About Decisiveness of Dynamic Probabilistic Models

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Abstract

Decisiveness of infinite Markov chains with respect to some (finite or infinite) target set of states is a key property that allows to compute the reachability probability of this set up to an arbitrary precision. Most of the existing works assume constant weights for defining the probability of a transition in the considered models. However numerous probabilistic modelings require the (dynamic) weight to also depend on the current state. So we introduce a dynamic probabilistic version of counter machine (pCM). After establishing that decisiveness is undecidable for pCMs even with constant weights, we study the decidability of decisiveness for subclasses of pCM. We show that, without restrictions on dynamic weights, decisiveness is undecidable with a single state and single counter pCM. On the contrary with polynomial weights, decisiveness becomes decidable for single counter pCMs under mild conditions. Then we show that decisiveness of probabilistic Petri nets (pPNs) with polynomial weights is undecidable even when the target set is upward-closed unlike the case of constant weights. Finally we prove that the standard subclass of pPNs with a regular language is decisive with respect to a finite set whatever the kind of weights.

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1 Introduction

Infinite Markov chains. Since the 1980's, finite-state Markov chains have been considered for the modeling and analysis of probabilistic concurrent finite-state programs [27]. More recently this approach has been extended to the verification of the infinite-state Markov chains obtained from probabilistic versions of automata extended with unbounded data (like stacks, channels, counters, clocks). The problem of *Computing the Reachability Probability up to an arbitrary precision* (CRP) is a central problem in quantitative verification and it has been studied by many authors [23, 16, 3, 12].

Computing the probability of reachability. There are (at least) two strategies to solve the CRP problem.

The first one is to consider the Markov chains associated with a particular class of probabilistic models like probabilistic pushdown automata (pPDA) or probabilistic Petri nets (pPN) and some specific target sets and to exploit the properties of these models to design a CRP-algorithm. For instance in [12], the authors exhibit a PSPACE algorithm for pPDA and PTIME algorithms for single-state pPDA and for one-counter automata.

The second one consists in exhibiting a property of Markov chains that yields a generic algorithm for solving the CRP problem and then looking for models that generate Markov chains that fulfill this property. *Decisiveness* of Markov chains is such a property (a Markov chain is decisive w.r.t. a target if almost surely a random path either reaches the target or the target becomes unreachable) and it has been shown that pLCS are decisive and that probabilistic Petri nets (pPN) are decisive when the target set is upward-closed [3].

Two limits of the previous approaches. In most of the works, the probabilistic models associate a constant (also called *static*) weight for transitions and get transition probabilities by normalizing these weights among the enabled transitions in the current state (except for some semantics of pLCS like in [19] where transition probabilities depend on the state due to the possibility of message losses). This forbids to model phenomena like congestion in networks (resp. performance collapsing in distributed systems) when the number of messages (resp. processes) exceeds some threshold leading to an increasing probability of message arrivals (resp. process creations) before message departures (resp. process terminations). In order to handle them, one needs to consider *dynamic* weights i.e., weights depending on the current state.

Dynamic weights. The usual formalism for performance evaluation is the model of continuous time Markov chain (CTMC) (see for instance the book "Continuous-Time Markov Chains: An Applications-Oriented Approach". William J. Anderson). In this model, the transitions are labelled by a rate (of a negative exponential distribution). The underlying discrete time model (which is enough to study some important properties) is a DTMC obtained by normalizing the rates viewed as weights. To emphasize the relevance of dynamic and more specifically polynomial weights, let us recall few examples which are recurrent patterns of CTMCs:

- in queuing networks, the policy of a server may be the infinite server policy leading to linear weights;
- in biological and epidemiological models, the rate of some "synchronization" between two instances of some species is quadratic w.r.t. the size of the species.

Probabilistic models. Generally given some probabilistic model and some kind of target set of states, it may occur that some instances of the model are decisive and some others are not. This raises the issue of the decidability status of the decisiveness problem.

The first definition of pPDA seems to be given by Eugene S. Santos [24] in 1972. The CRP and more generally, the qualitative and quantitative model checking has been shown decidable for pPDA (see surveys of Kucera et al. [16] and Brazdil et al. [12]). As we did in the paper, Brazdil et al. [12] and Lin [21] studied the same subclasses of pPDA (called there stateless pPDA and POC). Interestingly, the decidability of the decisiveness property has only be studied and shown decidable for pPDA with constant weights [16]: static pPDA, and even static one-counter automata, are not decisive w.r.t. regular languages but decisiveness is decidable (w.r.t regular languages).

Probabilistic lossy channel systems (pLCS) have been introduced by Iyer et al. [19] in 1997 where they prove that the CRP and the quantitative model checking against a fragment of LTL is decidable. See also Baier et al. [6], Abdulla et al. [1], Rabinovich [23], Bertrand et al. [10], Abdulla et al. [2] for pLCS with modified semantics of losses. Observe that depending on the selected semantics, the model of pLCS leads either to static or dynamic weights. In particular the decisiveness property of pLCS is ensured by a particular case of dynamic weights.

Probabilistic counter machines (pCM) have been studied in [13]. For example, static Probabilistic Petri nets (pPN) are decisive w.r.t. upward-closed sets but it is unknown whether decisiveness is decidable w.r.t. finite sets or w.r.t. dynamic weights.

Our contributions.

- In order to unify analysis of decisiveness, we introduce a dynamic probabilistic version of counter machine (pCM) and we first establish that decisiveness is undecidable for pCMs even with constant weights.
- Then we study the decidability of decisiveness of one-counter pCMs. We show that, without restrictions on dynamic weights, decisiveness is undecidable for one-counter pCM even with a single state. On the contrary, with polynomial weights, decisiveness becomes decidable for a large subclass of one-counter pCMs, called homogeneous probabilistic counter machine (pHM).
- Then we show that decisiveness of probabilistic Petri nets (pPNs) with polynomial weights is undecidable when the target set is finite or upward-closed (unlike the case of constant weights). Finally we prove that the standard subclass of pPNs with a regular language is decisive with respect to a finite set whatever the kind of weights.
- Some of our results are not only technically involved but contain new ideas. In particular, the proof of undecidability of decisiveness for pPN with polynomial weights with respect to a finite or upward closed set is based on an original weak simulation of CM. Similarly the model of pHM can be viewed as a dynamic extension of quasi-birth–death processes well-known in the performance evaluation field [8].

Organisation. Section 2 recalls decisive Markov chains, presents the classical algorithm for solving the CRP problem and shows that decisiveness is somehow related to recurrence of Markov chains. In section 3, we introduce pCM and show that decisiveness is undecidable for static pCM. In section 4, we study the decidability status of decisiveness for probabilistic one-counter pCM and in section 5, the decidability status of decisiveness for pPN. Finally in Section 6 we conclude and give some perspectives to this work. All missing proofs can be found in [17].

2 Decisive Markov chains

As usual, $\mathbb{N}$ and $\mathbb{N}^*$ denote respectively the set of non negative integers and the set of positive integers. The notations $\mathbb{Q}$, $\mathbb{Q}_{\geq 0}$ and $\mathbb{Q}_{>0}$ denote the set of rationals, non-negative rationals and positive rationals. Let $F \subseteq E$; when there is no ambiguity about E, $\overline{F}$ will denote $E \setminus F$.

