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THE NUMBER OF LATTICE PATHS BELOW A CYCLICALLY SHIFTING BOUNDARY

J. IRVING AND A. RATTAN

ABSTRACT. We count the number of lattice paths lying under a cyclically shifting piecewise linear boundary of varying slope. Our main result can be viewed as an extension of well-known enumerative formulae concerning lattice paths dominated by lines of integer slope (*e.g.* the generalized ballot theorem). Its proof involves a classical "reflection" argument, and a straightforward refinement of our bijection allows for the counting of paths with a specified number of corners. We also show how the result can be applied to give elegant derivations for the number of lattice walks under certain periodic boundaries. In particular, we recover known expressions concerning paths dominated by a line of half-integer slope, and some new and old formulae for paths lying under special "staircases".

1. INTRODUCTION

The term *lattice path* is used throughout to refer to a path in the integer lattice $\mathbb{Z} \times \mathbb{Z}$ with unit steps up and to the right (*i.e.* steps $(0,1)$ and $(1,0)$, respectively).

Let $\mathbf{a} = (a_0, \ldots, a_{m-1})$ be a weak m-part composition of n (recall that this means the a_i are nonnegative integers summing to n). This paper concerns the enumeration of lattice paths from the origin that lie weakly under the piecewise linear boundary curve $\partial\mathbf{a}$ defined by

$$x = a_i y + \sum_{j=0}^{i-1} a_j, \quad \text{for } y \in [i, i+1].$$

Any point or path lying weakly under $\partial\mathbf{a}$) is said to be *dominated* by $\mathbf{a}$. For instance, the boundary $\partial\mathbf{a}$ corresponding to $\mathbf{a} = (1,2,3,2)$ is shown in Figure 1, along with a path it dominates.

Let $D(\mathbf{a})$ be the number of paths from $(0,0)$ to (n,m) dominated by $\mathbf{a}$. (For example, the numbers $D(\mathbf{a})$ for various 3-part compositions of 6 are given above their respective

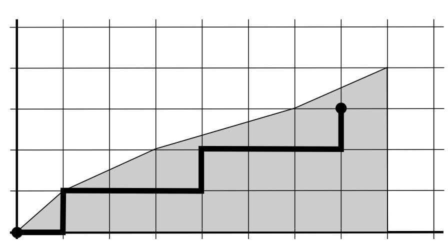

FIGURE 1. A path dominated by $\mathbf{a} = (1,2,3,2)$.

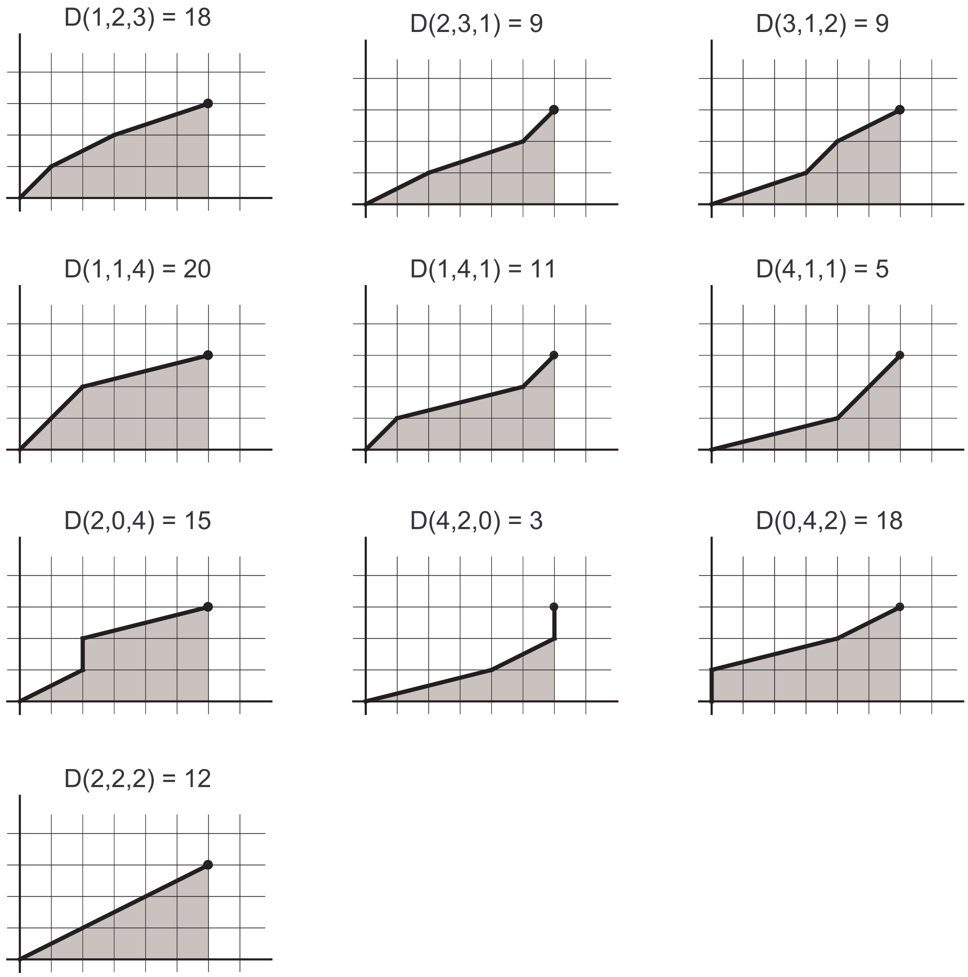

FIGURE 2. The number of paths dominated by cyclically shifted boundaries.

boundaries $\partial\mathbf{a}$ in Figure 2.) When all parts of $\mathbf{a}$ are the same, it is well-known [5, Exercise 5.3.5] that $D(\mathbf{a})$ is a generalized Catalan number. In particular, we have

$$D(\underbrace{a, a, \ldots, a}_{m \text{ copies}}) = \frac{1}{(a+1)m+1}\binom{(a+1)m+1}{m}, \tag{1}$$

where the case $a = 1$ corresponds with the classical Dyck paths counted by the usual Catalan numbers. However, for general $\mathbf{a}$, no simple formula for $D(\mathbf{a})$ is known. Indeed, it is unlikely that such a formula exists, though the Kreweras dominance theorem [5, Section 5.4.7] does give a determinantal expression. It is the purpose of this paper to show that simple enumerative formulae do hold provided we consider paths dominated by *all cyclic shifts* of an arbitrary composition.

Consider, for instance, the rows of Figure 2. Each row illustrates the boundaries corresponding to the three cyclic shifts of a given composition of 6. To be precise, for each integer j, let $\mathbf{a}^{\langle j\rangle}$ denote the j-th shift of $\mathbf{a}$, namely

$$\mathbf{a}^{\langle j\rangle} = (a_{-j}, a_{-j+1}, \ldots, a_{-j+m-1}), \tag{2}$$

where the indices are to be interpreted modulo m. Then the rows of Figure 2 illustrate the boundaries $\partial\mathbf{a}, \partial\mathbf{a}^{\langle 1\rangle}$, and $\partial\mathbf{a}^{\langle 2\rangle}$ for the compositions $\mathbf{a} = (1,2,3), (1,1,4), (2,0,4)$ and $(2,2,2)$.

Notice that in each of the top three rows of the figure there are a total of 36 dominated paths from $(0,0)$ to $(6,3)$. There are this many also in the bottom row if the three identical cyclic shifts of $\mathbf{a} = (2,2,2)$ are taken into account. That is, $D(\mathbf{a}) + D(\mathbf{a}^{\langle 1\rangle}) + D(\mathbf{a}^{\langle 2\rangle}) = 36$ for each of these 3-part compositions $\mathbf{a}$ of 6. This is a special case of a more general phenomenon, which we now explore.

Define a *lattice path boundary pair (LPBP)* to be an ordered pair $(\mathcal{P}, (\mathbf{a}, j))$, where $\mathcal{P}$ is a lattice path beginning at the origin, $\mathbf{a}$ is a weak m-part composition, and j is an integer with $0 \le j < m$. If $\mathcal{P}$ is dominated by $\mathbf{a}^{\langle j\rangle}$ then we say $(\mathcal{P}, (\mathbf{a}, j))$ is a *good pair*, otherwise it is a *bad pair*. Let $\mathscr{A}(\mathbf{a}, t)$ be the set of all LPBPs of the form $(\mathcal{P}, (\mathbf{a}, j))$, where $\mathcal{P}$ terminates at the point t. Let $\mathscr{B}(\mathbf{a}, t)$ and $\mathscr{G}(\mathbf{a}, t)$ be the subsets of $\mathscr{A}(\mathbf{a}, t)$ consisting of bad and good pairs, respectively. Clearly, $\mathscr{A}(\mathbf{a}, t) = \mathscr{B}(\mathbf{a}, t) \cup \mathscr{G}(\mathbf{a}, t)$, with the union disjoint.

The following theorem is our main result. After its discovery, we found an essentially equivalent conjecture in earlier work of Tamm [9]. Though Tamm's paper concerns paths under periodic boundaries (see Section 5 for further details), the conjecture itself is coarsely formulated in the language of *two-dimensional arrays*, with a proof only in the case $m = 2$.

