Coinductive Control of Inductive Data Types

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Abstract

We combine the theory of inductive data types with the theory of universal measurings. By doing so, we find that many categories of algebras of endofunctors are actually enriched in the corresponding category of coalgebras of the same endofunctor. The enrichment captures all possible partial algebra homomorphisms, defined by measuring coalgebras. Thus this enriched category carries more information than the usual category of algebras which captures only total algebra homomorphisms. We specify new algebras besides the initial one using a generalization of the notion of initial algebra.

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1 Introduction

In both the tradition of functional programming and categorical logic, one takes the perspective that most data types should be obtained as initial algebras of certain endofunctors (to use categorical language). For instance, the natural numbers are obtained as the initial algebra of the endofunctor $X \mapsto X + 1$, assuming that the category in question (often the category of sets) has a terminal object 1 and a coproduct $+$. Much theory has been developed around this approach, which culminated in the notion of W-types [5].

In another tradition, for k a field, it has been long understood (going back at least to Wraith, according to [3], and Sweedler [10]) that the category of k-algebras is naturally enriched over the category of k-coalgebras, a fact which has admitted generalization to several other settings (e.g. [3, 11, 8, 6]). In this paper, we extend this theory to the setting of an endofunctor on a category – in particular those endofunctors that are considered in the theory of W-types.

This work is thus the beginning of a development of an analogue of the theory of W-types – not based on the notion of initial objects in a *category* of algebras, but rather on generalized notions of initial objects in a *coalgebra enriched category* of algebras. Our main result (Theorem 31) states that the categories of algebras of endofunctors considered in the theory of W-types are often enriched in their corresponding categories of coalgebras. The hom-coalgebras of our enriched category carry more information than the hom-sets in the unenriched category that is usually considered in the theory of W-types. Because of our passage to the enriched setting, we have more precise control than in the unenriched setting,

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and we are able to specify more data types than just those which are captured by the theory of W-types. We do this by generalizing the notion of initial algebra, taking inspiration from the notion of weighted limits. This general theory is presented in Section 3.

But first, in Section 2, we begin our paper with an enlightening example which serves as an illustration of the relevance of our enriched theory and as a motivation for the more general setting. Therein, we provide explicit calculations for the case of algebras over the endofunctor $X \mapsto X + 1$ on Set. In that example, we illustrate that it is appropriate to interpret the elements of the hom-coalgebras as *partial homomorphisms*.

Indeed, in the classical Sweedler theory, the enrichment in coalgebras can also be understood as encoding a notion of partial homomorphism. Though we do not study k-algebras in this paper, we conclude this introduction with details of that classical theory. A *measuring* from a k-algebra A to a k-algebra B, in the sense of Sweedler [10], is a k-coalgebra C together with a linear homomorphism $\phi\colon C \otimes_k A \to B$ that is compatible with the multiplication and identities of A and B. A measuring from A to B is equivalently a k-coalgebra C together with a k-linear map $\phi : C \to A \to B$ such that

$$\phi_c(aa') = \sum_{i=1}^{n} \phi_{c_i^{(1)}}(a)\phi_{c_i^{(2)}}(a'), \quad \text{and} \quad \phi_c(1_A) = \varepsilon(c)1_B,$$

for all $c \in C$ and $a, a' \in A$ where $\Delta(c) = \sum_{i=1}^{n} c_i^{(1)} \otimes c_i^{(2)}$ is the comultiplication $\Delta\colon C \to C \otimes_k C$ and $\varepsilon\colon C \to k$ is the counit of C. Therefore the k-linear maps $\phi_c\colon A \to B$ can be regarded as partial algebra homomorphisms, and the elements $c \in C$ can be interpreted as measuring how far each partial homomorphism ϕ_c is from being a total homomorphism. For instance when $\Delta(c) = c \otimes c$, we have that $\phi_c\colon A \to B$ is a total algebra homomorphism. Now we proceed to tell an analogous story about endofunctors.

2 Illustrative example: id + 1

In this section, we illustrate our results in the context of one example: the endofunctor that sends $X \mapsto X + 1$ (the coproduct of X and a terminal set 1) in Set, the category of sets. The initial algebra of this endofunctor is $\mathbb{N}$, the natural numbers, and thus this endofunctor is one of the most basic and important examples in the theory of W-types.

This section is one very long worked example of our general, categorical theory which follows in Section 3.

We first review the classical story in Section 2.1, and afterwards our goal is to explain how the category of algebras is naturally enriched in the category of coalgebras of this functor and how we can use this extra structure to generalize the notion of *initial algebra* to capture more algebras than just $\mathbb{N}$. So, in Section 2.2 we explore by hand a notion of partial homomorphism between algebras that will be captured more formally later in the enrichment. Next, in Section 2.3, we explore the structure that this enrichment gives us. In Section 2.4, we introduce a computational tool and compute explicitly some of the hom-objects of our enrichment. Finally, in Section 2.5, we use this extra structure to generalize the notion of initial object, and we describe some of the algebras that can be specified in this way.

Note that many of the proofs in this paper were relegated to the appendices, which do not appear with this, published, version. Thus, we repeatedly reference proofs in the full version, [7].

2.1 Preliminaries

Here, we review the established theory regarding algebras and coalgebras of $\mathsf{id} + 1$ that we will use. See, for instance, [9, Ch. 3] for details.

We let Alg denote the category of algebras of $\mathsf{id} + 1$, and we let CoAlg denote the category of coalgebras of $\mathsf{id} + 1$. Recall that an algebra is a pair (A, α) of a set A together with a function $\alpha : A + 1 \to A$ (equivalently, a *successor* endofunction $\alpha|_A : A \to A$ and a *zero* $\alpha|_1 : 1 \to A$), and a coalgebra is a pair (C, χ) of a set C together with a function $\chi : C \to C + 1$, i.e., a partial endofunction. The initial object of Alg is $(\mathbb{N}, \alpha_\mathbb{N})$, where $\mathbb{N}$ is the usual natural numbers, $\alpha_\mathbb{N}|_\mathbb{N}$ is the usual successor function $x \mapsto x + 1$ and $\alpha_\mathbb{N}|_1$ picks out $0 \in \mathbb{N}$. The terminal object of CoAlg is $(\mathbb{N}^\infty, \chi_{\mathbb{N}^\infty})$ where $\mathbb{N}^\infty$ is the extended natural numbers $\mathbb{N} + \{\infty\}$, and the map $\chi_{\mathbb{N}^\infty} : \mathbb{N}^\infty \to \mathbb{N}^\infty + 1$ takes $0 \in \mathbb{N}^\infty$ to the element $\mathtt{t} \in 1$ and all other $x \in \mathbb{N}^\infty$ to $x - 1 \in \mathbb{N}^\infty$.

Note that because $\mathbb{N}$ is initial in Alg, any algebra (A, α_A) gets a function $!_A : \mathbb{N} \to A$, and thus it will be useful write n_A for $!_A(n)$. That is, 0_A is the zero of A, 1_A is the successor of 0_A, etc. For $a \in A$, we will often also write $a + 1$ as shorthand for $\alpha_A(a)$ (especially when the algebra structure morphism, here α_A, does not have an explicit name).

Dually, because $\mathbb{N}^\infty$ is terminal in CoAlg, there is a function $[\![-]\!] : C \to \mathbb{N}^\infty$ for any coalgebra (C, χ_A), and we will say that the *index* of a $c \in C$ is $[\![c]\!]$. Then the elements of C that have index 0 are those c such that $\chi_C(c) = \mathtt{t}$, those that have index 1 are all those other c such that $\chi_C^2(c) = \mathtt{t}$, etc. For $c \in C$ where $[\![c]\!] \neq 0$, we will also often write $c - 1$ to denote $\chi_C(c)$ (especially when the coalgebra structure morphism does not have an explicit name).

Besides $\mathbb{N}$, the *initial algebra*, we will often consider *preinitial algebras*, that is, algebras A for which $!_A : \mathbb{N} \to A$ is epic. The nontrivial preinitial algebras are of the form $\mathbb{n} := (\{0, 1, ..., n\}, \alpha_\mathbb{n})$ for any $n \in \mathbb{N}$. Here, $\alpha_\mathbb{n}$ is the algebra structure that $\{0, 1, ..., n\}$ inherits as the quotient of $\mathbb{N}$ in Set that identifies all $m \geq n$ (see [7, Example 39] and the preceding [7, Lemma 38]).

