Aczel-Mendler Bisimulations in a Regular Category

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Abstract

Aczel-Mendler bisimulations are a coalgebraic extension of a variety of computational relations between systems. It is usual to assume that the underlying category satisfies some form of axiom of choice, so that the theory enjoys desirable properties, such as closure under composition. In this paper, we accommodate the definition in a general regular category – which does not necessarily satisfy any form of axiom of choice. We show that this general definition 1) is closed under composition without using the axiom of choice, 2) coincides with other types of coalgebraic formulations under milder conditions, 3) coincides with the usual definition when the category has the regular axiom of choice. We then develop the particular case of toposes, where the formulation becomes nicer thanks to the power-object monad, and extend the formalism to simulations. Finally, we describe several examples in Stone spaces, toposes for name-passing, and modules over a ring.

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1 Introduction

Bisimulations and coalgebra have a rich literature and theory (see for example the textbook [13]). They cover a large variety of systems: non-deterministic, probabilistic [8, 7], quantum [1], name-passing [22], Kripke models [5], and so on. The reason for this success is that, if the underlying notions are on very different types of systems, those share common grounds: relations with logic, games, fixpoints, or even some form of decidability that have the same flavour. This suggested that those theories could be abstracted away into a meta-theory that would witness the essence of these common grounds.

In the present paper, we are interested in Aczel-Mendler bisimilarity [2], which defines a bisimulation as an abstract relation (that is, a subobject of a product) which itself carries a structure of coalgebra, and from which we can recover the coalgebra structures of the systems we are comparing by projections. This abstract notion has the privilege to be very close to usual notions of bisimulations in terms of relations, but this comes with the cost that they are too set-flavoured. For example, some basic properties (such as closure under composition, or their relation to bisimulation maps) only hold when the underlying category has some form of axiom of choice.

These issues prevent the usage of Aczel-Mendler bisimulations in some interesting categories. Regular categories, and particularly toposes, are a class of categories which enjoy very nice properties, and particularly that they have a very convenient theory of relations, crucial for abstract bisimulations. However, they do not satisfy the axiom of choice. This is the case for example of the effective topos [12], which encompass in a category concepts such as decidable sets and computable functions, or the topos of nominal sets [18] which models

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name-passing, or more generally, infinite systems that have some form of decidability. Being able to abstract bisimulations in such categories then becomes crucial, as a possible way to obtain some general decidability results.

The rest of the paper is organised as follows. In Section 2, we recall some necessary knowledge about relations in a general category and allegories, and particularly maps. In Section 3, we recall the definition of Aczel-Mendler bisimulations, and some of their properties that only hold under some form of the axiom of choice. We then extend them to *regular AM-bisimulations* that work nicely in any regular category. In Section 4, we describe a nicer reformulation of regular AM-bisimulations in toposes, thanks to the power-object monad. In Section 5, we extend this nicer formulation to simulations. Finally, in Section 6, we investigates examples of regular AM-bisimulations, for Stone spaces, toposes of name-passing, and for linear weighted systems.

Contributions

Our contributions can be summarised as follows: 1) An extension of the theory of Aczel-Mendler bisimulations that works in any regular category, without any usage of the axiom of choice. In particular, we prove that the closure under composition (Proposition 27) and the coincidence with other notions of coalgebraic bisimulations (Theorem 29) does not utilise the axiom of choice. 2) A nicer formulation in the case of toposes thanks to the power-object monads, whose connection to tabulations of coalgebra homomorphisms can be proved (Corollary 38), again without the axiom of choice. 3) We extend this nicer formulation to simulations in a topos (Section 5).

Background and Related Work

Section 2 is a summary of what is needed from the textbook [9] about allegories and particularly allegories of relations. Applications of allegories, and their extensions, to computer science cover fuzzy logic [25], compilation of logic programs [3], and generic programming [4]. Topos theory has a rich literature on different aspects. We recommend [14, 15] for a thorough reference on the matter. Coalgebra theory, and particularly bisimulations for them, also has a recent rich literature. Most of the development in this paper about bisimulations relies on concepts that can be found in the textbook [13]. A careful comparison between various notion of coalgebraic bisimilarities has been done in [23]. Aczel-Mendler bisimulations can be traced back to [2]. Simulations has been studied in the coalgebraic language in [11] for example. The connection with bisimulation and simulation maps in a categorical framework is also the core of the theory of open maps [16, 26].

Notations

Given two morphisms $f : X \longrightarrow Y$ and $g : X' \longrightarrow Y'$ in a category with binary products, we denote the pairing by $\langle f, g\rangle : X \longrightarrow Y \times Y'$ (if $X = X'$), and the product by $f \times g : X \times X' \longrightarrow Y \times Y'$.

2 Allegory of Relations

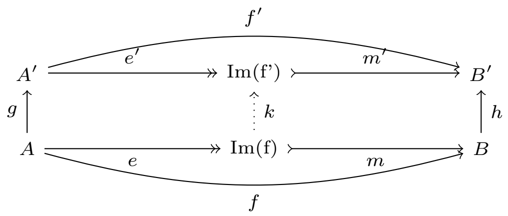
In this section, we cover the general notion of relations in a category, and in particular that they form a tabular allegory. Definitions, propositions, and proofs can be found in [9].

2.1 Subobjects and Factorisations

In this paper, subobjects will play a crucial role throughout. Let us then spend some time on their definition. Fix an object A of $\mathcal{C}$. There is a preorder on the class of monos of the form $m : X \rightarrowtail A$ defined by $m : X \rightarrowtail A \sqsubseteq m' : X' \rightarrowtail A$ if and only if there is a morphism $u : X \longrightarrow X'$ such that $m' \cdot u = m$. In this case, u is unique and is a mono. A *subobject of* A is then an equivalence class of monos with $m : X \rightarrowtail A \equiv m' : X' \rightarrowtail A$ if $m \sqsubseteq m'$ and $m \sqsupseteq m'$, that is, there are u and u' such that $m' \cdot u = m$ and $m \cdot u' = m'$. In this case, u and u' are inverse of each other. The preorder on the monos becomes a partial order on subobjects, also denoted by $\sqsubseteq$. Throughout the paper, when reasoning on subobjects, we will instead reason on a representing mono. This is harmless when dealing with notions such as pullbacks and factorisations that are unique only up to isomorphims.

▶ **Example 1.** In **Set**, since monos are injective functions, subobjects of a set are in bijection with its subsets. The order $\sqsubseteq$ then corresponds to the usual inclusion $\subseteq$ of sets.

Given a morphism $f : A \longrightarrow B$, there is a particular subobject of B called the *image of* f. In general, it is defined as the smallest (for $\sqsubseteq$) subobject $\mathrm{Im}(f)$ of B such that f can be factorised as $m \cdot e$, where m is any representing mono. The existence of the image is not guaranteed in general. It is however when the category $\mathcal{C}$ has a nice (epi,mono)-factorisation system, as it is the case for regular categories (and so for toposes). In a regular category, every morphism f can be uniquely (up to unique isomorphism) factorised as $m \cdot e$, where m is a mono and e is a regular epi, and furthermore, this factorisation is the image factorisation. In addition, this factorisation is functorial and is preserved by pullbacks, meaning that if we have a commutative diagram of the following form (outer rectangle):



there is a (dotted) morphism that makes the two squares commute, and if the outer rectangle is a pullback, then the rightmost square is also a pullback.

▶ Remark 2 (Pullbacks vs. weak-pullbacks). The preservation of images by pullbacks and the functoriality also imply the preservation of images by weak pullbacks, in the sense that, if the outer rectangle is a weak pullback, then the rightmost square is also a weak pullback.

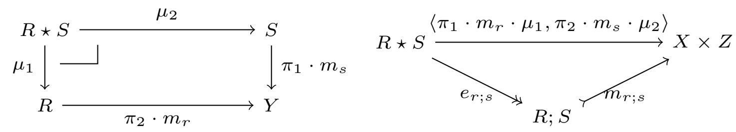
▶ **Example 3.** In **Set**, the image of a function is the usual notion of image, that is, the subset $\{f(a) \mid a \in A\}$ of B. Since **Set** is regular, and regular epis are surjective functions, the image factorisation is given by the (surjection,injection)-factorisation of the function f.