2.1 Markov chains: definitions and properties

Notations. A set S is *countable* if there exists an injective function from S to the set of natural numbers: hence it could be finite or countably infinite. Let S be a countable set of elements called states. Then $Dist(S) = \{\Delta : S \to \mathbb{Q}_{\geq 0} \mid \sum_{s \in S} \Delta(s) = 1\}$ is the set of *rational distributions* over S. Let $\Delta \in Dist(S)$, then $Supp(\Delta) = \Delta^{-1}(\mathbb{Q}_{>0})$.

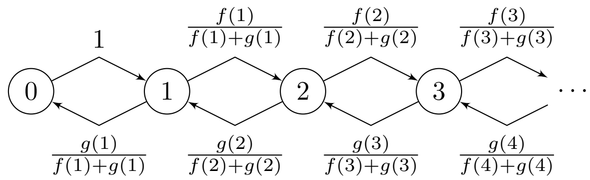


Figure 1 A Markov chain $\mathcal{M}_1$ with for all $n \in \mathbb{N}$, $0 < f(n)$ and $0 < g(n)$.

▶ **Definition 1** ((Effective) Markov chain). *A Markov chain $\mathcal{M} = (S, p)$ is a tuple where:*
- *S is a countable set of states;*
- *p is the transition function from S to $Dist(S)$.*

When for all $s \in S$, $Supp(p(s))$ is finite with both $Supp(p(s))$ and the function $s \mapsto p(s)$ being computable, one says that $\mathcal{M}$ is effective.

When S is countably infinite, we say that $\mathcal{M}$ is *infinite* and we sometimes identify S with $\mathbb{N}$. We also denote $p(s)(s')$ by $p(s, s')$ and $p(s, s') > 0$ by $s \xrightarrow{p(s,s')} s'$. A Markov chain is also viewed as a transition system whose transition relation $\rightarrow$ is defined by $s \rightarrow s'$ if $p(s, s') > 0$.

▶ **Example 2.** Let $\mathcal{M}_1$ be the Markov chain of Figure 1. In any state $i > 0$, the probability for going to the "right", $p(i, i+1) = \frac{f(i)}{f(i)+g(i)}$ and for going to the "left", $p(i, i-1) = \frac{g(i)}{f(i)+g(i)}$. In state 0, one goes to 1 with probability 1. $\mathcal{M}_1$ is effective if the functions f and g are computable.

We denote $\rightarrow^*$, the reflexive and transitive closure of $\rightarrow$ and we say that s' is *reachable from* s if $s \rightarrow^* s'$. We say that a subset $A \subseteq S$ is *reachable* from s if some $s' \in A$ is reachable from s and we denote $s \rightarrow^* A$. Let us remark that every finite path of $\mathcal{M}$ can be extended into (at least) one infinite path.

Given an initial state s_0, the *sampling* of a Markov chain $\mathcal{M}$ is an *infinite random sequence of states* (i.e., a path) $\sigma = s_0 s_1 \ldots$ such that for all $i \geq 0$, $s_i \rightarrow s_{i+1}$. As usual, the corresponding σ-algebra is generated by the finite prefixes of infinite paths and the probability of a measurable subset Π of infinite paths, given an initial state s_0, is denoted $\mathbf{Pr}_{\mathcal{M},s_0}(\Pi)$. In particular denoting $s_0 \ldots s_n S^\omega$ the set of infinite paths with $s_0 \ldots s_n$ as prefix $\mathbf{Pr}_{\mathcal{M},s_0}(s_0 \ldots s_n S^\omega) = \prod_{0 \leq i < n} p(s_i, s_{i+1})$.

Notations. From now on, $\mathbf{G}$ (resp. $\mathbf{F}$, $\mathbf{X}$) denotes the always (resp. eventual, next) operator of LTL, and $\mathbf{E}$ the existential operator of CTL* [7].

Let $A \subseteq S$. We say that *σ reaches A* if $\exists i \in \mathbb{N}\ s_i \in A$ which corresponds to $\sigma \models \mathbf{F}A$. Similarly $\sigma \models \mathbf{XF}A$ if $\exists i > 0\ s_i \in A$. The probability that starting from s_0, the path σ reaches A is thus denoted by $\mathbf{Pr}_{\mathcal{M},s_0}(\mathbf{F}A)$.

The next definition states qualitative and quantitative properties of a Markov chain.

▶ **Definition 3** (Irreducibility, recurrence, transience). *Let $\mathcal{M} = (S, p)$ be a Markov chain and $s \in S$. Then:*
- *$\mathcal{M}$ is* irreducible *if for all $s, s' \in S$, $s \rightarrow^* s'$;*
- *s is* recurrent *if $\mathbf{Pr}_{\mathcal{M},s}(\mathbf{XF}\{s\}) = 1$ otherwise s is* transient.

The next proposition states that in an irreducible Markov chain, all states are in the same category [20].

▶ **Proposition 4.** *Let $\mathcal{M} = (S, p)$ be an irreducible Markov chain and $s, s' \in S$. Then s is recurrent if and only if s' is recurrent.*

Thus an irreducible Markov chain will be said transient or recurrent depending on the category of its states (all states are in the same category). In the remainder of this section, we will relate this category with techniques for computing reachability probabilities.

▶ **Example 5.** $\mathcal{M}_1$ of Figure 1 is clearly irreducible. Let us define $p_n = \frac{f(n)}{f(n)+g(n)}$. Then (see [17] for more details), $\mathcal{M}_1$ is recurrent if and only if $\sum_{n\in\mathbb{N}} \prod_{1\leq m<n} \rho_m = \infty$ with $\rho_m = \frac{1-p_m}{p_m}$, and when transient, the probability that starting from i the random path reaches 0 is equal to $\frac{\sum_{i\leq n}\prod_{1\leq m<n}\rho_m}{\sum_{n\in\mathbb{N}}\prod_{1\leq m<n}\rho_m}$.

2.2 Decisive Markov chains

One of the goals of the quantitative analysis of infinite Markov chains is to approximately compute reachability probabilities. Let us formalize it. Given a finite representation of a subset $A \subseteq S$, one says that this representation is *effective* if one can decide the membership problem for A. With a slight abuse of language, we identify A with any effective representation of A.

The Computing of Reachability Probability (CRP) problem

- Input: an effective Markov chain $\mathcal{M}$, an (initial) state s_0, an effective subset of states A, and a rational $\theta > 0$.
- Output: an interval $[low, up]$ such that $up - low \leq \theta$ and $\mathbf{Pr}_{\mathcal{M},s_0}(\mathbf{F}A) \in [low, up]$.

In finite Markov chains, there is a well-known algorithm for computing exactly the reachability probabilities in polynomial time [7]. In infinite Markov chains, there are (at least) two possible research directions: (1) either using the specific features of a formalism to design such a CRP algorithm [16], (2) or requiring a supplementary property on Markov chains in order to design an "abstract" algorithm, then verifying that given a formalism this property is satisfied and finally transforming this algorithm into a concrete one. *Decisiveness*-based approach follows the second direction [3]. In words, decisiveness w.r.t. s_0 and A means that almost surely the random path σ starting from s_0 will reach A or some state s' from which A is unreachable.

▶ **Definition 6.** *A Markov chain $\mathcal{M}$ is* decisive *w.r.t. $s_0 \in S$ and $A \subseteq S$ if:*

$$\mathbf{Pr}_{\mathcal{M},s_0}(\mathbf{G}(\overline{A} \cap \mathbf{EF}A)) = 0$$

Then under the hypotheses of decisiveness w.r.t. s_0 and A and decidability of the reachability problem w.r.t. A, Algorithm 1 solves the CRP problem.

