Effective Continued Fraction Dimension Versus Effective Hausdorff Dimension of Reals

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Abstract

We establish that constructive continued fraction dimension originally defined using s-gales [20] is robust, but surprisingly, that the effective continued fraction dimension and effective (base-b) Hausdorff dimension of the same real can be unequal in general.

We initially provide an equivalent characterization of continued fraction dimension using Kolmogorov complexity. In the process, we construct an optimal lower semi-computable s-gale for continued fractions. We also prove new bounds on the Lebesgue measure of continued fraction cylinders, which may be of independent interest.

We apply these bounds to reveal an unexpected behavior of continued fraction dimension. It is known that feasible dimension is invariant with respect to base conversion [8]. We also know that Martin-Löf randomness and computable randomness are invariant not only with respect to base conversion, but also with respect to the continued fraction representation [20]. In contrast, for any $0 < \varepsilon < 0.5$, we prove the existence of a real whose effective Hausdorff dimension is less than ε, but whose effective continued fraction dimension is greater than or equal to 0.5. This phenomenon is related to the "non-faithfulness" of certain families of covers, investigated by Peres and Torbin [22] and by Albeverio, Ivanenko, Lebid and Torbin [1].

We also establish that for any real, the constructive Hausdorff dimension is at most its effective continued fraction dimension.

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1 Introduction

The concept of an individual random sequence, first defined by Martin-Löf using constructive measure [15], is well-established and mathematically robust - very different approaches towards the definition identify precisely the same sequences as random. These include Kolmogorov incompressibility (Levin [10], Chaitin [3]) and unpredictability by martingales [24]. While the theory of Martin-Löf randomness *classifies* sequences into random and non-random, it does not *quantify* the information rate in a non-random sequence. Lutz effectivized the classical notions of Hausdorff and packing dimensions [12], surprisingly extending it to individual infinite binary sequences [13], yielding a notion of information density in sequences. This definition also has several equivalent definitions in terms of Kolmogorov compression

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rates [16], unpredictability by s-gales, and using covers [12, 13]. These definitions have led to a rich variety of applications in various domains of computability and complexity theory (see for example, Downey and Hirschfeldt [5], Nies [21]).

Recently, settings more general than the Cantor space of infinite binary (or in general, infinite sequences from a finite alphabet) have been studied by Lutz and Mayordomo [14], and Mayordomo [17, 18]. Prominent among them is Mayordomo's definition of effective Hausdorff dimension for a very general class of metric spaces [17, 18]. Nandakumar and Vishnoi [20] and Vishnoi [28] define the notion of effective dimension of continued fractions, which involves a countably infinite alphabet, and is thus a setting which cannot be studied using Mayordomo's framework. This latter setting is interesting topologically since the space of continued fractions is non-compact, and interesting measure-theoretically since the natural shift invariant measure, the Gauss measure, is a non-product measure.

Nandakumar and Vishnoi [20] use the notion of an s-gale on the space of continued fractions to define effective dimension. Vishnoi [28] introduced the notion of Kolmogorov complexity of finite continued fraction strings using a one to one binary encoding. Vishnoi [28] also shows that the notion of Kolmogorov complexity is invariant under computable 1-1 encodings, upto an additive constant.

In this work, we first establish the mathematical robustness of the notion of effective dimension, by proving an equivalent characterization using Kolmogorov complexity of continued fractions. The characterization achieves the necessary equivalence by choosing a binary encoding of continued fractions which has a compelling geometric intuition, and then applying Mayordomo's characterization of effective (binary) Hausdorff dimension using Kolmogorov complexity [16] . In the process, analogous to the notion of an optimal constructive supergale on the Cantor space defined by Lutz [13], we provide the construction of a lower semi-computable s-gale that is optimal for continued fractions. We also prove new bounds on the Lebesgue measure of continued fraction cylinders using the digits of the continued fraction expansion, a result which may be of independent interest.

The topological and measure-theoretic intricacies involved in this setting imply that some, but not all, "natural" properties of randomness and dimension carry over from the binary setting. For example, while Martin-Löf and computable randomness are invariant with respect to the conversion between the base-b and continued fraction expansion of the same real [19, 20], Vandehey [26] and Scheerer [23] show that other notions of randomness like absolute normality and normality for continued fractions are not identical.

Staiger [25] showed that the Kolmogorov complexity of a base b expansion of a real $\alpha, 0 \leq \alpha \leq 1$, is independent of the chosen base b. Aligning with this, Hitchcock and Mayordomo [8] establish that feasible dimension of a real is the same when converting between one base to another. Hitherto, it was unknown whether effective dimension is invariant with respect to conversion between base-b and continued fraction representations. Since we can convert between the representations efficiently, it is possible that these are equal. We show this is true in one direction, that the effective base b dimension is a lower bound for effective continued fraction dimension.

However, using the technique of diagonalization against the optimal lower semicomputable continued fraction s-gale and using set covering techniques used in recent works by Peres and Torbin [22], Albeverio, Ivanenko, Lebid and Torbin [1] and Albeverio, Kondratiev, Nikiforov and Torbin [2] to show the "non-faithfulness" of certain families of covers, we show that the reverse direction does not hold, in general. We prove the following result: for every $0 < \varepsilon < 0.5$, there is a real whose effective (binary) Hausdorff dimension is less than ε while its effective continued fraction dimension is at least 0.5. By the result of Hitchcock and

Mayordomo [8], this also implies that the effective base-b dimension of this real is less than ε in every base-b, $b \geq 2$. Thus, surprisingly, there is a sharp gap between the effective (base-b) dimension of a real and its effective continued fraction dimension, highlighting another significant difference in this setting.

2 Preliminaries

We denote the binary alphabet by Σ. The set of strings of a particular length n is denoted Σ^n. The set of all finite binary strings is denoted Σ^* and infinite binary sequences is denoted Σ^∞. For a binary string $v \in \Sigma^n \setminus \{0^n \cup 1^n\}$, $v - 1$ denotes the string occurring just before v lexicographically, and $v + 1$ the string occurring just after v lexicographically. We use $\mathbb{N}$ to denote the set of positive integers. The set of finite continued fractions is denoted $\mathbb{N}^*$ and the set of all infinite continued fractions, as $\mathbb{N}^\infty$.