Theorem 1. *Let* $\mathbf{a}$ *be a weak* m*-part composition of* n *and let* $t = (k, l)$*, with* $0 \le k \le n$*,* $0 \le l \le m$*. If the point* $(k+1, l)$ *lies weakly to the right of* $\partial\mathbf{a}^{\langle j\rangle}$ *for all* j*, then*

$$|\mathscr{A}(\mathbf{a}, t)| = m\binom{k+l}{l}, \tag{3}$$

$$|\mathscr{B}(\mathbf{a}, t)| = n\binom{k+l}{l-1}, \tag{4}$$

and

$$|\mathscr{G}(\mathbf{a}, t)| = |\mathscr{A}(\mathbf{a}, t)| - |\mathscr{B}(\mathbf{a}, t)| = \frac{m(k+1) - nl}{k+1}\binom{k+l}{l}. \tag{5}$$

□

That is, we have the surprising fact that the total number of paths dominated by all cyclic shifts of a piecewise linear boundary does not depend on the specific parts of its defining composition $\mathbf{a}$. Instead, allowing all shifts of the boundary acts as an averaging process with a very pleasant enumerative outcome.

Clearly the hypotheses of Theorem 1 are satisfied by any terminus (k,l) that is dominated by all cyclic shifts of $\mathbf{a}$. In particular, setting $(k,l)=(n,m)$ in the theorem explains our previous observation that there are $36=\binom{6+3}{3-1}$ dominated paths for each row of Figure 2.

Corollary 2. *For any weak m-part composition $\mathbf{a}$ of n, we have*

$$D(\mathbf{a})+D(\mathbf{a}^{\langle 1\rangle})+\cdots+D(\mathbf{a}^{\langle m-1\rangle})=\binom{n+m}{m-1}.$$

□

Now consider the composition $\mathbf{a}=(a,a,\ldots,a)$ of $n=ma$. Observe that $\mathbf{a}^{\langle i\rangle}=\mathbf{a}$ for all i, while $\partial\mathbf{a}$ is simply the line $x=ay$. Applying Theorem 1 and dividing by m to remove the effect of boundary rotation yields the following well-known result, often referred to as the *generalized ballot theorem*. (See the survey article [7] for more information.)

Corollary 3. *If $k\geq al$, then there are*

$$\frac{k-al+1}{k+1}\binom{k+l}{l}$$

lattice paths from $(0,0)$ to (k,l) that lie weakly below the line $x=ay$. □

In the next section we give a bijective proof of Theorem 1. Of course, since (3) is trivial, this amounts to proving (4). We do so by showing that bad paths are in bijection with a less restrictive set of paths, in the spirit of André's [1] reflection principle. In fact, our proof is a generalization of the bijection used in [6] to prove Corollary 3. (Our bijection reduces to that of [6] in the case when all parts of $\mathbf{a}$ are the same, though an allowance must be made for the cyclically shifting boundary.)

Section 3 contains a brief account of an alternative derivation of Theorem 1 using the Cycle Lemma. In Section 4 we present a refinement of the theorem that counts paths with a specified number of corners. Finally, Section 5 illustrates a handful of applications to the enumeration of lattice paths lying under periodic boundaries. Interestingly, each of these applications pivots on the fact that the hypotheses of Theorem 1 require only $(k+1,l)$, and not the terminus (k,l), to be weakly right of $\mathbf{a}$.

2. A PROOF OF THEOREM 1

Throughout this section we have in mind a fixed weak composition $\mathbf{a}=(a_0,\ldots,a_{m-1})$ of n and its corresponding boundary $\partial\mathbf{a}$. For arbitrary $j\in\mathbb{Z}$ we interpret the symbol a_j to mean $a_{j \bmod m}$.

For any lattice point $p=(x,y)$ with $0\leq x<n$ and $1\leq y\leq m$, and for any integer j, define the *j-th shift of p (relative to $\mathbf{a}$)* to be the point

$$p^{\langle j\rangle}=(x+a_{-1}+a_{-2}+\cdots+a_{-j} \mod n,\ y+j \mod m), \tag{6}$$

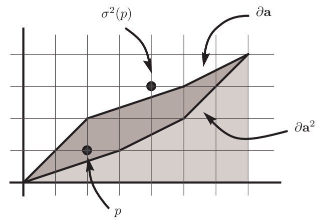

FIGURE 3. A point p and its shift $p^{\langle 2\rangle}$.

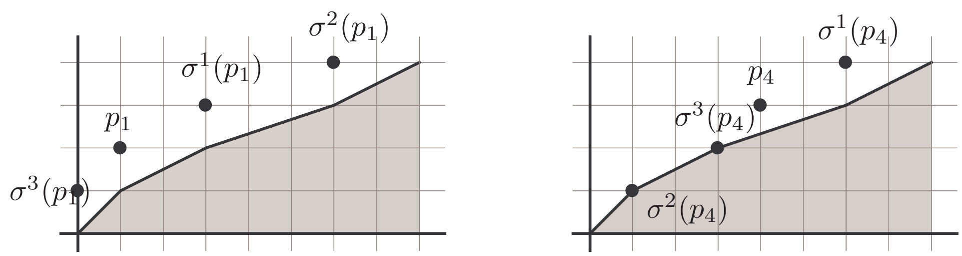

FIGURE 4. The sets B_1 and B_4 relative to the composition $\mathbf{a} = (1, 2, 3, 2)$.

where the modular reductions in the first and second coordinate are understood to yield representatives in $\{0, 1, \ldots, n-1\}$ and $\{1, 2, \ldots, m\}$, respectively. Informally, $p^{\langle j\rangle}$ is in the same position relative to $\partial\mathbf{a}^{\langle j\rangle}$ as p is to $\partial\mathbf{a}$. (See Figure 3.)

Finally, we define the relations $\leq$, $\lesssim$, and $<$ on lattice points as follows:

- $(x_1, y_1) \leq (x_2, y_2) \Leftrightarrow x_1 \leq x_2$ and $y_1 \leq y_2$
- $(x_1, y_1) \lesssim (x_2, y_2) \Leftrightarrow x_1 \leq x_2$ and $y_1 < y_2$
- $(x_1, y_1) < (x_2, y_2) \Leftrightarrow x_1 < x_2$ and $y_1 < y_2$

For $0 \leq i < n$, let $p_i = (i, y_i)$, where y_i is the least integer such that the point (i, y_i) lies strictly above $\partial\mathbf{a}$. Define

$$B_i := \{p_i, p_i^{\langle 1\rangle}, p_i^{\langle 2\rangle}, \ldots, p_i^{\langle m-1\rangle}\}. \tag{7}$$

to be the set of lattice points having the same relative positions to the boundary curves $\partial\mathbf{a}, \partial\mathbf{a}^{\langle 1\rangle}, \ldots, \partial\mathbf{a}^{\langle m-1\rangle}$ as the point p_i has to $\mathbf{a}$. (See Figure 4.) Note that the sets $B_0, \ldots, B_{m-1}$ are not disjoint.

Let $\mathscr{B}_i(\mathbf{a}, t)$ be the set of all bad LPBPs of the form $(\mathcal{P}, (\mathbf{a}, j))$, where the path $\mathcal{P}$ terminates at t and its first *bad* step (*i.e* the first step crossing $\partial\mathbf{a}^j$) lands at the point $p_i^{\langle j\rangle} \in B_i$. Then clearly $\mathscr{B}(\mathbf{a}, t) = \bigcup_{i=0}^{n-1} \mathscr{B}_i(\mathbf{a}, t)$, with the union being disjoint. We shall prove Theorem 1 by showing that $|\mathscr{B}_i(\mathbf{a}, t)|$ is independent of i.

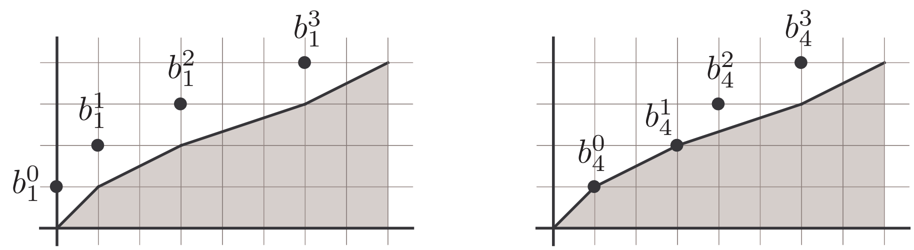

FIGURE 5. Construction of $B_1 = \{b_1^j\}$ and $B_4 = \{b_4^j\}$ relative to $\mathbf{a} = (1,2,3,2)$.

Observe that no two points in any given set B_i can have the same y coordinates. In fact, let $s_i = m + 1 - y_i$, so that $p_i^{\langle s_i \rangle}$ has y-coordinate 1, and define

$$b_i^j = p_i^{\langle s_i + j \rangle}, \qquad \text{for } 0 \le j < m. \tag{8}$$

Then

$$B_i = \{b_i^0, b_i^1, \ldots, b_i^{m-1}\}, \tag{9}$$

where the y-coordinate of b_i^j is $j+1$ and $b_i^0 \lesssim b_i^1 \lesssim \cdots \lesssim b_i^{m-1}$. For example, for the sets B_1 and B_4 of Figure 4 we have $s_1 = 3$ and $s_4 = 2$, respectively, and the appropriate relabellings are shown in Figure 5.