Let us explain Algorithm 1. If A is unreachable from s_0, then it returns the singleton interval $[0, 0]$. Otherwise it maintains a lower bound *pmin* (initially 0) and an upper bound *pmax* (initially 1) of the reachability probability and builds some prefix of the infinite execution tree of $\mathcal{M}$. It also maintains the probability to reach a vertex in this tree. There are three possible cases when examining the state s associated with the current vertex along a path of probability q: (1) either $s \in A$ and the lower bound is incremented by q, (2) either A is unreachable from s and the upper bound is decremented by q, (3) or it extends the prefix of the tree by the successors of s. The lower bound always converges to the searched probability while due to the decisiveness property, the upper bound also converges to it ensuring termination of the algorithm. For the sake of termination, a fair extraction policy is required such as a FIFO one.

▶ **Proposition 7** ([3]). *Algorithm 1 terminates and computes an interval of length at most θ containing $\mathbf{Pr}_{\mathcal{M},s_0}(\mathbf{F}A)$ when applied to a decisive Markov chain $\mathcal{M}$ w.r.t. s_0 and A with a decidable reachability problem w.r.t. A.*

Algorithm 1 can be applied to probabilistic Lossy Channel Systems (pLCS) since they are decisive (Corollary 4.7 in [3] and see [5] for the first statement) and reachability is decidable in LCS [4]. It can be also applied to pVASSs w.r.t. upward closed sets because Corollary 4.4 in [3] states that pVASSs are decisive w.r.t. *upward closed sets*.

Observations. The test $pmin = 0$ is not necessary but adding it avoids to return 0 as lower bound, which would be inaccurate since entering this loop means that A is reachable from s_0. Extractions from the front are performed in a way that the execution tree will be covered (for instance by a breadth first exploration).

Algorithm 1 Framing the reachability probability in decisive Markov chains.

```
CompProb(M, s0, A, θ)
if not s0 →* A then return (0,0)
pmin ← 0; pmax ← 1; Front ← ∅
Insert(Front, (s0, 1))
while pmax − pmin > θ or pmin = 0 do
    (s, q) ← Extract(Front)
    if s ∈ A then pmin ← pmin + q
    else if not s →* A then pmax ← pmax − q
    else
        for s' ∈ Supp(p(s)) do
            Insert(Front, (s', qp(s, s')))
        end
    end
end
return (pmin, pmax)
```

Let $\mathcal{M}$ be a Markov chain. One denotes $Post^*_{\mathcal{M}}(A)$, the set of states that can be reached from some state of A and $Pre^*_{\mathcal{M}}(A)$, the set of states that can reach A. While decisiveness has been used in several contexts including uncountable probabilistic systems [9], its relation with standard properties of Markov chains has not been investigated. This is the goal of the next definition and proposition. In words, $\mathcal{M}_{s_0,A}$ consists of reachable states not in A but that can reach A with an additional state $s_\perp$ corresponding to states of $\mathcal{M}$ that are either in A or cannot reach A. The probabilities are defined similarly as those of $\mathcal{M}$ except that the transition probabilities to $s_\perp$ are the sums of the transition probabilities to the corresponding states in $\mathcal{M}$.

▶ **Definition 8.** *Let $\mathcal{M}$ be a Markov chain, $s_0 \in S$ and $A \subseteq S$ such that $s_0 \in Pre^*_{\mathcal{M}}(A) \setminus A$. The Markov chain $\mathcal{M}_{s_0,A} = (S_{s_0,A}, p_{s_0,A})$ is defined as follows:*

- *$S_{s_0,A}$ is the union of (1) the smallest set containing s_0 and such that for all $s \in S_{s_0,A}$ and $s' \in Pre^*_{\mathcal{M}}(A) \setminus A$ with $s \to s'$, one have:*
  $s' \in S_{s_0,A}$ and (2) $\{s_\perp\}$ where $s_\perp$ is a new state;
- *for all $s, s' \neq s_\perp$, $p_{s_0,A}(s,s') = p(s,s')$ and $p_{s_0,A}(s,s_\perp) = \sum_{s' \notin \in Pre^*_{\mathcal{M}}(A)\setminus A} p(s,s')$;*
- *$p_{s_0,A}(s_\perp, s_0) = 1$.*

▶ **Proposition 9.** *Let $\mathcal{M} = (S, p)$ be a Markov chain, $s_0 \in S$ and $A \subseteq S$ such that $s_0 \in Pre^*_{\mathcal{M}}(A) \setminus A$. Then $\mathcal{M}_{s_0,A}$ is irreducible. Furthermore $\mathcal{M}$ is decisive w.r.t. s_0 and A if and only if $\mathcal{M}_{s_0,A}$ is recurrent.*

Proof. Let $s \in S_{s_0,A} \setminus \{s_\perp\}$. Then s is reachable from s_0 and A is reachable from s in $\mathcal{M}$ implying that $s \to^* s_\perp$ in $\mathcal{M}_{s_0,A}$ (using a shortest path for reachability). Since $s_\perp \to s_0$, $s_\perp \to^* s$. Thus $\mathcal{M}_{s_0,A}$ is irreducible.
$\mathcal{M}_{s_0,A}$ is recurrent iff $\mathbf{Pr}_{\mathcal{M}_{s_0,A},s_\perp}(\mathbf{XF}\{s_\perp\}) = 1$ iff $\mathbf{Pr}_{\mathcal{M}_{s_0,A},s_0}(\mathbf{F}\{s_\perp\}) = 1$ iff $\mathbf{Pr}_{\mathcal{M},s_0}(\mathbf{F}A \cup \overline{Pre^*_{\mathcal{M}}(A)}) = 1$ iff $\mathcal{M}$ is decisive w.r.t. s_0 and A. ◀

The equivalence between decisiveness of $\mathcal{M}$ w.r.t. $s_0 \in S$ and $A \subseteq S$ and recurrence of $\mathcal{M}_{s_0,A}$ allows to apply standard criteria for recurrence in order to check decisiveness. For instance in Section 4, we will use the criterion presented in Example 5 for the Markov chain of Figure 1.

3 Probabilistic counter machines

We now introduce *probabilistic Counter Machines (pCM)* in order to study the decidability of the decisiveness property w.r.t. several relevant subclasses of pCM.

▶ **Definition 10** (pCM). *A* probabilistic counter machine (pCM) *is a tuple $\mathcal{C} = (Q, P, \Delta, W)$ where:*
- *Q is a finite set of control states;*
- *$P = \{p_1, \dots, p_d\}$ is a finite set of counters (also called places);*
- *$\Delta = \Delta_0 \uplus \Delta_1$ where Δ_0 is a finite subset of $Q \times P \times \mathbb{N}^d \times Q$ and Δ_1 is a finite subset of $Q \times \mathbb{N}^d \times \mathbb{N}^d \times Q$;*
- *W is a computable function from $\Delta \times \mathbb{N}^d$ to $\mathbb{N}^*$.*

Notations. A transition $t \in \Delta_0$ is denoted $t = (q_t^-, p_t, \mathbf{Post}(t), q_t^+)$ and also $q_t^- \xrightarrow{p_t,\mathbf{Post}(t)} q_t^+$. A transition $t \in \Delta_1$ is denoted $t = (q_t^-, \mathbf{Pre}(t), \mathbf{Post}(t), q_t^+)$ and also $q_t^- \xrightarrow{\mathbf{Pre}(t),\mathbf{Post}(t)} q_t^+$. Let t be a transition of $\mathcal{C}$. Then $W(t)$ is the function from $\mathbb{N}^d$ to $\mathbb{Q}_{>0}$ defined by $W(t)(\mathbf{m}) = W(t, \mathbf{m})$. A polynomial is *positive* if all its coefficients are non-negative and there is a positive constant term. When for all $t \in T$, $W(t)$ is a positive polynomial whose variables are the counters, we say that $\mathcal{C}$ is a *polynomial* pCM.