We adopt the notation $[a_1, a_2, \dots]$ for the continued fraction

$$\cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}$$

and similarly, $[a_1, a_2, \dots, a_n]$ for finite continued fractions.

If a finite binary string x is a prefix of a finite string z or an infinite binary sequence Z, then we denote this by $x \sqsubseteq z$ or $x \sqsubseteq Z$ respectively. If x is a proper prefix of a finite string z, we denote it by $x \sqsubset z$. We adopt the same notation for denoting that a finite continued fraction v is a prefix of another continued fraction. For a $v \in \mathbb{N}^*$, the *cylinder set* of v, denoted C_v, is defined by $C_v = \{Y \in \mathbb{N}^\infty \mid v \sqsubset Y\}$. For a $w \in \Sigma^*$, C_w is defined similarly. For a continued fraction string $v = [a_1 \dots a_n]$, $P(v)$ denotes the string $[a_1, \dots a_{n-1}]$. λ denotes the empty string and we define $P(\lambda) = \lambda$.

For $v \in \mathbb{N}^*$, $\mu(v)$ refers to the Lebesgue measure of the continued fraction cylinder C_v. $\gamma(v)$ refers to the Gauss measure of the continued fraction cylinder C_v, defined by $\gamma(v) = \int_{C_v} \frac{1}{1+x}\, dx$. We use the same notation for a binary cylinder $w \in \Sigma^*$. It is well-known that the Gauss measure is absolutely continuous with respect to the Lebesgue measure, and is invariant with respect to the left-shift transformation on continued fractions (see for example, [4], or [6]). Wherever there is no scope for confusion, for a $v \in \mathbb{N}^*$, we use $\mu(v)$ and $\gamma(v)$ to represent $\mu(C_v)$ and $\gamma(C_v)$ respectively. The same holds for a $v \in \Sigma^*$. We also use the notation $\mu^s(v)$ and $\gamma^s(v)$ to denote $(\mu(v))^s$ and $(\gamma(v))^s$ respectively. For a continued fraction string $v = [a_1, \dots, a_n]$, we call n as the rank of v, and we denote it using $rank(v)$. $[v, i]$ denotes the continued fraction $[a_1, \dots a_n, i]$. For an infinite continued fraction string $Y = [a_1, a_2, \dots]$, $Y \upharpoonright n$ denotes the continued fraction string corresponding to the first n entries of Y, that is $Y \upharpoonright n = [a_1, a_2 \dots a_n]$. For $k \in \mathbb{N}$, $\mathbb{N}^{\leq k}$ refers to the set of continued fraction strings having rank less than or equal to k. All logarithms in the work have base 2, unless specified otherwise. For any sets A and B, $A \Delta B$ denotes the symmetric set difference operator, defined by $(A \setminus B) \cup (B \setminus A)$. In this work, for ease of notation, $Y \in \mathbb{N}^*$ denotes an infinite continued fraction and $X \in \Sigma^\infty$ denotes an infinite binary sequence.

2.1 Constructive dimension of binary sequences

Lutz [13] defines the notion of effective (equivalently, constructive) dimension of an individual infinite binary sequence using the notion of the success of s-gales.

▶ **Definition 1** (Lutz [13]). *For $s \in [0,\infty)$, a binary s-gale is a function $d : \Sigma^* \to [0,\infty)$ such that $d(\lambda) < \infty$ and for all $w \in \Sigma^*$, $d(w)[\mu(C_w)]^s = \sum_{i\in\{0,1\}} d(wi)[\mu(C_{wi})]^s$.*

The success set *of d is $S^\infty(d) = \left\{X \in \mathbb{N}^\infty \mid \limsup_{n\to\infty} d(X \upharpoonright n) = \infty\right\}$.*

For $\mathcal{F} \subseteq [0,1]$, $\mathcal{G}(\mathcal{F})$ denotes the set of all $s \in [0,\infty)$ such that there exists a lower semicomputable binary s-gale d with $\mathcal{F} \subseteq S^\infty(d)$.

The constructive dimension *or* effective Hausdorff dimension *of $\mathcal{F} \subseteq [0,1]$ is $\mathrm{cdim}(\mathcal{F}) = \inf \mathcal{G}(\mathcal{F})$ and the constructive dimension of a sequence $X \in \Sigma^\infty$ is $\mathrm{cdim}(X) = \mathrm{cdim}(\{X\})$.*

3 Effective Continued Fraction Dimension using s-gales

Nandakumar and Vishnoi [20] formulate the notion of effective dimension of continued fractions using the notion of lower semicomputable continued fraction s-gales. Whereas a binary s-gale bets on the digits of the binary expansion of a number, a continued fraction s-gales places bets on the digits of its continued fraction expansion.

▶ **Definition 2** (Nandakumar, Vishnoi [20]). *For $s \in [0,\infty)$, a continued fraction s-gale is a function $d : \mathbb{N}^* \to [0,\infty)$ such that $d(\lambda) < \infty$ and for all $w \in \mathbb{N}^*$, the following holds.*

$$d(w)[\gamma(C_w)]^s = \sum_{i\in\mathbb{N}} d(wi)[\gamma(C_{wi})]^s.$$

The success set *of d is $S^\infty(d) = \left\{Y \in \mathbb{N}^\infty \mid \limsup_{n\to\infty} d(Y \upharpoonright n) = \infty\right\}$.*

In this paper, we deal with the notion of effective or equivalently, constructive dimension. In order to effectivize the notion of s-gales, we require them to be *lower semicomputable.*

▶ **Definition 3.** *A function $d : \mathbb{N}^* \longrightarrow [0,\infty)$ is called* lower semicomputable *if there exists a total computable function $\hat{d} : \mathbb{N}^* \times \mathbb{N} \longrightarrow \mathbb{Q} \cap [0,\infty)$ such that the following two conditions hold.*

- ***Monotonicity**: For all $w \in \mathbb{N}^*$ and for all $n \in \mathbb{N}$, we have $\hat{d}(w,n) \le \hat{d}(w,n+1) \le d(w)$.*
- ***Convergence**: For all $w \in \mathbb{N}^*$, $\lim_{n\to\infty} \hat{d}(w,n) = d(w)$.*

For $\mathcal{F} \subseteq [0,1]$, $\mathcal{G}_{CF}(\mathcal{F})$ denotes the set of all $s \in [0,\infty)$ such that there exists a lower semicomputable continued fraction s-gale d with $\mathcal{F} \subseteq S^\infty(d)$.