2.2 Relations in a Regular Category

From now on, let us assume that the category $\mathcal{C}$ is regular, that is, it has finite limits and a pullback-stable (regular epi,mono)-factorisation as described in the previous section. Everything in this section can be done in locally regular category, but less conveniently. In general:

▶ **Definition 4.** *A* relation from X to Y *is a subobject of* $X \times Y$*.*

Objects of $\mathcal{C}$ and relations between them form a category, denoted by $\mathbf{Rel}(\mathcal{C})$. The composition is defined as follows. Let $m_r : R \rightarrowtail X \times Y$ and $m_s : S \rightarrowtail Y \times Z$ be two monos, representing two relations, r from X to Y and s from Y to Z. Form the following pullback and (regular epi,mono)-factorisation:



The composition $r; s$ from X to Z is then the subobject represented by the mono part $m_{r;s}$.

▶ Remark 5 (Pullbacks vs. weak pullbacks, continued). In the definition of the composition, we chose to form a pullback, because we know it exists. However, the definition is unchanged if we take any weak pullback instead.

The *identity relation* Δ_X is represented by the diagonal $\langle \mathrm{id}, \mathrm{id} \rangle : X \rightarrowtail X \times X$.

▶ **Proposition 6.** $\mathbf{Rel}$*(*$\mathcal{C}$*) is a category.*

▶ **Example 7.** In **Set**, the composition of relations is the usual one:

$$R; S = \{(x, z) \in X \times Z \mid \exists y \in Y. (x, y) \in R \wedge (y, z) \in S\},$$

while the identity relation is the usual diagonal $\Delta_X = \{(x, x) \mid x \in X\}$.

Of course, $\mathbf{Rel}(\mathcal{C})$ has much more structure. First, since subobjects are naturally ordered by $\sqsubseteq$, and that this order is compatible with the composition, $\mathbf{Rel}(\mathcal{C})$ has a structure of locally ordered 2-category. Furthermore, it comes equipped with an anti-involution which makes it into a *dagger 2-poset*. This means there is a functor $(\_)^\dagger : \mathbf{Rel}(\mathcal{C})^{op} \longrightarrow \mathbf{Rel}(\mathcal{C})$ such that for every relation R, $R^{\dagger\dagger} = R$ and for every other relation S with $R \sqsubseteq S$, $R^\dagger \sqsubseteq S^\dagger$. This involution is given by the inverse of a relation, as follows. If the relation r is represented by the mono $m_r : R \rightarrowtail X \times Y$, then $r^\dagger$ is represented by $m_{r^\dagger} = \langle \pi_2, \pi_1 \rangle \cdot m_r : R \rightarrowtail Y \times X$. Finally, the meet of two relations for the partial order $\sqsubseteq$ is defined and is called the intersection. Given $m_r : R \rightarrowtail X \times Y$ and $m_s : S \rightarrowtail X \times Y$ representing r and s respectively, the intersection $r \cap s$ is then represented by the pullback of m_r and m_s. Altogether:

▶ **Theorem 8.** $\mathbf{Rel}$*(*$\mathcal{C}$*) is an allegory, meaning that all this data satisfies the modular law:*

$$(R; S) \cap T \sqsubseteq (R \cap (T; S^\dagger)); S.$$

▶ **Example 9.** In **Set**, $R^\dagger$ is the usual inverse of the relation R: $R^\dagger = \{(y, x) \mid (x, y) \in R\}$. The intersection $\cap$ is the intersection of relations as sets. Finally, the modular law is trivial in $\mathbf{Rel}(\mathbf{Set})$. More generally, this law is crucial to make adjoints in an allegory behave like direct/inverse images, (see next section, and the Frobenius reciprocity [17]).

2.3 Maps in Allegories

From an allegory (intuitively of relations), it is possible to recover the morphisms of the original category through the notion of *maps*. In a general allegory $\mathcal{A}$, a map is a morphism which is a left adjoint (in the 2-categorical sense). Maps form a subcategory of $\mathcal{A}$ denoted by $\mathbf{Map}(\mathcal{A})$. In the case of an allegory of relations:

▶ **Theorem 10.** $\mathbf{Map}$*(*$\mathbf{Rel}$*(*$\mathcal{C}$*)) is isomorphic to* $\mathcal{C}$*.*

The reason for it is that maps (left adjoints) in $\mathbf{Rel}(\mathcal{C})$ are precisely the relations represented by a mono of the form $\langle \mathrm{id}, f\rangle$ for some morphism f of $\mathcal{C}$, justifying the remark from Example 9 that left adjoints in an allegory behave like direct images. Similarly, their right adjoints are relations represented by $\langle f, \mathrm{id}\rangle$, corresponding to inverse images. This also implies that $\mathbf{Rel}(\mathcal{C})$ is *tabular*, that is, it is generated by maps in the following sense. A *tabulation* of a morphism $\phi : X \longrightarrow Y$ in an allegory is a pair of maps $\psi : Z \longrightarrow X$ and $\xi : Z \longrightarrow Y$ such that $\phi = \psi^\dagger;\xi$ and $\psi;\psi^\dagger \cap \xi;\xi^\dagger = \mathrm{id}_Z$.

▶ **Theorem 11.** *In an allegory of relations, the tabulations of a relation R are exactly those pairs of relations (S, T) represented by monos of the form $\langle id, f\rangle$ and $\langle id, g\rangle$ respectively, with f and g jointly monic, and such that $R = S^\dagger;T$. In particular, every relation has a tabulation, that is, $\mathbf{Rel}(\mathcal{C})$ is tabular.*

The intuition of this theorem is that relations are precisely jointly monic spans.

▶ **Example 12.** In **Set**, maps are graphs of functions, that is, relations of the form $\{(x, f(x)) \mid x \in X\}$ for some function $f : X \longrightarrow Y$. Consequently, every relation R is the same as the span of $f : R \longrightarrow X$ $(x, y) \mapsto x$ and $g : R \longrightarrow Y$ $(x, y) \mapsto y$, that is, $R = \{(f(r), g(r)) \mid r \in R\}$.

3 Aczel-Mendler Bisimulations, in Regular Categories

We now start investigating our original problem: a nice general theory of bisimulations in terms of relations. The development of this section will start with the notion of Aczel-Mendler bisimulations [2], where systems are described as coalgebras. We will witness that one bottleneck of this theory is the role of the axiom of choice that is necessary to prove even some basic properties of this notion of bisimulation. This prevents to use this notion in most regular categories. We will then show that we can fix this issue by a careful use of relations.

3.1 Systems as Coalgebras

In this section, we briefly recall coalgebras, and how to model systems with them. For a more complete introduction, see for example [13].

Coalgebras require two ingredients: a category $\mathcal{C}$ that describes the type of state spaces of our systems and an endofunctor F on $\mathcal{C}$ that describes the type of allowed transitions. A *coalgebra* is then a morphism of type $\alpha : X \longrightarrow FX$. Intuitively, X is the state space of the system and α maps a state to the collection of transitions from this state.

▶ **Example 13.** For example, deterministic transition systems labelled in the alphabet Σ can be modelled with the **Set**-functor $X \mapsto \Sigma \Rightarrow X$, mapping X to the set of functions from Σ to X. A coalgebra for this functor is a function $X \to \Sigma \Rightarrow X$. It maps a state to a function from Σ to X, describing what is the next state after reading a particular letter. Non-deterministic labelled transition systems can be described using the functor $X \mapsto \mathcal{P}(\Sigma \times X)$. A coalgebra then maps a state to a set of transitions, given by a letter and a state, describing the states we can reach from another state reading a particular letter. Another typical example are probabilistic systems, that can be described using the distribution functor $\mathcal{D}$. A transition for those systems is then a distribution on the states, describing what is the probability to reach a state in the next step.

A *morphism of coalgebras* from $\alpha : X \longrightarrow FX$ to $\beta : Y \longrightarrow FY$ is a morphism $f : X \longrightarrow Y$ of $\mathcal{C}$ such that $\beta \cdot f = Ff \cdot \alpha$. Coalgebras on F and morphisms of coalgebras form a category, which we denote by $\mathbf{CoAlg}(F)$.