Let $0 \le k \le n$ and $0 \le l \le m$. We say B_i is *complete with respect to the point* $t = (k,l)$ if b_i^{l-1} is weakly to the left of t. For instance, the set B_1 in Figure 5 is complete with respect to $t = (3,3)$, while B_4 is not. The motivation behind this definition will be made clear in the proof of the following lemma.

Lemma 4. *If B_i is complete with respect to $t = (k,l)$ then*

$$|\mathscr{B}_i(\mathbf{a}, t)| = \binom{k+l}{l-1}.$$

Proof. We shall give a reflection-type correspondence between $\mathscr{B}_i(\mathbf{a}, t)$ and the set $\mathscr{U}$ of all lattice paths from $(-1,1)$ to t.

Let $\mathscr{B}_i^j \subseteq \mathscr{B}_i(\mathbf{a}, t)$ consist of those LPBPs in which the first bad step lands at the point b_i^j. Since the y-coordinate of b_i^j is $j+1$, clearly $\mathscr{B}_i^j = \emptyset$ for $j \ge l$. Thus $\mathscr{B}_i(\mathbf{a}, t) = \bigcup_{j=0}^{l-1} \mathscr{B}_i^j$, with the union disjoint.

Since B_i is complete with respect to t, we have $b_i^0 \lesssim \cdots \lesssim b_i^{l-1} \le t$. That is, on each of the lines $y = 1, \ldots, y = l$ there is a point in B_i that is weakly to the left of t. It follows that any path from $(-1,1)$ to t must intersect one of these points. For $0 \le j < l$, let $\mathscr{U}^j$ be the set of paths from $(-1,1)$ to t that avoid the points $b_i^0, b_i^1, \ldots, b_i^{j-1}$ but meet b_i^j. Then, by our previous comment, $\mathscr{U} = \bigcup_{j=0}^{l-1} \mathscr{U}^j$, with the union disjoint.

We now define a mapping $\psi_j : \mathscr{B}_i^j \longrightarrow \mathscr{U}^j$ for each $j = 0, \ldots, l-1$. Given $L \in \mathscr{B}_i^j$, construct $\psi_j(L)$ as follows. (See Figure 6 for an illustration of the construction.)

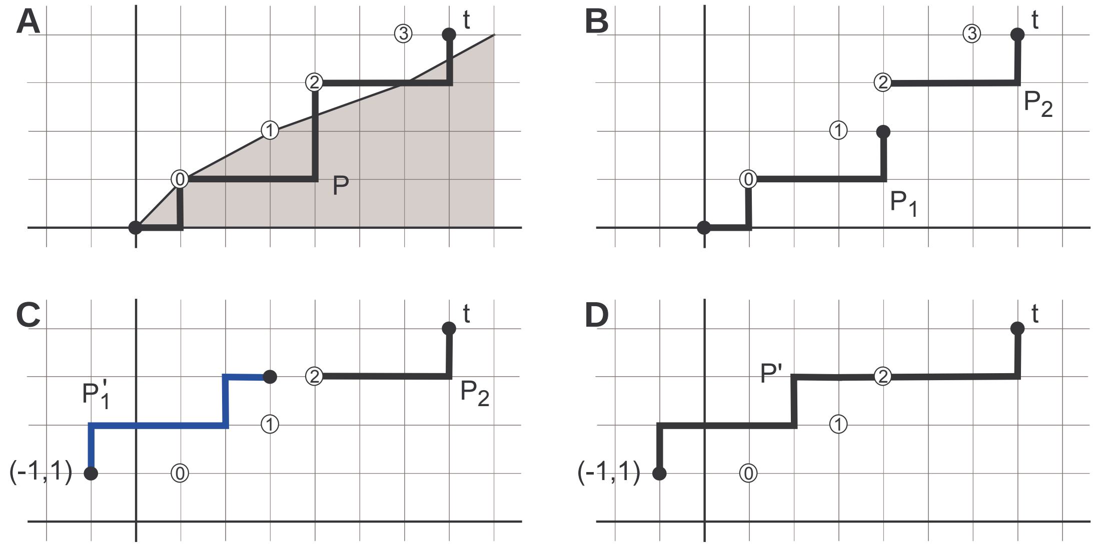

FIGURE 6. An illustration of the map $\psi_j : \mathscr{B}_i^j \longrightarrow \mathscr{U}^j$. Here $\mathbf{a} = (1,2,3,2)$, $i = 4$, $j = 2$, and $s_i = 2$. The boundary $\partial\mathbf{a}^{\langle s_i+j\rangle} = \partial\mathbf{a}^{\langle 4\rangle} = \partial\mathbf{a}$ is shown in panel A. Throughout, the point b_i^j is indicated with a circled j.

A. We have $b_i^j = (x, j+1)$ for some x, and $L = (\mathcal{P}, (\mathbf{a}, s_i + j))$ for some path $\mathcal{P}$ whose first bad step is an up-step from (x, j) to b_i^j.
B. Remove this step to break $\mathcal{P}$ into two parts: the first, $\mathcal{P}_1$, is a path from $(0,0)$ to (x, j), and the second, $\mathcal{P}_2$, is a path from $(x, j+1)$ to (k, l).
C. Rotate $\mathcal{P}_1$ through 180° and translate to obtain a new path $\mathcal{P}_1'$ beginning at $(-1,1)$ and terminating at $(x-1, j+1)$.
D. Join $\mathcal{P}_1'$ and $\mathcal{P}_2$ by adding a right-step from $(x-1, j+1)$ to b_i^j, thus creating a path $\mathcal{P}'$ from $(-1,1)$ to (k,l). Finally, set $\psi_j(L) = \mathcal{P}'$.

To ensure ψ_j is well defined we must check that indeed $\mathcal{P}' \in \mathscr{U}^j$. The only contentious issue here is whether $\mathcal{P}'$ avoids the points $b_i^0, \ldots, b_i^{j-1}$. To see why this is the case, consider the piecewise linear curve $\mathcal{C}$ obtained by joining the points $b_i^{-1}, b_i^0, \ldots, b_i^j$, where

$$b_i^{-1} := (-a_{-s_i} + (i + a_{-1} + a_{-2} + \cdots + a_{-s_i} \mod n),\ 0). \tag{10}$$

See Figure 7A for an illustration.

Since (6) and (8) give

$$b_i^r = (i + a_{-1} + a_{-2} + \cdots + a_{-(s_i+r)} \mod n,\ r+1), \qquad \text{for } 0 \le r \le j,$$

the slope of the line segment from b_i^r to b_i^{r+1} is $\frac{\Delta x}{\Delta y} = a_{-(s_i+r+1)}$ for $-1 \le r \le j$. That is, the $j+1$ segments of $\mathcal{C}$ have slopes $a_{-s_i}, a_{-(s_i+1)}, a_{-(s_i+2)}, \ldots, a_{-(s_i+j)}$, listed in order from left to right. Since $\mathbf{a}^{\langle s_i+j\rangle} = (a_{-(s_i+j)}, a_{-(s_i+j-1)}, \ldots, a_{-(s_i+j-m+1)})$, this identifies $\mathcal{C}$ as the first $j+1$ segments of the boundary curve $\partial\mathbf{a}^{\langle s_i+j\rangle}$ rotated 180° and translated.

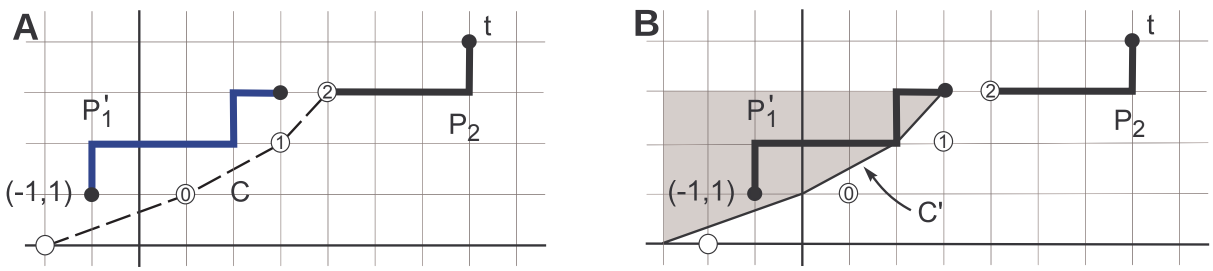

FIGURE 7. Proving that the map $\psi_2 : \mathscr{B}_4^2 \longrightarrow \mathscr{U}^2$ illustrated in Figure 6 is well defined. The point b_i^{-1} is indicated with an open circle. The shaded region in panel B fits perfectly into that of Figure 6A after a 180° rotation.