▶ **Definition 4** (Nandakumar, Vishnoi [20]). *The* effective continued fraction dimension *of $\mathcal{F} \subseteq [0,1]$ is*

$$\mathrm{cdim}_{CF}(\mathcal{F}) = \inf \mathcal{G}_{CF}(\mathcal{F}).$$

The effective continued fraction dimension *of a sequence $Y \in \mathbb{N}^\infty$ is defined by $\mathrm{cdim}_{CF}(\{Y\})$, the effective continued fraction dimension of the singleton set containing Y.*

3.1 Conversion of binary s-gales into continued fraction s-gales

In this subsection, from a continued fraction s'-gale $d : \mathbb{N}^* \to [0,\infty)$, for any $s > s'$, we construct a binary s-gale $h : \Sigma^* \to [0,\infty)$ which succeeds on all the reals on which d succeeds. The construction proceeds in multiple steps. We first mention some technical lemmas which we use in the proof.

The following lemma is an easy consequence of the fact that the Gauss measure is absolutely continuous with respect to the Lebesgue measure (see for example, Nandakumar and Vishnoi [20]).

▶ **Lemma 5.** *For any interval $B \subseteq (0,1)$, we have*

$$\frac{1}{2\ln 2}\mu(B) \leq \gamma(B) \leq \frac{1}{\ln 2}\mu(B).$$

In the construction that follows, we formulate betting strategies on binary cylinders based on continued fraction cylinders. In order to do this conversion, we require the following bounds on the relationships between the lengths of continued fraction cylinders and binary cylinders.

▶ **Lemma 6** (Nandakumar, Vishnoi [20])**.** *For any $0 \leq a < b \leq 1$, let $\left[\frac{m}{2^k}, \frac{m+1}{2^k}\right)$, where $0 \leq m \leq 2^k - 1$ be one of the largest dyadic intervals which is a subset of $[a,b)$, then $\frac{1}{2^k} \geq \frac{1}{4}(b-a)$.*

▶ **Lemma 7** (Falconer [7])**.** *For any $0 \leq a < b \leq 1$, let $\left[\frac{m}{2^k}, \frac{m+1}{2^k}\right)$, $\left[\frac{m+1}{2^k}, \frac{m+2}{2^k}\right)$, where $0 \leq m \leq 2^k - 2$, be the smallest consecutive dyadic intervals whose union covers $[a,b)$. Then $\frac{1}{2^k} \leq 2(b-a)$.*

The following lemma is a generalization of the Kolmogorov inequality for continued fraction martingales (Vishnoi [27]) to s-gales. The lemma states that an equality holds in the case of decompositions using prefix-free subcylinder sets upto a finite depth.

▶ **Lemma 8.** *Let $d : \mathbb{N}^* \to [0,\infty)$ be a continued fraction s-gale. Let $v \in \mathbb{N}^*$ and for some $k \in \mathbb{N}$, let A be a prefix free set of elements in $\mathbb{N}^{\leq k}$ such that $\cup_{w\in A} C_w = C_v$. Then, we have $d(v)\gamma^s(v) = \sum_{w\in A} d(w)\gamma^s(w)$.*

In the construction of a binary s-gale from continued fraction gales, The first step is the following decomposition of a binary cylinder into a set of prefix free continued fraction cylinders.

▶ **Lemma 9** (Vishnoi [27])**.** *For every $w \in \Sigma^*$, there exists a set $I(w) \subseteq \mathbb{N}^*$ and a constant $k \in \mathbb{N}$ such that,*

1. $y \in \mathbb{N}^{\leq k}$ *for every* $y \in I(w)$.
2. $(\cup_{y\in I(w)} C_y)\Delta C_w \subseteq \{\inf(C_w), \sup(C_w)\}$
3. $I(w0) \cup I(w1) = I(w)$
4. $I(w0) \cap I(w1) = \phi$

Moreover, given $w \in \Sigma^*$, Vishnoi [27] gives a division algorithm to compute $I(w)$. It is also clear from the division algorithm that for all $w \in \Sigma^*$, there exists a $u \in I(w)$ such that for all $v \in (I(w0) \cup I(w1)) \setminus I(w)$, we have $u \sqsubset v$. This $u \in I(w)$ is the continued fraction cylinder for which the mid point of w, $m(w)$ is an interior point in C_u and therefore gets divided.

From a continued fraction martingale, Vishnoi [27] uses the decomposition $I(w)$ to construct a binary martingale that places the same aggregate bets on an interval. We generalize this construction to the setting of s-gales. Given a continued fraction s'-gale $d : \mathbb{N}^* \to [0,\infty)$, using the decomposition $I(w)$, we construct a binary s'-gale H_d from d.

▶ **Definition 10.** *Given any continued fraction s'-gale $d : \mathbb{N}^* \to [0,\infty)$, define the* Proportional binary s'-gale *of d, $H_d : \Sigma^* \to [0,\infty)$ as follows:*

$$H_d(w) = \sum_{y\in I(w)} d(y)\left(\frac{\gamma(y)}{\mu(w)}\right)^{s'}.$$

For a $w \in \Sigma^*$, let $I'(w) = I(w0) \cup I(w1)$. Then we have,

$$H_d(w0) + H_d(w1) = 2^{s'} \sum_{y \in I'(w)} d(y) \Big(\frac{\gamma(y)}{\mu(w)} \Big)^{s'}.$$

Let $u \in I(w)$ such that for all $v \in I'(w) \setminus I(w)$, $u \sqsubset v$. Hence, by Lemma 8, it follows that $\sum_{y \in I'(w)} d(y)\gamma^{s'}(y) = \sum_{y \in I(w)} d(y)\gamma^{s'}(y)$. Therefore, we have $H_d(w0)+H_d(w1) = 2^{s'} H_d(w)$, so H_d is an s'-gale. Also as $\gamma(\lambda) = 1$, we have that $H_d(\lambda) = d(\lambda)$.

As $I(w)$ is computably enumerable, it follows that H_d is lower semicomputable if d is lower semicomputable.

The construction by Vishnoi [27] proceeds using the *savings-account trick* for martingales. In the setting of s-gales, however, the concept of a savings account does not work directly. Therefore, we require additional constructions in this setting.

