Spartan Bipartite Graphs Are Essentially Elementary

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── Abstract ──

We study a two-player game on a graph between an attacker and a defender. To begin with, the defender places guards on a subset of vertices. In each move, the attacker attacks an edge. The defender must move at least one guard across the attacked edge to defend the attack. The defender wins if and only if the defender can defend an infinite sequence of attacks. The smallest number of guards with which the defender has a winning strategy is called the eternal vertex cover number of a graph G and is denoted by $evc(G)$. It is clear that $evc(G)$ is *at least* $mvc(G)$, the size of a minimum vertex cover of G. We say that G is *Spartan* if $evc(G) = mvc(G)$. The characterization of Spartan graphs has been largely open. In the setting of bipartite graphs on $2n$ vertices where every edge belongs to a perfect matching, an easy strategy is to have n guards that always move along perfect matchings in response to attacks. We show that these are essentially the only Spartan bipartite graphs.

1 Introduction

A *vertex cover* of a graph G is a subset S of vertices such that every edge in G has at least one of its endpoints in S. An *optimal* vertex cover of a graph G is a vertex cover of the smallest possible size and the size of this optimal vertex cover is denoted by $mvc(G)$. A *bipartite graph* $G = (V, E)$ is a graph whose vertex set can be partitioned into two independent sets, say $V = (A \cup B)$, that is every edge is between a vertex in A and one in B. Clearly, both A and B are vertex covers of G. If a bipartite graph $G = (A \cup B, E)$ is connected and its *only* optimal vertex covers are A and B, then we say that G is *elementary*. If every connected component of a bipartite graph is elementary, then we call it *essentially elementary*.

Klostermeyer and Mynhardt [8] introduced the notion of the *eternal vertex cover* number of a graph G. There are two players – an *attacker* and a *defender*, who are playing on a simple, undirected graph G. In the beginning, the defender can choose to place guards on some of the vertices of G. The attacker's move involves choosing an edge to "attack". The defender is able to "defend" this attack if she can move the guards along the edges of the graph in such a way that at least one guard moves along the attacked edge. If both the

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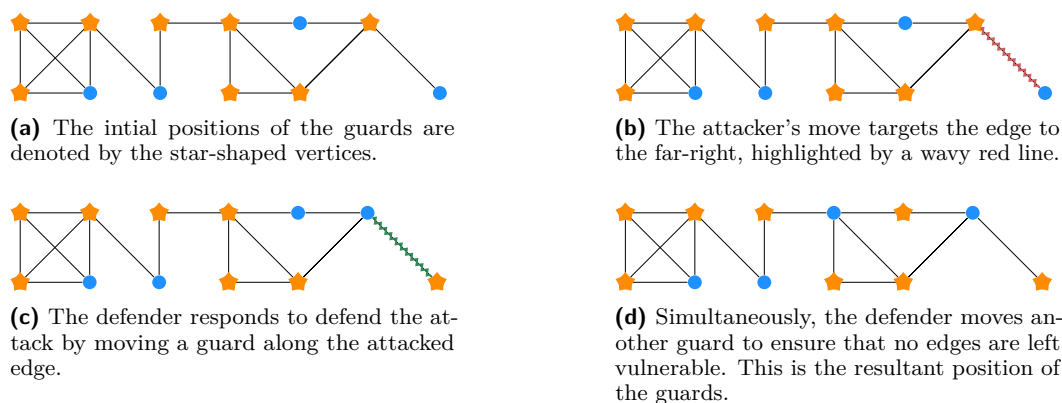Editors: Jérôme Leroux, Sylvain Lombardy, and David Peleg; Article No. 68; pp. 68:1–68:15
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endpoints of the attacked edge have guards on them, the guards can simply exchange their places and the attack is defended as well as the original configuration of guards is restored. Note that any number of guards can move after an attack, but each guard can move only one "step", i.e., just along a single edge. If such a movement is not possible, then the attacker wins. If the defender can defend the graph against an infinite sequence of attacks, then the defender wins (see Figure 1). The minimum number of guards with which the defender has a winning strategy is called the *eternal vertex cover number* of the graph G and is denoted by $evc(G)$.

There are two variations of this game, one where only one guard can be present per vertex at a given time and a variant where more than one guard can be present on a vertex at a given time. Our results apply to both the variants of the game.

(a) The intial positions of the guards are denoted by the star-shaped vertices.

(b) The attacker's move targets the edge to the far-right, highlighted by a wavy red line.

(c) The defender responds to defend the attack by moving a guard along the attacked edge.

(d) Simultaneously, the defender moves another guard to ensure that no edges are left vulnerable. This is the resultant position of the guards.

Figure 1 An attack that is defended by moving two guards.

If S_ℓ is the subset of vertices that have guards on them after the defender has played her ℓ-th move, and S_ℓ is not a vertex cover of G, then the attacker can target any of the uncovered edges to win the game. Therefore, when the defender has a winning strategy, it implies that she can always "reconfigure" one vertex cover into another in response to any attack, where the reconfiguration is constrained by the rules of how the guards can move and the requirement that at least one of these guards needs to move along the attacked edge. It follows that $evc(G) \geqslant mvc(G)$, where $mvc(G)$ is the size of the smallest vertex cover for any graph G.

We call graphs G which enjoy $evc(G) = mvc(G)$ *Spartan*, to indicate that these graphs can manage attacks without using "additional" guards. Understanding Spartan graphs for several special cases have been addressed in the literature (see, for instance, [3]). However, the characterization of bipartite graphs that are Spartan has been left open. We show that bipartite graphs are Spartan if and only if they are essentially elementary.

We also consider a natural extension of the eternal vertex cover problem: where the guards are allowed to move two or more steps on each turn instead of one, without retracing. An attack is defended only if the guard moves along the attacked edge on their first step. We show that with just one additional step, we need either just as many guards as the size of the smallest vertex cover or one more: so in this variant all graphs are "almost" Spartan. We also show that the extra guard is needed only if the graph has a degree one vertex not contained in any minimum vertex cover. Finally, we show that allowing for more than one extra step is the same as allowing exactly one extra step.

Related Work.   Among characterizations related to Spartan graphs, we have the following. Let the graph class $\mathcal{F}$ denote the class of all connected graphs G for which each minimum vertex cover of G that contains all the cut vertices of G induces a connected subgraph in G.

(A cut vertex is a vertex whose removal disconnects the graph.) Let $G(V, E)$ be a graph that belongs to $\mathcal{F}$, with at least two vertices, and $X \subset V$ be the set of cut vertices of G. Then Babu et al. [2] showed that G is Spartan if and only if for every vertex $v \in V \setminus X$, there exists a minimum vertex cover S_v of G such that $X \cup \{v\} \subset S_v$. Klostermeyer and Mynhardt [7] also study graphs for which the eternal vertex cover number coincides with the eternal domination number, a closely related notion. This is a similar game, except that the attacks happen on vertices and if the attacked vertex does not have a guard already, then a guard from a neighbouring vertex must come to the attacked vertex. All other guards can stay on their initial position or move to a neighbouring vertex. The minimum number of guards required to protect the graph from an infinite sequence of attacks is called the *eternal domination number*.