Shift $\mathcal{C}$ to the left one unit to obtain a new curve $\mathcal{C}'$ that terminates at $(x-1, j+1)$. (See Figure 7B.) Since, by definition, $\mathcal{P}$ remains weakly below $\partial\mathbf{a}^{\langle s_i+j\rangle}$, so too does the subpath $\mathcal{P}_1$. Since $\mathcal{P}_1'$ and $\mathcal{C}'$ are obtained by rotating $\mathcal{P}_1$ and $\partial\mathbf{a}^{\langle s_i+j\rangle}$, respectively, it follows that $\mathcal{P}_1'$ must remain weakly above $\mathcal{C}'$. But $b_i^0, \ldots, b_i^{j-1}$ lie on $\mathcal{C}$, so they lie strictly below $\mathcal{C}'$, and therefore $\mathcal{P}_1'$ avoids these points. The same is clearly true of $\mathcal{P}'$, and this establishes that ψ_j is well defined.

We claim $\psi_j : \mathscr{B}_i^j \longrightarrow \mathscr{U}^j$ is a bijection. Observe that this establishes Lemma 4, since the sets $\mathscr{B}_i(\mathbf{a}, t) = \bigcup_{j=0}^{l-1} \mathscr{B}_i^j$ and $\mathscr{U} = \bigcup_{i=0}^{l-1} \mathscr{U}^j$ are then equinumerous and the cardinality of $\mathscr{U}$ is clearly $\binom{k+l}{l-1}$.

To prove ψ_j is bijective, we construct the inverse map $\phi_j : \mathscr{U}^j \longrightarrow \mathscr{B}_i^j$. (See Figure 8 for an illustration.) Suppose $\mathcal{P}' \in \mathscr{U}^j$. The first point $\mathcal{P}'$ intersects amongst $b_i^0, \ldots, b_i^j$ is b_i^j and it is clear that the step landing at b_i^j is horizontal. Remove this step to split $\mathcal{P}'$ into two paths: Call the left part $\mathcal{P}_1'$ and the right part $\mathcal{P}_2$. Let $\mathcal{C}'$ be the piecewise linear curve obtained by joining the points $b_i^{-1}, b_i^0, \ldots, b_i^j$ (where b_i^{-1} is given by (10)) and shifting the result one unit to the left. Then, as above, the segments of $\mathcal{C}'$ have slopes $a_{-(s_i+j)}, a_{-(s_i+j-1)}, \ldots, a_{-(s_i+1)}, a_{-s_i}$, so that $\mathcal{C}'$ is simply the first $j+1$ segments of $\partial\mathbf{a}^{\langle s_i+j\rangle}$ rotated 180° and translated. Since $\mathcal{P}_1'$ lies weakly above $\mathcal{C}'$, the curve $\mathcal{P}_1$ obtained by rotating $\mathcal{P}_1'$ by 180° and translating its origin to $(0,0)$ must lie weakly under $\partial\mathbf{a}^{\langle s_i+j\rangle}$. Attach $\mathcal{P}_1$ to $\mathcal{P}_2'$ by a vertical step to form a new path $\mathcal{P}$. Then $L = (\mathcal{P}, (\mathbf{a}, s_i + j))$ is a bad LPBP in which $\mathcal{P}$ terminates at t and has its first bad step landing at b_i^j. Set $\phi_j(\mathcal{P}') = L$, so that clearly $\phi_j \circ \psi_j(L) = L$ and $\psi_j \circ \phi_j(\mathcal{P}') = \mathcal{P}'$, as required. $\square$

Theorem 1 now follows immediately from Lemma 4 and the following result:

Lemma 5. *Suppose $0 \le k \le n$, $0 \le l \le m$, and the point $(k+1, l)$ is weakly right of $\partial\mathbf{a}^{\langle j\rangle}$ for all j. Then each of the sets $B_0, \ldots, B_{n-1}$ is complete with respect to $t = (k, l)$.*

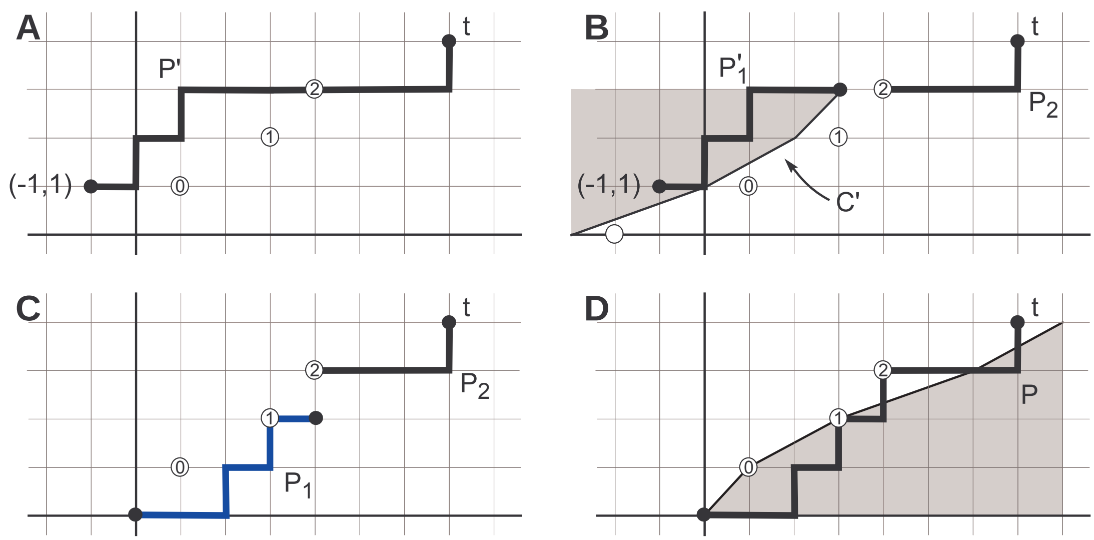

FIGURE 8. Construction of $\phi_j : \mathscr{U}^j \longrightarrow \mathscr{B}_i^j$, with $\mathbf{a} = (1,2,3,2)$, $i = 4$, $j = 2$, and $s_i = 2$. The point b_i^j is indicated with a circled j, and b_i^{-1} with an open circle. The boundary curve in panel D is $\partial\mathbf{a}^{\langle s_i+j\rangle} = \partial\mathbf{a}^{\langle 4\rangle}$.

Proof. The set B_i is not complete with respect to t if and only if it contains some point of the form $(k+\delta, l)$ with $\delta \geq 1$. But B_i consists of those points that lie immediately above $\partial\mathbf{a}^{\langle j\rangle}$ for some j. Thus $(k+\delta, l) \in B_i$ for some $\delta \geq 1$ precisely when $(k+1, l)$ is strictly left of some boundary $\partial\mathbf{a}^{\langle j\rangle}$. The result follows. □

3. A Cycle Lemma Proof of Theorem 1

We now sketch an alternative proof of Theorem 1 using the cycle lemma [4]. The formulation most applicable here is the following:

Cycle Lemma. *Let $\mathbf{i} = (i_0, \ldots, i_m)$ be a sequence with integral entries $i_j \leq 1$ having positive sum $k = i_0 + \cdots + i_m$. Then there are exactly k cyclic shifts of $\mathbf{i}$ with all partial sums positive.* □

The following result is the key to our alternative proof of Theorem 1.

Lemma 6. *Let $\mathbf{a}$ be a weak m-part composition of n, and let $t = (n, l)$, where $0 \leq l < m$. Let $\mathscr{G}^*(\mathbf{a}, t)$ be the set of good LPBPs of the form $(\mathcal{P}, (\mathbf{a}, j))$ where $\mathcal{P}$ is a path from $(0,0)$ to t that terminates with a right step. Then*

$$|\mathscr{G}^*(\mathbf{a}, t)| = \binom{n+l-1}{l}(m-l).$$

Proof: Let $\mathscr{W}$ be the set of words of length $n+l$ on the alphabet $\{\mathsf{R}, \mathsf{U}\}$ that contain l U's and n R's and end with an R. We give a bijection $\Omega : \mathscr{W} \times [m-l] \longrightarrow \mathscr{G}^*(\mathbf{a}, t)$, where $[m-l] = \{1, \ldots, m-l\}$. The construction is illustrated in Example 7, below.

Let $(w,k) \in \mathscr{W} \times [m-l]$. Factor w into m blocks $w = w_0 \cdots w_{m-1}$ as follows: Suppose $\mathbf{a} = (a_0, \ldots, a_{m-1})$, and parse w from left to right letting $w_0, \ldots, w_{m-1}$ in turn be maximal contiguous substrings such that

- w_i is empty if $a_i = 0$,
- w_i contains a_i R's and ends with an R if $a_i > 0$.

Observe that this decomposition of w is unique.

Consider the integer sequence $\mathbf{u} = (1, -u_0, 1, -u_1, \ldots, 1, -u_{m-1})$, where u_i is the number of U's in w_i. The entries of $\mathbf{u}$ sum to $m - (u_0 + \ldots + u_{m-1}) = m - l > 0$, so the cycle lemma implies there are exactly $m-l$ cyclic shifts of $\mathbf{u}$ whose partial sums are all positive. Clearly such shifts must be of the form $\mathbf{u}^{\langle -2s \rangle}$, where $0 \le s < m$. (See (2) for the definition of $\mathbf{u}^{\langle -2s \rangle}$.) Suppose the good shifts are $\mathbf{u}^{\langle -2s_1 \rangle}, \ldots, \mathbf{u}^{\langle -2s_{m-l} \rangle}$, where $s_1 < \cdots < s_{m-l}$. Set $j = s_k$ and form the word $w' = w_j w_{j+1} \cdots w_{j+m-1}$, where the indices are to be interpreted modulo m. From w', construct a lattice path $\mathcal{P}$ originating at $(0,0)$ by treating R and U as right and up steps, respectively.

