SEMIGROUP IDENTITIES OF TROPICAL MATRIX SEMIGROUPS

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In this thesis, we answer several questions relating to semigroup identities and tropical matrix semigroups. To begin, we look at two finiteness properties: weak permutability and strong permutability. We show that all tropical matrix semigroups are weakly permutable and that full and upper triangular tropical matrix semigroups are not strongly permutable for dimensions greater than 1. We then introduce and give a classification of truncated tropical semirings and fully describe which full matrix semigroups over truncated tropical semirings are strongly permutable.

Next, we construct minimal and irredundant generating sets for upper triangular and unitriangular matrix semigroups over commutative semirings. We then give minimal and irredundant generating sets for the full matrix semigroup over the tropical integer semiring for dimensions 2 and 3, showing that the full matrix semigroup is finitely generated in dimension 2 but not in dimension 3. In addition to this, we construct finite presentations for upper triangular matrix semigroups over the tropical integers in every dimension.

Turning towards the growth, we find new bounds on the degree of the polynomial growth of finitely generated subsemigroups of matrix semigroups over commutative bipotent semirings. In particular, for matrices over the tropical rational semiring, the bound of the degree of the polynomial growth is bounded only in the dimension of matrix semigroup, independent of the number of generators.

We then explore the semigroup identities satisfied by tropical matrix semigroups and the plactic monoid of rank 4. We find a condition to show that a semigroup identity is not satisfied by the upper triangular tropical matrix semigroup of dimension $n+1$, and use this to construct semigroup identities satisfied by the upper triangular tropical matrix semigroup of dimension n but not by dimension $n+1$ for all $n \in \mathbb{N}$. For full tropical matrix semigroups, we construct semigroup identities that are satisfied in dimension $p-1$ but are not satisfied in dimension p for p prime. For the plactic monoid of rank 4, we find a new set of semigroup identities satisfied by the monoid, allowing us to deduce that the plactic monoid of rank 4 generates a different semigroup variety than the semigroup of upper triangular tropical matrices of dimension 5.

In the final chapter, for all $n \in \mathbb{N}$, we construct a faithful representation of the stylic monoid of rank n by unitriangular tropical matrices of dimension $n+1$. We then show that the stylic monoid of rank n satisfies the exact same semigroup identities as the semigroup of unitriangular tropical matrices of dimension $n+1$. Next, we consider involution semigroups, showing that the faithful morphism extends to involution semigroups. We show that the stylic monoid of rank n with involution is finitely based if and only $n = 1$. Finally, we show that, in contrast to the non-involution case, the stylic monoid of rank n with involution and the semigroup of unitriangular tropical matrices of dimension $n+1$ with involution satisfy different involution semigroup identities.

Declaration

No portion of the work referred to in the thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

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Chapter 1

Introduction

1.1 Background

Tropical mathematics is a rapidly growing branch of mathematics that focuses on studying the *tropical semiring*, $\mathbb{T}$. The tropical semiring is the real numbers, augmented with a $-\infty$ element, with two operations, tropical addition, and tropical multiplication, which are given by maximum and classical addition respectively. Despite tropical mathematics being a relatively new area of research, only being introduced in 1962 by Cuninghame-Green [CG62], and in 1978 by Imre Simon [Sim78], it has seen a large amount of interest [Pin98, Sim88, Sim94].

The tropical semiring has influenced many fields of mathematics including, scheduling [Kri17], optimisation [Kri15], and cryptography [APM21]. However, its most significant impact has been in algebraic geometry, where tropical geometry has emerged as a substantial area of research, with many new developments and applications [MS21, RGST05, Spe05]. Tropical geometry allows one to construct a "combinatorial shadow" of an algebraic variety, which retains important combinatorial and geometric information, providing many tools to understand and study the algebraic variety.

In this thesis, we study the properties of multiplicative semigroups of matrices over semirings, with a particular emphasis on matrices over the tropical semiring. An interesting question to ask about a semigroup is, which semigroup identities are satisfied by the semigroup? A semigroup variety is a class of all semigroups which satisfy some given set of semigroup identities. So equivalently, we can ask, which semigroup varieties a semigroup is contained in? One reason these questions are of significant interesting is due to a key result, Birkhoff's HSP Theorem, which says the following.

Theorem 1.1.1 (Birkhoff's HSP Theorem [Bir35])**.** *A class of semigroups is a semigroup variety if and only if it is closed under taking homomorphic images, subsemigroups, and arbitrary direct products.*

This theorem is more general; it can be extended to apply to abstract algebras rather than just semigroups. However, in this thesis, we mainly only consider varieties over semigroups so we give the restricted version.

From the above theorem, we can see that for a semigroup of tropical matrices, understanding the semigroup identities that it satisfies and the variety it generates allows us to understand the homomorphic images, subsemigroups, and direct products of the semigroup. In this way, semigroup identities provide us with a means to determine which semigroups can be faithfully represented by a given tropical matrix semigroup.

Research into the topic of representations, identities, and varieties for tropical matrix semigroups began in [IM09], in which Izhakian and Margolis showed that the bicyclic monoid has a faithful semigroup representation by upper triangular matrices over the tropical semiring [IM09, Theorem 4.4]. This is particularly interesting as the bicyclic monoid cannot be faithfully represented by matrices over a field. For those with a background in the representation theory of semigroups, this can be seen as the full matrix semigroup over any field is semisimple [CP61, Theorem 5.5].

In the same paper [IM09], Izhakian and Margolis also showed that not only is the bicyclic monoid represented by $UT_2(\mathbb{T})$, the monoid of 2×2 upper triangular matrices over the tropical semiring, but the monoid $UT_2(\mathbb{T})$ satisfies Adian's identity $abba\ ab\ abba = abba\ ba\ abba$, which was shown to be satisfied by the bicyclic monoid by Adian [Adi66, Chapter IV, Theorem 2]. Further, they showed that replacing each a and b in Adian's identity with a^2 and b^2 respectively gives an identity satisfied by $M_2(\mathbb{T})$.

This paper sparked the idea that the multiplicative semigroup of matrices over the tropical semiring could be used as a carrier for semigroup representations of semigroups that cannot be faithfully represented by matrices over any field, like the bicyclic monoid. Since then there have been many more semigroups and monoids shown to be representable by matrices over the tropical semiring, most notably the *plactic monoid* of rank n for each $n \in \mathbb{N}$ [JM21].

The plactic monoid was first described implicitly by Schensted as a way of finding the maximal length of a nondecreasing subsequence of a given word [Sch61]. Knuth then found a set of defining relations for the plactic monoid, referred to as the Knuth relations [Knu70] and subsequently Lascoux and Schützenberger [LS81] carried out a systematic study of the plactic monoid. Since then, it has been shown to be a relevant algebraic structure to many different mathematical fields, and as such has been widely studied [CGM15, HM17, Lop16].

In this thesis, we will only be concerned with plactic monoids of finite rank, that is, the finitely generated plactic monoids. The plactic monoid of rank n can be defined in many different ways, the simplest of which is as follows. For $n \in \mathbb{N}$, we define the plactic monoid of rank n, $\mathbb{P}_n$, to be the monoid generated by the set $\{1, \ldots, n\}$ which satisfies the Knuth relations:

$$bca = bac \text{ for all } 1 \leq a < b \leq c \leq n,$$
$$cab = acb \text{ for all } 1 \leq a \leq b < c \leq n.$$

There also exists a more combinatorial description of the plactic monoid of rank n. We say a *Young diagram* is a finite left-aligned array of equally-sized boxes in which the row above has an equal number or fewer boxes than the row below. Note that in some research areas, such as representation theory, the rows instead have equal or more boxes than the row below. However, we give the standard definition used in the literature on the plactic monoid.

From a Young diagram, we define a *semistandard Young tableau* to be a Young diagram with a number from $\{1, \ldots, n\}$ in each box such that the columns are strictly decreasing from top-to-bottom and the rows are weakly increasing from left-to-right. Given a Young tableau, the column reading of a tableau is the word obtained by reading the letters in the tableau, reading the columns from left-to-right and each column from top-to-bottom. For example,

5		
3	4	4
1	2	3

has the column reading 5314243.

Given a word over the alphabet $\Sigma = \{1, \ldots, n\}$, we can use *Schensted's insertion algorithm* [Sch61] on that word to construct the corresponding Young tableau. We then say the plactic monoid of rank n is the quotient of the free monoid Σ^* by the relation $\equiv$ where $u \equiv v$ if and only if u and v construct the same semistandard Young tableau when Schensted's insertion algorithm is applied to them. We then define the multiplication of Young tableaux T and T' to be given by applying *Schensted's insertion algorithm* to the concatenation of the column readings of T and T'.

The study of the semigroup identities satisfied by plactic monoids has only recently begun. It was shown by Jaszuńska and Okniński [JO11] that the Chinese monoid of rank 2, which is isomorphic to the plactic monoid of rank 2, satisfied Adian's identity, $abbaababba = abbabaabba$, and Kubat and Okniński [KO15] showed that the plactic monoid of rank 3 satisfied the semigroup identity $uvvuvu = uvuvvu$ where u and v are the left and right side of Adian's identity respectively.

Refocusing on the semigroup of upper triangular matrices over the tropical semiring, independently Okniński [Okn15], and Izhakian [Izh13, Izh16a, Izh16b], found (different) sets of semigroup identities satisfied by $UT_n(\mathbb{T})$ for each $n \in \mathbb{N}$. Then, Daviaud, Johnson, and Kambites [DJK18] found an algorithm to check whether a semigroup identity is satisfied by $UT_n(\mathbb{T})$ running in time polynomial in the length of the identity and size of the alphabet.

Importantly the identity shown to be satisfied by the plactic monoid of rank 3 by Kubat and Okniński [KO15], is of a similar form to the identity for $UT_3(\mathbb{T})$ found by Okniński [Okn15], and can be easily shown to be satisfied by $UT_3(\mathbb{T})$. This inspired the question of whether there exists a faithful tropical representation of the plactic monoid of rank 3 by $UT_3(\mathbb{T})$.

This was answered, independently by Izhakian [Izh19] and Cain, Klein, Kubat, Malheiro, and Okniński [CKK+17], as they found different faithful representations of the plactic monoid of rank 3 by $UT_3(\mathbb{T}) \times UT_3(\mathbb{T})$. However, in both cases, the obvious generalisation to a representation of the plactic monoid of rank 4 by $UT_4(\mathbb{T}) \times UT_4(\mathbb{T})$ is not faithful. Johnson and Kambites [JM21] produced a faithful representation of the plactic monoid of rank n for all $n \in \mathbb{N}$, by $M_{2^n}(\mathbb{T})$. This representation was used to show that every identity satisfied by the plactic monoid of rank n is satisfied by

$UT_n(\mathbb{T})$ and the plactic monoid of rank n satisfies every identity satisfied by $UT_d(\mathbb{T})$ where $d = \left\lfloor \frac{n^2}{4} + 1 \right\rfloor$. Cain, Klein, Kubat, Malheiro, and Okniński [CKK$^+$17] showed that the shortest identity satisfied by the plactic monoid of rank n has length greater than n, and hence no single semigroup identity is satisfied by every plactic monoid of finite rank. Thus, using the representation of the plactic monoid given by Johnson and Kambites [JM21], we can deduce that there is no single semigroup identity satisfied by $UT_n(\mathbb{T})$ or $M_n(\mathbb{T})$ for all $n \in \mathbb{N}$.

Recently, a number of "plactic-like" monoids have been shown to be representable by matrices over semirings. Generally, the term "plactic-like" is used for monoids that are defined by an algorithm that takes a word and constructs some combinatorial object, where we say that two words over some alphabet are equal in the monoid if they construct the same combinatorial object when the algorithm is applied to them. For example, Cain, Johnson, Kambites and Malheiro [CJKM22] found representations for the following finite rank plactic-like monoids; the hypoplactic monoid; the stalactic monoid; the taiga monoid; the sylvester monoid; the baxter monoid; and the right patience sorting monoid.

Initially, the study of the semigroup identities satisfied by semigroups of matrices over the tropical semiring was restricted to the semigroup of upper triangular matrices and the full matrix semigroup in dimension 2. However, this has now been extended to other semigroups of matrices over the tropical semiring, such as unitriangular matrices and the full matrix semigroup. In the unitriangular case, Johnson and Fenner [JF19] extended the results of Daviaud, Johnson, and Kambites [DJK18] to unitriangular matrices, showing that the semigroup of unitriangular tropical matrices, $U_n(\mathbb{T})$, exactly satisfies semigroup identities in which both sides contain the exact same set of subsequences of length at most $n-1$. For the full matrix semigroup of tropical matrices, Shitov [Shi18] showed that $M_3(\mathbb{T})$ satisfied a semigroup identity. Building on this work, Izhakian and Merlet [IM18], showed that $M_n(\mathbb{T})$ satisfies semigroup identities for each $n \in \mathbb{N}$.

Recently, the study of semigroup identities of tropical matrices has been extended to matrices over the supertropical semiring, $\mathbb{ST}$, a non-idempotenet generalisation of $\mathbb{T}$. It was shown, by Izhakian and Merlet [IM22], that for all $n \in \mathbb{N}$, $UT_n(\mathbb{ST})$ and $UT_n(\mathbb{T})$ generate the same semigroup variety and that $M_n(\mathbb{ST})$ and $M_n(\mathbb{T})$ generate the same semigroup variety. However, by Johnson and Fenner [JF19], it is known that $U_n(\mathbb{ST})$ and $U_n(\mathbb{T})$ generate different semigroup varieties as the multiplicative identity elements of $\mathbb{ST}$ and $\mathbb{T}$ generate non-isomorphic semirings.

1.2 Structure of the Thesis

This thesis develops the theory of representations of semigroups over the tropical semiring by studying properties of semigroups of matrices, including: weak permutability, strong permutability, presentations, growth, and semigroup identities. To do this we break down this thesis into 8 chapters, including this introduction, with each chapter investigating a number of these properties that help in understanding representations over the tropical semiring and the semigroup identities satisfied by semigroups of matrices over the tropical semiring. The results in Chapters 3 and 4 were published together in a paper co-authored with Kambites [AK22], Chapters 5 and 6 were solo-authored and plan to be submitted to publication, Chapter 7 was published in a solo-authored paper [Air22], and Chapter 8 was published in a paper co-authored with Ribeiro [AR23]. In this section we identify the main contributions of the thesis and the overall structure.

We begin in Chapter 2 by giving the necessary notation and definitions for this thesis. In Chapter 3, we look at the finiteness properties *weak permutability* and *strong permutability.* These properties are preserved by subsemigroups and homomorphic images. Hence, suppose a semigroup of matrices is weakly permutable or strongly permutable then, if there exists a faithful representation of a semigroup T by the semigroup of matrices, we must have that T is weakly permutable or strongly permutable respectively. In the weak permutability case, we show the following.

Proposition. *Let S be a commutative bipotent semiring. Then $M_n(S)$ is weakly permutable for all $n \in \mathbb{N}$.*

Thus, an instant corollary of this is that only weakly permutable semigroups can be faithfully represented by matrices over any commutative bipotent semiring. For strong permutability, if we add the additional requirement that the semiring S has elements of unbounded multiplicative order, then we obtain that $M_n(S)$ is only strongly permutable in trivial cases.

Theorem. *Let S be a bipotent semiring with elements of unbounded multiplicative order. Then, the semigroups $M_n(S)$ and $UT_n(S)$ are not strongly permutable for $n \geq 2$. The semigroup $U_n(S)$ is strongly permutable if and only if $n \leq 2$.*

The matrix semigroups $M_1(S)$ and $UT_1(S)$ are isomorphic to the multiplicative semigroup of S, so if we additionally assume that S is commutative, we obtain that $M_n(S)$ and $UT_n(S)$ are permutable if and only if $n = 1$. This shows the stark contrast between weak and strong permutability.

In Chapter 4, we introduce a new class of semirings, *truncated tropical semirings*, and give a complete description of when two truncated tropical semirings are isomorphic. As discussed above, if S is a commutative bipotent semiring with unbounded multiplicative order, then we know exactly when $M_n(S)$ and $UT_n(S)$ are strongly permutable. However, the truncated tropical semirings $\mathbb{T}_{[x,y]}$ only has unbounded multiplicative order if $x = 0$. So, in this section, we completely classify when the full matrix semigroup over a truncated tropical semiring is strongly permutable, as summarised in the following theorem.

Theorem. *The semigroups $M_n(\mathbb{T}_{[x,y]})$ and $UT_n(\mathbb{T}_{[x,y]})$ are strongly permutable if and only if one of the following holds*

(i) $n = 1$;

(ii) $n = 2$ *and* $x \neq 0$;

(iii) $n \geq 3$ *and* $0 \neq 2x \leq y$.

In Chapter 5, we focus on generating sets and presentations. We show that, for all $n \in \mathbb{N}$, $UT_n(\mathbb{Z}_{\max})$ is a finitely presented monoid, contrasting to the full matrix case, where $M_3(\mathbb{Z}_{\max})$ is not finitely generated. More generally, we prove the following theorem.

Theorem. *Let S be an infinite commutative anti-negative unital semiring with a zero and no zero-divisors. Then, the monoid $M_3(S)$ is not finitely generated.*

It follows from this that if S is an anti-negative semifield with a zero then $M_3(S)$ is finitely generated if and only if S is the two-element boolean semifield.

In Chapter 6, we look at the growth of finitely generated subsemigroups. We produce an upper bound on the growth of finitely generated subsemigroups of matrix semigroups allowing us to deduce that we cannot represent any finitely generated semigroup with growth greater than the bound. In particular, we consider the case of finitely generated subsemigroups of $M_n(S)$ where S is a commutative bipotent semiring, and show the following.

Proposition. *Let S be a commutative bipotent semiring and $T = \langle X \rangle$ be a finitely generated subsemigroup of $M_n(S)$. If the growth of the multiplicative semigroup generated by the entries of the matrices in X is bounded above by a polynomial of degree $l \in \mathbb{N}_0$, then, the growth function of T is bounded above by a polynomial of degree ln^2.*

It follows from this that in the case where $S = \mathbb{Q}_{\max}$ and T is a finitely generated subsemigroup of $M_n(\mathbb{Q}_{\max})$, the growth function of T is bounded above by a polynomial of degree n^2.

In Chapter 7, we focus on semigroup identities satisfied by $UT_n(\mathbb{T})$, $M_n(\mathbb{T})$, and $\mathbb{P}_4$. To understand the semigroup identities satisfied by a semigroup, it is equally important to understand the semigroup identities not satisfied by the semigroup. To this end, we prove a key lemma which allows us to easily construct semigroup identities that are not satisfied by $UT_{n+1}(\mathbb{T})$. Importantly, the conditions of this lemma are broad enough that we are able to construct semigroup identities satisfied by $UT_n(\mathbb{T})$ which also satisfy the conditions of the lemma and hence are not satisfied by $UT_{n+1}(\mathbb{T})$. Therefore, from this, we can deduce that $UT_n(\mathbb{T})$ and $UT_{n+1}(\mathbb{T})$ generate different semigroup varieties for all $n \in \mathbb{N}$.

Theorem. *For all $n \in \mathbb{N}$, there exists an identity satisfied by $UT_n(\mathbb{T})$ but not satisfied by $UT_{n+1}(\mathbb{T})$.*

The semigroup identities satisfied by $M_n(\mathbb{T})$ are much less well understood, so we are unable to prove a theorem as general as above. However, if we only consider matrix semigroups of prime dimension, we can show that if p is a prime, then $M_{p-1}(\mathbb{T})$ and $M_p(\mathbb{T})$ generate different semigroup varieties.

Theorem. *Let p be a prime. Then there exists an identity satisfied by $M_{p-1}(\mathbb{T})$ but not by $M_p(\mathbb{T})$.*

By a result of Johnson and Kambites [JM21], the variety generated by $UT_4(\mathbb{T})$ is contained in the variety generated by $\mathbb{P}_4$ which is contained in the variety generated by $UT_5(\mathbb{T})$. It remains an open question whether $UT_n(\mathbb{T})$ and $\mathbb{P}_n$ generate the same semigroup variety.

So, we consider the semigroup identities satisfied by the plactic monoid of rank 4. In contrast to the techniques used above, we show that $\mathbb{P}_4$ generates a different variety to $UT_5(\mathbb{T})$ by finding new semigroup identities satisfied by the plactic monoid of rank 4 which are shorter than those previously known. We can then show that we are able to construct a semigroup identity of this form which is not satisfied by $UT_5(\mathbb{T})$.

Corollary. *There exists an identity satisfied by $\mathbb{P}_4$ but not satisfied by $UT_5(\mathbb{T})$.*

Thus, we have shown that the variety generated by $\mathbb{P}_4$ is strictly contained in the variety generated by $UT_5(\mathbb{T})$.

Finally, in Chapter 8, we focus on a plactic-like monoid recently introduced, the *stylic monoid* [AR22]. We show that the stylic monoid of rank n, styl_n, can be faithfully represented by $(n+1) \times (n+1)$ unitriangular matrices over the tropical semiring $U_{n+1}(\mathbb{T})$, which provides a complete classification of the semigroup identities satisfied by styl_n:

Theorem. *For each $n \in \mathbb{N}$, styl_n and $U_{n+1}(\mathbb{T})$ satisfy the exact same set of semigroup identities, that is, $u = v$ is satisfied by styl_n if u and v contain the exact same subsequences of length at most n. Thus, $\mathcal{V}(\mathsf{styl}_n) \subsetneq \mathcal{V}(\mathsf{styl}_{n+1})$ for all $n \in \mathbb{N}$.*

We then introduce the concept of an involution semigroup and show that we are able to extend the faithful representation of styl_n by $U_{n+1}(\mathbb{T})$, to involution semigroups, that is, there is a faithful morphism from $(\mathsf{styl}_n, ^*)$ to $(U_{n+1}(\mathbb{T}), ^\star)$. However, in contrast to the semigroup case, we show that for $n \geq 2$, $(\mathsf{styl}_n, ^*)$ and $(U_{n+1}(\mathbb{T}), ^\star)$ satisfy different involution semigroup identities.

Theorem. *For each $n \geq 2$, there exists an identity satisfied by $(\mathsf{styl}_n, ^*)$ but not satisfied by $(U_{n+1}(\mathbb{T}), ^\star)$.*

Chapter 2

Preliminaries

In this section, we give a basic introduction to semigroups, semirings, and universal algebra as applied to semigroups and monoids. We write $\mathbb{N}$ for the set of natural numbers excluding 0 and $\mathbb{N}_0$ for the natural numbers including 0. For $n \in \mathbb{N}$, we write $[n]$ for the discrete interval $\mathbb{N} \cap [1, n]$, and $\mathcal{S}_n$ for the symmetric group on the set $[n]$.

2.1 Semigroups and Semirings

Semigroups and semirings are the fundamental algebraic structure on which this thesis is based. Both semigroups and semirings have been defined in many different ways which have evolved over time. Here we give a modern definition of a semigroup which we use throughout.

Definition 2.1.1. A *semigroup* $(S, \cdot)$ is a non-empty set S with an associative binary operation $\cdot$ on S.

Definition 2.1.2. A *monoid* is a semigroup $(S, \cdot)$ with an identity element, that is, in which there exists a unique $1 \in S$ such that $s \cdot 1 = 1 \cdot s = s$ for all $s \in S$.

Definition 2.1.3. A *group* is a monoid $(S, \cdot)$ in which every element has an inverse, that is, for all $s \in S$ there exists a unique $s^{-1} \in S$ such that $s \cdot s^{-1} = s^{-1} \cdot s = 1$.

By the three previous definitions, we can see that semigroups and monoids can be seen to be generalisations of groups. We will often use concatenation to denote multiplication in a semigroup.

Similarly to groups, rings have been generalised in many ways, in this thesis we focus on semirings, which we define in the following way.

Definition 2.1.4. A *semiring* $(S, +, \cdot)$ is a set S with two associative binary operations $+$ and $\cdot$ such that $(S, +)$ is a commutative semigroup and $(S, \cdot)$ is a semigroup, and they satisfy the following distributive property: for all $a, b, c \in S$

$$a(b + c) = ab + ac \text{ and } (b + c)a = ba + ca.$$

Note that this definition of semirings differs from some other definitions of a semiring in the literature, in that we do not require our semirings to have an identity element or a zero element. However, it is sometimes useful to specify when a semiring has an identity or a zero.

Definition 2.1.5. We say a semiring $(S, +, \cdot)$ is *unital* if $(S, \cdot)$ is a monoid and say S has a *zero* if there exists $0 \in S$ such that $0 \cdot s = s \cdot 0 = 0$ and $s + 0 = 0 + s = s$ for all $s \in S$.

We denote the identity and zero of S, where they exist, by 1_S and 0_S respectively.

Definition 2.1.6. A *subsemiring* is a subset of a semiring closed under addition and multiplication; note that even if S has zero and/or identity elements, subsemirings are not required to contain them.

Definition 2.1.7. For a semiring S, we write $S^* = S$ if S does not have a zero, and $S^* = S \setminus \{0_S\}$ otherwise.

Definition 2.1.8. A *semiring* $(S, +, \cdot)$ is *commutative* if $(S, \cdot)$ is a commutative semigroup.

Definition 2.1.9. A *semifield* $(S, +, \cdot)$ is a commutative semiring, in which $(S^*, \cdot)$ is a group.

As stated above, we do not require semirings to have a zero or an identity element. However, it is sometimes useful to adjoin a zero or identity element to our semiring.

Definition 2.1.10. Given a semiring S, let S^0 be the *semiring with a zero obtained by adjoining a zero if necessary*, that is, $S^0 = S$ if S has a zero element and $S^0 = S \cup \{0\}$ otherwise, where $0s = s0 = 0$ and $s + 0 = 0 + s = s$ for all $s \in S^0$.

As in the above definition, it is easy to adjoin a zero to any semiring, however the same cannot be said for adjoining an identity element. In fact, there exist semirings in which no identity can be adjoined as every assignment of $s+1$ and $1+s$ gives a contradiction. Notwithstanding the impossibility in general of adjoining an identity element, it is sometimes convenient to introduce "the identity" as a *purely notational* device.

Definition 2.1.11. Given a semiring S, let $S^1 = S$ if S has an identity element and $S^1 = S \cup \{1\}$ otherwise, where $1s = s1 = s$ for all $s \in S^1$ and $1+s$ and $s+1$ are undefined for all $s \in S$ unless S has a zero, then we define $1+0 = 0+1 = 1$. Note that this is not a semiring, as addition is only partially defined.

For a semiring S, we write (S^{01}) to denote $(S^0)^1$. We now introduce two properties of semirings which hold in many of the semirings which we discuss throughout this thesis.

Definition 2.1.12. A semiring S is called *anti-negative* if for all $x, y \in S^0$, $x+y = 0_S$ if and only if $x = 0_S$ and $y = 0_S$.

If we restrict to anti-negative semifields rather than semirings, then these are exactly the semifields which are not fields [GJN20, Lemma 2.1].

Definition 2.1.13. A semiring S is called *bipotent* if for all $x, y \in S$, $x + y \in \{x, y\}$.

All bipotent semirings are anti-negative, but the converse is not true even in the case of semifields, as such, there exist semifields which are not fields and are not bipotent. We now give some examples of bipotent semirings, which we use heavily throughout.

Definition 2.1.14. Let $\mathbb{T} = (\mathbb{R} \cup \{-\infty\}, \max, +)$ be the *tropical semifield*, that is, the real numbers augmented with $-\infty$ with binary operations maximum as its addition and addition as its multiplication.

The tropical semiring admits isomorphic manifestations as the *min-plus semifield* (the real numbers augmented with $+\infty$ under *minimum* and classical addition) and the *max-times semifield* (the non-negative real numbers augmented with $-\infty$ under maximum and classical *multiplication*).

Moreover, we define the *tropical integer semiring* $\mathbb{Z}_{\max} = \mathbb{T} \cap (\mathbb{Z} \cup \{-\infty\})$ and the *tropical rational semiring* $\mathbb{Q}_{\max} = \mathbb{T} \cap (\mathbb{Q} \cup \{-\infty\})$.

Definition 2.1.15. Let $\mathbb{B} = (\{0,1\}, \max, \min)$ be the *boolean semifield*, that is, $\{0,1\}$ with binary operations maximum as its addition and minimum as its multiplication.

Note that $\mathbb{B}$ is isomorphic to the subsemiring of $\mathbb{T}$ given by $\{-\infty, 0\}$.

Given a semiring, we can define a matrix semigroup. To do this, we take a set of matrices with entries from the semiring which is closed under matrix multiplication and take matrix multiplication to be the semigroup operation. We introduce three main examples of matrix semigroups that can be defined over a semiring which we use throughout.

Definition 2.1.16. Let S be a semiring. Let $M_n(S)$ denote the semigroup of all $n \times n$ matrices with operation given by matrix multiplication. We call $M_n(S)$ the full matrix semigroup of $n \times n$ matrices over S.

Definition 2.1.17. Let S be a semiring. Let $UT_n(S)$ denote the semigroup of $n \times n$ upper triangular matrices with operation given by matrix multiplication. More precisely, this is the subsemigroup of $M_n(S^0)$, of matrices A with $A_{ij} = 0$ if $i > j$ and $A_{ij} \in S$ for $i \leq j$. We call $UT_n(S)$ the matrix semigroup of $n \times n$ upper triangular matrices over S.

Note that here, even if S does not have a zero element we can still define the upper triangular matrix semigroup over S by using an adjoined zero below the diagonal.

Similarly to how we defined upper triangular matrix semigroups, we can define the semigroup of unitriangular matrices over semirings with or without an identity or a zero. We do this by adjoining a zero and an identity to S as we described above. Note that despite defining over a structure with not all operations defined, enough are defined in order to compute the matrix multiplication of unitriangular matrices.

Definition 2.1.18. Let S be a semiring. Let $U_n(S)$ denote the semigroup of $n \times n$ unitriangular matrices with operation given by matrix multiplication. More precisely, this is the subsemigroup of $M_n(S^{01})$, of matrices A with $A_{ij} = 0$ for $i > j$, $A_{ij} = 1$ for $i = j$, and $A_{ij} \in S$ for $i \leq j$. We call $U_n(S)$ the matrix semigroup of $n \times n$ unitriangular matrices over S. (Note that even when S^{01} is not a semiring, $UT_n(S)$ still forms a semigroup.)

2.2 Universal Algebra

We now introduce the definitions relating to words, semigroup identities, and semigroup varieties, giving a basic introduction to universal algebra as it relates to semigroups and monoids.

Definition 2.2.1. Let Σ be a set of letters, we say Σ is an *alphabet*, and we call a finite (possibly empty) string of letters in Σ a *word*.

Definition 2.2.2. Denote the *free monoid on* Σ by Σ^*, that is, the monoid of all (possibly empty) words over Σ under concatenation and denote the *free semigroup on* Σ by Σ^+, that is, the semigroup of all non-empty words over Σ under concatenation.

Here we introduce a number of definitions and notations that we use when referring to words.

Definition 2.2.3. Let Σ be an alphabet. For $u, v \in \Sigma^*$ and $a \in \Sigma$, we write

(i) $|u|$ for the *length* of u,

(ii) $|u|_a$ for the number of times the letter a appears in u,

(iii) $u_{(i)}$ to denote the ith letter of u,

(iv) $\operatorname{supp}(u)$ for *support* of u, that is, the subset of Σ of letters which occur in u,

(v) u is a *suffix* of v if there exists $v_1 \in \Sigma^*$ such that $v_1 u = v$,

(vi) u is a *prefix* of v if there exists $v_2 \in \Sigma^*$ such that $uv_2 = v$,

(vii) $v_{\leq i}$ for the prefix of the first i letters of v,

(viii) u is a *factor* of v if there exists $v_1, v_2 \in \Sigma^*$ such that $v = v_1 u v_2$,

(ix) u is a *subsequence* of v if there exist $u_1, \ldots, u_n \in \Sigma$ and $v'_1, \ldots, v'_{n+1} \in \Sigma^*$ such that $u = u_1 \cdots u_n$ and $v = v'_1 u_1 v'_2 \cdots v'_n u_n v'_{n+1}$ and denote the subsequence $u = u_1 \cdots u_n$ by its sequence of letters $u_1, \ldots, u_n$.

Definition 2.2.4. Let S be a semigroup and $w \in \{a, b\}^*$; then for $x, y \in S$, we write $w(a \mapsto x, b \mapsto y)$ to denote $\phi(w)$ where $\phi\colon \{a, b\}^* \to S$ is the semigroup morphism defined by sending $a \mapsto x$ and $b \mapsto y$. In the case where $S = \Omega^+$, for some alphabet Ω, we write $w[a \mapsto x, b \mapsto y]$, rather than $w(a \mapsto x, b \mapsto y)$, to indicate that $w[a \mapsto x, b \mapsto y]$ is again a word.

Definition 2.2.5. Let $u, v \in \Sigma^+$, such that $u \neq v$. We say "$u = v$" is a (non-trivial) *semigroup identity.* The identity $u = v$ is *satisfied by* a semigroup $\mathcal{S}$, if $\phi(u) = \phi(v)$, for all semigroup morphisms $\phi \colon \Sigma^+ \to \mathcal{S}$.

The following definition is from universal algebra, but we state it only in the case of semigroups, as this is the main domain of application in this thesis.

Definition 2.2.6. The class of all semigroups that satisfy a given set of semigroup identities is called a *semigroup variety.* We say the *semigroup variety generated by a semigroup* $\mathcal{S}$, denoted $\mathcal{V}(\mathcal{S})$, is the set of all semigroups that satisfy all the semigroup identities satisfied by $\mathcal{S}$.

For example, we say that a semigroup $\mathcal{S}$ is contained in a variety generated by a semigroup $\mathcal{T}$, if $\mathcal{S}$ satisfies every semigroup identity satisfied by $\mathcal{T}$.

Chapter 3

Permutability of Matrices over Bipotent Semirings

3.1 Introduction

Recall, a semiring is called *bipotent* if $x + y$ is always either x or y. Commutative bipotent semirings appear naturally in many areas of mathematics; for example, the *boolean* semiring has important applications in computer science [Gol99], while *tropical* and related semirings have found applications in areas as diverse as algebraic geometry, geometric group theory, automata and formal languages, and combinatorial optimization and control theory [BBRT12, CGQ99, Mik03]. Many of the problems which arise naturally in these areas involve finite systems of linear (over the semiring) equations and can therefore be formulated in terms of matrix operations; understanding the structure of matrix algebra over these semirings is thus vital for applications, and much recent research has been devoted to this topic.

In this chapter, we focus on two algebraic finiteness conditions for semigroups of matrices over bipotent semirings: weak permutability and permutability. A semigroup S is called *weakly permutable* if there exists a $k \geq 2$ such for any $s_1, \ldots, s_k \in S$ there exist permutations $\sigma \neq \tau$ of $\{1, \ldots, k\}$ such that $s_{\sigma(1)} s_{\sigma(2)} \cdots s_{\sigma(k)} = s_{\tau(1)} s_{\tau(2)} \cdots s_{\tau(k)}$. A semigroup S is called *permutable* (or sometimes *strongly permutable*) if there exists a $k \geq 2$ such for any $s_1, \ldots, s_k \in S$ there exists a non-identity permutation σ of $\{1, \ldots, k\}$ such that $s_{\sigma(1)} s_{\sigma(2)} \cdots s_{\sigma(k)} = s_1 s_2 \cdots s_k$. We note here a few key facts about these properties; for a comprehensive introduction the reader is directed to [Okn91, Chapter 19]. Notice that every strongly permutable semigroup S is weakly permutable by taking τ to be the identity permutation. Every finite semigroup is clearly strongly permutable, as is every commutative semigroup. Indeed, weak and strong permutability may be thought of as very weak commutativity conditions. It is easy to see that if a semigroup S is weakly [strongly] permutable then every subsemigroup of S and every homomorphic image of S is also weakly [strongly] permutable. Permutability conditions are of interest in general because of connections with polynomial identities in semigroup algebras [Okn91, Chapter 19], and are lent additional importance in these particular semigroups by interest in representations over semirings: any permutability condition satisfied by matrix semigroups poses an obstruction to faithfully representing semigroups not satisfying the condition.

We begin, in Section 3.2, by establishing some structural results about commutative bipotent semirings which will be useful in our subsequent analysis. These include a simple classification of the monogenic examples, which may be of independent interest.

In Section 3.3 we proceed to look at weak permutability, proving that every full matrix semigroup, and hence every matrix semigroup, over a commutative bipotent semiring is weakly permutable. This fact was first stated by d'Alessandro and Pasku [dP03] but there is an error (described below) in their proof.

In Section 3.4 we turn our attention to (strong) permutability. If the semiring has an element of infinite multiplicative order (or more generally, elements of unbounded multiplicative order) we prove (Theorem 3.4.6) that the full matrix semigroup and upper triangular matrix semigroups are not strongly permutable in any dimension greater than 1. This applies in particular to the tropical and many related semirings. On the other hand, semirings with bounded multiplicative order exhibit a range of behaviours, with apparently similar semirings sometimes differing quite dramatically. Matrix semigroups over *chain semirings*, which are multiplicatively as well as additively idempotent, are strongly permutable in all dimensions (Corollary 3.4.10).

This chapter is based on joint work with my supervisor, Mark Kambites [AK22].

3.2 Commutative Bipotent Semirings

A bipotent semiring admits a natural linear order defined by $x \leq y$ if and only if $x+y = y$, and the distributive laws mean exactly that multiplication respects this order, giving rise to a totally ordered semigroup. Conversely, every totally ordered semigroup gives rise to a bipotent semiring, by taking the semigroup operation as multiplication and defining the sum to be maximum with respect to the order. Bipotent semirings are thus, at one level, the same thing as totally ordered semigroups, but the two viewpoints lead naturally to rather different questions; in particular the semiring viewpoint leads to the study of linear algebra and matrices. Our main interest is in commutative bipotent semirings, although some of our results will extend to the non-commutative case.

As discussed previously, some authors insist that a semiring should have a zero and/or an identity element, but most of our results will not require these. In fact it is easy to see that any commutative bipotent semiring S without a zero element can have one "adjoined", that is, can be embedded in a commutative bipotent semiring with one extra element 0 which is a zero, S^0. On the other hand, the corresponding statement is not true for identity elements:

Proposition 3.2.1. *There exists a commutative bipotent semiring S without identity which cannot be embedded in any bipotent semiring with identity.*

Proof. Let $S = \{a, b, c\}$ be the commutative bipotent semiring such that $c \geq b \geq a$, all elements are multiplicatively idempotent, and all non-idempotent products are b. It is straightforward to verify that the given operations respect the associative and distributive laws. Now suppose we can embed S in a bipotent semiring with identity 1, and consider where 1 lies in the order. If $1 > b$, then $a(1 + b) = a1 = a$, but by the distributive law $a(1 + b) = a1 + ab = a + b = b$ giving a contradiction. On the other hand, if $1 < b$, then $c(1 + b) = cb = b$, but similarly by the distributive law $c(1 + b) = c1 + cb = c + b = c$, giving a contradiction. Thus, we cannot embed S into a bipotent semiring with identity. $\square$

The above proposition can be restated in terms of totally ordered commutative semigroups and totally ordered monoids as follows.

Corollary 3.2.2. *There exists a totally ordered commutative semigroup that does not embed in any totally ordered monoid.*

This example motivates the natural question to ask when one can adjoin an identity and obtain a commutative bipotent semoiring.

Question 3.2.3. *When can a commutative bipotent semiring S without identity be embedded in commutative bipotent semiring with identity?*

If $a \in S$ then we write $\langle a \rangle$ for the (*monogenic*) subsemiring of S generated by a (that is, the intersection of all subsemirings containing a). If S is bipotent then $\langle a \rangle$ coincides with the multiplicative subsemigroup of S generated by a, in other words, the set of positive powers of a. The (multiplicative) *order* of a is defined to be the cardinality of the set of positive powers of a, which when S is bipotent is the cardinality of $\langle a \rangle$.

We will consider in particular the following examples of commutative bipotent semirings; some of these merit study due to external applications, some arise naturally in the general theory, and others are included to illustrate the full range of possible behaviours:

- The *tropical* (or *max-plus*) *semifield* $\mathbb{T}$; it has applications in numerous areas including biology [BBRT12], control theory [CGQ99] and algebraic geometry [Mik03].

- The *tropical natural number semiring* $\mathbb{N}^*_{\max}$ is the subsemiring of $\mathbb{T}$ consisting of natural numbers; it has applications in areas such as formal language theory and automata theory [KLMP04]. The * is used to denote that the semiring does not contain a zero element.

- The *tropical negative natural number semiring* $\mathbb{N}^*_{\min}$ is the subsemiring of $\mathbb{T}$ consisting of the negative integers. (It is isomorphic to the natural numbers under *minimum* and classical addition.)

- For $k \in \mathbb{N}$ the *truncated tropical natural number semiring* $[k]^*_{\max}$ consists of the set $[k] = \{1, \ldots, k\}$ with operations maximum and *k-truncated addition* given by $ab = \min(a + b, k)$ where $+$ here denotes classical addition.

- For $k \in \mathbb{N}$ the *truncated tropical negative natural number semiring* $[k]^*_{\min}$ consists of the set $\{-k, \dots, -1\}$ with operations maximum and $(-k)$*-truncated addition* given by $ab = \max(a + b, -k)$. It is isomorphic to $[k]$ under *minimum* and k*-truncated addition*. (Note that $[1]^*_{\min}$ and $[1]^*_{\max}$ are both trivial and therefore isomorphic to each other.)

- Any linearly ordered set admits the structure of a commutative bipotent semiring, with maximum as addition and minimum as multiplication. We call these *chain semirings*. A prominent example is the 2-element chain semiring, the *boolean semifield*, which is isomorphic to the semiring with two elements *True* and *False* with operations "or" and "and", and has natural applications in logic and computer science [Gol99].

For any semiring S and $n \in \mathbb{N}$, our principal interest is in the structure of $UT_n(S)$ and $U_n(S)$ as multiplicative semigroups. Note that $M_1(S) = UT_1(S)$ is isomorphic to the multiplicative semigroup of S, while $U_1(S)$ is the trivial monoid and $U_2(S)$ is isomorphic to the additive semigroup of S. Note that $M_n(S)$ will typically be neither commutative nor bipotent (even when S is both).

Lemma 3.2.4. *Let S be a bipotent semiring. If an element $x \in S$ has finite multiplicative order (that is, has finitely many distinct powers) then it has period* 1 *(that is, $x^k = x^{k+1}$ for some $k \in \mathbb{N}$).*

Proof. Let $x \in S$ have finite multiplicative order. Then there exist $r, m \in \mathbb{N}$ such that $x^m = x^{m+r}$. If $r = 1$ we are done, so assume $r > 1$. As S is bipotent we have that the sum $x^m + \dots + x^{m+r-1} = x^k$ for some k between m and $m + r - 1$. But now by distributivity and commutativity of addition,

$$\begin{aligned} x^{k+1} &= x(x^m + \dots + x^{m+r-2} + x^{m+r-1}) \\ &= x^{m+1} + \dots + x^{m+r-1} + x^m \\ &= x^k. \end{aligned}$$ □

The following lemma describes all the possible bipotent semirings generated by a single element:

Lemma 3.2.5. *Let S be a bipotent semiring. If $a \in S$ and $\langle a \rangle$ is the monogenic subsemiring generated by a, then*

$$\langle a \rangle \cong \begin{cases} \mathbb{N}^*_{\max} & \text{if } a \text{ has infinite order and } a < a^2; \\ \mathbb{N}^*_{\min} & \text{if } a \text{ has infinite order and } a^2 < a; \\ [k]^*_{\max} & \text{if } a \text{ has order } k \in \mathbb{N} \text{ and } a \leq a^2; \\ [k]^*_{\min} & \text{if } a \text{ has order } k \in \mathbb{N} \text{ and } a^2 < a. \end{cases}$$

Proof. First suppose $a \leq a^2$. Define a map

$$\phi \ : \ \mathbb{N}^*_{\max} \to \langle a \rangle, \quad n \mapsto a^n.$$

This map is surjective (because of our observation that, in a bipotent semiring, $\langle a \rangle$ coincides with the multiplicative semigroup generated by a) and preserves multiplication because of basic properties of powers. Now let $n, m \in \mathbb{N}$ and suppose without loss of generality that $n \geq m$. Since $a \leq a^2$ we have $a^k \leq a^{k+1}$ for all k (because the total order is compatible with multiplication) and hence $a^m \leq a^n$ (because $m \leq n$ and the order is transitive). Therefore

$$\phi(\max(n, m)) \ = \ \phi(n) \ = \ a^n \ = \ a^n + a^m \ = \ \phi(n) + \phi(m).$$

If a has infinite order then ϕ is injective, and we have shown that it is an isomorphism from $\mathbb{N}^*_{\max}$ to $\langle a \rangle$.