3.2 Aczel-Mendler Bisimulations of Coalgebras

In this section, we follow closely the development of [13]. We recall the definition of Aczel-Mendler bisimulations and give some of their properties.

▶ **Definition 14.** *We say that a relation is an* Aczel-Mendler bisimulation *(AM-bisimulation for short) from the coalgebra* $\alpha : X \longrightarrow FX$ *to* $\beta : Y \longrightarrow FY$*, if for any mono* $r : R \rightarrowtail X \times Y$ *representing it, there is a morphism* $W : R \longrightarrow FR$*, called* witness*, such that:*

$$\begin{array}{ccccc} & & X\times Y & \xrightarrow{\alpha\times\beta} & F(X)\times F(Y)\\ R & {\nearrow^{r}}\;{\searrow_{W}} & & & \uparrow{\scriptstyle\langle F\pi_1,F\pi_2\rangle}\\ & & FR & \xrightarrow[Fr]{} & F(X\times Y)\end{array}$$

▶ **Example 15.** In the case of non-deterministic labelled transition systems, AM-bisimulations correspond to usual strong bisimulations. The function W maps a pair (x, y) of states of α and β to a subset of triples (a, x', y') such that $(a, x') \in \alpha(x)$, $(a, y') \in \beta(y)$, and $(x', y') \in R$. The commutation means that the set of transitions $\alpha(x)$ from x exactly corresponds to the set $\{(a, x') \mid \exists y'. (a, x', y') \in W(x, y)\}$, and similarly for y. This implies the property of a bisimulation: if there is a transition (a, x') from x, then there is a transition (a, y') from y with $(x', y') \in R$; and vice versa.

We show now that AM-bisimulations behave well under the regular axiom of choice:

▶ **Proposition 16.** *Assume that* $\mathcal{C}$ *has the regular axiom of choice, that is, every regular epi is split, and that* F *preserves weak pullbacks. Then the following is a dagger 2-poset, denoted by* $\mathbf{Bis}(F)$*: objects are coalgebras on* F*, morphisms are AM-bisimulations,* $\sqsubseteq$*, identities, composition, and* $(\_)^\dagger$ *are defined as in* $\mathbf{Rel}(\mathcal{C})$*. That is, diagonals are AM-bisimulations, and AM-bisimulations are closed under composition and inverse.*

Proof. Let us focus on proving that Aczel-bisimulations are closed under composition. We then have two witnesses:

$$\begin{array}{ccccc} & & X\times Y & \xrightarrow{\alpha\times\beta} & F(X)\times F(Y)\\ R_1 & {\nearrow^{r_1}}\;{\searrow_{W_1}} & & & \uparrow{\scriptstyle\langle F\pi_1,F\pi_2\rangle}\\ & & FR_1 & \xrightarrow[Fr_1]{} & F(X\times Y)\end{array} \qquad \begin{array}{ccccc} & & Y\times Z & \xrightarrow{\beta\times\gamma} & F(Y)\times F(Z)\\ R_2 & {\nearrow^{r_2}}\;{\searrow_{W_2}} & & & \uparrow{\scriptstyle\langle F\pi_1,F\pi_2\rangle}\\ & & FR_2 & \xrightarrow[Fr_2]{} & F(Y\times Z)\end{array}$$

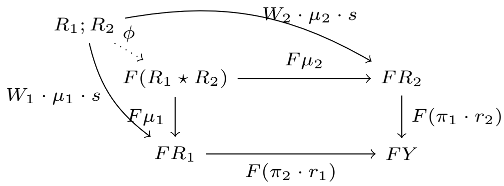
We then want to construct a morphism $W : R_1; R_2 \longrightarrow F(R_1; R_2)$ such that

$$\begin{array}{ccccc} & & X\times Z & \xrightarrow{\alpha\times\gamma} & F(X)\times F(Z)\\ R_1;R_2 & {\nearrow^{r_1;r_2}}\;{\searrow_{W}} & & & \uparrow{\scriptstyle\langle F\pi_1,F\pi_2\rangle}\\ & & F(R_1;R_2) & \xrightarrow[F(r_1;r_2)]{} & F(X\times Z)\end{array}$$

Since F preserves weak pullbacks and by definition of composition, we have the following weak pullback and (regular epi,mono)-factorisation:

$$\begin{array}{ccc} F(R_1\star R_2) & \xrightarrow{F\mu_2} & FR_2\\ {\scriptstyle F\mu_1}\downarrow & \lrcorner & \downarrow{\scriptstyle F(\pi_1\cdot r_2)}\\ FR_1 & \xrightarrow[F(\pi_2\cdot r_1)]{} & FY\end{array} \qquad \begin{array}{ccc} R_1\star R_2 & \xrightarrow{\langle \pi_1\cdot r_1\cdot\mu_1,\, \pi_2\cdot r_2\cdot\mu_2\rangle} & X\times Z\\ & {\searrow_{e_{r_1;r_2}}}\quad{\nearrow_{r_1;r_2}} & \\ & R_1;R_2 & \end{array}$$

Denote by s the section of $e_{r_1;r_2}$, which exists by the regular axiom of choice. By the universal property of weak pullbacks, we have $\phi : R_1;R_2 \longrightarrow F(R_1 \star R_2)$, such that



Now $W = Fe_{r_1;r_2} \cdot \phi$ is the expected witness:

$$\begin{aligned}
\langle F\pi_1, F\pi_2\rangle \cdot F(r_1;r_2) \cdot W &= \langle F\pi_1, F\pi_2\rangle \cdot F(r_1;r_2) \cdot Fe_{r_1;r_2} \cdot \phi \quad \text{(definition of } W\text{)} \\
&= \langle F\pi_1, F\pi_2\rangle \cdot F\langle \pi_1 \cdot r_1 \cdot \mu_1, \pi_2 \cdot r_2 \cdot \mu_2\rangle \cdot \phi \\
&\qquad \text{(definition of } r_1;r_2\text{)} \\
&= F(\pi_1 \cdot r_1) \times F(\pi_2 \cdot r_2) \cdot \langle F(\mu_1) \cdot \phi, F(\mu_2) \cdot \phi\rangle \\
&\qquad \text{(computation on products)} \\
&= F(\pi_1 \cdot r_1) \times F(\pi_2 \cdot r_2) \cdot \langle W_1 \cdot \mu_1 \cdot s, W_2 \cdot \mu_2 \cdot s\rangle \\
&\qquad \text{(definition of } \phi\text{)} \\
&= \alpha \times \gamma \cdot \langle \pi_1 \cdot r_1 \cdot \mu_1, \pi_2 \cdot r_2 \cdot \mu_2\rangle \cdot s \\
&\qquad \text{(definition of the } W_i \text{ and computation on products)} \\
&= \alpha \times \gamma \cdot (r_1;r_2) \quad \text{(definition of } s\text{)}
\end{aligned}$$

◀

▶ Remark 17. The preservation of weak pullbacks is a crucial property for a functor related to relations. More surprisingly, the dependence on the axiom of choice is necessary for proving the closure under composition. This was already observed in [13, 23].

In the proof we make the following usage of the regular axiom of choice: we need that the epi part $e_{r_1;r_2} : R_1 \star R_2 \longrightarrow\!\!\!\!\!\rightarrow R_1;R_2$ of a (regular epi,mono)-factorisation to be split, that is, has a section $s : R_1;R_2 \rightarrowtail R_1 \star R_2$. In **Set**, $R_1 \star R_2$ is given by triples (x, y, z) such that $(x, y) \in R_1$ and $(y, z) \in R_2$, so this section is then a choice of such a y for every (x, z) in the composition. This kind of choice is usual for example to prove that strong bisimulations are closed under composition: assuming that one has a transition (a, x') from x, to prove that one also has such a transition from z, one should pick an intermediate y, prove that there is such a transition for y using that R_1 is a bisimulation, then concluding using the fact that R_2 is a bisimulation.