Note that twice as many vertices as the $\mathfrak{mvc}(G)$ always suffice to defend against any sequence of attacks – by placing guards on both endpoints of any maximum matching to begin with and after any attack, reconfiguring the guards to obtain another maximum matching. Using this strategy, a $2-$approximation algorithm for ETERNAL VERTEX COVER was obtained by Fomin et al. [5]. This also implies $\mathfrak{mvc}(G) \leqslant evc(G) \leqslant 2\mathfrak{mvc}(G)$. [8] gave a characterization of the graphs for which the upper bound is achieved. The notion of elementary graphs was considered by Lovász and Plummer [9] and several useful characterizations were given by Hetyei [6]. We adapt these definitions to suit the context of bipartite graphs.

Methodology

Recall that a *matching* is a collection of vertex disjoint edges, and a matching which contains one edge incident to each vertex of a graph is called a *perfect matching*. We say that an edge is *allowed* if it is contained in some perfect matching. A graph G is said to be *elementary* if and only if it is connected and every edge is allowed.

It turns out that a connected bipartite graph $G = (A \cup B, E)$ is elementary if and only if its *only* optimal vertex covers are A and B, as we make explicit in the proposition below.

▶ **Proposition 1** ([6]). *The following are equivalent for any connected bipartite graph* $G = (A \cup B, E)$.
1. A *and* B *are the only minimum vertex covers of* G*, and in particular* $|A| = |B|$.
2. *Every edge in* G *is allowed.*

Notice that it is easy to see that for bipartite graphs, if G is elementary then it is Spartan (see also Lemma 2 and Araki, Fujito, and Inoue [1]). Indeed, we start by placing guards on all vertices of A. If the edge e is attacked, then we move guards to B along the edges of the perfect matching M that contains the edge e. Future attacks can be similarly defended, so the guards alternate between occupying A and B in response to edge attacks.

What we focus on demonstrating in this contribution is the converse, namely that if a connected bipartite graph G does not have this structure, then it is also not Spartan: in some sense, the trivial scenario is the only one in which we can hope to manage without employing any "extra" guards.

Spartan $\implies$ Perfect Matching. To begin with, notice that if $G = (A \cup B, E)$ is connected and does not have a perfect matching, we already need more than $\mathfrak{mvc}(G)$ guards (see also Proposition 4). In particular, let M be a maximum matching in G. Recall that $|M| = \mathfrak{mvc}(G)$ by König's theorem. Without loss of generality, let $b \in B$ be a vertex not incident to any edge of M. Since G is connected, b has some neighbor $a \in A$, which must belong to any vertex cover of G. In particular, the defender is forced to position a guard on

$\mathfrak{a}$ in the initial configuration. If the edge $\mathfrak{ab}$ is attacked, the guard on $\mathfrak{a}$ is forced to move to $\mathfrak{b}$. This creates a situation where $|M| - 1$ guards have to reposition themselves to protect all the edges of M – which is impossible since the edges of M are disjoint.

Therefore, a necessary condition for a connected bipartite graph to be Spartan is that it must admit a perfect matching. This is, however, evidently not sufficient: indeed, there are connected bipartite graphs with perfect matchings that are not Spartan (see also Figure 2).
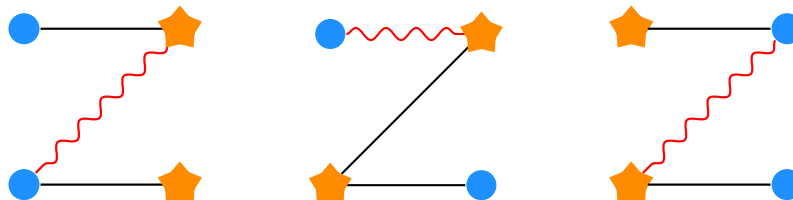

■ **Figure 2** A graph with a perfect matching where $\mathfrak{mvc}(G)$ many guards do not suffice. For each choice of $\mathfrak{mvc}(G)$ many guards at the initial configuration, attacking the dashed edge shown in the figure leads to an indefensible position in one or two steps.

Spartan $\implies$ No Degree One Vertices. We observe that a connected bipartite Spartan graph $G = (A \cup B, E)$ on at least three vertices cannot have a degree one vertex (see also Proposition 5 and Babu et al. [2]). Assume to the contrary that G has a degree one vertex, say $\mathfrak{a} \in A$. Let $\mathfrak{b}$ denote $\mathfrak{a}$'s unique neighbor in G. Since G has at least one vertex other than $\mathfrak{a}$ and $\mathfrak{b}$, and G is connected, $\mathfrak{b}$ must have another neighbor in G: say $\mathfrak{a}'$. Now, if the initial configuration has a guard on $\mathfrak{a}$, then attack the edge $\mathfrak{ba}'$; otherwise attack the edge $\mathfrak{ab}$ to force a guard on $\mathfrak{a}$ and then attack $\mathfrak{ba}'$. Both cases lead to a scenario where both endpoints of the edge $\mathfrak{ab}$ have guards on them. However, since G is Spartan, it has a perfect matching M. Further, since $\mathfrak{a}$ is a degree one vertex, M is forced to contain the edge $\mathfrak{ab}$. This in turn implies that at least $|A| - 1$ guards are required to defend the rest of the graph, contradicting the assumption that G is Spartan.

Proof by Structure-Preserving Contractions. So far, we have seen that a connected bipartite graph $G = (A \cup B, E)$ which is Spartan must have a perfect matching and no degree one vertices. Fix an arbitrary perfect matching M, and let $\mathfrak{ab} \in M$. Let P be a maximal walk (without repeating edges) starting at $\mathfrak{a}$, alternating between edges of M and $E \setminus M$. The terminal vertex of P is either $\mathfrak{a}$, or a vertex v which must have a neighbor on P, since G has no degree one vertices. So from P, we can derive a cycle whose edges alternate between M and $E \setminus M$ (see also Proposition 17).