If a has finite order k then let φ be the restriction of ϕ to the subset $[k]$. Clearly φ is a bijection. Since the semiring addition (in other words, the order) on $[k]^*_{\max}$ is the restriction of that on $\mathbb{N}^*_{\max}$, the fact that φ preserves semiring addition follows from the fact that ϕ does. Now let $n, m \in \mathbb{N}$ and suppose without loss of generality that $n \geq m$. Then

$$\varphi(n + m) \ = \ a^{n+m} \ = \ a^n a^m \ = \ \varphi(n)\varphi(m)$$

for all $n, m \in \mathbb{N}$. The first equality here holds because if $n + m \geq k$ then $a^{n+m} = a^k$, as a has period 1 by Lemma 3.2.4. Hence, φ is an isomorphism between $[k]^*_{\max}$ and $\langle a \rangle$.

Similarly if $a^2 \leq a$ then we define

$$\psi \ : \ \mathbb{N}^*_{\min} \to \langle a \rangle, \quad n \mapsto a^{-n}.$$

Again ψ is surjective. This time for negative integers $n \geq m$ we use $a^2 \leq a$ to deduce that $a^{-m} \leq a^{-n}$ so

$$\psi(\max(n,m)) = \psi(n) = a^{-n} = a^{-n} + a^{-m} = \psi(n) + \psi(m).$$

and ψ preserves semiring addition. If a has infinite order then ψ is injective and preserves the semiring multiplication, so it is an isomorphism between $\mathbb{N}^*_{\min}$ and $\langle a \rangle$. If a has finite order k then an entirely similar argument to that above shows that the restriction of ψ to $-[k]$ is an isomorphism between $[k]^*_{\min}$ and $\langle a \rangle$. □

3.3 Weak Permutability

In this section we briefly consider weak permutability, showing that any semigroup of matrices over a commutative bipotent semiring always has this property. This result was first stated by d'Allesandro and Pasku [dP03], but Taylor [Tay17] identified an error in their proof. The error and its consequences are discussed below. Our proof is, nonetheless, inspired by their method.

Proposition 3.3.1. *Let S be a commutative bipotent semiring. Then $M_n(S)$ is weakly permutable for all $n \in \mathbb{N}$.*

Proof. Fix $n \in \mathbb{N}$. Let Γ_n denote the complete directed graph (with loops) on the set $[n]$. We identify edges in Γ_n with pairs in $[n] \times [n]$ in the obvious way; in particular we will index the entries of $n \times n$ matrices by edges in Γ_n.

Let Π denote the set of $n \times n$ matrices whose entries are edges from Γ_n (that is, pairs from $[n] \times [n]$). Let $c = |\Pi| = n^{2n^2}$. Choose k large enough that $k! > c^k$.

Consider a finite sequence of k matrices of size $n \times n$ over the semiring S, say $M_1, \ldots, M_k$. For a permutation σ in the symmetric group $\mathcal{S}_k$, write

$$M_\sigma \;=\; M_{\sigma(1)} M_{\sigma(2)} M_{\sigma(3)} \cdots M_{\sigma(k)}.$$

We must show that there are distinct permutations $\sigma, \tau \in \mathcal{S}_k$ with $M_\sigma = M_\tau$.

We define a function $\pi \colon \mathcal{S}_k \to \Pi^{[k]}$ (where $\Pi^{[k]}$ denotes the set of functions from $[k]$ to Π) as follows. For each $\sigma \in \mathcal{S}_k$ and each $x, y \in [n]$, consider the (x, y) entry of the matrix M_σ. It follows from the definition of matrix multiplication and the fact S is bipotent that there is at least one path $p_1, \ldots, p_k$ of length k from x to y in Γ_n where p_i are edges in Γ_n such that this entry is given by

$$(M_\sigma)_{x,y} \;=\; \left(M_{\sigma(1)}\right)_{p_1}\left(M_{\sigma(2)}\right)_{p_2}\cdots\left(M_{\sigma(k)}\right)_{p_k}. \tag{3.1}$$

Choose any such path, and for each $i \in [k]$ define the (x, y) entry of $(\pi(\sigma))\,(i)$ to be the edge $p_{\sigma^{-1}(i)}$ (that is, the edge indexing the entry of M_i which contributes in the computation of the (x, y) entry of M_σ). Thus reordering the terms in (3.1) we have

$$(M_\sigma)_{x,y} = (M_1)_{(\pi(\sigma))(1)_{x,y}}(M_2)_{(\pi(\sigma))(2)_{x,y}}\cdots(M_k)_{(\pi(\sigma))(k)_{x,y}}$$

as S is commutative. But this means that M_σ is determined by $\pi(\sigma)$.

The domain $\mathcal{S}_k$ of π has cardinality $k!$ while the codomain $\Pi^{[k]}$ of π has cardinality $|\Pi|^k = c^k$. Since k was chosen such that $k! > c^k$ there must be distinct permutations $\sigma, \tau \in \mathcal{S}_k$ such that $\pi(\sigma) = \pi(\tau)$, which by the previous paragraph means that $M_\sigma = M_\tau$. □

We now discuss some details of proofs given in [dP03], referring to terminology and notation as in [dP03]. The mistake in [dP03] lies in the proof of the first part of [dP03, Proposition 3], where k is taken to be the smallest integer such that $\alpha k^\beta < k!$. The problem is that k was discussed prior to this point, and in fact played an implicit role in the definition of the set $\mathcal{C}$, the cardinality of which was in turn used to define α and β. Thus, one is not necessarily free to choose k at this point without also changing α and β. The claim that one may choose k with $\alpha k^\beta < k!$ implicitly assumes α and β to be constant, when in reality they are functions of k and there is no immediate reason to suppose that αk^β grows more slowly than $k!$.

We discuss briefly the impact upon the correctness of other results in [dP03]. The second part of [dP03, Proposition 3] (which establishes the very important result that finitely generated semigroups of tropical matrices have polynomial growth) is correct, even though the proof ostensibly employs the same argument as the first part; the erroneous section of the argument is not required in this part, and the values of α and β (and hence also of δ and γ) here are independent of k so that the growth bound obtained really is polynomial in k. The result [dP03, Proposition 4] is claimed to be proved by “a slight generalisation” of the (erroneous) proof of [dP03, Proposition 3]; we believe a variation on the above proof technique can be used to establish this result, but we do not do this here as it is (not being concerned with bipotent semirings) rather outside the scope of this thesis. The statement of [dP03, Proposition 5] is true: the main proof given relies on [dP03, Proposition 4] and is therefore incomplete, but the alternative proof via Gromov's polynomial growth theorem, outlined in [dP03, Remark 3], is valid.

3.4 Strong Permutability

In this section we turn our attention to the stronger version of permutability. We shall need the following result, which is trivial where the semiring S has a zero element but requires slightly more work when it does not. First, recall that for a sequence of k matrices $M_1, \dots, M_k$ and a permutation $\sigma \in \mathcal{S}_k$, we write $M_\sigma = M_{\sigma(1)} \cdots M_{\sigma(k)}$.

Proposition 3.4.1. *Let S be a bipotent semiring. If $M_n(S)$ is strongly permutable then $M_m(S)$ is strongly permutable for all $m < n$. If $UT_n(S)$ is strongly permutable then $UT_m(S)$ is strongly permutable for all $m < n$.*

Proof. Consider first the case of full matrix semigroups. Suppose, with the aim of obtaining a contradiction, that there is an $m < n$ such that for every $k \in \mathbb{N}$ there exist $m \times m$ matrices $M_1, \dots, M_k$ such that $M_\sigma \neq M_e$ for any non-trivial permutation σ. Fix k and let $M_1, \dots, M_k$ be as given. Let z be the smallest (with respect to the order on the semiring) entry of any of the matrices $M_1, \dots, M_k$. For each i let N_i be the $n \times n$ matrix obtained by taking M_i and adjoining $n - m$ rows at the bottom and $n - m$ columns at the right in which every entry is z.

Now consider the x, y entry of a product $N_{i_1} \cdots N_{i_k}$ for $x, y \leq m$. As S is bipotent this entry is equal to the maximum (with respect to the order in the semiring) across sequences $x = x_0, x_1, \dots, x_k = y$ of the term:

$$\prod_{j=1}^{k} (N_{i_j})_{x_{j-1}, x_j}.$$

If in such a sequence we have $x_j > m$ for some $1 \leq j < k$, then $(N_{i_j})_{x_{j-1},m} \geq z = (N_{i_j})_{x_{j-1},x_j}$ and $(N_{i_{j+1}})_{m,x_{j+1}} \geq z = (N_{i_{j+1}})_{x_j,x_{j+1}}$ by definition, so we may replace x_j by m in the sequence without reducing the resulting term. Thus, we may assume the above maximum is attained for a sequence with $x_j \leq m$ for all j, and it follows that the top-left $m \times m$ submatrix of the product is the product of the corresponding submatrices in the factors, in other words, the corresponding product of the M_is. In particular, for any permutation σ the top-left $m \times m$ submatrix of N_σ is exactly M_σ. Thus, $N_\sigma \neq N_e$ for any non-trivial permutation σ, which since k was chosen arbitrarily contradicts the assumption that $M_n(S)$ is permutable.

For the upper triangular case, there exists a surjective homomorphism from $UT_n(S)$ to $UT_m(S)$ for $m < n$ by only considering the first m rows and columns. Hence if $UT_n(S)$ is permutable then $UT_m(S)$ is permutable for all $m < n$. □

Our next objective is to show that matrix semigroups over a (not necessarily commutative) bipotent semiring with elements of infinite multiplicative order (or more generally, unbounded multiplicative order) are not, in general, permutable. A key tool is a result of Okniński [Okn91, Chapter 19, Lemma 22], stating that a finitely generated inverse semigroup with infinitely many idempotents cannot be permutable. In particular this means that the *bicyclic monoid* is not permutable. This will combine with a representation of the bicyclic monoid by tropical matrices, due to Izhakian and Margolis [IM09], to yield non-permutability results for tropical matrix monoids, and then with our classification of the monogenic bipotent semirings (Lemma 3.2.5) to obtain non-permutability results for matrix monoids over semirings with elements of infinite order. Some elementary model theory extends these results to semirings with unbounded order.

Theorem 3.4.2. *$M_n(\mathbb{N}^*_{\max})$, $M_n(\mathbb{N}^*_{\min}), UT_n(\mathbb{N}^*_{\max})$ and $UT_n(\mathbb{N}^*_{\min})$ are not strongly permutable for $n \geq 2$.*

Proof. Let $\mathcal{B} = \langle p, q \ : \ pq = 1 \rangle$ be the bicyclic monoid. Recall that every element of $\mathcal{B}$ can be written as $q^i p^j$ for some $i, j \in \mathbb{N} \cup \{0\}$. By [IM09] there is a semigroup embedding of $\mathcal{B}$ into $UT_2(\mathbb{T})$ given by

$$\rho\colon \mathcal{B} \to UT_2(\mathbb{T}), \quad q^i p^j \mapsto \begin{pmatrix} i-j & i+j \\ -\infty & j-i \end{pmatrix}.$$

Since the bicyclic monoid is not permutable [Okn91, Chapter 19, Lemma 22] and subsemigroups of permutable semigroups are permutable, we deduce that $UT_2(\mathbb{T})$ is not permutable. Indeed further, for every k there are upper triangular matrices

$M_1, \ldots, M_k \in UT_2(\mathbb{T})$ whose diagonal and above-diagonal entries are integers, with the property that $M_\sigma \neq M_e$ for every non-trivial permutation $\sigma \in \mathcal{S}_k$.

If we fix an integer λ strictly less then every integer appearing in these matrices, then the tropically scaled matrices $(-\lambda)M_1, \ldots, (-\lambda)M_k$ clearly also have this property. Replacing the $-\infty$ entry of these matrices with the zero element of $(\mathbb{N}^*_{\max})^0$ yields a sequence of matrices to show that $UT_2(\mathbb{N}^*_{\max})$ is not strongly permutable. Similarly, tropically scaling $M_1, \ldots, M_k$ by the negative of an integer strictly greater than every entry yields a sequence of matrices for each k showing that $UT_2(\mathbb{N}^*_{\min})$ is not strongly permutable.

It remains to establish the claims for full matrix semigroups. (Note that, since the semirings here lack zero elements, we do not have a natural embedding of each upper triangular matrix semigroup into the corresponding full matrix semigroup which would allow us to immediately deduce the remaining claims.)

Let $k > 1$ and $M_1, \ldots, M_k$ be as above. Choose a very large $\mu \in \mathbb{N}$, and let $N_1, \ldots, N_k \in M_n(\mathbb{N}^*_{\max})$ be obtained from $M_1, \ldots, M_k$ by scaling tropically by μ, and replacing the $-\infty$ below the diagonal with 1. Now consider the product N_σ for some $\sigma \in \mathcal{S}_k$, and in particular the computation of the (x, y) entry for some $(x, y) \neq (2, 1)$. A simple calculation shows that, provided μ was chosen large enough, the terms which do not feature the (2,1) entry of any N_i will all exceed those which do, from which it follows that $(N_\sigma)_{x,y} = k\mu + (M_\sigma)_{x,y}$. Thus, we conclude that $N_\sigma \neq N_e$. Since k and σ were arbitrary, this means that $M_n(\mathbb{N}^*_{\max})$ is not strongly permutable.

Finally, tropically scaling the matrices $N_1, \ldots, N_k$ by a sufficiently negative integer gives a sequence to show that $M_n(\mathbb{N}^*_{\min})$ is not strong permutable □

Lemma 3.4.3. *$U_n(\mathbb{N}^*_{\max})$ is strongly permutable if and only if $n \leq 2$.*

Proof. Remark that $U_1(\mathbb{N}^*_{\max})$ is trivial while $U_2(\mathbb{N}^*_{\max})$ is isomorphic to the (commutative) additive semigroup of the semiring, so both are strongly permutable. There exists a surjective morphism from $U_n(\mathbb{N}^*_{\max})$ to $U_3(\mathbb{N}^*_{\max})$ for all $n \geq 3$ by mapping to each matrix to its top-left corner 3 by 3 submatrix, so it suffices to show that $U_3(\mathbb{N}^*_{\max})$ is not strongly permutable.

So, we define the sequence of matrices $B_1, B_2, \ldots, B_m$ by

$$B_i = \begin{pmatrix} 0 & i & m \\ -\infty & 0 & m+1-i \\ -\infty & -\infty & 0 \end{pmatrix}$$

(Note that technically speaking $-\infty, 0 \notin \mathbb{N}^*_{\max}$; the "$-\infty$" and "0" featured here are technically the zero and identity elements adjoined in $(\mathbb{N}^*_{\max})^{01}$ which is used in the definition of the unitriangular matrix semigroup $U_3(\mathbb{N}^*_{\max})$, but because this is essentially the same as the subsemiring $\mathbb{N}^*_{\max} \cup \{0, -\infty\}$ of $\mathbb{T}$ it is clearer to denote them in this way.) A simple inductive argument shows that for each k,

$$\prod_{i=1}^{k} B_i = \begin{pmatrix} 0 & k & m \\ -\infty & 0 & m \\ -\infty & -\infty & 0 \end{pmatrix}$$

Now, suppose $\sigma \in \mathcal{S}_m$ is such that $B_\sigma := \prod_{i=1}^{m} B_{\sigma(i)} = \prod_{i=1}^{m} B_i$. By the definition of matrix multiplication, for all $j < k$ we must have

$$m \;=\; (B_\sigma)_{1,3} \;\geq\; (B_{\sigma(j)})_{1,2} + (B_{\sigma(k)})_{2,3} \;=\; \sigma(j) + m + 1 - \sigma(k)$$

and hence $\sigma(j) < \sigma(k)$. Since σ is a permutation, this can only happen if σ is the identity permutation. Further, as m was arbitrary no non-trivial permutations preserve this product for any $m \in \mathbb{N}$, so $U_3(\mathbb{N}^*_{\max})$ is not strongly permutable. □

Lemma 3.4.4. $U_n(\mathbb{N}^*_{\min})$ *is strongly permutable if and only if* $n \leq 3$.

Proof. Much as in the previous proof, $U_1(\mathbb{N}^*_{\min})$ is the trivial monoid while $U_2(\mathbb{N}^*_{\min})$ is isomorphic to the (commutative) additive semigroup of the semiring, so both are clearly strongly permutable, and there is a surjective morphism from $U_n(\mathbb{N}^*_{\min})$ to $U_3(\mathbb{N}^*_{\min})$ for all $n \geq 3$, so it suffices to show that $U_3(\mathbb{N}^*_{\min})$ is not strongly permutable.

To this end we define the sequence of matrices $C_1, \ldots, C_m$ given by

$$C_i = \begin{pmatrix} 0 & i-m-1 & -m-2 \\ -\infty & 0 & -i \\ -\infty & -\infty & 0 \end{pmatrix}$$

Once again, the $-\infty$ and 0 here are formally speaking the zero and identity elements in $(\mathbb{N}^*_{\min})^{01}$. The product of the first k such matrices is inductively seen to be

$$\prod_{i=1}^{k} C_i = \begin{pmatrix} 0 & k-m-1 & -m-2 \\ -\infty & 0 & -1 \\ -\infty & -\infty & 0 \end{pmatrix}$$

Now, if $\sigma \in \mathcal{S}_m$ is such that $C_\sigma := \prod_{i=1}^{m} C_{\sigma(i)} = \prod_{i=1}^{m} C_i$ then for all $j < k$,

$$(C_\sigma)_{1,3} \;=\; -m-2 \;\geq\; (C_{\sigma(j)})_{1,2} + (C_{\sigma(k)})_{2,3} \;=\; \sigma(j) - m - 1 - \sigma(k)$$

so that $\sigma(j) < \sigma(k)$. Since σ is a permutation, this can only happen if σ is the identity permutation. Further, as m was arbitrary no non-trivial permutations preserve this product for any $m \in \mathbb{N}$, so $U_3(\mathbb{N}^*_{\min})$ is not strongly permutable. □

Lemma 3.4.5. *Let S be a (not necessarily commutative) bipotent semiring. If S has an element of infinite multiplicative order, then $M_n(S)$ and $UT_n(S)$ are not strongly permutable for $n \geq 2$ and $U_n(S)$ is not strongly permutable if and only if $n \geq 3$.*

Proof. Suppose $a \in S$ has infinite order. Then by Lemma 3.2.5 we have that subsemiring generated by a is isomorphic to $\mathbb{N}^*_{\max}$ or $\mathbb{N}^*_{\min}$. Hence, $M_n(S)$ contains an embedded copy either of $M_n(\mathbb{N}^*_{\max})$ or of $M_n(\mathbb{N}^*_{\min})$; since neither of these are permutable for $n \geq 2$ by Theorem 3.4.2, $M_n(S)$ is not permutable for $n \geq 2$. Similarly, $UT_n(S)$ is not permutable for $n \geq 2$ using Theorem 3.4.2 and $U_n(S)$ is not permutable if and only if $n \geq 3$ using Lemma 3.4.3 and Lemma 3.4.4. □

A bipotent semiring (even a commutative one) may have elements of unbounded finite order, without having an element of infinite order. For example, we shall see below that the truncated tropical semiring $\mathbb{T}_{[0,1]}$ is such a semiring. Some basic model theory allows us to extend the above result to this case; we direct the reader unfamiliar with model theoretic techniques to [Kir19], for example.

Theorem 3.4.6. *Let S be a (not necessarily commutative) bipotent semiring with elements of unbounded multiplicative order (that is, such that for all $k \in \mathbb{N}$ there exists an $x \in S$ such that x has multiplicative order greater than k). Then the semigroups $M_n(S)$ and $UT_n(S)$ are not strongly permutable for $n \geq 2$. The semigroup $U_n(S)$ is not strongly permutable if and only if $n \geq 3$.*

Proof. Consider the set of first-order sentences in the language of semirings:

$$L = \{x^m \neq x^n \mid m, n \in \mathbb{N}, m \neq n\}$$

where x is a variable and x^m is shorthand for the product of m copies of x. Since S has elements of unbounded order, L is finitely satisfiable (every finite subset of L holds for some $x \in S$) which means that L is a 1-type of S.

By realisability of types (see for example [Kir19, Lemma 23.6]) there exists an elementary extension of S (a structure containing S and satisfying exactly the same first-order theory) in which L is satisfiable, that is, in which there is an element x satisfying all of the sentences in L. Let T be such a structure and $x \in T$ such an element. The axioms for a bipotent semiring are clearly all expressible as first-order sentences, so the structure T is itself a bipotent semiring. Moreover, since x satisfies all sentences in L, x is an element of infinite order, and so by Lemma 3.4.5 we deduce that $M_n(T)$ is not permutable for all $n \geq 2$.

Now suppose for a contradiction that $M_n(S)$ was strongly permutable for some $n \geq 2$. This means there exists an m such that

$$\forall X_1, \ldots, X_m \in M_n(S), \bigvee_{\sigma \in \mathcal{S}_m \setminus \{1_m\}} X_1 \cdots X_m = X_{\sigma(1)} \cdots X_{\sigma(m)}.$$

Since matrix multiplication is first-order definable in the language of semirings, this can clearly be re-expressed as a first-order sentence over S, featuring mn^2 universally quantified scalar variables corresponding to the entries of the m matrices. But T is elementary equivalent to S, so this sentence also holds in T, which contradicts the fact that $M_n(T)$ is not permutable.

Near-identical arguments show that $UT_n(S)$ is not permutable for $n \geq 2$ and that $U_n(S)$ is not permutable for $n \geq 3$. Finally, recall that $U_1(S)$ is trivial while $U_2(S)$ is isomorphic to the additive semigroup of S, which is always commutative and hence strongly permutable. □

Recall that $M_1(S) = UT_1(S)$ is isomorphic to the multiplicative semigroup of the semiring S. This may be permutable (for example when the semiring is commutative) or non-permutable (for example when S is a non-commutative free monoid with a bipotent addition given by the shortlex total ordering).

Corollary 3.4.7. *Let S be a commutative bipotent semiring with elements of unbounded multiplicative order. Then $M_n(S)$ (and $UT_n(S)$) are strongly permutable if and only if $n = 1$.*

Recall that a *semifield* is a commutative semiring, possibly without zero, where the non-zero elements form an abelian group with multiplication. In the case of semifields, we can now give an explicit description of when the matrix semigroups are permutable.

Corollary 3.4.8. *Let S be a bipotent semifield. Then $M_n(S)$ and $UT_n(S)$ are permutable for $n \geq 2$ (and $U_n(S)$ is permutable for $n \geq 3$) if and only if S is the 2-element boolean semifield.*

Proof. Since S is a bipotent semiring we have that every element has infinite order or period 1 by Lemma 3.2.4. However, S is a semifield, so the non-zero elements form a group with multiplication so the only possible elements of period 1 are the identity and the zero if there is one. Thus, non-identity, non-zero elements are of infinite order. Therefore if S is not the 2-element boolean semifield, it must have an element of infinite order and thus by Theorem 3.4.6 (or Lemma 3.4.5), $M_n(S)$ and $UT_n(S)$ are not permutable for $n \geq 2$ and $U_n(S)$ is not permutable for $n \geq 3$. If $\mathbb{B}$ is the 2-element boolean semifield then $M_n(\mathbb{B})$, $UT_n(\mathbb{B})$, and $U_n(\mathbb{B})$ are finite and hence permutable for all $n \in \mathbb{N}$. □

Theorem 3.4.9. *Suppose S is a (not necessarily commutative or bipotent) semiring with the following property: for every finite subset $X \subseteq S$, there exists a homomorphism to a finite semiring of order bounded by a function in the size of X such that each element of X occupies its own singleton kernel class. Then $M_n(S)$ is permutable for all $n \in \mathbb{N}$.*

Proof. Let k be such that for every subset X of S with $|X| = n^2$, there is a homomorphism from S to a finite semiring of size at most k such that each element of X occupies its own singleton kernel class. Let $m = k^{n^2} + 1$, and suppose

$$\Sigma = A_1 A_2 \cdots A_m = \begin{pmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{n,1} & \cdots & x_{n,n} \end{pmatrix}$$

for some $A_1, \dots, A_m \in M_n(S)$. Taking $X = \{x_{1,1}, \dots, x_{n,n}\}$, by assumption we may choose a semiring homomorphism ϕ mapping S into a semiring F of cardinality at most k, such that each $x_{i,j}$ occupies its own singleton kernel class. From this semiring homomorphism, we define a semigroup homomorphism ψ mapping $M_n(S)$ into $M_n(F)$ where

$$(\psi(A))_{i,j} = \phi(A_{i,j}) \text{ for all } i, j.$$

Notice that, since the entries of Σ each occupy their own singleton ϕ-kernel class, Σ occupies its own singleton ψ-kernel class. Since F has cardinality at most k, $M_n(F)$ has cardinality at most $k^{n^2} < m$, so there must exist distinct i and j with $\psi(A_i) = \psi(A_j)$. Let $\sigma \in \mathcal{S}_m$ be the transposition swapping i and j. Then clearly

$$\psi(A_{\sigma(1)} \cdots A_{\sigma(m)}) = \psi(A_{\sigma(1)}) \dots \psi(A_{\sigma(m)}) = \psi(A_1) \cdots \psi(A_m) = \psi(\Sigma),$$

which since Σ occupies its own singleton ψ-kernel class means that

$$A_{\sigma(1)} \cdots A_{\sigma(m)} \;=\; \Sigma \;=\; A_1 \cdots A_m,$$

as required to show that $M_n(S)$ is permutable. □

Recall that we say a binary relation $\cong$ on a semiring is a *congruence* if $\cong$ is an equivalence relation and if $a \cong b$ and $c \cong d$ together imply that $ac \cong bd$ and $a+c \cong b+d$.

Corollary 3.4.10. *Let S be a chain semiring (that is, a totally ordered set with operations maximum and minimum). Then $M_n(S)$ is permutable for all $n \in \mathbb{N}$.*

Proof. Let X be a finite subset of S. Define a binary relation $\equiv$ on S by $a \equiv b$ if and only if a and b either (i) are equal or (ii) are not in X and lie above exactly the same elements of X. Recalling that S is totally ordered, it is easy to see that $\equiv$ is an equivalence relation with at most $2|X| + 1$ classes (being the singleton sets containing elements of X, and the open order intervals above, below and between elements of X), in which each element of X occupies its own equivalence class. Further, it can be readily seen that $\equiv$ is a congruence. Hence, by the usual first isomorphism theorem for semirings, the natural morphism $S \to S/\equiv$ satisfies the conditions of Theorem 3.4.9. □

Chapter 4

Truncated Tropical Semirings

In this chapter, we introduce truncated tropical semirings. For $x, y \in \mathbb{R}$ with $0 \leq x < y$, the *truncated tropical semiring* $\mathbb{T}_{[x,y]}$ consists of the real interval $[x, y]$ augmented with 0 and $-\infty$ with operations maximum and *y-truncated addition* given by $ab = \min(a + b, y)$ where $+$ here denotes classical addition. These semirings are a new class of semirings which have not as of yet been well studied, but have many interesting properties. For example Kambites [Kam22] showed that while $UT_n(\mathbb{T}_{[0,1]})$ is locally finite for all $n \in \mathbb{N}$, the variety generated by $UT_n(\mathbb{T}_{[0,1]})$ is not locally finite for any $n \in \mathbb{N}$, that is, each variety contains a finitely generated infinite semigroup.

These semirings came about as an infinite generalisation of truncated tropical natural number semirings $[k]^*_{\max}$, (sometimes augmented with a zero or identity adjoined). These latter semirings have been very well studied, for example $[1]^*_{\max}$ with a zero adjoined is isomorphic to the boolean semiring.

In the first section of this chapter, we give a brief introduction to truncated tropical semirings, and then go on to give a complete classification of all isomorphisms between truncated tropical semirings. In the next section, we turn our attention back to strong permutability, where it transpires that the full matrix semigroups can be strongly permutable in all dimensions (Theorem 4.2.2), only in dimension 1 (Corollary 4.2.1) or, interestingly, only in dimensions 1 and 2 (Theorem 4.2.6). Similar results are obtained for the monoids of upper triangular and upper unitriangular matrices.

This chapter is based on joint work with my supervisor, Mark Kambites [AK22].

4.1 Classification of Truncated Tropical Semirings

To avoid confusion with classical operations, which we shall also need, we use the symbols $\oplus$ and $\otimes$ to denote the addition (maximum) and multiplication (truncated addition) operations in a truncated tropical semiring. The symbol + and juxtaposition will be used for standard arithmetic addition and multiplication of real numbers, respectively. We begin by observing that there are a number of isomorphisms between semirings in this class:

Theorem 4.1.1. *Let $y > x \geq 0$. Then*

$$\mathbb{T}_{[x,y]} \cong \begin{cases} \mathbb{T}_{[0,1]} & \text{if } x = 0; \\ \mathbb{T}_{[1,2]} & \text{if } x > 0 \text{ and } y \leq 2x; \\ \mathbb{T}_{[1,2.5]} & \text{if } x > 0 \text{ and } 2x < y < 3x; \\ \mathbb{T}_{[1,\frac{y}{x}]} & \text{if } x > 0 \text{ and } y \geq 3x. \end{cases}$$

The semirings $\mathbb{T}_{[0,1]}, \mathbb{T}_{[1,2]}, \mathbb{T}_{[1,2.5]}$ and $\mathbb{T}_{[1,y]}$ for $y \geq 3$ are pairwise non-isomorphic.

Proof. If $x = 0$, we define the map $\phi : \mathbb{T}_{[0,y]} \to \mathbb{T}_{[0,1]}$ by

$$\phi(-\infty) = -\infty \text{ and } \phi(z) = \frac{z}{y} \text{ for } z \in [0, y].$$

Using the fact that classical multiplication distributes over classical addition, and that $y > 0$ implies that ϕ is order preserving, it can be easily seen that ϕ is an isomorphism.

If $x > 0$ and $y \leq 2x$, we define the map $\phi : \mathbb{T}_{[x,y]} \to \mathbb{T}_{[1,2]}$ by

$$\phi(-\infty) = -\infty,\ \phi(0) = 0,\ \text{and } \phi(z) = \frac{z-x}{y-x} + 1 \text{ for } z \in [x, y].$$

Now, for $a, b \in [x, y]$, we have that

$$\phi(a) \otimes \phi(b) = \min\left(\frac{a-x}{y-x} + 1 + \frac{b-x}{y-x} + 1, 2\right) = 2 = \phi(a \otimes b)$$

as $a, b \geq x$. Moreover, as $y - x > 0$, ϕ is order preserving. Hence, it can be easily seen that ϕ is an isomorphism.

If $x > 0$ and $2x < y < 3x$, we define a piecewise linear map $\phi : \mathbb{T}_{[x,y]} \to \mathbb{T}_{[1,2.5]}$ by

$$\phi(z) = \begin{cases} \frac{z-2x}{2(y-2x)} + 2 & \text{if } 2x \leq z \leq y \\ \frac{z-(y-x)}{2(3x-y)} + 1.5 & \text{if } y - x < z < 2x \\ \frac{z-x}{2(y-2x)} + 1 & \text{if } x \leq z \leq y - x \\ 0 & \text{if } z = 0 \\ -\infty & \text{if } z = -\infty \end{cases}$$

Now, for $a \in [y - x, y]$ and $b \in [x, y]$, we have that

$$\phi(a) \otimes \phi(b) = 2.5 = \phi(y) = \phi(a \otimes b)$$

as $\phi(a) \geq 1.5$ and $\phi(b) \geq 1$. Finally, if $a, b \in [x, y - x]$ then

$$\begin{aligned} \phi(a) \otimes \phi(b) &= \min\left(\frac{a-x}{2(y-2x)} + 1 + \frac{b-x}{2(y-2x)} + 1, 2.5\right) \\ &= \min\left(\frac{(a+b)-2x}{2(y-2x)} + 2, \frac{y-2x}{2(y-2x)} + 2\right) \\ &= \frac{\min(a+b, y) - 2x}{2(y-2x)} + 2 \\ &= \phi(a \otimes b) \end{aligned}$$

as $a \otimes b \geq 2x$. Moreover, as $y - 2x > 0$ and $3x - y > 0$ this implies that ϕ is order preserving, and hence it can be easily seen that ϕ is an isomorphism.

If $x > 0$ and $y > 3$ then we define a map ϕ from $\mathbb{T}_{[x,y]}$ to $\mathbb{T}_{[1,\frac{y}{x}]}$ by

$$\phi(-\infty) = -\infty, \ \phi(0) = 0, \text{ and } \phi(z) = \frac{z}{x} \text{ for } z \in [x, y].$$

Using the fact that classical multiplication distributes over classical addition, and that $x > 0$ implies that ϕ is order preserving, it can be easily seen that ϕ is an isomorphism.

It remains to show that $\mathbb{T}_{[0,1]}, \mathbb{T}_{[1,2]}, \mathbb{T}_{[1,2.5]}$ and $\mathbb{T}_{[1,y]}$ for $y \geq 3$ are pairwise non-isomorphic. We can see that $\mathbb{T}_{[0,1]}$ is not isomorphic to any of the others, as it is the only one with unbounded multiplicative order. Similarly, $\mathbb{T}_{[1,y]}$ has no elements of multiplicative order 3 if and only if $y \leq 2$ (for $y > 2$ consider $1 + \frac{y-2}{3}$), so $\mathbb{T}_{[1,2]}$ is not isomorphic to the others. For $\mathbb{T}_{[1,2.5]}$, note that $\mathbb{T}_{[1,y]}$ has no elements of multiplicative order 4 if and only if $y \leq 3$ (for $y > 3$ consider $1+\frac{y-3}{4}$), so $\mathbb{T}_{[1,2.5]}$ can not be isomorphic to any of the others apart from perhaps $\mathbb{T}_{[1,3]}$.

For a contradiction, suppose that $\mathbb{T}_{[1,2.5]}$ is isomorphic to $\mathbb{T}_{[1,3]}$ and let $\phi : \mathbb{T}_{[1,2.5]} \to \mathbb{T}_{[1,3]}$ be an isomorphism. As ϕ is order-preserving, we have that $\phi(1) = 1$ and $\phi(2.5) = 3$. Similarly, as ϕ preserves the semiring multiplication, we can conclude that

$$\phi(2) = \phi(1) \otimes \phi(1) = 2 \text{ and } \phi(1.5) \otimes 1 = \phi(1.5) \otimes \phi(1) = \phi(2.5) = 3$$

and hence $\phi(1.5) \geq 2 = \phi(2)$ contradicting that ϕ is order-preserving. Hence, $\mathbb{T}_{[1,3]}$ and $\mathbb{T}_{[1,2.5]}$ are not isomorphic.

Finally, suppose $z \geq y \geq 3$ and let $\phi \colon \mathbb{T}_{[1,y]} \to \mathbb{T}_{[1,z]}$ be an isomorphism. From the fact that ϕ is a morphism and the definition of multiplication in the two semirings, we have $\phi(a + b) = \phi(a) + \phi(b)$ for all a, b with $a + b \leq y$, and $\phi(1) = 1$. Hence, $\phi(2) = \phi(1 + 1) = \phi(1) + \phi(1) = 2$, and for $1 \leq x \leq y - 1$,

$$\phi(x) = \phi(x + 1 - 1) = \phi(x + 1) - \phi(1) = \phi\left(\frac{x+1}{2}\right) + \phi\left(\frac{x+1}{2}\right) - 1.$$

We show by a simple inductive argument using this fact that $\phi(1 + 2^{-n}) = 1 + 2^{-n}$ for all $n \in \mathbb{N} \cup \{0\}$. Indeed, the base case is the fact that $\phi(2) = 2$, while if the claim holds for some n then taking $x = 1 + 2^{-n}$ we have $\frac{x+1}{2} = 1 + 2^{-(n+1)}$. Hence by the above $\phi(1 + 2^{-n}) = 2\phi(1 + 2^{-(n+1)}) - 1$, so

$$\phi(1 + 2^{-(n+1)}) = \frac{1}{2}(\phi(1 + 2^{-n}) + 1) = \frac{1}{2}(1 + 2^{-n} + 1) = 1 + 2^{-(n+1)}$$

and the claim holds for $n + 1$.

Note that for any a, b with $a + b \leq 1$ if $\phi(1 + a) = 1 + a$ and $\phi(1 + b) = 1 + b$ then $\phi(1 + a + b) = \phi(1 + a) + \phi(1 + b) - \phi(1) = 1 + a + b$. By another simple induction, we deduce that ϕ fixes all finite sums of negative powers of 2 (in other words, all dyadic rationals) in the interval $[1, 2]$. Since the dyadic rationals are dense in the order, it follows that ϕ fixes everything in the interval $[1, 2]$.

Finally, since ϕ preserves the multiplication in $\mathbb{T}_{[1,y]}$ and $y < z$, it preserves all finite sums which sum to y or less. Since every element in $[1, y]$ is a finite sum of values in $[1, 2]$, it follows that ϕ is the identity function on $[1, y]$. Since it is surjective, this means that $y = z$. □

Here, we defined the truncated tropical semirings on non-negative intervals, however, you can equally define truncated tropical semirings on non-positive intervals. So we pose the following question.

Question 4.1.2. *Let $x, y \in \mathbb{R}$ with $x < y \leq 0$. Is there a similar classification for $\mathbb{T}_{[x,y]}$?*

4.2 Permutability of Matrices over Truncated Tropical Semirings

Recall that, $M_n(\mathbb{T}_{[x,y]})$ is weakly permutable for all $n \in \mathbb{N}$ by Proposition 3.3.1. Thus, in this section, we look at strong permutability and illustrate some of the "wilder" behaviour which is possible in commutative bipotent semirings, by studying truncated tropical semirings.

To begin we observe that, as a consequence of our earlier results, there are examples of such semirings for which matrix semigroups are not permutable in any rank greater than 1:

Corollary 4.2.1. *The semigroup $M_n(\mathbb{T}_{[0,1]})$ is permutable if and only if $n = 1$.*

Proof. The semigroup $M_1(\mathbb{T}_{[0,1]})$ is commutative and therefore strongly permutable. For $n > 1$, it is easy to see that $\mathbb{T}_{[0,1]}$ has elements of unbounded multiplicative order (indeed, for any $j \in \mathbb{N}$ the element $1/j$ has order j), so $M_n(\mathbb{T}_{[0,1]})$ is not strongly permutable by Theorem 3.4.6. □

By Theorem 4.1.1, we can now always take truncated tropical semirings to be either of the form $\mathbb{T}_{[0,1]}$ or $\mathbb{T}_{[1,z]}$ for $z = 2$, 2.5 or $z \geq 3$. Corollary 4.2.1 gives a full description of when the matrix semigroups $M_n(\mathbb{T}_{[0,1]})$ are permutable, so we now focus on matrix semigroups of form $M_n(\mathbb{T}_{[1,z]})$ for some $z = 2$, 2.5 or $z \geq 3$.

Theorem 4.2.2. *$M_n(\mathbb{T}_{[1,2]})$ is strongly permutable for all $n \in \mathbb{N}$.*

Proof. We shall show that $\mathbb{T}_{[1,2]}$ satisfies the hypothesis of Theorem 3.4.9. Let $X = \{x_1, \ldots, x_k\}$ be a finite subset of $\mathbb{T}_{[1,2]}$ and $X' = X \cup \{0, -\infty\}$. Define a binary relation $\equiv$ on $\mathbb{T}_{[1,2]}$ by $a \equiv b$ if and only if a and b either (i) are equal or (ii) are not in X' and lie above exactly the same elements of X'. It is easy to see that $\equiv$ is an equivalence relation with at most $2|X| + 3$ classes, in which each element of X lies in a singleton equivalence class.

We must now show that $\equiv$ is a congruence. As $\mathbb{T}_{[1,2]}$ is commutative, we only have to show that $\equiv$ is a left congruence. Let $x, y \in \mathbb{T}_{[1,2]}$ and $x \equiv y$, so x and y lie above exactly the same elements of X'. Clearly, if $a = 0$ or $a = -\infty$, we have that $a \otimes x \equiv a \otimes y$ and $a \oplus x \equiv a \oplus y$. Moreover, if $x = y$, we have that $a \otimes x \equiv a \otimes y$ and $a \oplus x \equiv a \oplus y$. Hence, as $0, -\infty \in X'$, we can assume that $a, x, y \geq 1$, and thus $a \otimes x = 2 = a \otimes y$.

Further, if $a \geq \max(x, y)$ then $a \oplus x \equiv a \equiv a \oplus y$ and if $a \leq \min(x, y)$, then $a \oplus x \equiv x \equiv y \equiv a \oplus y$. On the other hand, if a lies between x and y in the order then since x and y lie above the same elements of X', we have that a, x, y, $a \oplus x$ and $a \oplus y$ all lie above exactly the same elements of X', giving that $a \oplus x \equiv a \oplus y$. Thus we conclude that $\equiv$ is a congruence.

Hence, by the usual first isomorphism theorem for semirings, the natural morphism $\mathbb{T}_{[1,2]} \to \mathbb{T}_{[1,2]}/\!\equiv$ satisfies the conditions of Theorem 3.4.9, and $M_n(\mathbb{T}_{[1,2]})$ is strongly permutable for all $n \in \mathbb{N}$. □

The rest of this section treats the remaining truncated tropical semirings, that is, those of the form $\mathbb{T}_{[1,z]}$ with $z > 2$. These will give examples of semirings S such that $M_2(S)$ is strongly permutable, but $M_n(S)$ is not strongly permutable for all $n \geq 3$. We use the notation $\lceil z \rceil$ to denote the smallest integer greater than or equal to $z \in \mathbb{R}$. We shall say that a semigroup S is k-permutable if for every $s_1, \ldots, s_k \in S$ there exists a non-trivial permutation $\sigma \in \mathcal{S}_k$ such that $s_{\sigma(1)} s_{\sigma(2)} \cdots s_{\sigma(k)} = s_1 s_2 \cdots s_k$. Note that if a semigroup S is k-permutable then S is j-permutable for all $j \geq k$.

Lemma 4.2.3. *For $z > 2$, let P and P' be subsemigroups of $M_2(\mathbb{T}_{[1,z]})$ given by*

$$P = \left\{ \begin{pmatrix} 0 & a \\ -\infty & b \end{pmatrix} : a, b \in \mathbb{T}_{[1,z]} \right\} \text{ and } P' = \left\{ \begin{pmatrix} 0 & -\infty \\ a & b \end{pmatrix} : a, b \in \mathbb{T}_{[1,z]} \right\}.$$

Then P and P' are both $(2\lceil z \rceil + 5)$-permutable.

