TESIS DE LA UNIVERSIDAD
DE ZARAGOZA

2023

140

Jesús Oliva Maza

Spectral study of operator semigroups and functional calculus. Applications

Estudio espectral de semigrupos de operadores y calculo funcional. Aplicaciones

Director/es
Miana Sanz, Pedro José
Galé Gimeno, José Esteban

ISSN 2254-7606

Prensas de la Universidad
Universidad Zaragoza

Servicio de Publicaciones

ISSN 2254-7606

Tesis Doctoral

SPECTRAL STUDY OF OPERATOR SEMIGROUPS AND FUNCTIONAL CALCULUS. APPLICATIONS

ESTUDIO ESPECTRAL DE SEMIGRUPOS DE OPERADORES Y CALCULO FUNCIONAL. APLICACIONES

Autor

Jesús Oliva Maza

Director/es

Miana Sanz, Pedro José
Galé Gimeno, José Esteban

UNIVERSIDAD DE ZARAGOZA
Escuela de Doctorado

Programa de Doctorado en Matemáticas y Estadística

2023

Universidad de Zaragoza
Departamento de Matemáticas – IUMA

Spectral study of operator semigroups and functional calculus. Applications

Estudio espectral de semigrupos de operadores y cálculo funcional. Aplicaciones

Memoria presentada por

Jesús Oliva Maza

para optar al Grado de

Doctor por la Universidad de Zaragoza

Dirigida por los doctores

José Esteban Galé Gimeno, Universidad de Zaragoza (España)

Pedro José Miana Sanz, Universidad de Zaragoza (España)

Quisiera expresar en estas líneas mi agradecimiento más profundo a todas aquellas personas que, de forma directa o indirecta, han contribuido a la realización de esta memoria.

El desarrollo de esta tesis habría sido imposible si no fuera por mis directores, José E. Galé y Pedro J. Miana, los cuales han sido una guía y apoyo fundamental en mi desarrollo como matemático. Gracias por su comprensión y su buen humor. En particular, quiero dar las gracias a Pedro por su entusiasmo desde el primer día para que este proyecto tan ambicioso saliera adelante, así como por su trabajo contra viento y marea para solucionar los numerosos obstáculos que han surgido a lo largo del mismo; a Pepe, por su incansable ambición y motivación, especialmente en los momentos de flaqueza, para que mi crecimiento como matemático e investigador fuera lo más profundo posible.

Desde el punto de vista académico (aunque también en el ámbito personal), he sido realmente afortunado por poder desarrollar mi tesis en el Área de Análisis Matemático de la Universidad de Zaragoza. Sus integrantes me arroparon desde el primer momento e hicieron que me sintiera como uno más. Les debo mucho por sus consejos y atención a lo largo de estos años. En particular a Luciano, con el que he pasado muchas horas investigando en el tramo final del doctorado, y a Ana y a Luis Carlos, por cómo me han facilitado las tareas de docencia durante la tesis. También estoy muy agradecido a Mahamadi Warma y a Daniel Beltiţă por su hospitalidad (y sus enseñanzas académicas) en sendas estancias de investigación en Fairfax y en Bucarest.

En lo personal, he tenido la suerte de estar rodeado por un grupo de personas increíbles que me han traído la felicidad durante los últimos años.

Gracias a mis compañeros y amigos por todos esos momentos de risas y despreocupación tan necesarios en nuestras vidas. En particular, gracias a mi grupo inseparable de amigos del instituto IES Eláios, a toda mi gente cercana del maravilloso municipio de Tardienta, y a mis estimados consejeros 'Peña' y 'Beto'.

Por supuesto, muchas gracias a mi familia. En particular, a mis padres, Jesús y Maite por su apoyo, de especial importancia durante la pandemia y durante las obras de mi piso; y a mis hermanas, Ana y Laura, por sus viajes e iniciativas para que descansara (aunque yo me negara) de la tesis. A mis abuelas Pilar y Tere (y a mis difuntos abuelos Pepe y Antonio), y a mis tíos y primos, a los que me hubiera gustado dedicarles mucho más tiempo que el que he podido durante estos últimos cinco años.

Y por último, muchas gracias a Marina, que se ha ganado el cielo por todo el apoyo y los buenos momentos que me ha dado durante esta etapa (y también por aguantarme).

Jesús Oliva Maza

Zaragoza, enero de 2023.

During the period of preparation of this work, the author was supported by a FPI grant from the University of Zaragoza (BES-2017-081552, MINECO). A fellowship from the same program allowed the author to carry out a stay in the 'Simion Stoilow' Institute of Mathematics of the Romanian Academy in Bucharest in February 2020, and a stay in the Mathematical Sciences Department of the George Mason University in Fairfax, Virginia, from September 2021 to December 2021. This research was also partially supported by the grants 'Análisis Matemático, Métodos Geométricos y Teoría de Operadores' (MTM2016-77710-P), and 'Operadores y Geometría en Análisis Matemático' (PID2019-105979GB-I00).

Esta memoria ha sido elaborada durante el periodo de disfrute de una ayuda para un contrato predoctoral de la Universidad de Zaragoza (BES-2017-081552, MINECO). Una ayuda del mismo programa permitió al autor realizar una estancia en el Instituto de Matemáticas 'Simion Stoilow' de la Academia de Rumanía en Bucarest en febrero 2020, y una estancia en el Departamento de Matemáticas de la Universidad de George Mason en Fairfax, Virginia, desde septiembre 2021 hasta diciembre 2021. Esta investigación también ha estado financiada por los proyectos 'Análisis Matemático, Métodos Geométricos y Teoría de Operadores' (MTM2016-77710-P), and 'Operadores y Geometría en Análisis Matemático' (PID2019-105979GB-I00).

Contents

Introducción

Los operadores integrales han sido objetos matemáticos de gran interés e importancia en el de análisis funcional y la teoría de operadores desde el mismo comienzo de ambos campos, y han sido estudiados desde diferentes puntos de vista a lo largo de los años. En esta memoria, estamos interesados en las relaciones de los operadores integrales con grupos y semigrupos de operadores. Veamos a continuación un ejemplo ilustrativo de dicha relación. Sea H un núcleo de Hardy en $(0,\infty)\times(0,\infty)$. En particular,

$$(\forall y>0) \quad H(\lambda x, \lambda y) = \lambda^{-1} H(x,y), \quad x,y>0.$$

Entonces, el operador integral T_H asociado a H está determinado por

$$(T_H f)(y) := \int_0^\infty H(x,y) f(x)\,dx,\ y>0,$$

siempre que $f:(0,\infty)\to\mathbb{C}$ sea una función adecuada. El operador T_H se puede expresar como una integral vectorial si aplicamos dos sencillos cambios de variable: para $y>0$,

$$\begin{aligned}(T_H f)(y) := \int_0^\infty H(x,y)f(x)dx &= \int_0^\infty y^{-1}H(y^{-1}x,1)f(x)dx \\ &= \int_0^\infty H(u,1)f(uy)du = \int_{-\infty}^\infty e^{-t}H(e^{-t},1)f(e^{-t}y)dt;\end{aligned}$$

es decir, el operador integral T_H se puede escribir como

$$T_H f = \int_{-\infty}^\infty e^{-t}H(e^{-t},1)E(t)f\,dt, \tag{0.1}$$

donde $(E(t))_{t\in\mathbb{R}}$ es el *grupo* de operadores dado por $E(t)f := f(e^{-t}(\cdot))$, $t\in\mathbb{R}$.

El ejemplo anterior se puede interpretar como un caso particular de subordinación de operadores, esto es, operadores $\mathcal{T}$ en un espacio de Banach X determinados por una integral vectorial (Bochner convergente) del tipo

$$\mathcal{T}f = \int_\Omega \varphi(t)T(t)f\ dt, \quad f\in X, \tag{0.2}$$

donde $\varphi:\Omega\to\mathbb{C}$ es una función apropiada, $\Omega=(0,\infty)$ o $\Omega=\mathbb{R}$, y $(T(t))_{t\in\Omega}$ es un C_0-semigrupo (si $\Omega=(0,\infty)$) o un C_0-grupo (si $\Omega=\mathbb{R}$) de operadores en X. En este contexto, una idea que ha resultado fructífera a lo largo de los años es la transferencia de información de propiedades desde el (semi)grupo $(T(t))$ hacia el operador $\mathcal{T}$. Para ello, resulta esencial la representación del operador $\mathcal{T}$ en términos del generador infinitesimal Δ del (semi)grupo $(T(t))$, lo cual nos permite interpretar (0.2) en términos de un cálculo funcional de funciones holomorfas. Nosotros adoptamos este punto de vista en la memoria, de modo que los grupos y semigrupos de operadores se encuentran en el núcleo del trabajo desarrollado aquí. De hecho, gran parte de los resultados e ideas expuestos en este documento están inspirados por las acciones de semigrupos en espacios de Banach.

Sea $(T(t))$ un C_0-semigrupo en un espacio de Banach X, y sea $\mathcal{T}$ un operador acotado en X obtenido por una subordinación con respecto a $(T(t))$ como en (0.2). El subespacio imagen $\operatorname{Ran}\mathcal{T} := \mathcal{T}(X)$ es un ejemplo de rango de operador. Los rangos de operadores han sido estudiados en numerosos artículos, como por ejemplo [ACG09; ACG13; FW71; Foi72; KT11; NRR+76; NRRR79], y su estudio se remonta al tratado seminal de Dixmier [Dix49]. Sin embargo, las posibles conexiones entre representaciones de grupos en espacios de Banach y los rangos de operadores no han sido exploradas con profundidad en la literatura. En esta dirección, parece natural la siguiente pregunta. Sea $(T(t))_{t\in\mathbb{R}}$ un grupo que es un subgrupo de un grupo estrictamente más grande G, tal que el rango de operador $\operatorname{Ran}\mathcal{T}$ es G-invariante. ¿Bajo que condiciones $(T(t))_{t\in\mathbb{R}}$ (ó G) inducen un C_0-grupo en el espacio rango $\operatorname{Ran}\mathcal{T}$? Esta pregunta está parcialmente originada con la finalidad de aplicar la teoría desarrollada en [BG14] a los espacios de derivación fraccionaria introducidos en [GMS21], y no resulta sencilla de responder. Dicha pregunta está estrechamente relacionada con la caracterización intrincada, en términos de descomposiciones espectrales, del subálgebra de operadores acotados que dejan un operador rango invariante, véase [NRRR79]. Con el objetivo de entender mejor este problema, se ha dado un primer paso en esta dirección en [Oli21a], en un marco abstracto de representaciones de grupos de Banach-Lie. Sin embargo, no proseguiremos con esta línea de investigación en la memoria. En vez de ello, centramos nuestro trabajo en las propiedades espectrales propias del operador $\mathcal{T}$. Para ello, notamos que en general no es posible describir el espectro de $\mathcal{T}$ directamente a partir del espectro del (semi)grupo $(T(t))$ dado en (0.2), ya que las igualdades no se dan en los teoremas de aplicación espectral en este contexto. Sin embargo, dadas las hipótesis adecuadas, es posible obtener información del espectro de $\mathcal{T}$ a partir del espectro del generador infinitesimal Δ de $(T(t))$ a través de una representación del tipo $\mathcal{T} = \int_\Omega \varphi(t)e^{t\Delta}\,dt$. Por tanto, es claro que un análisis detallado del espectro de los generadores infinitesimales de C_0-(semi)grupos es conveniente para nuestros fines. En particular, nos centramos en esta memoria en dos casos importantes de (semi)grupos de composición pesados que actúan en el disco unidad complejo $\mathbb{D}$.

$(E(t))_{t\in\mathbb{R}}$ es un ejemplo interesante de un (C_0-)grupo de operadores de composición sobre espacios de funciones en $(0,\infty)$, el cual está relacionado de una forma natural con los operadores de tipo Hardy, ver (0.1). Este grupo tiene aplicaciones en el análisis de la ecuación de Black-Scholes estudiada en [AP02]. Además, $(E(t))_{t\in\mathbb{R}}$ ha sido empleado para la representación de operadores de Cesàro-Hardy por medio de resolventes [AP10; AS13; LMPS14], véase también [GMS21]. A través de una transformada de Möbius pesada, el grupo $(E(t))_{t\in\mathbb{R}}$ es isomorfo al grupo de operadores de composición $(C_{\varphi_t})_{t\in\mathbb{R}}$ inducido por el flujo hiperbólico $(\varphi_t)_{t\in\mathbb{R}}$, dado por

$$\varphi_t(z) := \frac{(e^t+1)z+e^t-1}{(e^t-1)z+e^t+1}, \quad z\in\mathbb{D},\ t\in\mathbb{R};$$

es decir, $C_{\varphi_t}f := f\circ\varphi_t$, $f\in\mathcal{O}(\mathbb{D})$, donde $\mathcal{O}(\mathbb{D}) := \{f:\mathbb{D}\to\mathbb{C} \,:\, f \text{ holomorfa}\}$.

Uno de los objetivos de esta memoria es el estudio de C_0-grupos de composición pesados $(v_tC_{\psi_t})_{t\in\mathbb{R}}$, donde $(\psi_t)_{t\in\mathbb{R}}$ es un flujo de automorfismos hiperbólicos del disco $\mathbb{D}$ y $(v_t)_{t\in\mathbb{R}}$ es un cociclo para el flujo $(\psi_t)_{t\in\mathbb{R}}$. Para ello, llevamos a cabo un análisis detallado del espectro del generador infinitesimal Δ del grupo $(u_tC_{\varphi_t})_{t\in\mathbb{R}}$, donde $(u_t)_{t\in\mathbb{R}}$ es un cociclo adecuado para el flujo $(\varphi_t)_{t\in\mathbb{R}}$. Como consecuencia de este estudio, obtenemos una descripción precisa del espectro de operadores integrales promedio $\mathcal{T}$ en $\mathbb{D}$, obtenidos a partir de una subordinación como en (0.2) por grupos del tipo $(v_tC_{\psi_t})_{t\in\mathbb{R}}$. Los operadores así obtenidos son generalizaciones de otros dos operadores que están conectados con los operadores de tipo Cesàro. Uno de ellos es un operador integral introducido en [Sis86] por A. Siskakis, y el otro es el operador de la matriz reducida de Hilbert.

Nuestro estudio espectral para el generador infinitesimal Δ es llevado a cabo cuando el grupo $(u_tC_{\varphi_t})_{t\in\mathbb{R}}$ actúa en una familia de espacios de Banach entre los que se encuentran los espacios de Hardy, los espacios de Bergmann pesados, los espacios de Dirichlet pesados, los espacios little Bloch, los espacios little Korenblum y el álgebra del disco. Para ello, seguimos un enfoque unificado que nos permite obtener la descripción espectral de Δ siempre que el espacio de funciones holomorfas satisfaga ciertos axiomas. Por otro lado, los cociclos $(u_t)_{t\in\mathbb{R}}$ han de cumplir ciertas condiciones (poco restrictivas) que implican que $(u_t)_{t\in\mathbb{R}}$ se puede representar como $u_t = (\omega\circ\varphi_t)/\omega$, $t\in\mathbb{R}$, para cierta función holomorfa (también llamada peso) $\omega:\mathbb{D}\to\mathbb{C}$ que no presenta ceros en $\mathbb{D}$, véase [Kön90]. Uno de los resultados relevantes de esta tesis es que tal peso ω presenta ceros o singularidades de tipo polinomiales en los puntos Denjoy-Wolf de $(\varphi_t)_{t\in\mathbb{R}}$, esto es en -1 y en 1.

Además, el estudio presentado aquí para los grupos $(u_tC_{\varphi_t})_{t\in\mathbb{R}}$, o para los grupos más generales $(v_tC_{\psi_t})_{t\in\mathbb{R}}$, tienen aplicaciones en ciertas conjeturas sobre el espectro de operadores de composición de tipo hiperbólico con pesos invertibles vC_ψ; es decir, donde ψ es un automorfismo hiperbólico del disco $\mathbb{D}$ y donde v es un multiplicador invertible, véase [HLNS13; ELM16]. En efecto, respondemos de forma afirmativa a dichas conjeturas en el caso de que el operador vC_ψ pueda verse como un elemento de un grupo de operadores $(v_tC_{\psi_t})_{t\in\mathbb{R}}$.

Por otro lado, surge de forma natural el estudio de otros semigrupos de composición pesados usando técnicas inspiradas por las explicadas anteriormente para los automorfismos hiperbólicos. Más precisamente, consideramos semigrupos asociados a semiflujos cuyo punto de Denjoy-Wolf se encuentra en el disco $\mathbb{D}$, y que tienen un número finito de puntos repulsivos en la frontera $\partial\mathbb{D}$, esto es, en el círculo $\mathbb{T}$. Dichos semiflujos vienen inspirados como una versión abstracta del semiflujo analítico

$$\phi_{t,n}(z) := \frac{e^{-t}z}{((e^{-nt}-1)z^n+1)^{1/n}}, \quad z\in\mathbb{D}, t\geq 0, n\in\mathbb{N}.$$

Además, estos semiflujos aparecen, de una forma natural, relacionados con el estudio de operadores inducidos por matrices de Hausdorff en espacios de funciones holomorfas en $\mathbb{D}$. En este documento, hacemos uso del estudio espectral del generador de dichos semigrupos para obtener propiedades de acotación y propiedades espectrales de operadores generalizados de Hausdorff en el contexto de espacios de Hardy, espacios de Bergmann pesados, y espacios little Korenblum. Para ello, llevamos a cabo un análisis detallado y riguroso de funciones multivaluadas asociadas a ciertos semicociclos, las cuales también aparecen de forma natural al considerar pesos multivaluados asociados a los integrales promedio. Hemos de notar que, si bien el espectro de familias de operadores de composición pesados no invertibles se ha estudiado en [GLW20] en condiciones más generales que las dadas en este documento, el espectro de su generador infinitesimal (en el caso de que sean elementos de un semigrupo) no era conocido hasta la fecha.