At a high level (see Figure 3), our proof relies on "contracting" each such cycle to a single matching edge and arguing that the resulting graph remains Spartan provided the original graph was also Spartan. In particular, we show the following. If G is Spartan to begin with, and M is a perfect matching with S is a set of endpoints of $M' \subseteq M$ such that $G[S]$ is elementary, then "replacing" $G[S]$ with a single edge until this operation is no longer possible; keeps the graph Spartan (see also Lemma 13). Here it is useful that any bipartite graph obtained by adding edges to a cycle (while preserving the bipartite-ness) is elementary, a fact that we establish separately (see also Proposition 7) – thus we can simply keep contracting along cycles C. The idea of the proof is then the following. Consider what happens once we are stuck, i.e., we cannot contract any further: we either have an edge, or we have a graph that is not an edge but must have a degree one vertex (if not, note that we would not
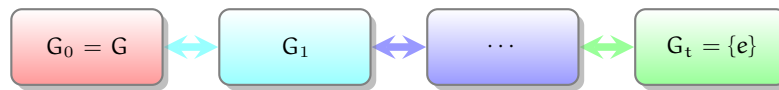
Figure 3 Proof idea: The sequence of forward contractions preserves "Spartan-ness" overall (i.e, G_t is Spartan if G_0 is Spartan) while the backward expansions preserve "elementary-ness".

be stuck). But from our previous discussion, we know that a connected bipartite Spartan graph on at least three vertices cannot have a degree one vertex. So in this situation, we have transformed the Spartan graph we started with into one that is not Spartan, and in particular, there is an attack that $\mathsf{mvc}(\cdot)$ many guards will not be able to defend on the "contracted" graph. It turns out that this attack can be mimicked on the graph we started with, contradicting our assumption that the original graph is Spartan. Thus, our process must end at a single edge.

To conclude the proof, we show that if we take an elementary bipartite graph G and "inject" an elementary graph into it by substituting an edge with an elementary graph, then the resulting graph is also elementary. This lets us run the contraction operations on G in reverse, starting from the edge that we ended up with; and since an edge is elementary, we conclude that G must have also been elementary (see also Lemma 15).

When G is not connected, the arguments we made can be made independently of every connected component of G. This leads to our final characterization: that a biparitite graph G is Spartan if and only if each of its components are elementary. Since it is straightforward to check if a bipartite graph $G = (A \cup B, E)$ is elementary (for instance, by checking if $G \setminus \{u, v\}$ admits a perfect matching for every $uv \in E$), our result also implies a polynomial-time recognition algorithm for bipartite Spartan graphs. We note that, in contrast to $\mathsf{mvc}(G)$, determining $\mathsf{evc}(G)$ is known to be NP-hard even for bipartite graphs.

In Section 2, we describe the proofs of the claims mentioned in the outline here in greater detail. Our notation largely follows conventions from Diestel [4]. The proofs for statements marked with a $(\star)$ can be found in the full version of this paper.

2 Spartan Bipartite Graphs are Essentially Elementary

We begin with the (easy) forward implication, which was also observed by [1] – we provide an explicit proof in the full version in the interest of completeness.

▶ **Lemma 2** ($\star$)**.** *If a bipartite graph* G *is essentially elementary,* $\mathsf{evc}(G) = |V(G)|/2 = \mathsf{mvc}(G)$ *,i.e.,* G *is Spartan.*

It turns out that we may assume, without loss of generality, that G is connected (we refer the reader to the full version for a more explicit explanation). Therefore, our goal amounts to showing the following.

▶ **Lemma 3** ($\star$)**.** *If* G *is a connected bipartite graph that is Spartan, then it is elementary.*

We first argue that any connected Spartan bipartite graph must have a perfect matching.

▶ **Proposition 4.** *If* G *is a Spartan connected bipartite graph, it admits a perfect matching.*

Proof. Let G be a connected Spartan bipartite graph with bipartition (A, B) where $A = \{a_1, \ldots, a_p\}$ and $B = \{b_1, \ldots, b_q\}$ (see also Figure 4). For the sake of contradiction, assume that G does *not* admit a perfect matching, i.e., without loss of generality let $q > k$. Let S be

a minimum vertex cover in G of size k such that the guards are placed on S in the initial configuration. Let M be any maximum matching. Without loss of generality, let the edges of M be given by:

$$\{e_1 = (a_1, b_1), \ldots, e_i = (a_i, b_i), e_{i+1} = (a_{i+1}, b_{i+1}), \ldots, e_k = (a_k, b_k)\},$$

where $S \cap A = \{a_1, \ldots, a_i\}$ and $S \cap B = \{b_{i+1}, \ldots, b_k\}$.

Since M is not a perfect matching, $|V(G)| > 2k$ i.e. there exists a vertex in G which is not an endpoint of any edge in M. Without loss of generality, let this vertex be b_q. Since b_q is not an isolated vertex, it must have a neighbour on the A side. Since S is a vertex cover, this neighbour must lie in $S \cap A$. Without loss of generality, let this neighbour be a_1. If the attacker attacks $a_1 b_q$, the guard on a_1 must come to b_q and cannot move further. Now there are $k-1$ guards on the vertices of M to protect k matching edges. So no matter how these guards arrange themselves, at least one edge of M must be vulnerable before the next attack. The attacker attacks this edge in the next move and wins thus contradicting the fact that G is Spartan. Therefore, the graph G must admit a perfect matching.   ◀
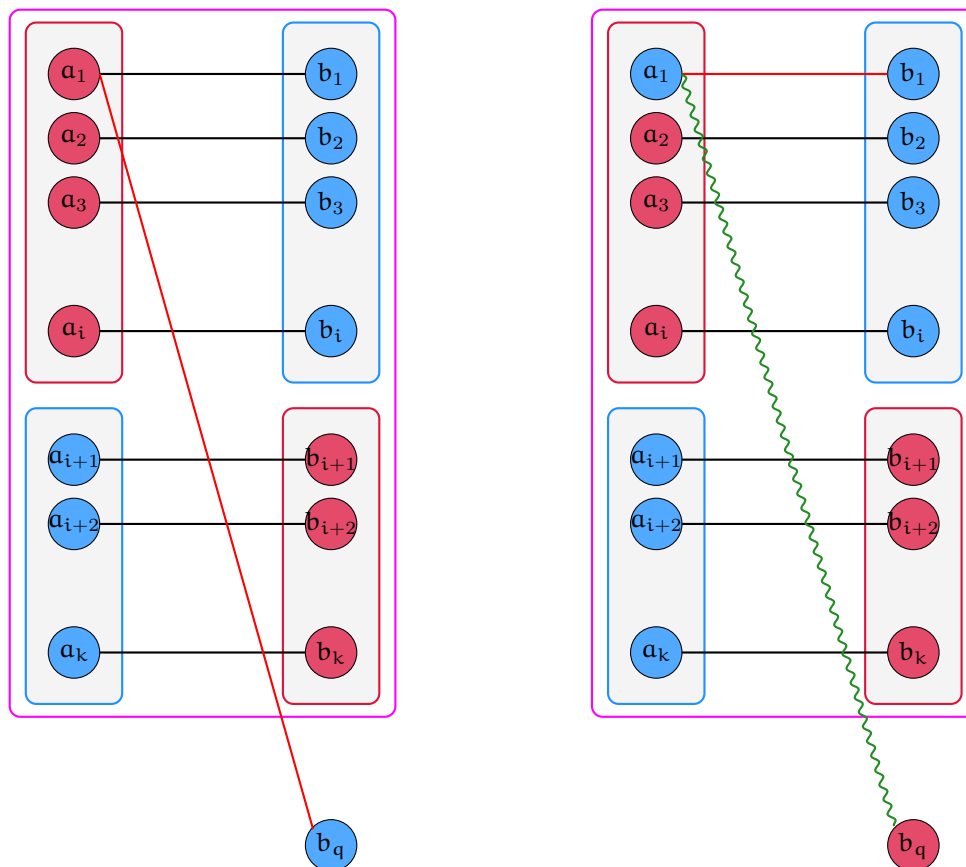
(a) A graph which does not have a perfect matching and the red edge is attacked.