Set $\Omega(w,k) := (\mathcal{P}, (\mathbf{a}, -j))$. Observe that indeed $\Omega(w,k) \in \mathscr{G}^*(\mathbf{a}, t)$, since $\mathcal{P}$ clearly terminates at (n,l) with a right step and

$$
\begin{aligned}
& \mathbf{u}^{\langle -2j \rangle} \text{ has all partial sums positive} \\
\iff \quad & u_j + \cdots + u_{j+d} < d+1, \text{ for } 0 \le d < m \\
\iff \quad & \mathcal{P} \text{ has at least } a_j + \cdots + a_{j+d} \text{ right steps before its } (d+1)\text{-st up step, for } 0 \le d < m \\
\iff \quad & \mathcal{P} \text{ is dominated by } \mathbf{a}^{\langle -j \rangle} = (a_j, a_{j+1}, \ldots, a_{j+m-1}).
\end{aligned}
$$

Moreover, this construction of $(\mathcal{P}, (\mathbf{a}, -j))$ from (w,k) can be reversed, as follows: (1) recover w' from $\mathcal{P}$, (2) parse w' as above, but relative to the composition $\mathbf{a}^{\langle -j \rangle}$, to obtain $w_j, w_{j+1}, \ldots, w_{j+d}$ and hence w, (3) retrieve $\mathbf{u}$ from $w_0, \ldots, w_{m-1}$, and (4) deduce k by applying the cycle lemma to $\mathbf{u}$.

Thus $\Omega : \mathscr{W} \times [m-l] \longrightarrow \mathscr{G}^*(\mathbf{a}, t)$ is bijective, and since $|\mathscr{W}| = \binom{n+l-1}{l}$, the result follows. $\square$

Example 7. *Let $n = 12$, $m = 7$, $l = 4$, $\mathbf{a} = (1,3,0,2,4,0,2)$ and take*

$$(w,k) = (\mathsf{RRRURRRRRURRUURR}, 3).$$

Then we have

$$w_0 = \mathsf{R}, \quad w_1 = \mathsf{RRUR}, \quad w_2 = \epsilon, \quad w_3 = \mathsf{RR}, \quad w_4 = \mathsf{RRURR}, \quad w_5 = \epsilon, \quad w_6 = \mathsf{UURR},$$

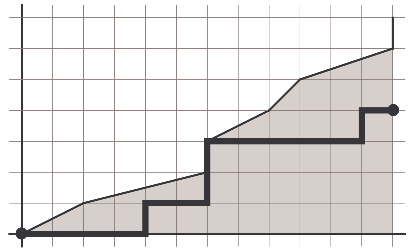

FIGURE 9. The path constructed in Example 7.

where ϵ denotes the empty string. This gives $\mathbf{u} = (1,0,1,-1,1,0,1,0,1,-1,1,0,1,-2)$*, and the* $m-l=3$ *cyclic shifts of* $\mathbf{u}$ *with all partial sums positive are seen to be*

$$\mathbf{u}^{\langle 0\rangle} = (1,0,1,-1,1,0,1,0,1,-1,1,0,1,-2)$$
$$\mathbf{u}^{\langle -4\rangle} = (1,0,1,0,1,-1,1,0,1,-2,1,0,1,-1)$$
$$\mathbf{u}^{\langle -6\rangle} = (1,0,1,-1,1,0,1,-2,1,0,1,-2,1,0).$$

Thus $s_1 = 0$, $s_2 = 2$, $s_3 = 3$, *so that* $j = s_3 = 3$ *and*

$$w' = w_3w_4w_5w_6w_0w_1w_2 = \mathsf{RRRRURRUURRRRRUR}.$$

Figure 9 shows the path $\mathcal{P}$ *corresponding to* w' *and the dominating boundary* $\partial\mathbf{a}^{\langle -j\rangle} = \partial\mathbf{a}^{\langle -3\rangle}$. □

It is now easy to establish Theorem 1 in the special case where the terminal point is $t = (n,l)$ for some $0 \le l \le m$. In particular, we clearly have $|\mathscr{G}(\mathbf{a},(n,l))| = \sum_{i=0}^{l} |\mathscr{G}^*(\mathbf{a},(n,i))|$, so Lemma 6 gives

$$|\mathscr{G}(\mathbf{a},(n,l))| = \sum_{i=0}^{l}(m-i)\binom{n+i-1}{i} = m\binom{n+l}{l} - n\binom{n+l}{l-1}, \tag{11}$$

in agreement with (5). Moreover, we have the usual lattice path recursion

$$|\mathscr{G}(\mathbf{a},(k+1,l))| = |\mathscr{G}(\mathbf{a},(k,l))| + |\mathscr{G}(\mathbf{a},(k+1,l-1))|$$

provided $(k+1,l)$ is weakly right of every shift of $\partial\mathbf{a}$. Using (11) as an initial condition and iterating the above recursion allows us to determine $|\mathscr{G}(\mathbf{a},(k,l))|$ for any terminal point (k,l) satisfying this same condition. Thus we have an inductive proof of (5). The details are more tedious than illuminating and are omitted here.

4. A REFINEMENT: COUNTING PATHS WITH A SPECIFIED NUMBER OF CORNERS

An *up-right corner* in a lattice path is a point at which an up step terminates and is immediately followed immediately by a right step. Observe that for any c points (X_j, Y_j) satisfying

$$(0,1) \le (X_1,Y_1) < (X_2,Y_2) < \cdots < (X_c,Y_c) \le (k-1,l),$$

there is a unique lattice path from $(0,0)$ to (k,l) having up-right corners at exactly these points. Since the the X_j's and Y_j's can be chosen in $\binom{k}{c}$ and $\binom{l}{c}$ ways, respectively, it follows that there are $\binom{k}{c}\binom{l}{c}$ lattice paths from $(0,0)$ to (k,l) with exactly c up-right corners.

Let $\mathscr{C}^c$ be the set of all LPBPs whose paths have exactly c up-right corners. The following theorem is a generalization of [7, Theorem 3.4.2], and our proof is inspired by that of [6, Theorem 5].

Theorem 8. *Let* $\mathbf{a}$ *be any weak* m*-part composition of* n *and let* $t = (k,l)$ *be a point dominated by all cyclic shifts of* $\mathbf{a}$*. Then*

$$|\mathscr{G}(\mathbf{a},t) \cap \mathscr{C}^c| = m\binom{k}{c}\binom{l}{c} - n\binom{k-1}{c-1}\binom{l+1}{c+1}.$$

□

Clearly this is a refinement of Theorem 1, and unsurprisingly our proof relies on a corresponding refinement of Lemma 4. Note that the hypothesis regarding the terminal point $t = (k,l)$ is slightly stronger than that of Theorem 1. That is, we require (k,l), rather than $(k+1,l)$, to be dominated by all $\mathbf{a}^{\langle j \rangle}$.

Indeed, our refinement of Lemma 4 requires a slightly stronger notion than completeness. With the sets B_i defined as in (7), we say B_i is *strongly complete* with respect to $t = (k,l)$ if the point b_i^{l-1} is *strictly* to the left of t.

Theorem 8 follows immediately from the following two results. We assume the notation of Section 2 throughout.

Lemma 9. *Suppose* $0 \le k \le n$*,* $0 \le l \le m$*, and the point* (k,l) *is dominated by* $\mathbf{a}^{\langle j \rangle}$ *for all* j*. Then each of the sets* $B_0, \dots, B_{n-1}$ *is strongly complete with respect to* $t = (k,l)$*.*

Proof. This is an obvious modification of Lemma 5. □

Lemma 10. *If* B_i *strongly complete with respect to* $t = (k,l)$*, then*

$$|\mathscr{B}_i(\mathbf{a},t) \cap \mathscr{C}^c| = \binom{k-1}{c-1}\binom{l+1}{c+1}.$$

Proof. We prove the lemma by giving a bijection between $\mathscr{B}_i(\mathbf{a},t) \cap \mathscr{C}^c$ and pairs of sequences $(\mathbf{X},\mathbf{Y}) \in \mathbb{Z}^c \times \mathbb{Z}^{c+1}$ satisfying

$$0 \le X_1 < \cdots < X_c = k-1 \qquad \text{and} \qquad 1 \le Y_1 < \ldots < Y_{c+1} \le l+1.$$

Fix such a pair $(\mathbf{X},\mathbf{Y})$. Since B_i is strongly complete with respect to (k,l), we have

$$x(b_i^{c-1}) \le k-1 = X_c,$$

where $x(p)$ denotes the x-coordinate of the point p. Let $r \le c$ be the smallest index for which $x(b_i^{r-1}) \le X_r$, and set $j = Y_r - 1$ so that $b_i^j = (x(b_i^j), Y_r)$. Since $Y_r \ge r$, we have

$j \geq r-1$, so the minimality of r implies either $r = 1$ or $X_{r-1} < x(b_i^{r-2}) \leq x(b_i^j)$. Thus we have a chain of points

$$(X_1, Y_1) < \cdots < (X_{r-1}, Y_{r-1}) < b_i^j \leq (x(b_i^j), Y_{r+1} - 1) < \\ (X_r + 1, Y_{r+2} - 1) < \cdots < (X_{c-1} + 1, Y_{c+1} - 1).$$

It is easy to verify that there is a unique path $\mathcal{P}$ from $(-1, 1)$ to (k, l) passing through all these points such that:

- $\mathcal{P}$ has $r-1$ right-up corners at $(X_1, Y_1) \ldots (X_{r-1}, Y_{r-1})$, and no further right-up corners strictly left of b_i^j,
- the steps of $\mathcal{P}$ terminating at b_i^j and originating at $(x(b_i^j), Y_{r+1} - 1)$ are horizontal,
- $\mathcal{P}$ has $c-r$ up-right corners at $(X_r + 1, Y_{r+2} - 1), \ldots, (X_{c-1} + 1, Y_{c+1} - 1)$, and no further up-right corners strictly right of b_i^j.