Con el objetivo de transferir mediante la fórmula de subordinación (0.2) información espectral del generador infinitesimal Δ a los operadores integrales $\mathcal{T}$ en los que estamos interesados, debemos aplicar un cálculo funcional apropiado (asociado a Δ), así como unos teoremas de aplicación espectral; es decir, igualdades del tipo

$$\widetilde{\sigma}(\mathcal{T}) = f(\widetilde{\sigma}(\Delta)),$$

donde f es una función en el dominio del cálculo funcional de Δ tal que $\mathcal{T} = f(\Delta)$, y donde $\widetilde{\sigma}$ representa el espectro extendido. Dicho teorema de aplicación espectral fue dado en [Haa05b] en el marco de operadores sectoriales y para funciones meromorfas f tales que: 1) $\sigma(\Delta)\setminus\{0\} \subset \mathrm{Dom}(f)$; 2) f tiene 'límites casi-logarítmicos' en $\widetilde{\sigma}(\Delta)\cap\{0,\infty\}$. En particular, dicho teorema de aplicación espectral es válido para operadores adecuados $\mathcal{T}$ que están subordinados a C_0-semigrupos. Para cubrir el caso en el que $\mathcal{T}$ está subordinado a un C_0-grupo, hemos de adaptar cuidadosamente los resultados de [Haa05b] al marco de operadores de tipo bisectorial. Es más, extendemos el teorema de aplicación espectral de modo que también sea aplicable a diversos subconjuntos espectrales $\widetilde{\sigma}_i$, todos ellos denominados como 'espectro esencial' en la bibliografía por diferentes autores (en particular, el usual espectro esencial $\widetilde{\sigma}_{ess}$ definido en términos de operadores de tipo Fredholm). Dicha extensión es un avance considerable con respecto a los teoremas de aplicación espectral para espectros esenciales dados en [GL71; GO85], ya que la clase de funciones considerada aquí es significativamente más amplia y grande que la considerada en [GL71; GO85]. Además, respondemos afirmativamente (incluso para todos los espectros esenciales considerados aquí) a una pregunta sugerida por Haase en [Haa05b]. Dicha pregunta plantea si el teorema de aplicación espectral se sigue cumpliendo si, en vez de 'límites casi-logarítmicos', la función f en cuestión tiene límites 'casi-regulares'.

En otra dirección, el operador de Cesàro, el cual puede ser representado con una subordinación en términos del grupo $(E(t))_{t\in\mathbb{R}}$ como en (0.2), está estrechamente relacionado con la ecuación de Black-Scholes estudiada en [AP02]. Es por ello que el estudio de las versiones fraccionarias de la ecuación de Black-Scholes inducidas por los operadores fraccionarios de Cesàro (los cuales también se pueden subordinar a $(E(t))_{t\in\mathbb{R}}$) surge de una forma natural. Dado $\alpha > 0$, estas ecuaciones fraccionarias vienen dadas por

$$\tag{0.3}
\begin{aligned}
\frac{\partial f}{\partial t} &= \frac{1}{\Gamma(\alpha+1)^2} D^\alpha(x^\alpha D^\alpha(x^\alpha f)) - \frac{2}{\Gamma(\alpha+1)} D^\alpha(x^\alpha f) + f,\\
\frac{\partial f}{\partial t} &= \frac{1}{\Gamma(\alpha+1)^2} x^\alpha W^\alpha(x^\alpha W^\alpha f),\\
\frac{\partial f}{\partial t} &= -\frac{1}{\Gamma(\alpha+1)^2} D^\alpha(x^{2\alpha} W^\alpha f) + \frac{1}{\Gamma(\alpha+1)} x^\alpha W^\alpha f,
\end{aligned}$$

donde D^α denota la derivada de orden fraccionario de Riemann-Liouville, y W^α denota la derivada de orden fraccionario de Weyl.

El estudio del buen planteamiento de los problemas (abstractos) de Cauchy asociados a las ecuaciones anteriores no resulta trivial en absoluto. En efecto, la aparición conjunta de derivadas de orden fraccionario, junto con la multiplicación por potencias de orden fraccionario, hace que dichas ecuaciones sean difíciles (quizá imposibles) de resolver usando métodos clásicos. Por otro lado, las ecuaciones (0.3) se pueden representar a través del cálculo funcional regularizado del generador infinitesimal Δ del grupo $(E(t))_{t\in\mathbb{R}}$. Es más, las funciones (pertenecientes al dominio de dicho cálculo funcional) que aparecen en dicha representación poseen singularidades de tipo polinomial en ∞. Por ello, resulta pertinente la propiedad de escalamiento, la cual nos garantiza que el operador fraccionario Δ^α induce un problema de Cauchy bien planteado para valores de α adecuados. En este sentido, demostramos (en un marco abstracto de operadores de tipo bisectorial y operadores sectoriales) una extensión de la propiedad de escalamiento que, en particular, implica que $f(\Delta)$ induce un problema de Cauchy bien planteado si f tiene singularidades de tipo polinomial en ∞. Esta extensión de la propiedad de escalamiento, junto con una serie de resultados auxiliares, nos permite demostrar el buen planteamiento de las ecuaciones fraccionarias de tipo Black-Scholes (0.3), y además de ello, obtener fórmulas explícitas de sus soluciones.

La memoria está organizada como se expone a continuación.

Tras esta introducción (y su versión en inglés), se encuentra un primer capítulo preliminar donde ciertas definiciones y resultados son recordados. La primera parte del Capítulo 2 está dedicada a la adaptación (de una forma estándar) del 'cálculo funcional regularizado' desde el marco de operadores sectoriales (véase [Haa05a; Haa06]) a operadores de tipo bisectorial, lo cual será necesario para el enfoque unificado que llevaremos a cabo. La segunda parte de este capítulo está dedicada a la demostración de los teoremas de aplicación espectral para los distintos espectros esenciales. Después, en el Capítulo 3, presentamos extensiones de la propiedad de escalamiento sobre potencias fraccionarias de operadores, y aplicamos dichas extensiones, junto con el cálculo funcional regularizado, para resolver ecuaciones generalizadas de Black-Scholes en espacios de interpolación. El contenido de los Capítulos 2 y 3 se puede encontrar en [Oli22b; OW23].

El Capítulo 4 contiene un análisis sobre funciones multivaluadas necesario para la representación de los semicociclos que aparecen en el Capítulo 5, donde presentamos nuestros resultados sobre operadores generalizados de Hausdorff y semigrupos de composición pesados inducidos por semiflujos con punto de Denjoy-Wolf en $\mathbb{D}$ y con un número finito de puntos repulsivos en $\mathbb{T}$. El contenido de estos capítulos es el objeto de estudio de un trabajo en desarrollo en colaboración con L. Abadías [AO23].

El Capítulo 6 está dedicado al estudio detallado del espectro del generador infinitesimal de grupos de composición pesados inducidos por flujos hiperbólicos, así como a las consecuencias de estos resultados en los operadores integrales subordinados mencionados anteriormente. Estos resultados se encuentran en [AGMO22].

Finalmente, en la Adenda A, señalamos diferentes resultados para los operadores de Hardy, resultados en direcciones similares a las mencionadas anteriormente. El material correspondiente a esta adenda se puede encontrar en [MO20; MO21; Oli21b; Oli22a].

Introduction

Integral operators form a class of mathematical objects of great importance in functional analysis and operator theory from the very beginning of these subjects. Such operators have been approached from different points of view over the years. Here in this memory we are interested in certain aspects of their relations with groups and semigroups. Assume for a moment that H is an integral kernel of Hardy's type on $(0,\infty)\times(0,\infty)$, which is to say

$$(\forall y>0)\quad H(\lambda x,\lambda y)=\lambda^{-1}H(x,y),\quad x,y>0.$$

Then the integral operator T_H defined by H is given by

$$(T_Hf)(y):=\int_0^\infty H(x,y)f(x)\,dx,$$

$y>0$, for suitable functions $f\colon(0,\infty)\to\mathbb{C}$. Two simple changes of variable enable us to express T_H as a vector-valued integral: for $y>0$,

$$\begin{aligned}(T_Hf)(y):=\int_0^\infty H(x,y)f(x)dx&=\int_0^\infty y^{-1}H(y^{-1}x,1)f(x)dx\\&=\int_0^\infty H(u,1)f(uy)du=\int_{-\infty}^\infty e^{-t}H(e^{-t},1)f(e^{-t}y)dt;\end{aligned}$$

that is, one can write formally the operator T_H as

$$T_Hf=\int_{-\infty}^\infty e^{-t}H(e^{-t},1)E(t)f\,dt, \tag{0.4}$$

where $(E(t))_{t\in\mathbb{R}}$ is the *group* of operators given by $E(t)f:=f(e^{-t}(\cdot))$, $t\in\mathbb{R}$.

The above example is a particular case of the general situation when one has a (Bochner-convergent) integral defining by subordination an operator $\mathcal{T}$ on a Banach space X of the form

$$\mathcal{T}f=\int_\Omega \varphi(t)T(t)f\;dt,\quad f\in X, \tag{0.5}$$

for suitable functions $\varphi\colon\Omega\to\mathbb{C}$, where $\Omega=(0,\infty)$ or $\Omega=\mathbb{R}$ and $(T(t))_{t\in\Omega}$ is a C_0-semigroup (when $\Omega=(0,\infty)$) or it is a C_0-group (if $\Omega=\mathbb{R}$). In such a case a fruitful idea consists of transferring information about the (semi)group $(T(t))$ to the operator

$\mathcal{T}$. Moreover, regarding the (semi)group $(T(t))$ in terms of its infinitesimal generator Δ, in short $T(t) = e^{t\Delta}$, the subordination of $\mathcal{T}$ to $e^{t\Delta}$ in (0.5) can be considered within the setting of the functional calculus. We adopt these views in the memory. It can be said in particular that groups and semigroups are in the core of this work, for the items we are going to deal with here arise from, or are inspired by, actions of the semigroups on Banach spaces.

Let $(T(t))$ be a C_0-semigroup on X and let $\mathcal{T}$ be the operator obtained by subordination to $(T(t))$ in (0.5). The subspace $\operatorname{Ran}\mathcal{T} := \mathcal{T}(X)$ of X is an example of so-called operator ranges. The class of operator ranges has been studied in a number of papers, begun with the seminal treatise of Dixmier [Dix49], as for example [ACG09; ACG13; FW71; Foi72; KT11; NRR+76; NRRR79]. However, it seems that the relations between group representations on Banach spaces and operator ranges remained unexplored in depth. In this direction, the following question is sensible to ask. Suppose that a group $(T(t))_{t\in\mathbb{R}}$ in (0.5) is a (one-parameter) subgroup of a larger group, say G, which acts on X, and assume that $\operatorname{Ran}\mathcal{T}$ is G-invariant. Under what conditions $(T(t))_{t\in\mathbb{R}}$ induces a C_0-group on $\operatorname{Ran}\mathcal{T}$? Such a question, which is partly motivated by the aim to apply the theory developed in [BG14] to spaces of fractional derivation introduced in [GMS21], is not simple to answer. In fact, it is related to an involved characterization of the algebra of operators leaving a given operator range invariant, provided in [NRRR79] via spectral decompositions. A first step in this direction, in order to get a suitable understanding of the posed problem, has been done in [Oli21a], in an abstract context concerning representations of Banach-Lie groups. We do not follow this line of research in the memory. Instead, we focus our investigation on the spectral properties that operators $\mathcal{T}$ keep inside. In this respect, we do notice that, usually, is not possible to describe spectra of $\mathcal{T}$ directly from spectra of the semigroup $(T(t))$ in the representation given in (0.5), since *equalities* do not hold in the spectral mappings theorems in general. However, by usage of functional calculi, it is possible to obtain information on the spectrum $\sigma(\mathcal{T})$ from the spectrum $\sigma(\Delta)$, through $\mathcal{T} = \int_\Omega \varphi(t)e^{t\Delta}dt$, quite often. Thus it is clear that a thorough analysis of the spectrum of infinitesimal generators of C_0-semigroups is in order. We do this in the present work for two important cases of weighted composition semigroups on the unit disc $\mathbb{D}$ in $\mathbb{C}$.

An interesting (C_0-)group of operators acting on functions on the positive real line as well as on functions defined on half-planes of $\mathbb{C}$ is $(E(t))_{t\in\mathbb{R}}$, which appears naturally associated with Hardy operators, see (0.4). It is of application in the study of (certain versions of) the Black-Scholes equation [AP02], and have been used to represent Cesàro-Hardy operators in the form of resolvents [AP10; AS13; LMPS14], see also [GMS21]. Via a weighted Möbius transform, the group $(E(t))_{t\in\mathbb{R}}$ becomes the hyperbolic composition group $(C_{\varphi_t})_{t\in\mathbb{R}}$ induced by the flow $(\varphi_t)_{t\in\mathbb{R}}$ defined by

$$\varphi_t(z) := \frac{(e^t+1)z + e^t - 1}{(e^t-1)z + e^t + 1}, \quad z \in \mathbb{D},\ t \in \mathbb{R};$$

that is, $C_{\varphi_t} f := f \circ \varphi_t$, $f \in \mathcal{O}(\mathbb{D})$, where $\mathcal{O}(\mathbb{D}) := \{f : \mathbb{D} \to \mathbb{C} :\ \text{f holomorphic}\}$.

We study here weighted C_0-groups $(v_t C_{\psi_t})_{t\in\mathbb{R}}$, where $(\psi_t)_{t\in\mathbb{R}}$ is a flow of hyperbolic automorphisms of $\mathbb{D}$ and $(v_t)_{t\in\mathbb{R}}$ is a cocycle for $(\psi_t)_{t\in\mathbb{R}}$, by carrying out a complete analysis of the spectrum of the infinitesimal generator Δ of the group $(u_t C_{\varphi_t})_{t\in\mathbb{R}}$, where $(u_t)_{t\in\mathbb{R}}$ is a cocycle for $(\varphi_t)_{t\in\mathbb{R}}$. As a consequence, a fairly good description of the spectrum of integral averaging operators on $\mathbb{D}$ is obtained. Such operators are generalizations of two others which can be seen as different versions of Cesàro-type operators. More precisely, one is an integral operator introduced in [Sis86] by A. Siskakis, and the other is the reduced Hilbert matrix operator.

Our spectral study of Δ is done for $(u_t C_{\varphi_t})_{t\in\mathbb{R}}$ acting on the members of a certain, large, class of Banach spaces which includes Hardy spaces, weighted Bergman spaces, weighted Dirichlet spaces, little Bloch spaces, little Korenblum classes and the disc algebra. We follow a unified approach to these questions by assuming conditions on the above spaces which are a bit more general than others in the subject. On the other hand, cocycles $(u_t)_{t\in\mathbb{R}}$ must also satisfy a number of (mild) conditions which imply in particular that $u_t = \omega^{-1}(\omega \circ u_t)$, $t \in \mathbb{R}$, for some non-vanishing function (so-called weight) in $\mathcal{O}(\mathbb{D})$, see [Kön90]. One of the relevant results of the thesis is that such a weight ω has zeroes or singularities of polynomial type in the Denjoy-Wolf points of (φ_t), namely -1 and 1.

The study given here on groups $(u_t C_{\varphi_t})_{t\in\mathbb{R}}$, or $(v_t C_{\psi_t})_{t\in\mathbb{R}}$ more generally, have application to certain conjectures on the characterization of the spectrum of single invertible weighted composition operators vC_ψ, where ψ is an hyperbolic automorphism of $\mathbb{D}$ and v is an invertible multiplier, see [HLNS13; ELM16]. In effect, we answer in the positive such conjectures in the case that the operator vC_ψ can be embedded in a group $(v_t C_{\psi_t})_{t\in\mathbb{R}}$, and for more spectra than the full spectrum.

Suggested by the above discussion concerning hyperbolic composition groups, we also deal with another interesting families of weighted semigroups defined by composition. Namely, we consider semigroups with the Denjoy-Wolf point in $\mathbb{D}$ and which have a finite number of repulsive points in the boundary of $\mathbb{D}$, that is, the circle $\mathbb{T}$. They are abstract versions of the semigroup induced by the analytic semiflow

$$\phi_{t,n}(z) := \frac{e^{-t}z}{((e^{-nt}-1)z^n+1)^{1/n}}, \quad z \in \mathbb{D}, t \geq 0, n \in \mathbb{N}.$$

Such semigroups arise in the study of Hausdorff matrix operators on spaces of analytic functions in $\mathbb{D}$. Here we investigate their boundedness properties as well as the spectrum of their infinitesimal generators when acting on Hardy spaces, weighted Bergman spaces and little Korenblum classes. To do this, we present a detailed analysis of multivalued coboundaries, which is necessary in order to make rigorous the use of certain multivalued functions appearing in formulas of the averaging operators that we consider. It is to be pointed out that the spectra of families of weighted composition operators more general than the ones considered here have been treated in [GLW20], but the spectrum of their generators remained unknown till now.

In order to transfer, by means of formula (0.5), information on the spectrum of Δ to the averaging operators we are interested in, one needs to apply a suitable functional calculus together with a spectral mapping theorem associated to such a functional calculus; that is, an identity of the type

$$\widetilde{\sigma}(\mathcal{T}) = f(\widetilde{\sigma}(\Delta)),$$

where f is a function in the domain of the functional calculus of Δ such that $\mathcal{T} = f(\Delta)$, and where $\widetilde{\sigma}$ denotes the extended spectrum. Such a spectral mapping theorem was given in [Haa05b] in the setting of sectorial operators and for meromorphic functions f such that: 1) $\sigma(\Delta) \setminus \{0\} \subset \mathrm{Dom}(f)$; 2) f has 'almost-logarithmic limits' at $\widetilde{\sigma}(\Delta) \cap \{0, \infty\}$. In particular, this spectral mapping theorem is valid for suitable operators $\mathcal{T}$ which are subordinated to C_0-semigroups. In order to cover the case when $\mathcal{T}$ is an operator subordinated to a C_0-group, we adapt carefully the results in [Haa05b] to the framework of bisectorial-like operators. Even more, we extend the spectral mapping theorem to several spectral sets $\widetilde{\sigma}_i$, all of them labeled as 'essential spectrum' in the literature by different authors (in particular, we consider the essential spectrum $\widetilde{\sigma}_{ess}$ defined in the sense of Fredholm operators). This extension is a significant step forward with respect to the spectral mapping theorems for essential spectra given in [GL71; GO85], in the sense that the class of functions considered here is considerable wider than the class considered in [GL71; GO85]. Also, we answer in the positive (even for all essential spectra considered here) a question posed by Haase in [Haa05b] regarding whether the spectral mapping still holds if, instead of 'almost-logarithmic limits', the function f has 'quasi-regular' limits at $\widetilde{\sigma}(\Delta) \cap \{0, \infty\}$. These results concerning spectral mapping theorems are collected in [Oli22b].