In this dagger 2-poset of bisimulations, we can also talk about maps and tabulations, as we did for relations. Furthermore, since the 2-categorical structure of $\mathbf{Bis}(F)$ is given by that of $\mathbf{Rel}(\mathcal{C})$, and particularly that the local posets of bisimulations are embedded in the corresponding local poset of relations, results from Section 2.3 can be used here. In particular, we can prove the following:

▶ **Theorem 18.** *$\mathbf{Map}(\mathbf{Bis}(F))$ is isomorphic to $\mathbf{CoAlg}(F)$.*

Using results from Section 2.3, proving this theorem boils down to proving that bisimulations that are maps are precisely graphs of coalgebra morphisms:

▶ **Proposition 19.** *A morphism $h : X \longrightarrow Y$ of $\mathcal{C}$ is a coalgebra morphism from α to β if and only if the mono $\langle id, h\rangle : X \rightarrowtail X \times Y$ represents an AM-bisimulation from α to β.*

Using this characterisation of maps for AM-bisimulations, and using the tabularity of the allegory of relations, we can prove that an AM-bisimulation can be described as a span of morphism of coalgebras, under some form of axiom of choice (see [13]). We can formulate this in terms of tabulations:

▶ **Proposition 20.** *If U is an AM-bisimulation from α to β, and if $f : Z \longrightarrow X$, $g : Z \longrightarrow Y$ is a tabulation of U, then there is a coalgebra structure γ on Z such that f is a coalgebra morphism from γ to α and g is a coalgebra morphism from γ to β.*

▶ **Corollary 21.** *Assume $\mathcal{C}$ has the regular axiom of choice. Assume given two coalgebras $\alpha : X \longrightarrow F(X)$ and $\beta : Y \longrightarrow F(Y)$, and two points $p : * \longrightarrow X$ and $q : * \longrightarrow Y$. There is an AM-bisimulation $r : R \rightarrowtail X \times Y$ from α to β, and a point $c : * \longrightarrow R$ such that $r \cdot c = \langle p, q\rangle$ if and only if there is a span $X \xleftarrow{f} Z \xrightarrow{g} Y$, an F-coalgebra structure γ on Z such that f is a coalgebra morphism from γ to α and g from γ to β, and a point $w : * \longrightarrow Z$ such that $f \cdot w = p$ and $g \cdot w = q$.*

▶ Remark 22. Here $*$ is usually the final object of $\mathcal{C}$, but it can be any object used to describe initial states in the systems under consideration.

3.3 Picking vs. Collecting: AM-Bisimulations for Regular Categories

We have seen that several results about AM-bisimulations depend on the regular axiom of choice, preventing its use in more exotic toposes and regular categories. Actually, the only occurrences are of similar flavour: one wants to prove some property of elements (x, z) in a composition of relations, and for that, one has to pick a witness y in between. The main idea of our proposal is that, instead of picking a witness (which would require the axiom of choice), it is enough to collect all the witnesses, prove properties about all of them, and make sure that there is enough of them. This can be done in any regular category as follows:

▶ **Definition 23.** *We say that a relation is a* regular AM-bisimulation *from the coalgebra $\alpha : X \longrightarrow FX$ to $\beta : Y \longrightarrow FY$, if for any mono $r : R \rightarrowtail X \times Y$ representing it, there is another relation represented by $w : W \rightarrowtail FR \times R$ such that $\pi_2 \circ w$ is a regular epi and:*

$$\begin{array}{ccccccc}
 & \xrightarrow{\pi_2 \circ w} & R & \xrightarrow{r} & X \times Y & \xrightarrow{\alpha \times \beta} & \\
W & & & & & & F(X) \times F(Y) \\
 & \xrightarrow{\pi_1 \circ w} & FR & \xrightarrow{Fr} & F(X \times Y) & \xrightarrow{\langle F\pi_1, F\pi_2\rangle} &
\end{array}$$

The intuition is as follows: W collects all the witnesses that R is a bisimulation. In particular, for a given pair (x, y) in R, there might be several witnesses. The fact $\pi_2 \circ w$ is a regular epi guarantees that every pair of R has at least one witness. Of course, we have to prove that this extends plain AM-bisimulations:

▶ **Proposition 24.** *If $\mathcal{C}$ is a regular category with the regular axiom of choice, then a relation is a regular AM-bisimulation if and only if it is a AM-bisimulation.*

Also, regular bisimulations are closed under composition. This requires a milder condition on F as already observed in [23].

▶ **Definition 25.** *We say that F* covers pullbacks *if for every pair of pullbacks:*

$$\begin{array}{ccc}
R & \xrightarrow{v} & Y \\
{\scriptstyle u}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\
X & \xrightarrow[f]{} & Z
\end{array}
\qquad\qquad
\begin{array}{ccc}
R' & \xrightarrow{v'} & FY \\
{\scriptstyle u'}\downarrow & \lrcorner & \downarrow{\scriptstyle Fg} \\
FX & \xrightarrow[Ff]{} & FZ
\end{array}$$

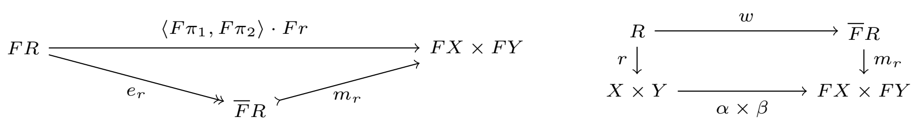
the unique morphism $\gamma : FR \longrightarrow R'$ such that $u' \circ \gamma = Fu$ and $v' \circ \gamma = Fv$ is a regular epi.

▶ Remark 26. When F preserves weak pullbacks, then F covers pullbacks. When $\mathcal{C}$ has the regular axiom of choice, then both notions coincide.

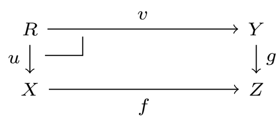
▶ **Proposition 27.** *When F covers pullbacks, then regular AM-bisimulations are closed under composition.*

In [23], Staton described conditions for several coalgebraic notions of bisimulations to coincide. In this picture, AM-bisimulations were quite weak, as they would coincide with other notions only under some form of axiom of choice (again). Here, we will show that the picture is much nicer with regular AM-bisimulations.

▶ **Definition 28.** *A relation from X to Y is a* Hermida-Jacobs bisimulation *(HJ-bisimulation for short) from $\alpha : X \longrightarrow FX$ to $\beta : Y \longrightarrow FY$ if if there is a mono $r : R \rightarrowtail X \times Y$ representing it and a morphism $w : R \longrightarrow \overline{F}R$ where $\overline{F}R$ is obtained by the (epi,mono)-factorisation on the left, and such that the square on the right commutes:*



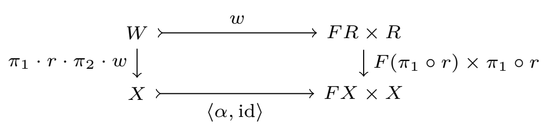
A relation is a behavioural equivalence *from $\alpha : X \longrightarrow FX$ to $\beta : Y \longrightarrow FY$ if it is represented by a pullback of coalgebra homomorphisms, that is, if there are a coalgebra $\gamma : Z \longrightarrow FZ$ and two coalgebra homomorphisms $f : \alpha \longrightarrow \gamma$ and $g : \beta \longrightarrow \gamma$ such that the mono $\langle u, v\rangle : R \rightarrowtail X \times Y$ obtained from their pullback in $\mathcal{C}$ represents it.*



▶ **Theorem 29.** *Assume that $\mathcal{C}$ is a regular category. Then:*

- *a relation is a regular AM-bisimulation if and only if it is a HJ-bisimulation,*
- *if $\mathcal{C}$ has pushouts, then a regular AM-bisimulation is included in a behavioural equivalence,*
- *if F covers pullbacks, then a behavioural equivalence is a regular AM-bisimulation.*

The last two bullets are a consequence of the first bullet and [23]. In Section 3.2, we described that AM-bisimilarity coincides with the existence of a span of coalgebra homomorphisms. This can also be formulated in the context of regular AM-bisimulations. The witness $w : W \rightarrowtail FR \times R$ can be seen as a coalgebra in $\mathbf{Rel}(\mathcal{C})$ (although F is technically not a functor on it). The coalgebra $\alpha : X \longrightarrow FX$ can also be seen as a coalgebra in $\mathbf{Rel}(\mathcal{C})$ as $\langle \alpha, \mathrm{id}\rangle : X \rightarrowtail FX \times X$. Then $\pi_1 \circ r$ can be seen as a coalgebra homomorphism from w to α, since the following diagram commutes



4 The Case of Toposes

Here, we investigate the particular case of toposes. The first part of this section recalls folklore about toposes and particularly power-objects, namely, that they form a commutative monad whose Kleisli category is isomorphic to the category of relations. Finally, we will show that regular AM-bisimulations can be formulated much more nicely in this context.