By construction, $\mathcal{P}$ avoids $b_i^0, \ldots, b_i^{j-1}$ but meets b_i^j. So we can apply the bijection ϕ_j (see the proof of Lemma 4) to get an LPBP $\phi_j(\mathcal{P}) = (\mathcal{P}', (\mathbf{a}, s_i + j)) \in \mathscr{B}_i(\mathbf{a}, t)$. Observe that the $r-1$ right-up corners of $\mathcal{P}$ to the left of b_i^j become up-right corners of $\mathcal{P}'$ through rotation, while the $c-r$ up-right corners of $\mathcal{P}$ to the right of b_i^j are preserved in $\mathcal{P}'$.

We now check for corners at b_i^j and $(x(b_i^j), Y_{r+1} - 1)$. There are two cases to consider. If $Y_{r+1} - 1 > Y_r$, then $\mathcal{P}'$ does not have an up-right corner at b_i^j but does at $(x(b_i^j), Y_{r+1} - 1)$. Otherwise $Y_{r+1} - 1 = Y_r$, in which case $b_i^j = (x(b_i^j), Y_{r+1} - 1)$ and $\mathcal{P}'$ has an up-right corner at this point.

In either case, $\mathcal{P}'$ has exactly $(r-1) + (c-r) + 1 = c$ up-right corners in total. That is, $\phi_j(\mathcal{P}) \in \mathscr{B}_i(\mathbf{a}, t) \cap \mathscr{C}^c$. Since ϕ_j is bijective, so too is the correspondence $(\mathbf{X}, \mathbf{Y}) \mapsto \phi_j(\mathcal{P})$ described here. This completes the proof. □

In analogy with up-right corners, we say a *right-up corner* is formed when a right step is followed immediately by an up step. It is convenient to treat an initial up step as a *virtual* right-up corner. Then, letting $\mathscr{C}_c$ be the set of all LPBPs whose paths have exactly c right-up corners (real or virtual), we have:

Theorem 11. *Let* $\mathbf{a}$ *be any weak m-part composition of n and let $t = (k, l)$ be a point dominated by all cyclic shifts of* $\mathbf{a}$. *Then*

$$|\mathscr{G}(\mathbf{a}, t) \cap \mathscr{C}_c| = m\binom{k+1}{c}\binom{l-1}{c-1} - n\binom{k}{c-1}\binom{l}{c}.$$

□

Consider the case $(k, l) = (n, m)$ in Theorems 8 and 11. Notice that the first and last corners of any good path are right-up corners. Since right-up corners and up-right corners must alternate, the number of good paths with c right-up corners is equal to the number

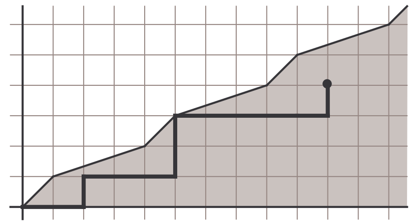

FIGURE 10. The boundary $\partial_{1,3}$ and a path it dominates.

of good paths with $c-1$ up-right corners. Indeed, Theorems 8 and 11 show this common number to be $\binom{n}{c-1}\binom{m}{c}$.

5. COUNTING PATHS DOMINATED BY PERIODIC BOUNDARIES

Let $\mathbf{a} = (a_0, \ldots, a_{m-1})$ be a weak composition of n, and let d be the least positive integer such that $\mathbf{a}^{\langle d \rangle} = \mathbf{a}$. Clearly d divides m. In the case that $d < m$ we say $\mathbf{a}$ is *periodic* with *period* d. For example $(3,1,2,3,1,2)$ has period 3.

If $\mathbf{a}$ has period d then $\mathbf{a}^{\langle i \rangle} = \mathbf{a}^{\langle i \bmod d \rangle}$. Thus

$$|\mathscr{G}(\mathbf{a},t)| = \frac{m}{d}\big(D(\mathbf{a},t) + D(\mathbf{a}^{\langle 1 \rangle},t) + \cdots + D(\mathbf{a}^{\langle d-1 \rangle},t)\big), \tag{12}$$

where $D(\mathbf{a},t)$ denotes the number of paths from $(0,0)$ to point t dominated by $\mathbf{a}$. The left-hand side of this equality can generally be evaluated by Theorem 1, and in certain special cases this allows us to deduce $D(\mathbf{a},t)$.

The case $d = 1$ is particularly straightforward. Here we have $\mathbf{a} = (a)^m = (a, a, \ldots, a)$, and (12) gives $D(\mathbf{a},t) = \frac{1}{m}|\mathscr{G}(\mathbf{a},t)|$. This is precisely our earlier proof of Corollary 3.

The case $d = 2$ involves compositions of the form $\mathbf{a} = (a,b,a,b,\ldots,a,b) = (a,b)^m$, where a, b are distinct nonnegative integers. In the following discussion it will be convenient to write $\partial_{a,b}$ for the "infinite" boundary curve $\partial(a,b,a,b,\ldots)$. See Figure 10 for an illustration of $\partial_{1,3}$ and one path that it dominates.

Theorem 12, below, gives explicit formulae for the number of paths under $\partial_{a,b}$ or $\partial_{b,a}$ to certain special endpoints. This result is also implicit in Tamm [9, Propositions 2,3], where it appears in generating series form. The proof given there follows a probabilistic argument (originally due to Gessel) reliant on Lagrange inversion, whereas our derivation is purely combinatorial.

Theorem 12. *Fix integers a, b with $0 \le a < b$ and set $c = a + b$. For $n \ge 0$ let*

$$p_n = (cn + b - a - 1, 2n) \qquad \textit{and} \qquad q_n = (cn + b - 1, 2n + 1).$$

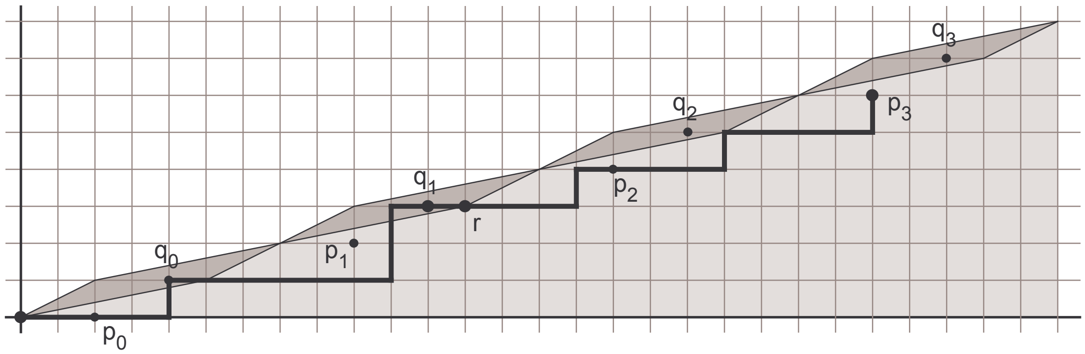

FIGURE 11. The points p_i, q_i in the case $a = 2, b = 5$ and a path $\mathcal{P} \in \mathscr{P}_3^{2,5}$.

Let $\mathscr{P}_n^{a,b}$ and $\mathscr{Q}_n^{a,b}$, respectively, be the sets of lattice paths from the origin to p_n and q_n that lie weakly under $\partial_{a,b}$. Define sets $\mathscr{P}_n^{b,a}$ and $\mathscr{Q}_n^{b,a}$ similarly, but for paths weakly under $\partial_{b,a}$. Then

$$|\mathscr{Q}_n^{a,b}| = M_n, \qquad |\mathscr{Q}_n^{b,a}| = 0, \tag{13}$$

$$|\mathscr{P}_n^{a,b}| = N_n + \frac{1}{2}\sum_{i=0}^{n-1} M_i M_{n-1-i}, \qquad |\mathscr{P}_n^{b,a}| = N_n - \frac{1}{2}\sum_{i=0}^{n-1} M_i M_{n-1-i}. \tag{14}$$

where

$$M_n = \frac{b-a}{cn+b}\binom{(c+2)n+b}{2n+1} \quad \text{and} \quad N_n = \frac{b-a}{cn+b-a}\binom{(c+2)n+b-a-1}{2n}.$$

Proof. A glance at Figure 11 will make the proof more clear. It illustrates several points p_i, q_i in the case $a = 2, b = 5$, along with the boundaries $\partial_{2,5}$, $\partial_{5,2}$ and a path $\mathcal{P} \in \mathscr{P}_3^{2,5}$.