A *configuration* of $\mathcal{C}$ is an item of $Q \times \mathbb{N}^d$. Let $s = (q, \mathbf{m})$ be a configuration and $t = (q_t^-, p_t, \mathbf{Post}(t), q_t^+)$ be a transition in Δ_0. Then t is *enabled* in s if $\mathbf{m}(p_t) = 0$ and $q = q_t^-$; its *firing* leads to the configuration $(q_t^+, \mathbf{m} + \mathbf{Post}(t))$. Let $t = (q_t^-, \mathbf{Pre}(t), \mathbf{Post}(t), q_t^+) \in \Delta_1$. Then t is *enabled* in s if $\mathbf{m} \geq \mathbf{Pre}(t)$ and $q = q_t^-$; its *firing* leads to the configuration $s' = (q_t^+, \mathbf{m} - \mathbf{Pre}(t) + \mathbf{Post}(t))$. One denotes the configuration change by: $s \xrightarrow{t} s'$. One denotes $En(s)$, the set of transitions enabled in s and $Weight(s) = \sum_{t \in En(s)} W(t, \mathbf{m})$. Let $\sigma = t_1 \dots t_n$ be a sequence of transitions. We define the enabling and the firing of σ by induction. The empty sequence is always enabled in s and its firing leads to s. When $n > 0$, σ is enabled if $s \xrightarrow{t_1} s_1$ and $t_2 \dots t_n$ is enabled in s_1. The firing of σ leads to the configuration reached by $t_2 \dots t_n$ from s_1. A configuration s is *reachable* from some s_0 if there is a firing sequence σ that reaches s from s_0. When Q is a singleton, one omits the control states in the definition of transitions and configurations.

We now provide the semantic of a pCM as a countable Markov chain.

▶ **Definition 11.** *Let $\mathcal{C}$ be a pCM. Then the Markov chain $\mathcal{M}_{\mathcal{C}} = (S, p)$ is defined by:*
- $S = Q \times \mathbb{N}^d$;
- *For all $s = (q, \mathbf{m}) \in S$, if $En(s) = \emptyset$ then $p(s, s) = 1$. Otherwise for all $s' \in S$:*

$$p(s, s') = Weight(s)^{-1} \sum_{s \xrightarrow{t} s'} W(t, \mathbf{m})$$

For establishing the undecidability results, we will reduce an undecidable problem related to *counter programs*, which are a variant of CM. Let us recall that a *d-counter program* $\mathcal{P}$ is defined by a set of d counters $\{c_1, \ldots, c_d\}$ and a set of $n + 1$ instructions labelled by $\{0, \ldots, n\}$, where for all $i < n$, the instruction i is of type
- either (1) $c_j \leftarrow c_j + 1;$ **goto** i' with $1 \leq j \leq d$ and $0 \leq i' \leq n$
- or (2) **if** $c_j > 0$ **then** $c_j \leftarrow c_j - 1;$ **goto** i', **else goto** i'' with $1 \leq j \leq d$ and $0 \leq i', i'' \leq n$

and the instruction n is **halt**. The program starts at instruction 0 and halts if it reaches the instruction n.

The halting problem for two-counter programs asks, given a two-counter program $\mathcal{P}$ and initial values of counters, whether $\mathcal{P}$ eventually halts. It is undecidable [22]. We introduce a subclass of two-counter programs that we call *normalized*. A normalized two-counter program $\mathcal{P}$ starts by resetting its counters and, on termination, resets its counters before halting.

Normalized two-counter program. The first two instructions of a normalized two-counter program reset counters c_1, c_2 as follows:
- $0:$ **if** $c_1 > 0$ **then** $c_1 \leftarrow c_1 - 1;$ **goto** 0 **else goto** 1
- $1:$ **if** $c_2 > 0$ **then** $c_2 \leftarrow c_2 - 1;$ **goto** 1 **else goto** 2

The last three instructions of a normalized two-counter program are:
- $n-2:$ **if** $c_1 > 0$ **then** $c_1 \leftarrow c_1 - 1;$ **goto** $n-2$ **else goto** $n-1$
- $n-1:$ **if** $c_2 > 0$ **then** $c_2 \leftarrow c_2 - 1;$ **goto** $n-1$ **else goto** n
- $n:$ **halt**

For $1 < i < n - 2$, the labels occurring in instruction i belong to $\{0, \ldots, n - 2\}$. In a normalized two-counter program $\mathcal{P}$, given any initial values v_1, v_2, $\mathcal{P}$ halts with v_1, v_2 if and only if $\mathcal{P}$ halts with initial values $0, 0$. Moreover when $\mathcal{P}$ halts, the values of the counters are null. The halting problem for normalized two-counter programs is also undecidable (see [17] for the proof).

We now show that decisiveness is undecidable even for *static* pCM, by considering only *static* weights: for all $t \in \Delta$, $W(t)$ is a constant function.

▶ **Theorem 12.** *Decisiveness w.r.t. a finite set is undecidable in (static) pCM.*

4 Probabilistic safe one-counter machines

We now study decisiveness for pCMs that only have one counter denoted c. We also restrict Δ_1: a single counter PCM is *safe* if for all $t \in \Delta_1$, $(\mathbf{Pre}(t), \mathbf{Post}(t)) \in \{1\} \times \{0, 1, 2\}$. In words, in a safe one-counter pCM, a transition of Δ_1 requires the counter to be positive and may either let it unchanged, or incremented or decremented by a unit.

4.1 One-state and one-counter pCM

We first prove that decisiveness is undecidable for the probabilistic version of one-state and one-counter machines. Then we show how to restrict the weight functions and Δ_1 such that this property becomes decidable. Both proofs make use of the relationship between decisiveness and recurrence stated in Proposition 9, in an implicit way.

▶ **Theorem 13.** *The decisiveness problem for safe one-counter pCM is undecidable even with a single state.*

Proof. We will reduce the Hilbert's tenth problem to decisiveness problems. Let $P \in \mathbb{Z}[X_1, \ldots X_k]$ be an integer polynomial with k variables. This problem asks whether there exist $n_1, \ldots, n_k \in \mathbb{N}$ such that $P(n_1, \ldots, n_k) = 0$.

We define $\mathcal{C}$ as follows. There are two transitions both in Δ_1:

- dec with $\mathbf{Pre}(dec) = 1$ and $\mathbf{Post}(dec) = 0$;
- inc with $\mathbf{Pre}(inc) = 0$ and $\mathbf{Post}(inc) = 1$.

The weight of dec is the constant function 1, i.e., $W(dec, n) = f(n) = 1$, while the weight of inc is defined by the following (non polynomial) function:

$$W(inc, n) = g(n) = \min(P^2(n_1, \ldots, n_k) + 1 \mid n_1 + \ldots + n_k \leq n)$$

This function is obviously computable. Let us study the decisiveness of $\mathcal{M}_\mathcal{C}$ w.r.t. $s_0 = 1$ and $A = \{0\}$. Observe that $\mathcal{M}_\mathcal{C}$ is the Markov chain $\mathcal{M}_1$ of Figure 1. Let us recall that in $\mathcal{M}_1$, the probability to reach 0 from i is 1 iff $\sum_{n\in\mathbb{N}} \prod_{1\leq m<n} \rho_m = \infty$ and otherwise it is equal to $\frac{\sum_{i\leq n} \prod_{1\leq m<n} \rho_m}{\sum_{n\in\mathbb{N}} \prod_{1\leq m<n} \rho_m}$ with $\rho_m = \frac{1-p_m}{p_m}$.

