SAT-Based Generation of Planar Graphs

Markus Kirchweger
Algorithms and Complexity Group, TU Wien, Austria

Manfred Scheucher
Institut für Mathematik, Technische Universität Berlin, Germany

Stefan Szeider
Algorithms and Complexity Group, TU Wien, Austria

Abstract

To test a graph's planarity in SAT-based graph generation we develop SAT encodings with dynamic symmetry breaking as facilitated in the SAT modulo Symmetry (SMS) framework. We implement and compare encodings based on three planarity criteria. In particular, we consider two eager encodings utilizing order-based and universal-set-based planarity criteria, and a lazy encoding based on Kuratowski's theorem. The performance and scalability of these encodings are compared on two prominent problems from combinatorics: the computation of planar Turán numbers and the Earth-Moon problem. We further showcase the power of SMS equipped with a planarity encoding by verifying and extending several integer sequences from the Online Encyclopedia of Integer Sequences (OEIS) related to planar graph enumeration. Furthermore, we extend the SMS framework to directed graphs which might be of independent interest.

2012 ACM Subject Classification Mathematics of computing → Solvers; Mathematics of computing → Graph enumeration; Theory of computation → Automated reasoning; Hardware → Theorem proving and SAT solving

Keywords and phrases SAT modulo Symmetry (SMS), dynamic symmetry breaking, planarity test, universal point set, order dimension, Schnyder's theorem, Kuratowski's theorem, Turán's theorem, Earth-Moon problem

Digital Object Identifier 10.4230/LIPIcs.SAT.2023.14

Supplementary Material *Software (Source Code)*: `https://github.com/markirch/sat-modulo-symmetries/`, archived at `swh:1:dir:c50f1ce7d22717e113b915ae9976da0cb0e67c1d`
Software (Documentation): `https://sat-modulo-symmetries.readthedocs.io/`

Funding M.K. and S.S. were supported by the Austrian Science Fund (FWF), project P32441, and from the Vienna Science and Technology Fund (WWTF), project ICT19-065. M.S. was supported by DFG Grant SCHE 2214/1-1.

1 Introduction

Graph generation is the problem of deciding whether a graph with a particular property exists. Many difficult problems in Combinatorics can be stated as graph generation problems. Over the last years, SAT-based approaches to graph generation have been proposed, yielding competitive alternatives to isomorphism-free exhaustive enumeration by canonical construction path, as implemented in tools like Nauty [34]. By combining the desired graph property with symmetry breaking, SAT-based approaches can avoid generating a prohibitively large number of candidate graphs for which the desired property needs to be checked. *SAT Modulo Symmetry (SMS)* [30] is a SAT-based approach that supports complete symmetry breaking performed by a special propagator that collaborates with a CDCL SAT solver [21].

In this article, we look into SAT-based graph generation where the given property entails the graph being planar.

26th International Conference on Theory and Applications of Satisfiability Testing (SAT 2023).
Editors: Meena Mahajan and Friedrich Slivovsky; Article No. 14; pp. 14:1–14:18
Leibniz International Proceedings in Informatics
Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

There are mainly two options for incorporating planarity into SAT-based graph search: (i) employing an "eager" encoding of planarity directly into a SAT formula or (ii) using a "lazy" encoding that incrementally adds clauses to ensure that the partially defined graph, as represented by the current partial assignment of the solver, is planar. Today, many criteria for planarity are known. Criteria that are *positive* in the sense that they state the existence of a planar embedding, are natural candidates for an eager SAT encoding because valid variable assignments are in correspondence with embeddings. Criteria that are *negative* in the sense that they state the existence of an obstruction against a planar embedding, are candidates for lazy encoding; once an obstruction has been found, the solver can exclude it by learning a corresponding clause.

In Section 3, we propose a lazy encoding based on forbidden graph minors and Kuratowski's theorem, and two eager encodings of planarity; one is based on Schnyder orders [40], and the other is based on universal point sets [13].

In Section 5, we compare the performance and scalability of these encodings on three problem settings.

The first problem setting is from extremal combinatorics [5] and seeks the maximum number of edges in graphs on n vertices that excludes a certain subgraph. The *Turán number* $\mathrm{ex}(n, H)$ for an integer n and a graph H is the maximum number of edges in an n-vertex graph G with no copy of H as a subgraph. Turán famously showed that $\mathrm{ex}(n, C_k) \leq (1 - \frac{1}{k-1})\frac{n^2}{2}$, where C_k denotes the cycle graph on k vertices ([44], see also [2, Chapter 27]). Dowden [14] studied the problem restricted to planar graphs G which gives rise to the *planar Turán number* $\mathrm{ex}_P(n, H)$. Very recently, planar Turán numbers for various graphs H have become the subject of intensive research in combinatorics [12, 15, 16, 24, 32]. With our SAT-based framework, we compute planar Turán numbers for $n \leq 18$ where the excluded subgraph is a cycle of length 4 or 5, as studied in [14].

The second problem setting is an extension of the planar map coloring problem, known as the *Earth-Moon problem* introduced by Ringel [37]. It seeks the chromatic number of a biplanar graph (a graph that can be formed as the union of two planar graphs). The name of the problem originates from the figurative statement of the problem, which asks for the minimum number of colors needed to properly color a map consisting of two separate spherical (planar) maps, an Earth map containing a collection of countries, and a Moon map containing a colony for each country on Earth. A proper coloring assigns the same color to a country and its lunar colony and different colors to countries and colonies with a common boundary. It is known that the number of required colors lies between 9 and 12 (cf. [27, p. 36] and [26, p. 199]). To encode biplanar graphs for the Earth-Moon problem, we extended the SMS framework from graphs to *directed* graphs: antiparallel edges indicate edges in one planar graph and the remaining edges indicate edges in the other. This extension might be of independent interest. With this approach, we are able to show the absence of biplanar graphs with certain order and at least a certain chromatic number.

As the third problem setting, we consider various *integer sequences* related to planar graph and digraph enumeration as listed in the On-Line Encyclopedia of Integer Sequences [35]. While existing graph enumeration tools such as plantri [8] mainly aim on *plane* graphs (i.e., planar graphs with a fixed embedding), our approach is the first for planar graphs (no embeddings involved). For several of the sequences we could verify and extend the known initial segments with a relatively minor effort. In particular, having common parameters implemented (such as bounds on degrees, clique/independence number, connectivity, etc.), a large variety of sequences can be simply tested from the command line just by combining the desired parameters. We see this as an indication that, in several cases, our approach is superior and easier to use than standard graph enumeration based approaches and for the versatility of our framework.

Related Work

Chimani, Hedtke and Wiedera [10] investigated the problem of finding a planar subgraph of a given graph with the maximum number of edges. They used encodings to integer linear programs and pseudo-boolean satisfiability based on various planarity criteria for that purpose. This problem setting is very different from ours since they work with a given input graph while we aim to generate graphs for which symmetry breaking plays a central role.

Plantri is the standard tool for the generation of certain types planar graphs and was developed by Brinkmann and McKay [8]. It enumerates non-isomorphic planar graphs with a fixed embedding (*plane* graphs). Since 3-connected planar graphs have a unique embedding, plantri can directly enumerate various subclasses of 3-connected planar graphs. However, in general planar graphs can have multiple (up to exponentially many) embeddings and therefore one must filter duplicates caused by distinct embeddings. With SMS, we can enumerate planar graphs of any connectivity directly.

2 Preliminaries

For positive integers n, we write $[n] := \{1, \ldots, n\}$.