(b) After defending the attack, there are only $k-1$ guards to protect k matching edges. So an edge will remain vulnerable.

Figure 4 Demonstrating that Spartan connected bipartite graphs must have perfect matching towards a proof of Proposition 4.

So we now have that connected Spartan bipartite graphs G have at least one perfect matching. From now on we denote the bipartitions of G by $A = \{a_1, a_2, \dots a_n\}$ and $B = \{b_1, b_2, \dots b_n\}$ where each a_i and b_i have an edge between them so that we have a perfect matching $M = \bigcup_{i=1}^{n}\{a_i b_i\}$. Since G is connected, these bipartitions are unique upto permutations. We now show that connected bipartite graphs that are Spartan do not have degree one vertices. This follows from known results (for example, as shown by [2]), however, our argument is more direct and we make it explicit to make our presentation self-contained.

▶ **Proposition 5** (No Degree One Vertex). *Let G be a connected bipartite graph with $|V(G)| > 2$ and with a perfect matching. If $\mathsf{evc}(G) = \mathsf{mvc}(G)$, then G has no degree one vertex.*

From Lemma 2 and Proposition 5, we have the following.

▶ **Corollary 6.** *Let G be a connected bipartite graph with more than two vertices. If G is elementary, then G cannot have a degree one vertex.*

Next, we show that a bipartite graph with a cycle that visits every vertex is elementary.

▶ **Proposition 7** (Adding edges to cycles). *Consider a connected bipartite graph $G = (A \cup B, E)$ which is a cycle $a_1 b_1 a_2 b_2 \dots a_k b_k a_1$ (where $k > 1$). The graph G' formed after adding any number of edges between A and B (preserving bipartiteness) will be elementary.*

Proof. Let G be a connected bipartite graph G which is a cycle $a_1 b_1 a_2 b_2 \dots a_k b_k a_1$, where $k > 1$ (see also Figure 5). Here the two sides of the bipartition are $A = \{a_1, a_2, \dots, a_k\}$ and $B = \{b_1, b_2, \dots, b_k\}$.

The graph G has two perfect matchings which are given by $M = \{a_1 b_1, a_2 b_2, \dots, a_k b_k\}$ and $M' = \{b_1 a_2, b_2 a_3, \dots, b_{k-1} a_k, b_k a_1\}$. It is easily seen that any edge of G lies in one of these two perfect matchings.

Suppose we add some edges from A to B. Notice that the vertex set of the new graph G' is same as the vertex set of G. In order to prove that G' is elementary, it is sufficient to show that each of the newly added edges belongs to a perfect matching of G'.

Consider a newly added edge $a_i b_j$. This clearly preserves bipartiteness. Here, $i \neq j, j+1$ because the edges $a_i b_i$ and $a_i b_{i-1}$ were already in G.

First, let $i > j + 1$. Consider the matching:

$$M_1 = \{a_i b_j, b_i a_{i+1}, b_{i+1} a_{i+2}, \dots, b_{k-1} a_k, b_k a_1, b_1 a_2, \dots, b_{j-1} a_j, a_{j+1} b_{j+1}, a_{j+2} b_{j+2}, \dots, a_{i-1} b_{i-1}\}.$$

Clearly M_1 is a perfect matching containing $a_i b_j$.

Now let $i < j$. Let $N_1 = \{a_i b_j, b_i a_{i+1}, b_{i+1} a_{i+2}, \dots, b_{j-1} a_j\}$; $N_2 = \{a_1 b_1, \dots, a_{i-1} b_{i-1}\}$; and $N_3 = \{a_{j+1} b_{j+1}, \dots, a_k b_k\}$.

Consider the matching M_2 given by $N_1 \cup N_2 \cup N_3$. Clearly M_2 is a perfect matching containing $a_i b_j$.

This concludes our argument showing that G' is elementary. ◀
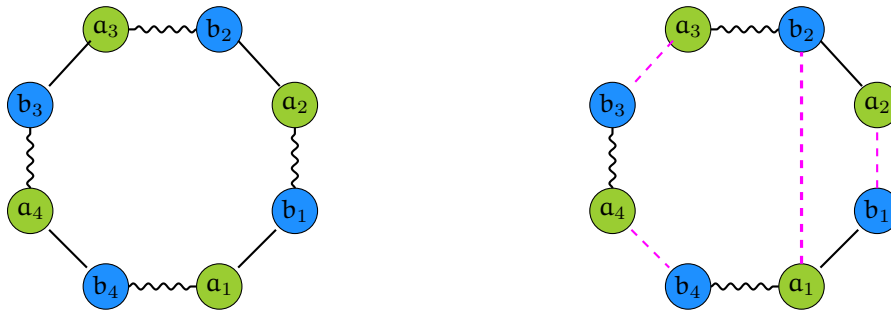
(a) This figure shows a cycle with $k = 4$.

(b) Each newly added edge is allowed: Suppose $a_1 b_2$ is the new edge, it can be combined with the existing edges to form the perfect matching denoted by the dashed purple lines.

Figure 5 An example of a bipartite graph with a spanning cycle.

We now introduce the terminology "special subset" to indicate that we are working with a subset of endpoints of some edges in a perfect matching.

▶ **Definition 8** (Special Subset). *Let G be a connected bipartite graph with a perfect matching* $M = \{a_1 b_1, a_2 b_2, \ldots, a_n b_n\}$. *Consider:* $S = \{a_{i_1}, b_{i_1}, a_{i_2}, b_{i_2}, \ldots, a_{i_k}, b_{i_k}\}$ *for some distinct* $i_1, i_2, \ldots, i_k \in [n]$ *and* $2 \leqslant k \leqslant n$. *Then S is said to be a **special subset** of* $V(G)$.

A "special induced subgraph" is a subgraph induced by a special subset.

▶ **Definition 9** (Special Induced Subgraph). *Let G be a connected bipartite graph with a perfect matching and S be a special subset of* $V(G)$. *The subgraph $G[S]$ induced by S is called a **special induced subgraph** of G.*

We now note that edges allowed in special induced subgraphs of a graph G are also allowed in G.