• Assume there exist $n_1, \ldots, n_k \in \mathbb{N}$ s.t. $P(n_1, \ldots, n_k) = 0$. Let $n_0 = n_1 + \cdots + n_k$. Thus for all $n \geq n_0$, $W(inc, n) = 1$, which implies that $p_\mathcal{C}(n, n-1) = p_\mathcal{C}(n, n+1) = \frac{1}{2}$. Thus due to the results on $\mathcal{M}_1$, from any state n, one reaches 0 almost surely and so $\mathcal{M}_\mathcal{C}$ is decisive.

• Assume there do not exist $n_1, \ldots, n_k \in \mathbb{N}$ s.t. $P(n_1, \ldots, n_k) = 0$. For all $n \in \mathbb{N}$, $W(inc, n) \geq 2$, implying that in $\mathcal{M}_1$, $\rho_n \leq \frac{1}{2}$. Thus $\mathcal{M}_\mathcal{C}$ is not decisive. ◀

Due to the negative result for single state and single counter pCM stated in Theorem 13, it is clear that one must restrict the possible weight functions.

▶ **Theorem 14.** *The decisiveness problem w.r.t. s_0 and finite A for polynomial safe one-counter pCM $\mathcal{C}$ with a single state is decidable in linear time.*

4.2 Homogeneous one-counter machines

We now study another interesting model that is considered as a generalization of the well-known model of quasi-birth-death process with dynamic weights. This model has been the topic of numerous theoretical results and modelings, see for instance [8].

Let $\mathcal{C}$ be a one-counter safe pCM. For all $q \in Q$, let $S_{q,1} = \sum_{t=(q,\mathbf{Pre}(t),\mathbf{Post}(t),q_t^+)\in\Delta_1} W(t)$ and $\mathbf{M}_\mathcal{C}$ be the $Q \times Q$ matrix defined by

$$\mathbf{M}_\mathcal{C}[q, q'] = \frac{\sum_{t=(q,\mathbf{Pre}(t),\mathbf{Post}(t),q')\in\Delta_1} W(t)}{S_{q,1}}$$

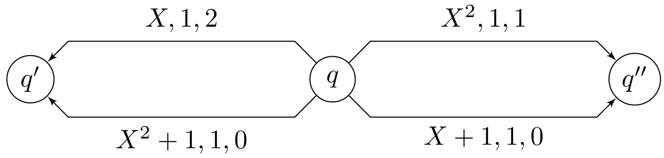
(thus $\mathbf{M}_\mathcal{C}[q, q']$ is a function from $\mathbb{N}$ to $\mathbb{Q}_{\geq}0$).

▶ **Definition 15** (pHM). *A* probabilistic homogeneous machine *(pHM) is a probabilistic safe one-counter machine $\mathcal{C} = (Q, \Delta, W)$ where:*

- *For all $t \in \Delta$, $W(t)$ is a positive polynomial in $\mathbb{N}[X]$;*
- *For all $q, q' \in Q$, $\mathbf{M}_\mathcal{C}[q, q']$ is constant.*

Observe that by definition, in a pHM, $\mathbf{M}_\mathcal{C}$ is a transition matrix.

▶ **Example 16.** Here $\mathbf{M}_\mathcal{C}[q, q'] = \mathbf{M}_\mathcal{C}[q, q''] = \frac{X^2+X+1}{2(X^2+X+1)} = \frac{1}{2}$ fulfilling the homogeneous requirement. Let us describe a possible transition: given a configuration (q, n) with $n > 0$ the probability to go to $(q', n+1)$ is equal to $\frac{X}{2(X^2+X+1)}$.



The family $(r_q)_{q\in Q}$ of the next proposition is independent of the function W and is associated with the qualitative behaviour of $\mathcal{C}$, i.e., its underlying transition system. For all $q \in Q$, r_q is an upper bound of the counter value for q, from which $Q \times \{0\}$ is reachable.

▶ **Proposition 17.** *Let $\mathcal{C}$ be a pHM. Then one can compute in polynomial time a family $(r_q)_{q\in Q}$ such that for all q, $r_q \in \{0, \dots, |Q| - 1\} \cup \{\infty\}$, and $Q \times \{0\}$ is reachable from (q, k) iff $k \le r_q$.*

▶ **Theorem 18.** *Let $\mathcal{C}$ be a pHM such that $\mathbf{M}_\mathcal{C}$ is irreducible. Then the decisiveness problem of $\mathcal{C}$ w.r.t. $s_0 = (q, n) \in Q \times \mathbb{N}$ and $A = Q \times \{0\}$ is decidable in polynomial time.*

Proof. With the notations of previous proposition, assume that there exist q with $r_q < \infty$ and q' with $r_{q'} = \infty$. Since $\mathbf{M}_\mathcal{C}$ is irreducible, there is a sequence of transitions in Δ_1 $q_0 \xrightarrow{1,v_1} q_1 \cdots \xrightarrow{1,v_m} q_m$ with $q_0 = q$ and $q_m = q'$. Let $sv = \min(\sum_{i\le j}(v_i - 1) | j \le m)$ and pick some $k > \max(r_q, -sv)$. Then there is a path in $\mathcal{M}_\mathcal{C}$ from (q, k) to $(q', k + \sum_{i\le m}(v_i - 1))$, which yields a contradiction since (q, k) cannot reach $Q \times \{0\}$ while $(q', k + \sum_{i\le m} v_i)$ can reach it. Thus either (1) for all $q \in Q$, $r_q < \infty$ or (2) for all $q \in Q$, $r_q = \infty$.

• First assume that for all $q \in Q$, $r_q < \infty$. Thus for all $k > r_q$, (q, k) cannot reach $Q \times \{0\}$ and thus $\mathcal{C}$ is decisive w.r.t. (q, k) and $Q \times \{0\}$. Now consider a configuration (q, k) with $k \le r_q$. By definition there is a positive probability say $p_{(q,k)}$ to reach $Q \times \{0\}$ from (q, k). Let $p_{\min} = \min(p_{(q,k)} \mid q \in Q \wedge k \le r_q)$. Then for all (q, k) with $k \le r_q$, there is a probability at least $p_{\min}$ to reach either $Q \times \{0\}$ or $\{(q, k) \mid q \in Q \wedge k > r_q\}$ by a path of length $\ell = \sum_{q\in Q}(r_q + 1)$. This implies that after $n\ell$ transitions the probability to reach either $Q \times \{0\}$ or $\{(q, k) \mid q \in Q \wedge k > r_q\}$ is at least $1 - (1 - p_{\min})^n$. Thus $\mathcal{C}$ is decisive w.r.t. (q, k) and $Q \times \{0\}$. Summarizing for all (q, k), $\mathcal{C}$ is decisive w.r.t. (q, k) and $Q \times \{0\}$.