Dually, besides $\mathbb{N}^\infty$, we will often consider *subterminal coalgebras*, that is, coalgebras C for which $[\![-]\!] : C \to \mathbb{N}^\infty$ is monic. The nontrivial ones are $\mathbb{n}^\circ$ with underlying subset $\{0, 1, ..., n\}$ of $\mathbb{N}^\infty$, $\mathbb{N}^-$ with underlying subset $\mathbb{N}$, and $\mathbb{I}$ with underlying subset $\{\infty\}$. These all inherit coalgebra structures from $\mathbb{N}^\infty$ (see [7, Example 43] and the preceding [7, Lemma 42]).

2.2 Partial homomorphisms

Consider algebras A and B. We are, first of all, most interested in algebra homomorphisms $f : A \to B$ (which we might call *total* algebra homomorphisms to distinguish them from the notion of *partial* algebra homomorphisms which we are about to introduce). This means that we have (H1) $f(0_A) = 0_B$ and (H2) $f(a + 1) = f(a) + 1$ for all $a \in A$. If A is $\mathbb{N}$, we know that there is a total algebra homomorphism $\mathbb{N} \to B$, and we can use (H1) and (H2) to inductively construct this homomorphism.

But depending on the nature of A and B, it might happen that we can only guarantee (H1) and (H2) hold for some, but not all, $a \in A$, and thus an attempt to construct a total algebra homomorphism $A \to B$ inductively might fail at some point. Perhaps A is a preinitial algebra $\mathbb{n}$ and B is $\mathbb{N}$. We can try to construct a total homomorphism, so we set $f(0_\mathbb{n}) := 0_\mathbb{N}$ following (H1), $f(1_\mathbb{n}) := 1_\mathbb{N}$ following (H2), etc. This works until we get to $n_\mathbb{n}$. Since $n_\mathbb{n}$ is the successor of both $(n-1)_\mathbb{n}$ and $n_\mathbb{n}$, (H2) tells us to send $n_\mathbb{n}$ both to $n_\mathbb{N}$ and $(n+1)_\mathbb{N}$. We might say that induction worked only up to the nth step, or that we can define a *n-partial homomorphism*.

We formalize this idea in the following way, in which we inductively construct partial homomorphisms in an attempt to approximate a total homomorphism. In our first attempt at formalizing this idea, Construction 1 below, we make the simplifying assumption that A is preinitial – simplifying because then there is at most one homomorphism $A \to B$. We will almost immediately drop this assumption in the more general Definition 2.

▶ **Construction 1** (Partial induction). *We seek to inductively approximate a homomorphism $A \to B$ when A is preinitial. We define a sequence of functions $f_c : A \to B$ as follows.*
Initial step (P1). *Define $f_0 \colon A \to B$ by $f_0(a) := 0_B$ for all $a \in A$.*
Inductive step. *Define $f_{c+1} \colon A \to B$ by:*
(P2) $f_{c+1}(0_A) := 0_B$*;*
(P3) $f_{c+1}(a + 1) := f_c(a) + 1$.
We stop when f_{c+1} is not well-defined.
*If we have defined f_c for all $c \in \mathbb{N}$, then we will say that we have defined an ∞-*partial homomorphism*. Otherwise, if we have only defined f_c for all $c \in \{0, ..., n\}$, we will say that we have defined an n-*partial homomorphism*.*

Since A is preinitial, it is of the form $\mathbb{n}$ or $\mathbb{N}$, and so every element is of the form x_A for $x \in \mathbb{N}$. Thus, $f_c(x_A)$ is x_B for $x \leq c$ and otherwise c_B. In particular, if $A = \mathbb{n}$, then $f_n = f_m$ for all $m \geq n$. Now we can see that there is an ∞-partial homomorphism $A \to B$ if and only if there is a total homomorphism $f : A \to B$. Indeed, if we have an ∞-partial homomorphism $A \to B$ consisting of an $f_c : A \to B$ for all $c \in \mathbb{N}$, then we can define a "diagonal" total homomorphism $f : A \to B$ by $f(x_A) := f_x(x_A)$. Conversely, if we have a total homomorphism $f : A \to B$, there is no obstruction to the inductive steps in defining a ∞-partial homomorphism. Thus, we can conflate the notions of a ∞-partial homomorphism and a total homomorphism $A \to B$ when A is preinitial.

Notice that in the term "n-partial homomorphism" in the above Construction 1, n takes values in $\mathbb{N}^\infty$, the terminal coalgebra of our endofunctor. In fact, the above construction follows the similar pattern of the measurings of algebras over a field that we mentioned in the introduction. So now we make the following definition in which we encode the coalgebra directly. This allows us to generalize Construction 1, dropping the hypothesis that A is preinitial.

▶ **Definition 2** (Measuring, cf. Definition 18 and Proposition 23). *Consider algebras A and B and a coalgebra C. A* measuring from A to B by C *is a function $f \colon C \to A \to B$ such that:*
(M1) $f_c(0_A) = 0_B$ *for all $c \in C$;*
(M2) $f_c(a + 1) = 0_B$ *for all $[\![c]\!] = 0$ and for all $a \in A$;*
(M3) $f_c(a + 1) = f_{c-1}(a) + 1$ *for $[\![c]\!] \geq 1$ and for all $a \in A$.*

We write $\mu_C(A, B)$ for the set of measurings from A to B. This defines a functor $\mu : \mathsf{CoAlg}^{\mathsf{op}} \times \mathsf{Alg}^{\mathsf{op}} \times \mathsf{Alg} \to \mathsf{Set}$.

*For a measuring f and an element $c \in C$, we call f_c a C-*partial homomorphism*.*

▶ **Example 3.** Suppose that A is preinitial, so that in particular every element of A is of the form 0_A or $a + 1$.

Then there is a measuring from A to B by $\mathbb{n}^\circ$ if the induction of Construction 1 creates an n-partial homomorphism, and in this case the functions of the form f_c constructed in Construction 1 are the same as those specified in Definition 2.

There is a measuring by $\mathbb{N}^-$ if the induction never fails, and again the functions f_c from Construction 1 and Definition 2 coincide. Now, note that exhibiting a measuring by $\mathbb{N}^\infty$ amounts to exhibiting a measuring by $\mathbb{N}^-$ together with a total algebra homomorphism f_∞. For such an A, then, exhibiting a measuring by $\mathbb{N}^-$ is equivalent to exhibiting one by $\mathbb{N}^\infty$.

The reader might wonder why Definition 2 does not require $(\mathsf{M2}')$ $f_c(x) = 0_B$ for any $[\![c]\!] = 0$ and any x, but rather only requires $f_c(x) = 0_B$ when $[\![c]\!] = 0$ and x is either the zero or a successor. In the previous example, when A is preinitial, every $x \in A$ is either the zero or a successor, so there is no difference between these two requirements. In the following example, we consider an algebra A where this is not the case, and illustrate why we only stipulate $(\mathsf{M2})$ and not $(\mathsf{M2}')$.

▶ **Example 4.** Consider the algebra A with underlying set $\mathbb{N} + \mathbb{N}$, where we will notate the elements of the first copy of $\mathbb{N}$ as n_A, and the elements of the second copy as n'. The zero of A is then 0_A and the successors are given by $n_A + 1 := (n + 1)_A$ and $n' + 1 := (n + 1)'$. Total homomorphisms $A \to \mathbb{N}$ are determined by the image of $0'$ in $\mathbb{N}$.

If we require $(\mathsf{M2}')$ instead of $(\mathsf{M2})$ in Definition 2, then in a measuring by $\mathbb{N}^\infty$, $f_0(0') = 0_{\mathbb{N}}$ by $(\mathsf{M2}')$, and in general $f_n(n') = n_{\mathbb{N}}$ by $(\mathsf{M3})$.

However, following Definition 2 as written, in a measuring by $\mathbb{N}^\infty$, $f_0(0')$ may be anything, say $z \in \mathbb{N}$ and then general $f_n(n') = (z + n)_{\mathbb{N}}$.

Thus, Definition 2 does generalize the idea of inductively approximating a total homomorphism $A \to \mathbb{N}$ from Construction 1.