4.1 Toposes, as Relation Classifiers

▶ **Definition 30.** *A topos is a finitely complete category with power objects. The latter condition means that for every object X, there is a mono $\in_X \colon E_X \rightarrowtail X \times \mathcal{P}X$ such that for every mono of the form $m : R \rightarrowtail X \times Y$ there is a unique morphism $\xi_m : Y \longrightarrow \mathcal{P}X$ such that there is a pullback diagram of the form:*

$$\begin{array}{ccc} R & \xrightarrow{\quad\theta_m\quad} & E_X \\ {\scriptstyle m}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \in_X} \\ X \times Y & \xrightarrow[\quad id \times \xi_m\quad]{} & X \times \mathcal{P}X \end{array}$$

This formulation passes to relations since $\xi_m = \xi_{m'}$ if and only if m and m' represent the same relation r. In that case, we will write ξ_r for $\xi_m = \xi_{m'}$. Another formulation of toposes uses sub-object classifiers which can be recovered as $\mathbb{T} = \in_{\mathbf{1}} \colon \mathbf{1} \simeq E_{\mathbf{1}} \to \mathbf{1} \times \mathcal{P}\mathbf{1} \simeq \mathcal{P}\mathbf{1} = \Omega$. The formulation by power-objects implies that a topos is cartesian closed, which is not the case of the sub-object classifier alone. Conversely, $\mathcal{P}X$ is equal to Ω^X and $\in_X$ is any mono corresponding to the evaluation morphism $X \times \Omega^X \to \Omega$ of the cartesian-closed structure.

▶ **Example 31.** In **Set**, $\mathcal{P}X$ is given by the usual power-set and E_X is the subset of $X \times \mathcal{P}X$ consisting of pairs (x, U) such that $x \in U$. In **Scha** – the Schanuel topos **Scha** [18], equivalent to the category of nominal sets and equivariant functions – $\mathcal{P}X$ is the nominal set of finitely supported subsets of X. In **Eff** – the effective topos [12], intuitively, the category of effective set and computable functions – $\mathcal{P}X$ is intuitively given by the set of decidable subsets of X (although the formal description is much harder).

4.2 The Power-Object Monad

The following is a folklore result about power-objects, that can be proved for example by noticing that the proof in **Set** does not use either the law of excluded-middle nor the axiom of choice and the fact that any such statement is true in any topos:

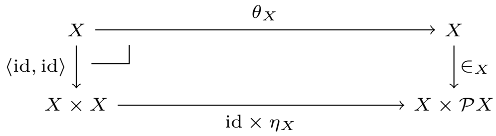
▶ **Theorem 32.** *In a topos $\mathcal{C}$, $\mathcal{P}$ extends to a commutative monad whose Kleisli category is isomorphic to the category of relations* $\mathbf{Rel}(\mathcal{C})$.

Let us describe some parts of this statement that will be useful in the following discussion. First, the structure of *covariant* functor (not to be confused with the more traditional contravariant structure) is given as follows. Given a morphism $f : X \longrightarrow Y$, $\mathcal{P}f : \mathcal{P}X \longrightarrow \mathcal{P}Y$ is defined as follows. Consider first the following (epi,mono)-factorisation:

$$\begin{array}{ccccc} E_X & & \xrightarrow{\quad (f \times \mathrm{id}) \cdot \in_X \quad} & & Y \times \mathcal{P}X \\ & {\scriptstyle e_f} \searrow & & \nearrow {\scriptstyle m_f} & \\ & & E_f & & \end{array}$$

Then $\mathcal{P}f : \mathcal{P}X \longrightarrow \mathcal{P}Y$ is the unique morphism corresponding to m_f.

The unit $\eta_X : X \longrightarrow \mathcal{P}X$ is defined as ξ_{Δ_X}, that is, the unique morphism such that there is a pullback of the form:

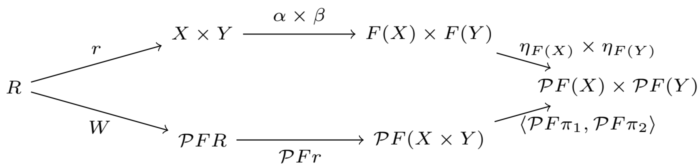


for some θ_X. The multiplication $\mu_X : \mathcal{P}\mathcal{P}X \longrightarrow \mathcal{P}X$ is defined as the unique morphism associated with the composition of relations $\in_X ; \in_{\mathcal{P}X}$.

4.3 AM-Bisimulations in a Topos

Since toposes are regular categories, the notion of regular AM-bisimulation makes sense. We show here that it can be reformulated as follows.

▶ **Definition 33.** *We say that a relation is a* toposal AM-bisimulation *from the coalgebra $\alpha : X \longrightarrow FX$ to $\beta : Y \longrightarrow FY$, if for any mono $r : R \rightarrowtail X \times Y$ representing it, there is a morphism $W : R \longrightarrow \mathcal{P}FR$ such that:*



In other words, an F-toposal AM-bisimulation between α and β is a $\mathcal{P}F$-AM-bisimulation between $\eta \cdot \alpha$ and $\eta \cdot \beta$. Intuitively, this means that toposal bisimulations look at systems as non-deterministic. This allows to *collect* witnesses as a morphism $W : R \longrightarrow \mathcal{P}FR$ instead of picking some, very much like regular AM-bisimulations.

We have to make sure that toposal and regular AM-bisimulations coincide.

▶ **Proposition 34.** *Assume that $\mathcal{C}$ is a topos. Then for every relation U from X to Y, every coalgebra $\alpha : X \longrightarrow FX$ and $\beta : Y \longrightarrow FY$, U is a toposal AM-bisimulation from α to β if and only if it is a regular AM-bisimulation between them.*

This nicer formulation allows us to prove a much nicer tabularity property, which could only be informally described for regular AM-bisimulations:

▶ **Proposition 35.** *Assume that $\mathcal{C}$ is a topos and that F covers pullbacks. Then the following is a dagger 2-poset: objects are coalgebras on F, morphisms are toposal AM-bisimulations, $\sqsubseteq$, identities, composition, and $(\_)^\dagger$ are defined as in $\mathbf{Rel}(\mathcal{C})$.*

▶ Remark 36. This Proposition is similar to Proposition 16, without the axiom of choice and assuming only that F covers pullbacks, but by replacing plain AM-bisimulations by toposal AM-bisimulations.

Obviously, the category of maps of the dagger 2-poset of toposal bisimulations is then not isomorphic to $\mathbf{CoAlg}(F)$, but to the category of F-coalgebras with $\mathcal{P}F$-coalgebra morphisms between them. Then tabularity can be formulated as follows:

▶ **Proposition 37.** *If U is a toposal bisimulation from the F-coalgebra α to the F-coalgebra β, and if $f : Z \longrightarrow X$, $g : Z \longrightarrow Y$ is a tabulation of U, then there is a $\mathcal{P}F$-coalgebra structure γ on Z such that f is a $\mathcal{P}F$-coalgebra morphism from γ to $\eta_X \cdot \alpha$ and g is a $\mathcal{P}F$-coalgebra morphism from γ to $\eta_Y \cdot \beta$.*

▶ **Corollary 38.** *Assume given two coalgebras $\alpha : X \longrightarrow F(X)$ and $\beta : Y \longrightarrow F(Y)$, and two points $p : * \longrightarrow X$ and $q : * \longrightarrow Y$. There is a toposal bisimulation $r : R \rightarrowtail X \times Y$ from α to β, and a point $c : * \longrightarrow R$ such that $r \cdot c = \langle p, q \rangle$ if and only if there is a span $X \xleftarrow{f} Z \xrightarrow{g} Y$, a $\mathcal{P}F$-coalgebra structure γ on Z, and a point $w : * \longrightarrow Z$ such that f is a $\mathcal{P}F$-coalgebra morphism from γ to $\eta_X \cdot \alpha$, g from γ to $\eta_Y \cdot \beta$, $f \cdot w = p$, and $g \cdot w = q$.*

5 From Bisimulations to Simulations

In this section, we extend the analysis of the previous sections to deal with *simulations*. Classically, simulations for coalgebras require a notion of order on morphisms of the form $X \longrightarrow FY$, to allow one to define that there is fewer transitions coming out of a state than another. This allows to easily modify the definition of AM-bisimulations to obtain *AM-simulations*. We will show that toposal bisimulations can also be extended to simulations in a nice way to mitigate those issues. The only reason we chose to stay in a topos and not in a general regular category is because theorems have a nicer formulation there, but most of the discussion here can be done in a regular category.

