mathcal{A}]$. Multiplication in $F[\mathcal{A}]$ is defined as $[u][v] = [uv]$. It is clear that associativity holds and that the equivalence class $[1]$ is both a left and a right hand identity. The inverse of the class $[u]$, denoted $[u]^{-1}$ is obtained by writing the string u in reverse order and changing the sign of each non-zero exponent. As an example, if $u = a^{-4}b^2a^3$, then $u^{-1} = a^{-3}b^{-2}a^4$. Therefore, the set $F[\mathscr{A}]$ is a group.

Each element of $F[\mathscr{A}]$ can be written in a variety of ways as some product of the elements of the set $\mathscr{A}$, for example, $[u] = [a^2b^{-4}c^3] = [a]^2[b]^{-4}[c]^3$. The elements $[a]$, $[b]$, $[c],\ldots,$ constitute a generating set of $F[\mathscr{A}]$, denoted $[\mathscr{A}]$.

__***Definition 2.1.3:***__ *A generating set E of elements of a group G is a* <u>*free basis*</u> *if given any group H, any function ϕ: E $\rightarrow$ H can be extended to a homomorphism of G into H.*

__***Definition 2.1.4:***__ *A* <u>*free group*</u> *is a group that has a free basis.*

The group $F[\mathscr{A}]$ is known as the free group on the free basis $[\mathscr{A}]$. If we have $F[\mathcal{I}]$, this is the trivial group, and a group $F[a]$ generated by one element is infinite cyclic.

__***Theorem 2.1.5:***__ *A group is free if and only if it is isomorphic to $F[\mathscr{A}]$ for some set $[\mathscr{A}]$.*

__***Theorem 2.1.6:***__ *Any group is a homomorphic image of some free group.*[15]

__***Theorem 2.1.7:***__ *Each equivalence class of words contains one and only one reduced word. Furthermore, any sequence of elementary contractions of the word u must lead to the same reduced word u_r.*[16]

Although the detail has been omitted, an algorithm does exist to perform the contractions to proceed from a word u to the reduced word u_r.[17] By comparing such reduced words it is then possible to determine if words are equivalent.

The importance of the free group $F[\mathscr{A}]$ is that in developing a group presentation, elements of $F[\mathscr{A}]$ are used as a framework upon which to build the presentation. Examples are presented in section three of this chapter.

Section Two

Generators and Defining Relations

The development of the presentation of a group using elements of the free group F will be made first in a general sense and then as application to the fundamental groups of specific knots, both tame and wild. First, let us be specific regarding the character of the defining relations and generators.

Let G be a group with a set $\{g_1, g_2, \ldots\} = [g_i]$ of generators of the group. In addition, a set of equations, called *defining relations* exist such that $f_1(g_1, g_2 \ldots) = 1$, $f_2(g_1, g_2 \ldots) = 1$. The free group $F[X]$ is the free group on the free basis $\{x_1, x_2, \ldots\}$ which is in a one to one correspondence with the set $[g_i]$.

Define a homomorphism $\varphi: F \to G$, such that $\varphi(x_i) = g_i$. For each defining relation $f_i(g_1, g_2, \ldots) = 1$, let $r_i = f_i(x_1, x_2 \ldots) \in F$. For example, if an equation is of the form $(g_2)^{-1} g_1 g_2 (g_1)^{-1} = 1$, then $(x_2)^{-1} x_1 x_2 (x_1)^{-1} = r_1$ which is an element of the free group F. Therefore, we have $\varphi(r_1) = f_i(g_1, g_2 \ldots) = 1$. The element r_1 is in the kernel of the homomorphism φ and r_1 is called a *relator*.

Definition 2.2.1: *Let $\{g_0, g_1, \ldots g_j\}$ be elements of an arbitrary group G and let there exist a group homomorphism*

$\lambda: G \to H$. *An element* g_0 *is called a* <u>*consequence*</u> *of other elements* $g_1, \ldots, g_j$, *if every homomorphism* λ *of* G *into* H *which maps the elements* $g_1, \ldots, g_j$, *into* 1 *also maps* g_0 *into* 1.