Now notice another difference between Construction 1 and Definition 2. In Construction 1 we continue the induction as far as we can, but there is nothing of this flavor in Definition 2. For instance, if there is a total algebra homomorphism $A \to B$, then in the process described by Construction 1, we will inductively construct f_c for all $c \in \mathbb{N}$. However, following Definition 2, we could say that $A \to B$ is measured by $\mathbb{0}$ (which only amounts to exhibiting f_0), without making any claim about it being measured by other coalgebras – it does not ask us to find any kind of maximum coalgebra C that measures $A \to B$. To remedy this, we make the following definition.

▶ **Definition 5** (Universal measuring, cf. Definition 20)**.** *Let A and B be algebras.*

We define the category of measurings from A to B to be the category whose objects are pairs $(C; f)$ of a coalgebra C and a measuring $f : C \to A \to B$, and whose morphisms $(C; f) \to (D; g)$ are coalgebra homomorphisms $d : C \to D$ such that $f = gd$.

The universal measuring *from A to B, denoted $(\underline{\mathsf{Alg}}(A, B); u)$, is the terminal object in the category of measurings from A to B. That is, if $(C; f)$ is a measuring from A to B, then there is a unique morphism $! : C \to \underline{\mathsf{Alg}}(A, B)$ that makes the following diagram commute.*

$$\begin{tikzcd}
C \arrow[r, "f"] \arrow[d, dashed, "!"] & A \to B \\
\underline{\mathsf{Alg}}(A, B) \arrow[ur, "u"'] &
\end{tikzcd}$$

▶ **Example 6.** Again, suppose that A is preinitial. In this case, the universal measuring is a subterminal coalgebra [7, Lemma 37]. We have shown that if the induction of Construction 1 creates an n-partial homomorphism, then the maximum subterminal coalgebra that measures $A \to B$ is $\mathbb{n}^\circ$, so this is the universal measuring. And if the induction of Construction 1 creates an total homomorphism, then the maximum subterminal coalgebra that measures A to B is $\mathbb{N}^\infty$ itself, so this is the universal measuring. We will also show this fact more directly (i.e., without reference to [7, Lemma 37]) in Section 2.4 below.

Since composing an arbitrary coalgebra homomorphism $C \to \underline{\mathsf{Alg}}(A, B)$ with u produces a measuring $C \to A \to B$, we obtain a bijection, natural in C, A, B,

$$\mu_C(A, B) \cong \mathsf{CoAlg}(C, \underline{\mathsf{Alg}}(A, B))$$

showing that $\mu_{-}(A, B)$ is represented by $\underline{\mathsf{Alg}}(A, B)$. In Theorem 25, we will see that $\underline{\mathsf{Alg}}(A, B)$ always exists (for this and other endofunctors of interest). The coalgebra $\underline{\mathsf{Alg}}(A, B)$ will constitute the hom-coalgebra from A to B of our enriched category of algebras (Theorem 31).

Now, note that a measuring by the coalgebra $\mathbb{I}$ is a total algebra homomorphism. Thus,

$$\mathsf{Alg}(A, B) \cong \mu_{\mathbb{I}}(A, B) \cong \mathsf{CoAlg}(\mathbb{I}, \underline{\mathsf{Alg}}(A, B))$$

and so we find the hom-sets of the category of algebras can be easily extracted from $\underline{\mathsf{Alg}}(A, B)$ – a statement that aligns with our intuition that $\mathsf{Alg}(A, B)$ is the set of total algebra homomorphisms and $\underline{\mathsf{Alg}}(A, B)$ is the coalgebra of partial algebra homomorphisms.

2.3 Composing partial homomorphisms

We will only prove that the universal measuring coalgebras form the hom-objects of our enriched category in Theorem 31, but we can already work out how this enrichment behaves. Thus, in this section, we describe the composition and identities of this enriched category.

Given algebras A, B, and T, we can always compose total homomorphisms $f : A \to B$ and $g : B \to T$ to form a total homomorphism $g \circ f : A \to T$. We wish to do the same for our partial homomorphisms. Consider coalgebras C and D, a C-partial homomorphism $f_c : A \to B$, and a D-partial homomorphism $g_d : B \to T$. We can compose g_d and f_c as functions to obtain $g_d \circ f_c : A \to T$, and we claim that this is a $(D \times C)$-partial homomorphism. Indeed $D \times C$ has a coalgebra structure where $[\![(d, c)]\!] = \min([\![d]\!], [\![c]\!])$ and $(d, c) - 1 = (d - 1, c - 1)$ if $[\![(d, c)]\!] > 0$ for $(d, c) \in D \times C$. This induces a symmetric monoidal structure on CoAlg for which $\mathbb{I}$ is the unit (Proposition 30), and one can verify that $g_d \circ f_c$ is a $(D \times C)$-partial homomorphism.

Now we have constructed a function

$$\begin{aligned} \mu_D(B, T) \times \mu_C(A, B) &\longrightarrow \mu_{D \times C}(A, T) \\ (g, f) &\longmapsto g \circ f \end{aligned}$$

where $g \circ f : (D \times C) \to A \to T$ is defined by $(g \circ f)_{(d,c)} = g_d \circ f_c$. Thus, by the universal property of $\underline{\mathsf{Alg}}$, we obtain a function

$$\mathsf{CoAlg}(D, \underline{\mathsf{Alg}}(B, T)) \times \mathsf{CoAlg}(C, \underline{\mathsf{Alg}}(A, B)) \longrightarrow \mathsf{CoAlg}(D \times C, \underline{\mathsf{Alg}}(A, T)).$$

Applying this function to $(\mathsf{id}_{\underline{\mathsf{Alg}}(B,T)}, \mathsf{id}_{\underline{\mathsf{Alg}}(A,B)})$, we obtain a composition morphism $\circ : \underline{\mathsf{Alg}}(B, T) \times \underline{\mathsf{Alg}}(A, B) \to \underline{\mathsf{Alg}}(A, T)$ such that for $(d, c) \in \underline{\mathsf{Alg}}(B, T) \times \underline{\mathsf{Alg}}(A, B)$, we have $u_d \circ u_c = u_{d \circ c}$ (where u is as in Definition 20).

Similarly, for any algebra A we might ask if there is an identity $\mathsf{id}_A : \mathbb{I} \to \underline{\mathsf{Alg}}(A, A)$. We showed above that $\mathsf{Alg}(A, A) \cong \mathsf{CoAlg}(\mathbb{I}, \underline{\mathsf{Alg}}(A, A))$. Thus, we take the image of $\mathsf{id}_A \in \mathsf{Alg}(A, A)$ under this bijection.

We leave it as an exercise for the interested reader to show by hand that this constitutes an enrichment of Alg in $(\mathsf{CoAlg}, \otimes, \mathbb{I})$, i.e., that all the axioms for an enriched category are satisfied by this choice of composition and identities. We will instead leave this result (Theorem 31) to the general setting.

2.4 The convolution algebra

We now give an alternative representation of $\mu_C(A, B)$ that can be directly defined and computed. In this section, we will be able to use it to compute $\underline{\mathsf{Alg}}(A, B)$ without appealing to [7, Lemma 37].

In Definition 2, we defined $\mu_C(A,B)$ to be a certain subset of $\mathsf{Set}(C,\mathsf{Set}(A,B)) \cong \mathsf{Set}(A,\mathsf{Set}(C,B))$. We now identify that subset as the subset of (total) algebra homomorphisms $A \to \mathsf{Set}(C,B)$ with a particular *convolution* algebra structure on $\mathsf{Set}(C,B)$.

▶ **Definition 7** (Convolution algebra, cf. Definition 27)**.** *Given a coalgebra (C,χ_C) and an algebra (B,α_B), define the* convolution algebra $[C,B]$ *to be the algebra whose underlying set is $\mathsf{Set}(C,B)$, whose zero is the constant function $C \to B$ at 0_B, and where $f+1$ is defined by*

$$(f+1)(c) = \begin{cases} 0_B & \textit{if } [\![c]\!] = 0; \\ f(c-1)+1 & \textit{if } [\![c]\!] > 0. \end{cases}$$

This defines a functor $[-,-] : \mathsf{CoAlg}^{\mathsf{op}} \times \mathsf{Alg} \to \mathsf{Alg}$.