5.1 Order-Structure on Functors, and Lax Coalgebra Morphisms

We want to be able to compare two morphisms of the form $X \longrightarrow FY$. So assuming a preorder $\leq$ on each Hom-set $\mathcal{C}(X, FY)$, we can define *lax morphisms of coalgebras*, as follows:

▶ **Definition 39.** *A lax morphism of coalgebras from $\alpha : X \longrightarrow FX$ to $\beta : Y \longrightarrow FY$ is a morphism $f : X \longrightarrow Y$ of $\mathcal{C}$ such that $Ff \cdot \alpha \leq \beta \cdot f$ in $\mathcal{C}(X, FY)$.*

Unfortunately, coalgebras and lax morphisms of coalgebras do not form a category in general, and some axioms are required for the interaction of $\leq$ with the composition.

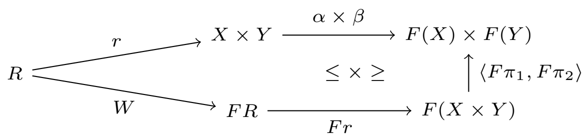
▶ **Definition 40.** *A* good order structure *on F is a preorder $\leq$ on each $\mathcal{C}(X, FY)$ such that: 1) if $\alpha \leq \beta$ in $\mathcal{C}(X, FY)$, $f : X' \longrightarrow X$, and $g : Y \longrightarrow Y'$, then $Fg \cdot \alpha \cdot f \leq Fg \cdot \beta \cdot f$ in $\mathcal{C}(X', FY')$; 2) if $h : X \longrightarrow FZ$, $k : X \longrightarrow FY$, $g : Y \longrightarrow Z$, and $h \leq Fg \cdot k$ in $\mathcal{C}(X, FZ)$, then there is $k' : X \longrightarrow FY$ such that $k' \leq k$ in $\mathcal{C}(X, FY)$ and $h = Fg \cdot k'$.*

▶ **Lemma 41.** *When $\leq$ is a good order structure on F, then coalgebras and lax morphisms of coalgebras form a category, denoted by $\mathbf{CoAlg}_{lax}(F)$.*

▶ **Example 42.** When F is the functor modelling non-deterministic labelled systems and $\leq$ is given by point-wise inclusion, lax morphisms are exactly morphisms in the sense of [16]. Those morphisms are intuitively morphisms whose graphs are simulations. More generally, we will see that lax morphisms are simulation maps.

5.2 AM-Simulations

▶ **Definition 43.** *We say that a relation is an* AM-simulation *from the coalgebra $\alpha : X \longrightarrow FX$ to $\beta : Y \longrightarrow FY$, if for any mono $r : R \rightarrowtail X \times Y$ representing it, there is a morphism $W : R \longrightarrow FR$ such that:*



meaning that $\alpha \cdot \pi_1 \cdot r \leq F\pi_1 \cdot Fr \cdot W$ and $\beta \cdot \pi_2 \cdot r \geq F\pi_2 \cdot Fr \cdot W$.

▶ **Proposition 44.** *When $\leq$ is a good order structure, it is equivalent to require that the left inequality is actually an equality $\alpha \cdot \pi_1 \cdot r = F\pi_1 \cdot Fr \cdot W$.*

▶ **Example 45.** When $F : X \mapsto \mathcal{P}(\Sigma \times X)$, AM-simulations correspond to strong simulations. The left part of the commutativity means that for every $(x, y) \in R$ and $(a, x') \in \alpha(x)$, there is y' such that $(a, (x', y')) \in W(x, y)$. The right part then implies that necessarily $(a, y') \in \beta(y)$.

Much as in the case of AM-bisimulations, diagonals (and actually all AM-bisimulations) are AM-simulations and AM-simulations are closed under composition, only under some conditions. However, they are not closed under inverse. These observations can be encompassed as follows:

▶ **Proposition 46.** *When $\mathcal{C}$ has the regular axiom of choice and F preserves weak pullbacks, then the following is a locally ordered 2-category: objects are F-coalgebras, morphisms are AM-simulations, identitites, compositions, and $\sqsubseteq$ are given by $\mathbf{Rel}(\mathcal{C})$. We denote this category by $\mathbf{Sim}(F)$.*

We can formalise the relationship between lax coalgebra morphisms and simulation maps:

▶ **Theorem 47.** *Maps in $\mathbf{Rel}(\mathcal{C})$ that are AM-simulations are precisely lax morphisms of coalgebra.*

Note that this theorem cannot have a form as nice as Theorem 18 because AM-simulations are not closed under inverse, and the right adjoint of a map has to be its inverse. At this point, we can also describe the tabulations of AM-simulations:

▶ **Proposition 48.** *If U is an AM-simulation from α to β, and if $f : Z \longrightarrow X$, $g : Z \longrightarrow Y$ is a tabulation of U then, there is a coalgebra structure γ on Z such that f is a coalgebra morphism from γ to α and g is a lax coalgebra morphism from γ to β.*

▶ **Corollary 49.** *Assume $\mathcal{C}$ has the regular axiom of choice. Assume given two coalgebras $\alpha : X \longrightarrow F(X)$ and $\beta : Y \longrightarrow F(Y)$, and two points $p : * \longrightarrow X$ and $q : * \longrightarrow Y$. There is an AM-simulation $r : R \rightarrowtail X \times Y$ from α to β, and a point $c : * \longrightarrow R$ with $r \cdot c = \langle p, q \rangle$ if and only if there is a span $X \xleftarrow{f} Z \xrightarrow{g} Y$, an F-coalgebra structure γ on Z such that f is a coalgebra morphism from γ to α and g is a lax coalgebra morphism from γ to β, and a point $w : * \longrightarrow Z$ such that $f \cdot w = p$ and $g \cdot w = q$.*

This formalises some observations that simulations are spans of a bisimulation map and a simulation map (see [24] for examples of this fact in the context of open maps).

5.3 Extending the Order-Structure

In Section 5.1, we started by assuming a relation $\leq$ on the Hom-sets of the form $\mathcal{C}(X, FY)$ satisfying some properties. This good order structure was necessary to prove the properties of Section 5.2. In the coming section, we will pass again from plain to toposal, by considering F-coalgebras as $\mathcal{P}F$-coalgebras. It is then needed to extend good order structures on F to good order structures on $\mathcal{P}F$.

Assume given a relation $\leq$ on all Hom-sets of the form $\mathcal{C}(X, FY)$. We define $\leq_{\mathcal{P}}$ on $\mathcal{C}(X, \mathcal{P}FY)$ as follows. A morphism $f : X \longrightarrow \mathcal{P}FY$ uniquely (up to isos) corresponds to a mono of the form $m_f : R_f \longrightarrow FY \times X$ by definition of $\mathcal{P}$. Then given two morphisms $f, g : X \longrightarrow \mathcal{P}FY$, $f \leq_{\mathcal{P}} g$ if there exist a morphism $u : Z \longrightarrow R_g$ and an epi $e : Z \twoheadrightarrow R_f$ such that: $\pi_1 \cdot m_f \cdot e \leq \pi_1 \cdot m_g \cdot u$ and $\pi_2 \cdot m_f \cdot e = \pi_2 \cdot m_g \cdot u$.