Proof. Transposing matrices is a semigroup anti-isomorphism between P and P', so it suffices to prove that P is $(2\lceil z \rceil + 5)$-permutable.

Let $m = 2\lceil z \rceil + 5$ and let $X_1, \ldots, X_m \in P$. If $(X_t)_{2,2} = -\infty$ for any $t > 2$ then, as X_t is a right zero of P, and we have that $X_1 X_2 \cdots X_m = X_2 X_1 \cdots X_m$. Thus we may assume $(X_t)_{2,2} \neq -\infty$ for all $t > 2$.

If $(X_t)_{1,2}, (X_{t+1})_{1,2} = -\infty$ for some $t < m$ then as diagonal matrices commute, we have $X_1 \cdots X_t X_{t+1} \cdots X_m = X_1 \cdots X_{t+1} X_t \cdots X_m$. Therefore, we may assume either $(X_2)_{1,2} \neq -\infty$ or $(X_3)_{1,2} \neq -\infty$. Combined with the assumption from the previous paragraph, this implies we may assume that $(X_1 \cdots X_m)_{1,2} \neq -\infty$.

If $(X_t)_{2,2}, (X_{t+1})_{2,2} = 0$ for some $t < m$ then, because 2×2 unitriangular matrices commute, we have $X_1 \cdots X_t X_{t+1} \cdots X_m = X_1 \cdots X_{t+1} X_t \cdots X_m$. Hence, we may assume that among every pair of every two consecutive matrices (except perhaps the first three) there is a matrix X_t with $(X_t)_{2,2} \geq 1$. Since $m = 2\lceil z \rceil + 5$ this means we have $(X_1 \cdots X_{m-2})_{1,2} = z$ and $(X_1 \cdots X_{m-2})_{2,2} \in \{z, -\infty\}$. In both of these cases $X_1 \cdots X_{m-2}$ acts as a left zero for all matrices M with $M_{2,2} \neq -\infty$. But we assumed $(X_t)_{2,2} \neq -\infty$ for $t > 2$, so we have

$$X_1 \cdots X_m = X_1 \cdots X_{m-2} X_{m-1} X_m = X_1 \cdots X_{m-2} X_m X_{m-1}.$$

Thus P, and hence also P', is $(2\lceil z \rceil + 5)$-permutable. □

Lemma 4.2.4. *Let $A_0 \in M_2(\mathbb{T}_{[1,z]})$ and m be the minimum finite entry of A_0 (or $m = z$ if A_0 if all entries are $-\infty$). Let $k \geq 17(16\lceil z \rceil + 45)$. Then for all $A_1, \ldots, A_k \in M_2(\mathbb{T}_{[1,z]})$, either*

$$(A_0 A_1 \cdots A_k)_{i,j} \neq m \text{ for all } i, j$$

or there exists a non-trivial $\sigma \in \mathcal{S}_k$ such that

$$A_0 A_1 A_2 \cdots A_k = A_0 A_{\sigma(1)} A_{\sigma(2)} \cdots A_{\sigma(k)}$$

Proof. Consider a product $A_0 A_1 \ldots A_k$. If $m = z$, then every entry of A_0 is either z or $-\infty$. There are 16 matrices in $M_2(\mathbb{T}_{[1,z]})$ in which every entry is either z or $-\infty$ and they form an ideal of $M_2(\mathbb{T}_{[1,z]})$. As $k \geq 3 \cdot 16 = 48$, by the pigeonhole principle, there exist $0 \leq k_1, k_2, k_3 \leq 48$ such that $A_0 \cdots A_{k_i} = M$ for $i = 1, 2, 3$ for $M \in M_2(\mathbb{T}_{[1,z]})$ with every entry being either z or $-\infty$. Finally, note that

$$A_0 \cdots A_{k_1}(A_{k_2+1} \cdots A_{k_3-1})(A_{k_1+1} \cdots A_{k_2}) A_{k_3} \cdots A_k = M$$

and hence, we have found a non-trivial $\sigma \in \mathcal{S}_k$ which preserves the product.

Now, suppose $m \neq z$. If no entry of the product is equal to m we are done. Moreover, as $M_2(\mathbb{T}_{[m,z]} \setminus \{0\})$ is an ideal of $M_2(\mathbb{T}_{[1,z]})$ for all $m \in [1, z]$, we may suppose every truncated product $A_0 A_1 \dots A_p$ with $0 \leq p \leq k$ has at least one entry equal to m.

By the pigeonhole principle there exists a sequence of indices $0 \leq i_0 < \cdots < i_n \leq k$ where $n = \lceil \frac{k}{4} \rceil - 1$ such that each product matrix $A_0 A_1 \cdots A_{i_j}$ has an m in the same position. If this is the (1,2) or the (2,1) position then note that swapping the rows of A_0 swaps the rows of the product $A_0 A_1 \cdots A_t$ for all $t \leq k$. Therefore if σ is a permutation that does not change the product, then σ will also preserve the product obtained by swapping A_0's rows. Hence, we can assume that the m's are in the (1,1) or (2,2) position. Moreover, by relabelling the rows and columns if necessary, we can assume without loss of generality that $A_0 A_1 \cdots A_{i_j}$ has m in the (1,1) position for all $0 \leq j \leq n$.

Now consider the matrices defined by

$$B = A_0 \cdots A_{i_0} \text{ and } B_j = A_{i_{j-1}+1} \cdots A_{i_j}$$

for $1 \leq j \leq n$. Any permutation of this sequence which does not change the product clearly yields a permutation of the original sequence which does not change the product, so it is enough to seek a non-trivial permutation of this sequence. We define the truncated products $\Pi_t := B B_1 \cdots B_t$ for $0 \leq t \leq n$. By the assumption of the previous paragraph, we have $(\Pi_t)_{1,1} = m$ for all $0 \leq t \leq n$.

First we consider any B_i whose entries are all either 0 or $-\infty$. There are only 16 distinct matrices of this form, so if more than 16 of the B_i have this form then the same matrix would appear twice in the sequence, resulting in a non-trivial permutation that preserves the product. Otherwise, since $n = \lceil \frac{k}{4} \rceil - 1 > 17(4\lceil z \rceil + 11)$ the B_i contain a subsequence of $4\lceil z \rceil + 11$ *consecutive* matrices not of this form, say $B_p, \dots, B_q$ where $q - p = 4\lceil z \rceil + 10$.

We now define five subsets of $M_2(\mathbb{T}_{[1,z]})$:

$$P = \left\{ \begin{pmatrix} 0 & a \\ -\infty & b \end{pmatrix} : a, b \in \mathbb{T}_{[1,z]} \right\}, \qquad P' = \left\{ \begin{pmatrix} 0 & -\infty \\ a & b \end{pmatrix} : a, b \in \mathbb{T}_{[1,z]} \right\},$$

$$T = \left\{ \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} : a, b \in \mathbb{T}_{[1,z]} \right\}, \qquad U = \left\{ \begin{pmatrix} -\infty & a \\ 0 & b \end{pmatrix} : a, b \in \mathbb{T}_{[1,z]} \right\},$$

$$V = \left\{ \begin{pmatrix} 0 & c \\ a & b \end{pmatrix} : a, b, c \in \mathbb{T}_{[1,z]} \right\}.$$

We shall show that the sequence $B_p, \ldots, B_q$ contains $2\lceil z \rceil + 5$ consecutive matrices either all in P or all in P'. From this it will follow by Lemma 4.2.3 that there is a permutation of the sequence which preserves the product, as required.

Note that $P, P', T \subseteq V$. For $p \leq t \leq q-1$, we have that $(\Pi_t)_{1,1} = (\Pi_{t+1})_{1,1} = m$. So, if $(\Pi_t)_{1,2} = -\infty$, then in order to ensure $(\Pi_t B_{t+1})_{1,1} = (\Pi_{t+1})_{1,1} = m$ we must have $(B_{t+1})_{1,1} = 0$, that is, $B_{t+1} \in V$. Similarly, if $(\Pi_t)_{1,2} = m$, then $B_{t+1} \in P, T$ or U. Otherwise, $(\Pi_t)_{1,2} > m$ and we have that $B_{t+1} \in P$.

If the matrices $B_p, \ldots, B_{p+2\lceil z \rceil+4}$ are all in P' then we are done. Otherwise, choose t with $p \leq t \leq p + 2\lceil z \rceil + 4$ such that $B_t \notin P'$. Since $(\Pi_{t-1})_{1,1} = m$ and $\Pi_t = \Pi_{t-1}B_t$, this means that $(\Pi_t)_{1,2} \neq -\infty$.

Now because $(\Pi_t)_{11}, (\Pi_t)_{1,2} \geq m$ and B_{t+1} lies in P, T or U with (because of the assumption that the entries of B_{t+1} are not all 0 and $-\infty$) either $(B_{t+1})_{1,2} \geq 1$ or $(B_{t+1})_{2,2} \geq 1$, we have that $(\Pi_{t+1})_{1,2} > m$ and of course by definition we have $(\Pi_{t+1})_{1,1} \geq m$. Continuing by induction we deduce that $(\Pi_i)_{1,2} > m$ for all i with $t+1 \leq i \leq q$. By the remarks in the last paragraph but one, this means that $B_j \in P$ for all $t+2 \leq j \leq q$, which means the matrices $B_{t+2}, \ldots, B_{t+1+2\lceil z \rceil+5}$ are all in P, as required. □

Theorem 4.2.5. *Let $z > 2$. Then $M_2(\mathbb{T}_{[1,z]})$ is strongly permutable.*

Proof. Consider a product of matrices $A_1 \cdots A_n$ for $n \geq 17(4\lceil z \rceil + 1)(16\lceil z \rceil + 45)$ and let m_t be the smallest finite entry in the product of the first t matrices $\Pi_t = A_1 \cdots A_t$. (If all entries of Π_t are $-\infty$, we define $m_t = z$). Note that $m_1 \leq \cdots \leq m_n$ as $M_2(\mathbb{T}_{[x,z]} \setminus \{0\})$ is an ideal of $M_2(\mathbb{T}_{[1,z]})$ for all $x \in [1, z]$. Further, let $k_1 = 1$ and $k_2, \ldots, k_s$ be all the indices such that $m_{k_j - 1} < m_{k_j}$. For a contradiction, suppose that there does not exist a non-trivial permutation $\sigma \in \mathcal{S}_n$ such that $A_1 \cdots A_n = A_{\sigma(1)} \cdots A_{\sigma(n)}$. Then, by letting $A_0 = \Pi_{k_{j-1}}$ and applying Lemma 4.2.4 to $A_0 A_{k_{j-1}+1} \cdots A_n$, we have that $s > 1$ and $k_j - k_{j-1} < 17(16\lceil z \rceil + 45)$ for all $j > 1$ as there is no permutation preserving the product $A_1 \cdots A_n$ by assumption.

We aim to show that $s \leq 4\lceil z \rceil + 1$, so suppose $s \geq 5$. Then, for any $1 \leq j < s-4$, consider the five values $m_{k_j} < m_{k_{j+1}} < m_{k_{j+2}} < m_{k_{j+3}} < m_{k_{j+4}}$ and suppose $m_{k_{j+4}} \neq z$. It is easy to see that each of these five values is either an entry of the matrix Π_{k_j}, or else exceeds m_{k_j} by at least 1. Since there are not five distinct entries in Π_{k_j} we must therefore have $m_{k_{j+4}} \geq m_{k_j} + 1$ or $m_{k_{j+4}} = z$. Thus, as $0 \leq m_t \leq z$ for all

t, we have that $s \leq 4\lceil z \rceil + 1$. So as $n \geq 17(4\lceil z \rceil + 1)(16\lceil z \rceil + 45)$ we have that $k_j - k_{j-1} \geq 17(16\lceil z \rceil + 45)$ for some $2 \leq j \leq s$, giving a contradiction. Therefore, $M_2(\mathbb{T}_{[1,2]})$ is strongly permutable. □

Theorem 4.2.6. *Let $z > 2$. Then $M_n(\mathbb{T}_{[1,z]})$, $UT_n(\mathbb{T}_{[1,z]})$, and $U_n(\mathbb{T}_{[1,z]})$ are strongly permutable if and only if $n \leq 2$.*

Proof. If $n \leq 2$ then $M_2(\mathbb{T}_{[1,z]})$ is strongly permutable by Theorem 4.2.5 and $UT_n(\mathbb{T}_{[1,z]})$ and $U_n(\mathbb{T}_{[1,z]})$ are strongly permutable as they are subsemigroups of $M_n(\mathbb{T}_{[1,z]})$. For the direct implication it suffices, by Proposition 3.4.1, to show that $U_3(\mathbb{T}_{[1,z]})$ is not permutable. We do this by a variation of the method used to prove Lemma 3.4.3 above.

Choose ε with $0 < \varepsilon < z - 2$. For a fixed m, we define a sequence of unitriangular matrices $B_1, B_2, \ldots, B_m$ by

$$B_i = \begin{pmatrix} 0 & 1 + \frac{i}{m}\varepsilon & 2 + \varepsilon \\ -\infty & 0 & 1 + \varepsilon - \frac{i-1}{m}\varepsilon \\ -\infty & -\infty & 0 \end{pmatrix}$$

By induction the product of the first k such matrices is given by

$$\prod_{i=1}^{k} B_i = \begin{pmatrix} 0 & 1 + \frac{k}{m}\varepsilon & 2 + \varepsilon \\ -\infty & 0 & 1 + \varepsilon \\ -\infty & -\infty & 0 \end{pmatrix}$$

Now, suppose $\sigma \in \mathcal{S}_m$ is such that $B_\sigma := \prod_{i=1}^{m} B_{\sigma(i)} = \prod_{i=1}^{m} B_i$. By the definition of matrix multiplication, for all $j < k$ we must have

$$2 + \varepsilon \;=\; (B_\sigma)_{1,3} \;\geq\; (B_{\sigma(j)})_{1,2} + (B_{\sigma(k)})_{2,3} \;=\; 2 + \varepsilon + \frac{\varepsilon}{m}(\sigma(j) - \sigma(k) + 1)$$

and hence $\sigma(j) < \sigma(k)$. Since σ is a permutation this means σ is the identity permutation. Further, as m was arbitrary $U_3(\mathbb{T}_{[1,z]})$ is not strongly permutable, and hence $UT_3(\mathbb{T}_{[1,z]})$ and $M_3(\mathbb{T}_{[1,z]})$ are also not strongly permutable. □

In Theorem 3.4.6, we showed that if S is a bipotent semiring with elements of unbounded multiplicative order, then $M_2(S)$ and $UT_2(S)$ are not strongly permutable. On the other hand, $\mathbb{T}_{[1,z]}$ with $z > 1$ is an example of a bipotent semiring with bounded multiplicative order and by Theorem 4.2.2 and Theorem 4.2.6, $M_2(\mathbb{T}_{[1,z]})$

and $UT_2(\mathbb{T}_{[1,z]})$ are strongly permutable. So, we pose the question of whether this is true for all bipotent semirings with bounded multiplicative order.

Question 4.2.7. *Let S be a bipotent semiring with bounded multiplicative order. Is $UT_2(S)$ strongly permutable? Is $M_2(S)$ strongly permutable?*

Chapter 5

Generating Sets and Presentations of Tropical Matrices

Constructing minimal and irredundant generating sets for semigroups is a widely studied area of research, see for example [AABK13, GR05]. This is related to the classical problem of calculating the *rank* of a semigroup, that is, the minimum cardinality of a generating set for a semigroup. This important invariant of a semigroup has again been widely researched, see for example [BGS19, Hui05].

Recently, there has been research into constructing minimal generating sets for matrix monoids, particularly semigroups of matrices over tropical semirings. East, Jonušas and Mitchell [EJM20] found generating sets for 2×2 full matrix monoids over the min-plus natural number semiring, the max-plus natural number semiring, the truncated tropical natural number semirings, and the truncated tropical negative natural number semirings. Subsequently, Hivert, Mitchell, Smith, and Wilson [HMSW21] found minimal generating sets for several submonoids of the monoid of boolean matrices and showed that the generating sets given in [EJM20] are minimal.

In this chapter, we construct minimal and irredundant generating sets for monoids of upper triangular and unitriangular matrices over commutative semirings and the monoid of 2×2 matrices over certain semifields. We have a particular focus on the tropical integer semiring, showing that the monoid of 3×3 matrices over the tropical integers is not finitely generated. Moving to monoid presentations, we show that the monoid of $n \times n$ upper triangular matrices over the tropical integer semiring is finitely presented for all $n \in \mathbb{N}$.

In addition to this introduction, this chapter comprises 4 sections. In Section 5.1, we introduce some notation and definitions that we use throughout the rest of this chapter.

In Section 5.2, we describe minimal and irredundant generating sets of the monoid of upper triangular matrices over a unital commutative semiring with a zero, showing that the monoid $UT_n(\mathbb{Z}_{\max})$ is finitely generated for all $n \in \mathbb{N}$. We then consider unitriangular matrices, describing the minimal and irredundant generating sets for the monoid of unitriangular matrices over a commutative semiring with a zero, and showing that $U_n(\mathbb{Z}_{\max})$ is not finitely generated for $n \geq 2$.

In Section 5.3, we turn our attention to full matrix monoids, first looking at the 2×2 full matrix monoid $M_2(\mathbb{Z}_{\max})$, showing that $M_2(\mathbb{Z}_{\max})$ is finitely generated, and constructing a minimal and irredundant generating set. Then, we look at the 3×3 full matrix monoid $M_3(S)$ over a semiring S and show that if S is an anti-negative semifield with a zero then $M_3(S)$ is finitely generated if and only if S is finite. We use this to show that $M_3(\mathbb{Z}_{\max})$ is not finitely generated. We then explicitly construct a minimal and irredundant generating set for $M_3(\mathbb{Z}_{\max})$ and show that the subsemigroup of $M_3(\mathbb{Z}_{\max})$ consisting of the matrices that can be expressed as products of regular matrices, is 4-generated.

In Section 5.4, we show that for all $n \in \mathbb{N}$, $UT_n(\mathbb{Z}_{\max})$ is finitely presented by showing that every word in the generators can be rewritten into a normal form over a finite alphabet. We then use this finite presentation to give a different finite presentation for $UT_n(\mathbb{Z}_{\max})$ using the minimal and irredundant generating set found in Section 5.2.

5.1 Preliminaries

For a semigroup $\mathcal{S}$, $X \subseteq \mathcal{S}$ is a *(semigroup) generating set* for $\mathcal{S}$, if for all $s \in \mathcal{S}$, there exists $x_1, \ldots, x_m \in X$ such that $s = x_1 \cdots x_m$. For a group $\mathcal{G}$, X is a *group generating set* for $\mathcal{G}$ if $X \cup X^{-1} \cup \{1_{\mathcal{G}}\}$ is a (semigroup) generating set for $\mathcal{G}$, where $X^{-1} = \{x^{-1} : x \in X\}$. We say a generating set X for $\mathcal{S}$ is *minimal* if $|X| \leq |Y|$ for any other generating set Y for $\mathcal{S}$ and say an element $x \in X$ is *irredundant* if $X \setminus \{x\}$ is not a generating set for $\mathcal{S}$. If every $x \in X$ is irredundant then we say X is *irredundant*. Moreover, we say S is *minimally generated by* X if X is a minimal generating set for

S and say S is *irredundantly generated by* X if X is an irredundant generating set for S. More generally, we say a set X is *minimal with a given property* if X has the property and $|X| \leq |Y|$ for any other set Y that has the property, and say a set is *irredundant with a given property* if X has the property and no proper subset of X has the property. We remark that, for a generating set X for $\mathcal{S}$, if X is minimal and finite then X is irredundant and if X is infinite and irredundant then X is minimal. Thus, in practice, it is often useful to discuss minimal generating sets in the finitely generated case and, if they exist, irredundant generating sets in the infinitely generated case.

We call $x \in \mathcal{S}$ a *unit* of a monoid $\mathcal{S}$ if there exists $x^{-1} \in \mathcal{S}$ such that $xx^{-1} = x^{-1}x = 1_{\mathcal{S}}$. Let $U(\mathcal{S})$ be the subgroup of $\mathcal{S}$ containing all units in $\mathcal{S}$; we call $U(\mathcal{S})$ the *group of units* of $\mathcal{S}$. Define a non-unit $x \in \mathcal{S}$ to be *prime* if for every product $x = uv$, exactly one of u or v is a unit. For a monoid $\mathcal{S}$, let $\mathcal{J}$ be Green's $\mathcal{J}$-relation, that is, the equivalence relation on $\mathcal{S}$ defined by $x \mathcal{J} y$ if and only if $\mathcal{S}x\mathcal{S} = \mathcal{S}y\mathcal{S}$. For $a \in \mathcal{S}$, denote the $\mathcal{J}$-class containing a by J_a and say J is a *prime* $\mathcal{J}$-class if every element of J is prime. It is easy to see that every generating set of $\mathcal{S}$ must contain a representative from each prime $\mathcal{J}$-class of $\mathcal{S}$.

For a unital semiring $(S, +, \cdot)$, let $U(S)$ be the *group of units* of $(S, \cdot)$. For a semiring S with a zero 0_S, we say $x \in S$ is *additively invertible* if there exists $y \in S$ such that $x + y = 0_S$ and define $V(S)$ to be the subset of additively invertible elements of S, i.e. the group of units of $(S, +)$. It is easy to see that for all $x, y \in V(S)$ and $z \in S$, $x + y \in V(S)$ and $zx, xz \in V(S)$. Note that $V(S)$ is a (possibly non-unital) ring and $V(S) = S$ if and only if $1_S \in V(S)$. Recall that, we say S is *anti-negative* if for all $x, y \in S$, $x + y = 0_S$ if and only if $x, y = 0_S$. Thus, S is anti-negative if and only if $V(S) = \{0_S\}$.

Finally, we standardise some notation for specific matrices. Let S be a semiring with a zero, then we let $I_n \in U_n(S)$ be the matrix with 1_S on the diagonal and 0_S everywhere else. Moreover, if S is unital, then $I_n \in UT_n(S) \subseteq M_n(S)$. Let $\lambda \in S$, then for $1 \leq i \leq n$, we let $A_i(\lambda) \in UT_n(S)$ be the diagonal matrix with 1_S on the diagonal apart from λ as the (i, i)th entry, and, for $1 \leq i < j \leq n$, let $E_{ij}(\lambda) \in UT_n(S)$ be the matrix where all diagonal entries are 1_S, $(E_{ij}(\lambda))_{ij} = \lambda$, and all other entries are 0_S. For convenience, we sometimes write E_{ij} to denote $E_{ij}(1_S)$. Recall that we denote the subsemiring of tropical integers by $\mathbb{Z}_{\max} = \mathbb{T} \cap (\mathbb{Z} \cup \{-\infty\})$.

To begin, we introduce two lemmas which we require for the following two sections.

Lemma 5.1.1. *Let S be a unital commutative semiring and $x, y \in S$. Then, xy is a unit if and only if x and y are units.*

Proof. If xy is a unit, then $xyt = 1_S$ for some $t \in S$. Then, by commutativity, $x(yt) = 1_S = (yt)x$ and $y(xt) = 1_S = (xt)y$, so x and y are units. If x and y are units, then $xs = sx = 1_S$ and $yt = ty = 1_S$ for some $s, t \in S$. Thus,

$$xy(ts) = xs = 1_S = ty = (ts)xy$$

and hence, xy is a unit. □

Lemma 5.1.2. *Let S be a unital commutative semiring with a zero and $X \in \mathcal{S}$ where $\mathcal{S} = M_n(S)$, $UT_n(S)$, or $U_n(S)$. If $X \mathcal{J} I_n$ in $\mathcal{S}$, then X is a unit in $\mathcal{S}$.*

Proof. If $X \mathcal{J} I_n$, then there exists $A, B \in \mathcal{S}$ such that $AXB = I_n$. The main theorem in [RS84] states that if S is a unital commutative semiring with a zero then, if $PQ = I_n$ then $QP = I_n$ for $P, Q \in M_n(S)$. Hence $XBA = BAX = I_n$, and thus $X \in U(S)$. □

5.2 Generating Sets for Upper Triangular and Unitriangular Matrix Monoids

In this section, we produce minimal generating sets for monoids of upper triangular matrices over commutative unital semirings with zeros and monoids of unitriangular matrices over commutative semirings with zeros. If an irredundant generating set exists, the given minimal generating sets will be irredundant. We also provide a more detailed form of the generating sets when we restrict to look at matrices over anti-negative semirings and anti-negative semifields.

5.2.1 Upper Triangular Matrix Monoids

To begin, we need the following lemma, which tells us when an upper triangular matrix over a commutative unital semiring with a zero is invertible.

Lemma 5.2.1. *Let S be a commutative unital semiring with a zero and $n \in \mathbb{N}$. Then, $X \in UT_n(S)$ is invertible in $UT_n(S)$ if and only if $X_{ii} \in U(S)$ for $1 \leq i \leq n$ and $X_{ij} \in V(S)$ for $1 \leq i < j \leq n$.*

Proof. By [LW16, Theorem 3.2] and [LW16, Theorem 4.2], we can see that $X \in UT_n(S)$ is invertible in $M_n(S)$ if and only if $X_{11}^2 \cdots X_{nn}^2 \in U(S)$ and $\sum_{k=1}^{n} X_{ki}X_{kj} \in V(S)$ for all $1 \leq i < j \leq n$. Thus, if $X \in UT_n(S)$ is such that $X_{ii} \in U(S)$ for $1 \leq i \leq n$ and $X_{ij} \in V(S)$ for $1 \leq i < j \leq n$, then X is invertible in $M_n(S)$.

So, suppose $X \in UT_n(S)$ is invertible in $M_n(S)$, then by Lemma 5.1.1, we have that $X_{ii} \in U(S)$ for $1 \leq i \leq n$. We can see that $X_{ij} \in V(S)$ by induction. Note that, for all $j > 1$, $\sum_{k=1}^{n} X_{k1}X_{kj} = X_{11}X_{1j}$ as X is upper triangular. Thus, $X_{1j} \in V(S)$ as $X_{11}X_{1j} \in V(S)$. So, for induction, suppose $X_{ij} \in V(S)$ for all $i < l$.

Now, $\sum_{k=1}^{n} X_{kl}X_{kj} = X_{ll}X_{lj} + \sum_{k=1}^{l-1} X_{kl}X_{kj}$ as X is upper triangular. Thus, $X_{ll}X_{lj} \in V(S)$ and hence $X_{lj} \in V(S)$ as $\sum_{k=1}^{l-1} X_{kl}X_{kj} \in V(S)$ by the inductive hypothesis. Therefore, $X_{ij} \in V(S)$ for $1 \leq i < j \leq n$.

We now need to show that the inverse of X in $M_n(S)$ lies in $UT_n(S)$. Let Y be the inverse of X, then $XY = I_n$. Suppose $Y \notin UT_n(S)$ and let i be the maximum such that there exists $j < i$ with $Y_{ij} \neq 0_S$. Then choose $j < i$ such that $Y_{ij} \neq 0_S$. Now,

$$X_{ii}Y_{ij} = (XY)_{ij} = (I_n)_{ij} = 0_S$$

where the first equality holds as $X_{ik} = 0_S$ for all $k < i$ and $Y_{kj} = 0_S$ for all $k > i$ by the maximality of i. Thus, $Y_{ij} = 0_S$ as $X_{ii} \in U(S)$ and hence is not a zero-divisor. This gives a contradiction, so $Y \in UT_n(S)$. □

Theorem 5.2.2. *Let S be a commutative unital semiring with a zero and $n \in \mathbb{N}$. Let $\mathcal{X}$ be a minimal semigroup generating set for the group of units of $UT_n(S)$, $\Omega \subseteq S$ be a minimal set such that $U(S)(\Omega \cup V(S))$ generates $(S, +)$, and $Y \subseteq S$ be a minimal set such that $Y \cup U(S)$ generates $(S, \cdot)$. Then, the monoid $UT_n(S)$ is minimally generated by $\mathcal{X} \cup E(\Omega) \cup A(Y)$ where*

$$E(\Omega) = \{E_{ij}(\omega) : \omega \in \Omega,\ 1 \leq i < j \leq n\}, \text{ and}$$

$$A(Y) = \{A_i(y) : y \in Y,\ 1 \leq i \leq n\}.$$

Moreover, if $\mathcal{X}$, Ω and Y are irredundant then $UT_n(S)$ is irredundantly generated by $\mathcal{X} \cup E(\Omega) \cup A(Y)$.

Proof. If $a \in U(S)$ then $A_i(a) \in \langle \mathcal{X} \rangle$ as $A_i(a)$ is invertible by Lemma 5.2.1. If $a \in S \setminus U(S)$, then we can write $a = x_1 \cdots x_s$ for some $x_1, \ldots, x_s \in Y \cup U(S)$.

Thus, $A_i(a) = A_i(x_1)\cdots A_i(x_s)$ and hence each $A_i(a)$ is generated by matrices from $A(Y) \cup \langle \mathcal{X} \rangle$. Thus, we can generate $A_i(a)$, for all $a \in S$ and $1 \leq i \leq n$.

Fix $a \in S$. Since $U(S)(\Omega \cup V(S))$ generates $(S, +)$ we can write $a = \sum_{t=1}^{m} l_t b_t$ where $l_t \in U(S)$ and $b_t \in \Omega \cup V(S)$. For all $i < j$, it is straightforward to verify that

$$E_{ij}(a) = \prod_{t=1}^{m} A_i(l_t) E_{ij}(b_t) A_i(l_t^{-1})$$

and if $b_t \in V(S)$, then by Lemma 5.2.1, $E_{ij}(b_t) \in \langle \mathcal{X} \rangle$, and thus each of these factors is in $E(\Omega) \cup \langle \mathcal{X} \rangle$. Further, define, for $1 \leq i \leq n$,

$$E_{ii}(a) = A_i(a).$$

Then, $E_{ij}(a)$ for all $a \in S$ and $1 \leq i \leq j \leq n$ can be expressed as a product of matrices from the given sets. Now, note that for any $M = (m_{ij}) \in UT_n(S)$,

$$M = \prod_{l=0}^{n-1} \prod_{k=l}^{n-1} E_{n-k,n-l}(m_{n-k,n-l}).$$

Therefore, every matrix in $UT_n(S)$ can be expressed as the product of matrices from the given sets.

We show this generating set is minimal by contradiction. Suppose that there exists a generating set Γ for $UT_n(S)$ such that $|\Gamma| < |\mathcal{X} \cup E(\Omega) \cup A(Y)|$. Let $\Gamma_1 \subseteq \Gamma$ be the set of all units in Γ. As any product containing a non-unit is a non-unit by Lemma 5.1.2, Γ_1 generates the group of units of $UT_n(S)$. Therefore, as $\mathcal{X}$ is a minimal generating set for the group of units, we have that $|\mathcal{X}| \leq |\Gamma_1|$, and hence $|\Gamma \setminus \Gamma_1| < |E(\Omega) \cup A(Y)|$.

Let $T = \langle \mathcal{X} \cup E(\Omega) \rangle$ and $\Gamma_2 \subseteq \Gamma \setminus \Gamma_1$ be all the matrices in $\Gamma \setminus \Gamma_1$ that are also in T. It can be easily seen that T is the set of all matrices where all the diagonal entries are in $U(S)$. Thus, we can see that $XY \in T$ if and only if $X \in T$ and $Y \in T$ by considering the diagonal entries with Lemma 5.1.1. Thus, $\langle \Gamma_1 \cup \Gamma_2 \rangle = T$. We show that $|\Gamma_2| \geq |E(\Omega)|$, by showing that in order to generate every $E_{ij}(x)$ with $x \in S \backslash V(S)$ and $i < j$, we need at least $|E(\Omega)|$ elements not in Γ_1.

Suppose $\prod_{t=1}^{m} N_t = E_{ij}(x)$ for some $x \in S \backslash V(S)$, $i < j$, and $N_1, \ldots, N_m \in UT_n(S)$. It follows from Lemma 5.1.1 that $(N_t)_{hh} \in U(S)$ for all t and all h, since $\prod_{t=1}^{m} (N_t)_{hh} = (\prod_{t=1}^{m} N_t)_{hh} = (E_{ij}(x))_{hh} = 1_S$. Let $k < l$ such that $(k, l) \neq (i, j)$. Then,

$$(\prod_{t=1}^{m} N_t)_{kl} = \sum_{(i_0, \ldots, i_m)} \prod_{s=1}^{m} (N_s)_{i_{s-1}, i_s} = (E_{ij}(x))_{kl} = 0_S.$$

where the sum ranges over $k = i_0 \leq \cdots \leq i_m = l$. Thus, for all $1 \leq t \leq m$,

$$(N_1)_{kk} \cdots (N_{t-1})_{kk}(N_t)_{kl}(N_{t+1})_{ll} \cdots (N_m)_{ll} \in V(S)$$

and hence, $(N_t)_{kl} \in V(S)$ as $(N_t)_{hh} \in U(S)$ for all $1 \leq h \leq n$.

Now, consider the (i, j) entry of $\prod_{t=1}^{m} N_t = E_{ij}(x)$, then

$$x = \sum_{i=i_0 \leq \cdots \leq i_m = j} (N_1)_{i_0,i_1}(N_2)_{i_1,i_2} \cdots (N_m)_{i_{m-1},i_m}.$$

By the previous paragraph, $(N_t)_{i_l,i_{l+1}} \in V(S)$ if $i_l < i_{l+1}$ and $(i_l, i_{l+1}) \neq (i, j)$. So, we can split this sum in products that contain an entry from $V(S)$ and those that do not. Let $t_1, \ldots, t_{m'}$ be all the indices such that $(N_{t_\alpha})_{ij} \in S \setminus V(S)$ when $1 \leq \alpha \leq m'$, and recalling that for $a, b \in V(S)$ and $c \in S$, $a + b$, ca, $ac \in V(S)$, x may be expressed as

$$x = v + \sum_{\alpha=1}^{m'} g_{t_\alpha}(N_{t_\alpha})_{ij}$$

for some $v \in V(S)$ and $g_{t_\alpha} \in U(S)$. (Since diagonal entries of all N_t have been shown to be units.)

Therefore, to generate $E_{ij}(x)$ for all $x \in S \setminus V(S)$, it is necessary to find a set $X \subseteq S$ such that for all $x \in S$, there exists $v \in V(S)$, $g_1, \ldots, g_{m_x} \in U(S)$, and $x_1, \ldots, x_{m_x} \in X$ for some $m_x \in \mathbb{N}_0$ such that $x = v + \sum_{t=1}^{m_x} g_t x_t$.

Thus, $U(S)X \cup V(S)$ generates $(S, +)$ and hence, by the definition of Ω, $|X| \geq |\Omega|$, as $U(S)(X \cup V(S)) = U(S)X \cup V(S)$. Moreover, as we have to generate $E_{ij}(x)$ for all $x \in S \setminus V(S)$ and $i < j$, we get that $|\Gamma_2| \geq \frac{n}{2}(n-1) \cdot |\Omega| = |E(\Omega)|$, and hence $|\Gamma_3| < |A(Y)|$, where $\Gamma_3 = \Gamma \setminus (\Gamma_1 \cup \Gamma_2)$.

For each $s \in S \setminus U(S)$ and $1 \leq i \leq n$, $A_i(s) \notin \langle \Gamma_1 \cup \Gamma_2 \rangle$, so consider a product $\prod_{t=1}^{m} N_t = A_i(s)$. Then

$$(\prod_{t=1}^{m} N_t)_{ii} = \prod_{t=1}^{m} (N_t)_{ii} = s \text{ and } (\prod_{t=1}^{m} N_t)_{hh} = \prod_{t=1}^{m} (N_t)_{hh} = 1_S$$

for all $1 \leq h \leq n$ with $h \neq i$. Thus, $(N_t)_{hh} \in U(S)$ for all t and $h \neq i$. Therefore, in order to generate each $A_i(s)$ for $s \in S \setminus U(S)$ we need to find a set Λ such that for all s there exist $\lambda_1, \ldots, \lambda_{m_s} \in \Lambda$ such that $s = g\lambda_1 \cdots \lambda_{m_s}$ for some $g \in U(S)$.

However, Y is the minimal set such that $Y \cup U(S)$ generates $(S, \cdot)$, so $|\Lambda| \geq |Y|$. Moreover, as we need to generate $A_i(s)$ for all $s \in S \setminus U(S)$ and for all $1 \leq i \leq n$, we get that $|\Gamma_3| \geq n|Y| = |A(Y)|$, giving a contradiction. Thus, $|\Gamma| \geq |\mathcal{X} \cup E(\Omega) \cup A(Y)|$ and hence the given set minimally generates $UT_n(S)$.

Assume $\mathcal{X}, \Omega$ and Y are irredundant. By Lemma 5.1.2, in $UT_n(S)$, any product containing a non-unit is a non-unit. Thus, all the elements of $\mathcal{X}$ are irredundant in the given generating set for $UT_n(S)$.

Suppose for a contradiction, $E_{ij}(\omega)$ is redundant for some $i < j$ and $\omega \in \Omega$. Then, in order to generate $E_{ij}(\omega)$, we have that there exists $v \in V(S)$, $g_1, \ldots, g_{m_\omega} \in U(S)$, and $x_1, \ldots, x_{m_\omega} \in \Omega \backslash \{\omega\}$ for some $m_\omega \in \mathbb{N}_0$ such that $\omega = v + \sum_{t=1}^{m_\omega} g_t x_t$ by above. This gives a contradiction as Ω is an irredundant set such that $U(S)(\Omega \cup V(S))$ generates $(S, +)$.

Now, suppose that $A_i(y)$ is redundant for some $y \in Y$. Then, in order to generate $A_i(y)$, we have that, there exist $g \in U(S)$ and $\lambda_1, \ldots, \lambda_{m_y} \in Y \setminus \{y\}$ such that $s = g\lambda_1 \ldots \lambda_{m_y}$ by above. This gives a contradiction as Y is an irredundant set such that $Y \cup U(S)$ generates $(S, \cdot)$. Thus, $UT_n(S)$ is minimally and irredundantly generated by $\mathcal{X} \cup E(\Omega) \cup A(Y)$. □

Let S be semiring with a zero, we define the function $\operatorname{diag}\colon S^n \to UT_n(S)$ by $\operatorname{diag}(x) = A_1(x_1) \cdots A_n(x_n)$ where $x = (x_1, \ldots, x_n) \in S^n$. We remark that diag is an injective homomorphism.

Corollary 5.2.3. *Let S be a commutative unital anti-negative semiring with a zero and $n \in \mathbb{N}$. Let X be a minimal generating set for $(U(S)^n, \cdot)$ and $\Omega \subseteq S$ be a minimal set such that $U(S)\Omega$ generates $(S^*, +)$, and $Y \subseteq S$ be a minimal set such that $Y \cup U(S)$ generates $(S, \cdot)$. The monoid $UT_n(S)$ is minimally generated by* $\operatorname{diag}(X) \cup E(\Omega) \cup A(Y)$ *where*

$$\operatorname{diag}(X) = \{\operatorname{diag}(x) : x \in X\},$$

$$E(\Omega) = \{E_{ij}(\omega) : \omega \in \Omega,\ 1 \leq i < j \leq n\},\ \textit{and}$$

$$A(Y) = \{A_i(y) : y \in Y,\ 1 \leq i \leq n\}.$$

Moreover, if X, Ω and Y are irredundant then $UT_n(S)$ is irredundantly generated by $\operatorname{diag}(X) \cup E(\Omega) \cup A(Y)$.

Proof. $V(S) = \{0_S\}$, so by Lemma 5.2.1, the invertible elements of $UT_n(S)$ are the diagonal matrices with entries in $U(S)$. Therefore the group of units of $UT_n(S)$ is isomorphic to $U(S)^n$ and hence minimally generated by $\operatorname{diag}(X)$. Moreover, as S is anti-negative, a set I minimally (and irredundantly) generates $(S^*, +)$ if and only if $I \cup \{0_S\}$ minimally (and irredundantly) generates $(S, +)$. □

Corollary 5.2.4. *Let S be an anti-negative semifield with a zero and $n \in \mathbb{N}$. Let X be a minimal generating set for $((S^*)^n, \cdot)$. The monoid $UT_n(S)$ is minimally generated by $\operatorname{diag}(X) \cup E(1_S) \cup A(0_S)$ where*

$$\operatorname{diag}(X) = \{\operatorname{diag}(x) : x \in X\},\ E(1_S) = \{E_{ij} : 1 \leq i < j \leq n\},\ \textit{and}$$

$$A(0_S) = \{A_i(0_S) : 1 \leq i \leq n\}.$$

Moreover, if X is irredundant then $UT_n(S)$ is irredundantly generated by $\operatorname{diag}(X) \cup E(1_S) \cup A(0_S)$.

Proof. As $U(S) = S^*$, we may take $\Omega = \{1_S\}$ and $Y = \{0_S\}$ in Corollary 5.2.3. □

Corollary 5.2.5. *Let $n \in \mathbb{N}$. The monoid $UT_n(\mathbb{Z}_{\max})$ is minimally and irredundantly generated by $A(1) \cup \{-1 \cdot I_n\} \cup E(0) \cup A(-\infty)$ where*

$$A(1) = \{A_i(1) : 1 \leq i \leq n\},\ E(0) = \{E_{ij} : 1 \leq i < j \leq n\},\ \textit{and}$$

$$A(-\infty) = \{A_i(-\infty) : 1 \leq i \leq n\}.$$

Recall that $1 \neq 1_{\mathbb{Z}_{\max}} = 0 \neq 0_{\mathbb{Z}_{\max}} = -\infty$ and that $-1 \cdot I_n$ is the diagonal matrix with -1 on the diagonal and $-\infty$ elsewhere.

Proof. For $1 \leq i \leq n$, let $a_i \in \mathbb{Z}^n$ be the element with 1 in the ith coordinate and 0 elsewhere, and $b \in \mathbb{Z}^n$ be the element with -1 in every coordinate. Clearly, $A_i(1) = \operatorname{diag}(a_i)$ and $-1 \cdot I_n = \operatorname{diag}(b)$.

As $\mathbb{Z}_{\max}$ is an anti-negative semifield, by Corollary 5.2.4, it suffices to show that $\{a_1, \ldots, a_n, b\}$ forms a minimal and irredundant generating set for $(\mathbb{Z}^n, \cdot)$ where $\cdot$ is the semiring multiplication of $(\mathbb{Z}^*_{\max})^n$, that is, coordinate-wise addition. It is clear that this is a generating set for $(\mathbb{Z}^n, \cdot)$. To see that it is minimal, observe that $|\{a_1, \ldots, a_n, b\}| = n + 1$ and $\mathbb{Z}^n$ is minimally $n + 1$ generated as a semigroup [BGS19, Corollary 4.3], so $\{a_1, \ldots, a_n, b\}$ is a minimal generating set. Moreover, as $\{a_1, \ldots, a_n, b\}$ is finite and minimal, it is irredundant. □

5.2.2 Unitriangular Matrix Monoids

Recall that, by the way we defined unitriangular matrices, we are free to consider unitriangular matrices over non-unital semirings. However, in this process, we work with matrices over a structure that is no longer a semiring. Thus, to proceed, we need analogous results to Lemma 5.1.2 and Lemma 5.2.1, which work for unitriangular matrices over non-unital semirings.

Lemma 5.2.6. *Let S be a commutative semiring with a zero and $n \in \mathbb{N}$. Then, for $X \in U_n(S)$, the following are equivalent*

(i) X is invertible in $U_n(S)$,

(ii) $X \mathcal{J} I_n$,

(iii) $X_{ij} \in V(S)$ for $1 \leq i < j \leq n$.