Given a coalgebra (C,χ_C) and an algebra (B,α_B), a function $m : C \to A \to B$ is a measuring if and only if the associated $\widetilde{m} : A \to C \to B$ (under the bijection $\widetilde{-}$: $\mathsf{Set}(C,\mathsf{Set}(A,B)) \to \mathsf{Set}(A,\mathsf{Set}(C,B))$) underlies a homomorphism $A \to [C,B]$ of algebras. Indeed, (M1) of Definition 2 for m is equivalent to (H1) $\widetilde{m}(0_A) = 0_{[C,B]}$ and criteria (M2) and (M3) for f are equivalent to (H2) $\widetilde{m}(a+1) = \widetilde{m}(a)+1$.

Therefore, we find the following string of bijections, natural in C, A, B,

$$\mu_C(A,B) \cong \mathsf{CoAlg}(C, \underline{\mathsf{Alg}}(A,B)) \cong \mathsf{Alg}(A,[C,B]) \tag{1}$$

so that we see that $\mu_C(-,B)$ is represented by $[C,B]$. We can even find a representation for $\mu_C(-,B)$, but we will leave this for the more general setting (Theorem 22). The interested reader is encouraged to calculate that other representation in this example.

In practice, when we want to compute $\underline{\mathsf{Alg}}(A,B)$, we will compute $[C,B]$ and then apply the universal property above. We do that now, computing some of the results of Example 6 without appealing to [7, Lemma 37].

▶ **Example 8.** We compute $\underline{\mathsf{Alg}}(\mathbb{n},B)$ using the right-hand bijection in Equation (1).

We first observe the following for any coalgebra Z.

$$\mathsf{Alg}(\mathbb{n},Z) \cong \begin{cases} * & \text{if } n_Z = (n+1)_Z \\ \emptyset & \text{otherwise} \end{cases}$$

Since we are considering $Z := [C,B]$, we need to understand when $n_{[C,B]} = (n+1)_{[C,B]}$. By definition, $0_{[C,A]}$ is the constant function at 0_A. Then $1_{[C,A]}$ is the function that takes every $c \in C$ of index 0 to 0_B, and every other $c \in C$ to 1_B. Inductively, we can show that $n_{[C,B]}(c) = \min([\![c]\!], n)_B$. Thus, $n_{[C,B]} = (n+1)_{[C,B]}$ means that $\min([\![c]\!], n)_B = \min([\![c]\!], n+1)_B$ for all $c \in C$, and this holds if and only if $[\![c]\!] \leq n$ for all $c \in C$ or $n_B = (n+1)_B$. Now we have the following.

$$\mathsf{CoAlg}(C, \underline{\mathsf{Alg}}(\mathbb{n},B)) \cong \mathsf{Alg}(\mathbb{n},[C,B]) \cong \begin{cases} * & \text{if } [\![c]\!] \leq n \text{ for all } c \in C \\ * & \text{if } n_B = (n+1)_B \\ \emptyset & \text{otherwise} \end{cases} \tag{2}$$

In the case that $n_B = (n+1)_B$, we find that $\underline{\mathsf{Alg}}(\mathbb{n},B)$ has the universal property of the terminal object, $\mathbb{N}^\infty$.

Now suppose that $n_B \neq (n+1)_B$. Since $\mathsf{CoAlg}(C,\mathbb{n}^\circ) = *$ if and only if $[\![c]\!] \leq n$ for all $c \in C$, $\underline{\mathsf{Alg}}(\mathbb{n},B)$ has the universal property of $\mathbb{n}^\circ$.

Now we have calculated the following

$$\underline{\mathsf{Alg}}(\mathbb{n}, B) = \begin{cases} \mathbb{N}^\infty \text{ if } n_B = (n+1)_B \\ \mathbb{n}^\circ \text{ otherwise} \end{cases}$$

This aligns with our expectations, since there is a total homomorphism $\mathbb{n} \to B$ if $n_B = (n{+}1)_B$ but there is only an n-partial homomorphism $\mathbb{n} \to B$ otherwise.

Finally, note that taking $B := \mathbb{N}$, we have calculated $\underline{\mathsf{Alg}}(\mathbb{n}, \mathbb{N})$, the *dual* (Definition 29) of $\mathbb{n}$, to be $\mathbb{n}^\circ$.

2.5 Generalizing initial objects

Now we turn to the question of specifying algebras other than $\mathbb{N}$ via a generalization of the notion of initial algebra.

The fact that $\mathbb{N}$ is the initial object in Alg means that the algebra structure on an algebra A specifies exactly one total algebra homomorphism $\mathbb{N} \to A$, and this can be constructed inductively. Now we have introduced the notion of partial homomorphism which can be constructed by partial induction (Construction 1). Thus, we might ask if we can formalize a notion of being initial with respect to partial homomorphisms and partial induction.

Our calculations in this section so far have perhaps given us the intuition that the algebra $\mathbb{n}$ represents n-partial homomorphisms in the way that $\mathbb{N}$ represents total homomorphisms. Indeed, from Equation (2), there is a unique measuring $f : \mathbb{n}^\circ \to \mathbb{n} \to B$ for any algebra B. Now we try to capture and elucidate this fact by rephrasing it to say that $\mathbb{n}$ is a certain kind of initial object with respect to such partial homomorphisms.

There are multiple equivalent definitions of initial object, and we choose the one that is amenable to generalization. We choose to define an initial object in a category $\mathcal{C}$ as an object $I \in \mathcal{C}$ such that there is a unique function $1 \to \mathcal{C}(I, X)$ for all $X \in \mathcal{C}$. Now we have brought to the surface a parameter, here 1, that we can vary, inspired by the theory of weighted limits.

▶ **Definition 9** (C-initial algebra, cf. Definition 35)**.** *For a coalgebra C, we define a C-*initial algebra *to be an algebra A such that there is a unique coalgebra morphism $C \to \underline{\mathsf{Alg}}(A, X)$ for all algebras X.*

▶ Remark 10. One may wonder what would happen if for a *set* S, we defined an S-initial algebra to be an algebra A such that there is a unique *function* $S \to \mathsf{Alg}(A, X)$ for all $X \in \mathsf{Alg}$. But every algebra is an $\emptyset$-initial algebra, and an S-initial algebra is an initial algebra for any $S \neq \emptyset$ (because functions $S \to T$ are unique only when $S = \emptyset$ or $T \cong 1$). Thus, we need to consider $\underline{\mathsf{Alg}}(A, X)$ and not just $\mathsf{Alg}(A, X)$ to obtain interesting C-initial algebras.

▶ **Example 11.** We have shown in Example 8 that $\mathbb{n}$ is an $\mathbb{n}^\circ$-initial algebra.

Since $\mathsf{Alg}(A, X) \cong \mathsf{CoAlg}(\mathbb{I}, \underline{\mathsf{Alg}}(A, X))$, the initial algebra $\mathbb{N}$ is the (only) $\mathbb{I}$-initial algebra. In fact, since $\underline{\mathsf{Alg}}(\mathbb{N}, X) = \mathbb{N}^\infty$ for all X by [7, Lemma 37] or by a similar computation to Example 8, we find that $\mathbb{N}$ is a C-initial algebra for any subterminal coalgebra (i.e., $\emptyset, \mathbb{n}^\circ, \mathbb{N}^-, \mathbb{N}^\infty$).

Now we see that for instance, both $\mathbb{n}$ and $\mathbb{N}$ are $\mathbb{n}^\circ$-initial algebras. Thus, $\mathbb{n}^\circ$-initial algebras are not determined up to isomorphism as initial algebras are. This captures the fact that for an algebra B, we can construct n-partial homomorphisms from both $\mathbb{n}$ and $\mathbb{N}$ to B.

▶ **Definition 12** (Terminal C-initial algebra, cf. Definition 35)**.** *Consider the category whose objects are C-initial algebras, and whose morphisms $A \to B$ are total algebra homomorphisms $A \to B$. Then we call the terminal object of this category the* terminal C-initial algebra*.*

▶ **Example 13.** Since the only $\mathbb{1}$-initial algebra is $\mathbb{N}$, it is also the terminal C-initial algebra.

We want to show that $\mathbb{n}$ is the terminal $\mathbb{n}^\circ$-initial algebra. However, we need another computational tool. This is in fact an alternate generalization of the notion of initial algebra.