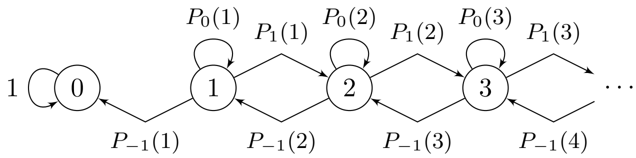
• Now assume that for all $(q, k) \in Q \times \mathbb{N}$, $Q \times \{0\}$ is reachable from (q, k). Thus the decisiveness problem boils down to the almost sure reachability of $Q \times \{0\}$.
Since the target of decisiveness is $Q \times \{0\}$, we can arbitrarily set up the outgoing transitions of these states (i.e., Δ_0) without changing the decisiveness problem. So we choose these transitions and associated probabilities as follows. For all q, q' such that $\mathbf{M}_\mathcal{C}[q, q'] > 0$, there is a transition $t = q \xrightarrow{c,0} q'$ with $W(t) = \mathbf{M}_\mathcal{C}[q, q']$.
Since $\mathbf{M}_\mathcal{C}$ is irreducible, there is a unique invariant distribution π_∞ (i.e., $\pi_\infty \mathbf{M}_\mathcal{C} = \pi_\infty$) fulfilling for all $q \in Q$, $\pi_\infty(q) > 0$.
Let $(Q_n, N_n)_{n\in\mathbb{N}}$ be the stochastic process defined by $\mathcal{M}_\mathcal{C}$ with $N_0 = k$ for some k and for all $q \in Q$, $\mathbf{Pr}(Q_0 = q) = \pi_\infty(q)$. Due to the invariance of π_∞ and the choice of transitions for $Q \times \{0\}$, one gets by induction that for all $n \in \mathbb{N}$:

- $\mathbf{Pr}(Q_n = q) = \pi_\infty(q)$;
- for all $k > 0$ and $v \in \{-1, 0, 1\}$, $\mathbf{Pr}(N_{n+1} = k + v - 1 | N_n = k) =$ $\sum_{q\in Q} \pi_\infty(q) \frac{\sum_{t=(q,1,v,q')\in\Delta_1} W(t,k)}{S_{q,1}(k)} = \frac{\sum_{q\in Q} \pi_\infty(q) \prod_{q'\neq q} S_{q',1}(k) \sum_{t=(q,1,v,q')\in\Delta} W(t,k)}{\prod_{q'\in Q} S_{q',1}(k)}$;
- $\mathbf{Pr}(N_{n+1} = 0 | N_n = 0) = 1$.

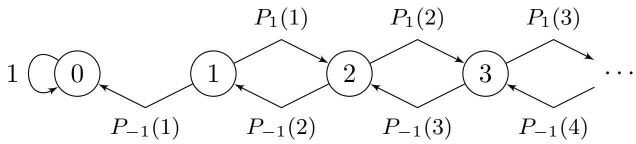
For $v \in \{-1, 0, 1\}$, let us define the polynomial P_v by:

$$\sum_{q\in Q} \pi_\infty(q) \prod_{q'\neq q} S_{q',1} \sum_{t=(q,1,v+1,q')\in\Delta_1} W(t)$$

Due to the previous observations, the stochastic process $(N_n)_{n\in\mathbb{N}}$ is the Markov chain defined below where the weights outgoing from a state have to be normalized:



Using our hypothesis about reachability, P_{-1} is a positive polynomial (while P_1 could be null) and thus the decisiveness of this Markov chain w.r.t. state 0 is equivalent to the decisiveness of the Markov chain below:



Due to Theorem 14, this problem is decidable (in linear time) and either (1) for all $k \in \mathbb{N}$ this Markov chain is decisive w.r.t k and 0 or (2) for all $k > 0$ this Markov chain is not decisive w.r.t k and 0. Let us analyze the two cases w.r.t. the Markov chain of the pHM.

Case (1). In the stochastic process $(Q_n, N_n)_{n\in\mathbb{N}}$, the initial distribution has a positive probability for (q, k) for all $q \in Q$. This implies that for all q, $\mathcal{C}$ is decisive w.r.t. (q, k) and $Q \times \{0\}$. Since k was arbitrary, this means that for all (q, k), $\mathcal{C}$ is decisive w.r.t. (q, k) and $Q \times \{0\}$.

Case (2). Choosing $k = 1$ and applying the same reasoning as for the previous case, there is some $(q, 1)$ which is not decisive (and so for all (q, k') with $k' > 0$). Let $q' \in Q$, since $\mathbf{M}_{\mathcal{C}}$ is irreducible, there is a (shortest) sequence of transitions in Δ_1 leading from q' to q whose length is at most $|Q| - 1$. Thus for all (q', k') with $k' \geq |Q|$ there is a positive probability to reach some (q, k) with $k > 0$. Thus (q', k) is not decisive.
Now let (q', k') with $k' < |Q|$. Then we compute by a breadth first exploration the configurations reachable from (q', k') until either (1) one reaches some (q'', k'') with $k'' \geq |Q|$ or (2) the full (finite) reachability set is computed. In the first case, there is a positive probability to reach some (q'', k'') with $k'' \geq |Q|$ and from (q'', k'') to some (q, k) with $k > 0$ and so (q', k') is not decisive. In the second case, it means that the reachable set is finite and from any configuration of this set there is a positive probability to reach $Q \times \{0\}$ by a path of length at most the size of this set. Thus almost surely $Q \times \{0\}$ will be reached and (q', k') is decisive. ◀

5 Probabilistic Petri nets

We now introduce probabilistic Petri nets as a subclass of pCM.

▶ **Definition 19** (pPN). *A probabilistic Petri net (pPN) $\mathcal{N}$ is a pCM $\mathcal{N} = (Q, P, \Delta, W)$ where Q is a singleton and $\Delta_0 = \emptyset$.*

Notations. Since there is a unique control state in a pPN, a configuration in a pPN is reduced to $\mathbf{m} \in \mathbb{N}^P$ and it is called a *marking*. As usual a marking $\mathbf{m}$ is also denoted as a bag $\sum_{p \in P} \mathbf{m}(p)p$ where the term $\mathbf{m}(p)p$ is omitted when $\mathbf{m}(p) = 0$ and the term $\mathbf{m}(p)p$ is rewritten p when $\mathbf{m}(p) = 1$. A pair $(\mathcal{N}, \mathbf{m}_0)$, where $\mathcal{N}$ is a pPN and $\mathbf{m}_0 \in \mathbb{N}^P$ is some (initial) marking, is called a *marked pPN*. As pPN is defined as a subclass of pCM, its formal semantics is the same as the one described in Section 3.

In previous works [3, 11] about pPNs, the weight function W is a *static* one: i.e., a function from Δ to $\mathbb{N}^*$. As above, we call these models *static* probabilistic Petri nets.

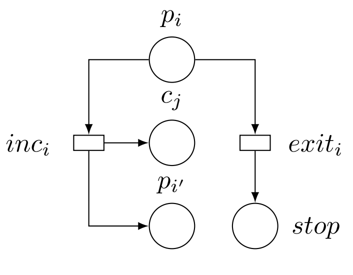


Figure 2 $i : c_j \leftarrow c_j + 1; \textbf{goto } i'$.

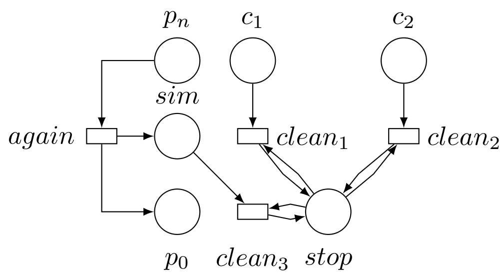


Figure 3 halt instruction and cleaning stage.