Using ideas from the construction given in Lemma 3.1 in Hitchcock and Mayordomo [8], we construct a "smoothed" s-gale $H_h : \Sigma^* \to [0, \infty)$ from the proportional s'-gale constructed in Definition 10.

▶ **Definition 11.** *For a $w \in \Sigma^*$, and an $n > |w|$, we define*

$$F_n(w) = \{u \in \{0^n \cup 1^n\} \mid w \sqsubseteq u\} \;\cup\; \{u \in \Sigma^n \setminus \{0^n \cup 1^n\} \mid w \sqsubseteq u+1 \text{ and } w \sqsubseteq u-1\},$$
$$H_n(w) = \{u \in \Sigma^n \mid w \sqsubseteq u \text{ or } w \sqsubseteq u+1 \text{ or } w \sqsubseteq u-1\} \setminus F_n.$$

▶ **Definition 12.** *Given an s'-gale $h : \Sigma^* \to [0, \infty)$, for any $s > s'$ and for each $n \in \mathbb{N}$, define:*

$$h_n(w) = \begin{cases} 2^{s|w|} \left(\sum\limits_{u \in H_n(w)} \frac{1}{2}\, h(u) \;+\; \sum\limits_{u \in F_n(w)} h(u) \right) & \text{if } |w| < n \\[2ex] 2^{(s-1)(|w|-n+1)} \;\; h_n(w[0 \ldots n-2]) & \text{otherwise.} \end{cases}$$

Define $S_h : \Sigma^* \to [0, \infty)$ by

$$S_h(w) = \sum_{n=0}^{\infty} 2^{-sn}\, h_n(w).$$

We call S_h as the smoothed s-gale of h*.*

Consider a string $w \in \Sigma^n$ other than 0^n and 1^n. In h_n, a factor of half the capital of w gets assigned to it's immediate parent w'. The other half is assigned to the neighbor of w' to which w is adjacent to.

It is straightforward to verify that each h_n is an s-gale. S_h is a combination of s-gales, and hence is a valid s-gale. Note that $h_n(\lambda) = \sum_{u \in \Sigma^n} h(u) = 2^{s'n}$. Therefore as $s > s'$, $S_h(\lambda) = \sum_{n \in \mathbb{N}} 2^{(s'-s)n}$ is finite. If h is lower semicomputable, it follows that S_h is lower semicomputable.

Combining the constructions given in the section, for any $s > s'$, we show the construction of a binary s-gale from a continued fraction s'-gale, satisfying certain bounds on the capital acquired.

This construction helps to establish a lower bound on effective continued fraction dimension using effective binary dimension. It is also central in formulating a Kolmogorov complexity characterization for continued fraction dimension.

▶ **Lemma 13.** *For $s' \in (0,\infty)$, let $d : \mathbb{N}^* \to [0,\infty)$ be a continued fraction s'- gale. Then, for any $s > s'$, there exists a binary s-gale $h : \Sigma^* \to [0,\infty)$ such that for any $v \in \mathbb{N}^*$ and for any $b \in \Sigma^*$ such that $C_b \cap C_v \neq \phi$ and $\frac{1}{16}\mu(v) \leq \mu(b) \leq 2\mu(v)$, we have*

$$h(b) \geq c_s d(v),$$

where c_s is a constant that depends on s. Moreover, if d is lower semicomputable, then h is lower semicomputable.

4 Kolmogorov Complexity characterization of Continued Fraction Dimension

Mayordomo [16] extended the result by Lutz [11] to show that effective dimension of a binary sequence $X \in \Sigma^\infty$ can be characterized in terms of the Kolmogorov complexity of the finite prefixes of X.

▶ **Theorem 14** (Mayordomo [16] and Lutz [11])**.** *For every $X \in \Sigma^\infty$,*

$$\mathrm{cdim}(X) = \liminf_{n\to\infty} \frac{K(X \upharpoonright n)}{n}.$$

We provide a similar characterization for effective continued fraction dimension. To obtain the Kolmogorov complexity of a continued fraction string, we use the Kolmogorov complexity of one of its binary encodings.

The idea of encoding a finite continued fraction using a 1-1 binary encoding is present in Vishnoi [28]. The author presents an invariance theorem stating that every computable binary 1-1 encoding of continued fractions defines the same Kolmogorov complexity, up to an additive constant. Hence in this work, we use a new binary encoding to define Kolmogorov complexity of continued fractions, which helps us establish the characterization of effective dimension of continued fractions in a fairly simple manner while having intuitive geometric meaning.

▶ **Definition 15** (Many-one binary encoding)**.** *For a continued fraction string $v \in \mathbb{N}^*$, let b_v be the leftmost maximal binary cylinder which is enclosed by C_v. We define $E(v) = b_v$.*

▶ **Lemma 16.** *For any $b \in \Sigma^*$, there exists at most three $v \in \mathbb{N}^*$ such that $E(v) = b$.*

Therefore for any $b \in \Sigma^*$, at most three continued fraction cylinders, say $[v]$, $[v,i]$ and $[v,i,j]$ get mapped to b. Therefore we pad two additional bits of information to $E(v)$, say $b_1(v).b_2(v)$ to identify the continued fraction cylinder that $E(v)$ corresponds to.

▶ **Definition 17** (One-one binary encoding)**.** *For $v \in \mathbb{N}^*$, let $\mathcal{E}(v) = E(v).b_1(v).b_2(v)$. This forms a one to one binary encoding of v.*

We define Kolmogorov complexity of continued fraction string $v \in \mathbb{N}^*$ as the Kolmogorov complexity of $\mathcal{E}(v)$.

▶ **Definition 18** (Kolmogorov complexity of continued fraction strings)**.** *For any $v \in \mathbb{N}^*$, define $K_{\mathcal{E}}(v) = K(\mathcal{E}(v))$.*

Notation. By the invariance theorem of Vishnoi [28], for any $v \in \Sigma^*$, $K_{\mathcal{E}}$ is at most an additive constant more than the complexity of v as defined in [28]. Hence, we drop the suffix and denote the above complexity as $K(v)$.

In the proof of Theorem 14, Mayordomo [16] provides the construction of an s-gale that succeeds on all X for which $s > s' > \liminf_{n\to\infty} \frac{K(X\restriction n)}{n}$. We extend the construction to the setting of continued fractions.