Above, we might have observed that an initial object can be characterized as the limit of the identity functor and then, following the theory of weighted limits, considered objects $\mathsf{lim}^C \mathsf{id}_{\underline{\mathsf{Alg}}}$ with the following universal property.

$$\mathsf{Alg}(A, \mathsf{lim}^C \mathsf{id}_{\underline{\mathsf{Alg}}}) \cong \mathsf{lim}_{X \in \mathsf{Alg}}\ \mathsf{CoAlg}(C, \underline{\mathsf{Alg}}(A, X))$$

We can immediately calculate (Proposition 36) that $\mathsf{lim}^C \mathsf{id}_{\underline{\mathsf{Alg}}}$ is $C^* := [C, \mathbb{N}]$, the dual of C (Definition 29). By the bijection above, there is a unique total algebra homomorphism from each C-initial object to $\mathsf{lim}^C \mathsf{id}_{\underline{\mathsf{Alg}}}$. This will help us understand the possible structure that a C-initial object can have. But first, we must understand the structure of C^*.

▶ **Example 14.** Let $C := \mathbb{n}^\circ$. Then elements of $[\mathbb{n}^\circ, \mathbb{N}]$ are sequences of $n+1$ natural numbers.

The successor of a sequence $(a_i)_{i=0}^n$ is $(b_i)_{i=0}^n$ where $b_0 = 0$ and $b_{i+1} = a_i + 1$. Notice that the successor of $(b_i)_{i=0}^n$ is $(c_i)_{i=0}^n$ where $c_0 = 0$, $c_1 = 1$ and otherwise $c_{i+2} = a_i + 2$. Thus, we can inductively show that the $(n+1)$-st successor of any element of $[\mathbb{n}^\circ, \mathbb{N}]$ is the sequence $(i)_{i=0}^n$, and the successor of this sequence is itself.

We claim that the unique morphism $!_{[\mathbb{n}^\circ, \mathbb{N}]} : \mathbb{N} \to [\mathbb{n}^\circ, \mathbb{N}]$ factors through $\mathbb{n}$. We have $m_{[\mathbb{n}^\circ, \mathbb{N}]} = (\min(i, m))_{i=0}^n$. Thus, the restriction of the map $!_{[\mathbb{n}^\circ, \mathbb{N}]}$ to $\{0, ..., n\} \subset \mathbb{N}$ is injective, and $n_{[\mathbb{n}^\circ, \mathbb{N}]} = m_{[\mathbb{n}^\circ, \mathbb{N}]}$ for all $m \geq n$.

▶ **Example 15.** Now we can show that $\mathbb{n}$ is the terminal $\mathbb{n}^\circ$-initial algebra. In this calculation, we use of the law of excluded middle for the only time in this paper.

Consider an $\mathbb{n}^\circ$-initial algebra A.

First, we show that every $a \in A$ is either the basepoint or a successor. So suppose that there is an element $a \in A$ that is not a basepoint or successor, and consider an algebra B with more than one element. Then for any $b \in B$ and any measure $f : \mathbb{n}^\circ \to A \to B$, we can form a measure $\tilde{f} : \mathbb{n}^\circ \to A \to B$ such that $\tilde{f}_n(a) = b$ and $\tilde{f}$ agrees with f everywhere else, since Definition 2 imposes no requirements on $\tilde{f}_n(a)$. Thus, there are multiple measures $\mathbb{n}^\circ \to A \to B$, equivalently total algebra homomorphisms $\mathbb{n}^\circ \to \underline{\mathsf{Alg}}(A, B)$, so we find a contradiction.

Now, we consider the unique map $A \to [\mathbb{n}^\circ, \mathbb{N}]$ and claim that this factors through the injection $\mathbb{n} \to [\mathbb{n}^\circ, \mathbb{N}]$, so that there is a unique $A \to \mathbb{n}$. Since every element of A is either a basepoint or a successor, every element of A is either of the form n_A or has infinitely many predecessors. The elements of the form n_A are mapped those to of the form $n_{[\mathbb{n}^\circ, \mathbb{N}]}$, and the elements who have infinitely many predecessors can only be mapped to the "top element" $n_{[\mathbb{n}^\circ, \mathbb{N}]} = (i)_{i=0}^n$, since this is the only element which has an m-th predecessor for any $m \in \mathbb{N}$. Thus, the unique $A \to [\mathbb{n}^\circ, \mathbb{N}]$ indeed factors through $\mathbb{n}$.

Thus we have shown how to specify algebras of the form $\mathbb{n}$ in a way analogous to the specification of $\mathbb{N}$ as an initial algebra. After determining an algebra structure on a set A, we obtain a unique n-partial algebra homomorphism $\mathbb{n} \to A$.

3 General theory

In this section, we now generalize the results of the previous section. So fix a symmetric monoidal category $(\mathcal{C}, \otimes, \mathbb{I})$ and a lax symmetric monoidal endofunctor (F, ∇, η) (defined below in Definition 16) on $\mathcal{C}$.

3.1 Measuring coalgebras

In this section, we define the general notion of *measuring* for F. Note that in Section 2.2 above, it was convenient to define a measuring to be a certain kind of function $C \to A \to B$, but here we first define the notion of measuring without requiring the monoidal structure to be closed. That is, we define a measuring to be a certain kind of function $C \otimes A \to B$.

▶ **Definition 16.** *That (F, ∇, η) is a* lax symmetric monoidal endofunctor *means that F is an endofunctor on $\mathcal{C}$ with*

(L1) *a natural transformation $\nabla_{X,Y} \colon F(X) \otimes F(Y) \longrightarrow F(X \otimes Y)$, for all $X, Y \in \mathcal{C}$; and*

(L2) *a morphism $\eta : \mathbb{I} \to F(\mathbb{I})$ in $\mathcal{C}$;*

such that (F, ∇, η) is associative, unital and commutative, as described in [7, Appendix A.2].

▶ **Example 17.** In Section 2, we considered the (cartesian closed) symmetric monoidal category $(\mathsf{Set}, \times, 1)$. For the endofunctor $\mathsf{id}+1$, we define $\nabla_{X,Y} : (X+1)\times(Y+1) \to (X\times Y)+1$ to take $(x, y) \mapsto (x, y)$, $(\mathtt{t}, y) \mapsto \mathtt{t}$, $(x, \mathtt{t}) \mapsto \mathtt{t}$, $(\mathtt{t}, \mathtt{t}) \mapsto \mathtt{t}$ for $x \in X, y \in Y, \mathtt{t} \in 1$. We define $\eta : 1 \to 1 + 1$ to be the inclusion into the first summand.

▶ **Definition 18** (Measuring, cf. Definition 2)**.** *Consider algebras (A, α) and (B, β), and a coalgebra (C, χ). We call a map $\phi : C \otimes A \to B$ a* measuring from A to B *if it makes the following diagram commute.*

$$\begin{tikzcd}
 & F(C)\otimes F(A) \arrow[r, "\nabla_{C,A}"] & F(C\otimes A) \arrow[r, "F(\phi)"] & F(B) \arrow[d, "\beta"] \\
C\otimes F(A) \arrow[ur, "\chi\otimes\mathsf{id}"] \arrow[r, "\mathsf{id}\otimes\alpha"'] & C\otimes A \arrow[rr, "\phi"] & & B
\end{tikzcd}$$

We denote by $\mu_C(A, B)$ the set of all measurings $C \otimes A \to B$.

If $\phi\colon C \otimes A \to B$ is a measuring, $a\colon (A', \alpha') \to (A, \alpha)$ and $b\colon (B, \beta) \to (B', \beta')$ are algebra homomorphisms, and $c\colon (C', \chi') \to (C, \chi)$ is a coalgebra homomorphism, then one can check that the composite

$$C' \otimes A' \xrightarrow{c\otimes a} C \otimes A \xrightarrow{\phi} B \xrightarrow{b} B'$$

is a measuring. Therefore, the assignment $C, A, B \mapsto \mu_C(A, B)$ underlies a functor

$$\mu\colon \mathsf{CoAlg}^{\mathsf{op}} \times \mathsf{Alg}^{\mathsf{op}} \times \mathsf{Alg} \longrightarrow \mathsf{Set}.$$

We shall see that this functor is representable in each of its variables under reasonable hypotheses.

▶ **Example 19.** The monoidal unit $\mathbb{I}$ of $\mathcal{C}$ is a coalgebra via the lax symmetric monoidal structure $\eta : \mathbb{I} \to F(\mathbb{I})$. Thus morphisms $A \to B$ in $\mathcal{C}$ are in bijection with morphisms $\mathbb{I} \otimes A \to B$ in $\mathcal{C}$, and one can check that a morphism $A \to B$ in $\mathcal{C}$ is an algebra homomorphism if and only if $\mathbb{I} \otimes A \to B$ is a measure. Thus, $\mu_{\mathbb{I}}(A, B) \cong \mathsf{Alg}(A, B)$.