Theorem 2.2.2: *Let* $\{g_1, g_2, \ldots\}$ *be a set of elements of a group* G *and let* ϕ *be a group homomorphism* $\phi: G \to H$. *Then* ϕ *maps the consequence of* $\{g_1, g_2, \ldots\}$ *onto the consequence of the set* $\{\phi(g_1), \phi(g_2), \ldots\}$ *of elements in* H.[18]

In the homomorphism defined earlier, $\varphi: F \to G$, let us denote by R the consequence of all such relators $\{r_1, r_2, \ldots\}$. The claim that the set of equations of the form $f_i(g_1, g_2, \ldots) = 1$ form a set of defining relations for the group G from which all other relations may be inferred, is equivalent to saying that the set R is the kernel of the homomorphism. Consequently, R is a normal subgroup of F.

In summary, the group G is determined by the free basis $\{x_1, x_2, \ldots\}$ and the set of elements $\{r_1, r_2, \ldots\}$ since G is isomorphic to its factor group $F(x)/R$.

Section Three

Group Presentations

Two components are necessary to obtain a *presentation of a group G.* An object called a group presentation and an isomorphism onto G are both required. The basic concept is that a known group, the free group $F[X]$, is mapped onto G to obtain a representation of the group G. This section details the process.

Let F be a free group with a free basis E. The group presentation, denoted $(X : r)$ consists of a subset X of E and a subset r of a subgroup $F[X]$ generated in F by X. The subsets X and r respectively constitute the sets of *generators* and *relators* of the group presentation.

Definition 2.3.1: *The* <u>*group of a presentation*</u> *is the factor group $F[X]/R$, denoted $X : r$,* (remembering that R is the consequence of $\{r_i\}$ in $F[X]$).

Definition 2.3.2: *A group presentation $(X : r)$, together with an isomorphism σ of the group of a presentation $X : r$ onto the group G is a* <u>*presentation of a group G*</u>.

A presentation of a group G is determined by a homomorphism ϕ, whose kernel is the consequence of $\{r_i\}$, of the free group $F[X]$ onto a group G. The following commutative diagram under the canonical homomorphism γ, the homomorphism

ϕ and the isomorphism σ, illustrates these relationships.

$$\begin{tikzcd} F(X) \arrow[d, "\gamma"'] \arrow[dr, "\phi"] & \\ X : r \arrow[r, "\sigma"] & G \end{tikzcd}$$
[19]

It is possible for a group to have a variety of presentations. For example, let us examine the group presentation

$$G = (X : r) = (x_1, x_2 : x_1x_2x_1(x_2)^{-1}(x_1)^{-1}(x_2)^{-1} = 1).$$

If we let $x_1x_2 = a$ and $x_2x_1x_2 = b$, then it is possible to obtain the group presentation $(a, b : a^3b^{-2} = 1)$. Obtaining this less complex presentation by substitution is known formally as the Tietze Transformations.[20] The calculations follow.

1) $x_1x_2x_1(x_2)^{-1}(x_1)^{-1}(x_2)^{-1} =$

2) $x_1x_2x_1\boldsymbol{[x_2(x_2)^{-1}]}(x_2)^{-1}(x_1)^{-1}(x_2)^{-1} =$

3) $x_1x_2x_1x_2\boldsymbol{[x_1(x_1)^{-1}]}(x_2)^{-1}(x_2)^{-1}(x_1)^{-1}(x_2)^{-1} =$

4) $x_1x_2x_1x_2x_1\boldsymbol{[x_2(x_2)^{-1}]}(x_1)^{-1}(x_2)^{-1}(x_2)^{-1}(x_1)^{-1}(x_2)^{-1} =$

5) $x_1x_2x_1x_2x_1x_2(x_2)^{-1}(x_1)^{-1}(x_2)^{-1}(x_2)^{-1}(x_1)^{-1}(x_2)^{-1} =$

6) $[x_1x_2][x_1x_2][x_1x_2][(x_2)^{-1}(x_1)^{-1}(x_2)^{-1}][(x_2)^{-1}(x_1)^{-1}(x_2)^{-1}] =$

7) $[x_1x_2][x_1x_2][x_1x_2][x_2x_1x_2]^{-1}[x_2x_1x_2]^{-1} =$

8) $[a][a][a][b^{-1}][b^{-1}] =$

9) $a^3b^{-2} = 1.$

Hence, the groups represented are isomorphic.[21]

CHAPTER THREE

Section One

The Wirtinger Presentation

Returning to the original goal of presenting the group of a knot K with respect to a basepoint p, this chapter is dedicated to the process of developing the presentation of a knot group from a knot diagram. Reference to the basepoint p is omitted since $\mathbb{R}^3 - K$ is pathwise connected and changes in basepoint result in groups which are isomorphic. The group of the knot K may be abbreviated as $\pi(\mathbb{R}^3 - K)$, although a basepoint is implied, and still further as $\pi(K)$.