Additionally, we take a convex combination of gales to remove the dependence of the s-gale on the parameter s'. Due to this, we obtain the notion of an optimal lower semicomputable continued fraction s-gale. This notion is crucial in the proofs we use in the upcoming sections.

▶ **Definition 19.** *Given* $0 < s' < s \leq 1$ *let*

$$G_{s'} = \{w \in \mathbb{N}^* \mid K(w) \leq -s' \log(\mu(w))\}.$$

Consider the following function $d_{s'} : \mathbb{N}^* \to [0,\infty)$ *defined by*

$$d_{s'}(v) = \frac{1}{\gamma^s(v)} \left(\sum_{w\in G_{s'};v\sqsubseteq w} \gamma^{s'}(w) + \sum_{w\in G_{s'};w\sqsubset v} \gamma^{s'}(w)\frac{\gamma(v)}{\gamma(w)} \right).$$

Now for each $i \in \mathbb{N}$, *let* $s_i = s(1-2^{-i})$. *Finally, define* $d^* : \mathbb{N}^* \to [0,\infty)$ *by*

$$d^*(v) = \sum_{i=1}^{\infty} 2^{-i} d_{s_i}(v).$$

We now go on to show that the function d^* given in Definition 19 is a lower semicomputable s-gale. Additionally, it succeeds on all continued fraction sequences Y for which the Kolmogorov complexity of its prefixes, $K(Y \restriction n)$ dips below $s \times -\log(\mu(Y \restriction n))$ infinitely often.

▶ **Lemma 20.** *For any* $0 < s \leq 1$, *there exists a lower semicomputable continued fraction* s*-gale* $d^* : \mathbb{N}^* \to [0,\infty)$ *that succeeds on all* $Y \in \mathbb{N}^\infty$ *such that* $\liminf\limits_{n\to\infty} \frac{K(Y\restriction n)}{-\log(\mu(Y\restriction n))} < s$.

We refer to Downey and Hirschfeldt's (Theorem 13.3.4 [5]) proof of the lower bound on constructive dimension using Kolmogorov complexity. The proof fundamentally uses properties of the universal lower semicomputable super-martingale.

For any real having continued fraction dimension less than s, we obtain a lower semicomputable binary s-gale that succeeds on it from Lemma 13. We use the success of this binary s-gale along with the same properties of the universal lower semicomputable super-martingale, to prove the following lemma.

▶ **Lemma 21.** *For any* $Y \in \mathbb{N}^\infty$ *and any* $s > \mathrm{cdim}_{CF}(Y)$, *we have* $\liminf\limits_{n\to\infty} \frac{K(Y\restriction n)}{-\log(\mu(Y\restriction n))} \leq s$.

Therefore, we have the following Kolmogorov complexity based characterization of effective continued fraction dimension.

▶ **Theorem 22.** *For any* $Y \in N^\infty$,

$$dim_{CF}(Y) = \liminf_{n\to\infty} \frac{K(Y\restriction n)}{-\log(\mu(Y\restriction n))}.$$

Proof. For any $Y \in \mathbb{N}^\infty$, let $s^* = \liminf\limits_{n\to\infty} \frac{K(Y\restriction n)}{-\log(\mu(Y\restriction n))}$.

For any $s > s^*$, from Lemma 20, it follows that there exists a lower semicomputable s-gale $\mathcal{D}$ that succeeds on Y. Hence $\dim_{CF}(Y) \leq s^*$.

For any $s > \dim_{CF}(Y)$, from Lemma 21, we have that $s^* \leq s$. Therefore, we have $s^* \leq \dim_{CF}(Y)$. ◀

4.1 Optimal gales and effective continued fraction dimension of a set

Lutz [13] utilizes the notion of the optimal constructive subprobability supermeasure $\mathbf{M}$ on the Cantor space [29] to provide a notion of an optimal constructive supergale.

We note that using Theorem 22, the gale that we obtain from Lemma 20 leads to an analogous notion in the continued fraction setting. We call the continued fraction s-gale d^* thus obtained as the *optimal lower semicomputable continued fraction s-gale.*

▶ **Lemma 23.** *For any $s > 0$, there exists a lower semicomputable continued fraction s-gale $d^* : \mathbb{N}^* \to [0, \infty)$ such that for all $Y \in \mathbb{N}^\infty$ with $\mathrm{cdim}_{CF}(Y) < s$, d^* succeeds on Y.*

Proof. For all $Y \in \mathbb{N}^*$ such that $\mathrm{cdim}_{CF}(Y) < s$, from Theorem 22, it follows that $\liminf\limits_{n\to\infty} \frac{K(Y\upharpoonright n)}{-\log(\mu(Y\upharpoonright n))} < s$. Now applying Lemma 20, we see that the given lemma holds. ◀

Lutz (Theorem 4.1 in [13]) shows that the effective dimension of a set is precisely the supremum of effective dimensions of individual elements in the set, that is for all $X \subseteq [0, 1]$, $\mathrm{cdim}(X) = \sup_{S\in X} \mathrm{cdim}(S)$. Using the notion of the optimal lower semicomputable continued fraction s-gale from Lemma 23, we extend this result to continued fraction dimension.

▶ **Theorem 24.** *For all $\mathcal{F} \subseteq [0, 1]$, $\mathrm{cdim}_{CF}(\mathcal{F}) = \sup_{Y\in\mathcal{F}} \mathrm{cdim}_{CF}(Y)$.*

Proof. For any $s > \mathrm{cdim}_{CF}(\mathcal{F})$, for all $Y \in \mathcal{F}$ there exists a lower semicomputable continued fraction s-gale that succeeds on Y. Thus we have $\sup_{Y\in\mathcal{F}} \mathrm{cdim}_{CF}(Y) \le s$.

Take any any $s > \sup_{Y\in\mathcal{F}} \mathrm{cdim}_{CF}(Y)$. It follows that for all $Y \in \mathcal{F}$, $\mathrm{cdim}_{CF}(Y) < s$. Therefore from Lemma 23, we have that there exists a lower semicomputable continued fraction s-gale $d^* : \mathbb{N}^* \to [0, \infty)$ that succeeds on all $Y \in \mathcal{F}$. Therefore, $\mathrm{cdim}_{CF}(\mathcal{F}) \le s$. ◀

5 Reals with unequal Effective Dimension and Effective Continued Fraction Dimension

In this section, we show that for any set of reals $\mathcal{F} \subseteq [0, 1]$, the effective Hausdorff effective dimension of $\mathcal{F}$ cannot exceed its effective continued fraction dimension. We show that this cannot be improved to an equality. Hence, this inequality is strict in general. We show this by proving the existence of a real such that its effective continued fraction dimension is strictly greater its effective dimension.