Proof. Clearly, (i) implies (ii). To see that (ii) implies (iii), let $T = S \times \mathbb{N}_0$ be the Dorroh extension of S by $\mathbb{N}_0$ [Gol99, p.3], where the operations on T are

$$(r, n) + (r', n') = (r + r', n + n') \text{ and } (r, n)(r', n') = (nr' + n'r + rr', nn')$$

where $nr = \sum_{i=1}^{n} r$ for $n \in \mathbb{N}_0$ and $r \in S$ where $0r = 0_S$. Then, T is a commutative unital semiring with a zero $(0, 0)$ and identity $(0, 1)$. Thus, S is a subsemiring of T as $S \cong S \times \{0\}$. Moreover, if we identify the identities of S^1 and T, then $U_n(S)$ is a subsemigroup of $U_n(T)$.

If $X \mathcal{J} I_n$ in $U_n(S)$, then $X \mathcal{J} I_n$ in $UT_n(T)$ so, by Lemma 5.1.2, X is a unit in $UT_n(T)$. Therefore, by Lemma 5.2.1, $X_{ij} \in V(T)$ for $1 \leq i < j \leq n$. However, by treating S as a subsemiring of T, it can be seen that $V(T) = V(S)$ as the only element in $\mathbb{N}_0$ with an multiplicative inverse is 0. Thus, $X_{ij} \in V(S)$ for $1 \leq i < j \leq n$.

Finally, to see (iii) implies (i), remark that by Lemma 5.2.1, X is invertible in $UT_n(T)$. Now, it suffices to show that the inverse of X lies in $U_n(S)$. So, let Y be the inverse of X in $UT_n(T)$, then $XY = I_n$. Then, for all $1 \leq i \leq n$,

$$Y_{ii} = X_{ii}Y_{ii} = (XY)_{ii} = (I_n)_{ii} = 1_S.$$

Therefore, $Y \in U_n(S)$. □

Theorem 5.2.7. *Let S be a commutative semiring with a zero and $n \in \mathbb{N}$. Let $\mathcal{X}$ be a minimal semigroup generating set for the group of units of $U_n(S)$. Let Ω be a minimal set such that $\Omega \cup V(S)$ generates $(S,+)$. The monoid $U_n(S)$ is minimally generated by $\mathcal{X} \cup E(\Omega)$ where*

$$E(\Omega) = \{E_{ij}(\omega) : \omega \in \Omega,\ 1 \leq i < j \leq n\}.$$

Moreover, if $\mathcal{X}$ and Ω are irredundant then $U_n(S)$ is irredundantly generated by $\mathcal{X} \cup E(\Omega)$.

Proof. Fix $a \in S$. Since $\Omega \cup V(S)$ generates $(S,+)$ there exists $b_1, \dots, b_m \in \Omega \cup V(S)$ such that $a = b_1 + \cdots + b_m$, and hence, for $i < j$, $E_{ij}(a) = E_{ij}(b_1) \cdots E_{ij}(b_m)$ and each of these factors is either in $\langle \mathcal{X} \rangle$ by Lemma 5.2.6, or in $E(\Omega)$. Thus, $E_{ij}(a)$ for any $a \in S$ and $i < j$, can be expressed as a product of matrices from $\mathcal{X} \cup E(\Omega)$. Now, note that for any $A = (a_{ij}) \in U_n(S)$,

$$A = \prod_{l=1}^{n-1} \prod_{i=1}^{n-l} E_{i,n+1-l}(a_{i,n+1-l})$$

where $E_{ij}(0_S) = I_n \in \langle \mathcal{X} \rangle$. Therefore, $U_n(S)$ is generated by the given sets of matrices.

We show that this is a minimal generating set by contradiction. Suppose that there exists a generating set Γ for $U_n(S)$ such that $|\Gamma| < |\mathcal{X} \cup E(\Omega)|$. Let $\Gamma = \Gamma_1 \cup \Gamma_2$ where Γ_1 is the set of all units in Γ. By Lemma 5.2.6, in $U_n(S)$ any product containing a non-unit is a non-unit, so Γ_1 generates the group of units of $U_n(S)$. Therefore, as $\mathcal{X}$ is a minimal generating set for the group of units, we have that $|\Gamma_1| \geq |\mathcal{X}|$ and hence $|\Gamma_2| < |E(\Omega)|$.

Suppose $\prod_{t=1}^m N_t = E_{ij}(x)$ for some $x \in S \setminus V(S)$, $i < j$, and $N_1, \dots, N_m \in U_n(S)$. Let $k < l$ such that $(k,l) \neq (i,j)$. Then,

$$(\prod_{t=1}^{m} N_t)_{kl} = \sum_{(i_0,\dots,i_m)} \prod_{s=1}^{m} (N_s)_{i_{s-1},i_s} = (E_{ij}(x))_{kl} = 0_S.$$

where $k = i_0 \leq \cdots \leq i_m = l$. Thus, for all $1 \leq t \leq m$,

$$(N_1)_{kk} \cdots (N_{t-1})_{kk} (N_t)_{kl} (N_{t+1})_{ll} \cdots (N_m)_{ll} = (N_t)_{kl} \in V(S)$$

as $(N_t)_{hh} = 1_S$ for all $1 \leq h \leq n$.

Now, consider the (i,j) entry of $\prod_{t=1}^{m} N_t = E_{ij}(x)$, then

$$x = \sum_{i=i_0 \leq \cdots \leq i_m = j} (N_1)_{i_0,i_1}(N_2)_{i_1,i_2} \cdots (N_m)_{i_{m-1},i_m}.$$

By the previous paragraph, $(N_t)_{i_l,i_{l+1}} \in V(S)$ if $i_l < i_{l+1}$ and $(i_l, i_{l+1}) \neq (i,j)$. So, we can split this sum in products that contain an entry from $V(S)$ and those that do not.

Let $t_1, \ldots, t_{m'}$ be all the distinct values such that $(N_{t_\alpha})_{ij} \in S \setminus V(S)$ where $1 \leq \alpha \leq m'$, and recalling that for $a, b \in V(S)$ and $c \in S$, $a + b, ca, ac \in V(S)$, x may be expressed as

$$x = v + \sum_{\alpha=1}^{m'} (N_{t_\alpha})_{ij}$$

where $v \in V(S)$.

Therefore, to generate $E_{ij}(x)$ for all $x \in S \setminus V(S)$, it is necessary to find a set X such that for all $x \in S$, there exists $v \in V(S)$ and $x_1, \ldots, x_{m_x} \in X$ such for some $m_x \in \mathbb{N}_0$ such that $x = v + \sum_{t=1}^{m_x} x_t$.

Thus, $X \cup V(S)$ generates $(S, +)$ and hence, by the definition of Ω, $|X| \geq |\Omega|$. Moreover, as we have to generate $E_{ij}(x)$ for all $x \in S \setminus V(S)$ and $i < j$, we get that $|\Gamma_2| \geq \frac{n}{2}(n-1) \cdot |\Omega| = |E(\Omega)|$ giving a contradiction. Thus, $|\Gamma| \geq |\mathcal{X} \cup E(\Omega)|$ and hence the given set minimally generates $U_n(S)$.

Assume $\mathcal{X}$ and Ω are irredundant. Then, by Lemma 5.2.6, in $U_n(S)$ any product containing a non-unit is a non-unit. Thus, all the elements of $\mathcal{X}$ are irredundant in the given generating set for $U_n(S)$.

Now, suppose that $E_{ij}(\omega)$ is redundant for some $i < j$ and $\omega \in \Omega$. Then, in order to generate $E_{ij}(\omega)$, we have that there exists $v \in V(S)$ and $x_1, \ldots, x_{m_\omega} \in \Omega \setminus \{\omega\}$ for some $m_\omega \in \mathbb{N}_0$ such that $\omega = v + \sum_{t=1}^{m_\omega} x_t$ by above. This gives a contradiction as Ω is an irredundant set such that $\Omega \cup V(S)$ generates $(S, +)$. Thus, $U_n(S)$ is minimally and irredundantly generated by $\mathcal{X} \cup E(\Omega)$. □

Corollary 5.2.8. *Let S be a commutative anti-negative semiring with a zero and $n \in \mathbb{N}$. Let Ω be a minimal generating set of $(S^*, +)$. The monoid $U_n(S)$ is minimally generated by $\{I_n\} \cup E(\Omega)$ where*

$$E(\Omega) = \{E_{ij}(\omega) : \omega \in \Omega, 1 \leq i < j \leq n.\}$$

Moreover, if Ω is irredundant then $U_n(S)$ is irredundantly generated by $I_n \cup E(\Omega)$.

Proof. The group of units of $U_n(S)$ is $\{I_n\}$, since $V(S) = \{0_S\}$. Moreover, as S is anti-negative, a set X minimally (and irredundantly) generates $(S^*, +)$ if and only if $X \cup \{0_S\}$ minimally (and irredundantly) generates $(S, +)$. □

Corollary 5.2.9. *Let $n \in \mathbb{N}$. The monoid $U_n(\mathbb{Z}_{\max})$ is minimally and irredundantly generated by $\{I_n\} \cup E(\mathbb{Z})$ where*

$$E(\mathbb{Z}) = \{E_{ij}(z) : z \in \mathbb{Z}, 1 \leq i < j \leq n\}.$$

Proof. Recall that $\mathbb{Z}_{\max}$ is a bipotent semiring, that is, $\max(x, y) \in \{x, y\}$ for all $x, y \in \mathbb{Z}_{\max}$. Thus, the minimal and irredundant generating set for $(\mathbb{Z}, \max)$ is $\mathbb{Z}$. □

5.3 Full Matrix Monoid Generating Sets

We now move on to finding generating sets of full matrix monoids over anti-negative semifields. In particular, we look at the monoids of matrices over the tropical semiring of dimensions 2 and 3 and provide minimal and irredundant generating sets for them.

We define two functions which we use throughout the rest of this section. Let S be an anti-negative semiring with a zero and no zero-divisors and $\mathbb{B}$ be the Boolean semiring. Define $\psi : S \to \mathbb{B}$ to be the map that sends 0_S to 0 and all non-zero elements to 1. Then, define $\phi_n : M_n(S) \to M_n(\mathbb{B})$ to be the map that sends A to $\phi_n(A)$ where $\phi_n(A)_{ij} = \psi(A_{ij})$. For all $n \in \mathbb{N}$, ψ and ϕ_n are surjective morphisms and hence the cardinality of a minimal generating set for $M_n(S)$ is at least the cardinality of a minimal generating set for $M_n(\mathbb{B})$.

5.3.1 2×2 Full Matrix Monoids

First, we consider the minimal generating sets for the semigroup of all 2×2 matrices over an anti-negative semifield S such that $x \leq y$ or $y \leq x$ for all $x, y \in S^*$.

For a semiring S, we say a matrix $M \in M_n(S)$ is a *monomial* matrix if it has exactly one entry from S^* in each row and column, and say M has *underlying permutation of* $\sigma \in \mathcal{S}_n$ if $M_{ij} \in S^*$ if and only if $j = \sigma(i)$ for all $1 \leq i \leq n$. Moreover, we say M is the *permutation matrix of* $\sigma \in \mathcal{S}_n$ if M has underlying permutation of σ and $M_{ij} = 1_S$ if and only if $j = \sigma(i)$ for all $1 \leq i \leq n$. We denote the group of units of $M_n(S)$ by $GL_n(S)$.

To continue we require the following proposition which gives equivalent conditions to a matrix being invertible in $M_n(S)$ when S is a commutative anti-negative unital semiring with a zero.

Proposition 5.3.1 (Corollary 3.3 [Tan13]). *Let S be a commutative anti-negative unital semiring with a zero. Then, the following statements are equivalent.*

(1) A is invertible in $M_n(S)$.

(2) $\sum_{\sigma\in\mathcal{S}_n} A_{1\sigma(1)}\cdots A_{n\sigma(n)} \in U(S)$ and $A_{ij}A_{ik} = 0$ for all $i,j,k \in [n]$ with $j \neq k$.

(3) $\sum_{\sigma\in\mathcal{S}_n} A_{1\sigma(1)}\cdots A_{n\sigma(n)} \in U(S)$ and $A_{ij}A_{kj} = 0$ for all $i,j,k \in [n]$ with $i \neq k$.

We now apply the previous proposition to the case where S has no zero-divisors.

Lemma 5.3.2. *Let S be a commutative anti-negative unital semiring with a zero and no zero-divisors. Then, the invertible matrices of $M_n(S)$ are exactly the monomial matrices in which every non 0_S entry is in $U(S)$.*

Proof. First note that monomial matrices in which every non 0_S entry is in $U(S)$ satisfy the conditions of Proposition 5.3.1 and hence are invertible. So, now suppose that $X \in M_n(S)$ is invertible. Then, by Proposition 5.3.1(2–3), we can see that $X_{ij}X_{ik} = 0 = X_{ji}X_{ki}$ for all $1 \leq i,j,k \leq n$ with $j \neq k$. Thus, as S has no zero-divisors, X has at most one non 0_S entry per row and column. Finally, observe that, by Proposition 5.3.1(2), X must have at least one non 0_S entry per row and column and every non 0_S is in $U(S)$. □

Lemma 5.3.3. *Let $m \geq 1$ and $X = \{x_1, \ldots, x_m\}$ be a minimal group generating set for $\mathbb{Z}^m$. Then $X \cup \{x_0\}$ where $x_0 = x_1^{-1}\cdots x_m^{-1}$ is a minimal semigroup generating set for $\mathbb{Z}^m$.*

Proof. Note that $\mathbb{Z}^m$ is minimally generated as a semigroup by a set of cardinality $m+1$ [BGS19, Corollary 4.3], so it suffices to generate x_i^{-1} as products of $x_0, \ldots, x_m$ for $1 \leq i \leq m$. Observe, for all $1 \leq i \leq m$,

$$x_0 \cdots x_{i-1}x_{i+1}\cdots x_m = x_1^{-1}\cdots x_m^{-1}x_1\cdots x_{i-1}x_{i+1}\cdots x_m = x_i^{-1}$$

as $\mathbb{Z}^m$ is commutative. Thus, X is a minimal generating set for $\mathbb{Z}^m$. □

For the following theorem, we introduce the notation that for a semiring S and $x, y \in S$, $x \leq y$ if and only if there exists $t \in S$ such that $t + x = y$. We remark that this agrees with the usual order of $\mathbb{Z}$ in $\mathbb{Z}_{\max}$ and is the reverse of Green's $\mathcal{J}$-preorder in the additive monoid of S.

Theorem 5.3.4. *Let S be an anti-negative semifield with a zero such that for all $x, y \in S^*$, $x \leq y$ or $y \leq x$. Let X be a minimal (semigroup) generating set for $(S^*, \cdot)$ with $x_0 \in X$. If $(S^*, \cdot)$ is non-trivial, we can choose X such that $x_0^{-1} \in \langle X \setminus \{x_0\} \rangle$. Then, the monoid $M_2(S)$ is minimally generated by the matrices:*

$$A = \begin{pmatrix} 0_S & x_0 \\ 1_S & 0_S \end{pmatrix}, \ B(x) = \begin{pmatrix} x & 0_S \\ 0_S & 1_S \end{pmatrix} \ \textit{for all } x \in X \setminus \{x_0\},$$

$$C = \begin{pmatrix} 0_S & 0_S \\ 0_S & 1_S \end{pmatrix}, \textit{ and } D = \begin{pmatrix} 1_S & 1_S \\ 1_S & 0_S \end{pmatrix}$$

Proof. First, we show that if $(S^*, \cdot)$ is non-trivial we can find a minimal generating set X for $(S^*, \cdot)$ with $x_0 \in X$ such that $x_0^{-1} \in \langle X \setminus \{x_0\} \rangle$. As S is an anti-negative semifield, every element but 0_S and 1_S has infinite multiplicative order [GJN20, Lemma 2.1(ii)]. Thus, if $|X| = m$ for some $m \in \mathbb{N}$, then $(S^*, \cdot)$ is a finitely generated torsion-free abelian group and therefore isomorphic to $\mathbb{Z}^{m-1}$, as $\mathbb{Z}^{m-1}$ is m-generated as a semigroup [BGS19, Corollary 4.3]. For $m \geq 2$, by Lemma 5.3.3, we can choose X be a minimal generating set such that $x_0^{-1} \in \langle X \setminus \{x_0\} \rangle$. If $m = 1$, then $(S^*, \cdot)$ is trivial and if X is infinite, then let X' be a minimal generating set for $(S^*, \cdot)$ and $x_0 = 1_S$. Then, $X = X' \cup \{x_0\}$ is a minimal generating set for $(S^*, \cdot)$ with the property that $x_0^{-1} = 1_S \in \langle X' \rangle$.

When $(S^*, \cdot)$ is non-trivial, there exists $y_1, \ldots, y_s \in (X \setminus \{x_0\})$ such that $y_1 \cdots y_s = x_0^{-1}$. Thus, $B(x_0^{-1}) = B(y_1) \cdots B(y_s)$ and when $(S^*, \cdot)$ is trivial, $A^2 = B(x_0^{-1}) = I_2$. Thus, we can therefore generate,

$$F = \begin{pmatrix} 0_S & 1_S \\ 1_S & 0_S \end{pmatrix} = B(x_0^{-1})A \text{ and } B(x_0) = \begin{pmatrix} x_0 & 0_S \\ 0_S & 1_S \end{pmatrix} = AB(x_0^{-1})A.$$

For all $x \in S^*$, there exist $z_1, \ldots, z_t \in X$ such that $x = z_1 \cdots z_t$ and hence $B(x) = B(z_1) \cdots B(z_t)$. Therefore, $B(x)$ can be expressed as the product of generators as each $B(z_i)$ is a generator or can be expressed as a product of generators. Moreover, pre-multiplying any matrix X by F swaps the rows and post-multiplying by F swaps the columns, so in order to prove we can generate every matrix as a product of the given matrices, it suffices to express each matrix up to rearranging rows and columns.

Observe that,

$$\begin{pmatrix} 0_S & 0_S \\ 0_S & 0_S \end{pmatrix} = CFC, \text{ and } \begin{pmatrix} 0_S & 0_S \\ x & 0_S \end{pmatrix} = CFB(x) \text{ for all } x \in S^*.$$

Thus, from the given matrices, we are able to generate any matrix containing three or four 0_S entries. For the case where a matrix contains one or two 0_S entries, let $x, y, z \in S^*$, then

$$\begin{pmatrix} 0_S & 0_S \\ x & y \end{pmatrix} = CFB(y)DB(y^{-1}x), \qquad \begin{pmatrix} 0_S & x \\ 0_S & y \end{pmatrix} = B(x)FB(y)DFC,$$
$$\begin{pmatrix} 0_S & x \\ y & 0_S \end{pmatrix} = B(x)FB(y), \text{ and } \qquad \begin{pmatrix} 0_S & x \\ y & z \end{pmatrix} = B(x)FB(z)DFB(z^{-1}y).$$

Hence, every matrix with at least one 0_S entry can be expressed as a product of matrices from the given matrices.

Finally, for $a, b, c, d \in S^*$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1_S & 1_S \\ db^{-1} & ca^{-1} \end{pmatrix} B(b)FB(a),$$

So, it suffices to express, for all $x, y \in S^*$, $\left(\begin{smallmatrix} 1_S & 1_S \\ x & y \end{smallmatrix}\right)$ as a product of matrices with at least one 0_S entry. Without loss of generality, we may suppose $y \leq x$ as if $x \leq y$, then we can multiply by F on the right to swap to x and y in $\left(\begin{smallmatrix} 1_S & 1_S \\ x & y \end{smallmatrix}\right)$. So, as $y \leq x$, there exists $t \in S$ such that $t + y = x$ and

$$\begin{pmatrix} 1_S & 1_S \\ x & y \end{pmatrix} = \begin{pmatrix} 0_S & 1_S \\ y & y \end{pmatrix} \begin{pmatrix} y^{-1}t & 0_S \\ 1_S & 1_S \end{pmatrix}.$$

Thus, the above matrix, and hence all matrices with no 0_S entries, can be expressed as a product of matrices which contain at least one 0_S entry. Hence, all matrices can be generated by the given matrices.

Now, we show that the given generating set is minimal. The invertible matrices are the monomial matrices with entries in S^*, by Lemma 5.3.2. Let perm : $GL_2(S) \to (S^*, \cdot)$ be the surjective morphism that maps A to $A_{1,\sigma(1)}A_{2,\sigma(2)}$ where σ is the underlying permutation of A. By assumption, $(S^*, \cdot)$ is minimally generated by $|X|$ elements, so $GL_2(S)$ is minimally generated by at least $|X|$ matrices as there exists a surjective morphism from $GL_n(S)$ to $(S^*, \cdot)$. However, $GL_2(S)$ is generated by the $|X|$ matrices A and $B(x)$ for $x \in X \setminus \{x_0\}$, so these matrices minimally generate $GL_2(S)$. Moreover, in $M_2(S)$, any product containing a non-invertible matrix is not invertible by Lemma 5.1.2, hence any minimal generating set for $M_2(S)$ must contain a generating set for $GL_2(S)$.

If X is infinite, then the generating set is minimal as every generating set has to contain at least $|X| = |X|+2$ elements. If X is finite, then for a contradiction, suppose there exists a generating set Γ of size $|X|+1$ for $M_2(S)$. By above $|X|$ elements of Γ are in $GL_2(S)$. Let Γ' be all the elements of Γ in $GL_2(S)$ and $\Gamma \setminus \Gamma' = \{\gamma\}$. Consider $\phi_2(\Gamma)$, this is a generating set for $M_2(\mathbb{B})$ as ϕ_2 is a surjective morphism. Moreover, as $\phi_2(\Gamma')$ generates $GL_2(\mathbb{B})$ which is generated by $\phi_2(A)$, we can see that $\phi_2(A) \cup \phi_2(\gamma)$ is a generating set for $M_2(\mathbb{B})$. However, this gives a contradiction as $M_2(\mathbb{B})$ is minimally generated by 3 matrices [HMSW21]. Therefore, $M_2(S)$ is minimally generated by these $|X|+2$ matrices. □

We can now apply the above Theorem to $\mathbb{Z}_{\max}$, by noting that for all $x, y \in S^*$, $x \leq y$ or $y \leq x$.

Corollary 5.3.5. *The monoid $M_2(\mathbb{Z}_{\max})$ is minimally generated by the matrices:*

$$A = \begin{pmatrix} -\infty & -1 \\ 0 & -\infty \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -\infty \\ -\infty & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} -\infty & -\infty \\ -\infty & 0 \end{pmatrix}, \text{ and } D = \begin{pmatrix} 0 & 0 \\ 0 & -\infty \end{pmatrix}$$

Proof. $X = \{-1, 1\}$ is a generating set for $(\mathbb{Z}, +)$ such that $(-1)^{-1} = 1$. □

5.3.2 3×3 Full Matrix Monoids

In this section, we focus on semigroups of 3×3 matrices and show that $M_3(S)$ is not finitely generated when S is an infinite anti-negative semifield with a zero. We then construct an (infinite) minimal and irredundant generating set for $M_3(\mathbb{Z}_{\max})$.

For matrices $X, Y \in M_n(S)$, we say that X is a *permutation* of Y if X can be obtained by permuting the rows and permuting the columns of Y. Equivalently, $X = PYP'$ for some permutation matrices $P, P' \in M_n(S)$.

In order to show that $M_3(\mathbb{Z}_{\max})$ is not finitely generated, we show that for any infinite commutative anti-negative unital semiring S with a zero and no zero-divisors, we can find an infinite set of matrices which are all prime in $M_3(S)$ and each one is contained in a different $\mathcal{J}$-class of $M_3(S)$.

Lemma 5.3.6. *Let S be a commutative anti-negative unital semiring with a zero and no zero-divisors. For $s \in S^*$, let*

$$X_s = \begin{pmatrix} 0_S & 1_S & s \\ 1_S & 0_S & 1_S \\ 1_S & 1_S & 0_S \end{pmatrix}.$$

Then X_s is prime in $M_3(S)$, and if $X_t \mathcal{J} X_s$ for some $t \in S^*$ then $t = s$ or $ts = 1_S$.

Proof. Note that $\phi_3(X_s)$ is prime in $M_3(\mathbb{B})$ [DCG80, Theorem 1], so if $AB = X_s$, for some $A, B \in M_3(S)$, then $\phi_3(A)$ or $\phi_3(B)$ is a unit in $M_3(\mathbb{B})$. Suppose $\phi_3(A)$ is a unit in $M_3(\mathbb{B})$. Hence, $\phi_3(A)$ is a permutation matrix by Lemma 5.3.2 and so, A is a monomial matrix. Suppose that A is not a unit, then A has a non 0_S, non-invertible entry by Lemma 5.3.2. Thus, some row of AB is a scaling of a row of B by a non-invertible element of S. However, each row of X_s contains a 1_S entry, giving a contradiction by Lemma 5.1.1. Therefore, A is a unit. If $\phi_3(B)$ is a unit a dual argument holds, since each column of X_s contains a 1_S entry. Hence, X_s is prime in $M_3(S)$.

Suppose $X_s \mathcal{J} X_t$ for some $s, t \in S$, then $UX_sV = X_t$ where $U, V \in GL_3(S)$ as X_t is prime. We may write $U = PD$ and $V = D'P'$ for permutation matrices P and P' and diagonal matrices with entries in $U(S)$, D and D' by Lemma 5.3.2. Suppose P and P' are the permutation matrix of σ and τ respectively for some $\sigma, \tau \in \mathcal{S}_3$. Then, for $1 \leq i \leq 3$, the $(\sigma^{-1}(i), \tau(i))$ entry of UX_sV is given by

$$P_{\sigma^{-1}(i),i} D_{i,i} (X_s)_{i,i} D'_{i,i} P'_{i,\tau(i)} = 0_S$$

as $(X_s)_{i,i} = 0_S$. Therefore, $(X_t)_{\sigma^{-1}(i),\tau(i)} = 0_S$, but $(X_t)_{i,j} = 0_S$ if and only if $i = j$, so $\tau = \sigma^{-1}$ and $P' = P^{-1}$.

Thus, $PDX_sD'P^{-1} = X_t$. Moreover, for $i \neq j$ and $(i, j) \neq (1, 3)$,

$$P_{i,\sigma(i)} D_{\sigma(i),\sigma(i)} (X_s)_{\sigma(i),\sigma(j)} D'_{\sigma(j),\sigma(j)} P^{-1}_{\sigma(j),j} = (X_t)_{ij} = 1_S \tag{5.1}$$

and for $(i,j) = (1,3)$,

$$P_{1,\sigma(1)} D_{\sigma(1),\sigma(1)} (X_s)_{\sigma(1),\sigma(3)} D'_{\sigma(3),\sigma(3)} P^{-1}_{\sigma(3),3} = (X_t)_{13} = t. \tag{5.2}$$

Suppose $\sigma = 1_{\mathcal{S}_3}$, then by (5.2) $D_{11} s D'_{33} = t$. In order to satisfy the equalities (5.1), we have that

$$D_{11} = (D')^{-1}_{22} = D_{33} = (D')^{-1}_{11} = D_{22} = (D')^{-1}_{33}.$$

Thus, $s = t$ by (5.2).

If $\sigma \neq 1_{\mathcal{S}_3}$, then taking $i = \sigma^{-1}(1)$ and $j = \sigma^{-1}(3)$ in (5.1) gives $D_{11} s D'_{33} = 1_S$, whilst (5.2) gives $D_{\sigma(1),\sigma(1)} D'_{\sigma(3),\sigma(3)} = t$, so $s, t \in U(S)$. Since $(\sigma(1), \sigma(3)) \neq (1,3)$ and is not of the form (h,h).

If $\sigma = (1,2,3)$, then

$$D_{11} = (D')^{-1}_{22} = D_{33} = (D')^{-1}_{11} = t^{-1} D_{22} = t^{-1} (D')^{-1}_{33},$$

as $D_{22} D'_{11} = t$ by (5.2). If $\sigma = (1,3,2)$, then

$$D_{11} = (D')^{-1}_{22} = t^{-1} D_{33} = t^{-1} (D')^{-1}_{11} = t^{-1} D_{22} = t^{-1} (D')^{-1}_{33},$$

as $D_{33} D'_{22} = t$ by (5.2). Thus, if $\sigma \neq 1_{\mathcal{S}_3}$ is an even permutation, $D_{11} s D'_{33} = t^{-1} s = 1_S$, so $s = t$.

If $\sigma = (1,2)$, then

$$D_{11} = (D')^{-1}_{22} = D_{33} = (D')^{-1}_{11} = D_{22} = t (D')^{-1}_{33},$$

as $D_{22} D'_{33} = t$ by (5.2). If $\sigma = (1,3)$, then

$$D_{11} = (D')^{-1}_{22} = D_{33} = t (D')^{-1}_{11} = t D_{22} = t (D')^{-1}_{33},$$

as $D_{33} D'_{11} = t$ by (5.2). If $\sigma = (2,3)$, then

$$D_{11} = t (D')^{-1}_{22} = t D_{33} = t (D')^{-1}_{11} = t D_{22} = t (D')^{-1}_{33},$$

as $D_{11} D'_{22} = t$ by (5.2). Thus, if σ is an odd permutation, $D_{11} s D'_{33} = ts = 1_S$. Therefore, if $X_t \mathrel{\mathcal{J}} X_s$, then $t = s$ or $ts = 1_S$. □

Theorem 5.3.7. *Let S be an infinite commutative anti-negative unital semiring with a zero and no zero-divisors. Then, the monoid $M_3(S)$ is not finitely generated.*

Proof. Let $\tilde{S}$ be an infinite subset of S such that if $x \in \tilde{S}$ and x^{-1} exists and $x^{-1} \neq x$, then $x^{-1} \notin \tilde{S}$. Consider the matrices X_s for $s \in \tilde{S}$ where

$$X_s = \begin{pmatrix} 0_S & 1_S & s \\ 1_S & 0_S & 1_S \\ 1_S & 1_S & 0_S \end{pmatrix}.$$

By Lemma 5.3.6, we have that X_s is prime for all $s \in \tilde{S}$. Moreover, if s^{-1} exists and $s \neq s^{-1}$, then $s^{-1} \notin \tilde{S}$, so $J_{X_t} \cap J_{X_s} = \emptyset$ for any $s \neq t \in \tilde{S}$.

Thus, as any generating set for $M_3(S)$ must contain a matrix $\mathcal{J}$-related to X_s for each $s \in \tilde{S}$, $M_3(S)$ is not finitely generated. □

We remark that anti-negative semirings with a zero and no zero-divisors, are exactly the trivial ring and the semirings attained from adjoining a zero to a semiring. Hence, we have the following immediate corollary.

Corollary 5.3.8. *Let S be an infinite commutative unital semiring. Then $M_3(S^0)$ is not finitely generated*

Corollary 5.3.9. *The monoid $M_3(\mathbb{Z}_{\max})$ is not finitely generated.*

Lemma 5.3.10. *Let S be a commutative anti-negative unital semiring with a zero and no zero-divisors, and $X = \{x_1, x_1^{-1}, x_2, \ldots, x_m\}$ be a generating set for $(U(S), \cdot)$. Then, for $n \geq 2$, $GL_n(S)$ is generated by the following matrices:*

$$A = A_1(x_1) \cdot P_{(1,\ldots,n-1)}, \; B = A_1(x_1^{-1}) \cdot P_{(1,\ldots,n)}$$

$$\textit{and } A_1(x) \textit{ for } x \in X \setminus \{x_1, x_1^{-1}\}$$

where P_σ, for $\sigma \in \mathcal{S}_n$, is the permutation matrix of σ.

Recall $A_i(x)$ is the diagonal matrix where the (i,i) entry is x and all other diagonal entries are 1_S.

Proof. Clearly $A^{n-1} = A_1(x_1) \cdots A_{n-1}(x_1)$. Now, remark that for all $1 \leq i \leq n-1$,

$A_i(x_1^{-1})P_{(1,\dots,n)} = P_{(1,\dots,n)}A_{i+1}(x_1^{-1})$. Hence, we have

$$\begin{aligned}
B^{n-2}A^{n-1}B &= (A_1(x_1^{-1})P_{(1,\dots,n)})^{n-2}A_1(x_1)\cdots A_{n-1}(x_1)B \\
&= P_{(1,\dots,n)}^{n-2}A_{n-1}(x_1^{-1})\cdots A_2(x_1^{-1})\cdot A_1(x_1)\cdots A_{n-1}(x_1)B \\
&= P_{(1,\dots,n)}^{n-2}A_1(x_1)B \\
&= P_{(1,\dots,n)}^{n-2}A_1(x_1)A_1(x_1^{-1})P_{(1,\dots,n)} \\
&= P_{(1,\dots,n)}^{n-1}.
\end{aligned}$$

Therefore, $(B^{n-2}A^{n-1}B)^{n-1} = P_{(1,\dots,n)}$ as $(n-1)^2 \equiv n^2-2n+1 \equiv 1 \mod n$. Moreover,

$$\begin{aligned}
B(B^{n-2}A^{n-1}B)A &= A_1(x_1^{-1})P_{(1,\dots,n)}P_{(1,\dots,n)}^{n-1}A_1(x_1)P_{(1,\dots,n-1)} \\
&= A_1(x_1^{-1})A_1(x_1)P_{(1,\dots,n-1)} \\
&= P_{(1,\dots,n-1)}.
\end{aligned}$$

Finally, note that

$$P_{(1,\dots,n)}^{-2}P_{(1,\dots,n-1)}P_{(1,\dots,n)} = P_{(1,2)}.$$

Thus, as $\mathcal{S}_n$ can be generated by the permutations $(1,2)$ and $(1,\dots,n)$ [Rot12, Exercise 2.9(iii)], every permutation matrix can be expressed as a suitable product of A and B. Furthermore,

$$A_i(x_1) = P_{(1,i)}AP_{(1,\dots,n-1)}^{n-2}P_{(1,i)},\ A_i(x_1^{-1}) = P_{(1,i)}BP_{(1,\dots,n)}^{n-1}P_{(1,i)}$$

$$\text{and } A_i(x) = P_{(1,i)}A_1(x)P_{(1,i)} \text{ for } x \in X \setminus \{x_1, x_1^{-1}\}.$$

Thus, $A_i(x)$ for $x \in X$ and $1 \le i \le n$ can be generated by the given set of matrices. Hence, as each diagonal matrix can be expressed as a product using the matrices $A_i(x)$ for $x \in X$ and $1 \le i \le n$, they can be generated by A, B, and $A_1(x)$ for $x \in X \setminus \{x_1, x_1^{-1}\}$.

Finally, by Lemma 5.3.2, every invertible matrix is a monomial matrix in which every non 0_S entry is in $U(S)$. Therefore, each invertible matrix, and hence every matrix in $GL_n(S)$, can be expressed as diagonal matrix with entries from $U(S)$ multiplied by a permutation matrix. □

Corollary 5.3.11. *Let $n \ge 2$. The group $GL_n(\mathbb{Z}_{\max})$ is minimally generated (as a semigroup) by the following matrices:*

$$A = A_1(1)\cdot P_{(1,\dots,n-1)} \text{ and } B = A_1(-1)\cdot P_{(1,\dots,n)}$$

where P_σ, for $\sigma \in \mathcal{S}_n$, is the permutation matrix of σ.

Proof. By Lemma 5.3.10, A and B generate $GL_n(\mathbb{Z}_{\max})$. To show minimality, observe that $GL_n(\mathbb{Z}_{\max})$ is non-abelian for $n \geq 2$ and hence not 1-generated. Thus, A and B minimally generate $GL_n(\mathbb{Z}_{\max})$. □

Lemma 5.3.12. *Let $n \geq 2$. The submonoid $M \subseteq M_n(\mathbb{Z}_{\max})$ generated by the following matrices contains $UT_n(\mathbb{Z}_{\max})$:*

$$A = A_1(1) \cdot P_{(1,\ldots,n-1)},\ B = A_1(-1) \cdot P_{(1,\ldots,n)},\ E_{12},\ \textit{and}\ A_1(-\infty).$$

where P_σ, for $\sigma \in \mathcal{S}_n$, is the permutation matrix of σ.

Proof. If we are able to generate the generators from Corollary 5.2.5 from the given matrices, we are done. First, $-1 \cdot I_n, A_1(1), \ldots, A_n(1)$ can be generated as they are units in $M_n(\mathbb{Z}_{\max})$ and A and B generate $GL_n(\mathbb{Z}_{\max})$ by Corollary 5.3.11. So, it suffices to show that we can generate $A_i(-\infty)$ for $1 \leq i \leq n$ and E_{ij} for $1 \leq i < j \leq n$. Observe that,

$$A_i(-\infty) = P_{(i1)} A_1(-\infty) P_{(1i)}, \text{ and } E_{ij} = \begin{cases} P_{(j2)} E_{12} P_{(2j)} & \text{if } i = 1, \\ P_{(21j)} E_{12} P_{(2j1)} & \text{if } i = 2, \\ P_{(i1j2)} E_{12} P_{(2j1i)} & \text{if } 2 < i < j. \end{cases}$$

Thus, as $P_\sigma \in GL_n(\mathbb{Z}_{\max})$ for all $\sigma \in \mathcal{S}_n$, $UT_n(\mathbb{Z}_{\max}) \subseteq M$. □

Theorem 5.3.13. *The monoid $M_3(\mathbb{Z}_{\max})$ is minimally and irredundantly generated by the following matrices:*

$$A = A_1(1) \cdot P_{(1,2)},\ B = A_1(-1) \cdot P_{(1,2,3)},\ E_{12},\ A_1(-\infty),\ \textit{and}$$

$$X_i = \begin{pmatrix} -\infty & 0 & i \\ 0 & -\infty & 0 \\ 0 & 0 & -\infty \end{pmatrix} \textit{ for } i \in \mathbb{N}_0$$

where P_σ, for $\sigma \in \mathcal{S}_3$, is the permutation matrix of σ.

Proof. Consider the following matrices:

$$A' = \begin{pmatrix} 0 & -\infty & -\infty \\ -\infty & -\infty & -1 \\ -\infty & 0 & -\infty \end{pmatrix},\ B' = \begin{pmatrix} 0 & -\infty & -\infty \\ -\infty & 1 & -\infty \\ -\infty & -\infty & 0 \end{pmatrix},$$

$$C' = \begin{pmatrix} 0 & -\infty & -\infty \\ -\infty & -\infty & -\infty \\ -\infty & -\infty & 0 \end{pmatrix}, \ D' = \begin{pmatrix} 0 & -\infty & -\infty \\ -\infty & 0 & 0 \\ -\infty & 0 & -\infty \end{pmatrix}.$$

By Lemma 5.3.10, A and B generate $GL_3(\mathbb{Z}_{\max})$. So, as $A', B', P_\sigma \in GL_3(\mathbb{Z}_{\max})$ for all $\sigma \in \mathcal{S}_3$, they can be generated by the given matrices. Hence, C' and D' can also be generated as $C' = P_{(1,2)}A_1(-\infty)P_{(1,2)}$ and $D' = P_{(1,3,2)}E_{12}P_{(1,3)}$. By considering the second and third rows and columns of A', B', C', and D', we get a generating set for $M_2(\mathbb{Z}_{\max})$ by Lemma 5.3.5, as these matrices are block diagonal and the $(1,1)$ entry of each matrix is 0. Hence, by multiplying by $A_1(x)$ for $x \in \mathbb{Z}_{\max}$, we can generate any block diagonal matrix with a 1×1 block and then a 2×2 block, as $A_1(x) \in GL_3(\mathbb{Z}_{\max})$ when $x \in \mathbb{Z}$. Moreover, as permutation matrices are in $GL_3(\mathbb{Z}_{\max})$, we can generate any permutation of this matrix.

If a matrix has at least four $-\infty$ entries then it contains a row and column with at least two $-\infty$ entries. Thus, every matrix with at least four $-\infty$ entries is either a permutation of an upper triangular matrix, which can be generated by Lemma 5.3.12, or a block diagonal matrix with a 2×2 block, which can be generated by the above argument.

Next, we show that we can generate all matrices with three $-\infty$ entries. It suffices to check up to permutation of the rows and columns. Moreover, as $A^T, B^T, (E_{12})^T$ and $A_1(-\infty)^T$ have more than four $-\infty$ entries and $X_i^T = P_{(1,3)}X_iP_{(1,3)}$, we can see that the transposes of the generators can be generated and hence, we only have to check we can generate all matrices up to transposition.

Note that, for $a, b, c, d, e, f, g \in \mathbb{Z}_{\max}$,

$$\begin{pmatrix} a & b & c \\ d & e & f \\ -\infty & -\infty & g \end{pmatrix} = \begin{pmatrix} 0 & -\infty & c \\ -\infty & 0 & f \\ -\infty & -\infty & g \end{pmatrix} \begin{pmatrix} a & b & -\infty \\ d & e & -\infty \\ -\infty & -\infty & 0 \end{pmatrix}.$$

Thus, as the above matrix can be expressed as a product of matrices with at least four $-\infty$ entries, any matrix with at least two $-\infty$ entries in the same row or column can be generated. For the final matrix with three $-\infty$ entries, we can assume some entries are 0 by multiplying by a diagonal matrix.

$$\begin{pmatrix} -\infty & a & b \\ c & -\infty & d \\ e & f & -\infty \end{pmatrix} = \begin{pmatrix} a & -\infty & -\infty \\ -\infty & d & -\infty \\ -\infty & -\infty & e \end{pmatrix} \begin{pmatrix} -\infty & 0 & b-a \\ c-d & -\infty & 0 \\ 0 & f-e & -\infty \end{pmatrix}.$$

We split this matrix into two cases, first if $x+y+z = i \geq 0$, then

$$\begin{pmatrix} -\infty & 0 & x \\ y & -\infty & 0 \\ 0 & z & -\infty \end{pmatrix} = \begin{pmatrix} 0 & -\infty & -\infty \\ -\infty & i-x & -\infty \\ -\infty & -\infty & z \end{pmatrix} X_i \begin{pmatrix} -z & -\infty & -\infty \\ -\infty & 0 & -\infty \\ -\infty & -\infty & x-i \end{pmatrix},$$

and if $x+y+z = -i \leq 0$, then

$$\begin{pmatrix} -\infty & 0 & x \\ y & -\infty & 0 \\ 0 & z & -\infty \end{pmatrix} = \begin{pmatrix} -\infty & -\infty & 0 \\ -\infty & -x & -\infty \\ z & -\infty & -\infty \end{pmatrix} X_i \begin{pmatrix} -\infty & -\infty & x \\ -\infty & 0 & -\infty \\ -z-i & -\infty & -\infty \end{pmatrix}.$$

Thus, we have now shown that we can generate all matrices with at least three $-\infty$ entries.

Now, let $a, b, c, d, e, f, g \in \mathbb{Z}$ and $x \in \mathbb{Z}_{\max}$ and split into two cases. If $a+e \geq b+d$, then

$$\begin{pmatrix} a & b & c \\ d & e & -\infty \\ f & x & g \end{pmatrix} = \begin{pmatrix} 0 & b-e & -\infty \\ -\infty & 0 & -\infty \\ -\infty & -\infty & 0 \end{pmatrix} \begin{pmatrix} a & -\infty & c \\ d & e & -\infty \\ f & x & g \end{pmatrix},$$

and if $b+d \geq a+e$, then

$$\begin{pmatrix} a & b & c \\ d & e & -\infty \\ x & f & g \end{pmatrix} = \begin{pmatrix} 0 & a-d & -\infty \\ -\infty & 0 & -\infty \\ -\infty & -\infty & 0 \end{pmatrix} \begin{pmatrix} -\infty & b & c \\ d & e & -\infty \\ x & f & g \end{pmatrix}.$$

We have already shown that we can generate all matrices with two $-\infty$ entries in the same row or column. By taking $x = -\infty$ above, we can see that we can generate all matrices with two $-\infty$ entries in different rows and columns as they are expressible as the product of matrices with at least three $-\infty$ entries. Taking $x \in \mathbb{Z}$ shows that we can express any matrix with one $-\infty$ entry as the product matrices with at least two $-\infty$ entries and therefore a product of the given matrices.