We use standard notation for *CNF formulas* (propositional formulas in conjunctive normal form), propositional variables, literals, and clauses [36].

We use standard notation for graphs and digraphs [6, 45], in particular, all considered graphs and digraphs are finite and simple. A *graph* G consists of a finite set $V(G)$ of *vertices* and a set $E(G) \subseteq \{\{u, v\} : u \neq v \in V(G)\}$ of *edges*. Similarly, a *directed graph* G (or *digraph*) G consists of a finite set $V(G)$ of *vertices* and a set $E(G) \subseteq \{(u, v) : u \neq v \in V(G)\}$ of *directed edges* or *arcs*. The *underlying graph* $\underline{G}$ of a digraph G has the vertex set $\underline{G}(V) := G(V)$ and edge set $E(\underline{G}) := \{\{u, v\} : (u, v) \in E(G)\}$.

An *edge-subdivision* operation deletes an edge $\{u, v\}$ from a graph G, and adds two new edges $\{u, w\}, \{w, v\}$ and a new vertex $w \notin V(G)$. A graph G is a *subdivision* of another graph H if G can be obtained from H by successively performing edge-subdivisions.

A graph G is *connected* if there exists a path between any two vertices $u, v \in V(G)$, that is, there exists a sequence of vertices $u = w_0, w_1, \ldots, w_k = v$ with $\{w_i, w_{i+1}\} \in E(G)$. Moreover, G is k*-connected* if $|V(G)| \geq k + 1$ and the deletion of any $k - 1$ vertices results in a connected graph. The *connectivity* $\kappa(G)$ denotes the largest integer k for which G is k-connected. Pause to note that the terms "k-connected" and "connectivity k" must not be confused as the class of k-connected graphs consists all graphs G with connectivity $\kappa(G) \geq k$. A digraph is *weakly* k*-connected* if its underlying graph is k-connected.

To define planarity, some auxiliary terminology is required. A *simple curve* in the plane (resp. on the sphere) is the image of a injective continuous mapping $\phi : [0, 1] \to \mathbb{R}^2$ (resp. $\phi : [0, 1] \to \mathbb{S}^2$). The points $\phi(0), \phi(1)$ are the curve's *ends* and the remaining points of the curve form the curve's *interior*. A graph is *planar* if there exists a mapping of the vertex to the plane (resp. to the sphere) and a mapping of each edge to a simple curve connecting the two corresponding vertices such the interiors of any two curves is disjoint. Such a mapping is called *embedding*. In general, one does not distinguish between embedding in the plane and embedding on the sphere since any embedding on the sphere can be transferred into the plane via a stereographic projection, and vice versa.

A *plane* graph is a planar graph with a fixed embedding. If a graph is 3-connected then it has a combinatorially unique embedding on the sphere [46], that is, the cyclic order of the incident edges around any vertex coincide in every embedding. However, graphs that are not 3-connected can have multiple embeddings, hence one planar graph can correspond to several plane graphs; see e.g. Figure 1.

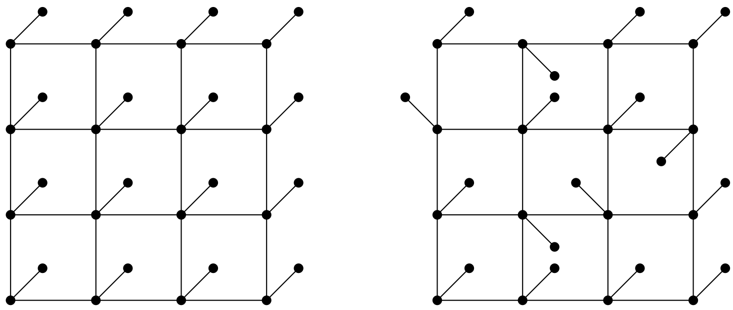

Figure 1 Two embeddings of the same planar graph. The graph is constructed by adding an edge to every vertex of the grid graph. Since every edge can be drawn in multiple cells, the graph corresponds to exponentially many non-isomorphic plane graphs.

A *k-coloring* of a graph G is mapping $c : V(G) \to [k]$ such that for every edge $\{u, v\} \in E(G)$ it holds $c(u) \neq c(v)$. A graph G is *k-colorable* if there exists a k-coloring of G and the *chromatic number* $\chi(G)$ of G denotes the smallest integer k such that G is k-colorable. The famous four-color theorem states that if G is planar then $\chi(G) \leq 4$ [3, 38].

For a graph G and permutation of the vertices $\pi : V(G) \to V(G)$, we denote the relabeled graph by $\pi(G)$, that is, $V(\pi(G)) = V(G)$ and $E(\pi(G)) = \{\{\pi(u), \pi(v)\} : \{u, v\} \in E(G)\}$.

During SAT-based graph generation, we encounter partially defined graphs and digraphs. In a *partially defined* (di)graph G, the edge set $E(G)$ is partitioned into the set $D(G)$ of *defined edges* and the set $U(G)$ of *undefined edges*. The (di)graph G is *fully defined* if $U(G) = \emptyset$. A partially defined (di)graph G can be *extended* to a fully defined (di)graph G' if $V(G') = V(G)$ and $D(G) \subseteq E(G') \subseteq D(G) \cup U(G)$. If not stated otherwise, graphs are undirected and fully defined.

3 SAT Encodings for Planarity

There are many different criteria in the literature for a graph being planar. In this section, we select three of them and implement and benchmark these encodings.

In the context of SAT-based graph generation and enumeration, the graph is not know during search, so we design the planarity encoding based on the variables describing the combinatorial object. In other words, we don't construct formulas for a given input graph, but rather for all graphs implicitly described by certain propositional variables. For a fixed number of vertices n, we use the propositional variables $e_{u,v}$, whose truth values indicate whether the edge $\{u, v\}$ is present.

3.1 Encoding Based on Kuratowski's Theorem

The famous theorem by Kuratowski asserts that a graph is planar if and only if it does not contain $K_{3,3}$ or K_5 as a topological minor, which means it does not contain a subdivision of the complete bipartite graph $K_{3,3}$ or the complete graph K_5 as a subgraph. This planarity criterion is *negative* in the sense that it is based on the non-existence of a certain object, and hence is not well suited for an eager SAT encoding.

Towards a lazy SAT encoding, note that the existence of a topological $K_{3,3}$ or K_5 minor can be checked in linear time [7, 47]. Thus we can efficiently test whether a partially defined graph can be extended to a planar graph. We can carry out such a test during the CDCL procedure, whenever an edge variable has been decided, similarly to the SMS minimality check [30]. Whenever we determine that the current partially defined graph cannot be extended to a planar graph, we add a clause preventing that the search on this partially defined graph with possible further edges continues.

For that we proceed as follows. First, we construct a partially defined graph G given by the partial assignment of the propositional edge variables. For the sake of planarity testing, we consider the fully defined graph G' with $V(G') := V(G)$, $E(G') := D(G)$, and $U(G') := \emptyset$. Since G' is a subgraph of all extensions of G, its non-planarity implies the non-planarity for all extensions of G. We apply the Boyer-Myrvold planarity testing algorithm [7] to G', a linear time planarity algorithm based on edge additions to compute a planar embedding. If it concludes that the graph G' is not planar the algorithm returns a subgraph H of G', which is a subdivision of $K_{3,3}$ or K_5. Adding the clause

$$\bigvee_{\{u,v\} \in E(H)} \neg e_{u,v}$$

blocks this specific subgraph.