▶ **Definition 20** (Universal measuring, cf. Definition 5)**.** *Let A and B be algebras.*

We define the category of measurings from A to B to be the category whose objects are pairs $(C; f)$ of a coalgebra C and a measuring $f : C \otimes A \to B$, and whose morphisms $(C; f) \to (D; g)$ are coalgebra homomorphisms $d : C \to D$ such that $f = g(d \otimes A)$.

The universal measuring *from A to B, denoted $(\underline{\mathsf{Alg}}(A, B), \mathsf{ev})$, is the terminal object (if it exists) in the category of measurings from A to B. That is, if $(C; f)$ is a measuring from A to B, then there is a unique morphism $! : C \to \underline{\mathsf{Alg}}(A, B)$ that makes the following diagram commute.*

$$\begin{tikzcd} C \otimes A \arrow[r, "f"] \arrow[d, dashed, "!\otimes A"'] & B \\ \underline{\mathsf{Alg}}(A, B) \otimes A \arrow[ur, "\mathsf{ev}"'] & \end{tikzcd}$$

If a universal measuring $(\underline{\mathsf{Alg}}(A, B), \mathsf{ev})$ exists, then we obtain a representation $\underline{\mathsf{Alg}}(A, B)$ for $\mu_{-}(A, B) : \mathsf{CoAlg}^{\mathsf{op}} \to \mathsf{Set}$. That is, we have the following bijection, natural in C, A, B.

$$\mu_C(A, B) \cong \mathsf{CoAlg}(C, \underline{\mathsf{Alg}}(A, B)).$$

In the following sections, we will show that if $\mathcal{C}$ is closed and locally presentable and F is accessible, then the universal measuring always exists.

3.2 Local presentability, accessibility, and the measuring tensor

We will now usually require that $\mathcal{C}$ be *locally presentable* and F is *accessible* [2, Def. 1.17 & 2.17]. Then Alg and CoAlg are also locally presentable, the forgetful functor $\mathsf{Alg} \to \mathcal{C}$ has a left adjoint Fr, and the forgetful functor $\mathsf{CoAlg} \to \mathcal{C}$ has a right adjoint Cof [2, Cor. 2.75 & Ex. 2.j]. We will also use that these categories, as locally presentable categories, are complete and cocomplete.

▶ **Example 21.** Set is locally presentable and $\mathsf{id} + 1$ is accessible.

If $\mathcal{C}$ is locally presentable and F is accessible, then for a coalgebra (C, χ), and algebras (A, α) and (B, β), a map $\phi\colon C \otimes A \to B$ uniquely determines an algebra homomorphism $\phi'\colon \mathsf{Fr}(C \otimes A) \to (B, \beta)$. Notice then that a map $\phi\colon C \otimes A \to B$ is a measuring if and only if both composites from $\mathsf{Fr}(C \otimes FA)$ to (B, β) coincide in the following diagram.

$$\begin{tikzcd} \mathsf{Fr}(C \otimes FA) \arrow[r, shift left, "\mathsf{Fr}(\mathsf{id}_C \otimes \alpha)"] \arrow[r, shift right, "f"'] & \mathsf{Fr}(C \otimes A) \arrow[r, "\phi'"] & (B, \beta) \end{tikzcd}$$

In the above, f is obtained as adjunct under the free-forgetful adjunction of the composition

$$C \otimes FA \xrightarrow{\chi \otimes \mathsf{id}} FC \otimes FA \xrightarrow{\nabla_{C,A}} F(C \otimes A) \xrightarrow{F(i)} F(\mathsf{Fr}(C \otimes A)) \xrightarrow{\alpha_{\mathsf{Fr}}} \mathsf{Fr}(C \otimes A),$$

in which i is the unit of the free-forgetful adjunction and α_{Fr} is the algebra structure on the free algebra $\mathsf{Fr}(C \otimes A)$. We have now shown the following.

▶ **Theorem 22.** *Suppose that $\mathcal{C}$ is locally presentable and F is accessible. Consider a coalgebra C and an algebra A. Then the coequalizer of the following diagram in Alg exists, and we denote it by $C \triangleright A$ and call it the* measuring tensor *of C and A.*

$$\begin{tikzcd} \mathsf{Fr}(C \otimes FA) \arrow[r, shift left] \arrow[r, shift right] & \mathsf{Fr}(C \otimes A) \arrow[r, dashed, "\mathsf{coeq}"] & C \triangleright A. \end{tikzcd}$$

Given any algebra B, a measuring $\phi : C \otimes A \to B$ uniquely corresponds to an algebra homomorphism $C \triangleright A \to B$. In other words, we obtain a natural identification

$$\mu_C(A, B) \cong \mathsf{Alg}(C \triangleright A, B).$$

That is, the functor $\mu_C(A, -) : \mathsf{Alg} \to \mathsf{Set}$ is represented by $C \triangleright A$.

In the following sections, we will also construct representing objects for $\mu_C(-, B)$ and $\mu_-(A, B)$.

3.3 Measurings as partial homomorphisms

Now we will often assume that the symmetric monoidal structure of $\mathcal{C}$ is closed. Whenever we do, we will denote the internal hom by $\underline{\mathcal{C}}(-, -)$. In this section, we provide a dual description of measurings when $\mathcal{C}$ is closed, generalizing Definition 2.

Note that since F is lax monoidal, it is also *lax closed*: that is, there is a map

$$\widetilde{\nabla}_{X,Y} : F\left(\underline{\mathcal{C}}(X, Y)\right) \longrightarrow \underline{\mathcal{C}}(FX, FY)$$

natural in $X, Y \in \mathcal{C}$. Indeed, this is the adjunct under the adjunction $- \otimes FX \dashv \underline{\mathcal{C}}(FX, -)$ of the composition

$$F(\underline{\mathcal{C}}(X, Y)) \otimes F(X) \xrightarrow{\nabla_{\underline{\mathcal{C}}(X,Y),X}} F(\underline{\mathcal{C}}(X, Y) \otimes X) \xrightarrow{F(\mathsf{ev}_X)} F(Y),$$

in which ev_X is the counit of the adjunction $- \otimes X \dashv \underline{\mathcal{C}}(X, -)$.

Given a closed monoidal structure, we can connect the notion of measuring with our notion of partial homomorphism from Section 2.

▶ **Proposition 23** (cf. Definition 2)**.** *Suppose that $\mathcal{C}$ is closed. Given algebras (A, α) and (B, β) and a coalgebra (C, χ), a map $\phi : C \otimes A \to B$ is a measuring if and only if its adjunct $\widetilde{\phi} : C \to \underline{\mathcal{C}}(A, B)$ fits in the following commutative diagram*

$$\begin{tikzcd}
 & F(C) \arrow[r, "F(\widetilde{\phi})"] & F(\underline{\mathcal{C}}(A,B)) \arrow[r, "\widetilde{\nabla}_{A,B}"] & \underline{\mathcal{C}}(FA, FB) \arrow[d, "\beta_*"] \\
C \arrow[ur, "\chi"] \arrow[r, "\widetilde{\phi}"'] & \underline{\mathcal{C}}(A,B) \arrow[rr, "\alpha^*"] & & \underline{\mathcal{C}}(FA, B)
\end{tikzcd}$$

where α^* denotes precomposition by α and β_* denotes postcomposition by β. We shall also refer to the pair $(C; \widetilde{\phi})$ as a measuring*.*

▶ **Example 24.** Note that the cartesian monoidal structure on Set is closed, and that the above recovers Definition 2.