Let $\mathbf{a} = (a,b)^{n+1}$ and $\mathbf{b} = (b,a)^{n+1}$, so a path lies under $\partial_{a,b}$ (respectively, $\partial_{b,a}$) if and only if it is dominated by $\mathbf{a}$ (respectively, $\mathbf{b}$).

Clearly $|\mathscr{Q}_n^{b,a}| = 0$ since q_n is not dominated by $\partial_{b,a}$. Moreover, since $\mathbf{a}^{\langle 2j\rangle} = \mathbf{a}$ and $\mathbf{a}^{\langle 2j+1\rangle} = \mathbf{b}$ for all j, we have

$$|\mathscr{G}(\mathbf{a}, q_n)| = (n+1)\big(|\mathscr{Q}_n^{a,b}| + |\mathscr{Q}_n^{b,a}|\big) = (n+1)|\mathscr{Q}_n^{a,b}|.$$

The point $r := (cn+b, 2n+1)$ one unit right of q_n is dominated by both $\mathbf{a}$ and $\mathbf{b}$. Theorem 1 may therefore be applied, and it gives $|\mathscr{G}(\mathbf{a}, q_n)| = (n+1)M_n$. This establishes (13). A similar analysis yields

$$|\mathscr{P}_n^{a,b}| + |\mathscr{P}_n^{b,a}| = \frac{1}{n+1}|\mathscr{G}(\mathbf{a}, p_n)| = 2N_n. \tag{15}$$

Observe that $\mathscr{P}_n^{b,a} \subset \mathscr{P}_n^{a,b}$. In fact, a path is dominated by $\mathbf{b}$ if and only if it is dominated by $\mathbf{a}$ and misses each of the points $q_0, \ldots, q_n$. Now consider a path $\mathcal{P} \in \mathscr{P}_n^{a,b} \setminus \mathscr{P}_n^{b,a}$, and let i be the largest index so that $\mathcal{P}$ meets q_i. Then $\mathcal{P}$ exits q_i with a right-step to r, and removal of this step splits $\mathcal{P}$ into two paths, $\mathcal{P}'$ and $\mathcal{P}''$, with $\mathcal{P}' \in \mathscr{Q}_i^{a,b}$ and $\mathcal{P}''$ dominated

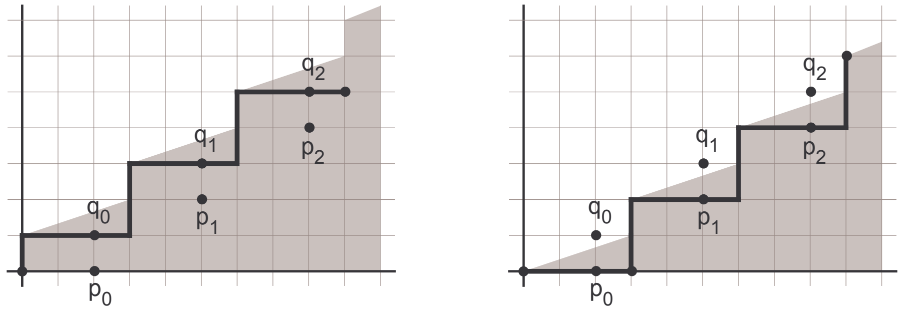

FIGURE 12. The boundaries $\partial_{0,3}$ and $\partial_{3,0}$ and corresponding staircases $\mathsf{U}(\mathsf{R}^3\mathsf{U}^2)^2\mathsf{R}^3$ and $(\mathsf{R}^3\mathsf{U}^2)^3$.

by $\partial_{a,b}$. (See Figure 11.) But $p_n - r = q_{n-i-1}$, so we effectively have $\mathcal{P}'' \in \mathscr{Q}^{a,b}_{n-1-i}$. Hence

$$|\mathscr{P}^{a,b}_n| = |\mathscr{P}^{b,a}_n| + |\mathscr{P}^{a,b}_n \setminus \mathscr{P}^{b,a}_n| = |\mathscr{P}^{b,a}_n| + \sum_{i=0}^{n-1} |\mathscr{Q}^{a,b}_i||\mathscr{Q}^{a,b}_{n-i-1}|.$$

Formulae (14) now follow from (13) and (15). $\square$

When $a = \frac{c-1}{2}$, $b = \frac{c+1}{2}$ for an odd positive integer c, observe that a path is dominated by $\partial_{b,a}$ if and only if it lies weakly under the line $cx = 2y$. So Theorem 12 can be applied to give the following enumeration of paths under a line of half-integer slope. An equivalent result also appears as [9, Theorem 1].

Corollary 13. *Let c be an odd positive integer. The number of lattice paths from $(0,0)$ to $(cn, 2n)$ that lie weakly below the line $cx = 2y$ is given by*

$$\frac{1}{cn+1}\binom{(c+2)n}{2n} - \frac{1}{2}\sum_{i=0}^{n-1} M_i M_{n-1-i}, \qquad \textit{where} \quad M_i = \frac{1}{2i+1}\binom{(c+2)i + \frac{c+1}{2}}{2i}.$$

$\square$

Another special case of Theorem 12 worth mentioning is that when $a = 0$, where we count paths from $(0,0)$ to $p_n = (b(n+1)-1, 2n)$ or $q_n = (b(n+1)-1, 2n+1)$ dominated by $\partial_{b,0}$ or $\partial_{0,b}$. Observe that a path to either point is bounded by $\partial_{0,b}$ if and only if it lies weakly under the "staircase" $\mathsf{U}(\mathsf{R}^b\mathsf{U}^2)^n\mathsf{R}^b$. (See Figure 12.) Similarly, the paths dominated by $\partial_{b,0}$ are precisely those that lie weakly under $(\mathsf{R}^b\mathsf{U}^2)^n$. For $n \geq 1$, such paths begin with b right steps, and removing these puts $\mathscr{P}^{b,0}_n$ in correspondence with the set of paths from $(0,0)$ to $(bn-1, 2n)$ that lie weakly beneath $(\mathsf{U}^2\mathsf{R}^b)^n$.

When $a = 0, b = 2$, the various quantities in Theorem 12 can be compactly expressed in terms of the Catalan numbers. In particular, we obtain simple formulae for the number of paths from $(0,0)$ to any point on the boundary $\mathsf{U}(\mathsf{R}^2\mathsf{U}^2)^n\mathsf{R}$. Note that the usual recursions for lattice paths then give similar expressions for paths to any point near the boundary.

Corollary 14. *Let $C_n = \frac{1}{n+1}\binom{2n}{n}$ be the n-th Catalan number. There are*

- $2C_{2n+1}$ *paths from* $(0,0)$ *to* $(2n+1, 2n+1)$*, and*
- $2^{2n+1}C_n - C_{2n+1}$ *paths from* $(0,0)$ *to* $(2n, 2n)$ *or* $(2n, 2n \pm 1)$

that lie weakly under $\mathsf{U}(\mathsf{R}^2\mathsf{U}^2)^n\mathsf{R}$*. Moreover, there are* $2^{2n+1}C_n - C_{2n+1}$ *paths from* $(0,0)$ *to* $(2n-1, 2n)$ *lying weakly under* $(\mathsf{U}^2\mathsf{R}^2)^n$*.*

Proof. The desired result rests upon the convolution identity

$$\sum_{i=0}^{n-1} C_{2i+1}C_{2n-(2i+1)} = C_{2n+1} - 2^{2n}C_n. \tag{16}$$

This is easily seen to be equivalent to the functional equation

$$D(x)^2 = \frac{1}{x}D(x) - C(4x^2), \tag{17}$$

where $C(x) = \sum_n C_n x^n$ is the Catalan generating series and $D(x) = \frac{1}{2}(C(x) - C(-x))$ is its odd part. To establish (17), expand $D(x)^2 = \frac{1}{4}(C(x) - C(-x))^2$ and substitute

$$\begin{aligned} C(x)^2 + C(-x)^2 &= \tfrac{2}{x}D(x) \\ C(x)C(-x) &= 2C(4x^2) - \tfrac{1}{x}D(x), \end{aligned}$$

which themselves are readily derived from the well-known identities $C(x) = 1 + xC(x)^2$ and $C(x) = \frac{1}{2x}(1 - \sqrt{1-4x})$, respectively.