Let K be a polygonal knot in regular position, divided into the two categories of connected, closed segments called overpasses and underpasses previously discussed in chapter one, page 2. The diagram of a knot K with n crossings consists of the union of a finite number of subarcs $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ (subscripts are taken $mod\ n$) in $\mathbb{R}^2$.

The algorithm requires a fixed orientation on the knot diagram achieved by placing a directional arrow which corresponds to the numerical order of the subscripts of the subarcs $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. A basepoint p in $\mathbb{R}^3 - K$ is chosen. Next, imagine traversing the knot according to the chosen orientation and under each $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ fix one arrow which proceeds from right to left. This arrow enables us to trace an oriented *p-based loop* from the chosen basepoint p to the tail of the arrow, from there to the head of the arrow and back to the basepoint. This *p-based loop* is labeled with an element x_i with subscript index matching that of the overpass subarc α_i. Let $F[X]$ be the free group with free generating set $[X] = \{x_1, x_2, \ldots, x_n\}$. The knot group $\pi(\mathbb{R}^3 - K)$ will be a quotient of $F[X]$. Figure 3.1

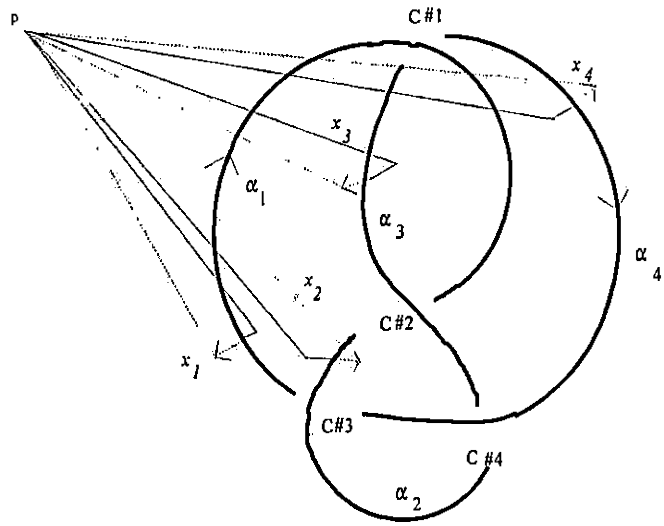
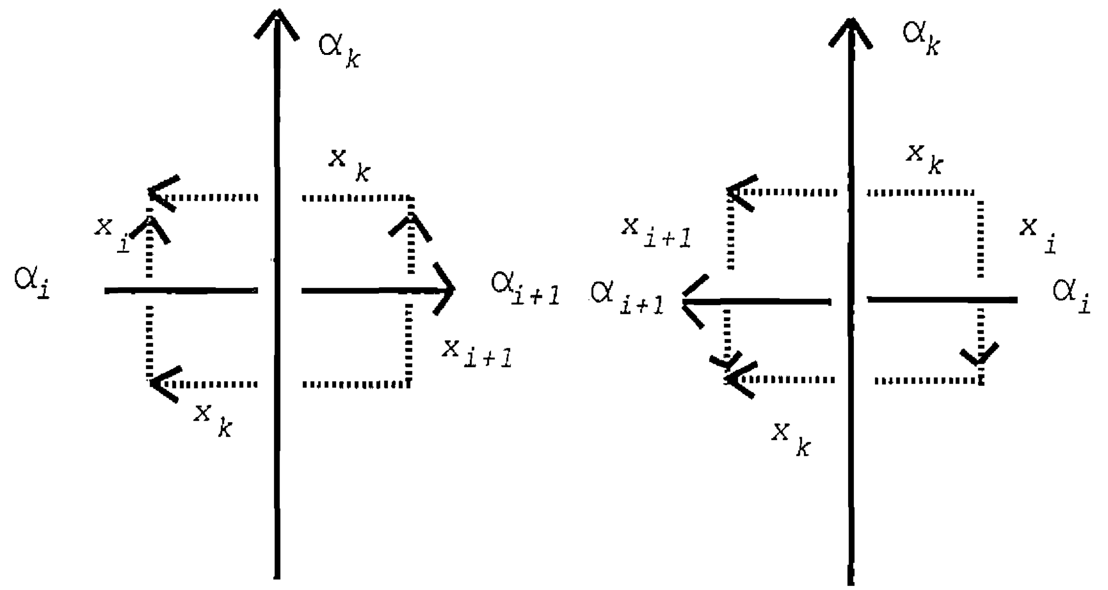