Finally, for matrices without $-\infty$ entries, we may scale the columns so that the top row only contains 0 entries. So, we only need to consider matrices of the form

$$\begin{pmatrix} 0 & 0 & 0 \\ a & b & c \\ d & e & f \end{pmatrix}$$

where $a, b, c, d, e, f \in \mathbb{Z}$. Further, we may rearrange the columns to assume $a \leq b, c$ and $e \leq f$. Now, observe that if $d \leq e$, then

$$\begin{pmatrix} 0 & 0 & 0 \\ a & b & c \\ d & e & f \end{pmatrix} = \begin{pmatrix} 0 & -\infty & -\infty \\ a & b & c \\ d & e & f \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ -\infty & 0 & -\infty \\ -\infty & -\infty & 0 \end{pmatrix}$$

as $a \leq b, c$, and $d \leq e \leq f$. If $e \leq d$, then

$$\begin{pmatrix} 0 & 0 & 0 \\ a & b & c \\ d & e & f \end{pmatrix} = \begin{pmatrix} 0 & -b & -d \\ c & 0 & -\infty \\ f & -\infty & 0 \end{pmatrix} \begin{pmatrix} -\infty & -\infty & 0 \\ a & b & -\infty \\ d & e & -\infty \end{pmatrix}$$

as $a - b \leq 0$ and $e - d \leq 0$. Thus, as every matrix in the products above contains a $-\infty$ entry we can generate every matrix without $-\infty$ entries. Therefore, every matrix can be expressed as a product of the given matrices.

To show that this generating set is irredundant, note that $\phi_3(A)$, $\phi_3(B)$, $\phi_3(E_{12})$, $\phi_3(A_1(-\infty))$, and $\phi_3(X_0)$ provide a generating set for $M_3(\mathbb{B})$ as ϕ_3 is a surjective morphism and $\phi_3(X_i) = \phi_3(X_0)$ for all $i \in \mathbb{N}_0$. However, $M_3(\mathbb{B})$ is minimally and irredundantly generated by 5 matrices [HMSW21, Table 1]. Thus, A, B, E_{12}, and $A_1(-\infty)$ are irredundant. Now, note that each X_i is in a different prime $\mathscr{J}$-class by Lemma 5.3.6, and as any generating set for $M_3(\mathbb{Z}_{\max})$ must require each at least one representative from each, all the X_i matrices are irredundant.

Therefore, all matrices in the generating set are irredundant, so the generating set is irredundant. The generating set is minimal by Corollary 5.3.9. □

For a semigroup $\mathcal{S}$, we say $x \in \mathcal{S}$ is *regular* if there exists $y \in \mathcal{S}$ such that $xyx = x$. In 1968, Devadze [Dev68] showed that the size of minimal generating sets for $M_n(\mathbb{B})$ grows at least exponentially as $n \to \infty$. However, Kim and Roush [KR77] showed that there exists a semigroup generated by four matrices from $M_n(\mathbb{B})$ which contains all regular matrices in $M_n(\mathbb{B})$. The following corollary shows that a similar result holds for $M_n(\mathbb{Z}_{\max})$ when $n \leq 3$.

Corollary 5.3.14. *The submonoid of $M_3(\mathbb{Z}_{\max})$ generated by the following matrices:*

$$A = A_1(1) \cdot P_{(1,2)},\ B = A_1(-1) \cdot P_{(1,2,3)},$$

$$E_{12},\ \textit{and}\ A_1(-\infty)$$

contains all regular matrices in $M_3(\mathbb{Z}_{\max})$.

Proof. By the proof of Theorem 5.3.13, the matrices $\mathcal{J}$-related to some X_i are exactly those with one $-\infty$ entry in each row and column. We can see this as every matrix with this property is generated by multiplying X_i by units, and any matrix $\mathcal{J}$-related to X_i has this property as each X_i is prime in $M_3(\mathbb{Z}_{\max})$ by Lemma 5.3.6. Moreover, in the proof of Theorem 5.3.13, matrices $\mathcal{J}$-related to some X_i are not used to generate any matrix not $\mathcal{J}$-related to some X_j. Thus, we can use the matrices above to generate every matrix not $\mathcal{J}$-related to X_i for some $i \in \mathbb{N}_0$.

We show that every matrix $\mathcal{J}$-related to X_i is not regular. By Lemma 5.3.6, each X_i is prime in $M_3(\mathbb{Z}_{\max})$, hence every matrix $\mathcal{J}$-related to X_i is prime. Let $M \mathrel{\mathcal{J}} X_i$ for some $i \in \mathbb{N}_0$, so in particular M is prime. Suppose M is regular, then there exists Y such that $MYM = M$. Hence, YM is an idempotent, and as M is prime, YM is a unit. Thus, $YM = I_3$, giving a contradiction by Lemma 5.1.2, as $M \notin GL_3(\mathbb{Z}_{\max})$. Therefore, the submonoid of $M_3(\mathbb{Z}_{\max})$ generated by A, B, E_{12}, and $A_1(-\infty)$ contains all regular matrices in $M_3(\mathbb{Z}_{\max})$. □

We now pose the question whether the theorem proved by Kim and Roush [KR77] is again true when applied to $M_n(\mathbb{Z}_{\max})$ rather than $M_n(\mathbb{B})$.

Question 5.3.15. *Is it the case that the matrices in the statement of Lemma 5.3.12 generate all regular matrices of $M_n(\mathbb{Z}_{\max})$?*

5.4 A Presentation for Upper Triangular Tropical Matrices

Presentations, especially finite presentations, are an important tool in the study of semigroups. In particular, they enable us to construct semigroup representations, as if one has a finite presentation of a semigroup, then we can define a representation of the semigroup by only defining the representation on the generators and verifying that the set of relations hold on these generators.

So far this chapter has been devoted to constructing minimal and irredundant generating sets for matrix monoids. Specifically, in Section 5.2.1, we established that $UT_n(\mathbb{Z}_{\max})$ is finitely generated. In this section, we construct a finite presentation of the monoid $UT_n(\mathbb{Z}_{\max})$ for all $n \in \mathbb{N}$ using the minimal generating set given in Corollary 5.2.5.

Let Σ be an alphabet and $R \subseteq \Sigma^* \times \Sigma^*$ be a set of relations. A *monoid presentation* is defined to be the ordered pair $\langle \Sigma \mid R \rangle$. We say that a monoid $\mathcal{S}$ is *presented* by $\langle \Sigma \mid R \rangle$ if $\mathcal{S} = \Sigma^*/\rho_R$ where ρ_R is the smallest congruence on Σ containing R. In other words, $\mathcal{S}$ is isomorphic to the free monoid of Σ subject to the relations given in R. We say $\mathcal{S}$ is *finitely presented* if there exists finite Σ and finite R such that $\mathcal{S}$ presented by $\langle \Sigma \mid R \rangle$.

We begin by constructing a finite presentation for $UT_n(\mathbb{Z}_{\max})$ using a generating set of cardinality $\frac{n(n+5)}{2}$. We remark that this is not a minimal generating set for $UT_n(\mathbb{Z}_{\max})$, as Corollary 5.2.5 establishes that $UT_n(\mathbb{Z}_{\max})$ is minimally and irreducibly generated by a generating set of cardinality $\frac{n(n+3)}{2} + 1$. Nonetheless, using this presentation allows for more concise proofs, and we can then use this presentation to show that a different presentation with a minimal generating set is also a presentation for $UT_n(\mathbb{Z}_{\max})$.

To construct our first presentation, we define the alphabet $\Omega_n = \{a_k, a_k^{-1}, c_k, d_{ij} : 1 \leq k \leq n,\ 1 \leq i < j \leq n\}$, and consider the following relations over Ω_n for $1 \leq i < j \leq n$ and $1 \leq k, l \leq n$:

$$a_i a_j = a_j a_i \tag{C1}$$

$$c_i c_j = c_j c_i \tag{C2}$$

$$c_k^2 = c_k \tag{C3}$$

$$d_{ij}^2 = d_{ij} \tag{C4}$$

$$a_l c_k = c_k a_l \tag{C5}$$

$$a_k d_{ij} = d_{ij} a_k \qquad i, j \neq k \tag{C6}$$

$$c_k d_{ij} = d_{ij} c_k \qquad i, j \neq k \tag{C7}$$

$$d_{ij} d_{st} = d_{st} d_{ij} \qquad j \neq s < t \neq i \tag{C8}$$

$$d_{ij} d_{jt} = d_{jt} d_{ij} d_{it} \qquad j < t \tag{C9}$$

$$d_{ij} a_i d_{ij} = a_i d_{ij} \tag{C10}$$

$$d_{ij} a_j d_{ij} = d_{ij} a_j \tag{C11}$$

$$a_i a_j d_{ij} = d_{ij} a_i a_j \tag{C12}$$

$$a_k c_k = c_k \tag{Z1}$$

$$c_i d_{ij} = c_i \tag{Z2}$$

$$d_{ij} c_j = c_j \tag{Z3}$$

$$a_k a_k^{-1} = \varepsilon \tag{I1}$$

$$a_k^{-1} a_k = \varepsilon \tag{I2}$$

where ε is the empty word in Ω_n^*. Define R' to be the collection of all these relations. Note that a_k and c_k commute with all the generators apart from d_{ik} or d_{kj}.

Throughout the rest of this section, we will use S to denote the monoid presented by $\langle \Omega_n \mid R' \rangle$, recall that this is the quotient of Ω_n^* by the smallest congruence on Ω_n containing the relations R'. We now aim to show that S is isomorphic to $UT_n(\mathbb{Z}_{\max})$. In order to do this, we require a number of technical lemmas.

We begin by showing that we are able to deduce a number of relations involving a_k^{-1} for $1 \leq k \leq n$ from the relations in R', and hence are satisfied by S.

Lemma 5.4.1. *The following relations are satisfied by S. For $1 \leq i < j \leq n$, and $1 \leq k, l \leq n$:*

$$a_l^{-1} a_k = a_k a_l^{-1} \tag{S1}$$

$$a_i^{-1} a_j^{-1} = a_j^{-1} a_i^{-1} \tag{S2}$$

$$a_l^{-1} c_k = c_k a_l^{-1} \tag{S3}$$

$$a_k^{-1} d_{i,j} = d_{i,j} a_k^{-1} \qquad i, j \neq k \tag{S4}$$

$$a_i^{-1} a_j^{-1} d_{i,j} = d_{i,j} a_i^{-1} a_j^{-1} \tag{S5}$$

$$d_{i,j}a_i^{-1}d_{i,j} = d_{i,j}a_i^{-1} \tag{S6}$$

$$d_{i,j}a_j^{-1}d_{i,j} = a_j^{-1}d_{i,j} \tag{S7}$$

$$a_k^{-1}c_k = c_k \tag{S8}$$

Proof. We show that these relations are satisfied by S by using (I1) and (I2) with the relations from R'.

(S1): $a_l^{-1}a_k =_S a_l^{-1}a_k a_l a_l^{-1} =_S a_l^{-1}a_l a_k a_l^{-1} =_S a_k a_l^{-1}$ by (I1), (C1) and (I2).

(S2): $a_i^{-1}a_j^{-1} =_S a_j^{-1}a_j a_i^{-1}a_j^{-1} =_S a_j^{-1}a_i^{-1}a_j a_j^{-1} =_S a_j^{-1}a_i^{-1}$ by (I2), (S1) and (I1).

(S3): $a_l^{-1}c_k =_S a_l^{-1}c_k a_l a_l^{-1} =_S a_l^{-1}a_l c_k a_l^{-1} =_S c_k a_l^{-1}$ by (I1), (C5) and (I2).

(S4): $a_k^{-1}d_{i,j} =_S a_k^{-1}d_{i,j}a_k a_k^{-1} =_S a_k^{-1}a_k d_{i,j}a_k^{-1} =_S d_{i,j}a_k^{-1}$ by (I1), (C6) and (I2).

(S5): $a_i^{-1}a_j^{-1}d_{i,j} =_S a_i^{-1}a_j^{-1}d_{i,j}a_j a_i a_i^{-1}a_j^{-1} =_S d_{i,j}a_i^{-1}a_j^{-1}$ by (I1), (C1), (C12) and (I2).

(S6): $d_{i,j}a_i^{-1}d_{i,j} =_S d_{i,j}a_j a_j^{-1}a_i^{-1}d_{i,j} =_S d_{i,j}a_j d_{i,j}a_j^{-1}a_i^{-1} =_S d_{i,j}a_i^{-1}$ by (I1), (S5) and (C11).

(S7): $d_{i,j}a_j^{-1}d_{i,j} =_S d_{i,j}a_j^{-1}a_i^{-1}a_i d_{i,j} =_S a_j^{-1}a_i^{-1}d_{i,j}a_i d_{i,j} =_S a_j^{-1}d_{i,j}$ by (I2), (S5) and (C10).

(S8): $a_k^{-1}c_k =_S a_k^{-1}a_k c_k =_S c_k$ by (Z1) and (I2).

□

Note that a_k^{-1} commutes with all the generators except d_{ik} or d_{kj}. For each $k \le n$, let $\Omega_{k,n} = \{a_i, a_i^{-1}, c_i, d_{i,j} : i \le k,\ 1 \le i < j \le n\} \subseteq \Omega_n$ and observe that $\Omega_{n,n} = \Omega_n$.

Lemma 5.4.2. *Let $k \le n$ and $w \in \Omega^*_{k-1,n}$. Then, $wa_k =_S a_k w_1$, $wa_k^{-1} =_S a_k^{-1}w_2$, and $wc_k =_S c_k w_3$ for some $w_1, w_2, w_3 \in \Omega^*_{k-1,n}$.*

Proof. Let $l \in \{1, -1\}$, to show $wa_k^l =_S a_k^l v$ for some $v \in \Omega^*_{k-1,n}$, note that

(i) $a_i a_k^l =_S a_k^l a_i$, for all $i < k$ by (C1) and (S1),

(ii) $a_i^{-1}a_k^l =_S a_k^l a_i^{-1}$, for all $i < k$ by (S1) and (S2),

(iii) $c_i a_k^l =_S a_k^l c_i$ for all $i < k$ by (C5) and (S3),

(iv) $d_{i,j}a_k^l =_S a_k d_{i,j}^l$ for all all $i < j \neq k$ with $i < k$ by (C6) and (S4).

(v) $d_{i,k}a_k =_S d_{i,k}a_k a_i a_i^{-1} =_S a_k a_i d_{i,k} a_i^{-1}$ for all $i < k$ by (I1), (C1), and (C12),

(vi) $d_{i,k}a_k^{-1} =_S d_{i,k}a_k^{-1}a_i^{-1}a_i =_S a_k^{-1}a_i^{-1}d_{i,k}a_i$ for all $i < k$ by (I2), (S2), and (S5).

Hence, using these rules, we can permute a_k^l to the left of w, possibly introducing copies of a_i and a_i^{-1} with $1 \leq i < k$. Thus, $wa_k =_S a_k w_1$ and $wa_k^{-1} =_S a_k^{-1} w_2$ for some $w_1, w_2 \in \Omega_{k-1,n}^*$.

To show $wc_k =_S c_k w_3$ for some $w_3 \in \Omega_{k-1,n}^*$, note that

(i) $d_{i,k}c_k =_S c_k$ for all $i < k$ by (Z3),

(ii) $a_i c_k =_S c_k a_i$ for all $i < k$ by (C5),

(iii) $a_i^{-1}c_k =_S c_k a_i^{-1}$ for all $i < k$ by (S3),

(iv) $c_i c_k =_S c_k c_i$ for all $i < k$ by (C2),

(v) $d_{i,j}c_k =_S c_k d_{i,j}$ for all $i < j \neq k$ with $i < k$ by (C7).

Hence, we can permute c_k to the left of w, removing any $d_{i,k}$ with $i < k$ in w. Thus, we have that $wc_k =_S c_k w_3$ for some $w_3 \in \Omega_{k-1,n}^*$ □

Lemma 5.4.3. *Let $k \leq n$ and $w \in (\Omega_{k-1,n} \cup \{a_k, a_k^{-1}\})^*$. Then, for each h with $k < h \leq n$, there exists $w' \in \Omega_{k-1,n}^*$ and $m \in \mathbb{Z}$ such that $wd_{k,h} =_S a_k^m d_{k,h} w'$.*

Proof. Since $w \in (\Omega_{k-1,n} \cup \{a_k, a_k^{-1}\})^*$ we can write $w = a_k^{m_1} u_1 a_k^{m_2} u_2 \cdots a_k^{m_t}$ where $m_i \in \mathbb{Z}$ and $u_i \in \Omega_{k-1,n}^*$. Applying Lemma 5.4.2 then gives $w =_S a_k^m v'$ where $m \in \mathbb{Z}$ and $v' \in \Omega_{k-1,n}^*$.

Now, note that we have the following equalities hold in S.

(i) $a_i d_{k,h} =_S d_{k,h} a_i$ for all $i < k$ by (C6),

(ii) $a_i^{-1} d_{k,h} =_S d_{k,h} a_i^{-1}$ for all $i < k$ by (S4),

(iii) $c_i d_{k,h} =_S d_{k,h} c_i$ for all $i < k$ by (C7),

(iv) $d_{s,t} d_{k,h} =_S d_{k,h} d_{s,t}$ for all $s < t \neq k$ with $s < k$ by (C8),

(v) $d_{s,k} d_{k,h} =_S d_{k,h} d_{s,k} d_{s,h}$ for all $s < k$ by (C9). (i.e for all $d_{s,k} \in \Omega_{k-1,n}$.)

Hence, for some $w' \in \Omega^*_{k-1,n}$,

$$wd_{k,h} =_S a_k^m v' d_{k,h} =_S a_k^m d_{k,h} w'$$

as we are able use the above rules to permute all letters in v' to the right of the $d_{k,h}$, possibly introducing some $d_{s,h} \in \Omega_{k-1,n}$ with $1 \leq s < k$. □

Finally, we can show that S is isomorphic to $UT_n(\mathbb{Z}_{\max})$.

Theorem 5.4.4. *$UT_n(\mathbb{Z}_{\max})$ is finitely presented by $\langle \Omega_n \mid R' \rangle$.*

Proof. Recall, we define S to be the monoid presented by $\langle \Omega_n \mid R' \rangle$. To show that $UT_n(\mathbb{Z}_{\max})$ is isomorphic to S and hence presented by $\langle \Omega_n \mid R' \rangle$, we first show that each $w \in S$ has a normal form in Ω_n^* given by

$$w =_S a_n(x_{n,n}) d_{n-1}(x_{n-1,n}) \cdots a_2(x_{2,2}) d_1(x_{1,2}, \ldots, x_{1,n}) a_1(x_{1,1})$$

where $x_{i,j} \in \mathbb{Z}_{\max}$ for $1 \leq i \leq j \leq n$, and for each i in the range $1 \leq i \leq n$,

$$a_i(x_{i,i}) = \begin{cases} a_i^{x_{i,i}} & \text{if } x_{i,i} \in \mathbb{Z}, \\ c_i & \text{if } x_{i,i} = -\infty, \end{cases}$$

while for each $1 \leq i < n$, $d_i(x_{i,i+1}, \ldots, x_{i,n}) = d_{i,i+1}(x_{i,i+1}) \cdots d_{i,n}(x_{i,n})$ where

$$d_{ij}(x_{i,j}) = \begin{cases} a_i^{x_{i,j}} d_{ij} a_i^{-x_{i,j}} & \text{if } x_{i,j} \neq -\infty, \\ \varepsilon & \text{if } x_{i,j} = -\infty. \end{cases}$$

We will then use this normal form to define an isomorphism between S and $UT_n(\mathbb{Z}_{\max})$. First, we show that given $w \in \Omega^*_{k,n}$, we can express w as

$$w =_S d_k(x_{k,k+1}, \ldots x_{k,n}) a_k(x_{k,k}) v$$

where $x_{k,j} \in \mathbb{Z}_{\max}$ for $k \leq j \leq n$ and $v \in \Omega^*_{k-1,n}$.

By using Lemma 5.4.2 and Lemma 5.4.3, we have that $w =_S uv$ for some $u \in (\Omega_{k,n} \setminus \Omega_{k-1,n})^*$ and $v \in \Omega^*_{k-1,n}$. Since $\Omega_{k,n} \setminus \Omega_{k-1,n} = \{a_k, a_k^{-1}, c_k, d_{k,h} : k < h \leq n\}$ and each $d_{k,h}$ is idempotent, we can write any word over this set as

$$u_1 d_{k,j_1} u_2 d_{k,j_2} \cdots u_{l'} d_{k,j_{l'}} u_{l'+1}$$

where $u_i \in \{a_k, a_k^{-1}, c_k\}^*$. Since the submonoid of S generated by $\{a_k, a_k^{-1}, c_k\}$ is commutative and c_k is a left zero for $\Omega_{k,n} \setminus \Omega_{k-1,n}$ it follows that we can express w as

$$w =_S \left(\prod_{i=1}^{l} a_k^{t_i} d_{k,j_i} \right) a_k^{t_{l+1}} c_k^{\varepsilon_k} v$$

where $l \in \mathbb{N}_0$, $t_1, \ldots, t_{l+1} \in \mathbb{Z}$, $k < j_1, \ldots, j_l \leq n$ and $\varepsilon_k \in \{0, 1\}$. In particular, for $k = n$, we may express w as $a_k(x)v$ for some $x \in \mathbb{Z}_{\max}$ and $v \in \Omega_{k-1,n}^*$. By the definition of $d_{k,j}(x)$, $a_k^x d_{k,j} = d_{k,j}(x) a_k^x$ for $x \in \mathbb{Z}$. Thus,

$$w =_S \left(\prod_{i=1}^{l} d_{k,j_i}(T_i) \right) a_k^{T_{l+1}} c_k^{\varepsilon_k} v.$$

where $T_i = \sum_{m=1}^{i} t_m$ for $1 \leq i \leq l+1$.

We are able to simplify the above expression, by noticing that, when $k < n, m$ and $n \neq m$, we can make the following commutation:

$$\begin{aligned}
d_{k,n}(x) d_{k,m}(y) &=_S a_k^x d_{k,n} a_k^{y-x} d_{k,m} a_k^{-y} \\
&=_S a_k^x d_{k,n} a_m^{x-y} a_m^{y-x} a_k^{y-x} d_{k,m} a_k^{-y} && \text{(I1), (I2)} \\
&=_S a_k^x a_m^{x-y} d_{k,n} d_{k,m} a_m^{y-x} a_k^{-x} && \text{(C6), (C12), (S4), (S5)} \\
&=_S a_k^x a_m^{x-y} d_{k,m} d_{k,n} a_m^{y-x} a_k^{-x} && \text{(C8)} \\
&=_S a_k^y d_{k,m} a_k^{x-y} a_m^{x-y} a_m^{y-x} d_{k,n} a_k^{-x} && \text{(C6), (C12), (S4), (S5)} \\
&=_S a_k^y d_{k,m} a_k^{x-y} d_{k,n} a_k^{-x} && \text{(I1), (I2)} \\
&=_S d_{k,m}(y) d_{k,n}(x).
\end{aligned}$$

When $n = m$, we can simplify in the following way.

$$\begin{aligned}
d_{k,n}(x) d_{k,n}(y) &=_S a_k^x d_{k,n} a_k^{y-x} d_{k,n} a_k^{-y} \\
&=_S \begin{cases} a_k^x d_{k,n} (\prod_{i=1}^{|y-x|} a_k d_{k,n}) a_k^{-y} & y \geq x \\ a_k^x d_{k,n} (\prod_{i=1}^{|y-x|} a_k^{-1} d_{k,n}) a_k^{-y} & y < x \end{cases} && \text{(C10), (S6)} \\
&=_S \begin{cases} a_k^x a_k^{y-x} d_{k,n} a_k^{-y} & y \geq x \\ a_k^x d_{k,n} a_k^{y-x} a_k^{-y} & y < x \end{cases} && \text{(C10), (S6)} \\
&=_S a_k^{\max(x,y)} d_{k,n} a_k^{-\max(x,y)} \\
&=_S d_{k,n}(\max(x, y)).
\end{aligned}$$

Now, we define the following variables. For $k < j$, let

$$x_{k,j} = \begin{cases} \max(\{T_m : j_m = j\}) & \text{if } j_m = j \text{ for some } m, \\ -\infty & \text{otherwise.} \end{cases}$$

$$x_{k,k} = \begin{cases} T_{l+1} & \text{if } \varepsilon_k = 0, \\ -\infty & \text{if } \varepsilon_k = 1. \end{cases}$$

Then, using the above facts about multiplying $d_{k,n}(x)$ and $d_{k,m}(y)$, we have that

$$w =_S d_k(x_{k,k+1}, \dots, x_{k,n}) a_k(x_{k,k}) v.$$

Thus, each $w \in \Omega^*_{k,n}$, can be expressed in the above form. Therefore, by applying this with $k = n, \dots, 1$ for $w \in \Omega^*_n = \Omega^*_{n,n}$, we have that

$$w =_S a_n(x_{n,n}) d_{n-1}(x_{n-1,n}) \cdots a_2(x_{2,2}) d_1(x_{1,2}, \dots, x_{1,n}) a_1(x_{1,1}).$$

Using this set of normal forms, we now construct an isomorphism between S and $UT_n(\mathbb{Z}_{\max})$. Define the map $\phi\colon \Omega^*_n \to UT_n(\mathbb{Z}_{\max})$, given by $a_i \to A_i(1)$, $a_i^{-1} \mapsto A_i(-1)$, $c_i \mapsto A_i(-\infty)$, $d_{ij} \mapsto E_{ij}$ and extending multiplicatively. Now, given a normal form

$$w = a_n(x_{n,n}) d_{n-1}(x_{n-1,n}) \cdots a_2(x_{2,2}) d_1(x_{1,2}, \dots, x_{1,n}) a_1(x_{1,1}),$$

a simple calculation shows that

$$\phi(w) = \begin{pmatrix} x_{1,1} & \cdots & x_{1,n} \\ & \ddots & \vdots \\ & & x_{n,n} \end{pmatrix}.$$

Thus, as $x_{i,j}$, for $1 \leq i \leq j \leq n$, can be any element from $\mathbb{Z}_{\max}$, we get that every matrix in $UT_n(\mathbb{Z}_{\max})$ is the image of exactly one normal form, and hence the normal forms are in bijection with $UT_n(\mathbb{Z}_{\max})$. So, it suffices to check that the images of the generators satisfy the relations in R'.

We can see that (C1–C3), (C5), (Z1), (I1), and (I2) hold as $\phi(a_k)$, $\phi(a_k^{-1})$, and $\phi(c_k)$ are diagonal for all $1 \leq k \leq n$, so the results follow instantly from the multiplication of $\mathbb{Z}_{\max}$.

For the rest of the relations, note that for $X \in UT_n(\mathbb{Z}_{\max})$,

$$(XE_{i,j})_{s,t} = \begin{cases} \max(X_{s,j}, X_{s,i}) & s \leq i \text{ and } t = j \\ X_{s,t} & \text{otherwise.} \end{cases}$$

$$(E_{i,j}X)_{s,t} = \begin{cases} \max(X_{i,t}, X_{j,t}) & s = i \text{ and } t \geq j \\ X_{s,t} & \text{otherwise.} \end{cases}$$

Then, (C4) holds as we can see that $E_{i,j}^2 = E_{i,j}$, (C6–C7) hold as $k \neq i, j$, and (Z2–Z3) hold as, for $s \leq i < j$ and $t \geq j > i$,

$$(A_i(-\infty)E_{i,j})_{s,j} = -\infty \text{ and } (E_{i,j}A_j(-\infty))_{i,t} = -\infty.$$

Similarly, (C8) holds as $(E_{i,j}E_{s,t})_{i,t} = -\infty = (E_{s,t}E_{i,j})_{s,j}$. For (C9), it suffices to check that $(E_{i,j}E_{j,t})_{j,k} = (E_{j,t}E_{i,j}E_{i,t})_{j,k}$ for $k \geq t$ and that $(E_{i,j}E_{j,t})_{i,t} = (E_{j,t}E_{i,j}E_{i,t})_{i,t}$.

For the final 3 relations, we can see that (C10) holds as for $t \geq j$,

$$(E_{i,j}A_i(1)E_{i,j})_{i,t} = (A_i(1)E_{i,j})_{i,t},$$

(C11) holds as for $s \leq i < j$,

$$(E_{i,j}A_j(1)E_{i,j})_{s,j} = (E_{i,j}A_j(1))_{s,j},$$

and finally (C12) holds as, for $s < i < j$ and $t > j > i$, we have that

$$\begin{aligned} (A_i(1)A_j(1)E_{i,j})_{i,t} &= -\infty = (E_{i,j}A_i(1)A_j(1))_{i,t}, \\ (A_i(1)A_j(1)E_{i,j})_{s,j} &= -\infty = (E_{i,j}A_i(1)A_j(1))_{s,j}, \\ (A_i(1)A_j(1)E_{i,j})_{i,j} &= 1 = (E_{i,j}A_i(1)A_j(1))_{i,j}. \end{aligned}$$

Thus, $UT_n(\mathbb{Z}_{\max})$ satisfies every relation in R', and is hence finitely presented by $\langle \Omega_n \mid R' \rangle$. □

It is well known that if a semigroup is finitely presented then it can be finitely presented with every finite generating set for the semigroup [Ruš95, Proposition 3.1]. So, we will now use the above Theorem to construct a finite presentation for $UT_n(\mathbb{Z}_{\max})$ using the minimal and irreducible generating set from Corollary 5.2.5. For this, we define an alphabet $\Sigma_n = \{a_k, b, c_k, d_{ij} : 1 \leq k \leq n,\ 1 \leq i < j \leq n\}$ and relations for $1 \leq i < j \leq n$, and $1 \leq k \leq n$:

$$a_k b = b a_k, \tag{R1}$$

$$d_{i,j} b = b d_{i,j}, \tag{R2}$$

$$a_1 \cdots a_n b = \varepsilon. \tag{R3}$$

We define R to be the collection of relations (C1–C11), (Z1–Z3), and (R1–R3). That is R' with (I1), (I2), and (C12) replaced with (R1–R3).

Theorem 5.4.5. *$UT_n(\mathbb{Z}_{\max})$ is finitely presented by $\langle \Sigma_n \mid R \rangle$.*

Proof. Let M be the monoid presented by $\langle \Sigma_n \mid R \rangle$ and recall that $S = UT_n(\mathbb{Z}_{\max})$ is the monoid presented by $\langle \Omega_n \mid R' \rangle$. We show that $M \cong S$. Define $\phi \colon M \to S$ to be the map given by $a_i \mapsto a_i,\ c_i \mapsto c_i,\ d_{i,j} \mapsto d_{i,j}$, and $b \mapsto a_1^{-1} \cdots a_n^{-1}$ and extending multiplicatively. To see that ϕ is a well-defined map, we must show that $\phi(\Sigma_n^*)$ satisfies the relations R. We see this by remarking that ϕ is the identity map on $\Sigma_n \setminus \{b\}$, and hence satisfies the relations (C1–C11) and (Z1–Z3). Thus, it suffices to check that $\phi(\Sigma_n^*)$ satisfies the relations (R1–R3). For $1 \leq i < j \leq n$ and $1 \leq k \leq n$,

$$\begin{aligned}
\phi(a_k)\phi(b) &= a_k a_1^{-1} \cdots a_n^{-1} =_S a_1^{-1} \cdots a_n^{-1} a_k = \phi(b)\phi(a_k) && \text{by (S1)},\\
\phi(d_{i,j})\phi(b) &= d_{i,j} a_1^{-1} \cdots a_n^{-1} =_S a_1^{-1} \cdots a_n^{-1} d_{i,j} = \phi(b)\phi(d_{i,j}) && \text{by (S2), (S4), (S5)},\\
\phi(a_1) \cdots \phi(a_n)\phi(b) &= a_1 \cdots a_n a_1^{-1} \cdots a_n^{-1} =_S \varepsilon = \phi(\varepsilon) && \text{by (C1), (I1)}.
\end{aligned}$$

Now, define $\psi \colon S \to M$ to be the map given by $a_i \mapsto a_i,\ c_i \mapsto c_i,\ d_{i,j} \mapsto d_{i,j}$, and $a_i^{-1} \mapsto a_1 \cdots a_{i-1} a_{i+1} \cdots a_n b$ and extending multiplicatively. To show that ψ is a well-defined map, we show that $\psi(\Omega_n^*)$ satisfies the relations R'. We see this by remarking that ψ is the identity map on $\Omega_n \setminus \{a_k^{-1} : 1 \leq k \leq n\}$, and hence satisfies the relations (C1–C11) and (Z1–Z3). Thus, it suffices to check that $\psi(\Omega_n^*)$ satisfies the relations (I1), (I2), and (C12). For $1 \leq i < j \leq n$ and $1 \leq k \leq n$,

$$\begin{aligned}
\psi(a_k)\psi(a_k^{-1}) &= a_k a_1 \cdots a_{k-1} a_{k+1} \cdots a_n b =_M \varepsilon = \psi(\varepsilon) && \text{(C1), (R3)},\\
\psi(a_k^{-1})\psi(a_k) &= a_1 \cdots a_{k-1} a_{k+1} \cdots a_n b a_k =_M \varepsilon = \psi(\varepsilon) && \text{(C1), (R1), (R3)},
\end{aligned}$$

$$\begin{aligned}
\psi(a_i)\psi(a_j)\psi(d_{i,j}) &= a_ia_jd_{i,j} \\
&=_M a_ia_jd_{i,j}a_1\cdots a_nb && \text{(R3)}, \\
&=_M a_1\cdots a_nbd_{i,j}a_ia_j && \text{(C1), (C6), (R1), (R2)}, \\
&=_M d_{i,j}a_ia_j && \text{(R3)}, \\
&= \psi(d_{i,j})\psi(a_i)\psi(a_j).
\end{aligned}$$

Thus, ψ is a well defined morphism. We now show that ϕ and ψ and mutually inverse morphisms. Clearly, $\psi\phi(a_i) = a_i, \psi\phi(c_i) = c_i$, $\psi\phi(d_{i,j}) = d_{i,j}$ and we can see that

$$\begin{aligned}
\psi\phi(b) &= \psi(a_1^{-1}\cdots a_n^{-1}) \\
&= (ba_2\cdots a_n)\cdots(ba_1\cdots a_{n-1}) \\
&=_M b^na_1^{n-1}\cdots a_n^{n-1} && \text{by (C1), (R1)}, \\
&=_M (a_1\cdots a_nb)^{n-1}b && \text{by (C1), (R1)}, \\
&=_M b && \text{by (R3)}.
\end{aligned}$$

Therefore, $\psi\phi\colon M \to M$ is the identity map on M. Similarly, $\phi\psi(a_i) = a_i, \phi\psi(c_i) = c_i$, and $\phi\psi(d_{i,j}) = d_{i,j}$, so finally note that, for $1 \leq k \leq n$,

$$\begin{aligned}
\phi\psi(a_k^{-1}) &= \phi(a_1\cdots a_{k-1}a_{k+1}\cdots a_nb) \\
&= a_1\cdots a_{k-1}a_{k+1}\cdots a_na_1^{-1}\cdots a_n^{-1} \\
&= a_k^{-1} && \text{by (C1), (S1), (I1)}.
\end{aligned}$$

Thus, $\phi\psi\colon S \to S$ is the identity map on S. Therefore, ϕ and ψ are mutually inverse morphisms and M and S are isomorphic. Hence, $\langle \Sigma_n \mid R\rangle$ is a finite presentation for $UT_n(\mathbb{Z}_{\max})$ with a minimal generating set. $\square$

The presentation given by the above theorem has $\frac{n(n+3)}{2}+1$ generators and $\frac{1}{8}(n^4 + 6n^3 + 15n^2 + 10n + 8)$ relations.

Chapter 6

Growth of Commutative Bipotent Matrices

The growth rate of a semigroup is an important invariant in geometric semigroup theory as it provides information about the geometry and structure of the semigroup [GK17]. For instance, a renowned theorem in this area is Gromov's theorem on groups of polynomial growth, which says that a finitely generated group has polynomial growth if and only if it has a nilpotent subgroup of finite index [Gro81]. This implies that every finitely generated group of polynomial growth satisfies a semigroup identity as Malcev [Mal53] showed that all nilpotent groups of class n satisfy a semigroup identity. However, the corresponding statement for semigroups is not true as, by Shneerson [Shn93], there exist finitely generated semigroups with polynomial growth which do not satisfy any semigroup identities.

In [dP03], d'Alessandro and Pasku investigated the growth of finitely generated subsemigroups of $M_n(S)$, when S is a commutative bipotent semiring. They showed that, for any such finitely generated subsemigroup, the growth function is bounded above by a polynomial. However, the degree of the polynomial is dependent on the dimension of the matrices and the number of unique entries in the matrices in the generating set. As a result, different generating sets for the same semigroup can give upper bounds for the growth functions with different polynomial degrees.

In Section 6.1, we introduce the notation and definitions required for the rest of this chapter. In Section 6.2, for S a commutative bipotent semiring, we provide a polynomial upper bound for the growth function of any finitely generated subsemigroup of

$M_n(S)$ where the degree of the polynomial is dependent both on n and the growth of the multiplicative semigroup generated by the entries of the matrices in the generating set. When we restrict our attention to finitely generated subsemigroups of $M_n(\mathbb{T})$, we obtain a polynomial upper bound for the growth function where the degree of the polynomial is only dependent on n and the *rank* of the free abelian subgroup which the finite entries in the matrices of the generating set generate as a group. The finite entries in the matrices in any generating set for a subsemigroup of $M_n(\mathbb{T})$ generate as a group the same free abelian subgroup of $(\mathbb{R}, +)$; hence the bound on the degree of the growth function of a semigroup is independent of the generating set. Moreover, when further restricted to $M_n(\mathbb{Q}_{\max})$, we show that the growth of any finitely generated subsemigroup is bounded above by a polynomial with a degree only dependent on n. We then consider $UT_n(S)$, where S is a commutative bipotent semiring, and find a different upper bound for any finitely generated subsemigroup $UT_n(S)$. Again, we show that if we restrict to $UT_n(\mathbb{Q}_{\max})$, then the growth of a finitely generated subsemigroup is bounded above by a polynomial with degree dependent only on n. Finally, for all $n \in \mathbb{N}$, we give examples of finitely generated subsemigroups of $M_n(\mathbb{T})$ and $UT_n(\mathbb{T})$ which attain these bounds, demonstrating that these bounds are sharp.

6.1 Preliminaries

Let S be a finitely generated semigroup and X be a finite generating set for S. The *growth function of S with respect to X* is given by $f_X(k) = |\bigcup_{i=1}^{k} X^i|$. We say the growth function of S with respect to X, $f_X(k)$, is *bounded above by a polynomial of degree n* if there exists $c_X > 0$ such that for all $k \in \mathbb{N}$, $f_X(k) \leq c_X k^n$.

In fact, if the growth function of S with respect to X is bounded above by a polynomial of degree n then the growth function with respect to any finite generating set is bounded above by a polynomial of degree n. This fact is well known, but we give a short proof here for completeness.

Proposition 6.1.1. *Let X and Y be finite generating sets for a semigroup S. Then, for all $n \in \mathbb{N}_0$, the growth function of S with respect to X is bounded above by a polynomial of degree n if and only if the growth function of S with respect to Y is bounded above by a polynomial of degree n.*

Proof. Let $X = \{x_1, \ldots, x_l\}$ and $Y = \{y_1, \ldots, y_s\}$ be generating sets for S. Suppose $f_X(k) \leq c_X k^n$ for some c_X and $n \in \mathbb{N}_0$. We can express the elements of Y as products of elements from X since X is a generating set for S. Thus, for each $1 \leq i \leq s$, there exists $1 \leq j_1, \ldots, j_{m_i} \leq l$ such that $y_i = x_{j_1} \cdots x_{j_{m_i}}$. Let $m = \max_i(m_i)$. Then

$$f_Y(k) \leq f_X(mk) \leq c_X (mk)^n = c_X m^n k^n \leq c_Y k^n$$

for some $c_Y \geq c_X m^n$. Therefore, as we chose X and Y arbitrarily, the growth function of S with respect to any finite generating set is bounded above by a polynomial of degree n. □

Thus, we say that the growth of a semigroup S is *bounded above by a polynomial of degree n* if there exists a generating set X for S and $c_X > 0$ such that $f_X(k) \leq c_X k^n$.

Recall, we denote the subsemiring of tropical rationals by $\mathbb{Q}_{\max} = \mathbb{T} \cap (\mathbb{Q} \cup \{-\infty\})$.

6.2 Growth of Commutative Bipotent Matrices

Recall that a semiring $(S, +, \cdot)$ is called bipotent if $x+y \in \{x, y\}$ for all $x, y \in S$. In this section, we find upper bounds for the growth of any finitely generated subsemigroup of $M_n(S)$ or $UT_n(S)$ when S is a commutative bipotent semiring.

Proposition 6.2.1. *Let S be a commutative bipotent semiring, T be a finitely generated subsemigroup of $M_n(S)$, and X be a finite generating set for T. If the growth of the multiplicative semigroup generated by the entries of the matrices in X is bounded above by a polynomial of degree $\ell \in \mathbb{N}_0$. Then, the growth function of T is bounded above by a polynomial of degree ℓn^2.*

Proof. For every $k \geq 1$, let C_k be the set of all the finite entries of the matrices in X^k. Let $c_k = |\bigcup_{i=1}^k C_i|$. As the growth of the semigroup generated by the entries of the matrices in X is bounded above by a polynomial of degree ℓ, we have that $c_k \leq \beta k^\ell$ for some $\beta > 0$ as S is bipotent.

Hence, as every matrix in X^k has entries in $C_k \cup \{-\infty\}$, we obtain for every $k \in \mathbb{N}$,

$$f_X(k) \leq (c_k + 1)^{n^2} \leq (\beta k^\ell + 1)^{n^2} \leq ((\beta + 1)k^\ell)^{n^2} = \delta k^{\ell n^2}$$

where $\delta = (\beta + 1)^{n^2}$. □

Since S is a commutative semiring, for every finitely generated subsemigroup of the multiplicative semigroup $(S, \cdot)$ there exists an $\ell \in \mathbb{N}_0$ such that the growth of S is bounded above by a polynomial of degree ℓ [Kho92, Kho95]. Thus, we may apply the above theorem to any finitely generated subsemigroup of $M_n(S)$ when S is a commutative bipotent semiring.

The previous upper bound on the degree of the growth, given by d'Alessandro and Pasku [dP03], is $(c-1)n^2+1$ where c is the number of distinct matrix entries in a generating set for T. Thus, the new bound given above is only worse when $n \geq 2$, no matrix in X contains 0_S or 1_S entries, and the growth of the multiplicative semigroup generated by the entries of the matrices in X is bounded above by a polynomial of degree c.

In order to achieve more explicit upper bound of the polynomial degree, we now restrict to the case where we consider finitely generated subsemigroups of $M_n(\mathbb{T})$. To do this, we first require the following lemma which gives the growth of finitely generated subsemigroups of the multiplicative semigroup of $\mathbb{T}$ in terms of the rank of free abelian subgroup they generate as a group.