Apply Theorem 12 with $a = 0, b = 2$, noting that $M_n = 2C_{2n+1}$, $N_n = C_{2n+1}$ and using (16) to simplify the results. This gives $|\mathscr{Q}_n^{0,b}| = 2C_{2n+1}$ paths to $(2n+1, 2n+1)$ and $|\mathscr{P}_n^{0,b}| = 3C_{2n+1} - 2^{2n+1}C_n$ paths to $(2n+1, 2n)$ under $\mathsf{U}(\mathsf{R}^2\mathsf{U}^2)^n\mathsf{R}$. Since paths to $(2n+1, 2n+1)$ pass through either $(2n+1, 2n)$ or $(2n, 2n)$, there are $|\mathscr{Q}_n^{0,b}| - |\mathscr{P}_n^{0,b}| = 2^{2n+1} - C_{2n+1}$ paths to $(2n, 2n)$ under $\mathsf{U}(\mathsf{R}^2\mathsf{U}^2)^n\mathsf{R}$. Clearly there are this same number of paths to $(2n, 2n \pm 1)$.

Finally, paths to $(2n-1, 2n)$ under $(\mathsf{U}^2\mathsf{R}^2)^n$ are in bijection with $\mathscr{P}_n^{b,0}$, and Theorem 12 yields $|\mathscr{P}_n^{b,0}| = 2^{2n+1}C_n - C_{2n+1}$. Alternatively, we could rotate and flip to view these as paths from $(0,0)$ to $(2n, 2n-1)$ dominated by $\mathsf{U}(\mathsf{R}^2\mathsf{U}^2)^n\mathsf{R}$. □

Let $\mathbf{a}$ be a composition of period d, and consider a terminus $t = (k, l)$ such that the point $(k+1, l)$ is dominated by all cyclic shifts of $\mathbf{a}$, but no shift except $\mathbf{a}$ itself dominates t. Then we clearly have $D(\mathbf{a}^{\langle i \rangle}, t) = 0$ for $i \geq 1$, so applying Theorem 1 in tandem with (12) gives a closed form expression for $D(\mathbf{a}, t)$. Indeed, the key to our proof of Theorem 12 was to determine $|\mathscr{Q}_n^{a,b}|$ in exactly this way.

As another interesting example we present the following result, also recently discovered independently by other authors [3]. (It appears there in a very slightly modified form. We shall make further comments below.)

Theorem 15. *Let s, t and n be positive integers. Then there are*

$$\frac{1}{n}\binom{(s+t)n-2}{tn-1}$$

lattice paths from $(0,0)$ to $(sn-1, tn-1)$ lying weakly beneath $\mathsf{U}^{t-1}(\mathsf{R}^s\mathsf{U}^t)^{n-1}\mathsf{R}^{s-1}$.

Proof. Let $\mathbf{a} = (0^{t-1}, s)^n$, so that $\mathbf{a}$ is a tn-part composition of sn with period t. Note that a path from $(0,0)$ to $(sn-1, tn-1)$ lies weakly beneath $\mathsf{U}^{t-1}(\mathsf{R}^s\mathsf{U}^t)^{n-1}\mathsf{R}^{s-1}$ precisely when it is dominated by $\mathbf{a}$. Furthermore, none of $\mathbf{a}^{\langle 1\rangle}, \ldots, \mathbf{a}^{\langle t-1\rangle}$ dominate $(sn-1, tn-1)$, whereas all of them dominate $(sn, tn-1)$. The result follows immediately from (12) after applying Theorem 1 with terminus $(k,l) = (sn-1, tn-1)$. $\square$

Setting $s = t = k$ in Theorem 15 yields the following elegant Catalan result, first appearing as [2, Theorem 8.3] with a proof based on the Cycle Lemma. Our need for the terminal point to be dominated by exactly one cyclic shift of the boundary sheds light on the observation of those authors that the ostensibly similar problem of counting paths to (nk, nk) dominated by $(\mathsf{U}^k\mathsf{R}^k)^n$ is in fact much more complicated.[1]

Corollary 16. *Let n and k be positive integers. Then there are kC_{nk-1} lattice paths from $(0,0)$ to $(nk-1, nk-1)$ lying weakly beneath $\mathsf{U}^{k-1}(\mathsf{R}^k\mathsf{U}^k)^{n-1}\mathsf{R}^{k-1}$.* $\square$

We conclude with some comments on recent work by Chapman *et al.* [3]. They consider lattice paths that remain strictly below the staircase boundary $\mathcal{S}_{s,t}$ beginning at $(0,t)$, moving to the right s steps, then up t steps, to the right s steps, *etc.* That is, $\mathcal{S}_{s,t}$ is described by $(\mathsf{R}^s\mathsf{U}^t)^n$, but is shifted t units upward to originate at $(0,t)$. Their main results concern the enumeration of two types of paths avoiding $\mathcal{S}_{s,t}$, namely those from $(0,0)$ to $(sn+1, tn)$, and those from $(1,0)$ to $(sn, tn-1)$. They employ a Cycle Lemma argument similar in structure to our proof of Lemma 6 to obtain compact expressions counting both types of paths, even allowing for the refined enumeration of paths with a specified number of corners. These same results can be obtained from our methods, as follows.

First observe that a path from $(1,0)$ to $(sn, tn-1)$ avoiding $\mathcal{S}_{s,t}$ can be shifted left one unit to give a path from $(0,0)$ to $(sn-1, tn-1)$ lying weakly below $\mathsf{U}^{t-1}(\mathsf{R}^s\mathsf{U}^t)^{n-1}\mathsf{R}^{s-1}$. Such paths are counted by Theorem 15, above, in agreement with [3, Corollary 4].

Now consider a path $\mathcal{P}$ from $(0,0)$ to $(sn+1, tn)$ lying strictly below $\mathcal{S}_{s,t}$. Clearly $\mathcal{P} = \mathsf{U}^j\mathsf{R}\cdot\mathcal{P}'$ for some $0 \le j \le t-1$ and some path $\mathcal{P}'$ from $(0,j)$ to (sn, tn). Let $\mathbf{a} = (0^{t-1}, s)^n$. Shift $\mathcal{P}'$ to the origin and append j up steps to create the path $\mathcal{P}'' = \mathcal{P}'\cdot\mathsf{U}^j$ from $(0,0)$ to (sn, tn). (See Figure 13.) It is easy to check that $\mathcal{P}''$ is dominated by $\mathbf{a}^{\langle -j\rangle}$, and that every

[1]Noy and de Mier [8] have recently introduced a very elegant approach to the enumeration of lattice paths from $(0,0)$ to (sn, tn) dominated by $(\mathsf{U}^t\mathsf{R}^s)^n$, for arbitrary s, t. They deduce generating series that are products of the fractional power series solutions of a certain functional equation dependent on s and t.

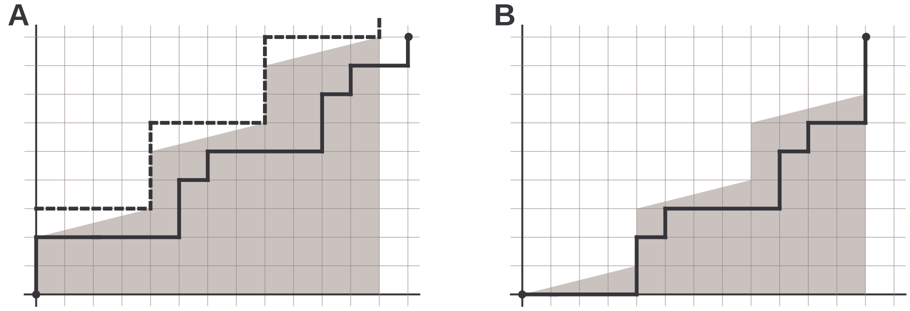

FIGURE 13. (A) The staircase $\mathcal{S}_{4,3}$ (dotted line), a path $\mathcal{P}$ that avoids it (solid line), and the area dominated by the associated composition $\mathbf{a} = (0^2, 4)^3$ (shaded). (B) The path $\mathcal{P}'$ is dominated by $\mathbf{a}^{\langle -2 \rangle} = (4, 0^2)^3$.

such path can be obtained in this way. Thus there are $\sum_{j=0}^{t-1} D(\mathbf{a}^{\langle -j \rangle})$ paths to $(sn+1, tn)$ that avoid $\mathcal{S}_{s,t}$. From (12) and Theorem 1, the sum evaluates to

$$\frac{1}{n}|\mathscr{G}(\mathbf{a}, (sn, tn))| = \frac{1}{n}\binom{(s+t)n}{tn-1},$$

again in accord with [3, Corollary 4].

In fact, [3, Theorem 3] gives formulae for the number of paths avoiding $\mathcal{S}_{s,t}$ with a specified number of corners. For instance, performing the analysis above, but replacing Theorem 1 with the more refined Theorem 8, shows that the number of paths from $(0,0)$ to $(sn+1, tn)$ that avoid $\mathcal{S}_{s,t}$ and have c up-right corners is

$$t\binom{sn}{c-1}\binom{tn}{c-1} - s\binom{sn-1}{c-2}\binom{tn+1}{c}.$$

Note that we have used $c-1$ instead of c in Theorem 8, since the mapping $\mathcal{P} \mapsto \mathcal{P}''$ described above reduces the number of up-right corners by 1.

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