3.2 Encoding Based on Schnyder Orders

Schnyder [40] proved that a graph G is planar if and only if its incidence order dimension is at most 3. Formally, there exist three partial orders $\prec_1, \prec_2, \prec_3$ (which we call *Schnyder orders*) such that for every edge $\{u,v\} \in E(G)$ and every vertex $w \in V(G) \setminus \{u,v\}$ there is some $i \in \{1,2,3\}$ such that $u \prec_i w$ and $v \prec_i w$. Since every partial order can be extended to a total order, one can assume without loss of generality that $\prec_1, \prec_2, \prec_3$ are total orders. We refer the interested reader to Chapter 2 of Felsner's book [18].

This results in a compact encoding for planarity. To enumerate all planar graphs on a vertex set $V = [n]$, we use variables $o_{u,v,i}$ to indicate whether $u \prec_i v$ and introduce the following constraints:

To ensure that $\prec_i$ is transitive, antisymmetric, and a total order, we require for $i \in \{1,2,3\}$ the following constraints.

$$\bigwedge_{\substack{u,v,w \in V \\ u \neq v \neq w \neq u}} \neg o_{u,v,i} \vee \neg o_{v,w,i} \vee o_{u,w,i}, \qquad \bigwedge_{\substack{u,v \in V \\ u \neq v}} \neg o_{u,v,i} \vee \neg o_{v,u,i}, \qquad \bigwedge_{\substack{u,v \in V \\ u \neq v}} o_{u,v,i} \vee o_{v,u,i}.$$

To ensure that $\prec_1, \prec_2, \prec_3$ form three Schnyder orders of the desired graph, we require

$$\bigwedge_{\substack{u,v \in V \\ u \neq v}} \Big(e_{u,v} \rightarrow \bigwedge_{w \in V \setminus \{u,v\}} \bigvee_{i \in \{1,2,3\}} (o_{u,w,i} \wedge o_{v,w,i}) \Big).$$

The formula is transformed in a CNF formula using the Tseitin transformation [43]. This leads to $O(n^3)$ variables and $O(n^3)$ clauses.

Solutions of the SAT encoding are in correspondence with planar graphs together with a witnessing triple of orders. Pause to note that, in contrast to the Kuratowski based encoding where non-planarity is witnessed, planarity is witnessed in this encoding.

One disadvantage of this encoding is that it is not propagating, i.e., if all variables $e_{u,v}$ are assigned and the graph is not planar then Boolean constraint propagation does not necessarily lead to a conflict. Further, for a given planar graph there are at least exponentially many different witnessing triples of orders $\prec_1, \prec_2, \prec_3$ [18].

3.3 Encoding Based on Universal Sets

A set S of points from the plane is *n-universal* if every planar n-vertex graph can be embedded such that vertices are mapped S as vertices and all edges are straight-line segments. For instance, a triangular subset of the $(n-1) \times (n-1)$ grid is n-universal [41], and there exist n-universal sets of size $\frac{1}{4}n^2 - O(n)$ [4]. In general, the existence of an n-universal set of subquadratic size remains one of the central open problems of graph drawing. However, n-universal sets of minimum size have been computed for $n \leq 11$ [9, 39] and for certain subclasses of planar graphs universal sets of subquadratic size exist [19].

We want to enumerate all planar graphs G with vertex set $V = [n]$ by testing whether the graph represented by edge variables embeds into a prescribed n-universal point set S of size $k = |S|$. Note that, since all edges are drawn as straight-line segments, the injective mapping $P : V \to S$ fully determines the embedding. We use variables $m_{v,p}$ to indicate whether $P(v) = p$ and use clauses to ensure that no two edges cross. To keep the number of constraints small, we introduce auxiliary variables $s_{p,q}$ for any distinct $p, q \in S$ to indicate whether the segment determined by p and q is present. Finally, we must forbid the presence of crossing segments, i.e., segments are only allowed to share a common endpoint. We can express these conditions by the following constraints: To ensure that P is a mapping and that the relation is injective, we require

$$\bigwedge_{v \in V} \bigvee_{u \in S} m_{v,u} \quad \text{and} \quad \bigwedge_{v_1, v_2 \in V,\ u \in S} \neg m_{v_1,u} \vee \neg m_{v_2,u}.$$

To determine the presence of certain segments, we require

$$\bigwedge_{u,v \in V,\ a,b \in S} (e_{u,v} \wedge m_{u,a} \wedge m_{v,b}) \to s_{a,b}.$$

Finally, for any $a, b, a', b' \in S$ such that the segments ab and $a'b'$ intersect in a non-shared endpoint, we require

$$\neg s_{a,b} \vee \neg s_{a',b'}.$$

The intersecting segments can be precomputed based on the point set S and don't have to be determined by the SAT encoding.

For the injective mapping, we use $O(n^2 \cdot k)$ clauses, for the presence of certain segments $O(n^2 \cdot k^2)$ clauses, and for avoiding intersecting segments we use up to $O(k^4)$ clauses, where $k = |S|$ is at least n. Hence, the encoding has $O(k^4)$ clauses and $O(k^2)$ variables. A variant of this encoding was already used in [39, Section 4.3] to find universal point sets for a prescribed list of graphs.

Using the currently best n-universal point set, which are of magnitude $O(n^2)$, this encoding renders itself useless even for relatively small n due to $O(k^4) = O(n^8)$ clauses. However, we will test this approach for $n \leq 11$ since for this cases there exist reasonably sized n-universal sets.

4 SAT Modulo Symmetries and Digraphs

In this section, we describe the basic ideas of SMS [30] and how we adapt it to digraphs.

SMS is a dynamic symmetry breaking method for excluding isomorphic copies of graphs during search. It is designed to keep canonical graphs in the search space and discard all non-canonical graphs by adding symmetry breaking clauses. The canonical version is given by the lexicographically minimal adjacency matrix among all relabelings of the graphs. More precisely, a graph G is canonical if the row wise concatenation of the adjacency matrix of G is either equal or lexicographically smaller than the adjacency matrix of any relabeling $\pi(G)$.

To add symmetry breaking clauses during search, we need to be able to decide whether the partially defined graph given by the current solver state can be extended to a canonical fully defined graph. For that a minimality check was designed which checks for some necessary conditions. More precisely, it checks whether there is a permutation such that $\pi(G') \prec G'$ for all extensions of the current partial defined graph, i.e., the graph can definitely not be extended to a lexicographically minimal graph. Such a permutation is called *witness*. If the minimality check finds a witness then a symmetry breaking clause based on the current assignment and the witness permutation is constructed. The clause holds for all lexicographically minimal graphs and therefore does not exclude any potential solutions. The construction of potential witness permutations by the minimality check is based on a branching algorithm by gradually building a permutation starting with the vertex of smallest index. It is crucial for good performance to have arguments for cutting of a branch early if it does not lead to a witness permutation.

To adapt SMS for digraphs, note that all definitions for graphs used in the original SMS article [30] can be adapted to digraphs in a straight forward way. We highlight some of the adaptions in the following.