▶ **Proposition 10** (Allowed Edge). *Let G be a connected bipartite graph with a perfect matching M. Let S be a special subset of G and $H = G[S]$ be the corresponding special induced subgraph. An edge which is allowed in H is also allowed in G.*

Proof. If $S = V(G)$, we are done. Otherwise without loss of generality, let S be the set $\{a_1, b_1, a_2, b_2, \ldots, a_k, b_k\}$. Consider an edge $a_i b_j$ which is allowed in H for some $i, j \in [k]$. Consider the matching M_H which is a perfect matching of H containing $a_i b_j$. Consider $M' = M_H \cup \{a_{i+1} b_{i+1}, \ldots, a_k b_k\}$. Clearly M' is a perfect matching of G containing $a_i b_j$. Thus, $a_i b_j$ is allowed in G. ◀

We now turn to a key definition: one of "contracting" a special elementary subset to a single edge. We refer the reader to Figure 6 for an example of an application of this operation.

▶ **Definition 11** (Matching Contraction Graph). *Consider a connected bipartite graph G with bipartition A and B and a perfect matching M. Let S and $H = G[S]$ be a special subset of $V(G)$ and the corresponding special induced subgraph of G. If H is elementary, replace $A \cap S$ by a new vertex α and $B \cap S$ by a new vertex β in G. Add an edge $\alpha\beta$ in the new graph. For an edge ab where $a \in A \backslash S$ and $b \in S$, add an edge $a\beta$ to the new graph. Similarly for an edge ab where $a \in S$ and $b \in B \backslash S$, add an edge αb to the new graph. Keep all the other edges and vertices of G unchanged. If there are any parallel edges, replace them by a single edge. The new graph G_S thus obtained is called the **matching contraction graph** of G w.r.t. S and the procedure of obtaining a matching contraction graph is called as **contraction**.*

(a) A graph G with special elementary subset $S = \{a_1, b_1, a_2, b_2, a_3, b_3\}$.

(b) The matching contraction graph of G w.r.t. S.
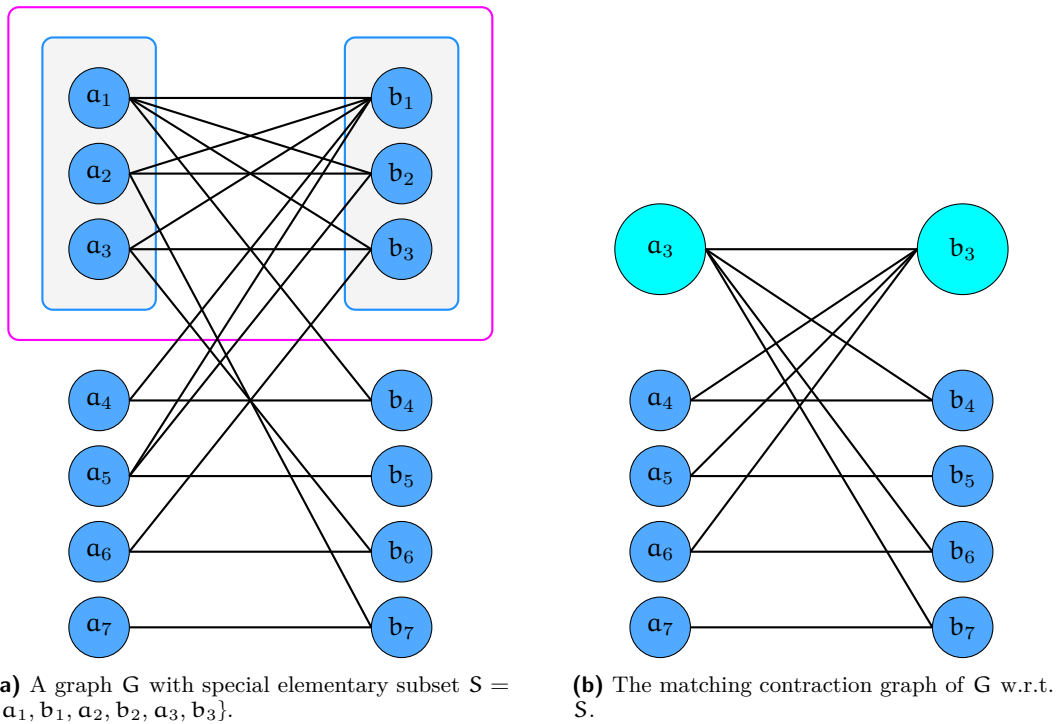
Figure 6 Demonstrating the operation of contracting a special elementary subset to a single edge. For a more elaborate example involving a sequence of contractions, see Figure 12.

Note that if G is a connected bipartite graph and S is a special subset, then the matching contraction graph G_S is also connected and bipartite.

It can be seen that if the special subset S has size $2k$, then the matching contraction graph has $2(n - k + 1)$ vertices, a perfect matching of size $n - k + 1$ and thus a minimum vertex cover of size $n - k + 1$.

▶ **Definition 12** (Special elementary subset). *Let G be a connected bipartite graph with a perfect matching. A special subset S such that $G[S]$ is elementary is called a* ***special elementary subset****.*

We consider a series of contractions over a graph G. Each time we contract a special elementary subset S of G. We rename each vertex in the new graph as the largest indexed vertex in the original graph. We associate a label with each vertex, which carries the names of the vertices of the original graph that were contracted to this new vertex. We keep on repeating this procedure until we reach a graph which cannot be contracted further. We denote this graph by $G_{\bar{S}}$. We fix a perfect matching M in G and in each contracted graph, we get a matching given by $M \setminus E(G[S]) \cup \{\alpha\beta\}$ where α and β denote the newly obtained vertices on the A and B side respectively. Whenever we say "matched partner" or "matching edge" in the new graph, unless mentioned otherwise, we will be referring to this matching.

▶ **Lemma 13** (Matching Contraction Graph Property). *Suppose that G is a connected bipartite graph with a perfect matching M. Then, if G is Spartan, then $G_{\bar{S}}$ is Spartan.*

Proof. If $G_{\bar{S}}$ is a single edge, then G was elementary to begin with, as each expansion preserves the "elementary-ness" by Lemma 15. And since a single edge is Spartan, $G_{\bar{S}}$ is Spartan in this case.

Consider the case where $G_{\bar{S}}$ is not a single edge. Now by Proposition 18, $G_{\bar{S}}$ must have a degree-1 vertex. Without loss of generality, we assume that this degree-1 vertex is on the B side and let us denote this vertex by β_1 and its matched partner by α_1. Also, note that $G_{\bar{S}}$ is not Spartan by Proposition 5. We show that G was also not Spartan in this case.