Lemma 6.2.2. *Let T be a finitely generated subsemigroup of the multiplicative semigroup of $\mathbb{T}$ and C be a finite generating set for T. Let ℓ be the rank of the free abelian subgroup of $(\mathbb{R}, +)$ generated as a group by the finite entries of C. Then the growth of T with respect to C is bounded above by a polynomial of degree ℓ*

Proof. Let G be the free abelian group generated as a group by $D = C \setminus \{-\infty\}$. Let $f(k)$ be the growth of T with respect to C and $g(k)$ be the growth of G with respect to $(D \cup D^{-1})$, where $D^{-1} = \{d^{-1} : d \in D\}$. Clearly, $f(k) \leq g(k) + 1$, where the $+1$ accounts for the case where $-\infty \in C$. So, it suffices to show that $g(k) \leq ck^\ell$ for some $c > 0$.

As G is a free abelian group of rank ℓ, G has growth upper bounded by a polynomial of degree ℓ [Wol68, Theorem 3.2]. Thus,

$$f(k) \leq g(k) + 1 \leq c'k^\ell$$

for some $c' > 0$. $\square$

Corollary 6.2.3. *Let T be a finitely generated subsemigroup of $M_n(\mathbb{T})$ and ℓ be the rank of the free abelian subgroup of $(\mathbb{R}, +)$ generated as a group by the finite entries of* *the matrices in T. Then, the growth function of T is bounded above by a polynomial of degree ℓn^2.*

Proof. As $\mathbb{T}$ is bipotent, the finite entries of the matrices in T and the finite entries of the matrices in any generating set for T generate, as a group, the same free abelian subgroup of $(\mathbb{R}, +)$. Thus, the result follows immediately from Proposition 6.2.1 and Lemma 6.2.2 □

If we restrict to matrices over $\mathbb{Q}_{\max}$ rather than matrices over $\mathbb{T}$ or an arbitrary commutative bipotent semiring, then the group generated, as a group, by any finite subset of $\mathbb{Q}$ containing a non-zero entry is isomorphic to $\mathbb{Z}$, so we can simplify the result in this case.

Corollary 6.2.4. *Let T be a finitely generated subsemigroup of $M_n(\mathbb{Q}_{\max})$. Then, the growth function of T is polynomially upper bounded of degree n^2.*

Proof. All finitely generated subgroups of $(\mathbb{Q}, +)$ are either trivial or isomorphic to $(\mathbb{Z}, +)$, [Rob96, Exercise 4.2.6]. □

Again comparing to the bound by d'Alessandro and Pasku [dP03], we can see in this case, the new bound is only worse when $n \geq 2$ and S is generated by a matrix in which every entry is the same non-zero entry of $\mathbb{Q}$.

We now provide similar results for the semigroup of upper triangular matrices over commutative bipotent semirings and $\mathbb{Q}_{\max}$.

Proposition 6.2.5. *Let S be a commutative bipotent semiring, T be a finitely generated subsemigroup of $UT_n(S)$, and X be a finite generating set for T. If the growth of the multiplicative semigroup generated by the entries of the matrices in X is bounded above by a polynomial of degree $\ell \in \mathbb{N}_0$. Then, the growth function of T is bounded above by a polynomial of degree $\frac{\ell n(n+1)}{2}$.*

Proof. For every $k \geq 1$, let C_k be the set of the finite entries of the matrices in the set X^k. Let $c_k = |\cup_{i=1}^{k} C_i|$. As the growth of the semigroup generated by the entries of the matrices in X is bounded above by a polynomial of degree ℓ, we have that $c_k \leq \beta k^\ell$ for some $\beta > 0$ as S is bipotent.

Hence, as every matrix in X^k has entries in $C_k \cup \{-\infty\}$, we obtain for every $k \in \mathbb{N}$,

$$f_X(k) \leq (c_k + 1)^{\frac{n(n+1)}{2}} \leq (\beta k^\ell + 1)^{\frac{n(n+1)}{2}} \leq ((\beta + 1)k^\ell)^{\frac{n(n+1)}{2}} = \delta k^\gamma$$

where $\delta = (\beta+1)^{\frac{n(n+1)}{2}}$ and $\gamma = \frac{\ell n(n+1)}{2}$. □

Again, we restrict to the cases where the commutative bipotent semiring is $\mathbb{T}$ or $\mathbb{Q}_{\max}$ to give explicit bounds on the growth of finitely generated subsemigroups of $UT_n(\mathbb{T})$ and $UT_n(\mathbb{Q}_{\max})$.

Corollary 6.2.6. *Let T be a finitely generated subsemigroup of $UT_n(\mathbb{T})$ and ℓ be the rank of the free abelian subgroup of $(\mathbb{R}, +)$ generated as a group by the finite entries of the matrices in T. Then, the growth function of T is bounded above by a polynomial of degree $\frac{\ell n(n+1)}{2}$.*

Proof. As $\mathbb{T}$ is bipotent, the finite entries of the matrices in T and the finite entries of the matrices in any generating set for T generate, as a group, the same free abelian subgroup of $(\mathbb{R}, +)$. Thus, the result follows immediately from Proposition 6.2.5 and Lemma 6.2.2. □

Corollary 6.2.7. *Let T be a finitely generated subsemigroup of $UT_n(\mathbb{Q}_{\max})$. Then, the growth function of T is bounded above by a polynomial of degree $\frac{n(n+1)}{2}$.*

Proof. All finitely generated subgroups of $(\mathbb{Q}, +)$ are either trivial or isomorphic to $(\mathbb{Z}, +)$, [Rob96, Exercise 4.2.6]. □

6.3 The Bounds Are Sharp

In the previous section, we showed that for a given finitely generated subsemigroup T of $M_n(\mathbb{T})$ or $UT_n(\mathbb{T})$, the growth function of T is bounded above by a polynomial of degree dependent on n and ℓ, the rank of the free abelian group generated as a group by the finite entries of the matrices in T. We now show that for all $n \in \mathbb{N}$ and $\ell \in \mathbb{N}_0$, there exist finitely generated subsemigroups of $M_n(\mathbb{T})$ and $UT_n(\mathbb{T})$ such that the finite entries generate, as a group, a free abelian group of of rank ℓ with growth functions bounded below by polynomials of degrees ℓn^2 and $\frac{\ell n(n+1)}{2}$ respectively, that is, the upper bounds given in Corollary 6.2.3 and Corollary 6.2.6.

Theorem 6.3.1. *Let $n \in \mathbb{N}$ and $\ell \in \mathbb{N}_0$. Then, there exists a finite generating set X for a subsemigroup of $UT_n(\mathbb{T})$, such that the growth function of X is bounded below by $ck^{\frac{\ell n(n+1)}{2}}$ for some $c > 0$ where ℓ is the rank of the free abelian subgroup of $(\mathbb{R}, +)$ generated as a group by the finite entries of the matrices generated by X.*

Proof. The proof is immediate if $\ell = 0$, so we may assume $\ell \geq 1$. Let $I = \{\gamma_1, \dots, \gamma_\ell\} \subset (\mathbb{R}, +)$ be a minimal group generating set for a free abelian group of rank ℓ. Consider the set of matrices $C_k \subseteq UT_n(\mathbb{T})$ such that the entries on and above the diagonal of the matrices are the tropical product of at most $\lfloor \frac{k-n}{2n} \rfloor$ elements from I. Now, let X be the set of all $n \times n$ upper triangular matrices with entries from $\{\gamma_1, \dots, \gamma_\ell, -\gamma_1, \dots, -\gamma_\ell, 0, -\infty\}$. We will show that for all $k \in \mathbb{N}$, $C_k \subseteq X^k$ and there exists $c > 0$ such that $|C_k| \geq ck^{\frac{\ell n(n+1)}{2}}$. Thus, showing that the growth function of X is bounded below by $ck^{\frac{\ell n(n+1)}{2}}$.

Let $A \in C_k$. Let $L_m, R_m \in UT_n(\mathbb{T})$ be the diagonal matrices such that $(L_m)_{ii} = A_{im}$ for all $i \leq m$ and 0 otherwise and $(R_m)_{ii} = -A_{im}$ for all $i < m$ and 0 otherwise. Let E'_m be the upper triangular matrix where all diagonal entries are 0 and $(E'_m)_{im} = 0$ for all $i \leq m$ and $-\infty$ elsewhere. To show that $A \in X^k$, let

$$\Sigma = \prod_{m=0}^{n-1} L_{n-m} E'_{n-m} R_{n-m} \in UT_n(\mathbb{T}).$$

Note that $(L_m E'_m R_m)_{ii} = 0$ for $i \neq m$. Thus, for $i \leq j$, we can see that

$$\Sigma_{ij} = (L_j)_{ii} (E'_j)_{ij} (R_j)_{jj} = A_{ij} + 0 + 0 = A_{ij}.$$

Therefore, $A = \Sigma$. Note that since $A \in C_k$, for all $i \leq j$, A_{ij} can be expressed as the product of at most $\lfloor \frac{k-n}{2n} \rfloor$ entries from I, so the diagonal matrices L_m and R_m can be expressed as the product of at most $\lfloor \frac{k-n}{2n} \rfloor$ matrices from X. Therefore, for each $1 \leq m \leq n$, $L_m E'_m R_m$ can be expressed as the product of $2 \lfloor \frac{k-n}{2n} \rfloor + 1 \leq \lfloor \frac{k}{n} \rfloor$ matrices from X. Hence, A can be expressed as the product of $n \lfloor \frac{k}{n} \rfloor \leq k$ matrices from X, and thus $A \in X^k$.

Now, as $\{\gamma_1, \dots, \gamma_\ell\}$ is a minimal group generating set for a free abelian group, the monoid generated by $\{\gamma_1, \dots, \gamma_\ell\}$ is a free commutative monoid of rank ℓ and has a growth function bounded below by $c'k^\ell$ for some $c' > 0$. Thus, there are at least $(c'(\lfloor \frac{k-n}{2n} \rfloor)^\ell)^{\frac{n(n+1)}{2}}$ matrices in $C_k \subseteq X^k$. Hence the growth function of X is bounded below by $ck^{\frac{\ell n(n+1)}{2}}$ for some $c > 0$. ◻

Theorem 6.3.2. *Let $n \in \mathbb{N}$ and $\ell \in \mathbb{N}_0$. Then, there exists a finite generating set X for a subsemigroup of $M_n(\mathbb{T})$, such that the growth function of X is bounded below by $ck^{\ell n^2}$ for some $c > 0$ where ℓ is the rank of the free abelian subgroup of $(\mathbb{R}, +)$ generated as a group by the finite entries of the matrices generated by X.*

Proof. The proof is immediate if $\ell = 0$, so we may assume $\ell \geq 1$. Let $I = \{\gamma_1, \ldots, \gamma_\ell\} \subset (\mathbb{R}, +)$ be a minimal group generating set for a free abelian group of rank ℓ such that $1 \leq \gamma_i \leq 2$ for each i, and consider the set of matrices $C_k \subseteq M_n(\mathbb{T})$ such that the diagonal entries of the matrices are the tropical product of between $\lfloor \frac{2k-4n}{16n+3} \rfloor$ and $\lfloor \frac{3k-6n}{16n+3} \rfloor$ elements from I and the off diagonal entries are the tropical product of between 0 and $\lfloor \frac{k-2n}{16n+3} \rfloor$ elements from $\{-\gamma_1, \ldots, -\gamma_\ell\}$. Let X be the set of all $n \times n$ matrices with entries from $\{\gamma_1, \ldots, \gamma_\ell, -\gamma_1, \ldots, -\gamma_\ell, 0, -\infty\}$. We will show that $C_k \subseteq X^k$ and $|C_k| \geq ck^{\ell n^2}$ for some $c > 0$.

Let $A \in C_k$. Let $L_m, R_m \in M_n(\mathbb{T})$ be the diagonal matrix with entries $(L_m)_{ii} = A_{im} - A_{mm}$ if $i \geq m$ and 0 otherwise and $(R_m)_{ii} = A_{im} - A_{ii}$ if $i \leq m$ and 0 otherwise. Let E_m be the matrix where all diagonal entries are 0 and $(E_m)_{im} = 0$ for all $i \geq m$ and all other entries are $-\infty$. Similarly, let E'_m be the matrix where all diagonal entries are 0 and $(E'_m)_{im} = 0$ for all $i \leq m$ and all other entries are $-\infty$. Let Λ be the diagonal matrix where $\Lambda_{ii} = A_{ii}$ for all $1 \leq i \leq n$. To show that $A \in X^k$, let

$$\Sigma = \left(\prod_{m=1}^{n} L_m E_m L_m^{-1} \right) \Lambda \left(\prod_{m=0}^{n-1} R_{n-m} E'_{n-m} R_{n-m}^{-1} \right).$$

Note that $(L_m E_m L_m^{-1})_{ii} = 0$ and $(R_m E'_m R_m^{-1})_{ii} = 0$. Thus, for $1 \leq i, j \leq n$, we can see that

$$\Sigma_{ij} = \max_{m \leq i,j} ((L_m)_{ii} + \Lambda_{mm} + (R_j)_{mm}) = \max_{m \leq i,j} (A_{im} + A_{mj} - A_{mm}) = A_{ij}$$

as if $m < \min(i, j)$ then $A_{im} + A_{mj} - A_{mm} \leq 0 + 0 - \lfloor \frac{2k-4n}{16n+3} \rfloor \leq A_{ij}$ as $1 \leq \gamma_s \leq 2$ for each s. Thus, we have that $\Sigma = A$.

Now, for all $1 \leq m \leq n$, $E_m, E'_m \in X$ and L_m and R_m (and therefore also L_m^{-1} and R_m^{-1}) can be expressed as the product of $\lfloor \frac{3k-6n}{16n+3} \rfloor + \lfloor \frac{k-2n}{16n+3} \rfloor$ matrices from X. Similarly, Λ can be expressed as the product of $\lfloor \frac{3k-6n}{16n+3} \rfloor$ matrices from X. Thus, Σ can be expressed as the product of

$$\begin{aligned} 4n \left\lfloor \frac{3k-6n}{16n+3} \right\rfloor + 4n \left\lfloor \frac{k-2n}{16n+3} \right\rfloor + 2n + \left\lfloor \frac{3k-6n}{16n+3} \right\rfloor &\leq \frac{16n(k-2n)}{16n+3} + 2n + \frac{3k-6n}{16n+3} \\ &= k \end{aligned}$$

matrices from X, and hence $A \in X^k$.

Now, as $\{\gamma_1, \ldots, \gamma_\ell\}$ is a minimal group generating set for a free abelian group, the monoid generated by $\{\gamma_1, \ldots, \gamma_\ell\}$ is a free commutative monoid of rank ℓ and has a growth function bounded below by $c'k^{\ell}$ for some $c' > 0$. Thus, there are at least $(c'(\lfloor \frac{k-2n}{16n+3} \rfloor)^{\ell})^{n^2}$ matrices in X^k. Hence the growth function of X is bounded below by $ck^{\ell n^2}$ for some $c > 0$. □

Corollary 6.3.3. *The bounds given in Corollary 6.2.3 and Corollary 6.2.6 are sharp. i.e. For all $n \in \mathbb{N}$ and $\ell \in \mathbb{N}_0$, there exist finitely generated subsemigroups of $M_n(\mathbb{T})$ and $UT_n(\mathbb{T})$ such that their growth function is bounded below by a polynomial of degree ℓn^2 and $\frac{\ell n(n+1)}{2}$ respectively where ℓ is the rank of the free abelian group generated as a group by the finite entries of the matrices in subsemigroup.*

Chapter 7

Identities of tropical matrix semigroups and the plactic monoid of rank 4

The semigroup identities satisfied by semigroups of matrices over the tropical semiring have been widely studied in recent years [DJK18, Izh16b, IM18]. Birkhoff's HSP Theorem (Theorem 1.1.1) allows us to connect semigroup identities satisfied by a semigroup with its homomorphic images, subsemigroups and direct products. Thus, this research has led to interest in the semigroup identities satisfied by the semigroups which are representable by tropical matrices [AR23, CKK$^+$17]. In this chapter, we focus on the problem of showing when semigroup identities are not satisfied by these semigroups, allowing us to deduce when a given semigroup variety is strictly contained in another.

In particular, we prove a conjecture posed by Johnson and Kambites [JM21, Conjecture 3.5]. That is, we show that for every positive integer n there is a semigroup identity satisfied by $UT_n(\mathbb{T})$ but not by $UT_{n+1}(\mathbb{T})$. Moreover, Johnson and Kambites also asked [JM21, Question 4.8] whether the variety generated by $\mathbb{P}_4$, the plactic monoid of rank 4, is equal to the variety generated by $UT_4(\mathbb{T})$ and/or the variety generated by $UT_5(\mathbb{T})$, and in Section 7.4 we show that $\mathbb{P}_4$ satisfies semigroup identities not satisfied by $UT_5(\mathbb{T})$ and hence, the variety generated by $\mathbb{P}_4$ is strictly contained in the variety generated by $UT_5(\mathbb{T})$. It is known that the variety generated by $UT_2(\mathbb{T})$ is equal to the variety generated by $\mathbb{P}_2$ and similarly the variety generated by $UT_3(\mathbb{T})$

is equal to the variety generated by $\mathbb{P}_3$. It remains open if the variety generated by $UT_4(\mathbb{T})$ is equal to the variety generated by $\mathbb{P}_4$.

In addition to this introduction, this chapter comprises 4 sections. In Section 7.1, we introduce some notations and definitions that we use throughout the rest of the chapter.

In Section 7.2, we introduce a necessary requirement for a semigroup identity to be satisfied by the semigroup of $n \times n$ upper triangular matrices, $UT_n(\mathbb{T})$. We then use this to show that for all $n \in \mathbb{N}$ we can construct semigroup identities satisfied by $UT_n(\mathbb{T})$ but not $UT_{n+1}(\mathbb{T})$ proving the conjecture given by Johnson and Kambites [JM21, Conjecture 3.5].

In Section 7.3, we turn our attention to the full matrix semigroup, $M_n(\mathbb{T})$. We show that there exists a semigroup identity satisfied by $M_3(\mathbb{T})$ but not $M_4(\mathbb{T})$ and additionally show that there exists a semigroup identity satisfied by $M_{p-1}(\mathbb{T})$ but not $M_p(\mathbb{T})$ when p is prime. The question of if $M_{p-1}(\mathbb{T})$ and $M_p(\mathbb{T})$ generate different varieties for non-prime p remains open.

In Section 7.4, we look at the plactic monoid and find a new set of semigroup identities that is satisfied by $\mathbb{P}_4$ but not by $UT_5(\mathbb{T})$, partially answering the question posed by Johnson and Kambites [JM21, Question 4.8] by showing that the variety generated by $\mathbb{P}_4$ is strictly contained in the variety generated by $UT_5(\mathbb{T})$.

This chapter is based on the paper [Air22].

7.1 Preliminaries

When writing matrices over the tropical semiring, we use blank entries for $-\infty$ when it is clear.

For a matrix $A = (A_{ij}) \in M_n(\mathbb{T})$, we write $G_A = (V, E)$ for the weighted digraph associated to A, that is, the digraph with vertex set $\{1, \ldots, n\}$ and edge set $E(G_A)$ containing, for all $1 \leq i, j \leq n$ such that $A_{ij} \neq -\infty$, a directed edge (i, j) with weight A_{ij}. Similarly, for $A, B \in M_n(\mathbb{T})$, we write $G_{A,B}$ for the labelled-weighted digraph with vertex set $\{1, \ldots, n\}$ and edge set $E(G_A) \cup E(G_B)$ with the edges from G_A labelled by A and the edges from $E(G_B)$ labelled by B.

A path γ on a digraph is a series of edges $(i_1, j_1), \ldots, (i_m, j_m)$ such that $j_k = i_{k+1}$

for all $1 \leq k < m$. We say g is a *node* of γ if an edge starting or ending at g is in γ, and call an edge a *loop* if it starts and ends at the same node. A path γ is said to have *length* m if γ contains m edges (counted with multiplicity), written $|\gamma| = m$, and has *simple length* m if γ contains m *non-loop* edges (again counted with multiplicity). A path is called *simple* if it does not contain any loops. For any word $w \in \{A, B\}^+$ and γ a path in $G_{A,B}$, we say γ *is labelled* w if $|\gamma| = |w|$ and, for all $1 \leq r \leq |\gamma|$, the edge (i_r, j_r) is labelled $w_{(r)}$, the rth letter of w. It can be easily seen that powers of a matrix A correspond to the maximal weights of paths in G_A, that is, the (i, j)th entry of A^m is given by the maximal weight of all paths from i to j of length m in G_A.

Moreover, it is well known that, for $w \in \{a, b\}^*$, the product $w(a \mapsto A, b \mapsto B)$ corresponds to the maximal weight of paths in $G_{A,B}$, where the (i, j)th entry of $w(a \mapsto A, b \mapsto B)$ is given by the maximal weight of all paths from i to j labelled by the word $w[a \mapsto A, b \mapsto B]$ in $G_{A,B}$, that is, all paths of length $|w|$ where, for $1 \leq k \leq |w|$, the kth edge in the path is labelled by $w[a \mapsto A, b \mapsto B]_{(k)}$, the kth letter of $w[a \mapsto A, b \mapsto B]$.

7.2 Upper Triangular Matrix Semigroups

In this section we restrict our attention to the subsemigroup of upper triangular tropical matrices and show that upper triangular tropical matrix semigroups of different dimensions generate different semigroup varieties.

We begin by proving a lemma which we will use to falsify semigroup identities, which hold in $UT_n(\mathbb{T})$, by matrices in $UT_{n+1}(\mathbb{T})$.

Lemma 7.2.1. *Suppose $u, v, w \in \{a, b\}^*$ are words such that w has length n and is a factor of u but not v. Then there exists $A, B \in UT_{n+1}(\mathbb{T})$ such that $u(a \mapsto AB, b \mapsto BA) \neq v(a \mapsto AB, b \mapsto BA)$.*

Proof. Let $w \in \{a, b\}^*$ be a word of length n. We recursively define $n+1$ parameters $c_1, \cdots, c_{n+1} \in \mathbb{T}$, using the structure of the word w. Let $c_1 = 0$ and for $2 \leq k \leq n+1$ let

$$c_k = \begin{cases} c_{k-1} - 1 \text{ if } w_{(k-1)} = a \\ c_{k-1} + 1 \text{ if } w_{(k-1)} = b. \end{cases}$$

From these parameters, we can define matrices $A_w, B_w \in UT_{n+1}(\mathbb{T})$ to be

$$A_w = \begin{pmatrix} c_1 & & & & \\ & c_2 & & & \\ & & \ddots & & \\ & & & c_n & \\ & & & & c_{n+1} \end{pmatrix} B_w = \begin{pmatrix} -c_1 & 0 & & & \\ & -\infty & \ddots & & \\ & & \ddots & \ddots & \\ & & & -\infty & 0 \\ & & & & -c_{n+1} \end{pmatrix},$$

where $(A_w)_{kk} = c_k$ for $1 \leq k \leq n+1$ and $-\infty$ otherwise; $(B_w)_{11} = -c_1$, $(B_w)_{n+1,n+1} = -c_{n+1}$, $(B_w)_{k,k+1} = 0$ for $1 \leq k \leq n$ and $-\infty$ otherwise.

Let $A = A_w$ and $B = B_w$. We aim to show that if w is a factor of u but not v, then $u(a \mapsto AB, b \mapsto BA) \neq v(a \mapsto AB, b \mapsto BA)$. Note that AB and BA are given by the following matrices

$$AB = \begin{pmatrix} 0 & c_1 & & & \\ & -\infty & c_2 & & \\ & & \ddots & \ddots & \\ & & & -\infty & c_n \\ & & & & 0 \end{pmatrix} BA = \begin{pmatrix} 0 & c_2 & & & \\ & -\infty & c_3 & & \\ & & \ddots & \ddots & \\ & & & -\infty & c_{n+1} \\ & & & & 0 \end{pmatrix}.$$

Consider the labelled-weighted digraph $G_{AB,BA}$; nodes 1 and $n+1$ each have two loops of weight 0 labelled AB and BA and for each $1 \leq i \leq n$ there are two edges from i to $i+1$ of weight c_i and c_{i+1} labelled AB and BA respectively. Moreover, we define a function f_w by

$$f_w : \{a,b\}^* \to \mathbb{T}, \quad t \mapsto t(a \mapsto AB, b \mapsto BA)_{1,n+1}.$$

Recall, that $t(a \mapsto AB, b \mapsto BA)_{1,n+1}$ is equal to the maximum weight of a path labelled by $t[a \mapsto AB, b \mapsto BA]$ from node 1 to $n+1$. Thus, we will now show that $f_w(u) > f_w(v)$ by considering the maximum weighted paths from node 1 to $n+1$ in $G_{AB,BA}$ labelled by $u[a \mapsto AB, b \mapsto BA]$ and $v[a \mapsto AB, b \mapsto BA]$.

By construction, we have that $c_i > c_{i+1}$ if $w_{(i)} = a$ and $c_i < c_{i+1}$ if $w_{(i)} = b$. As the only cycles in $G_{AB,BA}$ are loops of weight 0 at nodes 1 and $n+1$, the weight of any path from 1 to $n+1$ is bounded above by the weight of the unique path ρ of length n which takes the edge of largest weight from i to $i+1$ for each $1 \leq i \leq n$.

Moreover, ρ is labelled $w[a \mapsto AB, b \mapsto BA]$, and hence the upper bound is $f_w(w)$. So for any word t, we have that $f_w(t) \leq f_w(w)$. If $t = sws'$ is a word containing w as a factor, a path of maximal weight labelled $s[a \mapsto AB, b \mapsto BA]$ around the loops at 1, $w[a \mapsto AB, b \mapsto BA]$ along ρ, and $s'[a \mapsto AB, b \mapsto BA]$ around the loops at $n+1$, gives a path of weight $f_w(w)$ as the loops have weight 0. Hence, $f_w(t) = f_w(w)$. On the other hand, if t does not contain w as a factor, then a path from 1 to $n+1$ labelled t cannot contain the simple path ρ. It follows that at some step of the path we must traverse a non-maximal weight edge between two consecutive nodes. Thus, $f_w(t) < f_w(w)$ in this case.

Therefore, $f_w(u) = f_w(w) > f_w(v)$ as w is a factor of u but not v. Hence, letting $A = A_w$ and $B = B_w$, we have that there exists $A, B \in UT_{n+1}(\mathbb{T})$ such that $u(a \mapsto AB, b \mapsto BA) \neq v(a \mapsto AB, b \mapsto BA)$. □

The following corollary is a result by Izhakian [Izh16b, Theorem 4.5] applied to the semigroup $UT_n(\mathbb{T})$. This gives us a way of generating semigroup identities for $UT_n(\mathbb{T})$.

Corollary 7.2.2. *Let $w \in \{a,b\}^*$ be any word having as its factors all the words of length $n-1$ such that waw and wbw have no letter appearing n times sequentially. Then, the semigroup identity*

$$waw[a \mapsto ab, b \mapsto ba] = wbw[a \mapsto ab, b \mapsto ba]$$

is satisfied by $UT_n(\mathbb{T})$.

Example 7.2.3. For $n = 3$, $w = ab^2a^2b$ has all words of length 2 as a factor, and neither waw nor wbw has a^3 or b^3 as a factor. Therefore, by the above corollary, $waw[a \mapsto ab, b \mapsto ba] = wbw[a \mapsto ab, b \mapsto ba]$ is an identity that holds in $UT_3(\mathbb{T})$. We will use this example later in this chapter.

Theorem 7.2.4. *For all $n \in \mathbb{N}$, there exists an identity satisfied by $UT_n(\mathbb{T})$ but not satisfied by $UT_{n+1}(\mathbb{T})$.*

Proof. As matrix multiplication is commutative if and only if $n = 1$, the identity $ab = ba$ is satisfied by $UT_1(\mathbb{T})$, but not $UT_2(\mathbb{T})$. It is known [IM09] that $UT_2(\mathbb{T})$ satisfies the Adian identity, $ab^2a^2bab^2a = ab^2aba^2b^2a$. Note that this identity can be written in the form $u[a \mapsto ab, b \mapsto ba] = v[a \mapsto ab, b \mapsto ba]$ where $u = abaab$ and

$v = abbab$, and since a^2 is a factor of u but not v then the identity is not satisfied by $UT_3(\mathbb{T})$ by Lemma 7.2.1.

For $n = 3$, we can see by Example 7.2.3, that the following identity

$$u_3 := ab^2a^2bbab^2a^2b[a \mapsto ab, b \mapsto ba] = ab^2a^2baab^2a^2b[a \mapsto ab, b \mapsto ba] =: v_3$$

is satisfied by $UT_3(\mathbb{T})$. Note that bab is a factor of u_3 but not v_3. Thus, $u_3 = v_3$ is falsified in $UT_4(\mathbb{T})$ by Lemma 7.2.1.

Now let $n \geq 4$ and define w to be the word of length n given by $w = a^2b^{n-2}$.

We aim to construct a word $\overline{w} \in \{a, b\}^*$ such that for $u = \overline{w}a\overline{w}$ and $v = \overline{w}b\overline{w}$ we have that u and v do not have any letter appearing n times sequentially; the word $\overline{w}$ contains sufficiently many factors for Corollary 7.2.2 to apply, so that the identity $u[a \mapsto ab, b \mapsto ba] = v[a \mapsto ab, b \mapsto ba]$ is satisfied by $UT_n(\mathbb{T})$; and that the word w is a factor of u but not of v, so that $u(a \mapsto AB, b \mapsto BA) \neq v(a \mapsto AB, b \mapsto BA)$ for some $A, B \in UT_{n+1}(\mathbb{T})$ by Lemma 7.2.1.

Let $w_1, \ldots, w_m$ be a complete list of words in $\{a, b\}^{n-1}$ taken in some arbitrary but fixed order. For $1 \leq i \leq m$, we define w'_i to be the word obtained from w_i by removing the prefix b if possible, and the suffix a if possible. Now, we let

$$\overline{w} = ab^{n-2}(abw'_1ab)(abw'_2ab) \cdots (abw'_{m-1}ab)(abw'_mab).$$

By construction, $\overline{w}$ contains each word of length $n-1$ as a factor and each bracketed expression (abw'_iab) does not contain a^n or b^n as a factor. Likewise, it can be seen that $\overline{w}$ does not contain a^n or b^n as each bracketed expression starts and ends with ab and each w'_i contains at most $n-2$ copies of a or b in a row. Furthermore, since $\overline{w}$ begins and ends with ab, it follows that $u = \overline{w}a\overline{w}$ and $v = \overline{w}b\overline{w}$ do not contain a^n or b^n. This shows that Corollary 7.2.2 applies, so that $u[a \mapsto ab, b \mapsto ba] = v[a \mapsto ab, b \mapsto ba]$ is satisfied by $UT_n(\mathbb{T})$.

Similarly, a^2b^{n-2} is not a factor of the bracketed expressions (abw'_iab) as $n \geq 4$ and as each bracketed expression starts and ends with ab, a^2b^{n-2} is not a factor of $\overline{w}$.

Thus, we can see that w is a factor of $u = \overline{w}a\overline{w}$ as $a\overline{w} = aab^{n-2}\cdots = w\cdots$ but not a factor of v. Therefore, by Lemma 7.2.1, $u[a \mapsto ab, b \mapsto ba] = v[a \mapsto ab, b \mapsto ba]$ is falsified in $UT_{n+1}(\mathbb{T})$. □

Another possible approach to Theorem 7.2.4 would be to use knowledge about the free objects in these varieties discussed by Kambites in [Kam22] to show that the free objects in the varieties generated by $UT_n(\mathbb{T})$ and $UT_{n+1}(\mathbb{T})$ are not isomorphic for all $n \in \mathbb{N}$.

7.3 Full Tropical Matrix Semigroups

We introduce the notation that $\overline{n} = \text{lcm}\{1, \dots, n\}$ and, for $u, v \in \{a, b\}^*$, write $\langle u, v\rangle$ for the identity $u = v$ and $\langle u, v\rangle [a \mapsto x, b \mapsto y]$ for the semigroup identity $u[a \mapsto x, b \mapsto y] = v[a \mapsto x, b \mapsto y]$. Recall, we say a matrix $A \in M_n(\mathbb{T})$ has the underlying permutation of $\sigma \in \mathcal{S}_n$ if $A_{ij} \neq -\infty$ if and only if $j = \sigma(i)$. A matrix $A \in M_n(\mathbb{T})$ is invertible if and only if A has an underlying permutation by Lemma 5.3.2. The following theorem of Izhakian and Merlet allows us to produce semigroup identities satisfied by $M_n(\mathbb{T})$.

Theorem 7.3.1. *[IM18, Theorem 3.6] For any $t \geq (n-1)^2+1$ and any identity $u = v$ satisfied by $M_{n-1}(\mathbb{T})$, where $u, v \in \{a, b\}^+$, the following holds:*

(i) If $q = r$ is an identity satisfied by $UT_n(\mathbb{T})$, then $M_n(\mathbb{T})$ satisfies the identity

$$\langle ua, va\rangle \left[a \mapsto (qr)^t \left[a \mapsto a^{\overline{n}}, b \mapsto b^{\overline{n}}\right], b \mapsto (qr)^t r \left[a \mapsto a^{\overline{n}}, b \mapsto b^{\overline{n}}\right]\right],$$

where $q, r \in \{a, b\}^+$.

(ii) If $pqp = prp$ is an identity satisfied by $UT_n(\mathbb{T})$, then $M_n(\mathbb{T})$ satisfies the identity

$$\langle ua, va\rangle \left[a \mapsto wqp \left[a \mapsto a^{\overline{n}}, b \mapsto b^{\overline{n}}\right], b \mapsto wrp \left[a \mapsto a^{\overline{n}}, b \mapsto b^{\overline{n}}\right]\right],$$

where $w = (pqprp)^t$ and $p, q, r \in \{a, b\}^+$.

Theorem 7.3.2. *There exists an identity satisfied by $M_3(\mathbb{T})$ that is not satisfied by $M_4(\mathbb{T})$.*

Proof. We apply Theorem 7.3.1 in the case $n = 3$. Set $u = a^2b^3a^3babab^3a^2$ and $v = a^2b^3ababa^3b^3a^2$. Then $u = v$ holds in $M_2(\mathbb{T})$ by [DJ17]. Let $w = ab^2a^2b$, $q = wbw[a \mapsto ab, b \mapsto ba]$, and $r = waw[a \mapsto ab, b \mapsto ba]$. Then $q = r$ holds in $UT_3(\mathbb{T})$ by Example 7.2.3. Let $t = 5$ and note that since $\overline{n} = 6$ when $n = 3$, Theorem 7.3.1(i) now yields the identity of length 29328 satisfied by $M_3(\mathbb{T})$

$$\langle s, t\rangle := \langle ua, va\rangle \left[a \mapsto (qr)^5 \left[a \mapsto a^6, b \mapsto b^6\right], b \mapsto (qr)^5 r \left[a \mapsto a^6, b \mapsto b^6\right]\right],$$

Now, let $X, Y \in M_4(\mathbb{T})$ be given by

$$X = \begin{pmatrix} -\infty & 3 & -\infty & -\infty \\ -\infty & -\infty & 3 & -\infty \\ -\infty & -\infty & -\infty & 0 \\ 2 & -\infty & -\infty & -\infty \end{pmatrix} \quad Y = \begin{pmatrix} 1 & -\infty & -\infty & -\infty \\ -\infty & -\infty & 1 & -\infty \\ 3 & 0 & -\infty & -\infty \\ -\infty & -\infty & -\infty & 2 \end{pmatrix}.$$

Then, a computation (run on the GAP computer algebra system [GAP21]) gives $s(a \mapsto X, b \mapsto Y) \neq t(a \mapsto X, b \mapsto Y)$ and hence we have constructed an identity satisfied by $M_3(\mathbb{T})$ but not by $M_4(\mathbb{T})$. Note that the matrices X and Y were also found using code run on GAP [GAP21]. $\square$

In order to prove that $M_{p-1}(\mathbb{T})$ and $M_p(\mathbb{T})$ generate different semigroup varieties we must first prove a number of lemmas, the first of which shows that we are able to falsify an identity satisfied by $M_2(\mathbb{T})$ using two matrices from $M_n(\mathbb{T})$ when $n \geq 3$ is odd.

Lemma 7.3.3. *Let $n \geq 3$ be odd, and $A, B \in M_n(\mathbb{T})$ be invertible matrices such that A has the underlying permutation of an n-cycle and B is a non-scalar diagonal matrix. Then, there exists an identity satisfied by $M_2(\mathbb{T})$, $u_2 = v_2$, such that $u_2(a \mapsto A, b \mapsto B) \neq v_2(a \mapsto A, b \mapsto B)$.*

Proof. Let $u_2 = a^2b^4a^2\ a^2b^2\ a^2b^4a^2$, $v_2 = a^2b^4a^2\ b^2a^2\ a^2b^4a^2$. Then $u_2 = v_2$ is an identity satisfied by $M_2(\mathbb{T})$, [IM09, Theorem 3.9]. Now, let $A, B \in M_n(\mathbb{T})$ be such that A has the underlying permutation of an n-cycle σ and B is a diagonal matrix. Then, as A and B are invertible matrices, we can see that $u_2(a \mapsto A, b \mapsto B) = v_2(a \mapsto A, b \mapsto B)$ if and only if $A^2B^2 = B^2A^2$, by cancelling $A^2B^4A^2$ from both sides of $u_2(a \mapsto A, b \mapsto B)$ and $v_2(a \mapsto A, b \mapsto B)$. However,

$$\begin{aligned} (A^2B^2)_{i,\sigma^2(i)} &= A_{i\sigma(i)}A_{\sigma(i)\sigma^2(i)}B^2_{\sigma^2(i),\sigma^2(i)}, \\ (B^2A^2)_{i,\sigma^2(i)} &= B^2_{ii}A_{i\sigma(i)}A_{\sigma(i)\sigma^2(i)}. \end{aligned}$$

Moreover, as $(A^2B^2)_{ij} = -\infty = (B^2A^2)_{ij}$ if $j \neq \sigma^2(i)$, we get that $A^2B^2 = B^2A^2$ if and only if $B_{ii} = B_{\sigma^2(i),\sigma^2(i)}$ for all $1 \leq i \leq n$. That is, as σ is an n-cycle and n is odd, if and only if B is a scalar matrix. Therefore, $u_2(a \mapsto A, b \mapsto B) \neq v_2(a \mapsto A, b \mapsto B)$ if A has the underlying permutation of an n-cycle and B is a non-scalar diagonal matrix. $\square$

The following lemma allows us to construct an identity satisfied by $M_n(\mathbb{T})$ in the form of two words, a_2 and b_2, substituted into any identity satisfied by $M_2(\mathbb{T})$. Thus, by using the previous lemma we may, in the n odd case, simplify the problem of falsifying an identity for $M_n(\mathbb{T})$ to showing that there exists $X, Y \in M_n(\mathbb{T})$ such that $a_2(a \mapsto X, b \mapsto Y)$ has the underlying permutation of an n-cycle and $b_2(a \mapsto X, b \mapsto Y)$ is a non-scalar diagonal matrix.

Lemma 7.3.4. *For each k in the range $3 \leq k \leq n$, let $q_k = r_k$ be an identity satisfied by $UT_k(\mathbb{T})$, where $q_k, r_k \in \{a,b\}^+$, and let $t \geq (n-1)^2 + 1$ be a fixed integer. Let $a_n = a$, $b_n = b$, and for $k = n, \ldots, 3$ recursively define*

$$a_{k-1} = (q_k r_k)^t \left[a \mapsto a_k^{\overline{k}}, b \mapsto b_k^{\overline{k}}\right] \text{ and } b_{k-1} = (q_k r_k)^t r_k \left[a \mapsto a_k^{\overline{k}}, b \mapsto b_k^{\overline{k}}\right].$$

Then, for any identity satisfied by $M_2(\mathbb{T})$, $u_2 = v_2$, we have that

$$u_2[a \mapsto a_2, b \mapsto b_2]a_2 a_3 \cdots a_{n-1} = v_2[a \mapsto a_2, b \mapsto b_2]a_2 a_3 \cdots a_{n-1}$$

is an identity satisfied by $M_n(\mathbb{T})$.

Proof. For each $3 \leq k \leq n$, we construct the identity $u_k = v_k$ which holds in $M_k(\mathbb{T})$ using Theorem 7.3.1(i), as follows

$$u_k = (u_{k-1}a)\left[a \mapsto (q_k r_k)^t[a \mapsto a^{\overline{k}}, b \mapsto b^{\overline{k}}], b \mapsto (q_k r_k)^t r_k[a \mapsto a^{\overline{k}}, b \mapsto b^{\overline{k}}]\right];$$
$$v_k = (v_{k-1}a)\left[a \mapsto (q_k r_k)^t[a \mapsto a^{\overline{k}}, b \mapsto b^{\overline{k}}], b \mapsto (q_k r_k)^t r_k[a \mapsto a^{\overline{k}}, b \mapsto b^{\overline{k}}]\right].$$

By expressing a_{k-1} as $a[a \mapsto a_{k-1}, b \mapsto b_{k-1}]$, and substituting the definitions of a_{k-1}, b_{k-1} and the definition of u_k, we have that the following equalities hold for $3 \leq k \leq n$

$$\begin{aligned} u_{k-1}[a \mapsto a_{k-1}, b \mapsto b_{k-1}]a_{k-1}a_k \cdots a_{n-1} \\ &= (u_{k-1}a)[a \mapsto a_{k-1}, b \mapsto b_{k-1}]a_k \cdots a_{n-1} \\ &= (u_{k-1}a)\Big[a \mapsto (q_k r_k)^t\left[a \mapsto a_k^{\overline{k}}, b \mapsto b_k^{\overline{k}}\right], \\ &\quad b \mapsto (q_k r_k)^t r_k\left[a \mapsto a_k^{\overline{k}}, b \mapsto b_k^{\overline{k}}\ \right]\Big]a_k \cdots a_{n-1} \\ &= u_k[a \mapsto a_k, b \mapsto b_k]a_k \cdots a_{n-1}, \end{aligned}$$

where the product $a_k \cdots a_{n-1}$ is taken to be the empty word when $k = n$. Similarly, it can be shown that

$$v_{k-1}[a \mapsto a_{k-1}, b \mapsto b_{k-1}]a_{k-1} \cdots a_{n-1} = v_k[a \mapsto a_k, b \mapsto b_k]a_k \cdots a_{n-1}$$

for $3 \leq k \leq n$. So, through the equalities given above, we have that

$$u_2[a \mapsto a_2, b \mapsto b_2]a_2a_3 \cdots a_{n-1} = u_n[a \mapsto a_n, b \mapsto b_n], \text{ and}$$
$$v_2[a \mapsto a_2, b \mapsto b_2]a_2a_3 \cdots a_{n-1} = v_n[a \mapsto a_n, b \mapsto b_n].$$

Thus, as $u_n = v_n$ is an identity satisfied by $M_n(\mathbb{T})$, we have that the identity $u_2[a \mapsto a_2, b \mapsto b_2]a_2a_3 \cdots a_{n-1} = v_2[a \mapsto a_2, b \mapsto b_2]a_2a_3 \cdots a_{n-1}$ is satisfied by $M_n(\mathbb{T})$. □

In preparation for the remainder of this section, we include two technical lemmas. The first is an elementary result in real linear algebra.