Figure 3.1 Figure eight knot

illustrates the first step of the algorithm. Each crossing must either be a left handed or right handed crossing. Consequently, the relationship among the x_i's must be either that of Figure 3.2 a) or b). One relation is obtained for each crossing by traversing the loop under the crossing as indicated.



a) left hand crossing
r_i: $x_k x_i (x_k)^{-1}(x_{i+1})^{-1} = 1$
$x_k x_i = x_{i+1} x_k$

b) right hand crossing
r_i: $(x_k^{-1}) x_i x_k (x_{i+1})^{-1} = 1$
$x_i x_k = x_k x_{i+1}$

Figure 3.2 Left and right handed crossings

For a knot with *n* crossings, *n defining relations* will be established. The relation corresponding to the *ith* crossing will be denoted as r_i. These are the defining relations required for the Wirtinger Presentation of the knot group. Specific examples for the unknot, trefoil and figure eight knot will be given later.

Theorem 3.1.3: *The fundamental group $\pi(\mathbb{R}^3\text{-}K,\ p)$ of the knot K, with respect to the basepoint p is generated by the homotopy classes of x_i and has presentation*

$$\pi(\mathbb{R}^3 - K,\ p) = (x_1,\ \ldots,\ x_n : r_1,\ \ldots,\ r_n).$$ [22]

In addition, it is always possible to eliminate one defining relation since any one of the r_i's may be rewritten as a consequence of the other n-1 defining relations.

Now we apply the algorithm and Theorem 3.1.2 to the figure eight knot *K* of Figure 3.1 to obtain the Wirtinger Presentation of $\pi(\mathbb{R}^3$ *- figure eight)*, abbreviated as $\pi(F8)$, as indicated in Figure 3.3.