In another direction, the Cesàro operator, which can be subordinated in terms of the group $(E(t))_{t\in\mathbb{R}}$ as in (0.5), is closely related to the Black-Scholes equation as it was noted in [AP02]. Then, by considering the fractional versions of the Cesàro operator (which can also be subordinated by $(E(t))_{t\in\mathbb{R}}$), the following fractional versions of the Black-Scholes equation arise in a natural way: for $\alpha > 0$,

$$
\begin{aligned}
\frac{\partial f}{\partial t} &= \frac{1}{\Gamma(\alpha+1)^2} D^\alpha(x^\alpha D^\alpha(x^\alpha f)) - \frac{2}{\Gamma(\alpha+1)} D^\alpha(x^\alpha f) + f, \\
\frac{\partial f}{\partial t} &= \frac{1}{\Gamma(\alpha+1)^2} x^\alpha W^\alpha(x^\alpha W^\alpha f), \\
\frac{\partial f}{\partial t} &= -\frac{1}{\Gamma(\alpha+1)^2} D^\alpha(x^{2\alpha} W^\alpha f) + \frac{1}{\Gamma(\alpha+1)} x^\alpha W^\alpha f,
\end{aligned}
\tag{0.6}
$$

where D^α denotes the Riemann-Liouville fractional derivative, and W^α denotes the Weyl fractional derivative.

However, the study of the well-posedness of the abstract Cauchy problem associated to any of the above equations is far from trivial. Indeed, the combination of fractional derivatives and multiplication by fractional powers makes equations (0.6) difficult (maybe even not possible) to be solved by classical methods. On the other hand, it turns out that the equations (0.6) can be represented by the aforementioned regularized functional calculus of the infinitesimal generator Δ of $(E(t))_{t\in\mathbb{R}}$. Even more, the functions (of the domain of the regularized functional calculus) involved in such representations have singularities of polynomial type at ∞. In this sense, recall that, by the scaling property, Δ^α defines well-posed a Cauchy problem for small enough α. Then, we prove (in an abstract setting concerning bisectorial-like operators and sectorial operators) an extension of the scaling property which, in particular, implies that $f(\Delta)$ defines a well-posed

Cauchy problem whenever f has singularities of polynomial type. This extension, together with some auxiliary results, implies the well-posedness of the fractional equations (0.6), and moreover they enable us to obtain explicit representations of their solutions.

The organization of the memory is as follows.

After this introduction there is a first chapter on preliminaries where some definitions and results are reminded. The first part of Chapter 2 is devoted to the (standard) adaptation of the so-called regularized functional calculus from the setting of sectorial operators to the setting of bisectorial-like operators. This adaptation is required for our unified approach carried out here. In the second part of Chapter 2, we prove new theorems on spectral mappings on essential spectra that we will need in our spectral analysis. Then, in Chapter 3, we give extensions of the scaling property about fractional powers of operators, and apply them, together with that functional calculus, to solve generalized Black-Scholes on interpolation spaces. The material in Chapters 2 and 3 are included in [Oli22b; OW23].

Chapter 4 contains the aforementioned analysis of multivalued coboundaries and Chapter 5 is then devoted to establish our results about weighted composition semigroups induced by semiflows with Denjoy-Wolf point in $\mathbb{D}$ and with a finite number of repulsive points in $\mathbb{T}$. The content of these chapters will be the object of an ongoing joint work with L. Abadías [AO23].

Chapter 6 contains the detailed study of the spectrum of the infinitesimal generator of weighted hyperbolic composition groups, and its consequences on averaging operators and the conjectures mentioned above. These results are included in [AGMO22].

Finally, in Addendum A, we point out several facts, in the same direction as above, for Hardy operators. The corresponding material is to be found in [MO20; MO21; Oli21b; Oli22a].

Basic notation

Number sets

Through this memory, we use the symbol

- $\mathbb{N}$ to denote the set of natural numbers, i.e., $\{1, 2, 3, \ldots\}$;
- $\mathbb{N}_0$ to denote the set of non-negative integers, i.e., $\mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, \ldots\}$;
- $\mathbb{Z}$ to denote the set of integer numbers;
- $\mathbb{R}$ to denote the set of real numbers;
- $\mathbb{C}$ to denote the set of complex numbers;
- $\mathbb{C}_\infty$ to denote the Riemann sphere, i.e., $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$.

For $\lambda \in \mathbb{C}$, $\mathfrak{Re}\,\lambda$ and $\mathfrak{Im}\,\lambda$ denote the real part and the imaginary part, respectively, of λ.

Function-related notation

Let A be a measurable subset of $\mathbb{R}$ (or of $\mathbb{C}$). For $p \in [1, \infty)$, we denote by $L^p(A)$ the Banach space of (equivalence classes of almost everywhere equal) complex-valued Lebesgue-measurable functions f on A such that

$$\|f\|_{L^p(A)} := \left(\int_A |f(t)|^p \, dt \right)^{1/p} < \infty,$$

where dt denotes the restriction of the Lebesgue measure on A. In addition, let ess sup denote the essential supremum. Then, by $L^\infty(A)$ we denote the Banach space of functions f for which $\|f\|_{L^\infty(A)} := \operatorname{ess\,sup}_{x \in A} |f(x)| < \infty$, endowed with the norm given by $\| \cdot \|_{L^\infty(A)}$.

On the other hand, given an open subset Ω of $\mathbb{C}_\infty$, we denote by $\mathcal{O}(\Omega)$ ($\mathcal{M}(\Omega)$) the algebra of holomorphic (meromorphic) functions with domain Ω.

Also, given a set Y and two functions $f, g : Y \to [0, \infty]$, we use throughout this memory the notation $f \lesssim g$ to denote that there exists a constant $M > 0$ such that

$f(y) \leq Mg(y)$ for all $y \in Y$. In addition, by $f \sim g$ we mean that $f \lesssim g \lesssim f$. If Y is a topological space (all topological spaces considered here are first-countable) and $y \in Y$, by $f(y') \lesssim g(y')$ as $y' \to y$ we mean that there exist a neighborhood V of y and a constant $M > 0$ such that $f(y') \leq Mg(y')$ for all $y' \in V$. Similarly, by $f(y') \sim g(y')$ as $y' \to y$ we mean that both $f(y') \lesssim g(y')$ as $y' \to y$ and $g(y') \lesssim f(y')$ as $y' \to y$ are true.

Operator theory

Let X be a complex Banach space. We use the symbol $C(X)$ to denote the set of closed operators on X. For $T \in C(X)$, let $\mathrm{Dom}(T)$ denote the domain of T. Also, set $\mathrm{Ran}(T)$ to be the range space of T, i.e., $\mathrm{Ran}(T) = \{Tx : x \in \mathrm{Dom}(T)\}$; and set $\mathcal{N}(T)$ to be the null space of T, i.e., $\mathcal{N}(T) = \{x \in \mathrm{Dom}(T) : Tx = 0\}$. Moreover, we denote the dimension of the null space or *nullity* of T by $\mathrm{nul}(T)$, and the codimension of the range or *defect* of T by $\mathrm{def}(T)$.

We use the symbol $L(X)$ to denote the Banach algebra of bounded linear operators on X, with the usual operator norm $\|\cdot\|_{L(X)}$. Also, we use the symbol $\mathcal{K}(X)$ to denote the ideal of compact operators on X. We denote by X^* the dual space of X and, given $T \in C(X)$, we denote by T^* the adjoint operator of T.

Miscellaneous

Finally, we consider one-parameter families of different mathematical objects several times through the memory. These objects are usually parameterized over the real line, i.e., $(a(t))_{t\in\mathbb{R}}$, or over the positive real line, i.e., $(a(t))_{t\geq 0}$. We often omit the indexing set, and write $(a(t))$ to refer to such a one-parameter family.

Chapter 1

Definitions and starting properties

In this chapter, we give some definitions and some well-known results as they are to be used in the memory.

1.1 Sectorial operators and bisectorial-like operators

Given $\theta \in (0, \pi)$, let S_θ denote the sector

$$S_\theta := \Big\{ z \in \mathbb{C} : |\arg(z)| < \theta \Big\},$$

and set $S_0 := (0, \infty)$.

Definition 1.1.1. Let $\theta \in [0, \pi)$. We say that a closed linear operator A in a Banach space X is *s*ectorial (of angle θ) if the following holds

- $\sigma(A) \subset \overline{S_\theta}$,
- for every $\theta' \in (\theta, \pi)$, one has

$$\sup\{\|\lambda(\lambda - A)^{-1}\|_{L(X)} \,:\, \lambda \in \mathbb{C} \setminus S_{\theta'}\} < \infty.$$

Given a Banach space X, $\mathrm{Sect}(\theta)$ denotes the set of sectorial operators of angle θ in X. Also, if A is a sectorial operator, we set $M_A := \widetilde{\sigma}(A) \cap \{0, \infty\}$.

Now, for $\omega \in (0, \pi/2]$ and $a \geq 0$, let $BS_{\omega,a}$ be the bisectorial-like set given by

$$BS_{\omega,a} := \begin{cases} (-a + S_{\pi-\omega}) \cap (a - S_{\pi-\omega}) & \text{if } \omega < \pi/2 \text{ or } a > 0, \\ i\mathbb{R} & \text{if } \omega = \pi/2 \text{ and } a = 0. \end{cases}$$

Definition 1.1.2. Let $(\omega, a) \in (0, \pi/2] \times [0, \infty)$. We say that a closed linear operator A in a Banach space X is a *bisectorial-like operator* (of angle ω and half-width a) if the following holds:

- $\sigma(A) \subset \overline{BS_{\omega,a}}$.
- For all $\omega' \in (0,\omega)$, A satisfies the resolvent bound

$$\sup\left\{ \min\{|\lambda - a|, |\lambda + a|\}\|(\lambda - A)^{-1}\|_{L(X)} \,:\, \lambda \notin \overline{BS_{\omega',a}} \right\} < \infty.$$

Given a Banach space X, $\mathrm{BSect}(\omega, a)$ denotes the set of all bisectorial-like operators in X of angle ω and half-width a. We also set $M_A := \widetilde{\sigma}(A) \cap \{-a, a, \infty\}$.

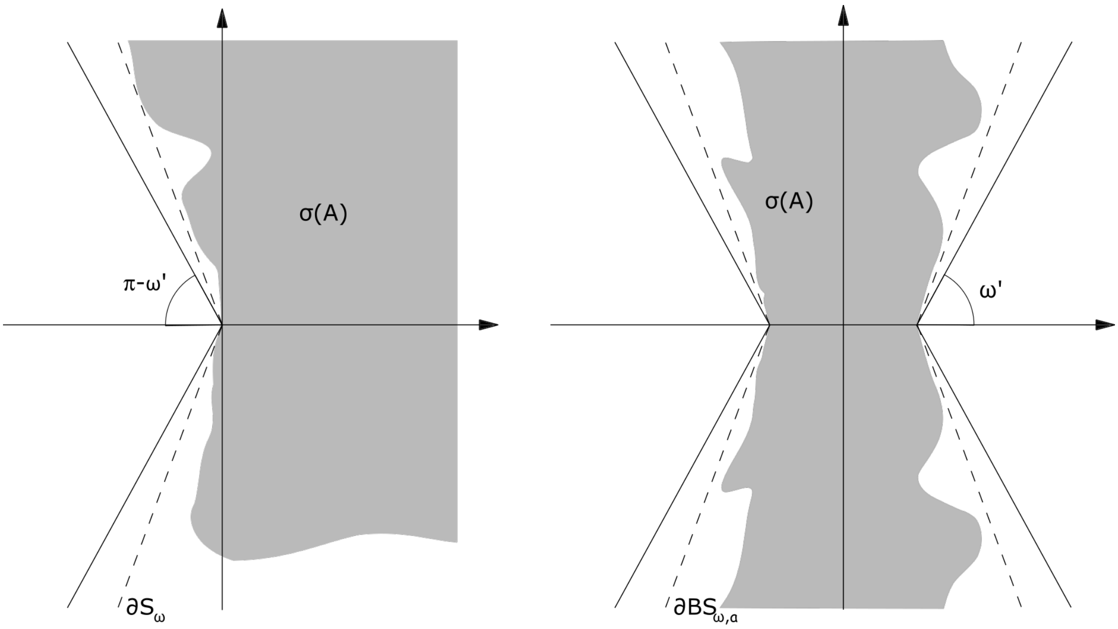


Figure 1.1: Illustration of the spectrum of a sectorial operator (left) and bisectorial-like operator (right).

The main reason for which we consider here (sectorial) bisectorial-like operators is that every generator of an exponentially bounded (semi)group is a (sectorial) bisectorial-like operator. Notice that a closed operator A is in $\mathrm{BSect}(\omega, a)$ if and only if both $a + A$ and $a - A$ are sectorial operators of angle $\pi - \omega$. Most properties of sectorial operators have an analogue in the bisectorial-like operator framework. For more background in sectorial operators, we refer the reader to the monograph [Haa06].

On the other hand, bisectorial-like operators of half-width $a = 0$ are usually referred as *bisectorial* operators in the literature. Such operators play a central role in the theory of abstract inhomogeneous first order differential equations on the whole real line, like

$$u'(t) = Au(t) + f(t), \qquad t \in \mathbb{R},$$

where A is a bisectorial operator on a Banach space X, see [Mie87; Sch00].

1.2 Operator semigroups

A family of operators $(T(t))_{t\geq 0} \subset L(X)$ is said to be an *exponentially bounded semigroup* of (bounded) operators on a Banach space X if:

1. the mapping $t \mapsto T(t)$ is strongly continuous on $(0, \infty)$;
2. $T(0)$ is the identity mapping, and the semigroup property $T(s+t) = T(s)T(t)$ holds for all $t, s \geq 0$;
3. there exist constants $M > 0, \omega \in \mathbb{R}$ such that $\|T(t)\|_{L(X)} \leq Me^{\omega t}$ for all $t \geq 0$.

Given an exponentially bounded semigroup $(T(t))$, the number

$$\omega_0(T) := \inf\{\omega \in \mathbb{R} : \exists M > 0 \text{ such that } \|T(t)\|_{L(X)} \leq Me^{\omega t}\,(t \geq 0)\}$$

is called the growth bound of $(T(t))$.

Note that the Laplace transform $\mathcal{L}T$ of $(T(t))$, given by

$$\mathcal{L}T(\lambda)x = \int_0^\infty e^{-\lambda t} T(t)x\,dt, \qquad x \in X,$$

defines a bounded operator on X for every λ in the half-plane $\{\mathfrak{Re}\,\lambda > \omega_0(T)\}$. It is readily seen that $\mathcal{L}T$ is a pseudo-resolvent, that is, $\mathcal{L}T$ satisfies the resolvent identity. Hence, there exists a unique closed multi-valued operator A such that

$$(\lambda - A)^{-1} = \mathcal{L}T(\lambda), \qquad \mathfrak{Re}\,\lambda > \omega_0(T). \tag{1.1}$$

A is said to be the *generator* of $(T(t))$. Note that, by the injectivity of the Laplace transform, the semigroup is determined by its generator. The semigroup $(T(t))$ is said to be *non-degenerate* if A is single-valued.

Given an exponentially bounded semigroup $(T(t))$, the space of strong continuity of $(T(t))$ is defined by

$$\mathbb{D}_T := \left\{x \in X : \lim_{t\to 0^+} T(t)x = x\right\}.$$

It is well known that $\mathbb{D}_T$ is precisely the closure of the domain of its generator, that is, $\mathbb{D}_T = \overline{\mathrm{Dom}(A)}$. We say that $(T(t))$ is a C_0*-semigroup* if $\mathbb{D}_T = X$. One has that every C_0-semigroup is non-degenerate, and that its generator A is given by

$$\begin{cases} \mathrm{Dom}(A) = \left\{x \in X : \lim_{t\downarrow 0} \dfrac{T(t)x - x}{t} \text{ exists}\right\} \\ Ax = \lim_{t\downarrow 0} \dfrac{T(t)x - x}{t}, \quad x \in \mathrm{Dom}(A). \end{cases} \tag{1.2}$$

Now, we introduce the notion of bounded holomorphic semigroup.

Definition 1.2.1. Let $\delta \in (0, \pi/2]$. A family $(T(z)_{z\in S_\delta} \subset L(X)$ is called an *exponentially bounded holomorphic semigroup* (of angle δ) if it satisfies the following:

1. The semigroup property: $T(z)T(z') = T(z+z')$ for all $z, z' \in S_\delta$.
2. The mapping $T : S_\delta \to L(X)$ is holomorphic.
3. For every $\delta' \in (0, \delta)$, there exist constants $M_{\delta'} > 0, \omega_{\delta'} \in \mathbb{R}$ such that
$$\|T(z)\|_{L(X)} \leq M_{\delta'} e^{\omega_{\delta'} \mathfrak{Re}\, z}, \qquad z \in S_{\delta'}.$$

If $\omega_{\delta'}$ can be taken as 0 for each $\delta' \in (0, \delta)$, we say that $(T(z))$ is a *bounded holomorphic semigroup*.

It is readily seen that item 3 above is equivalent to the fact that, for each $\delta' \in (0, \delta)$, $\sup\{\|T(z)\|_{L(X)} : z \in S_{\delta'}, |z| \leq 1\} < \infty$. It is also clear that every exponentially bounded holomorphic semigroup, when restricted to $[0, \infty)$ is an exponentially bounded semigroup.

The following result, given in [Haa06, Prop. 3.4.4] connects bounded holomorphic semigroups and sectorial operators.

Proposition 1.2.2. *A closed linear multivalued operator A is the generator of a bounded holomorphic semigroup $(T(z))_{z \in S_\delta}$ if and only if $-A$ is a sectorial operator of angle $\pi/2 - \delta$. If this is the case, A is single-valued if and only if $T(z)$ is injective for all $z \in S_\delta$.*

We refer the reader to [ABHN11; EN00; Haa06; HP57] for more background on exponentially bounded semigroups.

1.3 Spectral sets

Given a closed operator A on a Banach space X, we denote by $\sigma(A)$ the spectrum of A, that is,
$$\sigma(A) := \{\lambda \in \mathbb{C} : \lambda - A \text{ is not invertible}\}.$$
The extended spectrum $\widetilde{\sigma}(A)$ of A is defined as $\widetilde{\sigma}(A) := \sigma(A)$ if $A \in L(X)$ (that is, $\mathrm{Dom}(A) = X$), and $\widetilde{\sigma}(A) := \sigma(A) \cup \{\infty\}$ otherwise. Note that $\widetilde{\sigma}(A)$ is a closed subset of the Riemann sphere $\mathbb{C}_\infty$.