5.1 Effective Hausdorff dimension is at most the effective continued fraction dimension

▶ **Theorem 25.** *For any $\mathcal{F} \subseteq [0, 1]$, $\mathrm{cdim}(\mathcal{F}) \le \mathrm{cdim}_{CF}(\mathcal{F})$.*

Proof. Let $s > s' > \mathrm{cdim}_{CF}(\mathcal{F})$. By definition, there exists a lower semicomputable continued fraction s'-gale $d : \mathbb{N}^* \to [0, \infty)$ such that $\mathcal{F} \subseteq S^\infty[d]$.

Take any $Y \in S^\infty[d]$. Let $X \in \Sigma^\infty$ be the corresponding binary representation of Y. By definition, for any $m \in \mathbb{N}$, there exists an $n \in \mathbb{N}$ such that $d(Y \upharpoonright n) > m$. Let $v = Y \upharpoonright n$.

Using Lemma 7, we get two binary cylinders w_1 and w_2 such that $C_v \subseteq C_{w_1} \cup C_{w_2}$ such that $\mu(w_1) = \mu(w_2) \le 2\mu(v)$. We have that since $v \sqsubseteq Y$, $w1 \sqsubseteq X$ or $w2 \sqsubseteq X$. Without loss of generality assume that $w_1 \sqsubseteq X$.

From Lemma 13, we obtain a lower semicomputable s-gale h such that $h(w_1) \ge c_s.d(v) \ge c_s.m$ for some positive constant c_s.

Since m is arbitrary, we see that h succeeds on X. ◀

5.2 Reals with unequal effective Hausdorff and effective continued fraction dimensions

We now provide the main construction of the paper, utilizing the results we show in previous sections.

We first require some technical lemmas about the estimation of Lebesgue measure of a continued fraction cylinder in terms of digits of the continued fraction. Some of the bounds derived in this section may be of independent interest.

In combinatorial arguments, the Gauss measure is often difficult to deal with directly, and it is convenient to use the Lebesgue measure, and derive inequalities.

The following equation, Proposition 1.2.7 in Kraaikamp and Iosifescu [9], is extremely useful in deriving an estimate for the Lebesgue measure of continued fraction cylinders. We derive consequences of this Lemma below, and these are crucial in estimating the dimension of the continued fraction we construct in Section 5. Note that the bounds for Gauss measure are not simple to derive directly.

▶ **Lemma 26** (Kraaikamp, Iosifescu [9]). *For any $v = [a_1, \dots a_n]$ and $i \in \mathbb{N}$,*

$$\frac{\mu([v,i])}{\mu([v])} = \frac{s_n + 1}{(s_n + i)(s_n + i + 1)}$$

where $s_n = [a_n, \dots a_1]$ is the rational corresponding to the reverse of string v.

The lemma given above gives the following bounds on the Lebesgue measure of a continued fraction cylinder in terms of the digits of the continued fraction.

▶ **Lemma 27.** *For any $v = [a_1, \dots a_k] \in \mathbb{N}^k$ we have*

$$\prod_{i=1}^{k} \frac{1}{(a_i + 1)(a_i + 2)} \;\le\; \mu(v) \;\le\; \prod_{i=1}^{k} \frac{2}{a_i\ (a_i + 1)}$$

▶ **Lemma 28.** *Let $v = [a_1 \dots a_k] \in \mathbb{N}^*$. Then for any $a, b \in \mathbb{N}$ such that $b > a$,*

$$\mu\left(\bigcup_{i=a}^{b} [v, i]\right) \le \frac{2}{a} \prod_{i=1}^{k} \frac{2}{a_i\ (a_i + 1)}$$

The following lemma is a direct constructive extension of the proof by Lutz [12]. Using this technique, we convert a set of computably enumerable prefix free binary covers into a lower semicomputable binary s-gale.

▶ **Lemma 29** (Lutz [12]). *For all $n \in \mathbb{N}$, and $\mathcal{F} \subseteq [0, 1]$, if there is a computably enumerable prefix free binary cover $\{B_i^n\}$ of $\mathcal{F}$, such that $\sum_i |B_i^n|^s < 2^{-n}$, then there exists a lower semicomputable binary s-gale that succeeds on $\mathcal{F}$.*

We now proceed to show the construction of the set $\mathcal{F}$. The definition uses a parameter $\mathbf{s}$. We later go on to show that for all such $\mathcal{F}$, there exists an element $Y \in \mathcal{F}$ such that the effective continued fraction dimension of Y is greater than 0.5. We also go on to show that $cdim(F) \le \mathbf{s}$.

We first provide the stage-wise construction of a set $\mathcal{F}_k \subseteq [0, 1]$, such that for each $k \in \mathbb{N}$ $\mathcal{F}_{k+1} \subseteq \mathcal{F}_k$. We then define the set $\mathcal{F}$ using an infinite intersection of the sets $\mathcal{F}_k$.

▶ **Definition 30.** *Let $0 < \mathbf{s} < 0.5$. Define $a_1 = 1$. For any $k \in \mathbb{N}$, such that $k > 1$, define a_k inductively as:*

$$a_k = 2\left(k \prod_{i=1}^{k-1} 100a_i\right)^{1/\mathbf{s}}.$$

For any $k \in \mathbb{N}$, define $b_k = 50.a_k$. Take $\mathcal{F}_0 = \lambda$.
Let $\mathcal{F}_k = \{[v_1 \dots v_k] \in \mathbb{N}^k$ such that $v_i \in [a_i, b_i]$ for $i \in 1$ to $k\}$. Finally define

$$\mathcal{F} = \bigcap_{k=1}^{\infty} \mathcal{F}_k.$$

We use the bounds obtained from Lemma 26, along with basic properties of harmonic numbers to prove the following property of measures of continued fraction sub cylinders.