▶ **Example 50.** The order $\leq_{\mathcal{P}}$ seems complicated but can be interpreted easily in **Set**, when the order structure on $\mathcal{C}(X, FY)$ is a point-wise order, assuming that FY itself is preordered. Indeed, given two functions $f, g : X \longrightarrow \mathcal{P}FY$, $f \leq_{\mathcal{P}} g$ if and only if for every $x \in X$, and every $a \in f(x) \subseteq FY$ there is $b \in g(x)$ such that $a \leq b$ in $F(Y)$.

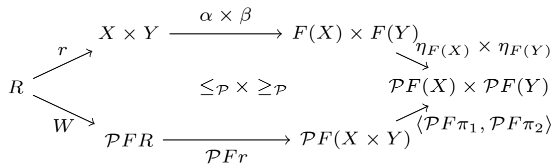
To make it consistent with the previous section, we show that this preserves goodness:

▶ **Proposition 51.** *$\leq_{\mathcal{P}}$ is a good order structure if $\leq$ is.*

5.4 Toposal AM-Simulations

With all those ingredients, we can easily deduce the right notion of *AM toposal-simulations*:

▶ **Definition 52.** *We say that a relation is a* toposal AM-simulation *from the coalgebra $\alpha : X \longrightarrow FX$ to $\beta : Y \longrightarrow FY$, if for any mono $r : R \rightarrowtail X \times Y$ representing it, there is a morphism $W : R \longrightarrow \mathcal{P}FR$ such that:*



Plain and toposal AM-simulations also coincide under the axiom of choice:

▶ **Proposition 53.** *Assume that $\mathcal{C}$ has the regular axiom of choice. Then for every relation U from X to Y, every coalgebra $\alpha : X \longrightarrow FX$ and $\beta : Y \longrightarrow FY$, U is an AM-simulation from α to β if and only if it is a toposal AM-simulation between them.*

Finally, we can prove the closure under composition and the characterisation with spans without the axiom of choice:

▶ **Proposition 54.** *Proposition 46 holds without regular axiom of choice when replacing AM-simulations by toposal AM-simulations.*

▶ **Theorem 55.** *Assume given two coalgebras $\alpha : X \longrightarrow F(X)$ and $\beta : Y \longrightarrow F(Y)$, and two points $p : * \longrightarrow X$ and $q : * \longrightarrow Y$. There is a toposal AM-simulation $r : R \rightarrowtail X \times Y$ from α to β, and a point $c : * \longrightarrow R$ such that $r \cdot c = \langle p, q \rangle$ if and only if there is a span $X \xleftarrow{f} Z \xrightarrow{g} Y$, a $\mathcal{P}F$-coalgebra structure γ on Z such that f is a $\mathcal{P}F$-coalgebra morphism from γ to $\eta_X \cdot \alpha$ and g a lax $\mathcal{P}F$-coalgebra morphism from γ to $\eta_Y \cdot \beta$, and a point $w : * \longrightarrow Z$ such that $f \cdot w = p$ and $g \cdot w = q$.*

6 Examples

In this section, let us develop some examples in different regular categories.

6.1 Vietoris Bisimulations

In [5], Bezhanishvili et al. are studying bisimulations for the Vietoris functor – the functor mapping a topological space to it set of closed subspaces equipped with a suitable topology – in the category **Stone** of Stone spaces and continuous functions. More concretely, they show that so-called descriptive models coincide with coalgebras of the form $X \to \mathcal{V}(X) \times A$ where $\mathcal{V}$ is the Vietoris functor and A is some fixed Stone space (usually, $A = \mathcal{P}S = \prod_{s \in S}\{0, 1\}$

equipped with the product topology and $\{0,1\}$ equipped with the discrete topology). They are interested in describing relation liftings (much as the one defining HJ-bisimulations) that coincide with behavioural equivalences. They actually proved that in this case AM-bisimilarity does not coincide with behavioural equivalence, and that the main reason is because the Vietoris functor does not preserves weak-pullbacks. In [23], Staton proved that the Vietoris functor is a so-called $\mathcal{S}$-powerset functor, and that in particular it covers pullbacks. Together with the (well-known) fact that the category of Stone spaces is regular and has pushouts, Theorem 29 holds in this case, and all three notions – regular AM-bisimulations, HJ-bisimulations, and behavioural equivalences – coincide.

Let us develop the counter-examples described in [5]. Let $\overline{\mathbb{N}}$ being $\mathbb{N}\cup\{\infty\}$, obtained as the Alexandroff-compactification of $\mathbb{N}$ equipped with the discrete topology. Concretely, the open sets of $\overline{\mathbb{N}}$ are $\{U \subseteq \mathbb{N}\} \cup \{U \cup \{\infty\} \mid U \subseteq \mathbb{N} \wedge \exists n \in U. \forall m \geq n. m \in u\}$. Denote $\overline{\mathbb{N}} \oplus \overline{\mathbb{N}} \oplus \overline{\mathbb{N}}$, the coproduct of three copies of $\overline{\mathbb{N}}$, by $3\overline{\mathbb{N}}$. Let us also consider $A = \mathcal{P}(\mathbb{N}\times\{+,-\})$ as above. Define the continuous function $\tau : 3\overline{\mathbb{N}} \longrightarrow \mathcal{V}(3\overline{\mathbb{N}})$ as follows: $\tau(i_1) = \{i_2, i_3\}$ and $\tau(i_2) = \tau(i_3) = \varnothing$, where i_j denotes the j-th copy of $i \in \overline{\mathbb{N}}$. Define two continuous functions $\lambda, \lambda' : 3\overline{\mathbb{N}} \longrightarrow A$ $\lambda(i_1) = \lambda'(i_1) = \{\}$ for all $i \in \overline{\mathbb{N}}$; $\lambda(\infty_j) = \lambda'(\infty_j) = \{\}$ for $j \in \{2,3\}$; $\lambda(i_2) = \lambda'(i_2) = \{i+\}$, $\lambda(i_3) = \lambda'(i_3) = \{i-\}$, for i odd; $\lambda(i_2) = \lambda'(i_3) = \{i+\}$, $\lambda(i_2) = \lambda'(i_3) = \{i-\}$ for i even. Altogether, this defines two coalgebras $\alpha = \langle\tau, \lambda\rangle$ and $\beta = \langle\tau, \lambda'\rangle$. In [5], they proved that the following relation (for Stone spaces, relations coincide with closed subspaces of a product):

$$R = \{(i_1, i_1) \mid i \in \overline{\mathbb{N}}\} \cup \{(i_2, i_2), (i_3, i_3) \mid i \in \mathbb{N} \text{ odd}\} \cup \{(i_2, i_3), (i_3, i_2) \mid i \in \mathbb{N} \text{ even}\} \\ \cup \{(\infty_j, \infty_k) \mid j, k \in \{2,3\}\}$$

is a Vietoris bisimulation but not an AM-bisimulation. We can reformulate this as:

▶ **Theorem 56.** *R is a regular AM-bisimulation but not an AM-bisimulation.*

For the second part of this statement, this means that there is no continuous function $W : R \longrightarrow \mathcal{V}(R) \times A$ satisfying the requirement of an AM-bisimulation. However, there is a relation $W \subseteq R \times \mathcal{V}(R) \times A$ that satisfies the requirement of a regular AM-bisimulation as:

$$\begin{aligned} W = \; & \{((i_1, i_1), \{(i_2, i_2), (i_3, i_3)\}, \{\}) \mid i \in \mathbb{N} \text{ odd}\} \\ & \cup \{((i_1, i_1), \{(i_2, i_3), (i_3, i_2)\}, \{\}) \mid i \in \mathbb{N} \text{ even}\} \\ & \cup \{((\infty_1, \infty_1), \{(\infty_2, \infty_2), (\infty_3, \infty_3)\}, \{\}), ((\infty_1, \infty_1), \{(\infty_2, \infty_3), (\infty_3, \infty_2)\}, \{\})\} \\ & \cup \{((i_j, i_k), \varnothing, \lambda(i_j)) \mid i \in \overline{\mathbb{N}} \wedge (i_j, i_k) \in R\} \end{aligned}$$

The interesting part is that (∞_1, ∞_1) is related to two elements, and that if one of them is removed, then W is not closed anymore, and so not a relation in **Stone**. This explains why this relation cannot be restricted to the graph of a continuous function.