Let us now define some spectral sets associated with A. By $\sigma_{point}(A)$ we denote the point spectrum of A, i.e., those $\lambda \in \mathbb{C}$ for which $\lambda - A$ is not injective. By $\sigma_{ap}(A)$, we denote the approximate point spectrum of A, which is given by those $\lambda \in \mathbb{C}$ for which there exists an approximate eigenvector of λ for A. Recall that a sequence $(x_n) \subset X$ is said to be an approximate eigenvector of λ for A if
$$\|x_n\|_X = 1,\, n \in \mathbb{N}, \quad \text{and} \quad \lim_{n \to \infty} \|Ax_n - \lambda x_n\|_X = 0.$$
It is clear that $\sigma_{point}(A) \subset \sigma_{ap}(A)$. In addition, one has
$$\sigma_{ap}(A) = \{\lambda \in \mathbb{C} : \lambda - A \text{ is not injective or } \mathrm{Ran}(\lambda - A) \text{ is not closed}\}.$$

One also has that $\sigma_{ap}(A)$ is a closed subset of $\mathbb{C}$, while $\sigma_{point}(A)$ may not be closed. Moreover, the topological boundary of $\sigma(A)$ lies in $\sigma_{ap}(A)$.

On the other hand, the residual spectrum of A, $\sigma_{res}(A)$, is given by

$$\sigma_{res}(A) := \{\lambda \in \mathbb{C} \,:\, \mathrm{Ran}(\lambda - A) \text{ is not dense in } X\}.$$

It is readily seen that $\sigma_{res}(A) = \sigma_{point}(A^*)$.

Another spectral set of interest is the essential spectrum (in the Fredholm sense) of A, which we denote by $\sigma_{ess}(A)$. It is given by

$$\sigma_{ess}(A) := \{\lambda \in \mathbb{C} \,:\, \mathrm{nul}(A), \mathrm{def}(A) < \infty\},$$

that is, $\lambda \in \mathbb{C}$ belongs to $\sigma_{ess}A$ if and only if $\lambda - A$ is not a Fredholm operator. (Recall that a (bounded) operator A is said to be Fredholm if its nullity $\mathrm{nul}(A)$ and its defect $\mathrm{def}(A)$ are finite.)

The essential spectrum is invariant under compact perturbations. As a matter of fact, an equivalent definition of $\sigma_{ess}(A)$ is $\sigma_{ess}(A) := \sigma(p(A))$, where $p(A)$ denotes the projection of A in the Calkin algebra $L(X)/\mathcal{K}(X)$.

Finally, for $A \in L(X)$, let $r(A)$ denote the spectral radius of A, i.e., $r(A) := \sup_{\lambda\in\sigma(A)} |\lambda|$. By the spectral radius formula, we have $r(A) = \lim_{n\to\infty} \|A^n\|_{L(X)}^{1/n}$.

For the proofs of the properties described above, and more spectral properties, we refer the reader to the monographs [EE87; EN00].

1.4 Fractional integrals

We conclude this chapter with some notes on fractional integrals and fractional derivatives. Let $\alpha > 0$ be a real number and f a 'suitable' (in the sense that the integrals below are well defined for a.e. $x > 0$) function defined on $(0, \infty)$. The Riemann-Liouville fractional integral of order α of f, denoted $D^{-\alpha}f$, and the Weyl fractional integral of order α of f, denoted $W^{-\alpha}f$, are respectively given by

$$(D^{-\alpha}f)(x) := \frac{1}{\Gamma(\alpha)} \int_0^x (x-y)^{\alpha-1} f(y)\,dy, \quad x > 0, \tag{1.3}$$

and

$$(W^{-\alpha}f)(x) := \frac{1}{\Gamma(\alpha)} \int_x^\infty (y-x)^{\alpha-1} f(y)\,dy, \quad x > 0, \tag{1.4}$$

where Γ denotes the usual Euler-Gamma function.

The Riemann-Liouville fractional derivative of order α of f, denoted by $D^\alpha f$, and the Weyl fractional derivative of order α of f, denoted by $W^\alpha f$, are respectively given by

$$(D^\alpha f)(x) := \left(\frac{d^n}{dx^n}(D^{-(n-\alpha)}f)\right)(x), \quad x > 0, \tag{1.5}$$

and

$$(W^\alpha f)(x) := (-1)^n \left(\frac{d^n}{dx^n}(W^{-(n-\alpha)} f)\right)(x), \quad x > 0, \tag{1.6}$$

where n is the smallest integer greater than or equal to α. We refer to the monographs [MR93; SKM93] and references therein for the class of functions for which the above expressions (1.3)-(1.6) exist. For instance, all such fractional integrals and fractional derivatives induce isomorphisms on the Schwartz class on $(0, \infty)$. We notice that D^α is usually denoted by I^α in the literature on fractional calculus.

In particular, we are interested in fractional calculus because of the following (fractional) versions of the Cesàro operator. For $\alpha > 0$, set

$$(\mathcal{C}_\alpha f)(x) = \frac{\alpha}{x^\alpha}\int_0^x (x-y)^{\alpha-1} f(y)\, dy = \frac{\Gamma(\alpha+1)}{x^\alpha}(D^{-\alpha} f)(x), \quad x > 0 \tag{1.7}$$

and

$$(\mathcal{C}_\alpha^* f)(x) = \alpha \int_x^\infty \frac{(y-x)^{\alpha-1}}{y^\alpha} f(y)\, dy = \Gamma(\alpha+1)(W^{-\alpha}((\cdot)^{-\alpha} f))(x), \quad x > 0. \tag{1.8}$$

These operators are injective in all the functional spaces considered through this monograph since operators $D^{-\alpha}, W^{-\alpha}$ are also injective. For convenience, we denote by $\mathcal{D}^\alpha$, $\mathcal{W}^\alpha$ the inverse operators $(\mathcal{C}_\alpha)^{-1}$, $(\mathcal{C}_\alpha^*)^{-1}$.

As a final note, the fractional operators $\mathcal{C}_\alpha$, $\mathcal{C}_\alpha^*$ can be expressed by subordination to the group $(E(s))_{s\in\mathbb{R}}$ given by $(E(s)f)(x) = f(e^{-s}x)$, $x > 0$, $s \in \mathbb{R}$:

$$\begin{aligned}(\mathcal{C}_\alpha f)(x) &= \alpha \int_0^\infty e^{-s}(1-e^{-s})^{\alpha-1}(E(s)f)(x)\, ds, \quad x > 0,\\ (\mathcal{C}_\alpha^* f)(x) &= \alpha \int_{-\infty}^0 (1-e^{s})^{\alpha-1}(E(s)f)(x)\, ds, \quad x > 0,\end{aligned} \tag{1.9}$$

see [GMS21; LMPS14].

Chapter 2

Regularized functional calculus for bisectorial-like operators and spectral mapping theorems for essential spectra

We collect here the results given in [Oli22b].

Let T be a bounded self-adjoint operator on a Hilbert space $\mathcal{H}$. A spectral singularity $\lambda \in \sigma(T)$, where $\sigma(T)$ denotes the spectrum of T, is in the essential spectrum of T if and only if λ is not an isolated eigenvalue of finite multiplicity, see [Wol59].

As we commented in Section 1.3, if T is not self-adjoint, or if T is an operator on a Banach space X, most modern texts define the essential spectrum $\sigma_{ess}(T)$ of T in terms of Fredholm operators, that is, $\lambda \in \sigma_{ess}(T)$ if and only if $\lambda - T$ is not a Fredholm operator. One of the main useful properties of the essential spectrum (on the above sense) is that it is invariant under compact perturbations. (Recall that $\sigma_{ess}(T) = \sigma(p(T))$, where $p(T)$ is the projection of T in the Calkin algebra.)

However, several different definitions for the essential spectrum were introduced in the 50s and 60s, especially in the framework of differential operators. For instance, for G^l (G^r) the semigroup of left (right) regular elements in the Calkin algebra $L(X)/\mathcal{K}(X)$, the spectral sets of $T \in L(X)$ given by $\sigma_2(T) := \{\lambda \in \mathbb{C} : \lambda - p(T) \notin G^l\}$ and $\sigma_3(T) := \{\lambda \in \mathbb{C} : \lambda - p(T) \notin G^r\}$ are studied in [Yoo51]. Indeed, it is shown in [Yoo51] that semigroups $p^{-1}(G^l), p^{-1}(G^r)$ are characterized by:

$$p^{-1}(G^l) = \{T \in L(X) : \operatorname{nul}(T) < \infty \text{ and } \operatorname{Ran}(T) \text{ is complemented}\},$$
$$p^{-1}(G^r) = \{T \in L(X) : \operatorname{def}(T) < \infty \text{ and } \mathcal{N}(T) \text{ is complemented}\}.$$

Alternatively, spectral sets associated with semi-Fredholm operators have also been referred to as essential spectra. More precisely, let Φ^+, Φ^- be given by

$$\Phi^- := \{T \in L(X) : \operatorname{nul}(T) < \infty \text{ and } \operatorname{Ran}(T) \text{ is closed}\},$$
$$\Phi^+ := \{T \in L(X) : \operatorname{def}(T) < \infty\}.$$

In [GW69], the term essential spectra is used for the spectral sets $\sigma_4(T) := \{\lambda \in \mathbb{C} : \lambda - T \notin \Phi^-\}$, $\sigma_5(T) := \{\lambda \in \mathbb{C} : \lambda - T \notin \Phi^+\}$, while in [Kat66], they considered the spectral set $\sigma_6(T) := \sigma_4(T) \cap \sigma_5(T)$, i.e., $\lambda \in \sigma_6(T)$ if and only if $\lambda - T$ is not in $\Phi^- \cup \Phi^+$. Note that $\sigma_2(T) = \sigma_4(T)$ and $\sigma_3(T) = \sigma_5(T)$ if T is an operator on a Hilbert space, but these equalities are not true in general in the framework of Banach spaces, see [Pie60].

In another direction, the essential spectrum $\sigma_8(T)$ of T was defined in [Bro61] as the set of spectral values of T which are not isolated eigenvalues of finite multiplicity either of T or of the adjoint operator T^* (cf. [Lay68]). It turns out that $\lambda \notin \sigma_8(T)$ if and only if λ is a pole of the resolvent of finite rank [Bro61, Lemma 17]. With this in mind, the following essential spectrum was considered in [GL71]:

$$\sigma_9(T) := \{\lambda \in \mathbb{C} : \text{the resolvent of } T \text{ is not meromorphic at } \lambda\}.$$

Nevertheless, the essential spectrum $\sigma_8(T)$ fails to satisfy the property of invariance under compact perturbations. In this regard, in [Sch66], the essential spectrum $\sigma_7(T)$ of T is defined as the larger subset of $\sigma(T)$ which is invariant under compact perturbations. Equivalently $\lambda \notin \sigma_7(T)$ if and only if $\lambda - T$ is Fredholm with index zero, i.e., $\mathrm{nul}(\lambda - T) = \mathrm{def}(\lambda - T) < \infty$.

In this chapter, we deal with spectral mapping theorems for the different essential spectra described above, that is, identities of the form

$$\sigma_i(f(T)) = f(\sigma_i(T)). \tag{2.1}$$

There, f is a function in the domain of a functional calculus of a (possibly unbounded) operator T.

At this point, the first approach to a Banach space functional calculus of unbounded operators is the so-called Dunford-Taylor calculus. For this calculus, one considers functions f which are holomorphic in an open set containing the extended spectrum $\widetilde{\sigma}(T)$ of T. Then, $f(T)$ is determined by

$$f(T) := f(\infty) + \frac{1}{2\pi i} \int_\Gamma f(z)(z - T)^{-1}\, dz, \tag{2.2}$$

where Γ is a suitable finite cycle that avoids $\widetilde{\sigma}(T)$. Moreover, the Dunford-Taylor formula above (2.2) still works if the curve Γ touches $\widetilde{\sigma}(T)$ at some points $a_1, \ldots, a_n$ and f is not holomorphic at $a_1, \ldots, a_n$, as long as f tends to a finite number at each point $a_1, \ldots, a_n$ fast enough to deal with the size of the resolvent at these points. In this case, we say that f has regular limits (at $a_1, \ldots, a_n$) and denote it by $\mathcal{E}(T)$.

Furthermore, in the setting of strip-type operators, a 'regularization trick' was introduced in [Bad53] in order to define $f(T)$ for functions which do not grow too fast at ∞. This 'regularization trick' was further developed, in the framework of sectorial operators, in [CDMY96; Haa05a; McI86]. In particular, fractional powers and/or logarithms can be defined for suitable operators with this 'regularization trick'.

Here, we consider the 'regularized' functional calculus of meromorphic functions developed in [Haa05a], which is based on the following idea. A meromorphic function f is in the domain of the regularized functional calculus of T, which we denote by $f \in \mathcal{M}(T)$, if there exists a function $e \in \mathcal{E}(T)$ such that $e(T)$ is injective and $ef \in \mathcal{E}(T)$. In this case, one defines

$$f(T) := e(T)^{-1}(ef)(T), \tag{2.3}$$

which is a (possibly unbounded) closed operator on X.

Going back to the spectral mapping theorem, identities (2.1) were proven in [GL71] in the setting of the Dunford-Taylor calculus for most (extended) essential spectra described here, see Section 2.2 for their definitions. Moreover, in [GO85], the authors used a unified approach and gave simpler proofs for these spectral mapping theorems. Their proofs are based on the following observations:

1) A closed operator T with non-empty resolvent set is essentially invertible if and only if the (bounded) operator $T(b-T)^{-1}$ is essentially invertible [GO85, Lemma 1],

2) For f, g in the domain of the Dunford-Taylor calculus of T, one has $(fg)(T) = f(T)g(T) = g(T)f(T)$. As a consequence, $(fg)(T)$ is essentially invertible if/only if (depending on the essential spectrum $\sigma_i(T)$ considered, see [GO85, Lemma 3]) both $f(T), g(T)$ are essentially invertible,

where we say that an operator T is essentially invertible (regarding the essential spectrum σ_i) if $0 \notin \sigma_i(T)$, and

3) if f is in the domain of the Dunford-Taylor calculus of T, then one can assume that f has a finite number of zeroes of finite multiplicity.

It sounds sensible to ask whether these spectral mapping theorems can be extended to cover the functions in the domain of the regularized functional calculus given by (2.3). This is partly motivated by potential applications in Fredholm theory, in particular in connection with fractional powers or logarithms of unbounded operators. For instance, we make use, in Chapter 6, of the results presented here to provide the essential spectrum of several integral operators acting on spaces of holomorphic functions. However, there are two main difficulties for such an extension of the spectral mapping theorem. First, for $f, g \in \mathcal{M}(T)$, it is not true in general that $(fg)(T) = f(T)g(T) = g(T)f(T)$, so item 2) above fails. Indeed, one only has the inclusions $f(T)g(T), g(T)f(T) \subseteq (fg)(T)$, where $S \subseteq T$ means that $\mathrm{Dom}(S) \subseteq \mathrm{Dom}(T)$ with $Sx = Tx$ for every $x \in \mathrm{Dom}(S)$. Secondly, since the function f may not be holomorphic at the points $a_1, \dots, a_n$ where the integration path touches $\sigma(T)$, item 3) above also fails (in general) to be true.

Nevertheless, in the setting of sectorial operators, these two problems were successfully dealt with in [Haa05b] for the usual extended spectrum $\widetilde{\sigma}$. In particular, they obtained the spectral mapping theorem

$$\widetilde{\sigma}(f(T)) = f(\widetilde{\sigma}(T)), \tag{2.4}$$

for a meromorphic function f in the domain of the regularized functional calculus, i.e., $f \in \mathcal{M}(T)$, such that f has almost logarithmic limits at the points $a_1, \dots, a_n$ where the integration path Γ touches $\widetilde{\sigma}(T)$. This 'almost logarithmic' condition on the behavior of the limits of f is stronger than asking f to have regular limits at $a_1, \dots, a_n$. As a matter of fact, they left open the question whether the hypothesis of f having regular limits is sufficient to obtain the spectral mapping theorem, see [Haa05b, Remark 5.4].

Still, it is far from trivial to extend the spectral mapping theorem (2.4) from the usual extended spectrum to the (extended) essential spectra described here. This extension is the main contribution of the chapter. Even more, we obtain spectral mapping theorems for the essential spectra described above and for functions f with regular limits lying in the domain of the regularized functional calculus of meromorphic functions (2.3), answering in the positive the conjecture on regular limits explained above.

To obtain these results, on the one hand, we provide a slightly simpler proof for the spectral inclusion of the usual extended spectrum, i.e., $f(\widetilde{\sigma}(A)) \subseteq \widetilde{\sigma}(f(A))$, than the one given in [Haa05b]. As a matter of fact, we no longer make use of the composition rule of the functional calculus, which in the end allow us to weaken the condition on the function f from almost logarithmic limits to the (quasi-)regular limits, cf. [Haa05b, Remark 5.4].

On the other hand, we address the items 2) and 3) above to cover all the essential spectra described above. First, we provide a commutativity property in Lemma 2.2.4, which is a refinement of [Haa05b, Lemma 4.2], and which is crucial to deal with regularized functions $f \in \mathcal{M}(T)$ which are not in $\mathcal{E}(T)$. Moreover, since item 3) is not true if f has a zero at the points $a_1, \dots, a_n$ of $\widetilde{\sigma}(T)$ (recall that f may not be meromorphic there), another issue of importance is whether the points $f(a_1), \dots, f(a_n)$ belong to $\widetilde{\sigma}_i(f(T))$. To solve this, we apply different techniques depending on the topological properties (relative to $\widetilde{\sigma}(T)$) of $a_1, \dots, a_n$. If these points are isolated points of $\widetilde{\sigma}(T)$, we provide useful properties of the spectral projections associated with such points in Lemmas 2.2.13 and 2.2.14. If otherwise, these points are limit points of $\widetilde{\sigma}(T)$, we make use of a mixture of topological properties shared by all the essential spectra considered here, and a mixture of algebraic properties of the regularized functional calculus in Propositions 2.2.15 and 2.2.16.

In this work, we use the model case of bisectorial-like operators, which is a family of operators that slightly generalizes the one of bisectorial operators, see for instance [AD06; Mie87]. This is partly motivated by two reasons. On the one hand, we want our results to cover the case when T is the generator of an exponentially bounded group. This is because, as explained above, we obtain spectral properties of certain integral operators via subordination of such operators in terms of an exponentially bounded group in Chapter 6. On the other hand, the another incentive to do this is the fact that the regularized functional calculus for bisectorial-like operators is easily constructed by mimicking the regularized functional calculus of sectorial operators [Haa05a; Haa06]. Finally, bisectorial operators play an important role in the field of abstract inhomogeneous differential equations over the real line, so we are confident that our results have applications of interest in that topic.

Nevertheless, the proofs presented here are generic and are valid for every regularized functional calculus of meromorphic functions (in the sense of [Haa05a; Haa06]) satisfying the properties collected in Lemmas 2.1.4, 2.1.5, 2.1.6 and 2.1.7. For instance, our proofs work for the regularized functional calculus of sectorial operators and the regularized functional calculus of strip-type operators, as we point out in Subsection 2.2.F.