Let B_1 be the set of vertices in G in the label of β_1, i.e, which were contracted to get β_1 and similarly A_1 be the set of vertices in A which were contracted to get α_1 and let $|A_1| = |B_1| = k$. Note that this contraction may not have happened in one step, but over a series of steps. Notice that $G \setminus \{A_1 \cup B_1\}$ is non-empty, as $G_{\bar{S}}$ is not a single edge. Note that no vertex in B_1 can be adjacent to a vertex in $A \setminus A_1$. Suppose some vertex $b \in B_1$ is adjacent to a vertex $a \in A \setminus A_1$, this will result in an edge $a\beta_1$ in $G_{\bar{S}}$ which contradicts the fact that β_1 is a degree-1 vertex. Also, we know that after a series of contractions, $G[A_1 \cup B_1]$ got contracted to a single edge. By Lemma 15, $G[A_1 \cup B_1]$ is elementary. Therefore, any minimum sized vertex cover of G will contain all the k vertices of A_1 and no vertex from B_1 or all the k vertices of B_1 and no vertex from A_1. Now consider any initial configuration of guards on G with n guards. As seen above, either all the vertices of B_1 have guards or all the vertices of A_1 have guards and only one of these can happen. If all the vertices of A_1 are occupied by guards, the attacker attacks a matched edge $a_i b_i$ such that $a_i \in A_1$. The guard on a_i is forced to move to b_i and cannot move any further. If all the guards cannot reconfigure to form a vertex cover, then G is not Spartan and hence we are done. Otherwise, all the guards must reconfigure in such a way that all the vertices in B_1 have guards and all the vertices in A_1 do not have guards. Thus without loss of generality, we can assume that all the vertices in B_1 have guards and all the vertices in A_1 do not have guards.

Now recall that no vertex in B_1 has a neighbour in $A \setminus A_1$. Since G is connected, there exists $a_p \in A_1$ and $b_q \in B \setminus B_1$ such that $a_p b_q \in E(G)$. Since a_p does not have a guard, b_q must have a guard as the guards are occupying a vertex cover. Now suppose the attacker attacks the edge $a_p b_q$, the guard on b_q must move to a_p. Since all the vertices in B_1 have guards but none of them has a neighbour in $A \setminus A_1$ and no vertex in A_1 had a guard before this guard just moved to a_p, now no guard can move from $\{A_1 \cup B_1\}$ to $V(G) \setminus A_1$. Thus there will be $k + 1$ guards on k matching edges in $G[A_1 \cup B_1]$. Therefore, some edge in $G[V(G) \setminus \{A_1 \cup B_1\}]$ will be vulnerable and can be attacked in the next move. Hence, G is not Spartan. ◄

▶ **Remark.** We have actually shown that any bipartite graph G which is not "essentially elementary" can be destroyed by the attacker in at most three moves when the defender has $mvc(G)$ many guards.

▶ **Definition 14** (Maximal Contraction Graph). *Let G be a connected bipartite graph with a perfect matching and there does not exist any special elementary subset S of $V(G)$. Then G is said to be a* **maximal contraction graph.**

Note that by definition the size of a special subset is always more than 2 and hence if $|V(G)| = 2$, i.e., G is a single edge, then G is a maximal contraction graph as G has no special subset and hence no special elementary subset. Note that the sequence of contractions above ends in a maximal contraction graph that is in fact an edge.

▶ **Lemma 15** (Maximal Contraction Graph). *Let G be a connected bipartite graph with a perfect matching M. Let S be a special elementary subset of $V(G)$ and G_S be the corresponding matching contraction graph. Then, if G_S is elementary, then G is also elementary.*

Proof. Let $G = (A \cup B, E)$. To show that G is elementary, we show that A and B are the only optimal vertex covers of G. We assume that the edges of the perfect matching in G are $\{a_1 b_1, a_2 b_2, \cdots, a_n b_n\}$ and that the special subset S is induced by the vertices based on the edges $\{a_1 b_1, \cdots, a_k b_k\}$. Let α and β denote the endpoints of the edge in G_S created by the contraction of S. Finally, let (A', B') denote the partition of G_S.

Note that $G[S]$ is assumed to be elementary. If X is an optimal vertex cover for G, then $X \cap S$ is a vertex cover for $G[S]$ of size k, since $G \setminus S$ has a matching of size $(n - k)$. Observe that:

- either $X \cap A = \{a_1, \cdots, a_k\}$ and $X \cap B = \emptyset$, or
- $X \cap A = \emptyset$ and $X \cap B = \{b_1, \cdots, b_k\}$,

because any other subset of k vertices that forms a valid vertex cover for $G[S]$ would contradict the assumption that $G[S]$ is elementary.

Now assume that G has a vertex cover X such that $X \neq A$ and $X \neq B$. Without loss of generality, assume that $X \cap A = \{a_1, \cdots, a_k\}$ (the argument for the other case is symmetric). We let $X_A := X \cap \{a_{k+1}, \cdots, a_n\}$ and $X_B = X \cap \{b_{k+1}, \cdots, b_n\}$. Note that $X_B \neq \emptyset$ because $X \neq A$.

We now claim that $X' := X_A \cup X_B \cup \{\alpha\}$ is an optimal vertex cover in G_S that is different from both A' and B'. It is clear that $|X'| = (n - k + 1)$, thus the size of X' is $\mathrm{mvc}(G_S)$. Further, since $\alpha \in A'$ and $X_B \neq \emptyset$, $X' \neq A'$ and $X' \neq B'$. It remains to be shown that X' covers all edges in G_S.

Note that if any edge $a_i b_j$ in $G_S[\{a_{k+1}, \cdots, a_n\} \cup \{b_{k+1}, \cdots, b_n\}]$ is not covered by X', then the $a_i b_j$ is also not covered by X in G. All edges incident on α are also covered. Now suppose an edge of the form βa_ℓ is not covered by X' for some $\ell \in \{a_{k+1}, \cdots, a_n\}$. Note that for this edge to be present in G_S, by the definition of the contraction operation, there must have been an edge of the form $b_i a_\ell$, for some $i \in [k]$. Note that if βa_ℓ is not covered by X' then $b_i a_\ell$ is not covered by X in G either, which contradicts our assumption that X was a vertex cover in G.