▶ **Lemma 31.** *For any $x \in \mathbb{N}^*$, $s \le 0.5$ and $a_k, b_k \in N$ such that $b_k = 50.a_k$, $\sum_{i=a_k}^{b_k} \gamma^s([x, i]) > c\gamma^s([x])$ for some $c > 1$.*

Using the bound derived above, we show that for $s = 0.5$, the optimal s-gale d^* formulated in Section 4.1 does not succeed on some sequence in $Y \in \mathcal{F}$. Using this we establish that $\mathrm{cdim}_{CF}(Y) \ge 0.5$.

▶ **Lemma 32.** *There exists a $Y \in \mathcal{F}$ such that $\mathrm{cdim}_{CF}(Y) \ge 0.5$.*

Proof. Let $s = 0.5$. Consider the continued fraction s-gale d^* from Lemma 23. It follows that for all $Y \in \mathbb{N}^*$, if d^* does not succeed on any $Y \in \mathbb{N}^*$, then $\mathrm{cdim}_{CF}(Y) \ge s$.

Consider any $v \in \mathbb{N}^*$, let $rank(v) = k$. From lemma 31, we have that for some $c > 1$, $\sum_{i=a_k}^{b_k} \gamma^s([v, i]) > c.\gamma^s([v])$.

Now if $\forall i \in [a_k, b_k]$, $d^*([v, i]) \ge \frac{1}{c}.d^*(v)$, from the s-gale condition it follows that $d^*(v)\gamma^s(v) \ge \frac{d^*(v)}{c} \sum_{i=a_k}^{b_k} \gamma^s([v, i]) > d^*(v)\gamma^s(v)$, which is a contradiction.

Therefore, it follows that for all $v \in \mathbb{N}^*$, there exists some $n_v \in [a_k, b_k]$ such that $d^*([v, i]) < \frac{1}{c}.d^*([v])$.

Let $v_0 = \lambda$, for each $i \in \mathbb{N}$, we define $v_i = [v_{i-1}, n_{v_{i-1}}]$. Take $Y = \cap_{i=1}^{\infty} C_{v_i}$, it follows that $Y \in F$. Taking $d^*(\lambda) = k$ we get $d^*(Y \upharpoonright n) < \frac{k}{c^n}$. Therefore the continued fraction s-gale d^* does not succeed on Y. Hence $\mathrm{cdim}_{CF}(Y) \ge 0.5$. ◀

From this, it follows that the constructive dimension of the entire set $\mathcal{F}$ is also greater than or equal to 0.5.

▶ **Lemma 33.** $\mathrm{cdim}_{CF}(\mathcal{F}) \ge 0.5$.

Proof. From Theorem 24, we get that $\mathrm{cdim}_{CF}(\mathcal{F}) = \sup_{Y \in \mathcal{F}} \mathrm{cdim}_{CF}(Y)$. From Lemma 32, it follows that there exists a $Y \in F$ such that $\mathrm{cdim}_{CF}(Y) \ge 0.5$. Combining these two, we get that $\mathrm{cdim}_{CF}(\mathcal{F}) \ge 0.5$. ◀

Now we show that the effective Hausdorff dimension of all points in the set $\mathcal{F}$ is less than $\mathbf{s}$. Using ideas from [2], we devise a set of covers S_k for $\mathcal{F}$, by combining adjacent continued fraction cylinders into a single cover.

Using the bounds derived on Lebesgue measure of continued fraction cylinders, we show that for the set of covers S_k for $\mathcal{F}$, the $\mathbf{s}$-dimensional Hausdorff measure shrinks arbitrarily small.

▶ **Lemma 34.** *For $k \in \mathbb{N}$, let $S_k = \{ \bigcup_{i=a_k}^{b_k} [v, i]$ for $v \in \mathcal{F}_{k-1}\}$. Then, $\sum_{S \in S_k} \mu^{\mathbf{s}}(S) \leq 1/k$.*

Proof. The largest element in S_k is $I = [(a_1 \dots a_{k-1}, a_k), [(a_1 \dots a_{k-1}, b_k)]$. The number of elements in S_k equals $\prod_{i=1}^{k-1}(b_i - a_i)$. Additionally, we have $b_i = 50a_i$ for all $i \in \mathbb{N}$. Therefore,

$$\sum_{S \in S_k} \mu^{\mathbf{s}}(S) \leq \mu^{\mathbf{s}}(I) \prod_{i=1}^{k-1} 50a_i.$$

From Lemma 28, it follows that $\mu(I) \leq \frac{2}{a_k} \prod_{i=1}^{k-1} \frac{2}{a_i\ (a_i+1)}$. Therefore,

$$\begin{aligned}\sum_{S \in S_k} \mu^{\mathbf{s}}(S) &\leq \Big(\frac{2}{a_k} \prod_{i=1}^{k-1} \frac{2}{a_i^2}\Big)^{\mathbf{s}} \prod_{i=1}^{k-1}(50a_i) \\ &\leq \frac{2^{\mathbf{s}}}{a_k^{\mathbf{s}}} \prod_{i=1}^{k-1} 100a_i\end{aligned}$$

Since $a_k = 2(k \prod_{i=1}^{k-1} 100a_i)^{1/\mathbf{s}}$, this value is less than $1/k$. ◀

To show that the constructive dimension of $\mathcal{F}$ is less than $\mathbf{s}$, we construct a lower semicomputable binary $\mathbf{s}$-gale that succeeds on $\mathcal{F}$. Using standard techniques, we first convert the covers obtained in Lemma 34 to a set of binary covers of $\mathcal{F}$. Finally applying Lemma 29, we convert the binary covers into a semicomputable $\mathbf{s}$-gale that succeeds on $\mathcal{F}$.

▶ **Lemma 35.** $\mathrm{cdim}(\mathcal{F}) \leq \mathbf{s}$.

Proof. Given $k \in \mathbb{N}$, from Lemma 34, we have that for $S_k = \{ \bigcup_{i=a_k}^{b_k} [v, i]$ for $v \in \mathcal{F}_{k-1}\}$, $\sum_{S \in S_k} \mu^{\mathbf{s}}(S) \leq 1/k$. For each $S \in S_k$, using Lemma 7, we get that for the two smallest consecutive binary cylinders say $b_1(S)$ and $b_2(S)$ that cover S, we have that $\mu(b_1) = \mu(b_2) \leq 2\mu(C)$.