The chapter is organized as follows. The regularized functional calculus for bisectorial-like operators is detailed in Section 2.1. Section 2.2 deals with the spectral mapping theorems, and it is divided in several subsections. In Subsection 2.2.A, we give the spectral mapping theorems for a bisectorial-like operator A in the case that the integration path Γ does not touch any point of $\widetilde{\sigma}(T)$. The general case is dealt with in Subsection 2.2.B. We give applications of the results here in Subsections 2.2.C, 2.2.D, 2.2.E and 2.2.F, such as the answer in the positive to Haase's conjecture [Haa05b, Remark 5.4].

2.1 Regularized functional calculus

Now we turn to the definition of the regularized functional calculus of bisectorial-like operators. Its construction is completely analogous to the one of the regularized functional calculus of sectorial operators given in [Haa05a; Haa06], and the adaptation of it from the sectorial operators to the bisectorial-like operators is straightforward.

For the rest of the chapter, ω denotes a number in $(0, \pi/2]$ and a a number in $[0, \infty)$. Recall that we denote by $\mathcal{O}(\Omega), \mathcal{M}(\Omega)$ the sets of holomorphic functions and meromorphic functions defined in an open subset $\Omega \subseteq \mathbb{C}$, respectively. For $A \in \mathrm{BSect}(\omega, a)$, let $U_A := \{-a, a, \infty\} \setminus \widetilde{\sigma}(A)$. If $\sigma(A) \neq \emptyset$, set

$$r_d := \begin{cases} \operatorname{dist}\{d, \sigma(A)\}, & \text{if } d \in \{-a, a\}, \\ r(A)^{-1}, & \text{if } d = \infty, \end{cases} \qquad d \in U_A,$$

where $\operatorname{dist}\{\cdot, \cdot\}$ denotes the distance between two sets, and $r(A)$ the spectral radius of A. If $\sigma(A) = \emptyset$ (so $\widetilde{\sigma}(A) = \{\infty\}$ and $\infty \notin U_A$), set $r_a = r_{-a} := \infty$.

For $d \in U_A$ suppose that $s_d \in (0, r_d)$. Then, for $\varphi \in (0, \omega)$, set $\Omega(\varphi, (s_d)_{d \in U_A})$ as follows. If $U_A = \emptyset$ (i.e., $M_A = \{-a, a, \infty\}$), we set $\Omega_\varphi := BS_{\varphi,a}$. Otherwise, for each $d \in U_A$, let $B_d(s_d)$ be a ball centered at d of radius s_d, where $B_\infty(r_\infty) = \{z \in \mathbb{C} \mid |z| > r_\infty^{-1}\}$. Then, we set $\Omega(\varphi, (s_d)_{d \in U_A}) := BS_{\varphi,a} \setminus (\bigcup_{d \in U_A} \overline{B_d(s_d)})$. Note that, if $\varphi < \varphi' < \omega$ and $s_d < s'_d < r_d$ for each $d \in U_A$, then the inclusion $\Omega(\varphi', (s'_d)_{d \in U_A}) \subseteq \Omega(\varphi, (s_d)_{d \in U_A})$ holds. Thus we can form the inductive limits

$$\mathcal{O}[\Omega_A] := \bigcup \left\{ \mathcal{O}(\Omega(\varphi, (s_d)_{d \in U_A})) \,\middle|\, 0 < \varphi < \omega,\ 0 < s_d < r_d \text{ for } d \in U_A \right\},$$
$$\mathcal{M}[\Omega_A] := \bigcup \left\{ \mathcal{M}(\Omega(\varphi, (s_d)_{d \in U_A})) \,\middle|\, 0 < \varphi < \omega,\ 0 < s_d < r_d \text{ for } d \in U_A \right\}.$$

Hence, $\mathcal{O}[\Omega_A], \mathcal{M}[\Omega_A]$ are algebras of holomorphic functions and meromorphic functions (respectively) defined on an open set containing $\widetilde{\sigma}(A) \setminus M_A$. Next, we define the following notion of regularity at M_A.

Definition 2.1.1. Let $f \in \mathcal{M}[\Omega_A]$. We say that f is regular at $d \in \{-a, a\} \cap M_A$ if $\lim_{z\to d} f(z) =: c_d \in \mathbb{C}$ exists and, for some $\varphi \in (0, \omega)$

$$\int_{\partial BS_{\varphi',a}, |z-d|<\varepsilon} \left| \frac{f(z) - c_d}{z - d} \right| |dz| < \infty, \quad \text{for some } \varepsilon > 0 \text{ and for all } \varphi' \in \left(\varphi, \frac{\pi}{2}\right].$$

If $\infty \in M_A$, we say that f is regular at ∞ if $\lim_{z\to\infty} f(z) =: c_\infty \in \mathbb{C}$ exists and

$$\int_{\partial BS_{\varphi',a}, |z|>R} \left| \frac{f(z) - c_\infty}{z} \right| |dz| < \infty, \quad \text{for some } R > 0 \text{ and for all } \varphi' \in \left(\varphi, \frac{\pi}{2}\right].$$

We say that f is quasi-regular at $d \in M_A$ if f or $1/f$ is regular at d. Finally, we say that f is (quasi-)regular at M_A if f is (quasi-)regular at each point of M_A.

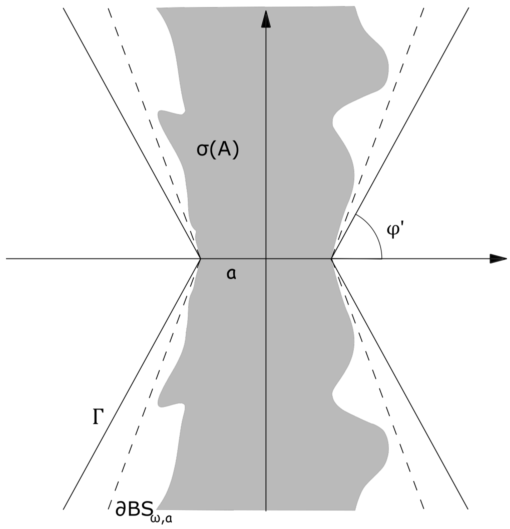


Figure 2.1: Spectrum of a bisectorial-like operator and integration path of the functional calculus.

Remark 2.1.2. Note that if f is regular at M_A with every limit being not equal to 0, then $1/f$ is also regular at M_A. If f is quasi-regular at M_A, then $\mu - f$ and $1/f$ are also quasi-regular at M_A for each $\mu \in \mathbb{C}$. A function f which is quasi-regular at M_A has well-defined limits in $\mathbb{C}_\infty$ as z tends to each point of M_A.

Next, let $\mathcal{E}(A)$ be the subset of functions of $\mathcal{O}[\Omega_A]$ which are regular at M_A. Then, for any $b \in \mathbb{C} \backslash \overline{BS_{\varphi,a}}$, the set equality

$$\mathcal{E}(A) = \mathcal{E}_0(A) + \mathbb{C}\frac{1}{b+z} + \mathbb{C}\frac{1}{b-z} + \mathbb{C}\mathbf{1},$$

holds true, where $\mathbf{1}$ is the constant function with value 1, and

$$\mathcal{E}_0(A) := \left\{ f \in \mathcal{O}[\Omega_A] \,:\, f \text{ is regular at } M_A \text{ with } \lim_{z\to d} f(z) = 0 \text{ for all } d \in M_A \right\}.$$

Given a bisectorial operator $A \in \mathrm{BSect}(\omega, a)$, we define the algebraic homomorphism $\Phi : \mathcal{E}(A) \to L(X)$ determined by setting $\Phi\left(\frac{1}{b+z}\right) = (b+A)^{-1}$, $\Phi\left(\frac{1}{b-z}\right) = (b-A)^{-1}$, $\Phi(\mathbf{1}) = I$, and

$$\Phi(f) := f(A) := \frac{1}{2\pi i}\int_\Gamma f(z)(z-A)^{-1}\,dz, \quad f \in \mathcal{E}_0(A),$$

where Γ is the positively oriented boundary of $\Omega(\varphi', (s'_d)_{d\in U_A})$ with $\varphi < \varphi' < \omega$ and $s_d < s'_d < r_d$, where $f \in \mathcal{O}(\Omega(\varphi, (s_d)_{d\in U_A}))$. Note that the above integral is well defined in the Bochner sense since $z \mapsto (z-A)^{-1}$ is analytic, so continuous, and $\int_\Gamma |f(z)| \|(z-A)^{-1}\|\,|dz| < \infty$. It is readily seen that $f(A)$ is well defined for $f \in \mathcal{E}(A)$ (that is, that $f(A)$ does not depend on the choice of $\varphi', (s'_d)_{d\in U_A}$).

Next, we follow the regularization method given in [Haa05a] to extend the functional calculus Φ to a regularized functional calculus (also denoted by Φ), which involves meromorphic functions.

Definition 2.1.3. Let $a \geq 0$, $0 < \omega \leq \frac{\pi}{2}$ and $A \in \mathrm{BSect}(\omega, a)$. A function $f \in \mathcal{M}[\Omega_A]$ is called regularizable by $\mathcal{E}(A)$ if there exists $e \in \mathcal{E}(A)$ such that

- $e(A)$ is injective,
- $ef \in \mathcal{E}(A)$.

For any regularizable $f \in \mathcal{M}[\Omega_A]$ with regularizer $e \in \mathcal{E}(A)$, we set

$$\Phi(f) := f(A) := e(A)^{-1}(ef)(A).$$

By [Haa05a, Lemma 3.2], one has that this definition is independent of the regularizer e, and that $f(A)$ is a well-defined closed operator. We denote by $\mathcal{M}(A)$ the subset of functions of $\mathcal{M}[\Omega_A]$ which are regularizable by $\mathcal{E}(A)$. As in the case of sectorial operators [Haa05a, Theorem 3.6], this regularized functional calculus satisfies the properties given in the lemma below.

Lemma 2.1.4. *Let $A \in \mathrm{BSect}(\omega, a)$ and $f \in \mathcal{M}(A)$. Then*

1. *If $T \in L(X)$ commutes with A, that is, $TA \subseteq AT$, then T also commutes with $f(A)$, i.e. $Tf(A) \subseteq f(A)T$.*
2. *$\zeta(A) = A$, where $\zeta(z) = z$, $z \in \mathbb{C}$.*
3. *Let $g \in \mathcal{M}(A)$. Then*

$$f(A) + g(A) \subseteq (f+g)(A), \qquad f(A)g(A) \subseteq (fg)(A).$$

Furthermore, $\mathrm{Dom}(f(A)g(A)) = \mathrm{Dom}((fg)(A)) \cap \mathrm{Dom}(g(A))$*, and one has equality in these relations if $g(A) \in L(X)$.*

4. *Let $\lambda \in \mathbb{C}$. Then*

$$\frac{1}{\lambda - f(z)} \in \mathcal{M}(A) \iff \lambda - f(A) \text{ is injective.}$$

If this is the case, $(\lambda - f(z))^{-1}(A) = (\lambda - f(A))^{-1}$. In particular, $\lambda \in \rho(A)$ if and only if $(\lambda - f(z))^{-1} \in \mathcal{M}(A)$ with $(\lambda - f(A))^{-1} \in L(X)$.

Proof. The statement follows by straightforward applications of the Cauchy's theorem, the resolvent identity, and [Haa05a, Section 3]. □

Lemma 2.1.5. *Let $A \in \mathrm{BSect}(\omega, a)$ and $f \in \mathcal{M}(A)$. Then $f(A)x = f(\lambda)x$ for any $x \in \mathcal{N}(\lambda - A)$.*

Proof. See [Haa05b, Proposition 3.1] for the analogous result for sectorial operators. □

Lemma 2.1.6. *Let $A \in \mathrm{BSect}(\omega, a)$, $f \in \mathcal{M}(A)$ and $\lambda \in \tilde{\sigma}(A) \backslash M_A$ such that $f(\lambda) \neq \infty$. There is a regularizer $e \in \mathcal{E}(A)$ for f with $e(\lambda) \neq 0$.*

Proof. The proof is analogous to the case of sectorial operators, see [Haa05b, Lemma 4.3]. □

Lemma 2.1.7. *Let $A \in \mathrm{BSect}(\omega, a)$ and $f \in \mathcal{M}[\Omega_A]$. Assume that f is regular at M_A and that the poles of f are contained in $\mathbb{C}\backslash\sigma_p(A)$. Then, $f \in \mathcal{M}(A)$. Moreover, if every pole of f is contained in $\rho(A)$, then $f(A) \in L(X)$.*

Proof. The proof is the same as in the case of sectorial operators, see [Haa05b, Lemma 6.2]. We include it here since we need it in the proof of Theorem 2.2.20.

Let $f \in \mathcal{M}[\Omega_A]$ be as required. That is, there exists $\varphi \in (0, \omega)$ and $s_d \in (0, r_d)$ for each $d \in U_A$ such that $f \in \mathcal{M}(\Omega(\varphi, (s_d)_{d \in U_A}))$. Since f has finite limits at M_A, we can assume that f has only finitely many poles by making $\varphi, (s_d)_{d \in U_A}$ bigger. Thus, let λ_j for $j \in \{1, ..., N\}$ be an enumeration of those poles of f and let $n_j \in \mathbb{N}$ be the order of pole of f located at λ_j, for $j \in \{1, ..., N\}$. Then, the function $g(z) := f(z) \prod_{j=1}^{N} \frac{(\lambda_j - z)^{n_j}}{(b - z)^{n_j}}$ has no poles, i.e. $g \in \mathcal{O}[\Omega_A]$, and is regular at M_A. Hence $g \in \mathcal{E}(A)$. Moreover, setting $r(z) := \prod_{j=1}^{N} \frac{(\lambda_j - z)^{n_j}}{(b - z)^{n_j}}$, one has that $r(A) = \prod_{j=1}^{N} (\lambda_j - A)^{n_j} (b - A)^{-n_j}$ is bounded and injective, since by assumption $\{\lambda_1, ..., \lambda_n\} \subseteq \mathbb{C} \setminus \sigma_p(A)$. In short, f is regularized by r, so $f \in \mathcal{M}(A)$.

Now, assume that the poles of f lie inside $\rho(A)$. Then the operator $r(A)$ is not only bounded and injective, but invertible too, from which follows that $f(A) = r(A)^{-1}(rf)(A) \in L(X)$. □

2.2 Spectral mapping theorems for essential spectra

Let us fix (and recall) the notation through the paper. Let X denote an infinite dimensional complex Banach space. Let $L(X)$, $C(X)$ denote the sets of bounded operators and closed operators on X, respectively. The *ascent* of T, $\alpha(T)$, is the smallest integer n such that $\mathcal{N}(T^n) = \mathcal{N}(T^{n+1})$, and the *descent* of T, $\delta(T)$, is the smallest integer n such that $\operatorname{Ran}(T^n) = \operatorname{Ran}(T^{n+1})$.

Now we recall the definition of the different essential spectra described above. Following the notation and terminology of [GO85; GL71], set

$$
\begin{aligned}
\Phi_0 &:= \{T \in C(X) \,|\, \operatorname{nul}(T) = \operatorname{def}(T) = 0\},\\
\Phi_1 &:= \{T \in C(X) \,|\, \operatorname{nul}(T), \operatorname{def}(T) < \infty\},\\
\Phi_2 &:= \{T \in C(X) \,|\, \operatorname{nul}(T) < \infty,\ \operatorname{Ran}(T) \text{ complemented}\},\\
\Phi_3 &:= \{T \in C(X) \,|\, \operatorname{def}(T) < \infty,\ \mathcal{N}(T) \text{ complemented}\},\\
\Phi_4 &:= \{T \in C(X) \,|\, \operatorname{nul}(T) < \infty,\ \operatorname{Ran}(T) \text{ closed}\},\\
\Phi_5 &:= \{T \in C(X) \,|\, \operatorname{def}(T) < \infty\},\\
\Phi_6 &:= \Phi_4 \cup \Phi_5,\\
\Phi_7 &:= \{T \in C(X) \,|\, \operatorname{nul}(T) = \operatorname{def}(T) < \infty\},\\
\Phi_8 &:= \{T \in \Phi_7 \,|\, \alpha(T) = \delta(T) < \infty\},\\
\Phi_9 &:= \{T \in C(X) \,|\, \alpha(T), \delta(T) < \infty\}.
\end{aligned}
$$

We notice that these operator families satisfy the following spectral inclusions

$$
\Phi_0 \subseteq \Phi_8 \subseteq \Phi_7 \subseteq \Phi_1
\begin{array}{c}
\rotatebox{30}{$\subseteq$} \quad \Phi_3 \subseteq \Phi_5 \quad \rotatebox{-30}{$\subseteq$}\\
\qquad\qquad\qquad\quad \Phi_6\\
\rotatebox{-30}{$\subseteq$} \quad \Phi_2 \subseteq \Phi_4 \quad \rotatebox{30}{$\subseteq$}
\end{array}
\qquad \text{and} \qquad \Phi_0 \subseteq \Phi_8 \subseteq \Phi_9,
$$

Then, the respective spectra $\sigma_i(T)$ are defined in terms of the above families by

$$
\sigma_i(T) := \{\lambda \in \mathbb{C} \,|\, \lambda - T \notin \Phi_i\} \quad \text{for} \quad i \in \{0, 1, ..., 9\}.
$$

Note that $\sigma_0(T)$ is the usual spectrum $\sigma(T)$ and most modern texts use the term essential spectrum to denote the set $\sigma_1(T)$. In [GO85; GL71], it is also considered the essential spectrum $\sigma_{10}(T)$ defined in terms of normally solvable operators, i.e., operators with closed range, see [DS63]. There exist bounded operators on Hilbert spaces for which $\sigma_{10}(T^2) \nsubseteq (\sigma_{10}(T))^2$ and $\sigma_{10}(S^2) \nsupseteq (\sigma_{10}(S))^2$, see [GL71, Section 5], whence we are not interested in this spectral set.

Next we define the extended essential spectra $\widetilde{\sigma}_i(T)$.