6.2 Toposes for Name-Passing

In [23], Staton studies models of name-passing and their bisimulations. Three toposes and functors are presented to model different parts of the theory. The first topos is the category of name substitution, which is the category of presheaves over non-empty finite subsets of a fixed countable set, together with all functions between them. It comes with a functor combining non-determinism and name-binding. This functor satisfies strong properties: in particular, AM-bisimulations coincide with HJ-bisimulations, and the largest AM-bisimulation coincide with the largest behavioural equivalence. This framework is already nice as AM-bisimulations describe precisely open bisimulations [20].

The second topos is a refinement of the first one, as the category of functors over all finite subsets of the given countable set, together with injections. The proposed functor in this case is less nice: it does not preserve weak-pullbacks and AM-bisimulations do not coincide with HJ-bisimulations anymore. However, it is nice enough in our theory: it covers pullbacks, and the category is a topos, so regular and with pushouts, then HJ-bisimulations coincide with regular AM-bisimulations, and their existence coincides with the existence of a behavioural equivalence.

For this topos, it is remarked in [23] that if a relation is a HJ-bisimulation (so a regular/toposal AM-bisimulation), then its ¬¬-completion is an AM-bisimulation, which means in particular that this framework for name-passing is much nicer when restricting to ¬¬-sheaves. One main reason for that is that the sheaf topos for the ¬¬-topology satisfies the axiom of choice when the base topos is a presheaf topos over a poset [19], which is the case here.

6.3 Weighted Linear Systems

In [6], Bonchi et al. are studying linear weighted systems, that is, coalgebras for the endofunctor $X \mapsto K \times X^A$ on $K\mathbf{Vect}$, in the category of K-vector spaces, with K a field, and A a set. The following discussion can also be made in the category of modules over a ring. The category $K\mathbf{Vect}$ is abelian, and so is regular and has pushouts. The endofunctor actually preserves pullbacks, so the three notions of bisimilarity coincide by Theorem 29. In this paper, they are interested in linear bisimulations, which coincide with behavioural equivalence, and so to the other two notions of bisimilarities.

In perspective, usual weighted systems are described in the category **Set**, with the functor $X \mapsto A \Rightarrow K^{(X)}$ where $K^{(X)}$ is the set of functions from X to K which takes finitely many non-zero values. In this context, this functor does not even cover pullbacks in general, and they actually prove that AM-bisimilarity (and so regular AM-bisimilarity since **Set** has the regular axiom of choice) does not coincide with behavioural equivalence.

7 Conclusion

This paper introduces some foundations of the theory of bisimulations and simulations in a general regular category, mitigating some known issues about Aczel-Mendler bisimulations. The relations and power objects are the key ingredients for this mitigation: if the axiom of choice allows to pick some witnesses of bisimilarity, the relations and power objects allow to collect them up without need to choose. This paves the way to the study of such bisimulations in more exotic regular categories and toposes.

One direction of future work is to investigate regular AM-bisimulations for probabilistic systems, compared to what is done in [8, 7] for behavioural equivalences. The main challenge is to find a suitable *regular* category of "probabilistic space" and a "probabilistic distribution functor" that *covers pullbacks*. For the first property, the work on Quasi-Borel spaces [10], producing a quasi-topos, is of interest. For the second one, looking at categories of σ-frames (see for example [21]), for which pullbacks do not coincide with pullbacks in the category of measurable spaces is a solution under investigation.

References

1 Samson Abramsky. Coalgebras, Chu spaces, and representations of physical systems. *Journal of Philosophical Logic*, 42:551–574, 2013.

2 Peter Aczel and Nax Mendler. A final coalgebra theorem. In *Proceedings of Category Theory and Computer Science*, volume 389 of *Lecture Notes in Computer Science*. Springer, 1999.

3 Emilio Jesús Gallego Arias and James B. Lipton. Logic Programming in Tabular Alegories. In *Technical Communications of the 28th International Conference on Logic Programming (ICLP'12)*, Leibniz International Proceedings in Informatics, pages 334–347. Schloss Dagstuhl, 2012.

4 Roland Backhouse and Paul Hoogendijk. Final dialgebras: from categories to allegories. *Theoretical Informatics and Application*, 33:401–426, 1999.

5 Nick Bezhanishvili, Gaelle Fontaine, and Yde Yenema. Vietoris bisimulations. *Journal of Logic and Computation*, 20(5):1017–1040, 2010.

6 Filippo Bonchi, Marcello Bonsangue, Michele Boreale, Jan Rutten, and Alexandra Silva. A coalgebraic perspective on linear weighted automata. *Information and Computation*, 211:77–105, 2012.

7 Vincent Danos, Josée Desharnais, Francois Laviolette, and Prakash Panangaden. Bisimulation and cocongruence for probabilistic systems. *Information and Computation*, 204(4):503–523, 2006. `doi:10.1016/j.ic.2005.02.004`.

8 Josée Desharnais, Abbas Edalat, and Prakash Panangaden. Bisimulation for Labelled Markov Processes. *Information and Computation*, 179(2):163–193, 2003. `doi:10.1006/inco.2001.2962`.

9 Peter Freyd and Andre Scedrov. *Categories, Allegories*, volume 39 of *Mathematical Library*. North-Holland, 1990.

10 Chris Heunen, Ohad Kammar, Sam Staton, and Hongseok Yang. A convenient category for higher-order probability theory. In *Proceedings of the 32nd Annual ACM/IEEE Symposium on Logic in Computer Science (LICS'17)*, 2017.

11 Jesse Hughes and Bart Jacobs. Simulations in coalgebra. *Theoretical Computer Science*, 327(1-2):71–108, 2004.

12 J. Martin E. Hyland. The effective topos. In *The L.E.J. Brouwer Centenary Symposium*, volume 110 of *Studies in Logic and the Foundations of Mathematics*, pages 165–216, 1982.

13 Bart Jacobs. *Introduction to Coalgebra: Towards Mathematics of States and Observation*, volume 29 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, 2017.

14 Peter T. Johnstone. *Sketches of an Elephant: A Topos Theory Compendium*, volume 1. Oxford University Press, 2002.

15 Peter T. Johnstone. *Sketches of an Elephant: A Topos Theory Compendium*, volume 2. Oxford University Press, 2002.

16 André Joyal, Mogens Nielsen, and Glynn Winskel. Bisimulation from Open maps. *Information and Computation*, 127:164–185, 1996.

17 F. William Lawvere. Equality in hyperdoctrines and comprehension schema as an adjoint functor. In *Proceedings of the AMS Symposium on Pure Mathematics XVII*, pages 1–14, 1970.

18 F. William Lawvere. Qualitative Distinctions between some Toposes of Generalized Graphs. *Contemporary Mathematics*, 92, 1989.

19 Saunders MacLane and Ieke Moerdijk. *Sheaves in Geometry and Logic: A First Introduction to Topos Theory*. Springer New-York, 1992.

20 Davide Sangiorgi. A theory of bisimulations for the π-calculus. *Acta Informatica*, 33:69–97, 1996.

21 Alex Simpson. Measure, randomness and sublocales. *Annals of Pure and Applied Logic*, 163(11):1642–1659, 2012.

22 Sam Staton. *Name-passing process calculi: operational models and structural operational semantics*. PhD thesis, University of Cambridge, 2008.

23 Sam Staton. Relating Coalgebraic Notions of Bisimulation. *Logical Methods in Computer Science*, 7:1–21, 2011.

24 Paulo Tabuada. Open Maps, Alternating Simulations and Control Synthesis. In *Proceedings of the 15th International Conference in Concurrency Theory (CONCUR'04)*, volume 3170 of *Lecture Notes in Computer Science*, pages 466–480. Springer, 2004.

25 Michael Winter. *Goguen Categories: A Categorical Approach to L-fuzzy Relations*. Number 25 in Trends in Logic. Springer, 2007.

26 Thorsten Wißman, Jérémy Dubut, Shin-ya Katsumata, and Ichiro Hasuo. Path Category for Free: Open Morphisms from Coalgebras with Non-deterministic Branching. In *Proceedings of the 22nd International Conference on Fondations of Software Science and Computation Structures (FoSSaCS'19)*, volume 11425 of *Lecture Notes in Computer Science*. Springer, 2019.