Hence the set $B_k = \{\{b_1(S)\} \cup \{b_2(S)\}$ such that $S \in S_k\}$ forms a binary cover of S_k. Also from Lemma 7, we have that $\sum_{b \in B_k} \mu^{\mathbf{s}}(b) \leq 2^{1+\mathbf{s}} \sum_{S \in S_k} \mu^{\mathbf{s}}(S) \leq 2^{1+\mathbf{s}}/k$.

Note that the set S_k is computable as a_k and b_k are computable for all k. Given any interval S, $b_1(S)$ and $b_2(S)$ are also computable. Hence the set B_k is computable.

Since B_k is a finite set, we can remove all $v \in B_k$ such that $u \sqsubset v$ for some $u \in B_k$, to make B_k prefix free.

For an $n \in \mathbb{N}$, taking $k = \lceil 2^{1+\mathbf{s}}.2^n \rceil$, the set B_k forms a computably enumerable prefix free binary cover of $\mathcal{F}$ such that $\sum_{b \in B_k} \mu^{\mathbf{s}}(b) \leq 2^{-n}$.

Applying Lemma 29, we get that there exists a lower semicomputable $\mathbf{s}$-gale that succeeds on $\mathcal{F}$. Hence the lemma holds. ◀

We sum up the results from Lemma 32 and Lemma 35 into the following theorem.

▶ **Theorem 36.** *Given any $0 < \varepsilon < 0.5$, there exists a $Y \in \mathbb{N}^*$ such that $\mathrm{cdim}_{CF}(Y) \geq 0.5$ and $\mathrm{cdim}(Y) \leq \varepsilon$.*

Proof. Given $0 < \varepsilon < 0.5$, taking $\mathbf{s} = \varepsilon$, construct the set $\mathcal{F}$ given in Definition 30.

From Lemma 32, it follows that there exists a $Y \in \mathcal{F}$ such that $\mathrm{cdim}_{CF}(Y) \geq 0.5$.

From Lemma 35, it follows that for all $X \in F$, $cdim(X) \leq \varepsilon$. Hence $cdim(Y) \leq \varepsilon$. ◀

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A Appendix

A.1 Proof of Lemma 13

Proof. Given an s'-gale $d : \mathbb{N}^* \to [0, \infty)$, let $h' = H_d$, the proportional binary s'-gale of d given in Definition 10.

For a $v \in \mathbb{N}^*$, consider the smallest $w_1, w_2 \in \Sigma^*$ such that $C_v \subseteq C_{w_1} \cup C_{w_2}$. Also we can see that there exists a $S \subseteq I(w_1) \cup I(w_2)$ such that $C_v = \cup_{u \in S} C_u$.

Therefore from Lemma 8, we get $h'(w_1) + h'(w_2) \geq d(v)\frac{\gamma^{s'}(v)}{\mu^{s'}(w_1)}$. From Lemma 7, we get that $\mu(w_1) = \mu(w_2) \leq 2\mu(v)$. Also from Lemma 5, we have that $\gamma(v) \leq (ln2)^{-1}\mu(v)$. Therefore, $h'(w_1) + h'(w_2) \geq (2ln2)^{-s'}d(v)$. Now for any $s > s'$, we have that $h'(w_1) + h'(w_2) \geq c_1.d(v)$, where $c_1 = 1/(2ln2)^s$.

Now for any $s > s'$, consider the smoothed s-gale $h = S_{H_d}$ of the s'-gale H_d given in Definition 11.

Let $|w_1| = n$, and let $W_1 = P(P(w_1))$ be the parent cylinder of parent of w_1. Similarly let $W_2 = P(P(w_2))$. We see that for any $W \in \{W_1, W_2\}$, $h_n(W) \geq 2^{s(n-2)}\frac{h'(w_1)+h'(w_2)}{2} \geq c_2.2^{sn}.d(v)$, where $c_2 = 2^{-(2s+1)}c_1$.

Take any any $b \in \Sigma^*$ such that $C_b \cap C_v \neq \phi$ and $2.\mu(v) \geq \mu(b) \geq \frac{1}{16}\mu(v)$. Since $\mu(b) \leq 2\mu(v)$ and $\mu(v) \leq 2.\mu(w_1)$, it follows that for some $W \in \{W_1, W_2\}$, $W \sqsubseteq b$. Also since $\mu(b) \geq \frac{1}{16}\mu(v)$, we have that, $\mu(b) \geq \frac{1}{32}\mu(w_1)$.

Therefore, we have that $h_n(b) \geq 2^{5(s-1)}h_n(W) \geq c_3.2^{sn}.d(v)$, where $c_3 = c_2.2^{5(s-1)}$.

Since $h(b) \geq 2^{-sn}h_n(b)$, we have that $h(b) \geq c_3.d(v)$. ◀

A.2 Proof of Lemma 31

Proof. From Lemma 26, it follows that

$$\begin{aligned}\sum_{a_n}^{b_k} \frac{\mu^s([x,i])}{\mu^s([x])} &= \sum_{a_n}^{b_k}\left(\frac{s_k+1}{(s_k+i)(s_k+i+1)}\right)^s \\ &\geq \sum_{a_k}^{b_k}\left(\frac{1}{(i+1)(i+2)}\right)^s\end{aligned}$$

The second inequality follows from the fact that $s_k \in [0,1]$.

Using Lemma 5, we get that

$$\sum_{a_n}^{b_k} \frac{\gamma^s([x,i])}{\gamma^s([x])} \geq \sum_{a_k}^{b_k} \left(\frac{1}{2(i+1)(i+2)} \right)^s$$

Putting $b_k = 50a_k$ and $s \leq 0.5$, we get

$$\begin{aligned} \sum_{a_n}^{b_k} \frac{\gamma^s([x,i])}{\gamma^s([x])} &\geq \frac{1}{2} \sum_{a_k}^{50a_k} \frac{1}{i+2} \\ &= 0.5(H(50a_k+2) - H(a_k+1)). \end{aligned}$$

where H_n is the n^{th} Harmonic number. From the fact that $\ln n \leq H_n \leq \ln n + 1$, we have

$$\begin{aligned} H(50a_k+2) - H(a_k+1) &\geq \ln(50.a_k) - \ln(2.a_k) - 1 \\ &= \ln(25) - 1. \end{aligned}$$

The lemma holds as $0.5(\ln 25 - 1)$ is greater than 1. ◀