Definition 2.2.1. Let $T \in C(X)$. We define

$$\widetilde{\sigma}_i(T) := \begin{cases} \sigma_i(T) \text{ if } \begin{cases} \mathrm{Dom}(T) = X, \text{ for } i \in \{0,7,8\}, \\ \mathrm{codim}(\mathrm{Dom}(T)) < \infty, \text{ for } i \in \{1,3,5\}, \\ \mathrm{Dom}(T) \text{ closed}, \text{ for } i \in \{4,6\}, \\ \mathrm{Dom}(T) \text{ complemented}, \text{ for } i = 2, \\ \mathrm{Dom}(T^n) = \mathrm{Dom}(T^{n+1}) \text{ for some } n \in \mathbb{N}, \text{ for } i = 9, \end{cases} \\ \sigma_i(T) \cup \{\infty\}, \text{ otherwise.} \end{cases}$$

Note that $\widetilde{\sigma}_0(T)$ is the usual extended spectrum $\widetilde{\sigma}(T)$. If the resolvent set $\rho(T)$ is not empty, $\widetilde{\sigma}_i(T)$ coincides with the extended essential spectrum introduced in [GO85], which satisfies that $\infty \in \widetilde{\sigma}_i(T)$ if and only if $0 \in \sigma_i((\mu - T)^{-1})$ for any $\mu \in \rho(T)$. In particular, if T has non-empty resolvent set, $\widetilde{\sigma}_i(T)$ are non-empty compact subsets of $\mathbb{C}_\infty$ except for $i = 9$ (see [GL71]), where $\mathbb{C}_\infty$ denotes the Riemann sphere $\mathbb{C} \cup \{\infty\}$. If T has empty resolvent set, $\sigma_i(T)$ is a closed subset of $\mathbb{C}$ for $i \in \{0,1,2,4,5,6,7\}$, see [EE87, Section I.3] and [Yoo51]. We do not know if $\widetilde{\sigma}_i(T)$ or $\sigma_i(T)$ are closed in the other cases.

2.2.A Case $M_A = \emptyset$

For $A \in \mathrm{BSect}(\omega, a)$, the spectral mapping theorems (2.1) given in [GO85; GL71] for $i = 0,1,2,3,4,5,8$, are applicable to every $f \in \mathcal{E}(A)$ whenever $M_A = \emptyset$. This section is devoted to extend these spectral mapping theorems for all $f \in \mathcal{M}(A)$.

First, we proceed to state the spectral mapping inclusion of the spectrum $\widetilde{\sigma}$.

Proposition 2.2.2. *Let $A \in \mathrm{BSect}(\omega, a)$, $f \in \mathcal{M}(A)$, and assume that f is quasi-regular at M_A. Then*

$$\widetilde{\sigma}(f(A)) \subseteq f(\widetilde{\sigma}(A)).$$

Proof. The proof runs along the same lines as in the case of sectorial operators, see [Haa05b, Prop. 6.3]. As in Lemma 2.1.7, we include the proof here since it is required in the proof of Theorem 2.2.20.

Take $\mu \in \mathbb{C}$ such that $\mu \notin f(\widetilde{\sigma}(A))$. Then $\dfrac{1}{\mu - f} \in \mathcal{M}[\Omega_A]$ is regular in M_A, and all of its poles are contained in $\rho(A)$. By Lemma 2.1.7, we conclude that $(\mu - f)^{-1} \in \mathcal{M}(A)$ with $(\mu - f)^{-1}(A)$ is a bounded operator. Thus, it follows that $\mu - f(A)$ is invertible, hence $\mu \notin \widetilde{\sigma}(f(A))$.

Assume now that $\mu = \infty \notin f(\widetilde{\sigma}(A))$. Then f is regular at M_A and its poles are contained in $\rho(A)$. Another application of Lemma 2.1.7 yields that $f(A)$ is a bounded operator, so $\infty \notin \widetilde{\sigma}(f(A))$. $\square$

Next, we give some technical lemmas.

Lemma 2.2.3. *Let $A \in \mathrm{BSect}(\omega, a)$, $e \in \mathcal{M}(A)$ with $e(A) \in L(X)$ injective, $\lambda, b \in \mathbb{C}$ with $b \in \rho(A)$. Assume that there is $c \in \mathbb{C} \setminus \{0\}$ such that*

$$f(z) := \frac{b-z}{\lambda - z}(e(z) - c) \in \mathcal{M}(A) \qquad \text{with } f(A) \in L(X).$$

Then $\mathrm{Ran}(\lambda - A) = \mathrm{Ran}\left((\lambda - A)(b-A)^{-1}e(A)^{-1}\right) = \mathrm{Ran}\left(e(A)^{-1}(\lambda - A)(b-A)^{-1}\right)$.

Proof. Note that $\mathrm{Ran}(\lambda - A) = \mathrm{Ran}\left((\lambda - A)(b-A)^{-1}\right) = \mathrm{Ran}\left((\lambda - A)(b-A)^{-1}e(A)^{-1}\right)$ since $e(A)^{-1}$ is surjective. The inclusion $(\lambda - A)(b-A)^{-1}e(A)^{-1} \subseteq e(A)^{-1}(\lambda - A)(b-A)^{-1}$ implies

$$\mathrm{Ran}\left((\lambda - A)(b-A)^{-1}e(A)^{-1}\right) \subseteq \mathrm{Ran}\left(e(A)^{-1}(\lambda - A)(b-A)^{-1}\right).$$

Thus, all that is left to prove is the reverse inclusion.

Let $u \in \mathrm{Ran}\left(e(A)^{-1}(\lambda - A)(b-A)^{-1}\right)$, so there is $x \in X$ such that $e(A)u = (\lambda - A)(b-A)^{-1}x$. Since $e(z) = \frac{\lambda - z}{b-z}f(z) + c$, one has $u = \frac{1}{c}(\lambda - A)(b-A)^{-1}(x - f(A)u)$. Thus, $u \in \mathrm{Ran}(\lambda - A) = \mathrm{Ran}\left((\lambda - A)(b-A)^{-1}e(A)^{-1}\right)$, and the claim follows. □

Lemma 2.2.4. *Let $f \in \mathcal{M}(A)$ and $\lambda \in \sigma(A) \setminus M_A$ with $f(\lambda) = 0$, and let $b \in \rho(A)$. If $g(z) := \dfrac{b-z}{\lambda - z}f(z)$, then $g \in \mathcal{M}(A)$, $\mathrm{Dom}(g(A)) = \mathrm{Dom}(f(A))$ and*

$$f(A) = (\lambda - A)(b-A)^{-1}g(A) = g(A)(\lambda - A)(b-A)^{-1}.$$

Proof. Let $e \in \mathcal{E}(A)$ be a regularizer for f with $e(\lambda) \neq 0$, see Lemma 2.1.6. The fact that eg has the same behavior as ef at M_A implies that $eg \in \mathcal{E}(A)$, that is, e is a regularizer for g and $g \in \mathcal{M}(A)$, so $g(A)$ is well defined.

On the one hand, it follows by Lemma 2.1.4 (3) that

$$f(A) = g(A)(\lambda - A)(b-A)^{-1} \supseteq (\lambda - A)(b-A)^{-1}g(A),$$

with $\mathrm{Dom}\left((\lambda - A)(b-A)^{-1}g(A)\right) = \mathrm{Dom}(f(A)) \cap \mathrm{Dom}(g(A))$. By the definition of composition of closed operators,

$$\begin{aligned}\mathrm{Dom}\left((\lambda - A)(b-A)^{-1}g(A)\right) &= \mathrm{Dom}(g(A)) \cap g^{-1}\left(\mathrm{Dom}\left((\lambda - A)(b-A)^{-1}\right)\right) \\ &= \mathrm{Dom}(g(A)),\end{aligned}$$

since $\mathrm{Dom}\left((\lambda - A)(b-A)^{-1}\right) = X$. As a consequence, $\mathrm{Dom}(g(A)) \subseteq \mathrm{Dom}(f(A))$.

Now, let $x \in \mathrm{Dom}(f(A))$ and set $\tilde{x} := (eg)(A)x$. One has

$$f(A) = e(A)^{-1}\left(\frac{\lambda - z}{b-z}(eg)(z)\right)(A) = e(A)^{-1}(\lambda - A)(b-A)^{-1}(eg)(A),$$

so $\tilde{x} \in \mathrm{Dom}\left(e(A)^{-1}(\lambda - A)(b-A)^{-1}\right)$. An application of Lemma 2.2.3 with $c = e(\lambda) \neq 0$ shows that there is $v \in \mathrm{Ran}(e(A))$ with $(\lambda - A)(b-A)^{-1}e(A)^{-1}v = e(A)^{-1}(\lambda - A)(b-A)^{-1}\tilde{x}$. By composing with $e(A)$ one gets $\tilde{x} - v \in \mathcal{N}\left((\lambda - A)(b-A)^{-1}\right) = \mathcal{N}(\lambda - A)$.

Moreover, $\mathcal{N}(\lambda - A) \subseteq \mathrm{Ran}(e(A))$ since $y = \frac{1}{e(\lambda)} e(A)y$ for any $y \in \mathcal{N}(\lambda - A)$, see Lemma 2.1.5. Hence, $\widetilde{x} = (\widetilde{x} - v) + v \in \mathrm{Ran}(e(A))$, that is $x \in \mathrm{Dom}(g(A))$, so $\mathrm{Dom}(g(A)) = \mathrm{Dom}(f(A))$. Lemma 2.1.7(3) shows that

$$f(A) = (\lambda - A)(b - A)^{-1} g(A) = g(A)(\lambda - A)(b - A)^{-1},$$

and the commutativity property follows. □

Remark 2.2.5. Let $T \in C(X)$ with non-empty resolvent set, and $\alpha(T), \delta(T) < \infty$. Then $\alpha(T) = \delta(T) =: p_T$ and $X = \mathcal{N}(T^{p_T}) \oplus \mathrm{Ran}(T^{p_T})$, see for example [TL58, Th. V.6.2].

Lemma below is inspired by [GL71, Lemma 5].

Lemma 2.2.6. *Let $S, T \in C(X)$ with non-empty resolvent set and such that $ST = TS$. One has the following*

(a) If $S, T \in \Phi_9$, then $ST \in \Phi_9$. If $\mathrm{Ran}(T) \subseteq \mathrm{Dom}(S)$ and $S, T \in \Phi_8$, then $ST \in \Phi_8$.

(b) Assume that T is injective and $\mathrm{Dom}(S) \subseteq \mathrm{Ran}(T)$. If $ST \in \Phi_i$, then $S \in \Phi_i$ for $i \in \{8, 9\}$.

Proof. (a) Let $S, T \in \Phi_9$. By Remark 2.2.5, $\alpha(S) = \delta(S) =: p_S$, $\alpha(T) = \delta(T) =: p_T$, and $X = \mathrm{Ran}(S^{p_S}) \oplus \mathcal{N}(S^{p_S}) = \mathrm{Ran}(T^{p_T}) \oplus \mathcal{N}(T^{p_T})$. Let P_S be the projection onto $\mathcal{N}(S^{p_S})$ along $\mathrm{Ran}(S^{p_S})$, $Q_S := I - P_S$, and set the analogous projections P_T, Q_T. Since $ST = TS$, one has that $\mathcal{N}(T^n) \subseteq \mathrm{Dom}(S)$, $\mathcal{N}(S^n) \subseteq \mathrm{Dom}(T)$ for any $n \in \mathbb{N}$, and that P_S, Q_S, P_T, Q_T commute between themselves. Then $Q := Q_T Q_S$ is a bounded projection onto $\mathrm{Ran}(S^{p_S}) \cap \mathrm{Ran}(T^{p_T})$, and it is readily seen that ST is a (possibly unbounded) invertible operator when restricted to $Q(X)$. Since $I - Q = P_S + P_T - P_S P_T$, it is clear that $(I - Q)(X) \subseteq \mathcal{N}(ST)^{\max\{p_S, p_T\}}$. Then, $ST \in \Phi_9$ with $\alpha(ST) = \delta(ST) \leq \max\{p_S, p_T\} < \infty$ by [TL58, Problem V.6].

If in addition, $S, T \in \Phi_8 \subseteq \Phi_7$ with $\mathrm{Ran}(T) \subseteq \mathrm{Dom}(S)$, then $ST \in \Phi_7$, see [EE87, Th. I.3.16] (although this result is stated for bounded operators, its proof is purely algebraic). Hence, we conclude that $ST \in \Phi_8$.

(b) It follows by induction that $\mathrm{Dom}(S^n) \subseteq \mathrm{Ran}(T^n)$ for $n \in \mathbb{N}$. Since $(ST)^n = S^n T^n = T^n S^n$ and T is injective, one has that $\alpha(ST) = \alpha(S)$ and $\delta(ST) = \delta(S)$, and the claim follows (note that $ST \in \Phi_1$ implies that $S, T \in \Phi_1$). □

Remark 2.2.7. Let $T \in C(X)$ with non-empty resolvent set, and let $b \in \rho(T)$ and $\lambda \in \mathbb{C}$. Then $\lambda - T \in \Phi_i$ if and only if $(\lambda - T)(b - T)^{-1} \in \Phi_i$ for $i \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, see for example [GO85, Lemma 1].

Lemma 2.2.8. *Let $A \in \mathrm{BSect}(\omega, a)$, $f, g \in \mathcal{M}(A)$ with f, g quasi-regular at M_A, $0 \notin g(\widetilde{\sigma}(A))$ and such that*

$$f(z) := g(z) \prod_{j=1}^{N} \left(\frac{\lambda_j - z}{b - z} \right)^{n_j},$$

for some $b \in \rho(A)$, $\lambda_j \in \sigma(A) \setminus M_A$, and $n_j \in \mathbb{N}$ for $j = 1, \ldots, N$. Then

(a) *If* $f(A) \in \Phi_i$, *then* $\lambda_j - A \in \Phi_i$ *for all* $j = 1, ..., N$ *and for* $i \in \{0,1,2,3,4,5,6,8,9\}$.

(b) *If* $\lambda_j - A \in \Phi_i$ *for all* $j = 1, ..., N$, *then* $f(A) \in \Phi_i$ *for* $i \in \{0,1,2,3,4,5,7,8,9\}$.

Proof. Set $q(z) := \prod_{j=1}^{N} \left(\frac{\lambda_j - z}{b - z}\right)^{n_j}$, so $q(A) \in L(X)$. Several applications of Lemma 2.2.4 imply that $\mathrm{Dom}(g(A)) = \mathrm{Dom}(f(A))$ and $f(A) = q(A)g(A)$. Moreover, Proposition 2.2.2 yields that $0 \notin \widetilde{\sigma}(g(A))$, so $g(A)$ is surjective and injective. Thus $g(A) : \mathrm{Dom}(g(A)) = \mathrm{Dom}(f(A)) \to X$ is an isomorphism when $\mathrm{Dom}(f(A))$ is endowed with the graph norm given by $f(A)$ (which is equivalent to the graph norm given by $g(A)$). Therefore, $f(A) \in \Phi_i$ if and only if $q(A)$. This follows by the very definition of Φ_i for all $i \in \{0,1,2,3,4,5,6,7\}$, and by Lemma 2.2.6 for $i \in \{8,9\}$.

Since the bounded operators $(\lambda_j - A)(b - A)^{-1}$ commute between themselves, we have:

1. if $q(A) \in \Phi_i$, then $(\lambda_j - A)(b - A)^{-1} \in \Phi_i$ for all $j = 1, ..., N$, and for $i \in \{0,1,2,3,4,5,6,8,9\}$,
2. If $(\lambda_j - A)(b - A)^{-1} \in \Phi_i$ for all $j = 1, ..., N$, then $q(A) \in \Phi_i$, for $i \in \{0,1,2,3,4,5,7,8,9\}$.

see for example [GO85, Lemma 3] and [GL71, Lemma 5(c)]. Hence, the claim follows from Remark 2.2.7.

□

We give now the main result of this subsection.

Proposition 2.2.9. *Let* $A \in \mathrm{BSect}(\omega, a)$, $f \in \mathcal{M}(A)$, *where* f *is quasi-regular at* M_A. *Then*

(a) $f(\widetilde{\sigma}_i(A)) \setminus f(M_A) \subseteq \widetilde{\sigma}_i(f(A))$ *for* $i \in \{0,1,2,3,4,5,6,8,9\}$.

(b) $\widetilde{\sigma}_i(f(A)) \subseteq f(\widetilde{\sigma}_i)(A) \cup f(M_A)$ *for* $i \in \{0,1,2,3,4,5,7,8,9\}$.

Proof. Take $i \in \{0,1,2,3,4,5,6,8,9\}$ and let $\mu \in \mathbb{C}$ be such that $\mu \in f(\widetilde{\sigma}_i(A)) \setminus f(M_A)$. By considering the function $f - \mu$ instead of f, we can assume without loss of generality that $\mu = 0$. As $0 \notin f(M_A)$, $f^{-1}(0) \cap \widetilde{\sigma}(A)$ must be finite. Let $\lambda_1, \ldots, \lambda_N$ be the points in $f^{-1}(0) \cap \widetilde{\sigma}(A)$ (so $\lambda_j \in \widetilde{\sigma}_i(A)$ for some $j \in \{1, ..., N\}$), and let n_j be the order of the zero of f at λ_j. Let $b \in \rho(A)$ and set

$$g(z) := f(z) \prod_{j=1}^{N} \left(\frac{b - z}{\lambda_j - z}\right)^{n_j}. \tag{2.5}$$

Then $0 \notin g(\widetilde{\sigma}(A))$ and is g quasi-regular at M_A. Several applications of Lemma 2.2.4 imply $g \in \mathcal{M}(A)$, and Lemma 2.2.8(a) yields $f(A) \notin \Phi_i$.

Take now $i \in \{0,1,2,3,4,5,7,8,9\}$ and let $\mu \in \mathbb{C}$ be such that $\mu \notin f(\widetilde{\sigma}_i(A)) \cup f(M_A)$. We prove that $\mu \notin \widetilde{\sigma}_i(f(A))$. We can assume $\mu = 0$. Again, $f^{-1}(0) \cap \sigma(A)$ has finite cardinal, so let g be as in (2.5). Since $\lambda_j - A \in \Phi_i$ for all $j = 1, ..., n$, applications of Lemma 2.2.4 and Lemma 2.2.8(b) yield that $f(A) \in \Phi_i$, as we wanted to show.