Let us start with defining a total order on the set of all digraphs $\mathcal{D}_n$ with vertex set $[n]$ for a fixed n. For that we first define an order on vertex pairs, naturally leading to an order of the digraphs. A vertex pair (v_1, v_2) is smaller than (u_1, u_2) (short $(v_1, v_2) \prec (u_1, u_2)$) if (i) $\min(v_1, v_2) < \min(u_1, u_2)$ or (ii) $\min(v_1, v_2) = \min(u_1, u_2)$ and $\max(v_1, v_2) < \max(u_1, u_2)$ or (iii) $\{v_1, v_2\} = \{u_1, u_2\}$ and $v_1 < u_1$. For example, for $n = 5$ we look at the following order of the non-diagonal elements of the $n \times n$ adjacency matrix:

–	1	2	3	4
5	–	9	10	11
6	12	–	15	16
7	13	17	–	19
8	14	18	20	–

The lexicographic order on $\mathcal{D}_n$ is given by comparing the string resulting from concatenating the entries in the adjacency matrix in the order given by $\preceq$. We use $G \prec H$ for denoting that G is lexicographically smaller than H. A digraph G is $\preceq$-minimal if $G \preceq \pi(G)$ holds for all relabelings.

As in the setting of undirected graphs, the minimality check for digraphs searches for witnessing permutations. The main difference is that, while in the undirected case the adjacency matrix is symmetric and only the lower triangular matrix has to be considered, in the directed case the entire adjacency matrix needs to be checked. However, the main idea of the algorithm is the same.

A formalism presented in previous work [29] based on object variables and object symmetries guaranties that adding symmetry breaking clauses with certain structure based on some permutations of the variables does preserve lexicographically minimal objects.

5 Experiments

We test our planarity encodings in three problem settings: planar Turán numbers, the Earth-Moon problem, and planar graph enumeration. To allow comparisons between the different encodings for each problem setting separately we ensure that the programs run on the same hardware. For all encodings we use Python scripts for the generation of the clauses and feed it to our SMS framework. The underlying SAT solver is an adaption of CaDiCal with the new interface IPASIR-UP that allows the solver to interact with a custom propagator [17]; this replaces the clingo solver used previously for SMS. For the Boyer-Myrvold planarity testing algorithm [7], we use the implementation provided in the C++ Boost libraries [1].

We have developed a Python layer over SMS to ease its usage and provide better readable code (SMS is written in C++ for performance reasons). In this Python layer, we have implemented various fundamental properties and invariants for graphs and digraphs such as the bounds on the connectivity, clique number, independence number, or degrees. In particular, we have implemented the planarity encodings based on Schnyder orders and universal sets in the Python layer (see Sections 3.2 and 3.3). The Kuratowski encoding, however, is implemented in C++.

With this Python layer, it should be reasonably easy also for non-programmers to run experiments from the command line and to add additional properties for graphs and digraphs. The source code and documentation is available at GitHub[1] and Read the Docs[2], respectively.

As preliminary results show, the encoding based on universal point performs much worse than the others (see Table 5). Also recall that n-universal sets of optimal size are hard to find in general and only have been computed for $n \leq 11$. Because of these two major drawbacks, we omitted this encoding on further benchmarks.

5.1 Planar Turán Numbers

Recall that the *planar Turán number* $\mathrm{ex}_P(n, H)$ for a graph H is the maximum number of edges in a planar n-vertex graph G with no copy of H as a subgraph. We are interested in planar Turán numbers $\mathrm{ex}_P(n, C_k)$, where C_k denotes the cycle graph of length k. The case $k = 3$ is rather straight-forward: since triangle-free graphs have at most $2n - 4$ edges and $K_{2,n-2}$ obtains this bound, it holds $\mathrm{ex}_P(n, C_3) = 2n - 4$ [14]. However, the situation for $k \geq 4$ get more complicated. The currently best estimates for $k \in \{4, 5\}$ are by Dowden [14], who proved the upper bounds $\mathrm{ex}_P(n, C_4) \leq \frac{15}{7}(n - 2)$ for $n \geq 4$ and $\mathrm{ex}_P(n, C_5) \leq \frac{12n-33}{5}$ for $n \geq 11$. These bounds are tight for infinitely many values of n. For example, for $k = 4$ the bound is tight for all $n \equiv 30 \pmod{70}$, and for $k = 5$ it is tight for all $n \equiv 9 \pmod{15}$.

Using our planarity encodings, we determine the exact values of $\mathrm{ex}_P(n, C_4)$ and $\mathrm{ex}_P(n, C_5)$ for small values of n. We construct a formula $F_{n,m,k}$ which is satisfiable if there is a C_k-free graph with at least m edges. To encode C_k-free graphs we explicitly, we add the clause

$$\neg e_{v_1,v_2} \vee \neg e_{v_2,v_3} \vee \cdots \vee \neg e_{v_{k-1},v_k} \vee \neg e_{v_k,v_1}$$

for any k distinct vertices $v_1, \ldots, v_k \in V$. Using sequential counters [42], we ensure that the number of edges is at least m.

Given the ideas in Section 3 for ensuring planarity of the generated graphs, we compute the exact values of $\mathrm{ex}_P(n, C_k)$. For a fixed n and k, this is done by testing $F_{n,m,k}$ enhanced with a planarity encoding for satisfiability, starting with $m = n$. We increment m until the formula is unsatisfiable. Our computational result are summarized by the following theorem.

[1] `https://github.com/markirch/sat-modulo-symmetries/`
[2] `https://sat-modulo-symmetries.readthedocs.io/`

Table 1 Result for computing $\mathrm{ex}_P(n, C_4)$. All computation times are given in seconds. The third column gives the upper bound by Dowden [14]. SMS also found a graph with 19 vertices and 35 edges within 14 seconds, but we are not aware if this example is extremal for $n = 19$.

			SAT		UNSAT	
n	$\mathrm{ex}_P(n, C_4)$	$\lfloor \frac{15}{7}(n-2) \rfloor$	Kura	Ord	Kura	Ord
4	4	4	0.00	0.00	0.00	0.00
5	6	6	0.00	0.00	0.00	0.00
6	7	8	0.00	0.00	0.00	0.01
7	9	10	0.01	0.01	0.01	0.02
8	11	12	0.01	0.02	0.02	0.03
9	13	15	0.03	0.04	0.05	0.05
10	16	17	0.04	0.07	0.07	0.06
11	18	19	0.16	0.44	0.16	0.23
12	20	21	0.27	0.98	0.56	2.29
13	22	23	0.23	0.14	1.96	15.27
14	24	25	0.20	0.44	6.46	340.11
15	27	27	1.00	0.85	21.39	294.07
16	29	30	5.87	24.90	172.90	31142.08
17	31	32	5.19	83.59	3479.65	t.o.
18	33	34	14.69	14.85	59862.72	t.o.

▶ Theorem 1. *It holds that*

$$\mathrm{ex}_P(n, C_4) = \left\lfloor \frac{15}{7}(n-2) \right\rfloor - \begin{cases} 0 & \text{for } n \in \{4, 5, 15\}, \\ 1 & \text{for } n \in [6, 8] \cup [10, 14] \cup [16, 18], \\ 2 & \text{for } n = 9, \end{cases}$$

and

$$\mathrm{ex}_P(n, C_5) = \left\lfloor \frac{12n - 33}{5} \right\rfloor + \begin{cases} 0 & \text{for } n \in \{9\} \cup [11, 18], \\ 1 & \text{for } n \in \{8, 10\}, \\ 2 & \text{for } n \in [5, 7]. \end{cases}$$

Moreover, based on our computational data, we conjecture that Dowden's upper bound for $k = 5$ is tight for all $n \geq 11$.

▶ Conjecture 2. $\mathrm{ex}_P(n, C_5) = \lfloor \frac{12n-33}{5} \rfloor$ *for* $n \geq 11$.