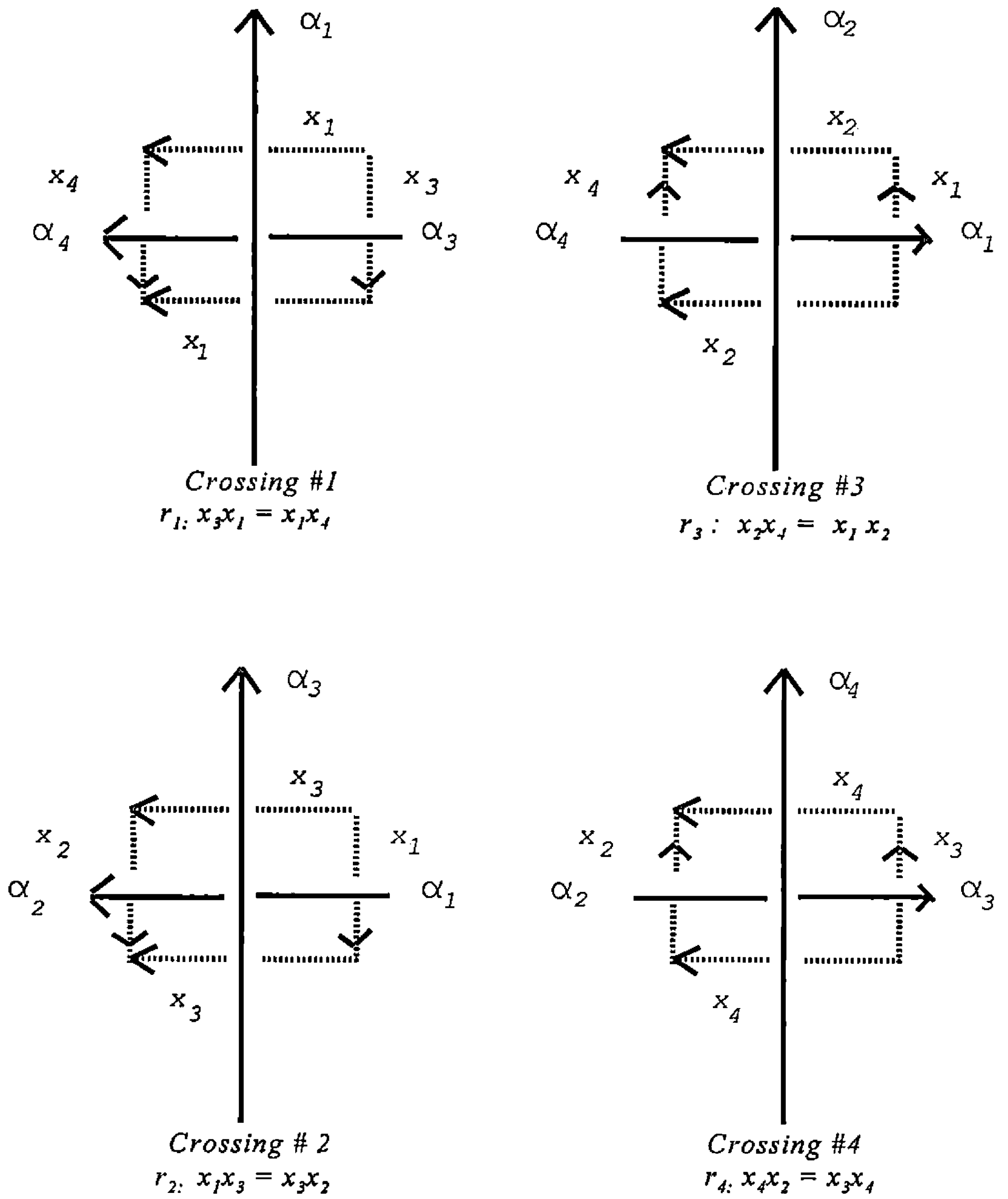


Figure 3.3 Defining relations for figure eight knot

Theorem 3.1.3 results in a presentation of the knot group of the figure eight knot

π(*figure eight knot*) = (x_1, x_2, x_3, x_4:

(1)	$x_3x_1 = x_1x_4,$
(2)	$x_1x_3 = x_3x_2,$
(3)	$x_2x_4 = x_1x_2,$
(4)	$x_4x_2 = x_3x_4).$

Any one of these defining relations may be rewritten in terms of the others. For example, using properties of the group multiplication *(1)*, *(3)* and *(4)* may be rewritten as

(5) $$x_1 = (x_3)^{-1}\, x_1x_4,$$

(6) $$x_1 = x_2x_4(x_2)^{-1}$$

(7) $$x_3 = x_4x_2(x_4)^{-1}$$

Next, substitute *(5)*, *(6)* and *(7)* into defining relation *(2)* and simplify using the group properties.

(8)
$$x_3x_1 = x_1x_4$$
$$x_3x_2x_4(x_2)^{-1} = x_2x_4(x_2)^{-1}x_4$$
$$x_4x_2(x_4)^{-1}x_2x_4(x_2)^{-1} = x_2x_4(x_2)^{-1}x_4$$

The processes of substitution, simplification and rewriting of the group presentation can be rigorously defined in terms of Tietze transformations, but this is beyond the scope of this paper. As a result of these calculations the first and third defining relations have been eliminated and the group presentation becomes the following equivalent presentation.

π(figure eight knot) = *(x_2,x_4:*

(9) $$x_4x_2(x_4)^{-1}x_2x_4(x_2)^{-1} = x_2x_4(x_2)^{-1}x_4$$

Clearly more manipulations and equivalent representations are possible.

Now let us derive the Wirtinger Presentation for the unknot, sometimes called the trivial knot. Consider the unknot which has only one overpass and no crossings resulting in one *p-based loop* x_1. No defining relations exist, since x_1 is the only generator, giving the presentation of

(10) $$\pi(\mathit{unknot}) = (x_1 :).$$

From this we observe that *π(unknot)* is infinite cyclic and hence abelian.

One method for showing that the unknot and another knot, in our next case the trefoil, are distinct knot types is to show that their fundamental groups are not isomorphic. In general, this can be an extremely difficult. However, when one of the knots is the unknot it suffices to show *π(trefoil)* is not abelian and therefore not infinite cyclic. First we find the Wirtinger presentation for *π(trefoil)*.