This approach allows us to reformulate the notion of measuring as certain coalgebra homomorphisms which we now describe. If $\mathcal{C}$ is locally presentable and F is accessible, then given a coalgebra (C, χ) and algebras (A, α) and (B, β), a map $\phi : C \to \underline{\mathcal{C}}(A, B)$ in $\mathcal{C}$ uniquely determines a coalgebra homomorphism $\phi' : (C, \chi_C) \to \mathsf{Cof}(\underline{\mathcal{C}}(A, B))$. A map $\phi : C \to \underline{\mathcal{C}}(A, B)$ is a measuring if and only if both composites from (C, χ_C) to $\mathsf{Cof}\big(\underline{\mathcal{C}}(FA, B)\big)$ in the following diagram coincide.

$$(C, \chi_C) \xrightarrow{\phi'} \mathsf{Cof}\big(\underline{\mathcal{C}}(A, B)\big) \underset{f}{\overset{\mathsf{Cof}(\underline{\mathcal{C}}(\alpha,B))}{\rightrightarrows}} \mathsf{Cof}\big(\underline{\mathcal{C}}(FA, B)\big)$$

In the above, f is the adjunct under the cofree-forgetful adjunction of the following composite.

$$\mathsf{Cof}(\underline{\mathcal{C}}(A,B)) \xrightarrow{\chi_{\mathsf{Cof}}} F\big(\mathsf{Cof}(\underline{\mathcal{C}}(A,B))\big) \xrightarrow{F(\varepsilon)} F\big(\underline{\mathcal{C}}(A,B)\big) \xrightarrow{\widetilde{\nabla}_{A,B}} \underline{\mathcal{C}}(F(A),F(B)) \xrightarrow{\beta_*} \underline{\mathcal{C}}(F(A),B).$$

Here χ_{Cof} is the coalgebraic structure on the cofree coalgebra, and ε is the counit of the cofree-forgetful adjunction.

Now we can use this to guarantee the existence of a universal measuring.

▶ **Theorem 25** (Proof in [7, Appendix A.3]). *Suppose that $\mathcal{C}$ is locally presentable and closed and that F is accessible. Given algebras A and B, then the universal measuring coalgebra $\underline{\mathsf{Alg}}(A,B)$ exists and is obtained as the following equalizer diagram in* CoAlg

$$\underline{\mathsf{Alg}}(A,B) \overset{\mathsf{eq}}{\dashrightarrow} \mathsf{Cof}\big(\underline{\mathcal{C}}(A,B)\big) \rightrightarrows \mathsf{Cof}\big(\underline{\mathcal{C}}(F(A),B)\big),$$

with $\widetilde{\mathsf{ev}}\colon \underline{\mathsf{Alg}}(A,B) \to \underline{\mathcal{C}}(A,B)$ obtained as the composition of the equalizer map eq *together with the counit $\mathsf{Cof}\big(\underline{\mathcal{C}}(A,B)\big) \to \underline{\mathcal{C}}(A,B)$ of the cofree-forgetful adjunction.*

▶ **Corollary 26.** *Suppose that $\mathcal{C}$ is locally presentable and closed and that F is accessible. Given algebras A and B, the functor $\mu_{-}(A,B) : \mathsf{CoAlg}^{\mathsf{op}} \to \mathsf{Set}$ is represented by $\underline{\mathsf{Alg}}(A,B)$.*

3.4 Measuring via the convolution algebra

We will now describe the last representable object for the measuring functor.

▶ **Definition 27** (Convolution algebra, cf. Definition 7). *Suppose that $\mathcal{C}$ is closed. Given a coalgebra (C,χ) and an algebra (A,α) in $\mathcal{C}$, we define an algebra structure on $\underline{\mathcal{C}}(C,A)$, called the* convolution algebra*, which is denoted $[(C,\chi),(A,\alpha)]$ or simply $[C,A]$, as follows. The algebra structure $F[C,A] \to [C,A]$ is the composition*

$$F(\underline{\mathcal{C}}(C,A)) \xrightarrow{\widetilde{\nabla}_{C,A}} \underline{\mathcal{C}}(FC,FA) \xrightarrow{\alpha_*\chi^*} \underline{\mathcal{C}}(C,A),$$

where $\alpha_*\chi^*$ denotes postcomposition by α and precomposition by χ. The convolution algebra construction lifts the internal hom to a functor

$$[-,-]\colon \mathsf{CoAlg}^{\mathsf{op}} \times \mathsf{Alg} \longrightarrow \mathsf{Alg}.$$

The convolution algebra provides a representing object for $\mu_C(-,B) : \mathsf{Alg}^{\mathsf{op}} \to \mathsf{Set}$. Indeed, we have the following bijection natural in C, A, B.

$$\mu_C(A,B) \cong \mathsf{Alg}(A,[C,B]).$$

In other words, a measuring $\phi\colon C \otimes A \to B$ corresponds to an algebra homomorphism $\phi' : A \to [C,B]$ under the bijection $\mathcal{C}(C\otimes A, B) \cong \mathcal{C}(A, \underline{\mathcal{C}}(C,B))$. Indeed, notice that ϕ' is a homomorphism if and only if the following diagram, adjunct to the one appearing in Definition 18, commutes.

$$\begin{tikzcd}
 & F([C,B]) \arrow[r, "\widetilde{\nabla}_{C,B}"] & {[F(C),F(B)]} \arrow[d, "\beta_*\chi^*"] \\
F(A) \arrow[ur, "F(\phi')"] \arrow[r, "\alpha"'] & A \arrow[r, "\phi'"] & {[C,B]}
\end{tikzcd}$$

▶ Remark 28. The convolution algebra also provides an alternative characterization of the algebra $C \triangleright A$ and coalgebra $\underline{\mathsf{Alg}}(A, B)$. As limits in Alg and colimits in CoAlg are determined in $\mathcal{C}$ [1] and the internal hom $\underline{\mathcal{C}}(-,-) : \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathcal{C}$ preserves limits, the functor $[-,-] : \mathsf{CoAlg}^{\mathsf{op}} \times \mathsf{Alg} \to \mathsf{Alg}$ also preserves limits. Moreover, fixing a coalgebra C, the induced functor $[C,-] : \mathsf{Alg} \to \mathsf{Alg}$ is accessible since filtered colimits in Alg are computed in $\mathcal{C}$ (see [1, 5.6]). Therefore, by the adjoint functor theorem [2, 1.66], the functor $[C,-]$ is a right adjoint. Its left adjoint is precisely $C \triangleright - : \mathsf{Alg} \to \mathsf{Alg}$. Indeed, for any algebras A and B, we obtain the following bijection, natural in C, A, B.

$$\mathsf{Alg}\big(C \triangleright A, B\big) \cong \mathsf{Alg}\big(A, [C, B]\big).$$

Notice we can also determine the universal measuring by using the adjoint functors. Fixing now an algebra B, the opposite functor $[-, B]^{\mathsf{op}} : \mathsf{CoAlg} \to \mathsf{Alg}^{\mathsf{op}}$ preserves colimits, where the domain is locally presentable and the codomain is essentially locally small. By the adjoint functor theorem [2, 1.66] and [4, 5.5.2.10], this functor is a left adjoint. Its right adjoint is precisely the functor $\underline{\mathsf{Alg}}(-, B) : \mathsf{Alg}^{\mathsf{op}} \to \mathsf{CoAlg}$. Indeed, for any algebra A and B and any coalgebra C, we have the following bijection, natural in C, A, B.

$$\mathsf{CoAlg}\big(C, \underline{\mathsf{Alg}}(A, B)\big) \cong \mathsf{Alg}\big(A, [C, B]\big).$$

Combining the identifications, we see that the measuring functor is representable in each factor:

$$\mu_C(A, B) \cong \mathsf{CoAlg}\big(C, \underline{\mathsf{Alg}}(A, B)\big) \cong \mathsf{Alg}\big(A, [C, B]\big) \cong \mathsf{Alg}\big(C \triangleright A, B\big).$$

In other words, for any algebra A and B and any coalgebra C, the following data are equivalent.

$C \otimes A \to B$	$C \to \underline{\mathcal{C}}(A, B)$	$C \to \underline{\mathsf{Alg}}(A, B)$	$A \to [C, B]$	$C \triangleright A \to B$
measuring	measuring	coalgebra homomorphism	algebra homomorphism	algebra homomorphism

▶ **Definition 29.** *Assuming that $\mathcal{C}$ is locally presentable and F is accessible,* Alg *has an initial object which we denote by N.*

Let $(-)^* : \mathsf{CoAlg}^{\mathsf{op}} \to \mathsf{Alg}$ denote the functor $[-, N]$, and call C^* the dual algebra of C *for any coalgebra C.*

Let $(-)^\circ : \mathsf{Alg}^{\mathsf{op}} \to \mathsf{CoAlg}$ denote the functor $\underline{\mathsf{Alg}}(-, N)$, and call A° the dual coalgebra of *A for any algebra A.*

These functors form a dual adjunction since we have the following bijection, natural in C, A:

$$\mathsf{Alg}(A, C^*) \cong \mathsf{CoAlg}(C, A^\circ).$$

3.5 Measuring as an enrichment

We now come to the main punchline of the general theory presented in this paper: that $\underline{\mathsf{Alg}}(-,-)$ gives the category of algebras an enrichment in coalgebras. First, we describe how to compose measurings.