This shows that G_S has an optimal vertex cover different from both A' and B', but this contradicts our assumption that G_S was elementary to begin with. Therefore it must be the case that G is also elementary. ◀

▶ **Definition 16** (Alternating cycle). *Let G be a connected bipartite graph with no degree 1 vertex and a perfect matching $M = \{a_1 b_1, a_2 b_2, \ldots a_n b_n\}$. We define a cycle $a_{i_1} b_{i_1} a_{i_2} b_{i_2} \ldots b_{i_k} a_{i_k} a_{i_1}$ in G as an **alternating cycle** where $i_1, i_2, \ldots, i_k \in [n]$ and $2 \leqslant k \leqslant n$.*

▶ **Proposition 17** (No alternating cycle). *Let G be a maximal contraction graph. Then G cannot contain any alternating cycle.*

Proof. Let G be a maximal contraction graph. It is clear that G is a connected bipartite graph with a perfect matching. Let $M = \{a_1 b_1, a_2 b_2, \ldots, a_n b_n\}$ be a perfect matching of G. If $n = 1$, G cannot contain any cycle and hence cannot contain any alternating cycle. Let $n \geqslant 2$ and let $C = a_{i_1} b_{i_1} a_{i_2} b_{i_2} \ldots b_{i_k} a_{i_k} a_{i_1}$ be an alternating cycle in G. Let $S = V(C)$. Clearly S is a special subset of $V(G)$ and by Proposition 7 and Proposition 10, S is a special elementary subset of $V(G)$. This contradicts the fact that G is a maximal contraction graph. Thus G cannot have an alternating cycle. ◀

▶ **Proposition 18** (Structure of the Maximal Contraction). *Let G be a maximal contraction graph. Then G must have a degree 1 vertex.*

Proof. Let G be a maximal contraction graph. It is clear that G is a connected bipartite graph with a perfect matching. Let $M = \{a_1 b_1, a_2 b_2, \ldots, a_n b_n\}$ be a perfect matching of G. If $n = 1$, then G is a single edge and thus G has a degree 1 vertex. Now let $n \geqslant 2$ and assume that G has no degree 1 vertices. Consider the set $S = \{a_1, b_1\}$. Since b_1 is not a degree 1 vertex, we are done. So there exits $a_{i_1} \neq a_1$ which is a neighbor of b_1. Add a_{i_1} and b_{i_1} to S. Now for each b_{i_p} for $p = 1, 2, \ldots$, there exists a_{p+1} which is a neighbor of b_{i_p} because G has no degree 1 vertex or alternating cycle. Add a_{p+1} and b_{p+1} to S. Thus S will be an infinite set. But $S \subset V(G)$ which is finite. Hence we have a contradiction. Thus G must have a degree 1 vertex.   ◀

To sum up the proof of Lemma 3: consider any connected bipartite graph G. First we have shown that if G is Spartan, it must have a perfect matching and no degree one vertices. Therefore, a maximal contraction graph derived from G must be an edge. But note that an edge is elementary, and running the contractions in reverse to recover G preserves the property of the graph being elementary, and we have the desired conclusion. We remark that our proof shows that every connected bipartite graph with a Hamiltnonian cycle is Spartan. However, it turns out that the converse is not true in the sense that there exist connected bipartite graphs that are Spartan but do not have Hamiltonian cycles: for instance, Figure 7 presents an example of a connected bipartite Spartan graph that is not Hamiltonian.
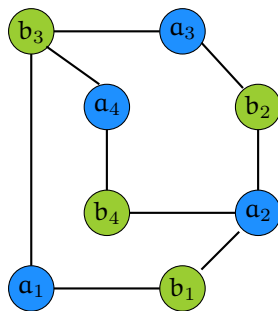
Figure 7 An example of an elementary bipartite graph where the two independent sets are $A = \{a_1, a_2, a_3, a_4\}$ and $B = \{b_1, b_2, b_3, b_4\}$ and all the edges belong to one of the three perfect matchings: $M_1 = \{a_1 b_1, a_2 b_2, a_3 b_3, a_4 b_4\}, M_2 = \{a_1 b_3, a_3 b_2, a_2 b_1, a_4 b_4\}$ or $M_3 = \{a_1 b_1, a_2 b_4, a_4 b_3, a_3 b_2\}$. It can be verified that this graph does not have a Hamiltonian cycle.

3   EVC with Extra Steps

Almost all of the proofs in the previous sections rely crucially on the fact that a guard can only move one step after each attack. This gives rise to the question that what happens if multiple moves are allowed? If retracing of steps is allowed, then any guard can clean up an attack and come back to their original position. Thus this problem will the same as the vertex cover problem and hence it is not so interesting.

We define a variant of the ETERNAL VERTEX COVER problem that we call *New Eternal Vertex Cover*. Again we have a two player game with one player as "the attacker" and the other player as "the defender". Just like the ETERNAL VERTEX COVER problem, the defender initially places the guards on some of the vertices of the graph (of his choice). The attacker attacks an edge. In response to the attack, the defender can move each guards for at most *two* steps without retracing. The constraint is however that the defense of the attacked edge must happen in the first move itself, i.e., after the attack, at least one guard who was present on one of the endpoints of the graph must move across the attacked edge. If such a movement is not possible after a finite sequence of attacks, the attacker wins; otherwise, if

the defender has a strategy to defend the graph G against an infinite sequence of attacks, the defender wins. There can only be one guard per vertex in the configuration before the attack and the configuration after the attack is defended. However, more than one guard can cross a vertex during the reconfiguration.

We define the *New Eternal Vertex Cover Number* of a graph G as the smallest number of guards required in this new setting such that the defender has a winning strategy. We denote this number by $\mathrm{nevc}(G)$. It is clear that since the defense must happen in the first step of a guard, if the vertices occupied by guards do not form a vertex cover, the attacker wins. Therefore, we have $\mathrm{mvc}(G) \leqslant \mathrm{nevc}(G)$ for any graph G. Also, since $\mathrm{evc}(G)$ many guards can defend an infinite sequence of attacks on a graph G with each guard moving just one step after each attack, it can be seen that for any graph G, we have $\mathrm{nevc}(G) \leqslant \mathrm{evc}(G)$.

▶ **Remark.** These two inequalities imply that for any Spartan graph G, we have $\mathrm{mvc}(G) = \mathrm{nevc}(G) = \mathrm{evc}(G)$. However, there are graph classes where these inequalities are not strict.

We show that computing the *New Eternal Vertex Cover* number for a given graph G is NP-hard in the next lemma. We use a reduction from the *Vertex Cover* problem. Given an input instance (G, k) (where $|V(G)| = n$ and $k < n - 1$) of the *Vertex Cover* problem, we construct an equivalent instance of the *New Eternal Vertex Cover* problem by adding a global vertex $\star$ to the graph G, i.e., make the new vertex $\star$ adjacent to every other vertex in G. It turns out that G has a vertex cover of size k if and only if the defender has a winning strategy in the new setting using $k + 1$ guards. The proof of equivalence is deferred to the full version due to lack of space.

▶ **Lemma 19** ($\star$)**.** *The* New Eternal Vertex Cover *problem is* NP*-hard.*

▶ **Lemma 20.** *For any connected graph* G*, the defender has a winning strategy using* $\mathrm{mvc}(G) + 1$ *guards, i.e.,* $\mathrm{nevc}(G) \leqslant \mathrm{mvc}(G) + 1$ *for any graph* G*.*

Proof. For any graph G, we give a winning strategy for the defender in the new setting using $\mathrm{mvc}(G) + 1$ guards.

Let $\mathrm{mvc}(G) = k$ and let $C_1, C_2, \ldots, C_q$ be the connected components of the subgraph induced by the vertices of a k-sized vertex cover of G. Place one guard on each of these vertices of this vertex cover (say S) and one guard in the independent set (say on the vertex w).