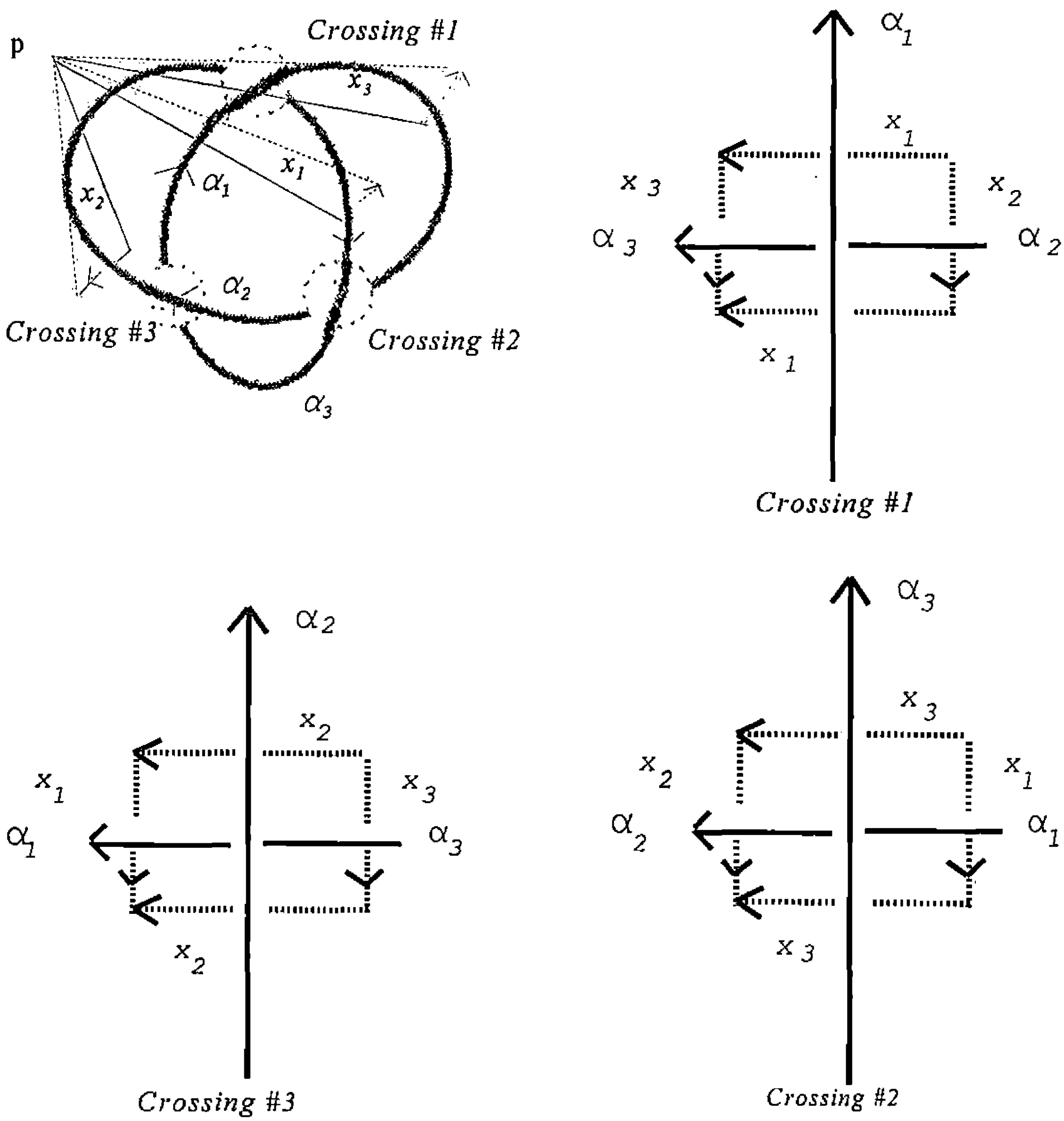


Figure 3.4 Trefoil with p-based loops

By theorem 3.31 and the above figures, the presentation of the group of the trefoil is therefore

$(x_1, x_2, x_3 :$

(1) $x_1(x_3)^{-1}(x_1)^{-1}x_2,$

(2) $x_3(x_2)^{-1}(x_3)^{-1}x_1,$

(3) $x_2(x_1)^{-1}(x_2)^{-1}x_3)$.

One of which can be dropped immediately. Using Tietze Transformations this can be simplified as follows. Solve (2) for x_1, substitute into (1) and (3) and simplify.