▶ **Proposition 30.** *The category* CoAlg *has a symmetric monoidal structure for which the forgetful functor* $\mathsf{CoAlg} \to \mathcal{C}$ *is strong symmetric monoidal.*

Proof. Suppose (C, χ_C) and (D, χ_D) are coalgebras. Then $C \otimes D$ has the following coalgebra structure.

$$C \otimes D \xrightarrow{\chi_C \otimes \chi_D} F(C) \otimes F(D) \xrightarrow{\nabla_{C,D}} F(C \otimes D)$$

The morphism $\eta : \mathbb{I} \to F(\mathbb{I})$ provides the coalgebraic structure on $\mathbb{I}$. One can verify that $(\mathsf{CoAlg}, \otimes, (\mathbb{I}, \eta))$ is a symmetric monoidal category (see details in [7, Appendix A.4]). ◀

Now we can prove our main theorem.

▶ **Theorem 31** (Proof in [7, Appendix A.5])**.** *Suppose that $\mathcal{C}$ is locally presentable and closed and that F is accessible. Then the category* Alg *is enriched, tensored, and powered over the symmetric monoidal category* CoAlg *respectively via*

$$\mathsf{Alg}^{\mathsf{op}} \times \mathsf{Alg} \xrightarrow{\underline{\mathsf{Alg}}(-,-)} \mathsf{CoAlg}, \quad \mathsf{CoAlg} \times \mathsf{Alg} \xrightarrow{-\triangleright-} \mathsf{Alg}, \quad \mathsf{CoAlg}^{\mathsf{op}} \times \mathsf{Alg} \xrightarrow{[-,-]} \mathsf{Alg}.$$

▶ **Example 32** (Details in [7, Appendix A.7])**.** Suppose that $\mathcal{C}$ is locally presentable and closed. The following endofunctors on $\mathcal{C}$ are accessible and lax symmetric monoidal.

(id) The identity endofunctor $\mathsf{id}_{\mathcal{C}}$.

(A) The constant endofunctor that sends each object to a fixed commutative monoid A in $\mathcal{C}$.

(GF) The composition GF of accessible, lax symmetric monoidal endofunctors F and G.

($F \otimes G$) The pointwise tensor product $F \otimes G$ of accessible, lax symmetric monoidal endofunctors F and G, assuming $\mathcal{C}$ is closed.

($F + G$) The pointwise coproduct $F + G$ of an accessible, lax symmetric monoidal endofunctor F and an accessible endofunctor G equipped with natural transformations $GX \otimes GY \to G(X \otimes Y)$, $\lambda : FX \otimes GY \to G(X \otimes Y)$, $\rho : GX \otimes FY \to G(X \otimes Y)$ satisfying the axioms described in [7, Appendix A.7], assuming $\mathcal{C}$ is closed.

(id^A) The exponential id^A for any object A of $\mathcal{C}$, assuming the monoidal product on $\mathcal{C}$ is cartesian closed.

(W-types) A polynomial endofunctor associated to a morphism $f : X \to Y$ in Set, given a commutative monoid structure on Y and an oplax symmetric monoidal structure on the preimage functor $f^{-1} : C \to \mathsf{Set}$.

(d.e.s.) A discrete equational system, assuming that the monoidal structure on $\mathcal{C}$ is cocartesian and that $\mathcal{C}$ has binary products that preserve filtered colimits.

On some occasions, the category of coalgebras of F can be interesting while its category of algebras is less so. For instance, given an alphabet Σ, coalgebras over the endofunctor $F(X) = 2 \times X^{\Sigma}$ in Set are automata but the initial algebra remains $\emptyset$. To remedy this, we can extend our main result into the following theorem.

▶ **Theorem 33** (Proof in [7, Appendix A.6])**.** *Suppose that $\mathcal{C}$ is locally presentable and closed and that F is also accessible. Let $G \colon \mathcal{C} \to \mathcal{C}$ be a $\mathcal{C}$-enriched functor that is accessible. Then* Alg_{GF} *is enriched, tensored and powered over* CoAlg_F.

▶ **Example 34.** If $F(X) = 2 \times X^{\Sigma}$, we could consider $G = \mathsf{id} + 1$, and thus Alg_{GF} has $\mathbb{N}$ as an initial object and remains enriched in automata.

The enrichment of algebras in coalgebras specify a pairing of coalgebras

$$\underline{\mathsf{Alg}}(B, T) \otimes \underline{\mathsf{Alg}}(A, B) \longrightarrow \underline{\mathsf{Alg}}(A, T),$$

regarded as an enriched composition, for any algebras A, B and T. In more details, the above coalgebra homomorphism is induced by the measuring of

$$\big(\underline{\mathsf{Alg}}(B,T)\otimes\underline{\mathsf{Alg}}(A,B)\big)\otimes A \xrightarrow{\mathsf{id}\otimes\mathsf{ev}_{A,B}} \underline{\mathsf{Alg}}(B,T)\otimes B \xrightarrow{\mathsf{ev}_{B,T}} T.$$

In other words, the enrichment is recording precisely that we can compose a measuring $C\otimes A\to B$ with $D\otimes B\to T$ to obtain a measuring $(D\otimes C)\otimes A\to T$. In particular, our above discussion shows that $\underline{\mathsf{Alg}}(A,A)$ is always a monoid object in the symmetric monoidal category $(\mathsf{CoAlg},\otimes,\mathbb{I})$.

3.6 General C-initial objects

Now we generalize Section 2.5. We can use the extra structure in the enriched category of algebras to specify more algebras than we could in the unenriched category of algebras.

▶ **Definition 35** (C-initial algebra, cf. Definition 9 and Definition 12). *Suppose that $\mathcal{C}$ is locally presentable and closed and that F is accessible.*

*Given a coalgebra C, we say an algebra A is a C-*initial algebra *if there exists a unique map $C\to\underline{\mathsf{Alg}}(C,X)$, for all algebras X.*

The terminal *C-initial algebra is the terminal object, if it exists, in the subcategory of* Alg *spanned by the C-initial algebras.*

We end with a result that helped us calculate some terminal C-initial algebras in Section 2.5.

▶ **Proposition 36** (Proof in [7, Appendix A.8]). *Suppose that $\mathcal{C}$ is locally presentable and closed and that F is accessible. There is a unique map from any C-initial algebra to C^*.*

4 Conclusions & Vista

In this paper, we have shown that given a closed symmetric monoidal category $\mathcal{C}$ and an accessible lax symmetric monoidal endofunctor F on $\mathcal{C}$, the category of algebras of F is enriched, tensored, and cotensored in the category of coalgebras of F. The algebras of such a functor are of central importance in theoretical computer science, and we hope that identifying such extra structure can shed light on these studies. Indeed, we have demonstrated one use case: we can now specify C-initial algebras in an analogous way to initial algebras. We identified a large class of examples of endofunctors that are encompassed by our theory. Thus, we have established the beginning of an *enriched* analogue of the theory of W-types. We have also worked out concretely the results for the endofunctor $\mathsf{id}+1$ on Set, which suggested a meaningful interpretation of the enrichment as partial algebra homomorphisms.

In future work, we will present similar meaningful interpretations for other endofunctors of our theory. Our future plans involve incorporating features, such as C-initial algebras, of this new enriched theory into concrete programming languages like Haskell or Agda.

We also seek to extend the results of Example 15 into more general settings and provide conditions for the existence of the terminal C-initial algebras. We will also develop more robust theory from Theorem 33. Our partial algebra homomorphisms remain total functions: it would be interesting to develop a theory that encodes maps that are partial both as a function and as algebra homomorphisms. Lastly in Example 32, when we consider the constant functor at an object A, we must choose a commutative monoid structure on A. What if we had two different monoidal structures on A? There are other such choices that are needed in Example 32, for instance in our motivating example of W-types. We seek to understand how these choices interact with one another.

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