Tables 1 and 2 summarize the computation times for both encodings. The times for solving $F_{n,\mathrm{ex}_P(n,C_k),k}$ are given by "SAT" and $F_{n,\mathrm{ex}_P(n,C_k)+1,k}$ given by "UNSAT". Computations not finished within three days are marked with "t.o." (timeout). The columns labeled "Kura" provide the times for the encoding based on Kuratowski's theorem with a propagator and the columns "Ord" provides the times for the encoding based on Schnyder orders.

In general, we see that the version excluding Kuratowski graphs performs much better, especially for unsatisfiable cases. For example for $n = 17, k = 4$ the Kuratowski based generation is over a hundred times faster than the encoding based on Schnyder orders.

Table 2 Result for computing $\mathrm{ex}_P(n, C_5)$. All computation times are given in seconds. The third column gives the upper bound by Dowden [14] for $n \geq 11$.

			SAT		UNSAT	
n	$\mathrm{ex}_P(n, C_5)$	$\lfloor \frac{12n-33}{5} \rfloor$	Kura	Ord	Kura	Ord
5	7	5	0.00	0.01	0.00	0.00
6	9	7	0.00	0.01	0.01	0.01
7	12	10	0.01	0.02	0.02	0.01
8	13	12	0.02	0.07	0.05	0.11
9	15	15	0.03	0.06	0.07	0.38
10	18	17	0.10	0.29	0.23	1.67
11	19	19	0.12	0.30	0.57	4.89
12	22	22	1.83	1.72	1.99	33.08
13	24	24	0.48	1.61	11.45	271.18
14	27	27	3.18	7.63	35.24	1174.85
15	29	29	2.24	10.82	277.78	15459.24
16	31	31	4.71	59.09	3172.27	235353.58
17	34	34	207.49	890.98	29023.55	t.o.
18	36	36	1851.84	1249.38	t.o.	t.o.

5.2 The Earth-Moon Problem

A graph G is *biplanar* if it can be partitioned into two planar graphs, that is, there exist two planar graphs G_1, G_2 with $E(G) = E(G_1) \cup E(G_2)$. In that case, we write $G = G_1 \uplus G_2$. Biplanar graphs are also known as graphs with *thickness* two. The Earth-Moon problem asks for the largest chromatic number a biplanar graph can have, denoted by χ_2. In 1973, Thom Sulanke constructed a biplanar graph on 11 vertices with chromatic number 9 by removing the edges of a C_5 from a K_{11}, improving an earlier lower bound by Ringel to $\chi_2 \geq 9$ [22]. On the other hand, using Euler's formula, one can derive that any biplanar graph must have a vertex of degree at most 11, which applied inductively shows that $\chi_2 \leq 12$. Despite of much research efforts, the estimates $9 \leq \chi_2 \leq 12$ could not be improved since then. Some have suggested that this problem is "as hard as two or three four-color theorems" [26, p. 199].

Searching for biplanar graphs and at least a certain chromatic number seems to be an extremely challenging problem. Indeed, the problem of deciding whether a graph is biplanar is NP-complete [33] and checking whether a graph has at least chromatic number χ for a fixed constant $\chi \geq 3$ is coNP-complete in general [28]. To admit partial progress, one can parameterized the Earth-Moon problem by the number n of vertices in the biplanar graph, denoting the highest chromatic number for a n-vertex biplanar graph by $\chi_2(n)$. Sulanke's lower bound $\chi_2(11) \geq 9$ carries over to $n > 11$ since adding isolated vertices to a biplanar graph does not change its chromatic number and keeps the graph biplanar.

Our goal is to show the absence or presence of biplanar graphs for given order n and chromatic number χ using SMS and planarity encodings.

One possibility of using SMS for biplanar graphs is applying the symmetry breaking directly at the graph G. This way, we would take edge variables $e_{u,v}$ describing the graph G. To encode the decomposition $G = G_1 \uplus G_2$, we introduce auxiliary variables $e^1_{u,v}$ and $e^2_{u,v}$ to indicate whether an edge $\{u, v\}$ belongs to $E(G_1)$ or $E(G_2)$, respectively. However, this way we don't break all symmetries. If π is an automorphism of G, i.e., $\pi(G) = G$, then it does not necessarily hold that $\pi(G_1) = G_1$ and $\pi(G_2) = G_2$. In other words, we will get different partitions representing isomorphic decompositions. This is a real problem in practice as some experiments on testing biplanarity of K_9 showed.

Hence, we propose a different and more efficient approach. Instead of encoding the biplanar graph G directly, we represent the decomposition $G_1 \uplus G_2$ as a directed graph H with $\underline{H} = G$. H represents the decomposition $G_1 \uplus G_2$ as follows.

- $\{u, v\} \in E(G_1)$ if and only if $(u, v) \in E(H)$ and $(v, u) \in E(H)$.
- $\{u, v\} \in E(G_2)$ if and only if either $(u, v) \in E(H)$ or $(v, u) \in E(H)$, but not both.

Now we can apply SMS for digraphs as discussed in Section 4. Consider two directed graphs H and H' that represent the decompositions $G_1 \uplus G_2 = \underline{H}$ and $G'_1 \uplus G'_2 = \underline{H'}$, respectively. We observe that if H and H' are isomorphic, then $\underline{H}$ and $\underline{H'}$ are isomorphic and G_i and G'_i are isomorphic, $i \in \{1, 2\}$. Consequently, it is sound to only consider lexicographically minimal digraphs H.

We further restrict the digraphs. W.l.o.g., we may assume that if $(u, v) \in E(H)$ for $u < v$ then also $(v, u) \in E(H)$. This is the case because $(u, v) \prec (v, u)$, hence replacing the arc (u, v) by (v, u) if (v, u) is not present leads to a strictly lexicographically smaller graph representing the same decomposition.

We note that the symmetry breaking on biplanar graphs using digraphs still has some potential room for improvement. There are non-isomorphic digraphs H, H' whose underlying graphs $\underline{H}, \underline{H'}$ are isomorphic, i.e., we have different representation for the same underlying graph. For example, if the underlying graph is $\underline{H} = K_5$ (the complete graph on 5 vertices), we can partition the graph $\underline{H} = G_1 \uplus G_2$ in almost all ways granted that both G_1 and G_2 contain at least one edge and none contains all edges. Further, the representation as digraphs doesn't exclude all isomorphic partitions, i.e., there are lexicographically minimal digraphs with the described restrictions representing isomorphic partitions. We plan to design a version of SMS avoiding these isomorphic copies in the future.

W.l.o.g., we may assume for a decomposition $G = G_1 \uplus G_2$ that G_1 is *maximal planar*, i.e., inserting any additional edge makes the graph non-planar, since we can move as many edges as possible from G_2 to G_1. We encode this by requiring that $|E(G_1)| = 3n - 6$, hence we can also require $|E(G_2)| \leq 3n - 6$.

Further, we restrict our search on *vertex-critical* graphs with respect to the chromatic number χ, i.e., deleting any vertex decreases the chromatic number of G. Hence we can assume that the minimum degree of G is $\geq \chi - 1$.

The following encoding describes the digraph H with vertex set $V = [n]$ that represents the decomposition of a χ-chromatic graph $\underline{H} = G$ into two planar subgraphs. We use directed edge variables $d_{u,v}$ to encode the existence of the directed edge $(u, v) \in E(H)$.