Now suppose some edge is attacked such that both its endpoints have guards, then the guards can exchange places and the configuration is restored.

Suppose some edge uv is attacked such that $u \in S$ and $v \in V(G) \setminus S$. Then the guard on u moves to v. Suppose w is adjacent to some vertex which lies in the same connected component of S as w, then the guards can each move one step along a path from w to u such that there is a guard on u and all the vertices of S at the end of this movement and now no guard on w. But we still have all the vertices of S with one guard each and one guard in the independent set (now on v). Thus we have a configuration just like the initial configuration and each strategy can be implemented here accordingly.

Suppose u and any neighbour of w do not lie in the same connected component of S. As G is connected, there is a path from w to u in G and this path contains some vertices of $V(G) \setminus S$. Let this path be $w = u_1 u_2 \ldots u_\ell = u$. We trigger a movement of guards along this path as follows: If $u_{\ell-1}$ has a guard, this guard moves to u, otherwise $u_{\ell-2}$ must have a guard because S is a vertex cover. This guard moves to $u_{\ell-1}$ and then to u. Similarly for each $u_i \in S$ where $i \in \{\ell - 2, \ell - 3, \ldots, 2\}$, if u_{i-1} has a guard, the guard on u_{i-1} moves to u_i. Otherwise u_{i-2} must have a guard (because S is a vertex cover) which moves to u_{i-1}

and then to u_i. It can be observed that at the end of this movement, all vertices in S have a guard and v has a guard and w does not have a guard. Therefore, we have the same situation as before where all the vertices in the vertex cover have a guard and one guard outside the vertex cover. Thus we have shown that $\mathrm{mvc}(G) + 1$ many guards are sufficient to defend G against any sequence of attacks in this new setting.   ◄

▶ **Lemma 21.** *If G is any graph without a degree-1 vertex, then $\mathrm{nevc}(G) = \mathrm{mvc}(G)$.*

Proof. It is sufficient to show that $\mathrm{nevc}(G) \leqslant \mathrm{mvc}(G)$, i.e., the defender has a winning strategy using $\mathrm{mvc}(G)$ many guards. Also it is sufficient to only look at connected graphs.

Let $\mathrm{mvc}(G) = k$ and let S be a minimum sized vertex cover of G. The defender places one guard on each vertex of S. If any edge with both endpoints in S is attacked, then the guards exchange their positions and we are back to the same configuration.

Suppose any edge uv such that $u \in S$ and $v \notin S$ is attacked. The guard on u moves to v. Since v is not a degree 1 vertex, it must have a neighbour w other than u. Depending on whether w and u lie in the same connected component in $G[S]$ or not, we can trigger a movement of guards just like the proof of Lemma 20 such that each guard moves at most two steps and there is no guard on v at the end of this movement and there is a guard on u. Thus the vertex cover S is restored and we have a winning strategy using $\mathrm{mvc}(G)$ many guards.   ◄

▶ **Lemma 22.** *For any graph G, $\mathrm{nevc}(G) = \mathrm{mvc}(G)$ if and only if for each degree-1 vertex v of G, there exists a minimum sized vertex cover S_v of G which contains v.*

Proof. Let G be a graph such that $\mathrm{mvc}(G) = k$ and there exists a degree-1 vertex v such that no k−sized vertex cover of G contains v. Let u be the neighbour of v, then any minimum sized vertex cover of G must contain u. Therefore, in the initial configuration there must be a guard on u and no guard on v. If the attacker attacks the edge uv, then the defender must move the guard on u to v. Since v has no other neighbour than u and retracing is not allowed, the guard on v cannot move anywhere else. Now since there is no vertex cover of size k which contains v, the guards cannot reconfigure to form a vertex cover, no matter how the other guards arrange themselves. Thus $\mathrm{mvc}(G)$ many guards are not sufficient to protect G and thus $\mathrm{nevc}(G) \neq \mathrm{mvc}(G)$.

Now suppose that for every degree-1 vertex v of G, there exists a minimum sized vertex cover S_v of G which contains v. Let the size of the minimum sized vertex cover of G be k. We now describe a strategy to defend an attack on G with k guards. Place each guard on a vertex of a minimum sized vertex cover (say S) of G. Without loss of generality, we assume that some edge uv such that $u \in S$ and $v \notin S$ is attacked. The guard on u is forced to move to v. If v is a vertex with degree 2 or more, then we mimic the strategy in Lemma 20 to get all the guards back on S. If v is a degree-1 vertex, we show that it is possible to transfer the guards from S to S_v (where S_v is the minimum sized vertex cover containing v).

Denote $S \cap S_v$ by P and $(V(G) \setminus S \cap S_v) \setminus \{v\}$ by Q. Here P are the vertices which need to retain a guard and Q are the vertices which need to gain a guard in order to reconfigure from S to S_v. Let $|P| = p$ and $|Q| = q$. Clearly $p + q + 1 = k$, i.e., $k - p = q + 1$. Therefore $|S \setminus P| = |Q \cup \{v\}|$. We show that there is a perfect matching between these two sets. Notice that there cannot be an edge with both endpoints in $S \setminus P$ because these vertices are not in S_v and S_v is a vertex cover. Suppose there is no perfect matching between $S \setminus P$ and $Q \cup \{v\}$, then there exists a set $A \in S \setminus P$ such that $|N(A) \cap (Q \cup \{v\})| < |A|$. But this is not possible as $(S \setminus P) \setminus A \cup N(A)$ will be a vertex cover of G of size less than k. Thus there exists a perfect matching between $S \setminus P$ and $Q \cup \{v\}$ which can be used to reconfigure the guards from S to S_v.   ◄

▶ **Corollary 23** (⋆). *There exists a polynomial time algorithm for finding the* New Eternal Vertex Cover *number on bipartite graphs.*

Note that if we allow the guards to move for an arbitrary number of steps without retracing after each attack, Lemma 22 still holds, i.e., there are families of graphs (for instance, star graphs) such that even allowing the guards to move for an arbitrary number of steps does not make them "Spartan" in the new setting. Thus the power of one extra step subsumes the power of any number of additional steps.

4 Concluding Remarks

We showed that a natural sufficient condition for when a graph is Spartan (i.e, when $evc(G) = mvc(G)$) is also necessary in the context of bipartite graphs. Motivated by our proof we extend the notion of eternal vertex cover to a variant where the guards are allowed to move more than one step on their turn, and completely characterize the number of guards needed in terms of mvc: indeed, we show that one extra guard suffices, and is needed only if the graph has a degree one vertex that is not contained in any minimum vertex cover.

It would be interesting to see what happens if the defense can happen in the second step of a guard. We also showed that while the new variant remains computationally hard, unlike the original problem, it is in fact solvable in polynomial time on bipartite graphs. Generalizing our structural results beyond bipartite graphs is an interesting direction for future work.

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