Assume now $\mu = \infty$. If $\rho(f(A)) \neq \emptyset$ take $b \in \rho(f(A))$. An application of which we have already proven for the function $\dfrac{1}{b - f(z)}$ shows the claim, see the paragraph below Definition 2.2.1. Hence, all that is left to prove is that we can assume without loss of generality that $\rho(f(A)) \neq \emptyset$. Take $\nu \in \mathbb{C} \setminus f(M_A)$, so $f^{-1}(\nu) \cap \widetilde{\sigma}(A)$ has finite cardinal. Let $\nu_1, \ldots, \nu_M$ be the points in $f^{-1}(\nu) \cap \sigma(A)$, and let m_j be the order of the zero of $f - \nu$ at ν_j. Let $b \in \rho(A)$ and set

$$h(z) := (f(z) - \nu) \prod_{j=1}^{M} \left(\frac{b - z}{\nu_j - z} \right)^{m_j} . \tag{2.6}$$

By Lemma 2.2.4, $h \in \mathcal{M}(A)$ with $\mathrm{Dom}(f(A)) = \mathrm{Dom}(h(A))$, and using (2.6) it is readily seen that $\mathrm{Dom}(f(A)^n) = \mathrm{Dom}(h(A)^n)$ for all $n \in \mathbb{N}$. In particular, $\infty \in \widetilde{\sigma}_i(f(A))$ if and only if $\infty \in \widetilde{\sigma}_i(h(A))$. Since $0 \notin h(\widetilde{\sigma}(A))$, Proposition 2.2.2 implies that $0 \in \rho(h(A))$. Therefore, we can assume that $\rho(f(A)) \neq \emptyset$, and the proof is done. □

2.2.B General case

In this section we deal with the case $M_A \neq \emptyset$. The difficulty of this setting arises from the fact that f is not necessarily either holomorphic or meromorphic at M_A, so the factorization techniques used in Subsection 2.2.A do not apply here.

First, we give some results involving M_A which are the key for the proof of the spectral mapping theorems.

Remark 2.2.10. Let $T \in C(X)$ with non-empty resolvent set, $d \in \widetilde{\sigma}(T)$ with d an accumulation point of $\rho(T)$, and $i \in \{1, 2, 3, 4, 5, 6, 7, 8\}$. The following assertions about the essential spectra are well known, see for example [EE87, Sections I.3 & I.4], [Kat66, Chapter 4§5] and [TL58, Section V.6].

(a) If d is also an accumulation point of $\widetilde{\sigma}(T)$, then $d \in \widetilde{\sigma}_i(T)$.

(b) If $d \in \widetilde{\sigma}_i(T)$ and d is not an accumulation point of $\widetilde{\sigma}_i(T)$, then there is a neighborhood Ω of d such that $\widetilde{\sigma}(T) \cap \Omega$ consists of d and a countable (possibly empty) set of eigenvalues of T with finite dimensional eigenspace, which are isolated between themselves.

(c) If $d \notin \widetilde{\sigma}_i(T)$, then d is an isolated point of $\widetilde{\sigma}(T)$. Moreover, $d \in \sigma_p(T)$ with $\mathrm{nul}(d - T) = \mathrm{def}(d - T) < \infty$, $\alpha(d - T) = \delta(d - T) < \infty$, and $\dim\left(\cup_{n \geq 1} \mathcal{N}((d - T)^n)\right) < \infty$.

Lemma 2.2.11. *Let $A \in \mathrm{BSect}(\omega, a)$, $d \in M_A$ and $i, j \in \{1, 2, 3, 4, 5, 6, 7, 8\}$. Then*

- *$d \in \widetilde{\sigma}_i(A)$ if and only if $d \in \widetilde{\sigma}_j(A)$,*
- *if $\infty \in \widetilde{\sigma}(A)$, then $\infty \in \widetilde{\sigma}_i(A)$.*

Proof. If $d \in \widetilde{\sigma}_6(A)$, then $d \in \widetilde{\sigma}_i(A)$ since $\widetilde{\sigma}_6(A) \subseteq \widetilde{\sigma}_i(A)$ for any $i \in \{1,2,3,4,5,6,7,8\}$. If $d \notin \widetilde{\sigma}_6(A)$, then Remark 2.2.10(c) implies that $d \notin \widetilde{\sigma}_i(A)$ for $i \in \{1,2,3,4,5,6,7,8\}$, and the claim follows.

□

Take $T \in C(X)$ with non-empty resolvent set, and let $\Lambda_1, \ldots, \Lambda_n$ be the connected components of $\widetilde{\sigma}(T)$. Let Λ be subset of $\widetilde{\sigma}(T)$ which is open and closed in the relative topology of $\widetilde{\sigma}(T)$ (i.e. Λ is the union of some connected components of $\widetilde{\sigma}(T)$). If $\infty \notin \Lambda$, the spectral projection P_Λ of T is given by

$$P_\Lambda := \int_\Gamma (z - A)^{-1}\, dz, \tag{2.7}$$

where Γ is a finite collection of paths contained in $\rho(T)$ such that Γ has index 1 with respect to every point in Λ, and has index 0 with respect to every point in $\sigma(T) \setminus \Lambda$. If $\infty \in \Lambda$, then the spectral projection P_Λ of T is given by $P_\Lambda := I - P_{\widetilde{\sigma}(T)\setminus\Lambda}$, where $P_{\widetilde{\sigma}(T)\setminus\Lambda}$ is as in (2.7).

We collect in the form of a lemma some well-known results about spectral projections, see for instance [DS63, Section V.9].

Lemma 2.2.12. *Let T, Λ be as above. Then*

1. *P_Λ is a bounded projection commuting with T;*
2. *$\widetilde{\sigma}(T_\Lambda) = \Lambda$, where $T_\Lambda : \mathrm{Ran}(P_\Lambda) \to \mathrm{Ran}(P_\Lambda)$ is the part of T in $\mathrm{Ran}(P_\Lambda)$.*

As a consequence, for $\lambda \in \Lambda \cap \mathbb{C}$, $\mathcal{N}(\lambda - T) \subseteq \mathrm{Ran}(P_\Lambda)$ and $\mathrm{Ran}(I - P_\Lambda) \subseteq \mathrm{Ran}(\lambda - T)$. Also, if $\infty \notin \Lambda$, then $\mathrm{Ran}(P_\Lambda) \subseteq \mathrm{Dom}(T)$.

We also need the following two lemmas.

Lemma 2.2.13. *Let $A \in \mathrm{BSect}(\omega, a)$, and let $\Lambda \subseteq \widetilde{\sigma}(A)$ be an open and closed subset in the relative topology of $\widetilde{\sigma}(A)$. Then $A_\Lambda \in \mathrm{BSect}(\omega, a)$ with $\mathcal{M}(A) \subseteq \mathcal{M}(A_\Lambda)$, and one has*

$$f(A_\Lambda) = f(A)|_{\mathrm{Ran}(P_\Lambda)}, \qquad \text{and} \qquad \widetilde{\sigma}_i(f(A_\Lambda)) \subseteq \widetilde{\sigma}_i(f(A)),$$

for every $f \in \mathcal{M}(A)$ and $i \in \{0,1,2,3,4,5,6,9\}$.

Proof. It follows by Lemma 2.2.12 that $\widetilde{\sigma}(A_\Lambda) = \Lambda \subseteq \overline{BS(\omega, a)} \cup \{\infty\}$. Moreover, it is readily seen that $(z - A_\Lambda)^{-1} = (z - A)^{-1}|_{\mathrm{Ran}(P_\Lambda)}$ for every $z \in \rho(A)$. As a consequence, one gets that A_Λ is indeed a bisectorial operator on $\mathrm{Ran}(P_\Lambda)$ of angle ω and half-width a, and that $\mathcal{E}(A) \subseteq \mathcal{E}(A_\Lambda)$ with $f(A)|_{\mathrm{Ran}(P_\Lambda)} = f(A_\Lambda)$ for all $f \in \mathcal{E}(A)$. Thus, if $e \in \mathcal{E}(A)$ is a regularizer for $f \in \mathcal{M}(A)$, then e is also a regularizer for f with respect to A_Λ, so $\mathcal{M}(A) \subseteq \mathcal{M}(A_\Lambda)$.

Now, we have $P_\Lambda f(A) \subseteq f(A) P_\Lambda$ for every $f \in \mathcal{M}(A)$ by Lemma 2.1.4. From this and the above properties, it is not difficult to get, for every $f \in \mathcal{M}(A)$, $f(A_\Lambda) = f(A)|_{\mathrm{Ran}(P_\Lambda)}$, with $\mathrm{Dom}(f(A_\Lambda)) = \mathrm{Dom}(f(A)) \cap \mathrm{Ran}(P_\Lambda)$, $\mathcal{N}(f(A_\Lambda)) = \mathcal{N}(f(A)) \cap$

$\mathrm{Ran}(P_\Lambda)$, $\mathrm{Ran}(f(A_\Lambda)) = \mathrm{Ran}(f(A)) \cap \mathrm{Ran}(P_\Lambda)$. Hence $\widetilde{\sigma}_i(f(A_\Lambda)) \subseteq \widetilde{\sigma}_i(f(A))$ for $i \in \{0,1,4,5,6\}$.

Thus, we only have to prove are the spectral inclusions for $\widetilde{\sigma}_2, \widetilde{\sigma}_3$ and $\widetilde{\sigma}_9$. Regarding the first case, assume $f(A) \in \Phi_2$, i.e., $\mathrm{Ran}(f(A)) \oplus W = X$ for some closed linear subspace W. Then $\mathrm{Ran}(f(A_\Lambda)) \oplus (\mathrm{Ran}(f(A)) \cap \mathrm{Ran}(I - P_\Lambda)) \oplus W = X$ since $\mathrm{Ran}(f(A)) = \mathrm{Ran}(f(A_\Lambda)) \oplus (\mathrm{Ran}(f(A) \cap \mathrm{Ran}(I-P_\Lambda))$. Thus, there exists a bounded projection $\tilde{P}$ from X onto $\mathrm{Ran}(f(A_\Lambda))$. Then, $\tilde{P}|_{\mathrm{Ran}(P_\Lambda)}$ is a bounded projection from $\mathrm{Ran}(P_\Lambda)$ onto $\mathrm{Ran}(f(A_\Lambda))$, that is, $\mathrm{Ran}(f(A_\Lambda))$ is complemented in $\mathrm{Ran}(P_\Lambda)$ and so $f(A_\Lambda) \in \Phi_2$. A similar argument for $\mathrm{Dom}(f(A))$ and $\mathrm{Dom}(f(A_\Lambda))$ shows that, if $\infty \in \widetilde{\sigma}_2(f(A_\Lambda))$, then $\infty \in \widetilde{\sigma}_2(f(A))$. Hence, the inclusion $\widetilde{\sigma}_2(f(B)) \subseteq \widetilde{\sigma}_2(f(A))$ holds for any $f \in \mathcal{M}(A)$.

Similar reasoning proves the inclusion for $\widetilde{\sigma}_3$. For $\widetilde{\sigma}_9$, the inclusion follows from $\alpha(f(A)) = \max\{\alpha(f(B)), \alpha(f(A)|_Z)\}$ and $\delta(f(A)) = \max\{\delta(f(B)), \delta(f(A)|_Z)\}$, see [TL58, Problem V.6]. □

Let $A \in \mathrm{BSect}(\omega, a)$. Recall that $M_A \setminus \widetilde{\sigma}_i(A) = M_A \setminus \widetilde{\sigma}_j(A)$ for $i, j \in \{1, 2, ..., 8\}$, see Lemma 2.2.11. Then, by Remark 2.2.10(c), $M_A \setminus \widetilde{\sigma}_i(A)$ is an open and closed subset of $\widetilde{\sigma}(A)$ (in the relative topology of $\widetilde{\sigma}(A)$) for $i \in \{1,2,3,4,5,6,7,8\}$. Also,

Lemma 2.2.14. *Let* $A \in \mathrm{BSect}(\omega, a)$*, and let* $\Lambda = \widetilde{\sigma}(A) \setminus (M_A \setminus \widetilde{\sigma}_i(A))$ *for any* $i \in \{1,2,3,4,5,6\}$. *Then* $M_{A_\Lambda} \subseteq \widetilde{\sigma}_i(A_\Lambda)$ *for* $i \in \{0,1,2,3,4,5,6\}$ *and*

$$\widetilde{\sigma}_i(f(A)) = \widetilde{\sigma}_i(f(A_\Lambda)), \qquad f \in \mathcal{M}(A), \quad i \in \{1,2,3,4,5,6\}.$$

Also, $\mathrm{codim}\, \mathrm{Ran}(P_\Lambda) < \infty$.

Proof. The inclusions $\widetilde{\sigma}_i(f(A_\Lambda)) \subseteq \widetilde{\sigma}_i(f(A))$ are given in Lemma 2.2.13. Let us show that the inclusions $\sigma_i(f(A)) \subseteq \sigma_i(f(B))$ also hold. To do this, we prove the following claims for all $f \in \mathcal{M}(A)$,

1. If $\mathrm{nul}(f(A_\Lambda)) < \infty$, then $\mathrm{nul}(f(A)) < \infty$.

2. If $\mathrm{def}(f(A_\Lambda)) < \infty$, then $\mathrm{def}(f(A)) < \infty$.

3. If $\mathrm{Ran}(f(A_\Lambda))$ is closed/complemented in $\mathrm{Ran}(P_\Lambda)$, then $\mathrm{Ran}(f(A))$ is closed/-complemented in X.

4. If $\mathcal{N}(f(A_\Lambda))$ is complemented in $\mathrm{Ran}(P_\Lambda)$, then $\mathcal{N}(f(A))$ is complemented in X.

Set $\Omega = \widetilde{\sigma}(A) \setminus \Lambda$, so $\Omega = M_A \setminus \widetilde{\sigma}_i(A)$ for any $i \in \{1, \ldots, 6\}$. Lemma 2.2.12 implies that $\widetilde{\sigma}_i(A|_\Omega) = \emptyset$ for $i \in \{1,2,3,4,5,6\}$, whence $\mathrm{Ran}(P_\Omega)$ is finite dimensional, i.e., $\mathrm{codim}\,\mathrm{Ran}(P_\Lambda) < \infty$ since $\mathrm{Ran}(P_\Lambda)$ and $\mathrm{Ran}(P_\Omega)$ are complementary subspaces. Since $\mathcal{N}(f(A_\Lambda)) = \mathcal{N}(f(A)) \cap \mathrm{Ran}(P_\Lambda)$ and $\mathrm{Ran}(f(A_\Lambda)) = \mathrm{Ran}(f(A)) \cap \mathrm{Ran}(P_\Lambda)$ (see the proof of Lemma 2.2.13), we conclude that claims (1) and (2) hold true.

For the claim regarding closedness in (3), assume $\mathrm{Ran}(f(A_\Lambda))$ is closed in $\mathrm{Ran}(P_\Lambda)$, so $\mathrm{Ran}(f(A_\Lambda))$ is closed in X too. Since $\mathrm{Ran}(f(A)) = \mathrm{Ran}(f(A_\Lambda)) \oplus \mathrm{Ran}(f(A_\Omega))$, we have that $\mathrm{Ran}(f(A))/\mathrm{Ran}(f(A_\Lambda))$ is finite-dimensional in $X/\mathrm{Ran}(f(A_\Lambda))$, hence closed. Thus $\mathrm{Ran}(f(A))$ is closed in X. For the claim regarding complementation in (3), assume

$\mathrm{Ran}(f(A_\Lambda)) \oplus U = \mathrm{Ran}(P_\Lambda)$ for some closed subspace U. Note that $\mathrm{Ran}(f(A_\Omega)) \oplus V = Z$ for some closed subspace since $\dim \mathrm{Ran}(P_\Omega) < \infty$. Therefore $\mathrm{Ran}(f(A)) \oplus (U \oplus V) = X$, and (3) follows. An analogous reasoning proves claim (4).

Now, a similar reasoning as above with subspaces $\mathrm{Dom}(f(A)), \mathrm{Dom}(f(A_\Lambda))$ shows that, if $\infty \in \widetilde{\sigma}_i(f(A))$, then $\infty \in \widetilde{\sigma}_i(f(A_\Lambda))$ for $i \in \{1,2,3,4,5,6\}$. Therefore, $\widetilde{\sigma}_i(f(A)) \subseteq \widetilde{\sigma}_i(f(A_\Lambda))$, as we wanted to show.

Finally, to prove that $M_{A_\Lambda} \subseteq \widetilde{\sigma}_i(A_\Lambda)$ (for $i \in \{1,2,3,4,5,6\}$), note that $M_A \setminus \widetilde{\sigma}_i(A) = \widetilde{\sigma}(A) \setminus \Lambda \subseteq \rho(A_\Lambda)$ by Lemma 2.2.12. □

We are now ready to prove the spectral mapping theorems for most of the extended essential spectra considered here. For the sake of clarity, we separate the proof of each inclusion into two different propositions.

Proposition 2.2.15. *Let $A \in \mathrm{BSect}(\omega, a)$ and let $f \in \mathcal{M}(A)$ be quasi-regular at M_A. Then*

$$\widetilde{\sigma}_i(f(A)) \subseteq f(\widetilde{\sigma}_i(A)), \qquad i \in \{0,1,2,3,4,5,7,8\}.$$

Proof. The inclusion for $\widetilde{\sigma}_0$ is already given in Proposition 2.2.2. For $i \in \{1,2,3,4,5\}$, we can assume $M_A \subseteq \widetilde{\sigma}_i(A)$ without loss of generality by Lemma 2.2.14. Thus $\widetilde{\sigma}_i(f(A)) \subseteq f(\widetilde{\sigma}_i(A))$ for $i \in \{1,2,3,4,5\}$ by Proposition 2.2.9(b).

Now, we show the inclusions for $\widetilde{\sigma}_7, \widetilde{\sigma}_8$. So take $i \in \{7,8\}$, and let $\mu \in \mathbb{C} \setminus f(\widetilde{\sigma}_i(A))$. Note that we can assume $\mu = 0$. If $0 \notin f(M_A)$, Proposition 2.2.9(b) implies $0 \notin \widetilde{\sigma}_i(f(A))$. So assume $0 \in f(M_A)$. As $0 \notin f(\widetilde{\sigma}_i(A))$, Lemma 2.2.11 and Remark 2.2.10(c) imply that $f^{-1}(0) \cap \widetilde{\sigma}(A)$ is a finite set. Let $\lambda_1, ..., \lambda_N$ be an enumeration of $(f^{-1}(0) \cap \widetilde{\sigma}(a)) \setminus M_A$, and let $n_1, ..., n_N$ be the multiplicity of f at $\lambda_1, \ldots, \lambda_N$ respectively. Set $q(z) := \prod_{j=1}^{N} \left(\frac{\lambda_j - z}{b - z}\right)^{n_j}$ for some $b \in \rho(A)$, and $g(z) = f(z)/q(z)$. Several applications of Lemma 2.2.8 yield that $g \in \mathcal{M}(A)$ and $f(A) = q(A)g(A) = g(A)q(A)$ with $\mathrm{Dom}(f(A)) = \mathrm{Dom}(g(A))$. Also, one has $g^{-1}(0) \cap \widetilde{\sigma}(A) \subseteq M_A \setminus \widetilde{\sigma}_i(A)$. Then, $g^{-1}(0) \cap \Lambda = \emptyset$ where $\Lambda := \widetilde{\sigma}(A) \setminus (M_A \setminus \widetilde{\sigma}_i(A))$. Thus, $g(A_\Lambda)$ is invertible by Proposition 2.2.2 and Lemma 2.2.13. On the other hand, $\dim \mathrm{Ran}(I - P_\Lambda) < \infty$ by Lemma 2.2.14 (see also Lemma 2.2.11), so $g(A_{\widetilde{\sigma}(A) \setminus \Lambda}) \in \Phi_i$. As a consequence, $g(A) \in \Phi_i$. Furthermore, $r(A)$ is a bounded operator and belongs to Φ_i. Therefore, $f(A) = r(A)g(A) = g(A)r(A) \in \Phi_i$ by [EE87, Th. I.3.16] for $i = 7$, and by Lemma 2.2.6 for $i = 8$, that is $0 \notin \widetilde{\sigma}_i(f(A))$.