(4) $x_1 = x_3x_2(x_3)^{-1}$
(5) $\mathbf{x_3x_2(x_3)^{-1}}(x_3)^{-1}\mathbf{(x_3x_2(x_3)^{-1})^{-1}}x_2$
(6) $x_2x_3x_2 = x_3x_2x_3$

(7) $x_2\mathbf{(x_3x_2(x_3)^{-1})^{-1}}(x_2)^{-1}x_3$
(8) $x_3x_2x_3 = x_2x_3x_2$

From (6) and (8) and using the property of symmetry, the new presentation is

$(x_2, x_3 :$

(9) $x_3x_2x_3 = x_2x_3x_2)$

To demonstrate $\pi(trefoil)$ is not abelian, hence not infinite cyclic, let us homomorphically map this group onto the nonabelian group S_3. The transpositions (12) and (23) generate the group and since

(8) (12)(23) = (132),
(9) (23)(12) = (123),
(10) (132) ≠ (123).

it is clear that S_3 is nonabelian. To define a homomorphism θ of $\pi(trefoil)$ onto the symmetric group S_3 let $\theta(x_2) = (23)$ and $\theta(x_3) = (12)$. To insure that θ extends to a homomorphism we must verify that the relation in $\pi(trefoil)$ is satisfied in S_3. We have $\theta(x_3x_2x_3) = \theta(x_3)\theta(x_2)\theta(x_3) =$

(12)(23)(12) = (13), and $\theta(x_2x_3x_2) = \theta(x_2)\theta(x_3)\theta(x_2) =$ *(23)(12)(23)* = *(13)*. Since π*(trefoil)* can be mapped onto the nonabelian group S_3 the fundamental groups π*(unknot)* and π*(trefoil)* are not isomorphic, and the unknot and the trefoil of different knot types. The unknot is known as being *unknotted* and the trefoil is considered *knotted.*

A similar process could be used to show π*(trefoil)* and π*(figure eight knot)* are not isomorphic, and therefore knot types are distinct, however it would be a much more difficult problem. In the next section a *wild knot* whose group is nonabelian and which is considered *almost unknotted* will be examined.

Section Two

A Wild Knot

In this section we will examine the simple closed curve Λ in Figure 3.5. This curve is considered wild since it is not equivalent to a polygonal knot. It also has the

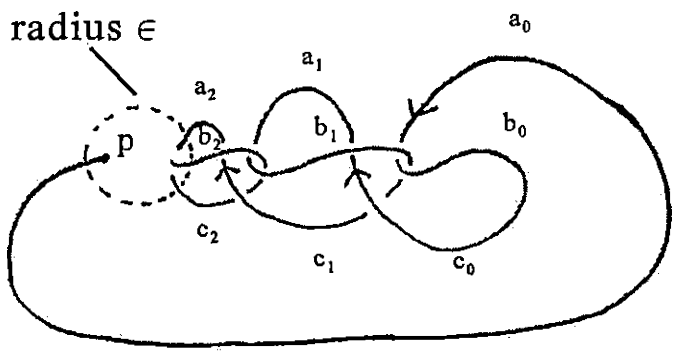


Figure 3.5 A wild knot with subarcs

quality of being *almost unknotted* according to the following definition.

Definition 3.1.1: *A simple closed curve is considered almost unknotted if for any neighborhood U of a given point p there is a neighborhood $V \subset U$ of p and a homeomorphism ϕ such that (i) $\phi(q) = q$ for every $q \in V$, and (ii) $\phi(\Lambda - V)$ is a subset of the plane in $\mathbb{R}$.*[23]

The generators of the curve Λ are a_n, b_n, and c_n with $n \geq 0$. One set of defining relations is

(1_n)	$a_0 =$	$c_n b_n^{-1} a_n$,
(2_n)	$a_{n+1} =$	$b_{n+1} c_n b_{n+1}^{-1}$,
(3_n)	$b_n =$	$b_{n+1}^{-1} a_n b_{n+1}$,
(4_n)	$c_{n+1} =$	$c_n b_n^{-1} b_{n+1} b_n c_n^{-1}$.[24]

To obtain a group presentation, the following calculations are made.

(1_0)	a_0	$=$	$c_0\ b_0^{-1}\ a_0$
	1	$=$	$c_0 b_0^{-1}$
	$\boldsymbol{b_0}$	$=$	$\boldsymbol{c_0}$
(4_0)	c_1	$=$	$c_0 b_0^{-1} b_1 b_0 c_0^{-1}$
	c_1	$=$	$c_0 c_0^{-1} b_1 c_0 c_0^{-1}$
	$\boldsymbol{c_1}$	$=$	$\boldsymbol{b_1}$
(4_1)	c_2	$=$	$c_1 b_1^{-1} b_2 b_1 c_1^{-1}$
	c_2	$=$	$c_1 c_1^{-1} b_2 b_1 b_1^{-1}$
	$\boldsymbol{c_2}$	$=$	$\boldsymbol{b_2}$
(4_2)	c_3	$=$	$c_2 b_2^{-1} b_3 b_2 c_2^{-1}$
	c_3	$=$	$c_2 c_2^{-1} b_3 c_2 c_2^{-1}$
	$\boldsymbol{c_3}$	$=$	$\boldsymbol{b_3}$
(3_0)	b_0	$=$	$b_1^{-1} a_0 b_1$
	$b_1 b_0$	$=$	$a_0 b_1$
	$\boldsymbol{b_1 b_0 b_1^{-1}}$	$=$	$\boldsymbol{a_0}$