To restrict the digraph, we require

$$\bigwedge_{\substack{v,u \in V \\ v<u}} d_{v,u} \to d_{u,v} \qquad \text{and} \qquad \bigwedge_{\substack{v,u \in V \\ v<u}} e^1_{v,u} \leftrightarrow (d_{v,u} \wedge d_{u,v});$$

this results in $e^1_{v,u} \leftrightarrow d_{v,u}$ for $v < u$. We further require

$$\bigwedge_{\substack{v,u \in V \\ v<u}} e^2_{v,u} \leftrightarrow (\neg d_{v,u} \wedge d_{u,v}) \qquad \text{and} \qquad e_{v,u} = e^1_{v,u} \wedge e^2_{v,u},$$

which can be simplified to $e_{v,u} \leftrightarrow d_{u,v}$ for $v < u$. Finally, we require

$$\sum_{\substack{v,u \in V \\ v<u}} e^1_{v,u} = 3n - 6 \qquad \text{and} \qquad \sum_{\substack{v,u \in V \\ v<u}} e^2_{v,u} \leq 3n - 6,$$

encoded with sequential counters [42].

Table 3 Computations for the Earth-Moon problem, given the number n of vertices and the chromatic number χ. If the timeout of 2 days is reached we write "t.o.".

	$\chi \geq 9$			$\chi \geq 10$		
n	#digraph	Kura	Ord	#digraph	Kura	Ord
9	0	0.55	18.14			
10	0	15.75	2028.21	0	1.95	112.43
11	5554	1709.49	t.o.	0	19.03	4543.13
12	-	t.o.	t.o.	0	837.14	t.o.
13	-	t.o.	t.o.	0	146484.00	t.o.

For ensuring at least a certain chromatic number χ, we add *coloring clauses* ensuring that the underlying graph cannot be colored with $\chi - 1$ colors. Let $\mathcal{P}_n$ be the set of all partitions of V. Then

$$\bigwedge_{\substack{P \in \mathcal{P}_n \\ |P| = \chi - 1}} \bigvee_{S \in P} \bigvee_{\substack{u,v \in V \\ u < v}} e_{u,v}$$

ensures that every $(\chi - 1)$-coloring is no proper coloring of the underlying graph for $\chi - 1 \geq n$, because at least one edge is monochromatic. Since the number of partitions $\mathcal{P}_n$ is exponential, this size of the encoding grows exponentially. However, as our experiments showed, this approach is still feasible for small values of n. We have also tried a lazy encoding which adds the clauses incrementally whenever there is a violation instead of adding all clauses right at the beginning. As it turned out, the results for this version were worse and hence we omit the results for the lazy version.

Table 3 shows the results and computation times of our experiments. For $\chi \geq 9$ and $n = 11$ the formula is satisfiable and the previously known results were confirmed. For $\chi \geq 10$ and $n \leq 13$ the formula is unsatisfiable. Therefore we have the following result.

▶ **Theorem 3.** *All biplanar graphs on $n \leq 13$ vertices are 9-colorable.*

Our experiments again show that the Kuratowski-based encoding is superior by orders of magnitudes. Table 4 summarizes the new results in context of what has been known so far.

In the literature, there are some potential candidates for the Earth-Moon Problem, which are known to have chromatic number 10, but haven't been shown to be biplanar yet [23]. One of these graphs is $G = C_5[4, 4, 4, 4, 3]$, i.e., a 5-cycle where the first four vertices of the cycle are inflated to a 4-clique, and the last to a 3-clique. The graph has 19 vertices and 99 edges. We can test whether this graph is biplanar using our planarity encodings. This can be done by adding constraints that ensure that the underlying graph of the resulting directed graph is the graph G:

$$\bigwedge_{u,v \in E(G), u < v} e_{v,u} \wedge \bigwedge_{u,v \in V(G), u,v \notin E(G)} \neg e_{v,u}.$$

By fixing some of the directed edges, SMS is not applicable anymore for all permutations. We only allow permuting vertices within the 4-clique and 3-clique, respectively, which preserves the underlying graph G. Within 12 hours, we are able to show that the graph is not biplanar, hence we can exclude the graph as a potential candidate.

Table 4 Current state of knowledge on the Earth-Moon problem for n-vertex biplanar graphs for $8 \leq n \leq 18$. Orange cells indicate that there doesn't exist an n-vertex biplanar graph with chromatic number χ, blue cells indicate the existence. The cells labeled "new" correspond to new results obtained in this paper, using the observations that removing an independent set decreases the chromatic number by at most 1 and since K_n with $n \geq 9$ is not biplanar, every biplanar graph with $n \geq 9$ vertices has an independent set of size two. With a minimality argument it is also possible to exclude the case with $n = 18$ and $\chi = 12$. If $n < \chi$, the problem is trivially unsatisfiable. If $\chi = n$, then the only potential n-vertex graph is the complete graph K_n; for $n \leq 8$, K_n is known to be biplanar, for $n \geq 9$ it is not biplanar. All biplanar graphs are known to have a chromatic number ≤ 12, hence all cells in the rightmost column are orange. The cases $n \geq 11$ and $\chi = 9$ are all satisfiable, as witnessed by Sulanke's graph.

	chromatic number χ					
n	8	9	10	11	12	13
8	K_8					
9		K_9				
10		new	K_{10}			
11		Sulanke	new	K_{11}		
12			new	new	K_{12}	
13			new	new	new	K_{13}
14			open	new	new	
15			open	new	new	
16			open	open	new	
17			open	open	new	
18			open	open	new	
19			open	open	open	

5.3 Integer Sequences Related to Planar Graphs and Digraphs

Many integer sequences featured in the *On-Line Encyclopedia of Integer Sequences (OEIS)* [35] give the number of non-isomorphic n-vertex graphs with a certain property, for $n \in \mathbb{N}$. The encyclopedia is very useful for research in combinatorics because a sequence can for instance be used to come up with a hypothetical closed formula for a sequence, or to check whether two graph classes coincide. Often, no closed formula for a sequence is known; therefore, only a finite prefix is reported on OEIS. In this section, we demonstrate the versatility of our SMS framework in conjunction with the new planarity encoding to almost effortlessly verify and extend sequences listed on OEIS. Moreover, it allows us to compute and add new natural sequences for which no suitable tools have existed.

In the following, we review some specific integer sequences that we could verify or extend with SMS. The list is not exhaustive and can certainly be improved by further optimization.

Let us start with the sequence for non-isomorphic planar n-vertex graphs OEIS/A5470; the precise numbers are known for up to $n = 12$. Table 5 shows the running times required to verify these numbers with SMS and the three planarity encodings. Since the encoding based on Kuratowski's theorem performs significantly better than the other two, we only used this encoding in the following.

Table 5 Enumeration of planar graphs with SMS.

n	# (OEIS/A5470)	Kura	Ord	Univ
2	2			
3	4			
4	11			
5	33			
6	142	0.02s	0.04s	0.02s
7	822	0.12s	0.27s	0.24s
8	6966	1.02s	3.48s	6.94s
9	79853	13.5s	1m16s	8m50s
10	1140916	5m56s	2h16m	116h
11	18681008	13h53m		
12	333312451			

OEIS/A49339 counts the number of n-vertex planar graphs with even degrees. With previous tools, the first 12 terms were computed (Brendan McKay gave 11). We verified these 12 terms with SMS and extended the sequence by the 13th and 14th terms (about 2 and 40 hours of computation time, respectively).

OEIS/A49339 is also the Euler transform of OEIS/A49365, which counts the number of *connected* n-vertex planar graphs with even degrees. Therefore, having n terms of one sequence, one can compute the n terms of the other. Surprisingly, SMS performed almost twice as fast for computing the 13th and 14th term on OEIS/A49339.