Lemma 7.3.5. *Let p be an odd prime, $X = (x_i) \in \mathbb{N}_0^p$ and $P_\sigma \in M_p(\mathbb{R})$ be the permutation matrix of a p-cycle σ. Then, $X, P_\sigma X, \ldots, P_\sigma^{p-1}X$ are linearly dependent over $\mathbb{Q}$ if and only if $x_i = x_j$ for all $1 \leq i, j \leq p$.*

Proof. Clearly, if $x_i = x_j$ for all $1 \leq i, j \leq p$ then $X = P_\sigma X$ for all $\sigma \in \mathcal{S}_p$ and hence linearly dependent over $\mathbb{Q}$. So, we only need to show the forward implication.

All permutation matrices of p-cycles are conjugate so we may, without loss of generality, suppose that σ is the p-cycle given by $\sigma(i) = i + 1 \mod p$. Suppose $X, P_\sigma X, \ldots, P_\sigma^{p-1}X$ are linearly dependent over $\mathbb{Q}$, then there exist $c_0, \ldots, c_{p-1} \in \mathbb{Q}$, not all zero, such that $\sum_{i=0}^{p-1} c_i P_\sigma^i X = 0$. By factorising out X, we can express this sum as $CX = 0$ where $C = \sum_{i=0}^{p-1} c_i P_\sigma^i \in M_p(\mathbb{R})$ is a circulant matrix [Laz95].

If $x_i = 0$ for all i, we are done. Suppose then that $x_i \neq 0$ for some i. Then, X is an eigenvector of C with eigenvalue 0. Let ω be a primitive pth root of unity, then by [Laz95, Theorem 0], C over $\mathbb{Q}[\omega]$ has (right) eigenvectors v_j with corresponding eigenvalues λ_j for $0 \leq j \leq p-1$ given by the column vectors

$$v_j = (1, \omega^j, \omega^{2j}, \ldots, \omega^{j(p-1)}) \text{ and } \lambda_j = c_0 + c_1\omega^j + \cdots + c_{p-1}\omega^{j(p-1)}.$$

Now, as we know C has 0 as an eigenvalue, suppose $\lambda_k = 0$ for some $k \geq 1$, then we have that

$$\lambda_k = c_0 + c_1\omega^k + \cdots + c_{p-1}\omega^{k(p-1)} = 0.$$

Then, as $\omega^k \neq 1$ is a pth root of unity, the above equality can only hold when $c_0 = \cdots = c_{p-1} = c$ for some $c \in \mathbb{Q}$ by [LL00, Theorem 2.2], as a non-constant solution in $\mathbb{Q}$ implies there exists a non-constant solution in $\mathbb{Z}$, giving a contradiction.

Therefore, we have that $(CX)_i = c(x_1 + \cdots + x_p) = 0$ for all $1 \leq i \leq p$ and hence either $c = 0$ or $x_1 + \cdots + x_p = 0$. However, we assumed that not all $c_0, \ldots, c_{p-1}$ were equal to zero, so we cannot have $c = 0$. So, we must have $x_1 + \cdots + x_p = 0$, but as $x_1, \ldots, x_p \in \mathbb{N}_0$ we must have $x_1 = \cdots = x_p = 0$ giving a contradiction as we supposed $x_i \neq 0$ for some i.

So, suppose $\lambda_k \neq 0$ for all $k \geq 1$. Then, $\lambda_0 = 0$, as X is an eigenvector of C with eigenvalue 0. Thus, as $\lambda_k \neq 0$ for all $k \geq 1$, X must be a scaling of the corresponding eigenvector $v_0 = (1, \ldots, 1)$ and hence, $x_i = x_j$ for all $1 \leq i, j \leq p$. □

Next, in order to prove that, for p prime, $M_{p-1}(\mathbb{T})$ and $M_p(\mathbb{T})$ generate different semigroup varieties, we require the following lemma. We introduce the notation $|w|_b^{a,k,p}$ to denote the total number of b's in w which have $k \mod p$ copies of a occurring before them when read left-to-right.

Lemma 7.3.6. *Let p be an odd prime and $A, B \in M_p(\mathbb{Q}_{\max})$ be invertible matrices such that A has the underlying permutation of a p-cycle and B is a diagonal matrix. Then, for $w \in \{a, b\}^*$, if $w(a \mapsto A, b \mapsto B)$ is a scalar matrix then, either*

(i) B is a scalar matrix, or

(ii) for all $1 \leq k \leq p$, $|w|_b^{a,k,p} = T$ for some fixed $T \in \mathbb{N}$.

Proof. We prove the contrapositive of this statement. First, suppose B is not a scalar matrix and that $|w|_b^{a,n,p} \neq |w|_b^{a,m,p}$ for some $1 \leq n < m \leq p$. If $w(a \mapsto A, b \mapsto B)$ is not a diagonal matrix then it is not a scalar matrix, so we may assume it is diagonal.

As A and B are invertible matrices, G_A and G_B have exactly one edge leaving each node. Thus, as $w(a \mapsto A, b \mapsto B)$ is diagonal, for $1 \leq i \leq p$, there is a unique path on $G_{A,B}$ from i to i labelled $w[a \mapsto A, b \mapsto B]$, call this path ρ_i, recall that the weight of ρ_i is $w(a \mapsto A, b \mapsto B)_{ii}$. As A has the underlying permutation of a p-cycle and G_B only contains loop edges, the total weight of all the edges labelled A in ρ_i is the same for all $1 \leq i \leq p$ as in order to start and end at i, the path must go through every edge of G_A the same number of times.

Note that the weight of an edge of ρ_i labelled B is entirely determined by the node in which the edge starts. Moreover, as A has the underlying permutation of a p-cycle and B is a diagonal matrix, the weight of an edge of ρ_i labelled B is entirely determined by the starting node of ρ_i, and the number of edges modulo p labelled by A occurring before the edge. That is, the number of a's modulo p in w occurring to the left of the b which corresponds to the B labelling the edge.

As all permutation matrices of p-cycles are conjugate, without loss of generality, we suppose that A has the underlying permutation of σ where $\sigma(i) = i + 1 \mod p$. Now, by the previous paragraphs we can see that the total weight of all the edges labelled B in ρ_i is given by

$$\begin{aligned} M_i &= \sum_{k=1}^{p} |w|_b^{a,k,p} B_{\sigma^k(i),\sigma^k(i)} \\ &= \sum_{k=1}^{p-i} |w|_b^{a,k,p} B_{i+k,i+k} + \sum_{k=p-i+1}^{p} |w|_b^{a,k,p} B_{i+k-p,i+k-p} \\ &= \sum_{k=1}^{p} |w|_b^{a,k-i,p} B_{k,k}. \end{aligned}$$

Now suppose ρ_i and ρ_j have the same weight for all i, j, then we have that $M_i = N$ for a fixed $N \in \mathbb{N}$ for all $1 \leq i \leq p$. However, as p is an odd prime, $|w|_b^{a,k,p} \in \mathbb{N}_0$ for all $1 \leq k \leq p$, and, by assumption, $|w|_b^{a,n,p} \neq |w|_b^{a,m,p}$ for some $1 \leq n < m \leq p$, the vectors given by

$$X_i = (|w|_b^{a,1-i,p}, \ldots, |w|_b^{a,p-i,p})$$

for $1 \leq i \leq p$ are linearly independent over $\mathbb{Q}$ by Lemma 7.3.5. Thus, there is at most one solution for the entries of B that gives $M_i = N$ for all $1 \leq i \leq p$, and we can see that $B_{ii} = \frac{N}{|w|_b}$ for all $1 \leq i \leq p$ is the solution. However, this gives a contradiction as we supposed B was not a scalar matrix. Thus, the weight of ρ_i is different than the weight of ρ_j for some i, j and hence $w(a \mapsto A, b \mapsto B)_{ii} \neq w(a \mapsto A, b \mapsto B)_{jj}$. Therefore, $w(a \mapsto A, b \mapsto B)$ is not a scalar matrix. □

Theorem 7.3.7. *Let p be a prime. Then there exists an identity satisfied by $M_{p-1}(\mathbb{T})$ but not by $M_p(\mathbb{T})$.*

Proof. As matrix multiplication is not commutative in dimension $p > 1$, $ab = ba$ is satisfied by $M_1(\mathbb{T})$ but not by $M_2(\mathbb{T})$ and by Lemma 7.3.3 there exists an identity satisfied by $M_2(\mathbb{T})$ but not by $M_3(\mathbb{T})$.

Suppose that p is a prime greater than 3. For each $3 \leq k < p$, let $q_k := w_k baw'_k$ and $r_k := w_k abw'_k$ for some $w_k, w'_k \in \{a, b\}^*$ such that $q_k = r_k$ is an identity satisfied by

$UT_k(\mathbb{T})$ with the property that $|q_k|_a, |r_k|_a \equiv -1 \mod p$. This can be done by starting with an identity satisfied by $UT_k(\mathbb{T})$, given by Corollary 7.2.2. In such an identity, the letter a occurs the same number of times on both sides and hence, by appending a power of a to the right of both sides of the identity, we can get that $|q_k|_a, |r_k|_a \equiv -1 \mod p$. Let $a_{p-1} = a$, $b_{p-1} = b$ and define words a_{k-1}, b_{k-1} for $3 \leq k < p$ recursively, as in Lemma 7.3.4, by

$$a_{k-1} = (q_k r_k)^t \left[a \mapsto a_k^{\overline{k}}, b \mapsto b_k^{\overline{k}}\right] \text{ and } b_{k-1} = (q_k r_k)^t r_k \left[a \mapsto a_k^{\overline{k}}, b \mapsto b_k^{\overline{k}}\right],$$

where $t = \frac{p^3-1}{2}$. Note that $t \geq (p-2)^2 + 1$.

Let $X \in M_p(\mathbb{Q}_{\max})$ be the permutation matrix of a p-cycle σ and $Y \in M_p(\mathbb{Q}_{\max})$ be an invertible non-scalar diagonal matrix. Then, for $2 \leq m \leq p-1$, let $A_m = a_m(a \mapsto X, b \mapsto Y) \in M_p(\mathbb{Q}_{\max})$ and $B_m = b_m(a \mapsto X, b \mapsto Y) \in M_p(\mathbb{Q}_{\max})$. We will now show A_m has the underlying permutation of a p-cycle and B_m is an invertible non-scalar diagonal matrix for every $2 \leq m \leq p-1$. This is true for A_{p-1} and B_{p-1} by definition. Proceeding by induction, suppose it is true for A_k and B_k and we show it is true for A_{k-1} and B_{k-1}.

For any $u \in \{a, b\}^*$, the matrix $u(a \mapsto A_k, b \mapsto B_k)$ is invertible as A_k and B_k are invertible. Moreover, as the underlying permutation of A_k, τ say, is a p-cycle, and B_k is a diagonal matrix, it follows that the underlying permutation of $u(a \mapsto A_k, b \mapsto B_k)$ depends only on the number of occurrences of a in u modulo p. If $|u|_a \equiv 0 \mod p$, then $u(a \mapsto A_k, b \mapsto B_k)$ is a diagonal matrix; otherwise it is a p-cycle as p is prime so τ^n is a p-cycle unless p divides n. Let $|a_k|_a = n$ for some $n \in \mathbb{N}_0$, and note that $|b_k|_a \equiv 0 \mod p$ as B_k is diagonal. Then, we can see that

$$|a_{k-1}|_a \equiv |(q_k r_k)^t|_a |a_k^{\overline{k}}|_a \equiv t|q_k r_k|_a n\overline{k} \equiv \frac{p^3-1}{2}(-2)n\overline{k} \equiv n\overline{k} \mod p$$
$$|b_{k-1}|_a \equiv |(q_k r_k)^t r_k|_a |a_k^{\overline{k}}|_a \equiv |(q_k r_k)^t|_a n\overline{k} + |r_k|_a n\overline{k} \equiv n\overline{k} - n\overline{k} \equiv 0 \mod p.$$

Thus, as p does not divide n as A_k has the underlying permutation of a p-cycle, and p does not divide $\overline{k}$, p does not divide $n\overline{k}$. Hence, A_{k-1} has the underlying permutation of a p-cycle and B_{k-1} is a diagonal matrix, as required.

Let $z = (q_k r_k)^t r_k$. Now, we must show that B_{k-1} is not a scalar matrix. To do this we will show that $|z|_b^{a,n,p} \neq |z|_b^{a,m,p}$ for some $n \neq m$ and then apply the contrapositive of Lemma 7.3.6.

Recall that $t = \frac{p^3-1}{2}$, $q_k = w_k x w'_k$, and $r_k = w_k y w'_k$, where $x = ba$ and $y = ab$. Consider,

$$z = (q_k r_k)^t r_k = (w_k x w'_k w_k y w'_k)^t w_k y w'_k.$$

Now, by only considering the factors w_k (resp. w'_k) of z which are prefixes (resp. suffixes) of q_k and r_k in z, we can see that there are p^3 copies of w_k (resp. w'_k) in z. There are $-1 \mod p$ occurrences of a from the start of any w_k (resp. w'_k) to the start of the next w_k (resp. w'_k) as $|q_k|_a, |r_k|_a \equiv -1 \mod p$, so we can see that z contains p^2 factors labelled w_k (resp. w'_k) with $l \mod p$ copies of a before them for each $1 \leq l \leq p$. Thus, the total number of b's which are contained in w_k or w'_k and have $l \mod p$ copies of a before them is the same for all $1 \leq l \leq p$; denote this number N.

Therefore, the difference in the $|z|_b^{a,l,p}$ for different l's is entirely due to the b's in the x's immediately to the right of w_k in q_k and in the y's immediately to the right of w_k in r_k. For clarity, we will now refer to the b in x as b_1 and the b in y as b_2.

For $j = 1, 2$, there are t copies of each b_j in $(q_k r_k)^t$ and $-2 \mod p$ copies of a between a b_j and the next b_j in $(q_k r_k)^t$ as $|q_k r_k|_a \equiv -2 \mod p$. Remark that, $t = \frac{p^2-1}{2} \cdot p + \frac{p-1}{2} \equiv \frac{p-1}{2} \mod p$ so, for $j = 1, 2$, the number of b_j's in $(q_k r_k)^t$ with $i \mod p$ copies of a before them is $\frac{p^2+1}{2}$ for $i \equiv l_j, l_j - 2, \ldots, l_j + 3 - p \mod p$ and $\frac{p^2-1}{2}$ for $i \equiv l_j + 1, l_j - 1, \ldots, l_j + 2 - p \mod p$, where l_j is the number of a's before the first occurrence of b_j.

Let $l = |w_k|_a$ and note that $|w'_k w_k|_a \equiv -2 \mod p$ as $|q_k|_a, |r_k|_a \equiv -1 \mod p$. Now, by considering $q_k r_k = (w_k b_1 a w'_k)(w_k a b_2 w'_k)$, we can see that there are $l \mod p$ copies of a before the first b_1 and $l \mod p$ copies of a before the first b_2 in $(q_k r_k)^t$, that is, $l_1 \equiv l_2 \equiv l \mod p$. Therefore, in total, b_1 and b_2 in $(q_k r_k)^t$ contribute $p^2 + 1$ copies of b with $l, l - 2, \ldots, l + 3 \mod p$ copies of a before them and $p^2 - 1$ copies of b with $l + 1, l - 1, \ldots, l + 2 \mod p$ copies of a before them.

Now, as $|(q_k r_k)^t|_a \equiv (-2) \cdot \frac{p-1}{2} \equiv 1 \mod p$, we can see that there are $l + 2 \mod p$ occurrences of a before the final b_2 in the final r_k in z. Thus,

$$|z|_b^{a,l,p} = N + p^2 + 1 \neq N + p^2 - 1 = |z|_b^{a,l+1,p}$$

and hence, by Lemma 7.3.6, B_{k-1} is not a scalar matrix as B_k, and therefore $B_k^{\overline{k}}$, is not a scalar matrix by the inductive hypothesis.

So, by induction, A_2 has the underlying permutation of a p-cycle and B_2 is a non-scalar diagonal matrix. Therefore, if we let $u_2 = v_2$ be the identity satisfied by $M_2(\mathbb{T})$ given by Lemma 7.3.3, then $u_2(a \mapsto A_2, b \mapsto B_2) \neq v_2(a \mapsto A_2, b \mapsto B_2)$ and hence

$$u_2(a \mapsto A_2, b \mapsto B_2)A_2 \cdots A_{p-2} \neq v_2(a \mapsto A_2, b \mapsto B_2)A_2 \cdots A_{p-2}$$

as $A_2, \ldots, A_{p-2} \in M_p(\mathbb{T})$ are invertible matrices. However, by Lemma 7.3.4,

$$u_2[a \mapsto a_2, b \mapsto b_2]a_2 \cdots a_{p-2} = v_2[a \mapsto a_2, b \mapsto b_2]a_2 \cdots a_{p-2}$$

is an identity satisfied by $M_{p-1}(\mathbb{T})$ and so we have constructed an identity satisfied by $M_{p-1}(\mathbb{T})$ that is falsified by $X, Y \in M_p(\mathbb{Q}_{\max}) \subseteq M_p(\mathbb{T})$. $\square$

Question 7.3.8. *For each $n \in \mathbb{N}$, does there exist a semigroup identity satisfied by $M_n(\mathbb{T})$ not satisfied by $M_{n+1}(\mathbb{T})$?*

7.4 Plactic Monoid of Rank 4 and Upper Triangular Matrix Semigroup of Rank 5

In this section we show the plactic monoid of rank 4, $\mathbb{P}_4$, does not generate the same variety as $UT_5(\mathbb{T})$. To do this we will use the faithful tropical representation of $\mathbb{P}_n$ given in [JM21]. We begin by recalling some notation used in the definition of this representation. For $S, T \in 2^{[n]}$, we write S^i for the ith smallest element of S, and say $S \leq T$ if $|S| \geq |T|$ and $S^i \leq T^i$ for each $i \leq |T|$. Moreover, for $P, Q \in 2^{[n]}$, we write $[P, Q]$ for the order interval from P to Q, and $\cup[P, Q]$ for the union of sets in the order interval.

The following theorem is given in greater generality in [JM21], but we only require the $n = 4$ case in what follows.

Theorem 7.4.1. *[JM21, Theorem 2.8] There exists a faithful semigroup morphism $\rho : \mathbb{P}_4 \to UT_{2^{[4]}}(\mathbb{T})$, where*

$$\rho(x)_{P,Q} = \begin{cases} -\infty & \text{if } |P| \neq |Q| \text{ or } P \not\leq Q; \\ 1 & \text{if } |P| = |Q| \text{ and } x \in \cup[P, Q]; \\ 0 & \text{otherwise.} \end{cases}$$

for each generator $x \in \mathbb{P}_4$, extending multiplicatively for products of generators and defining the identity element e as

$$\rho(e) = \begin{cases} -\infty & \text{if } |P| \neq |Q| \text{ or } P \nleq Q; \\ 0 & \text{otherwise.} \end{cases}$$

Note that in [JM21], the map ρ has codomain $M_{2^{[4]}}(\mathbb{T})$, however, given a natural choice of ordering of $2^{[4]}$ (any linear extension of $\leq$), the image of ρ is contained in $UT_{2^{[4]}}(\mathbb{T})$, so we have restricted the codomain in the above theorem to $UT_{2^{[4]}}(\mathbb{T})$.

Moreover, by considering [JM21, Example 2.1], we can see that for all $x \in \mathbb{P}_4$, $\rho(x)$ is a block matrix where the largest block is of size 6 by 6 and all simple paths in $G_{\rho(x)}$ have length at most 4.

Lemma 7.4.2. *Let $X, Y \in UT_m(\mathbb{T})$ and $u = v$ be an identity satisfied by $UT_n(\mathbb{T})$ where $n \leq m$. If there exists a path in $G_{X,Y}$ of simple length less than or equal to $n-1$ of maximal weight among all paths from i to j labelled $u[a \mapsto X, b \mapsto Y]$, then $u(a \mapsto X, b \mapsto Y)_{ij} \leq v(a \mapsto X, b \mapsto Y)_{ij}$.*

Proof. Let γ be a path of maximal weight of $G_{X,Y}$ labelled $u[a \mapsto X, b \mapsto Y]$ of simple length $k \leq n-1$ from i to j. Let $\overline{X}, \overline{Y} \in UT_{k+1}(\mathbb{T})$ be the matrices obtained from X and Y by removing rows and columns not indexed by the nodes of γ, with the rows and columns labelled by their original labelling. Let γ' be the path obtained by taking γ and replacing the labels X and Y by $\overline{X}$ and $\overline{Y}$ respectively. Then, as $G_{\overline{X},\overline{Y}}$ is the subgraph of $G_{X,Y}$ induced by the nodes of γ, the path γ' is a path in $G_{\overline{X},\overline{Y}}$ having maximal weight among all paths from i to j labelled by $u[a \mapsto \overline{X}, b \mapsto \overline{Y}]$, and so we have that $u(a \mapsto X, b \mapsto Y)_{ij} = u(a \mapsto \overline{X}, b \mapsto \overline{Y})_{ij}$. Moreover, we have that

$$\begin{aligned} u(a \mapsto X, b \mapsto Y)_{ij} = u(a \mapsto \overline{X}, b \mapsto \overline{Y})_{ij} &= v(a \mapsto \overline{X}, b \mapsto \overline{Y})_{ij} \\ &\leq v(a \mapsto X, b \mapsto Y)_{ij}, \end{aligned}$$

where the second equality holds since $u = v$ is an identity satisfied by $UT_n(\mathbb{T})$ and hence also for $UT_{k+1}(\mathbb{T})$ as $k+1 \leq n$, and the inequality follows from the construction of $\overline{X}$ and $\overline{Y}$. □

Theorem 7.4.3. *Let $u, v \in \{a,b\}^*$ be such that $u[a \mapsto ab, b \mapsto ba] = v[a \mapsto ab, b \mapsto ba]$ is a semigroup identity satisfied by $UT_4(\mathbb{T})$. Then the identity $abuab[a \mapsto ab, b \mapsto ba] = abvab[a \mapsto ab, b \mapsto ba]$ is satisfied by $\mathbb{P}_4$.*

Proof. Let $\rho : \mathbb{P}_4 \to UT_{2^{[4]}}(\mathbb{T})$ be the morphism given in Theorem 7.4.1, and let $x, y \in \mathbb{P}_4$. Recall that ρ is faithful and the semigroup identity $abuab[a \mapsto ab, b \mapsto ba] = abvab[a \mapsto ab, b \mapsto ba]$ is satisfied by $UT_4(\mathbb{T})$. Let $X = \rho(x), Y = \rho(y)$ and consider $abuab(a \mapsto XY, b \mapsto YX)$ and $abvab(a \mapsto XY, b \mapsto YX)$. Note that XY and YX are block diagonal matrices where each block is indexed by sets of a given size. Hence, every node of a path in $G_{XY,YX}$ is indexed by sets of the same size. In the subgraph labelled by sets of size two, the only simple path of length at most 4 is from $\{1,2\}$ to $\{3,4\}$. Thus, for all $(P,Q) \neq (\{1,2\},\{3,4\})$ a path from P to Q in $G_{XY,YX}$ has simple length at most 3, as all other subgraphs containing sets of a given size have at most 4 nodes and can therefore only contain simple paths of length at most 3. Hence we may apply Lemma 7.4.2 (in both directions) to obtain $abuab(a \mapsto XY, b \mapsto YX)_{P,Q} = abvab(a \mapsto XY, b \mapsto YX)_{P,Q}$ for all $(P,Q) \neq (\{1,2\},\{3,4\})$. Now by the fact that ρ is faithful, we have that

$$\begin{aligned} &abuab(a \mapsto xy, b \mapsto yx) = abvab(a \mapsto xy, b \mapsto yx) \text{ if and only if} \\ &\qquad abuab(a \mapsto XY, b \mapsto YX)_{\{1,2\},\{3,4\}} = abvab(a \mapsto XY, b \mapsto YX)_{\{1,2\},\{3,4\}}. \end{aligned} \tag{7.1}$$

Therefore, it suffices to check that $abuab[a \mapsto ab, b \mapsto ba] = abvab[a \mapsto ab, b \mapsto ba]$ holds for the $\{1,2\},\{3,4\}$ entry in the image of ρ.

It follows from the definition of ρ, we have that for $s \in \mathbb{P}_4$, $\rho(s)_{P,P}$ is the total number of occurrences of letters from the set P in some fixed word representing s. It follows from this that

$$\rho(s)_{\{1,2\},\{1,2\}} + \rho(s)_{\{3,4\}\{3,4\}} = \rho(s)_{\{1,3\},\{1,3\}} + \rho(s)_{\{2,4\},\{2,4\}}$$

for all $s \in \mathbb{P}_4$ as all words representing s have the same number of 1's, 2's, 3's and 4's and so both sides of the equality count the number of occurrences of 1's, 2's, 3's and 4's in some word representing s. Then for each s we have that, either

$$\rho(s)_{\{1,3\},\{1,3\}} \geq \rho(s)_{\{1,2\},\{1,2\}} \text{ or } \rho(s)_{\{2,4\},\{2,4\}} \geq \rho(s)_{\{3,4\}\{3,4\}}.$$

We now look at the graph $G_{XY,YX}$. This is the graph where, if $\rho(xy)_{ij} \neq -\infty$, there is an edge from i to j labelled XY with weight $\rho(xy)_{ij}$ and, if $\rho(yx)_{ij} \neq -\infty$, there is an edge from i to j labelled YX with weight $\rho(yx)_{ij}$. Note $\rho(xy)_{\{1,3\},\{1,3\}} \geq \rho(xy)_{\{1,2\},\{1,2\}}$ if and only if $\rho(yx)_{\{1,3\},\{1,3\}} \geq \rho(yx)_{\{1,2\},\{1,2\}}$ as $\rho(xy)$ and $\rho(yx)$ have the same diagonal entries. Suppose that $\rho(xy)_{\{1,3\},\{1,3\}} \geq \rho(xy)_{\{1,2\},\{1,2\}}$, and let γ be a path of maximal weight in $G_{XY,YX}$ from $\{1,2\}$ to $\{3,4\}$ labelled by the word $abuab[a \mapsto XY, b \mapsto YX]$.

We split into two cases:

(i) If γ does not contain an edge from $\{1,2\}$ to $\{1,3\}$. Then, γ is a path of simple length ≤ 3, so by Lemma 7.4.2,

$$abuab(a \mapsto XY, b \mapsto YX)_{\{1,2\},\{3,4\}} \leq abvab(a \mapsto XY, b \mapsto YX)_{\{1,2\},\{3,4\}}.$$

(ii) If γ contains an edge from $\{1,2\}$ to $\{1,3\}$. Then γ is of the form

$$\gamma = \lambda_{\{1,2\}} \circ \gamma_{\{1,2\},\{1,3\}} \circ \mu,$$

where $\lambda_{\{1,2\}}$ is a path made up of loop edges around node $\{1,2\}$, $\gamma_{\{1,2\},\{1,3\}}$ is the subpath of γ corresponding to an edge from $\{1,2\}$ to $\{1,3\}$ and μ is the rest of γ. Since, we have assumed $\rho(xy)_{\{1,3\},\{1,3\}} \geq \rho(xy)_{\{1,2\},\{1,2\}}$ (and similarly for $\rho(yx)$), each loop at $\{1,3\}$ has greater weight than its counterpart at $\{1,2\}$. Since γ is assumed to have maximal weight on the word $abuab[a \mapsto XY, b \mapsto YX]$, this means that the path $\lambda_{\{1,2\}}$ can be assumed to have length at most 1; it has length 0 if $\gamma_{\{1,2\},\{1,3\}}$ is labelled XY, and length 1 if $\gamma_{\{1,2\},\{1,3\}}$ is labelled YX.

Therefore, the edge $\gamma_{\{1,2\},\{1,3\}}$ is contained within the first two edges of γ corresponding to the first two letters of $abuab$ and hence by the definition of matrix multiplication in $UT_{2^{[4]}}(\mathbb{T})$ we have that

$$\begin{aligned} &abuab(a \mapsto XY, b \mapsto YX)_{\{1,2\},\{3,4\}} \\ &\qquad = ab(a \mapsto XY, b \mapsto YX)_{\{1,2\},P} + uab(a \mapsto XY, b \mapsto YX)_{P,\{3,4\}} \end{aligned}$$

for some $P \in 2^{[4]}$ such that $\{1,3\} \leq P \leq \{3,4\}$. Moreover, as each such path from P to $\{3,4\}$ (and hence the path of maximal weight) has simple length at most 3, we can apply Lemma 7.4.2 to get that

$$\begin{aligned} abuab(a \mapsto &XY, b \mapsto YX)_{\{1,2\},\{3,4\}} \\ &= ab(a \mapsto XY, b \mapsto YX)_{\{1,2\},P} + uab(a \mapsto XY, b \mapsto YX)_{P,\{3,4\}} \\ &\leq ab(a \mapsto XY, b \mapsto YX)_{\{1,2\},P} + vab(a \mapsto XY, b \mapsto YX)_{P,\{3,4\}} \\ &\leq abvab(a \mapsto XY, b \mapsto YX)_{\{1,2\},\{3,4\}}. \end{aligned}$$

We can now apply a similar case analysis to a maximal weight path labelled by $abvab[a \mapsto XY, b \mapsto YX]$ to get that

$$abuab(a \mapsto XY, b \mapsto YX)_{\{1,2\},\{3,4\}} \geq abvab(a \mapsto XY, b \mapsto YX)_{\{1,2\},\{3,4\}}.$$

Therefore, $abuab(a \mapsto XY, b \mapsto YX)_{\{1,2\},\{3,4\}} = abvab(a \mapsto XY, b \mapsto YX)_{\{1,2\},\{3,4\}}$, and by (7.1) we can conclude that $abuab(a \mapsto xy, b \mapsto yx) = abvab(a \mapsto xy, b \mapsto yx)$.

A similar argument in the case where $\rho(xy)_{\{2,4\},\{2,4\}} \geq \rho(xy)_{\{3,4\}\{3,4\}}$ applies to show that $abuab(a \mapsto xy, b \mapsto yx) = abvab(a \mapsto xy, b \mapsto yx)$. □

Corollary 7.4.4. *There exists an identity satisfied by* $\mathbb{P}_4$ *but not satisfied by* $UT_5(\mathbb{T})$.

Proof. Let $u = ba^3b^3aba\ b\ ba^3b^3aba$ and $v = ba^3b^3aba\ a\ ba^3b^3aba$. By Corollary 7.2.2, we have that $u[a \mapsto ab, b \mapsto ba] = v[a \mapsto ab, b \mapsto ba]$ is an identity satisfied by $UT_4(\mathbb{T})$. So by Theorem 7.4.3, $abuab[a \mapsto ab, b \mapsto ba] = abvab[a \mapsto ab, b \mapsto ba]$ is satisfied by $\mathbb{P}_4$. However, $abab$ is a factor of $abuab$ but not of $abvab$. So, by Lemma 7.2.1, we have that there exists $A, B \in UT_5(\mathbb{T})$ such that $abuab(a \mapsto AB, b \mapsto BA) \neq abvab(a \mapsto AB, b \mapsto BA)$, and thus $abuab[a \mapsto ab, b \mapsto ba] = abvab[a \mapsto ab, b \mapsto ba]$ is not satisfied by $UT_5(\mathbb{T})$. □

Johnson and Kambites [JM21, Question 4.8] asked if the plactic monoid of rank 4 generates the same semigroup variety as $UT_4(\mathbb{T})$ and/or $UT_5(\mathbb{T})$. By the above corollary, we have partially answered this question by showing that the variety generated by $\mathbb{P}_4$ is strictly contained in the variety generated by $UT_5(\mathbb{T})$. However, what remains to be answered is the following.

Question 7.4.5. *Is the semigroup variety generated by* $\mathbb{P}_4$ *equal to the semigroup variety generated by* $UT_4(\mathbb{T})$*? That is, does* $\mathbb{P}_4$ *satisfy the exact same set of semigroup identities as* $UT_4(\mathbb{T})$*?*

Chapter 8

Tropical Representation of the Stylic Monoid

While studying identities and varieties of semigroups and monoids, several important questions arise, such as the question of whether a semigroup admits a finite basis for its equational theory. This question is known as the finite basis problem [Sap14, Vol01], and it is well known that there are finite semigroups which are not finitely based [Per69]. Other questions regarding the variety generated by a semigroup are those of whether it contains only finitely generated subvarieties (see, for example, [Vol01]), or countably infinite subvarieties [Tra88]. These problems have also been considered for involution semigroups, that is, semigroups equipped with a unary operation * which satisfies the identities $(x^*)^* = x$ and $(xy)^* = y^* x^*$ (see [Lee20] for a collection of results on this subject). In particular, the finite basis problem for finite involution semigroups has received much attention, since, contrary to intuition, finite involution semigroups and their underlying semigroups need not necessarily be simultaneously finitely based (see, for example, [Lee16, Lee19]).

The stylic monoid of finite rank n, introduced by Abram and Reutenauer in [AR22] and denoted by styl_n, is a finite quotient of the plactic monoid of rank n, defined by the action of words, over a finite totally ordered alphabet with n letters, on the left of columns of semistandard Young tableaux, by Schensted left insertion. Its elements can be uniquely identified with so-called N-tableaux, and it is presented by the Knuth relations and the relations $a^2 \equiv a$, for each $a \in [n]$. As such, to the author's knowledge, it is the first finite plactic-like monoid to be studied. It is a finite $\mathcal{J}$-trivial monoid

([AR22, Theorem 11.1]), hence, by [Sim72], is in $\boldsymbol{\mathcal{J}}_k$, the pseudovariety in Simon's hierarchy of $\mathcal{J}$-trivial monoids which corresponds to the class of all piecewise testable languages of height k, in Eilenberg's correspondence ([Eil76, Pin86]), for some $k \in \mathbb{N}$. The pseudovariety $\boldsymbol{\mathcal{J}}_k$ is defined by the equational theory J_k of all identities $u = v$ such that u and v share the same subsequences of length $\leq k$. Blanchet-Sadri has studied these equational theories in depth ([BS89, BS93, BS94]), showing that J_k is finitely based if and only if $k \leq 3$. In this chapter, we show that the stylic monoid of rank n generates the pseudovariety $\boldsymbol{\mathcal{J}}_n$.

The chapter is organized as follows: Section 8.1 gives the necessary background on the subject matter, namely identities, varieties and pseudovarieties in Subsection 8.1.1; and the stylic monoid in Subsection 8.1.2. In Section 8.2, we give a faithful representation of styl_n by $U_{n+1}(\mathbb{T})$, thus proving that styl_n is in the variety generated by $U_{n+1}(\mathbb{T})$, and we follow up in Section 8.3 by showing that all identities satisfied by styl_n must also be in J_n, and therefore the equational theory of styl_n is J_n. From this, we deduce that the variety generated by styl_n, for $n \geq 3$, has uncountably many subvarieties. Finally, in Section 8.4, we look at the finite basis problem for the stylic monoid with involution $*$ induced by the unique order-reversing permutation of $[n]$, and show that $(\mathsf{styl}_n, *)$ is finitely based if and only if $n = 1$. We also show that $(\mathsf{styl}_n, *)$ and $(U_{n+1}(\mathbb{T}), \star)$, where $\star$ is the skew transposition, do not generate the same variety for $n \geq 2$, which contrasts with the results obtained in Section 8.3.

This chapter is based on joint work with Duarte Ribeiro [AR23].

8.1 Background

This chapter is the only place where we consider finite semigroups, so here we introduce some universal algebra specific to finite semigroups. For a general background on universal algebra, see [BS81]; on pseudovarieties, see [Alm94]. We also refer the reader to the survey [Vol01] on the finite basis problem for finite semigroups.

8.1.1 Identities and Varieties

The set of all identities Σ satisfied by a monoid M is called its *equational theory*. An identity $u = v$ is a *consequence* of a set of identities Σ if all monoids which satisfy all identities of Σ also satisfy $u = v$. An *equational basis*, or simply basis, $\mathcal{B}$ of an equational theory Σ is a subset of Σ such that each identity in Σ is a consequence of $\mathcal{B}$. We say an equational theory is *finitely based* if it admits a finite basis, and *non-finitely based* otherwise.

On the other hand, a class of finite monoids is a *pseudovariety* if it is closed under taking homomorphic images, submonoids and finitary direct products. A *subvariety* is a subclass of a variety which is itself a variety. We say a pseudovariety is *generated* by a finite monoid M if it is the smallest pseudovariety containing M.

An *equational pseudovariety* is a pseudovariety which consists of all the finite monoids in some variety (see, for example, [Alm94]). An equational pseudovariety is defined by its equational theory. We say that a variety or an equational pseudovariety is finitely based if its equational theory is finitely based, and that a monoid is finitely based if the variety it generates is finitely based.

For each $k \in \mathbb{N}$, we denote by $\boldsymbol{\mathcal{J}}_k$ the pseudovariety defined by J_k, the set of all identities $u = v$ such that u and v share the same subsequences of length $\leq k$. The increasing sequence

$$\boldsymbol{\mathcal{J}}_1 \subsetneq \boldsymbol{\mathcal{J}}_2 \subsetneq \cdots \subsetneq \boldsymbol{\mathcal{J}}_k \subsetneq \ldots,$$

whose union is the pseudovariety $\boldsymbol{\mathcal{J}}$ of all finite $\mathcal{J}$-trivial monoids, was introduced in [Sim72], and is known as *Simon's hierarchy of $\mathcal{J}$-trivial monoids*. Furthermore, a finite monoid is $\mathcal{J}$-trivial if and only if it is in $\boldsymbol{\mathcal{J}}_k$ if and only if it satisfies all identities in J_k, for some k. Regarding whether these equational theories admit finite bases, we have the following:

(I) ([BS94, folklore]) J_1 admits a finite basis, consisting of the following identities:

$$x^2 = x \quad \text{and} \quad xy = yx.$$

(II) ([Sim72]) J_2 admits a finite basis, consisting of the following identities:

$$xyxzx = xyzx \quad \text{and} \quad (xy)^2 = (yx)^2.$$

(III) ([BS89, Proposition 4.1.6] and [BS93]) J_3 admits a finite basis, consisting of the following identities:

$$xyx^2zx = xyxzx,$$
$$xyzx^2tz = xyxzx^2tx,$$
$$zyx^2ztx = zyx^2zxtx,$$
$$(xy)^3 = (yx)^3.$$

(IV) ([BS94, Theorem 3.4]) The equational theory J_k is non-finitely based, for $k \geq 4$.

8.1.2 The Stylic Monoid

The *stylic monoid of rank n*, denoted by styl_n, was first defined in [AR22, Section 5] as the monoid of endofunctions of the set of columns over $[n]$ obtained by a left action of words on columns [AR22, Section 4]. It is a finite quotient of the free monoid over $[n]$, and the corresponding *stylic congruence* of $[n]^*$ is denoted by $\equiv_{\mathsf{styl}}$. It is $\mathcal{J}$-trivial [AR22, Theorem 11.1], and therefore, by Simon's Theorem, there exists $k \in \mathbb{N}$ such that $\mathsf{styl}_n \in \boldsymbol{\mathcal{J}}_k$.

The stylic monoid of rank n can be defined in two other ways, which will be the ones used in this work: It is defined by the presentation $\langle [n] \,|\, \mathcal{R}_{\mathsf{styl}} \rangle$ [AR22, Theorem 8.1], where

$$\mathcal{R}_{\mathsf{styl}} = \mathcal{R}_{\mathsf{plac}} \cup \{(a^2, a) : a \in [n]\}.$$

and $\mathcal{R}_{\mathsf{plac}}$ is the set of Knuth relations. The defining relations are known as the *stylic relations*, and are the plactic relations together with generator idempotent relations. As such, the stylic monoid of rank n can be viewed as a quotient of the plactic monoid [AR22, Proposition 5.1], and two words in the same stylic class have the same support [AR22, Lemma 5.3].

For the other definition, we need a combinatorial object analogous to a Young tableau: An *N-tableau* is a Young tableau where each row is strictly increasing and contained in the row below [AR22, Subsection 6.1]. An example of an N-tableau is

5	6				
2	5	6			
1	2	3	4	5	6

As with Young tableaux and Schensted's algorithm, it is possible to associate each word $w \in [n]^*$ to a unique N-tableau, which we denote by $N(w)$, by using the *right N-algorithm*: Consider rows of an N-tableau as subsets of the alphabet. The *right N-insertion* of a letter $a \in [n]$ into a row $\mathcal{B} \subseteq [n]$ gives the row $\mathcal{B} \cup \{a\}$. If b is the smallest letter in B strictly greater than a, we say b is *bumped* (but b is not deleted in $\mathcal{B} \cup \{a\}$). The *right N-insertion* of a letter $a \in [n]$ into an N-tableau is recursively defined as follows: a is inserted into the first row, then, if a letter b is bumped, b is inserted into the row above. The algorithm stops when no letter is bumped. Inserting a letter into an N-tableau, using this algorithm, produces an N-tableau [AR22, Proposition 6.1]. The *right N-insertion* of a word $w \in [n]^*$ into an N-tableau is done by inserting the letters of w, one-by-one from left-to-right. The stylic congruence on $[n]^*$ is defined by

$$u \equiv_{\mathsf{styl}} v \iff N(u) = N(v),$$

for $u, v \in [n]^*$ [AR22, Theorem 7.1].

The stylic monoid of rank n has an absorbing element, which is the stylic class of the decreasing product of all letters in $[n]^*$ [AR22, Proposition 5.4]. This element corresponds to the N-tableau with n rows and the letters $\{i, \dots, n\}$ in the i-th row.

The following definitions are introduced in [AR22, Subsection 6.3]: For each subset $\mathcal{B}$ of $[n]$, and each letter $a \in [n]$, the element $a^{\uparrow}_{\mathcal{B}} \in \mathcal{B} \cup \{\varepsilon\}$ is the smallest letter in $\mathcal{B}$ which is strictly greater than a, or ε if such a letter does not exist. Define the mapping $\delta : [n]^* \to [n]^*$ as follows: for any word $w \in [n]^*$ and letter $a \in [n]$, $\delta(wa) = \delta(w) \cdot a^{\uparrow}_{\text{supp}(w)}$. Notice that the smallest letter in w is not in $\delta(w)$, hence $\text{supp}(\delta^k(w)) \subsetneq \text{supp}(\delta^{k-1}(w))$, for all $k \in \mathbb{N}$ such that $\text{supp}(\delta^{k-1}(w)) \neq \emptyset$.

Example 8.1.1. Let $w = 311321424543$. Then, $\delta(w) = 3332354$; this can be seen by applying the above algorithm to w which can be expressed in the following way:

$$\begin{array}{rcllllllllllll} w & = & 3 & 1 & 1 & 3 & 2 & 1 & 4 & 2 & 4 & 5 & 4 & 3 \\ \delta(w) & = & & 3 & 3 & & 3 & 2 & & 3 & & & 5 & 4 \end{array}$$

We introduce the following definition, which expands upon the "arrow" notation: For a word $w \in [n]^*$, and $k \in \mathbb{N}$, define the mapping $\uparrow^k_w : \{1, \dots, |w|\} \to \text{supp}(w)$ recursively, as follows: for $1 \leq l \leq k$,

$$\uparrow^0_w(i) = w_{(i)}$$
$$\uparrow^l_w(i) = \left(\uparrow^{l-1}_w(i)\right)^{\uparrow}_{\text{supp}(\delta^{l-1}(w_{\leq i}))}.$$

where $\delta^0(w_{\leq i}) = w_{\leq i}$. If $\uparrow_w^k(i) \neq \varepsilon$, then $\uparrow_w^k(i)$ is the letter which is bumped into the $(k+1)$-th row when $w_{(i)}$ is inserted into the N-tableau. As an example, consider the word 535234512345. Then,

$$\begin{array}{cccccccccccccl}
5 & 3 & 5 & 2 & 3 & 4 & 5 & 1 & 2 & 3 & 4 & 5 & = & w, \\
 & 5 & & 3 & 5 & 5 & & 2 & 3 & 4 & 5 & & = & \delta(w), \\
 & & & 5 & & & & 3 & 5 & 5 & & & = & \delta^2(w), \\
 & & & & & & & 5 & & & & & = & \delta^3(w),
\end{array}$$

and $\uparrow_w^3(8) = 5$, that is, the letter $w_{(8)} = 1$ bumps 5 to the fourth row of the N-tableau $N(535234512345)$.