From the above, it may be seen that all $b_n = c_n$. This information together with (1_n) $a_0 = c_n b_n^{-1} a_n$, implies that

$a_0 = c_n c_n^{-1} a_n$, (since all $b_n = c_n$) and $a_0 = a_n$. The result is the group of this curve is generated by $b_0, b_1, \ldots, b_i$, $i \geq 0$ with the following defining relations

$$b_1 b_0 b_1^{-1} = b_2 b_1 b_2^{-1} = b_3 b_2 b_3^{-1} = \ldots\ b_{n-1} b_n b_{n-1}^{-1} \ldots$$

Since b_n has the following representation it may be seen that the group of this remarkable simple closed curve is non-abelian. To define a homomorphism θ of $\pi(\Lambda)$ onto the non-abelian symmetric group S_3, let $\theta(b_n) = (12)$ *for n even* and $\theta(b_n) = (23)$ for *n odd*. To insure *that* θ extends to a homomorphism we must verify that the relation in $\pi(\Lambda)$ is satisfied in S_3.

$$\theta(b_1)\theta(b_0)\theta_{(}(b_1^{-1}) \quad = \quad (23)(12)(23)^{-1} =$$

$$\text{(for } n \text{ even)} \quad \theta(b_{n-1})\theta(b_n)\theta(b_{n-1}^{-1}) = (23)(12)(23)^{-1} =$$

$$\text{(for } n \text{ odd)} \quad \theta(b_{n-1})\theta(b_n)\theta(b_{n-1}^{-1}) = (12)(23)(12)^{-1} = (13).$$

Once again the transpositions (12) and (23) generate the group of $\pi(\Lambda)$ as they did for π*(trefoil)*. This illustrates the fact mentioned earlier that although equivalent knots are associated with isomorphic groups, isomorphic images do not guarantee equivalency of knots since the trefoil is *knotted* and the simple closed curved Λ is *almost unknotted.*

In conclusion, I have studied the relationship between algebra and topology to determine whether two knots are distinct. I observed in the case of the trefoil and the example of the wild knot in figure 3.5 that although these knots map to isomorphic fundamental groups, they are not equivalent since the trefoil is *knotted* and the wild knot is *almost unknotted*. However, the Wirtinger Presentation was useful in distinguishing between the unknot and the trefoil since the unknot mapped to a cyclic, abelian group and the trefoil mapped to the non-abelian non-cyclic group S_3.

One final point concerns the structure of the wild knot. If one were to perform an isotopy such that one of the loops were "pulled out", as if this were a chain stitch being dropped, we would end up with an object resembling the unknot. The study of this and other wild knots having a similar "dropped stitch" quality under various isotopic deformations provide further opportunities for study.

ENDNOTES

1. Kauffman, On Knots, edition 1990, p. 47.
2. Adams, The Knot Book, edition 1994, p. 64.
3. Crowell and Fox, An Introduction to Knot Theory, p. 16.
4. Fox, "A Remarkable Simple Closed Curve," pp. 264-265.
5. Crowell and Fox, p. 8.
6. Adams, p. 176.
7. Crowell and Fox, p. 11.
8. Adams, p. 11.
9. Crowell and Fox, p. 14.
10. Ibid., p. 21.
11. Ibid., pp. 22-23.
12. Ibid., p. 23.
13. Ibid., p. 24.
14. Ibid., p. 24.
15. Ibid., p. 33.
16. Ibid., p. 35.
17. Ibid., p. 37.
18. Ibid., p. 39.
19. Ibid., p. 40.
20. Ibid., p. 43.
21. Livingston, p. 103.
22. Rolfsen, p. 57.

23. Fox, "A Remarkable Simple Closed Curve," pp. 264-265.

24. Ibid., pg 265.

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