The sequences OEIS/A49369 to OEIS/A49373 count the number of planar graphs with minimum degree at least $k \in \{1, 2, 3, 4, 5\}$. Verifying all terms for $k = 3, 4, 5$ using SMS took about 3 hours, 1 hour, and 2 days, respectively. Moreover, we have extended OEIS/A49372 (the sequence for $k = 4$) by the 16th term, which was computed within 2 days, and OEIS/A49373 (the sequence for $k = 5$) by the 26th term, which was computed within 8 days,

OEIS/A255600 counts the number of connected planar regular graphs on $2n$ vertices with a girth of at least 4. Note that girth at least 4 is equivalent to C_3-free (a.k.a. *triangle-free*) and, as noted in the comments of that sequence, all such graphs are 3-regular. SMS can verify the previous 13 terms within 90 minutes. We have extended the sequence by the 14th and 15th term, for which the computations took 16 hours and 9 days, respectively.

OEIS/A58378 counts the number of 3-regular 2-connected planar $2n$-vertex graphs. SMS verified all known terms up to $n = 13$ (i.e., up to 26 vertices) within 5 days.

While plantri was used to enumerate k-connected planar graphs for up to $k = 4$, it is surprising that there was no OEIS entry yet for *5-connected planar graphs*. So we created OEIS/A361578.

There was no OEIS entry yet for *planar digraphs*, so we created it OEIS/A361366 for up to $n = 6$. Note that, when compared with the number of planar graphs, the two options for directing each edge cause an increase in the numbers exponentially. Table 6 gives an overview of k-connected graphs and weakly k-connected digraphs for $k \leq 5$ for both general and planar settings. Since planar (directed) graphs have connectivity at most 5, we here only discuss the case $k \leq 5$. For more information on higher connectivity on general graphs, we refer to the table in OEIS/A259862.

Only sequences for weakly k-connected digraphs with $k \in \{0, 1\}$ were known; hence we created sequences for $k \in \{2, 3\}$. We also created sequences for weakly k-connected planar directed graphs for all $k \in \{0, \ldots, 3\}$ and added them to OEIS. Surprisingly, when we

Table 6 Sequences for k-connected planar graphs and weakly k-connected planar digraphs. Entries marked with $*$ are new.

	graphs		digraphs	
k-connected	general	planar	general	planar
$k = 0$	OEIS/A88	OEIS/A5470	OEIS/A273	OEIS/A361366*
$k = 1$	OEIS/A1349	OEIS/A3094	OEIS/A3085	OEIS/A361368*
$k = 2$	OEIS/A2218	OEIS/A21103	OEIS/A361367*	OEIS/A361369*
$k = 3$	OEIS/A6290	OEIS/A944	OEIS/A361370*	OEIS/A361371*
$k = 4$	OEIS/A86216	OEIS/A7027	?	?
$k = 5$	OEIS/A86217	OEIS/A361578*	?	?

recently submitted our results to OEIS, Andrew Howroyd extended the sequence for weakly 2-connected graphs by using an approach based on generating functions and combinatorial species. This gives a beautiful example of how our contribution can stimulate research in enumerative combinatorics.

Last, we should mention the well-understood class of maximal planar graphs, known as triangulations. The entries OEIS/A109, OEIS/A7021, and OEIS/A111358 count 3, 4, and 5-connected triangulations, respectively.

6 Conclusion

We have presented a comprehensive study on SAT-based planar graph generation using encodings with dynamic symmetry breaking. Our experimental results compare the effectiveness and scalability of the Kuratowski-based and order-based encodings in solving combinatorial problems related to planar graphs. In particular, we provided progress concerning the computation of planar Turán numbers and the Earth-Moon problem. Furthermore, we have shown the potential of the SMS framework equipped with planarity encodings by verifying and extending several OEIS sequences related to planar graph enumeration.

Additionally, we suggest exploring the adaptation of the Kuratowski [18, Section 1.4] and Schnyder encodings [20] for outerplanar graphs, which presents an interesting application area for SMS.

For planar graphs, there exists a polynomial-time canonization algorithm [25, 31]. Cook's Theorem [11] allows us to translate this algorithm into a polynomially-sized SAT encoding for planar graph canonization. It would be interesting to see, whether such a symmetry breaking tailored to planar graphs outperforms the general symmetry breaking in a practical setting.

References

1 David Abrahams et al. Boost C++ Libraries. `http://www.boost.org/`.

2 Martin Aigner and Günter M. Ziegler. *Proofs from The Book*, chapter Chapter 41: Turán's graph theorem, pages 285–289. Springer, Berlin, sixth edition, 2018. `doi:10.1007/978-3-662-57265-8`.

3 Kenneth Appel and Wolfgang Haken. The solution of the four-color-map problem. *Sci. Amer.*, 237(4):108–121, 152, 1977. `doi:10.1038/scientificamerican1077-108`.

4 Michael J. Bannister, Zhanpeng Cheng, William E. Devanny, and David Eppstein. Superpatterns and universal point sets. *J. Graph Algorithms Appl.*, 18(2):177–209, 2014. doi:10.7155/jgaa.00318.

5 Béla Bollobás. *Extremal graph theory.* Academic Press, 1978.

6 John A. Bondy and Uppaluri S.R. Murty. *Graph Theory.* Springer Verlag, 1st edition, 2008.

7 John M. Boyer and Wendy J. Myrvold. On the Cutting Edge: Simplified $O(n)$ Planarity by Edge Addition. *J. Graph Algorithms Appl.*, 8(3):241–273, 2004. doi:10.7155/jgaa.00091.

8 Gunnar Brinkmann and Brendan D. McKay. Fast generation of planar graphs. *MATCH Communications in Mathematical and in Computer Chemistry*, 58(2):323–357, 2007.

9 Jean Cardinal, Michael Hoffmann, and Vincent Kusters. On universal point sets for planar graphs. In *Computational Geometry and Graphs*, pages 30–41. Springer, 2013. doi:10.1007/978-3-642-45281-9_3.

10 Markus Chimani, Ivo Hedtke, and Tilo Wiedera. Exact algorithms for the maximum planar subgraph problem: New models and experiments. *ACM J. Exp. Algorithmics*, 24(1):2.1:1–2.1:21, 2019. doi:10.1145/3320344.

11 Stephen A. Cook and Robert A. Reckhow. Time bounded random access machines. *Journal of Computer and System Sciences*, 7(4):354–375, 1973. doi:10.1016/S0022-0000(73)80029-7.

12 Daniel W. Cranston, Bernard Lidický, Xiaonan Liu, and Abhinav shantanam. Planar Turán numbers of cycles: a counterexample. *Electron. J. Combin.*, 29(3):Paper No. 3.31, 10, 2022. doi:10.37236/10774.

13 Hubert de Fraysseix, János Pach, and Richard Pollack. Small Sets Supporting Fary Embeddings of Planar Graphs. In *Proceedings of the Twentieth Annual ACM Symposium on Theory of Computing (STOC'88)*, pages 426–433. Association for Computing Machinery, 1988. doi:10.1145/62212.62254.

14 Chris Dowden. Extremal C_4-free/C_5-free planar graphs. *J. Graph Theory*, 83(3):213–230, 2016. doi:10.1002/jgt.21991.

15 Liangli Du, Bing Wang, and Mingqing Zhai. Planar Turán numbers on short cycles of consecutive lengths. *Bull. Iranian Math. Soc.*, 48(5):2395–2405, 2022. doi:10.1007/s41980-021-00644-1.

16 Longfei Fang, Bing Wang, and Mingqing Zhai. Planar Turán number of intersecting triangles. *Discrete Math.*, 345(5):Paper No. 112794, 10, 2022. doi:10.1016/j.disc.2021.112794.