Static-probabilistic VASS (and so pPNs) are *decisive* with respect to *upward closed sets* (Corollary 4.4 in [3]) but they may not be decisive w.r.t. an arbitrary finite set. Surprisingly, the decisiveness problem for Petri nets or VASS seems not to have been studied. We establish below that even for polynomial pPNs, decisiveness is undecidable.

▶ **Theorem 20.** *The decisiveness problem of polynomial pPNs w.r.t. a finite or upward closed set is undecidable.*

Proof. We reduce the reachability problem of normalized two-counter machines to the decisiveness problem of pPN. Let $\mathcal{C}$ be a normalized two-counter machine with an instruction set $\{0, \ldots, n\}$. The corresponding marked pPN $(\mathcal{N}_\mathcal{C}, \mathbf{m}_0)$ is built as follows. Its set of places is $P = \{p_i \mid 0 \leq i \leq n\} \cup \{q_i \mid i \text{ is a test instruction}\} \cup \{c_j \mid 1 \leq j \leq 2\} \cup \{sim, stop\}$. The initial marking is $\mathbf{m}_0 = p_0$.

The set Δ of transitions is defined by a pattern per type of instruction. The pattern for the incrementation instruction is depicted in Figure 2. The pattern for the test instruction is depicted in Figure 4. The pattern for the halt instruction is depicted in Figure 3 with in addition a *cleaning stage*. A place is depicted by a circle while a transition is depicted by a rectangle. There is an edge from place p to transition t (resp. from transition t to place p) labelled by $v = \mathbf{Pre}(t)(p)$ (resp. $v = \mathbf{Post}(t)(p)$) when $v > 0$; v is omitted when $v = 1$.

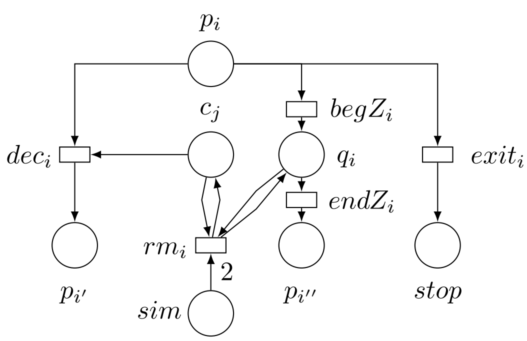


Figure 4 $i : \textbf{if } c_j > 0 \textbf{ then } c_j \leftarrow c_j - 1; \textbf{goto } i' \textbf{else goto } i''$.

Before specifying the weight function W, let us describe the qualitative behaviour of this net. $(\mathcal{N}_\mathcal{C}, \mathbf{m}_0)$ performs repeatedly a *weak* simulation of $\mathcal{C}$. As usual since the zero test does not exist in Petri nets, during a test instruction i, the simulation can follow the zero branch while the corresponding counter is non null (transitions $begZ_i$ and $endZ_i$). If the net has cheated then with transition rm_i, it can remove tokens from sim (two per two). In addition when the instruction is not **halt**, instead of simulating it, it can *exit* the simulation by putting a token in *stop* and then will remove tokens from the counter places including the simulation counter as long as they are not empty. The simulation of the **halt** instruction consists in restarting the simulation and incrementing the simulation counter *sim*.

Thus the set of reachable markings is included in the following set of markings $\{p_i + xc_1 + yc_2 + zsim \mid 0 \leq i \leq n, x, y, z \in \mathbb{N}\} \cup \{q_i + xc_1 + yc_2 + zsim \mid i \text{ is a test instruction}, x, y, z \in \mathbb{N}\} \cup \{stop + xc_1 + yc_2 + zsim \mid x, y, z \in \mathbb{N}\}$. By construction, the marking *stop* is always reachable. We will establish that $\mathcal{N}_\mathcal{C}$ is decisive w.r.t. $\mathbf{m}_0$ and $\{stop\}$ if and only if $\mathcal{C}$ does not halt.

Let us specify the weight function. For any incrementation instruction i, $W(inc_i, \mathbf{m}) = \mathbf{m}(sim)^2 + 1$. For any test instruction i, $W(begZ_i, \mathbf{m}) = \mathbf{m}(sim)^2 + 1$, $W(dec_i, \mathbf{m}) = 2\mathbf{m}(sim)^4 + 2$ and $W(rm_i, \mathbf{m}) = 2$. All other weights are equal to 1.

• Assume that $\mathcal{C}$ halts and consider its execution $\sigma_\mathcal{C}$ with initial values $(0, 0)$. Let $\ell = |\sigma_\mathcal{C}|$ be the length of this execution. Consider now σ the infinite sequence of $(\mathcal{N}_\mathcal{C}, \mathbf{m}_0)$ that infinitely performs the correct simulation of this execution. The infinite sequence σ never marks the place *stop*. We now show that the probability of σ is non null implying that $\mathcal{N}_\mathcal{C}$ is not decisive.

After every simulation of $\sigma_\mathcal{C}$, the marking of *sim* is incremented and it is never decremented since (due to the correctness of the simulation) every time a transition $begZ_i$ is fired, the corresponding counter place c_j is unmarked which forbids the firing of rm_i. So during the $(n+1)^{th}$ simulation of ρ, the marking of *sim* is equal to n.

So consider the probability of the correct simulation of an instruction i during the $(n+1)^{th}$ simulation.

- If i is an incrementation then the weight of inc_i is n^2 and the weight of $exit_i$ is 1. So the probability of a correct simulation is $\frac{n^2+1}{n^2+2} = 1 - \frac{1}{n^2+2} \geq e^{-\frac{2}{n^2+2}}$. [1]
- If i is a test of c_j and the marking of c_j is non null then the weight of dec_i is $2n^4 + 2$, the weight of $begZ_i$ is $n^2 + 1$ and the weight of $exit_i$ is 1. So the probability of a correct simulation is $\frac{2n^4+2}{2n^4+n^2+4} \geq \frac{2n^4+2}{2n^4+2n^2+4} = \frac{n^2+1}{n^2+2} = 1 - \frac{1}{n^2+2} \geq e^{-\frac{2}{n^2+2}}$.
- If i is a test of c_j and the marking of c_j is null then the weight of $begZ_i$ is $n^2 + 1$ and the weight of $exit_i$ is 1. So the probability of a correct simulation is $\frac{n^2+1}{n^2+2} = 1 - \frac{1}{n^2+2} \geq e^{-\frac{2}{n^2+2}}$.

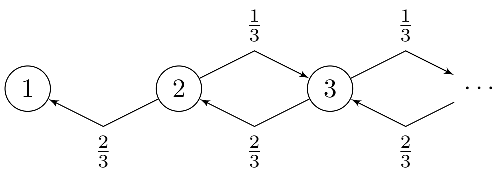
So the probability of the correct simulation during the $(n+1)^{th}$ simulation is at least $(e^{-\frac{2}{n^2+2}})^\ell = e^{-\frac{2\ell}{n^2+2}}$. Hence the probability of σ is at least $\prod_{n\in\mathbb{N}} e^{-\frac{2\ell}{n^2+2}} = e^{-\sum_{n\in\mathbb{N}} \frac{2\ell}{n^2+2}} > 0$, as the sum in the exponent converges.

• Assume that $\mathcal{C}$ does not halt (and so does not halt for any initial values of the counters). We partition the set of infinite paths into a countable family of subsets and prove that for all of them the probability to infinitely avoid to mark *stop* is null which will imply that $\mathcal{N}_\mathcal{C}$ is decisive. The partition is based on $k \in \mathbb{N} \cup \{\infty\}$, the number of firings of *again* in the path.