The following lemmas are consequences of the definition of $\uparrow_w^k$, and the right N-algorithm in the case of the first lemma:

Lemma 8.1.2. *Let $w \in [n]^*$, $a \in [n]$, and $k \in \mathbb{N}$. Then, a occurs in the k-th row of $N(w)$ if and only if there exists an index $j \leq |w|$ such that $\uparrow_w^{k-1}(j) = a$.*

Proof. By repeated application of [AR22, Lemma 6.3], $\operatorname{supp}(\delta^{k-1}(w))$ is the k-th row of $N(w)$, viewed as a subset of $[n]$. Moreover, by the definition of $\uparrow_w^{k-1}$, for $a \in [n]$, we have that $a \in \operatorname{supp}(\delta^{k-1}(w))$ if and only if $\uparrow_w^{k-1}(j) = a$ for some $j \leq |w|$. Thus, a is in the k-th row of $N(w)$ if and only if there is some j satisfying the previously mentioned condition. □

Lemma 8.1.3. *Let $w \in [n]^*$ and $k, s_k \in \mathbb{N}$ be such that $1 \leq k \leq s_k \leq |w|$. If $\uparrow_w^{k-1}(s_k) = a$, for some $a \in [n]$, then there exists a strictly decreasing subsequence $w_{(s_1)}, \ldots, w_{(s_k)}$ of w such that $w_{(s_1)} = a$ and $\uparrow_w^{l-1}(s_l) = a$ for $1 < l \leq k$.*

Proof. Since $\operatorname{supp}(\delta^l(w)) \subsetneq \operatorname{supp}(\delta^{l-1}(w))$, for all $l < k$, then $\uparrow_w^{k-1}(s_k) = a$ implies that $a \in \operatorname{supp}(\delta^l(w))$, for all $1 \leq l \leq k-1$, and $a \in \operatorname{supp}(w)$. Thus, there exist $1 \leq s_1, \ldots, s_{k-1} \leq |w|$ such that $w_{(s_1)} = a$ and $\uparrow_w^{l-1}(s_l) = a$ for $1 < l \leq k-1$.

Notice that $\uparrow_w^{l-1}(s_l) = a$ implies that there is a letter a to the left of $\uparrow_w^{l-2}(s_l)$ in $\delta^{l-2}(w)$, for all $2 < l \leq k$. Similarly, $\uparrow_w^1(s_2) = a$ implies that there is a letter a to the left of $w_{(s_2)}$ in w. As such, we can restrict the choice of $s_1, \ldots, s_{k-1}$ to have $s_1 < \cdots < s_k$.

We now prove that, since $\uparrow_w^{l-1}(s_l) = a$, there must exist $i \leq s_l$ such that $\uparrow_w^{l-2}(i) = a$ and $w_{(i)} > w_{(s_l)}$: In order to obtain a contradiction, take i such that $w_{(i)} \leq w_{(s_l)}$,

$$\uparrow_w^j(i) \leq \uparrow_w^j(s_l) < \uparrow_w^{j+1}(s_l) < \uparrow_w^{j+1}(i),$$

and $\uparrow_w^{j'}(i) \leq \uparrow_w^{j'}(s_l)$ for all $1 \leq j' \leq j$, such that j is minimal. In other words, when comparing the sequences of "arrows" of i and s_l, this choice of i gives us the sequence where there are the least number of elements which are less than or equal to the corresponding elements of the sequence of s_l, i.e.

$$\begin{array}{ccc} w_{(i)} & \leq & w_{(s_l)} \\ \uparrow_w^1(i) & \leq & \uparrow_w^1(s_l) \\ \vdots & & \vdots \\ \uparrow_w^j(i) & \leq & \uparrow_w^j(s_l) \\ \uparrow_w^{j+1}(i) & > & \uparrow_w^{j+1}(s_l) \\ \vdots & & \vdots \\ \uparrow_w^{l-2}(i) & > & \uparrow_w^{l-2}(s_l) \\ & & \uparrow_w^{l-1}(s_l) \end{array}$$

Then, we have that all occurrences of $\uparrow_w^{j+1}(s_l)$ must be to the right of $\uparrow_w^j(i)$ in $\delta^j(w_{\leq s_l})$, since $\uparrow_w^j(i)$ bumps $\uparrow_w^{j+1}(i)$ and not $\uparrow_w^{j+1}(s_l)$. But at least one occurrence of $\uparrow_w^{j+1}(s_l)$ in $\delta^j(w_{\leq s_l})$ will bump a to the $(l-2)$-th row. This contradicts the minimality of j, hence, we can choose $s_1, \ldots, s_l$ such that $s_1 < \cdots < s_k$ and $w_{(s_1)} > \cdots > w_{(s_l)}$.

Thus, we have found a strictly decreasing subsequence $w_{(s_1)}, \ldots, w_{(s_k)}$ of w, where $w_{(s_1)} = a$ and $\uparrow_w^{l-1}(s_l) = w_{(s_1)}$ for all $1 < l \leq k$. □

8.2 Tropical Representations of the Stylic Monoid

We first construct a faithful representation of the stylic monoid of finite rank n in the monoid of upper unitriangular $(n+1)\times(n+1)$ tropical matrices, for each $n \in \mathbb{N}$. Since, by [JF19, Corollary 3.3], this monoid generates the variety with equational theory J_n, we show that styl_n satisfies all identities in J_n.

Let $\overline{x} := n+1-x$ for all $x \in [n]$. We define the map $\rho_n \colon [n]^* \to U_{n+1}(\mathbb{T})$ as follows:

$$\rho_n(x)_{i,j} = \begin{cases} 0 & \text{if } i = j; \\ 1 & \text{if } i \leq \overline{x} < j; \\ -\infty & \text{otherwise.} \end{cases}$$

for each $x \in [n]$, extending multiplicatively to all of $[n]^*$ and defining the image of the empty word to be $\rho_n(\varepsilon) = I_{(n+1)\times(n+1)}$. For example, the images of 2 and of 4213 under ρ_4 are, respectively,

$$\begin{pmatrix} 0 & -\infty & -\infty & 1 & 1 \\ -\infty & 0 & -\infty & 1 & 1 \\ -\infty & -\infty & 0 & 1 & 1 \\ -\infty & -\infty & -\infty & 0 & -\infty \\ -\infty & -\infty & -\infty & -\infty & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 & 2 & 2 & 3 \\ -\infty & 0 & 1 & 1 & 2 \\ -\infty & -\infty & 0 & 1 & 2 \\ -\infty & -\infty & -\infty & 0 & 1 \\ -\infty & -\infty & -\infty & -\infty & 0 \end{pmatrix}$$

Notice that, for each $x \in [n]$, its image under ρ_n is a unitriangular tropical matrix where the only entries above the diagonal different from $-\infty$ are equal to 1.

Lemma 8.2.1. *Let $w \in [n]^*$. For $1 \leq i < j \leq n{+}1$ and $k \in \mathbb{N}$, we have that $\rho_n(w)_{i,j} = k$ if and only if k is the maximum length of any strictly decreasing subsequence of w only using letters between $\overline{j}+1$ and $\overline{i}$. On the other hand, $\rho_n(w)_{i,j} = -\infty$ if and only if w does not contain a for any $i \leq \overline{a} < j$.*

A remark about abuse of language: we say "only using letters between $\overline{j}+1$ and $\overline{i}$" in order to avoid the formal, but more cumbersome, statement "only using letters $a \in [n]$ such that $\overline{j}+1 \leq a \leq \overline{i}$".

Proof. Let $w \in [n]^*$ and $1 \leq i < j \leq n+1$. Suppose $\rho_n(w)_{i,j} = k$, for some $k \in \mathbb{N}$. Then, by the definition of tropical matrix multiplication, w admits a subsequence $w_{(s_1)}, \ldots, w_{(s_k)}$, of length k, and there exist $i = t_0 < \cdots < t_k = j$ such that $\rho_n(w_{(s_i)})_{t_{i-1},t_i} = 1$ for all $1 \leq i \leq k$. Furthermore, by the definition of ρ_n, $t_{i-1} \leq \overline{w_{(s_i)}} < t_i$. Therefore, $w_{(s_1)}, \ldots, w_{(s_k)}$ is a strictly decreasing subsequence of w such that $\overline{i} \geq w_{(s_1)} > \cdots > w_{(s_k)} \geq \overline{j}+1$ and hence, the maximum length of a strictly decreasing subsequence of w only using letters between $\overline{j}+1$ and $\overline{i}$ is greater than or equal to $\rho_n(w)_{i,j}$.

Suppose now that k is the maximum length of any strictly decreasing subsequence of w only using letters between $\overline{j}+1$ and $\overline{i}$. Let $w_{(s_1)}, \ldots, w_{(s_k)}$ be a strictly decreasing subsequence of w such that $\overline{i} \geq w_{(s_1)} > \cdots > w_{(s_k)} \geq \overline{j}+1$, then let $t_0 = i, t_k = j$ and $t_i = \overline{w_{(s_{i+1})}}$ for $1 \leq i < k$. Hence, by the definition of ρ_n, we have that $\rho_n(w_{(s_i)})_{t_{i-1},t_i} = 1$ for $1 \leq i \leq k$, and therefore

$$\rho_n(w)_{i,j} \geq \prod_{i=1}^{k} \rho_n(w_{(s_i)})_{t_{i-1},t_i} = k.$$

Thus, $\rho_n(w)_{i,j}$ is greater than or equal to the maximum length of a strictly decreasing subsequence only using letters between $\overline{j+1}$ and $\overline{i}$. Equality follows.

In the case where $\rho_n(w)_{i,j} = -\infty$, there is no $t \in \{1, \ldots, |w|\}$ such that $i \leq \overline{w_{(t)}} < j$, otherwise, $w_{(t)}$ would form a strictly decreasing subsequence of w (with just one letter), only using letters between $\overline{j+1}$ and $\overline{i}$, which would imply that $\rho_n(w)_{i,j} \geq 1$. Conversely, if $\overline{w_{(t)}} < i$ or $\overline{w_{(t)}} \geq j$ for all $1 \leq t \leq |w|$, then $\rho_n(w_{(t)})_{i,j'} = -\infty$ for all $i < j' \leq j$ and hence $\rho_n(w)_{i,j} = -\infty$. □

As an immediate corollary, notice that $\rho_n(w)_{i,j} \leq n$, for all $1 \leq i, j \leq n+1$. We also have the following:

Corollary 8.2.2. *Let $w \in [n]^*$. Then, any two finite adjacent entries in $\rho_n(w)$ must differ by at most 1, and are weakly increasing on columns and weakly decreasing on rows. In other words, for $1 \leq i \leq j \leq n+1$, if $\rho_n(w)_{i,j}$ and $\rho_n(w)_{i+1,j}$ are both finite, then $\rho_n(w)_{i+1,j} \leq \rho_n(w)_{i,j} \leq \rho_n(w)_{i+1,j} + 1$. Similarly, if $\rho_n(w)_{i,j}$ and $\rho_n(w)_{i,j+1}$ are both finite, then $\rho_n(w)_{i,j} \leq \rho_n(w)_{i,j+1} \leq \rho_n(w)_{i,j} + 1$.*

Proof. First, by noticing that any strictly decreasing subsequence only using letters between $\overline{j+1}$ and $\overline{i}$ is also a strictly decreasing subsequence only using letters between $\overline{j}$ and $\overline{i}$, and $\overline{j+1}$ and $\overline{i+1}$, we have that the entries of $\rho_n(w)$ weakly increase left-to-right on the columns and weakly decrease top-to-bottom on the rows.

Suppose, in order to obtain a contradiction, that there exist $1 \leq i \leq j \leq n$ and $0 \leq k < k' \leq n$ such that $\rho_n(w)_{i,j} = k$ and $\rho_n(w)_{i,j+1} = k'+1$. By the previous lemma, there exist maximum length strictly decreasing subsequences u and v of w, of length k and $k'+1$ and only using letters between $\overline{j+1}$ and $\overline{i}$ and between $\overline{j}$ and $\overline{i}$, respectively. Taking v and discarding its smallest letter gives us a strictly decreasing subsequence of w, of length k', only using letters between $\overline{j+1}$ and $\overline{i}$, which contradicts the maximality of the length of u. Similarly, we can prove that there are no $2 \leq i \leq j \leq n+1$ and $1 \leq k < k' \leq n$ such that $\rho_n(w)_{i,j} = k$ and $\rho_n(w)_{i-1,j} = k'+1$. □

Proposition 8.2.3. *The map $\rho_n \colon [n]^* \to U_{n+1}(\mathbb{T})$ induces a well-defined morphism from styl_n to $U_{n+1}(\mathbb{T})$.*

Proof. We show that ρ_n satisfies the stylic relations, that is $x^2 \equiv x$ for all $x \in [n]$ and the Knuth relations.

To show that $\rho_n(x^2) = \rho_n(x)$ for all $x \in [n]$, begin by observing that for all $i \leq j$, $\rho_n(x^2)_{i,j} = \rho_n(x)_{i,k} \cdot \rho_n(x)_{k,j}$ for some $i \leq k \leq j$. Suppose $\rho_n(x^2)_{i,j} \neq -\infty$. If there exists $i < k < j$ such that $\rho_n(x)_{i,k} \neq -\infty \neq \rho_n(x)_{k,j}$, then we have that $i \leq \overline{x} < k \leq \overline{x} < j$, giving a contradiction. Thus, we either have $i = k$ or $k = j$. In either case, as $\rho_n(x)_{i,i} = \rho_n(x)_{j,j} = 0$, we have that $\rho_n(x^2)_{i,j} = \rho_n(x)_{i,j}$. If $\rho_n(x^2)_{i,j} = -\infty$, then as $\rho_n(x^2)_{i,j} \geq \rho_n(x)_{i,j} \cdot \rho_n(x)_{j,j}$, we have that $\rho_n(x)_{i,j} = -\infty$.

For the Knuth relations, both sides of each relation have the same number of occurrences of each letter, and are of length 3. Let w be one side of a Knuth relation, then by Lemma 8.2.1, $\rho_n(w)_{i,j} \in \{-\infty, 0, 1, 2\}$ for all i, j, as w does not contain a strictly decreasing subsequence of length 3. Moreover, it is clear to see that $\rho_n(w)_{i,j} = 0$ if and only if $i = j$.

Let $u \equiv v$ be a Knuth relation. Then, $\rho_n(u)_{i,j} \neq -\infty$ if and only if $i = j$ or $i \leq \overline{u_{(t)}} < j$ for some $t \in \{1, 2, 3\}$. Thus, as u and v have the same content, $\rho_n(u)_{i,j} \neq -\infty$ if and only if $\rho_n(v)_{i,j} \neq -\infty$.

Finally, it suffices to show that $\rho_n(u)_{i,j} = 2$ if and only if $\rho_n(v)_{i,j} = 2$. Observe that, as $\rho_n(u)_{i,j} \leq 2$, then $\rho_n(u)_{i,j} = 2$ if and only if there exists $i \leq k \leq j$ such that $\rho_n(u_{(s_1)})_{i,k} = \rho_n(u_{(s_2)})_{k,j} = 1$ for some $1 \leq s_1 < s_2 \leq 3$ and hence, $i \leq \overline{u_{(s_1)}} < k \leq \overline{u_{(s_2)}} \leq j$.

By considering all the decreasing sequences in both sides of each Knuth relation, it suffices to show that if $\rho_n(ca)_{i,j} = 2$ then $\rho_n(ba)_{i,j} = 2$ for $a < b \leq c$ and $\rho_n(cb)_{i,j} = 2$ for any $a \leq b < c$.

Suppose $\rho_n(ca)_{i,j} = 2$ for $a < b \leq c$. Then, there exists k such that $\rho_n(c)_{i,k} = \rho_n(a)_{k,j} = 1$, with $i \leq \overline{c} < k \leq \overline{a} < j$. But then as $a < b \leq c$, there exists k' such that $i \leq \overline{b} < k' \leq \overline{a} < j$, hence $\rho_n(b)_{i,k'} = \rho_n(a)_{k',j} = 1$. Similarly, if $\rho_n(ca)_{i,j} = 2$ for $a \leq b < c$ then there exists k such that $\rho_n(c)_{i,k} = \rho_n(a)_{k,j} = 1$, with $i \leq \overline{c} < k \leq \overline{a} < j$. But then as $a \leq b < c$, there exists k' such that $i \leq \overline{c} < k' \leq \overline{b} < j$, hence $\rho_n(c)_{i,k'} = \rho_n(b)_{k',j} = 1$. Thus, ρ_n respects the Knuth relations. □

Let us denote by $\hat{\rho}_n$ the induced morphism from styl_n to $U_{n+1}(\mathbb{T})$. For example, the words 4213, 4214234 and 4241234 are in the same stylic class, and the image of

$[4213]_{\mathsf{styl}_4}$ under $\hat{\rho}_4$ is the same as that of 4213 under ρ_4, that is,

$$\hat{\rho}_4 : \begin{array}{|c|c|c|c|} \hline 4 \\ \cline{1-1} 2 & 4 \\ \cline{1-2} 1 & 2 & 3 & 4 \\ \hline \end{array} \quad \mapsto \quad \begin{pmatrix} 0 & 1 & 2 & 2 & 3 \\ -\infty & 0 & 1 & 1 & 2 \\ -\infty & -\infty & 0 & 1 & 2 \\ -\infty & -\infty & -\infty & 0 & 1 \\ -\infty & -\infty & -\infty & -\infty & 0 \end{pmatrix}$$

The following lemma allows us to deduce if a letter a occurs in the k-th row of $N(w)$, by looking at the image of w under ρ_n and seeing if, in line $\overline{a}$, the leftmost entry with value k (if it exists) has below it an entry with value $k-1$:

Lemma 8.2.4. *Let $w \in [n]^*$, $a \in [n]$, and $k \in \mathbb{N}$. Then, a occurs in the k-th row of $N(w)$ if and only if there exists $j \in \{1, \ldots, n+1\}$, with $\overline{a} < j$, such that $\rho_n(w)_{\overline{a},j} = k$, and $\rho_n(w)_{\overline{a}+1,j} = k-1$.*

Proof. Suppose for some $\overline{a} < j$, $\rho_n(w)_{\overline{a},j} = k$ and $\rho_n(w)_{\overline{a}+1,j} = k-1$. Then, by Lemma 8.2.1, there exists a strictly decreasing subsequence $w_{(s_1)}, \ldots, w_{(s_k)}$ of w such that $a \geq w_{(s_1)} > \cdots > w_{(s_k)} \geq \overline{j} + 1$.

Recall the "arrow" notation introduced in Subsection 8.1.2. We want to show that $\uparrow_w^{k-1}(s_k) = a$. As $w_{(s_1)}, \ldots, w_{(s_k)}$ is a strictly decreasing sequence of length k, $b := \uparrow_w^{k-1}(s_k) \neq \varepsilon$. Note that $b \leq a$ as $\uparrow_w^{l}(s_k) \leq w_{(s_{k-l})}$ by the definition of $\uparrow_w^l$. Thus, by Lemma 8.1.3, there is a strictly decreasing subsequence $w_{(s'_1)}, \ldots, w_{(s'_{k-1})}, w_{(s_k)}$ such that $w_{(s'_1)} = b$. However, as $\rho_n(w)_{\overline{a}+1,j} = k-1$, by Lemma 8.2.1, there is no strictly decreasing subsequence of length k only using letters between $\overline{j}+1$ and $a-1$. Hence, $a - 1 < w_{(s'_1)}$ or $w_{(s_k)} < \overline{j} + 1$. Thus, $b = w_{(s'_1)} > a - 1$ as $w_{(s_k)} \geq \overline{j} + 1$. Therefore, $a = b$, and hence, by Lemma 8.1.2, a occurs in the k-th row of $N(w)$.

Suppose now that a occurs in the k-th row of $N(w)$. Hence, by Lemma 8.1.2, there exists an index $s_k \leq |w|$ such that $\uparrow_w^{k-1}(s_k) = a$, and therefore, by Lemma 8.1.3, there exists a strictly decreasing subsequence $w_{(m_1)}, \ldots, w_{(m_k)}$ of w, where $w_{(m_1)} = a$ and hence, by Lemma 8.2.1, $\rho(w)_{\overline{a},n+1} \geq k$.

Choose j as the minimum index such that $\rho_n(w)_{\overline{a},j} = k$, which exists by Corollary 8.2.2, since $\rho_n(w)_{\overline{a},\overline{a}} = 0$. Suppose, in order to obtain a contradiction, that $\rho_n(w)_{\overline{a}+1,j} = k$. Let $b < a$ be such that $\rho_n(w)_{\overline{b},j} = k$ and $\rho_n(w)_{\overline{b}+1,j} = \rho_n(w)_{\overline{b},j-1} = k-1$. Notice that such a b exists, by Corollary 8.2.2. By Lemma 8.2.1, there exists a strictly decreasing $w_{(p_1)}, \ldots, w_{(p_k)}$ of w such that $b \geq w_{(p_1)} > \cdots > w_{(p_k)} \geq \overline{j} + 1$. By the same reasoning as given before, we can show that $\uparrow_w^{k-1}(p_k) = b$. Thus, by Lemma 8.1.3, there exists a strictly decreasing subsequence $w_{(r_1)}, \ldots, w_{(r_k)}$ of w such that $w_{(r_k)} = w_{(p_k)} \geq \overline{j} + 1$, $w_{(r_1)} = b$, and $\uparrow_w^{i-1}(r_i) = b$ for $1 < i \leq k$. Notice that, since $\rho_n(w)_{\overline{b},j-1} = k - 1$, then $w_{(r_k)} \leq \overline{j} + 1$ by Lemma 8.2.1, otherwise we would have a strictly decreasing subsequence of w of length k only using letters between $\overline{j} + 2$ and b, contradicting the minimality of j. Hence, $w_{(r_k)} = \overline{j} + 1$.

On the other hand, as a is in the k-th row of $N(w)$, by Lemma 8.1.2, there exists s_k such that $\uparrow_w^{k-1}(s_k) = a$, and hence, by Lemma 8.1.3, there exists a strictly decreasing sequence $w_{(s_1)}, \ldots, w_{(s_k)}$ where $w_{(s_1)} = a$, $\uparrow_w^{i-1}(s_i) = a$ for $1 < i \leq k$, and $w_{(s_k)} \leq \overline{j} + 1$, since $\rho_n(w)_{\overline{a},j-1} = k - 1$.

As $a = w_{(s_1)} > w_{(r_1)} = b$, we have that $r_1 \leq s_1$ as otherwise $w_{(s_1)}, w_{(r_1)}, \ldots, w_{(r_k)}$ would form a strictly decreasing sequence between a and $w_{(r_k)}$ of length $k+1$. Moreover, if we had $w_{(s_2)} < w_{(r_1)}$, then we would have $a = \uparrow_w^1(s_2) \leq w_{(r_1)} = b < a$ as $r_1 \leq s_2$. Thus, $w_{(s_2)} \geq w_{(r_1)}$.

By induction, we will show that $w_{(s_{i+1})} \geq w_{(r_i)}$, for all $1 \leq i \leq k - 1$. The base case was covered in the previous paragraph. Suppose that there is $1 \leq i \leq k - 2$ such that $w_{(s_{i+1})} \geq w_{(r_i)}$. Notice that, if $s_{i+1} < r_{i+1}$, then, by our assumption, $w_{(s_{i+1})} \geq w_{(r_i)} > w_{(r_{i+1})}$, and hence $w_{(s_1)}, \ldots, w_{(s_{i+1})}, w_{(r_{i+1})}, \ldots, w_{(r_k)}$ is a strictly decreasing sequence between a and $w_{(r_k)}$ of length $k + 1$, giving a contradiction. So, $r_{i+1} \leq s_{i+1}$. Since $w_{(s_{i+2})}$ occurs after $w_{(s_{i+1})}$, which was shown to occur after $w_{(r_{i+1})}$, we have that $w_{(s_{i+2})} < w_{(r_{i+1})}$ implies $\uparrow_w^1(s_{i+2}) \leq w_{(r_{i+1})}$ and hence $a = \uparrow_w^{i+1}(s_{i+2}) \leq \uparrow_w^i(r_{i+1}) = b < a$, giving a contradiction. Thus, $w_{(s_{i+2})} \geq w_{(r_{i+1})}$.

Therefore, we can conclude that

$$\overline{j} + 1 \geq w_{(s_k)} \geq w_{(r_{k-1})} > w_{(r_k)} = \overline{j} + 1,$$

which results in a contradiction. Thus, $\rho_n(w)_{\overline{a}+1,j} \neq k$, which, by Corollary 8.2.2, implies that $\rho_n(w)_{\overline{a}+1,j} = k - 1$. □

Theorem 8.2.5. *The morphism $\hat{\rho}_n \colon \mathsf{styl}_n \to U_{n+1}(\mathbb{T})$ is a faithful representation of styl_n.*

Proof. It suffices to show that we can construct $N(w)$ from $\rho_n(w)$. By the previous lemma, a letter a is in the k-th row of $N(w)$ if and only if there exists an index

j such that $\rho_n(w)_{\overline{a},j} = k$ and $\rho_n(w)_{\overline{a}+1,j} = k-1$. Since N-tableaux are uniquely determined by the support of each row (see [AR22, Subsection 6.1]), and ρ_n induces $\hat{\rho}_n$ by Proposition 8.2.3, we can recover, from $\hat{\rho}_n([w]_{\mathsf{styl}_n})$, all the information needed to construct $N(w)$. □

As an example, recall the image of $[4213]_{\mathsf{styl}_4}$ under $\hat{\rho}_4$, that is,

$$\hat{\rho}_4\colon \begin{array}{|c|ccc}\hline 4 & & & \\ \hline 2 & \multicolumn{1}{|c|}{4} & & \\ \hline 1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{4} \\ \hline \end{array} \quad \mapsto \begin{pmatrix} 0 & 1 & 2 & 2 & 3 \\ -\infty & 0 & 1 & 1 & 2 \\ -\infty & -\infty & 0 & 1 & 2 \\ -\infty & -\infty & -\infty & 0 & 1 \\ -\infty & -\infty & -\infty & -\infty & 0 \end{pmatrix}$$

Notice that $\rho_n(4213)_{1,5} = 3$ and $\rho_n(4213)_{2,5} = 2$, hence, 4 is in the third row of $N(4213)$. However, since $\rho_n(4213)_{2,4} = \rho_n(4213)_{3,4} = 1$ and $\rho_n(4213)_{2,5} = \rho_n(4213)_{3,5} = 2$, we have that 3 is neither in the second nor the third row of $N(4213)$; on the other hand, since $\rho_n(4213)_{2,3} = 1$, we can conclude that 3 is in the first row of $N(4213)$. Similarly, we can see that 2 is in the second, but not the third row, and 1 is only in the first row. With this information, we have all the necessary information to construct $N(4213)$.

Corollary 8.2.6. $\mathcal{V}(\mathsf{styl}_n) \subseteq \mathcal{V}(J_n)$ *for all* $n \in \mathbb{N}$.

Proof. Follows from the previous theorem, and [JF19, Corollary 3.3]. □

We now define two semirings: $\mathbb{N}_{0,\max} := \mathbb{T} \cap (\mathbb{N} \cup \{0, -\infty\})$; and $[n]_{0,\max} := \{x \in \mathbb{N}_{0,\max} \mid x \leq n\}$, for $n \in \mathbb{N}$, with operations max and n-truncated addition. These semirings can be seen to be $\mathbb{N}^*_{\max} \cup \{0, -\infty\}$ and $[n]^*_{\max} \cup \{0, -\infty\}$ respectively. Furthermore, we define the morphism $\varphi_{n+1}\colon U_{n+1}(\mathbb{N}_{0,\max}) \to U_{n+1}([n]_{0,\max})$ to be given by $\varphi_{n+1}(X)_{i,j} = \min(X_{i,j}, n)$.

Note that $\hat{\rho}_n(\mathsf{styl}_n) \subseteq U_{n+1}(\mathbb{N}_{0,\max})$. For the following corollary, treat $\hat{\rho}_n$ as a morphism with codomain $U_{n+1}(\mathbb{N}_{0,\max})$. Consider the morphism $\overline{\rho_n}\colon \mathsf{styl}_n \to U_{n+1}([n]_{0,\max})$ defined by $\overline{\rho_n}([x]_{\mathsf{styl}_n}) = (\varphi_{n+1} \circ \hat{\rho}_n([x]_{\mathsf{styl}_n}))$, for $x \in [n]^*$.

Corollary 8.2.7. *The morphism* $\overline{\rho_n}\colon \mathsf{styl}_n \to U_{n+1}([n]_{0,\max})$ *is a faithful representation of* styl_n.

Proof. We can see that $\overline{\rho_n}$ is a morphism as φ_{n+1} and $\hat{\rho}_n$ are both morphisms. Moreover, for $w_1, w_2 \in [n]^*$,

$$\overline{\rho_n}([w_1]_{\mathsf{styl}_n}) = \overline{\rho_n}([w_2]_{\mathsf{styl}_n}) \quad \text{if and only if} \quad \hat{\rho}_n([w_1]_{\mathsf{styl}_n}) = \hat{\rho}_n([w_2]_{\mathsf{styl}_n}),$$

as $\hat{\rho}_n([w]_{\mathsf{styl}_n})_{i,j} \leq n$ for all $1 \leq i, j \leq n+1$ and $w \in [n]^*$. Hence, $\overline{\rho_n}(\mathsf{styl}_n) \cong \hat{\rho}_n(\mathsf{styl}_n) \cong \mathsf{styl}_n$. □

Remark 8.2.8. *By [JF19, Proposition 3.2], $U_{n+1}(\mathbb{T})$, $U_{n+1}(\mathbb{N}_{0,\max})$ and $U_{n+1}([n]_{0,\max})$ satisfy the exact same set of monoid identities. Hence, we gain no more information about the monoid identities satisfied by styl_n by considering $\overline{\rho_n}$ rather than $\hat{\rho}_n$.*

8.3 Identities of the Stylic Monoid

We now show that styl_n and $U_{n+1}(\mathbb{T})$ satisfy the exact same set of monoid identities, thus proving that styl_n and $U_{n+1}(\mathbb{T})$ both generate the variety $\mathcal{V}(J_n)$, and that styl_n generates the pseudovariety $\boldsymbol{\mathcal{J}}_n$.

Theorem 8.3.1. *Let $n \in \mathbb{N}$ and let $u = v$ be a non-trivial identity satisfied by styl_n. Then, $u = v \in J_n$.*

Proof. We show the contrapositive of the statement. Let $\Sigma = \{x_1, \ldots, x_m\}$ be a set of variables, and let $u, v \in \Sigma^*$ be such that $u = v \notin J_n$. Without loss of generality, we can assume that there exist variables $a_1, \ldots, a_k \in \Sigma$ such that $a_k, \ldots, a_1$ form a subsequence of u but not of v, for some $k \leq n$.

Let $y_1, \ldots, y_m$ be strictly increasing words over $[k]$, defined as follows:

$$i \in \operatorname{supp}(y_j) \text{ if and only if } a_i = x_j,$$

for $1 \leq i \leq k$, $1 \leq j \leq m$. In other words, y_j is the strictly increasing product of indexes i such that a_i is the variable x_j.

Let $\phi \colon \Sigma^* \to [k]^*$ be the homomorphism given by $x_i \mapsto y_i$. Notice that, since $j \in \operatorname{supp}(\phi(a_j))$, then, for any $w \in \Sigma^*$, if w contains the subsequence $a_k, \ldots, a_1$, then $\phi(w)$ contains the subsequence $k, \ldots, 1$. On the other hand, if $\phi(w)$ contains the subsequence $k, \ldots, 1$, then each index i occurs in some y_{j_i}, such that $\phi(w)$ contains the subsequence $y_{j_k}, \ldots, y_{j_1}$. This implies that w contains the subsequence $x_{j_k}, \ldots, x_{j_1}$, which, by the definition of y_j, is the subsequence $a_k, \ldots, a_1$.

Hence, $\phi(u)$ contains the subsequence $k, \ldots, 1$, but $\phi(v)$ does not. Therefore, since this subsequence is the only strictly decreasing subsequence, of length k, whose first letter is k, that can occur in a word over $[k]$, we have that, by Lemmas 8.1.3 and 8.1.2, $N(\phi(u))$ contains k in the k-th row, but $N(\phi(v))$ does not. Hence $\phi(u) \not\equiv_{\mathsf{styl}} \phi(v)$ and therefore $u = v$ is not satisfied by styl_k. Since $k \leq n$, $u = v$ is not satisfied by styl_n. □

Therefore, the stylic monoid of rank n joins an increasing list of monoids (see [JF19, Vol04]) whose equational theory is J_n.

Corollary 8.3.2. *For each $n \in \mathbb{N}$, styl_n generates the variety $\mathcal{V}(J_n)$ and the pseudovariety $\boldsymbol{\mathcal{J}}_n$. Furthermore, $\mathcal{V}(\mathsf{styl}_n) \subsetneq \mathcal{V}(\mathsf{styl}_{n+1})$ for all $n \in \mathbb{N}$, and styl_n is finitely based if and only if $n \leq 3$.*

The following is an immediate consequence of [BFH+20, Section 3]:

Corollary 8.3.3. *$\mathcal{V}(\mathsf{styl}_n)$ has uncountably many subvarieties, for $n \in \mathbb{N}$ such that $n \geq 3$.*

Proof. For any $n \in \mathbb{N}$ such that $n \geq 3$, the word xyx is an *isoterm* for the equational theory of styl_n, that is, there is no non-trivial identity $u = v$ satisfied by styl_n, where u or v is the word xyx. Hence [Jac00, Theorem 3.2] applies. □

As such, styl_3 is the only stylic monoid which is simultaneously finitely based and which generates a variety with uncountably many subvarieties. Thus, it is finitely based but not *hereditarily finitely based*, that is, not all of its subvarieties are finitely based. On the other hand, since styl_1 and styl_2 are monoids with a zero and five or less elements, they are hereditarily finitely based [ELL10].

8.4 The Finite Basis Problem for the Stylic Monoid with Involution

Given a semigroup S, an *involution* on S is a unary operation * on S such that $(x^*)^* = x$ and $(xy)^* = y^*x^*$. An *involution semigroup* is a semigroup together with an involution, denoted $(S, {}^*)$. Given an involution semigroup $(S, {}^*)$, we say the *semigroup reduct* of $(S, {}^*)$ is the underlying semigroup S.

The definitions of *involution semigroup variety*, *finitely based involution semigroup*, *identities satisfied by involution semigroups*, and their corresponding involution monoid definitions are analogous to the ones given for semigroups in Chapter 2. For a formal definition of these terms, see [ADV12].

The unique order-reversing permutation on a finite ordered alphabet $[n]$, which we denote $\bar{\cdot}$, is an anti-automorphism of the free monoid over $[n]$, thus giving an involution. Furthermore, it induces an anti-automorphism of the stylic monoid of rank n, which is also an involution (see [AR22, Subsection 9.1]). We will denote this involution by $*$ and the stylic monoid with involution by $(\mathsf{styl}_n, *)$. Similarly, the operation of skew transposition, denoted $\star$, is an involution on the monoid of unitriangular matrices over the tropical semiring.

We can extend the tropical representation of the stylic monoid of rank n given in Section 8.2 to the involution case:

Proposition 8.4.1. *The morphism* $\hat{\rho}_n \colon \mathsf{styl}_n \to U_{n+1}(\mathbb{T})$ *is a faithful morphism from* $(\mathsf{styl}_n, *)$ *to* $(U_{n+1}(\mathbb{T}), \star)$*.*

Proof. It suffices to show that $\hat{\rho}_n(x)^\star = \hat{\rho}_n(x^*)$ for all $x \in [n]$. For $x \in [n]$, we have that $\hat{\rho}_n(x^*)_{i,j} = 1$ if and only if $i \leq n+1-\overline{x} < j$ and $(\hat{\rho}_n(x)^\star)_{i,j} = 1$ if and only if $n+1-j < \overline{x} \leq n+1-i$. Thus, $\hat{\rho}_n(x^*)_{i,j} = 1$ if and only if $(\hat{\rho}_n(x)^\star)_{i,j} = 1$, and hence, by the definition of $\hat{\rho}_n$, we have that $\hat{\rho}_n(x^*) = \hat{\rho}_n(x)^\star$. $\square$

In [HZL21, Section 5], it was shown that the involution monoid $(U_{n+1}(\mathbb{T}), \star)$ is non-finitely based, for $n \geq 3$. It was also shown that $(U_3(\mathbb{T}), \star)$ satisfies, for each $k \in \mathbb{N}$, the identity

$$xy_1y_1^*y_2y_2^* \cdots y_ky_k^*x^*zz^* = zz^*xy_1y_1^*y_2y_2^* \cdots y_ky_k^*x^*.$$

As such, $(\mathsf{styl}_2, *)$ must also satisfy these identities. Similarly, it was also shown that $(U_4(\mathbb{T}), \star)$ satisfies, for each $k \in \mathbb{N}$, the identity

$$x_1x_2 \cdots x_kx_1^*x_2^* \cdots x_k^*x_1x_2 \cdots x_k = x_k^*x_{k-1}^* \cdots x_1^*x_kx_{k-1} \cdots x_1x_k^*x_{k-1}^* \cdots x_1^*.$$

As such, $(\mathsf{styl}_3, *)$ must also satisfy these identities.

However, as with the case of $(U_{n+1}(\mathbb{B}), \star)$ where the involution is again given by skew transposition, we have that $(\mathsf{styl}_n, *)$ does not satisfy exactly the same identities as $(U_{n+1}(\mathbb{T}), \star)$, in contrast to the monoid reduct case:

Proposition 8.4.2. *For each $n \geq 2$, $(\mathsf{styl}_n, {}^*)$ satisfies the identity*

$$x^* x^{n-1} = x^* x^n, \tag{8.1}$$

while $(U_{n+1}(\mathbb{T}), {}^\star)$ does not.

Proof. By [HZL21, Theorem 5.2], we already know that $(U_{n+1}(\mathbb{T}), {}^\star)$ does not satisfy the identity (8.1). Let $\phi : X \to \mathsf{styl}_n$ be a map. If $\operatorname{supp}(\phi(x)) = [n]$, then $\operatorname{supp}(\phi(x^*)) = [n]$ and, as such, each side of the identity (8.1) has a word representative with a decreasing subsequence of all letters in $[n]$. As such, the evaluation of both sides of the identity are equal to $[n \cdots 1]_{\mathsf{styl}_n} = 0_{\mathsf{styl}_n}$.

On the other hand, suppose $\operatorname{supp}(\phi(x)) \neq [n]$. Then, $\phi(x)^{n-1} = \phi(x)^n$, since both elements have a word representative with the maximal decreasing subsequence of elements of its support, of length less than or equal to $n-1$. Equality follows. Therefore, $(\mathsf{styl}_n, {}^*)$ satisfies the identity (8.1). □

As such, $(\mathsf{styl}_n, {}^*)$ does not generate the same variety as $(U_{n+1}(\mathbb{T}), {}^\star)$, in contrast to the monoid reduct case. It remains open if $(\mathsf{styl}_n, {}^*)$ and $(U_{n+1}(\mathbb{B}), {}^\star)$ generate the same variety, where $\mathbb{B}$ is the boolean semiring.

Regarding the question of finite bases for the stylic monoids with involution, it is immediate that $(\mathsf{styl}_1, {}^*)$ is finitely based, since it is a two-element monoid with a zero. Hence, it admits a finite basis, consisting of the following identities:

$$x^2 = x \quad \text{and} \quad xy = yx \quad \text{and} \quad x^* = x.$$

We say an involution semigroup $(S, {}^*)$ is *twisted* if the variety $\mathcal{V}(S, {}^*)$ it generates contains the involution semilattice $(Sl_3, {}^*)$, where

$$Sl_3 = \{0, a, b\}$$

is a semilattice such that $ab = ba = 0$ and the involution is given by

$$0^* = 0, \quad a^* = b, \quad b^* = a.$$

Notice that any identity $u = v$ satisfied by $(Sl_3^1, {}^*)$, that is, $(Sl_3, {}^*)$ with an identity adjoined, is such that $\operatorname{supp}(u) = \operatorname{supp}(v)$. It can be easily seen that the variety generated by a twisted involution monoid also contains $(Sl_3^1, {}^*)$. Therefore, the identities satisfied by any twisted involution monoid must have the same support in both sides of the identity.

Lemma 8.4.3. *For each* $n \geq 2$, $(\mathsf{styl}_n, {}^*)$ *is twisted.*

Proof. Consider the quotient of the involution subsemigroup

$$\{[1]_{\mathsf{styl}_2}, [2]_{\mathsf{styl}_2}, [12]_{\mathsf{styl}_2}, [21]_{\mathsf{styl}_2}\}$$

of $(\mathsf{styl}_2, {}^*)$ by the congruence which identifies $[12]_{\mathsf{styl}_2}$ with $[21]_{\mathsf{styl}_2}$. This quotient is isomorphic to $(Sl_3, {}^*)$, hence $(\mathsf{styl}_2, {}^*)$ is twisted. Furthermore, since $(\mathsf{styl}_2, {}^*)$ embeds into $(\mathsf{styl}_n, {}^*)$, for each $n \geq 3$, we have that $(\mathsf{styl}_n, {}^*)$ is also twisted. □

By [Lee17, Theorem 4], we have that any twisted involution semigroup whose semigroup reduct is non-finitely based must also be non-finitely based. Since styl_n is non-finitely based for $n \geq 4$, by Corollary 8.3.2, the following is immediate:

Corollary 8.4.4. *For any* $n \geq 4$, $(\mathsf{styl}_n, {}^*)$ *is non-finitely based.*

Now, we look at the case of $(\mathsf{styl}_2, {}^*)$: The following proof was suggested by the anonymous referee for [AR23].

Proposition 8.4.5. $(\mathsf{styl}_2, {}^*)$ *is non-finitely based.*

Proof. It is easy to see that $(\mathsf{styl}_2, {}^*)$ is isomorphic to the Catalan monoid with involution $(\mathsf{Cat}_2, {}^*)$, of rank 2 and order 5, which was shown to be non-finitely based in [GZL20], hence, the result follows. □

Finally, we look at the case of $(\mathsf{styl}_3, {}^*)$: Again, the following proof was suggested by the anonymous referee for [AR23].

Proposition 8.4.6. $(\mathsf{styl}_3, {}^*)$ *is non-finitely based.*

Proof. In [Vol22], it is shown that styl_3 is a homomorphic image of the Kiselman monoid Kis_3 and the Catalan monoid Cat_3 is a homomorphic image of styl_3. It can be easily checked that these properties still hold when considering the mentioned monoids with involution. Since, in [GZL22], it was shown that $(\mathsf{Kis}_3, {}^*)$ and $(\mathsf{Cat}_3, {}^*)$ generate the same variety and said variety is non-finitely based, the result follows. □

Therefore, we obtain the following result:

Theorem 8.4.7. *The involution monoid* $(\mathsf{styl}_n, {}^*)$ *is finitely based if and only if* $n = 1$.

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