17 Katalin Fazekas, Aina Niemetz, Mathias Preiner, Markus Kirchweger, Stefan Szeider, and Armin Biere. IPASIR-UP: User propagators for CDCL. In Meena Mahajan and Friedrich Slivovsky, editors, *The 26th International Conference on Theory and Applications of Satisfiability Testing (SAT 2023), July 04–08, 2023, Alghero, Italy*, LIPIcs. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2023. to appear.

18 Stefan Felsner. *Geometric Graphs and Arrangements – Some Chapters from Combinatorial Geometry.* Advanced Lectures in Mathematics (ALM). Vieweg+Teubner, 2004. doi:10.1007/978-3-322-80303-0.

19 Stefan Felsner, Hendrik Schrezenmaier, Felix Schröder, and Raphael Steiner. Linear size universal point sets for classes of planar graphs, 2023. arXiv:2303.00109, to appear in Proceedings of the 39th International Symposium on Computational Geometry (SoCG 2023).

20 Stefan Felsner and William T. Trotter. Posets and planar graphs. *Journal of Graph Theory*, 49(4):273–284, 2005. doi:10.1002/jgt.20081.

21 Johannes K. Fichte, Daniel Le Berre, Markus Hecher, and Stefan Szeider. The silent (r)evolution of SAT. *Communications of the ACM*, 66(6):64–72, June 2023. doi:10.1145/3560469.

22 Martin Gardner. Mathematical games. *Sci. Amer.*, 242(1):22–33B, 1980.

23 Ellen Gethner. *To the Moon and Beyond*, pages 115–133. Springer Verlag, 2018. doi:10.1007/978-3-319-97686-0_11.

24 Debarun Ghosh, Ervin Győri, Ryan R. Martin, Addisu Paulos, and Chuanqi Xiao. Planar Turán number of the 6-cycle. *SIAM J. Discrete Math.*, 36(3):2028–2050, 2022. doi:10.1137/21M140657X.

25 J. E. Hopcroft and J. K. Wong. Linear time algorithm for isomorphism of planar graphs (preliminary report). In *Proceedings of the Sixth Annual ACM Symposium on Theory of Computing*, STOC '74, pages 172–184. Association for Computing Machinery, 1974. `doi:10.1145/800119.803896`.

26 Joan P. Hutchinson. Some conjectures and questions in chromatic topological graph theory. In *Graph theory—favorite conjectures and open problems. 1*, Probl. Books in Math., pages 195–210. Springer, 2016. `doi:10.1007/978-3-319-31940-7_12`.

27 Tommy R. Jensen and Bjarne Toft. *Graph coloring problems*. Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons, Inc., New York, 1995. A Wiley-Interscience Publication.

28 Richard M. Karp. Reducibility among combinatorial problems. In *Proceedings of a symposium on the Complexity of Computer Computations*, pages 85–103. Plenum, New York, 1972.

29 Markus Kirchweger, Manferd Scheucher, and Stefan Szeider. A SAT attack on Rota's basis conjecture. In *25th International Conference on Theory and Applications of Satisfiability Testing, SAT 2022*, volume 236 of *LIPIcs*, pages 4:1–4:18. Schloss Dagstuhl, 2022. `doi:10.4230/LIPIcs.SAT.2022.4`.

30 Markus Kirchweger and Stefan Szeider. SAT modulo symmetries for graph generation. In *27th International Conference on Principles and Practice of Constraint Programming (CP 2021)*, LIPIcs, pages 39:1–39:17. Dagstuhl, 2021. `doi:10.4230/LIPIcs.CP.2021.34`.

31 Jacek P. Kukluk, Lawrence B. Holder, and Diane J. Cook. Algorithm and experiments in testing planar graphs for isomorphism. *Journal of Graph Algorithms and Applications*, 8(3):313–356, 2004. `doi:10.7155/jgaa.00094`.

32 Yongxin Lan, Yongtang Shi, and Zi-Xia Song. Extremal H-free planar graphs. *Electron. J. Combin.*, 26(2):Paper No. 2.11, 17, 2019. `doi:10.37236/8255`.

33 Anthony Mansfield. Determining the thickness of graphs is NP-hard. *Math. Proc. Cambridge Philos. Soc.*, 93(1):9–23, 1983. `doi:10.1017/S030500410006028X`.

34 Brendan D. McKay and Adolfo Piperno. Practical graph isomorphism, II. *J. Symbolic Comput.*, 60:94–112, 2014. `doi:10.1016/j.jsc.2013.09.003`.

35 OEIS Foundation Inc. The On-Line Encyclopedia of Integer Sequences. Published electronically at `http://oeis.org`.

36 Steven D. Prestwich. CNF encodings. In Armin Biere, Marijn Heule, Hans van Maaren, and Toby Walsh, editors, *Handbook of Satisfiability – Second Edition*, volume 336 of *Frontiers in Artificial Intelligence and Applications*, pages 75–100. IOS Press, 2021. `doi:10.3233/FAIA200985`.

37 Gerhard Ringel. *Färbungsprobleme auf Flächen und Graphen*, volume 2 of *Mathematische Monographien*. Berlin: VEB Deutscher Verlag der Wissenschaften, 1959.

38 Neil Robertson, Daniel Sanders, Paul Seymour, and Robin Thomas. The four-colour theorem. *J. Combin. Theory Ser. B*, 70(1):2–44, 1997. `doi:10.1006/jctb.1997.1750`.

39 Manfred Scheucher, Hendrik Schrezenmaier, and Raphael Steiner. A note on universal point sets for planar graphs. *J. Graph Algorithms Appl.*, 24(3):247–267, 2020. `doi:10.7155/jgaa.00529`.

40 Walter Schnyder. Planar graphs and poset dimension. *Order*, 5:323–343, 1989. `doi:10.1007/BF00353652`.

41 Walter Schnyder. Embedding planar graphs on the grid. In *Proceedings of the First Annual ACM-SIAM Symposium on Discrete Algorithms*, SODA '90, pages 138–148. Society for Industrial and Applied Mathematics, 1990. `doi:10.5555/320176.320191`.

42 Carsten Sinz. Towards an Optimal CNF Encoding of Boolean Cardinality Constraints. In *11th International Conference on Principles and Practice of Constraint Programming (CP 2005)*, volume 3709 of *Lecture Notes in Computer Science*, pages 827–831. Springer Verlag, 2005.

43 Grigori S. Tseitin. Complexity of a derivation in the propositional calculus. *Zap. Nauchn. Sem. Leningrad Otd. Mat. Inst. Akad. Nauk SSSR*, 8:23–41, 1968. Russian. English translation in J. Siekmann and G. Wrightson (eds.) *Automation of Reasoning. Classical Papers on Computer Science 1967–1970*, Springer Verlag, 466–483, 1983.

44 Paul Turán. Eine Extremalaufgabe aus der Graphentheorie. *Középiskolai Matematikai és Fizikai Lapok*, 48:436–452, 1941.

45 Douglas B. West. *Introduction to Graph Theory*. Prentice Hall, 2nd edition, 2000.

46 Hassler Whitney. Non-separable and planar graphs. *Proceedings of the National Academy of Sciences of the United States of America*, 17(2):125–127, 1931. URL: `http://www.jstor.org/stable/86196`.

47 Stanley G. Williamson. Depth-First Search and Kuratowski Subgraphs. *Journal of the ACM*, 31(4):681–693, 1984. `doi:10.1145/1634.322451`.