[1] We use $1 - x \geq e^{-2x}$ for $0 \leq x \leq \frac{1}{2}$.

Case $k < \infty$. Let σ be such a path and consider the suffix of σ after the last firing of *again*. The marking of *sim* is at most k and can only decrease along the suffix. Consider a simulation of an increment instruction i. The weight of inc_i is at most is k^2+1 and the weight of $exit_i$ is 1. So the probability of avoiding $exit_i$ is at most $\frac{k^2+1}{k^2+2} = 1 - \frac{1}{k^2+2} \leq e^{-\frac{1}{k^2+2}}$. Consider the simulation of a test instruction i. Then the weight of dec_i is at most $2k^4+2$, the weight of $begZ_i$ is at most k^2+1 and the weight of $exit_i$ is 1. So the probability of avoiding $exit_i$ is at most $\frac{2k^4+k^2+2}{2k^4+k^2+4} \leq \frac{4k^4+1}{4k^4+2} = 1 - \frac{1}{4k^4+2} \leq e^{-\frac{1}{4k^4+2}}$.
Thus after n simulations of instructions in the suffix, the probability to avoid to mark *stop* is at most $e^{-\frac{n}{4k^4+2}}$. Letting n go to infinity yields the result.

Case $k = \infty$. We first show that almost surely there will be an infinite number of simulations of $\mathcal{C}$ with the marking of *sim* at most 1. Observe that all these simulations are incorrect since they mark p_n while $\mathcal{C}$ does not halt. So at least once per simulation some place q_i and the corresponding counter c_j must be marked and if the marking of *sim* is at least 2 with probability $\frac{2}{3}$ two tokens of *sim* are removed (recall that the weight of rm_i is 2 and the weight of $endZ_i$ is 1). Thus once the marking of *sim* is greater than 1, considering the successive random markings of *sim* after the firing of *again* until it possibly reaches 1, this behaviour is *stochastically bounded* by the following random walk:



In this random walk, one reaches the state 1 with probability 1. This establishes that almost surely there will be an infinite number of simulations of $\mathcal{C}$ with the marking of *sim* at most 1. Such a simulation must simulate at least one instruction. If this instruction is an incrementation, the exiting probability is at least $\frac{1}{3}$; if it is a test instruction the exiting probability is at least $\frac{1}{7}$. Thus after n such simulations of $\mathcal{C}$, the probability to avoid to mark *stop* is at most $(\frac{6}{7})^n$. Letting n go to infinity yields the result.
Observe that the result remains true when substituting the singleton $\{stop\}$ by the set of markings greater than or equal to *stop*. ◀

We deduce thus that decisiveness of extended (probabilistic) Petri nets is undecidable : in particular for Reset Petri nets [15], Post-Self-Modifying Petri nets [25], Recursive Petri nets, etc.

▶ **Definition 21.** *The* language *of a marked Petri net* $(\mathcal{N}, \mathbf{m}_0)$ *is defined by* $\mathcal{L}(\mathcal{N}, \mathbf{m}_0) = \{\sigma \in \Delta^* \mid \mathbf{m}_0 \xrightarrow{\sigma}\}$. *The marked Petri net* $(\mathcal{N}, \mathbf{m}_0)$ *is* regular *if* $\mathcal{L}(\mathcal{N}, \mathbf{m}_0)$ *is regular.*

Given a marked Petri net $(\mathcal{N}, \mathbf{m}_0)$, the problem who asks whether it is regular is decidable [18, 26] and belongs to EXPSPACE [14]. For establishing the next proposition, we only need the following result that holds for regular Petri nets: There exists a computable bound $B(\mathcal{N}, \mathbf{m}_0)$ such that for all markings $\mathbf{m}_1$ reachable from $\mathbf{m}_0$ and all markings $\mathbf{m}_2$ with some $p \in P$ fulfilling $\mathbf{m}_2(p) + B(\mathcal{N}, \mathbf{m}_0) < \mathbf{m}_1(p)$, $\mathbf{m}_2$ is unreachable from $\mathbf{m}_1$ ([18]).

▶ **Theorem 22.** *Let* $(\mathcal{N}, \mathbf{m}_0)$ *be a regular marked pPN and* $\mathbf{m}_1$ *be a marking. Then* $(\mathcal{N}, \mathbf{m}_0)$ *is decisive with respect to* $\mathbf{m}_0$ *and* $\{\mathbf{m}_1\}$.

Proof. Consider the following algorithm that, after computing $B(\mathcal{N}, \mathbf{m}_0)$, builds a finite graph whose vertices are some reachable markings and edges correspond to transition firings between markings:

- The initial vertex is $\mathbf{m}_0$ and push on the stack $\mathbf{m}_0$.
- While the stack is not empty, pop from the stack some marking $\mathbf{m}$. Compute the set of transition firings $\mathbf{m} \xrightarrow{t} \mathbf{m}'$. Add $\mathbf{m}'$ in the set of vertices, the firings $\mathbf{m} \xrightarrow{t} \mathbf{m}'$ to the set of edges and push on the stack $\mathbf{m}'$ if:
  1. $\mathbf{m}'$ is not already present in the set of vertices,
  2. and $\mathbf{m}' \neq \mathbf{m}_1$,
  3. and for all $p \in P$, $\mathbf{m}_1(p) + B(\mathcal{N}, \mathbf{m}_0) \geq \mathbf{m}'(p)$.

Due to the third condition, this algorithm terminates. From above, if $\mathbf{m}_1$ does not occur in the graph then $\mathbf{m}_1$ is unreachable from $\mathbf{m}_0$ and thus $\mathcal{N}$ is decisive w.r.t. $\mathbf{m}_1$.
Otherwise, considering the weights specified by W and adding loops for states without successors, this graph can be viewed as a finite Markov chain and so reaching some bottom strongly connected component (BSCC) almost surely. There are three possible cases: (1) the BSCC consisting of $\mathbf{m}_1$, (2) a BSCC consisting of a single marking $\mathbf{m}$ for which there exists some $p \in P$ fulfilling $\mathbf{m}_1(p) + B(\mathcal{N}, \mathbf{m}_0) < \mathbf{m}(p)$ and thus from which $\mathbf{m}_1$ is unreachable or (3) a BSCC that is also a BSCC of $\mathcal{M}_\mathcal{N}$ and thus from which one cannot reach $\mathbf{m}_1$. This establishes that $\mathcal{N}$ is decisive w.r.t. $\mathbf{m}_1$. ◀

In this particular case, instead of using Algorithm 1 to frame the reachability probability, one can use the Markov chain of the proof to exactly compute this probability.

6 Conclusion and perspectives

We have studied the decidability of decisiveness with respect to several subclasses of probabilistic counter machines. The results are summarized in the following table. When A is not mentioned it means that A is finite.

model	constant	polynomial	general
pHM	D	D [Th 14]	U [Th 13] even with a single state
pPN	?	U [Th 20] also w.r.t. upward closed sets[Th 20]	U but D when regular [Th 22]
pCM	U [Th 12]	U	U

In the future, apart for solving the left open problem in the above table, we plan to introduce sufficient conditions for decisiveness for models with undecidability of decisiveness like pPNs with polynomial weights. This could have a practical impact for real case-study modellings.

In another direction, we have established that the decisiveness and recurrence properties are closely related. It would be interesting to define a property related to transience in Markov chains. In fact we have identified such a property called divergence and the definition and analysis of this property will appear in a forthcoming paper.

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