Now, assume $\infty \notin f(\widetilde{\sigma}_i(A))$. Reasoning as at the end of the proof of Proposition 2.2.9, we can assume $\rho(f(A)) \neq \emptyset$ without loss of generality. So let $\nu \in \rho(f(A))$. An application of which we have already proven for the function $\dfrac{1}{\nu - f(z)}$ shows that $0 \notin \widetilde{\sigma}_i\left((\nu - f(A))^{-1}\right)$, that is $\infty \notin \widetilde{\sigma}_i(f(A))$ for $i \in \{8\}$, and the proof is finished. □

Proposition 2.2.16. *Let $A \in \mathrm{BSect}(\omega, a)$ and let $f \in \mathcal{M}(A)$ be quasi-regular at M_A. Then*

$$f(\widetilde{\sigma}_i(A)) \subseteq \widetilde{\sigma}_i(f(A)), \qquad i \in \{0,1,2,3,4,5,6,8\}.$$

Proof. Note that $\widetilde{\sigma}_6(f(A)) \subseteq \widetilde{\sigma}_i(f(A))$ for each $i \in \{0,1,2,3,4,5,6,8\}$. Thus, Lemma 2.2.11 yields that it is enough to prove the claim for $i \in \{0,6\}$. Hence, we assume $i \in \{0,6\}$ from now on.

Let $\mu \in f(\widetilde{\sigma}_i(A))$ with $\mu \neq \infty$, so we can assume $\mu = 0$ without loss of generality. If $0 \in f(\widetilde{\sigma}_i(A)) \backslash f(M_A)$, then $0 \in \widetilde{\sigma}_i(f(A))$ by Proposition 2.2.9(a). So assume $0 \in f(\widetilde{\sigma}_i(A))$ with $0 \in f(M_A)$. If any point in $f^{-1}(0) \cap \widetilde{\sigma}_i(A)$ is an an accumulation point of $\widetilde{\sigma}_i(A)$ (and we rule out the trivial case where f is constant), then 0 is an accumulation point of $f(\widetilde{\sigma}_i(A)) \setminus f(M_A) \subseteq \widetilde{\sigma}_i(f(A))$ (see Proposition 2.2.9(a)), thus $0 \in \widetilde{\sigma}_i(f(A))$ since $\sigma_i(T)$ is closed for any $T \in C(X)$. So assume that each point in $f^{-1}(0) \cap \widetilde{\sigma}_i(A)$ is an isolated point in $\widetilde{\sigma}_i(A)$, and set

$$V_A := \{d \in f^{-1}(0) \cap \widetilde{\sigma}_i(A) \,|\, d \text{ is not an isolated point of } \widetilde{\sigma}(A)\},$$

which is a finite set by Remark 2.2.10(c).

Assume first that V_A is not empty (thus $i = 6$). One has that, for each $d \in V_A$, there is some neighborhood Ω_d of d such that $\Omega_d \cap \widetilde{\sigma}(A) = \{d, \lambda_1^d, \lambda_2^d, ...\}$, where $\lambda_j^d \in \sigma_{point}(A) \setminus \sigma_i(A)$, each λ_j^d is an isolated point of $\sigma(A)$, and $\lambda_j^d \xrightarrow[j\to\infty]{} d$. Thus, $(f^{-1}(0) \cap \widetilde{\sigma}(A)) \setminus (\cup_{d\in V_A}\Omega_d)$ is a finite set. Let $\kappa_1, \ldots, \kappa_N$ be the elements of this set, let $n_1, \ldots, n_N$ be the multiplicity of the zero of f at $\kappa_1, \ldots, \kappa_N$ respectively, and set $g(z) := f(z) \prod_{j=1}^{N} \left(\frac{b-z}{\kappa_j - z}\right)^{n_j}$. Several applications of Lemma 2.2.4 yield that $g \in \mathcal{M}(A)$ with $\operatorname{Dom}(g(A)) = \operatorname{Dom}(f(A))$, and

$$f(A) = \left(\prod_{j=1}^{N} \left((\kappa_j - A)(b-A)^{-1}\right)^{n_j}\right) g(A) = g(A) \prod_{j=1}^{N} \left((\kappa_j - A)(b-A)^{-1}\right)^{n_j}. \tag{2.8}$$

where we regard $(\kappa_j - A)(b-A)^{-1}$ as bounded operators on $\operatorname{Dom}(f(A))$ in the last term. Let us show that $0 \in \widetilde{\sigma}_6(g(A))$, from which it follows $0 \in \widetilde{\sigma}_6(f(A))$, see for example [EE87, Theorem I.3.20]. Note that $g^{-1}(0) \cap \widetilde{\sigma}(A) \subset \cup_{d\in V_A}\Omega_d$, which is a countable set. As a consequence, 0 is an accumulation point of $\mathbb{C} \setminus g(\widetilde{\sigma}(A))$. Thus, Proposition 2.2.2 implies that 0 is an accumulation point of $\rho(g(A))$. If 0 is also an accumulation point of $\widetilde{\sigma}(g(A))$, then $0 \in \widetilde{\sigma}_6(g(A))$ by Remark 2.2.10. So assume that 0 is not an accumulation point of $\widetilde{\sigma}(g(A))$. Since $\sigma_p(g(A)) \subseteq g(\sigma_p(A))$ (Lemma 2.1.5), and $\lambda_j^d \in \sigma_p(A)$ with $\lambda_j^d \xrightarrow[j\to\infty]{} d$ for each $d \in V_A$, it follows that $g(\lambda_j^d) = 0$ for all but finitely many pairs $(j,d) \in \mathbb{N} \times V_A$. Hence, the set $g^{-1}(0) \cap \sigma_p(A)$ has infinite cardinal, so $\operatorname{nul}(g(A)) \geq \sum_{\lambda \in g^{-1}(0)\cap\sigma_p(A)} \operatorname{nul}(\lambda - A) = \infty$. Then Remark 2.2.10(c) yields that $0 \in \widetilde{\sigma}_6(g(A))$, as we wanted to prove.

Now, assume $V_A = \emptyset$, so each $d \in f^{-1}(0) \cap \widetilde{\sigma}_i(A)$ is an isolated point of $\widetilde{\sigma}(A)$. Set $\Lambda := f^{-1}(0) \cap M_A \cap \widetilde{\sigma}_i(A)$. Note that, in the case $i = 6$, then $\dim \operatorname{Ran}(P_\Lambda) = \infty$ as a consequence of Lemma 2.2.12. Since $f(\Lambda) = \{0\}$, we have $\widetilde{\sigma}(f(A_\Lambda)) \subseteq \{0\}$ by Proposition 2.2.2. Hence, $\widetilde{\sigma}_i(f(A_\Lambda)) = \{0\}$ since $\widetilde{\sigma}_i(f(A_\Lambda))$ cannot be the empty set (at least for any operator with non-empty resolvent set, see for example [GL71]). Therefore, $0 \in \widetilde{\sigma}_i(f(A))$ by Lemma 2.2.13, as we wanted to show.

Finally, we deal with the case $\mu = \infty$. Reasoning as at the end of the proof of Proposition 2.2.9, we can assume $\rho(f(A)) \neq \emptyset$. Take $\nu \in \rho(f(A))$, so that $\infty \in \widetilde{\sigma}_i(f(A))$ if and only if $0 \in \widetilde{\sigma}_i \left((\nu - f(A))^{-1}\right)$. But $0 \in \widetilde{\sigma}_i \left((\nu - f(A))^{-1}\right)$ by applying which we have already proven for the function $\dfrac{1}{\nu - f(z)}$. Hence, the proof is finished. □

As a consequence, we have the following

Theorem 2.2.17. *Let $A \in \mathrm{BSect}(\omega, a)$ and $f \in \mathcal{M}(A)$ quasi-regular at M_A. Then*

$$\begin{aligned} &\widetilde{\sigma}_i(f(A)) = f(\widetilde{\sigma}_i(A)), \qquad i \in \{0,1,2,3,4,5,8\}, \\ &f(\widetilde{\sigma}_6(A)) \subseteq \widetilde{\sigma}_6(f(A)), \\ &\widetilde{\sigma}_7(f(A)) \subseteq f(\widetilde{\sigma}_7(A)). \end{aligned}$$

Proof. Immediate consequence of Proposition 2.2.15 and Proposition 2.2.16. □

It is known that the spectral mapping theorem does not hold (in general) for $\widetilde{\sigma}_6, \widetilde{\sigma}_7$, see for instance [EE87, Section 3]. However, we do not know if it holds, for $\widetilde{\sigma}_9$ for the regularized functional calculus considered here, in general. It holds if $M_A = \emptyset$, see Proposition 2.2.9.

2.2.C Remarks on the bounded functional calculus

A natural question is whether or not the condition of quasi-regularity can be relaxed in Theorem 2.2.17. One possibility to relax such a condition could be asking for f to have well-defined limits at M_A. In order to prove a spectral mapping theorem for this wider class of functions we need to assume the following condition on A. The property of having a bounded (regularized) functional calculus is studied in [CDMY96; Haa06; Mor10].

Definition 2.2.18. Let $A \in \mathrm{BSect}(\omega, a)$. We say that the regularized calculus of A is bounded if $f(A) \in L(X)$ for every bounded $f \in \mathcal{M}(A)$.

Lemma 2.2.19. *Let $A \in \mathrm{BSect}(\omega, a)$ and $f \in \mathcal{M}(A)$. Then f is regular at $\sigma_{point}(A) \cap M_A$.*

Proof. The proof is analogous to the case of sectorial operators, see [Haa05a, Lemma 4.2]. □

Theorem 2.2.20. *Let $A \in \mathrm{BSect}(\omega, a)$ such that the regularized functional calculus of A is bounded, and let $f \in \mathcal{M}(A)$ with (possibly ∞-valued) limits at M_A. Then*

$$\begin{aligned} &\widetilde{\sigma}_i(f(A)) = f(\widetilde{\sigma}_i(A)), \qquad i \in \{0,1,2,3,4,5,8\}, \\ &f(\widetilde{\sigma}_6(A)) \subseteq \widetilde{\sigma}_6(f(A)), \\ &\widetilde{\sigma}_7(f(A)) \subseteq f(\widetilde{\sigma}_7(A)). \end{aligned}$$

Proof. The way to prove this claim is completely analogous to the way followed in this section to prove Theorem 2.2.17. Indeed, the quasi-regularity notion is only explicitly needed in the proofs of Lemma 2.1.7 and Proposition 2.2.2. All following results need the quasi-regularity assumption just to apply Proposition 2.2.2. Therefore, the claim is proven if we prove the following version of Proposition 2.2.2:

"Let $A \in \mathrm{BSect}(\omega, a)$ such that the regularized functional calculus of A is bounded, and let $f \in \mathcal{M}(A)$ with (possibly ∞-valued) limits at M_A. Then $\widetilde{\sigma}(f(A)) \subseteq f(\widetilde{\sigma}(A))$."

We outline the proof of this claim. Let $\mu \in \mathbb{C}_\infty$ with $\mu \notin f(\widetilde{\sigma})$, and set $f_\mu = \dfrac{1}{\mu - f}$ if $\mu \in \mathbb{C}$ or $f_\mu = f$ if $\mu = \infty$. Then we show that $f_\mu \in \mathcal{M}(A)$ with $f_\mu(A) \in L(X)$. Note that f_μ has finite limits at M_A. Even more, f_μ is regular at $\sigma_{point}(A) \cap M_A$ by Lemma 2.2.19. Proceeding as in the proof of Lemma 2.1.7, we can assume that f_μ has finitely many poles, all of them contained in $\rho(A)$. Let $q(z) := \prod_{j=1}^{n} \dfrac{(\lambda_j - z)^{n_j}}{(b - z)^{n_j}}$, where λ_j, n_j are the poles of f_μ and their order, respectively. Hence, qf_μ has no poles, is regular at $\sigma_{point}(A) \cap M_A$ and has finite limits at M_A, thus qf_μ is bounded. For any $b \in \rho(A)$, the function $h(z) := \dfrac{1}{b - z} \prod_{d \in \{-a, a\} \setminus \sigma_{point}(A)} \dfrac{z - a}{b - z}$ regularizes qf_μ, so $qf_\mu \in \mathcal{M}_A$. Since the regularized functional calculus of A is bounded, then $qf_\mu(A) \in L(X)$. Moreover, $q(A)$ is bounded and invertible. Therefore, hq regularizes f_μ with $f_\mu(A) = q(A)^{-1}(qf_\mu)(A) \in L(X)$, and the claim follows. $\square$

2.2.D Remarks on the point spectrum

To finish this paper, we give a spectral mapping theorem for the point spectrum. To prove it, we need to restrict to functions $f \in \mathcal{M}_A$ satisfying the following condition:

(**2.2.P1**) For each $d \in M_A$ such that $f(d) \notin f(\sigma_p(A)) \cup \{\infty\}$, there is some $\beta > 0$ for which

- if $d \in \mathbb{C}$, then $|f(z) - c_d| \gtrsim |z - d|^\beta$ as $z \to d$, or
- if $d = \infty$, then $|f(z) - c_d| \gtrsim |z|^{-\beta}$ as $z \to d$,

where c_d denotes the limit of $f(z)$ as $z \to d$.

Proposition 2.2.21. *Let $A \in \mathrm{BSect}(\omega, a)$ and $f \in \mathcal{M}(A)$ such that f is quasi-regular at M_A. Then*

$$f(\sigma_{point}(A)) \subseteq \sigma_{point}(f(A)) \subseteq f(\sigma_{point}(A)) \cup f(M_A).$$

*If, furthermore, f satisfies condition (**2.2.P1**), then $f(\sigma_{point}(A)) = \sigma_{point}(f(A))$.*

Proof. The proof of the inclusions $f(\sigma_{point}(A)) \subseteq \sigma_{point}(f(A)) \subseteq f(\sigma_{point}(A)) \cup f(M_A)$ runs the same as for sectorial operators, see [Haa05b, Prop. 6.5]. Regarding the second statement, all that is left to prove is that if $\mu \in f(M_A) \setminus f(\sigma_{point}(A))$, then

$\mu \notin \sigma_{point}(f(A))$. The statement is trivial if $\mu = \infty$, so assume $\mu \in \mathbb{C}\backslash f(\sigma_{point}(A))$, and consider the function $g := \frac{1}{\mu - f}$, which is quasi-regular at M_A. Note that poles of g are precisely $f^{-1}(\mu) \subseteq \mathbb{C}\backslash\sigma_{point}(A)$. Moreover, g is regular at $M_A \cap \sigma_{point}(A)$, since by assumption $\mu \notin f(\sigma_{point}(A))$. Let now

$$h_{l,m,n}(z) := \frac{(z-a)^m(z+a)^n}{(b-z)^{l+m+n}}, \qquad z \in \mathbb{C}, \quad l,m,n \in \mathbb{N}, \quad b > a.$$

Then, by the assumptions made on f, $h_{l,m,n}g$ is regular for some m,n,l large enough, and where $l,m,n = 0$ if $\infty, a, -a \notin \tilde{\sigma}(A)\backslash\sigma_p(A)$ respectively. Since $h_{l,m,n}(A) = (A-a)^m(A+a)^n R(b,A)^{m+n+l}$ is bounded and injective, $h_{l,m,n}$ regularizes g. Hence $g \in \mathcal{M}(A)$, which by Lemma 2.1.4(4) implies that $\mu - f$ is injective, as we wanted to show. □

2.2.E Functional calculus of generators of exponentially bounded groups

Important examples of bisectorial-like operators are the generators of exponentially bounded groups. Let $(T(t))_{t\in\mathbb{R}}$ be an exponentially bounded group on a Banach space X with $\|T(t)\|_{L(X)} \lesssim \exp(a|t|)$, $t \in \mathbb{R}$, for some $\alpha \geq 0$. Then, as pointed out in Section 1.2, the generator A of $(T(t))$ is in $\mathrm{BSect}(\pi/2, a)$.

In this subsection, we transfer, in Corollary 2.2.25, the spectral mapping theorem given in Section 2.2.B to the setting of subordinated operators of an exponentially bounded group. To do this, we give first some auxiliary results which are completely analogous to the ones given in [Bad53, Th. 5.2] for the primary functional calculus of strip operators; or in [Haa06, Section 3.3] for the regularized functional calculus of sectorial operators.

It should be mentioned that we only work here with the regularized functional calculus for a bisectorial-like operator A with $M_A = \{-a, a, \infty\}$. The reason for this is that, in order to successfully apply some identities, we need that the integration paths of the regularized functional calculus leave the spectrum of A completely on one side. This is enough to cover all the results used here.

For $a \geq 0$, let $M_a(\mathbb{R})$ be the set of Borel measures μ on $\mathbb{R}$ for which $e^{a|t|}$ is μ-integrable. It is readily seen that $M_a(\mathbb{R})$ is closed under translation and convolution. Moreover, for any $\mu \in M_a(\mathbb{R})$, one can define its Fourier transform $\mathcal{F}$ given by

$$\mathcal{F}\mu(z) = \int_{-\infty}^{\infty} e^{-zt}\,\mu(dt), \qquad z \in BS_{\pi/2,a}.$$

Lemma 2.2.22. *Let $a \geq 0$ and $f \in \mathcal{E}[BS_{\pi/2,a}] \oplus \mathbb{C}\mathbf{1}$. Then, there exists a (unique) measure $\mu_f \in M_a(\mathbb{R})$ such that $f(z) = \mathcal{F}\mu_f(-z)$ for all $z \in BS_{\pi/2,a}$, which is given by $\mu_f(dt) = \psi_f(t)dt + c\delta_0(dt)$, where $c = f(\infty)$ and*

$$\psi_f(t) := \begin{cases} \dfrac{-1}{2\pi i}\displaystyle\int_{J_-} e^{-zt} f(z)\,dz, & t < 0, \\ \dfrac{1}{2\pi i}\displaystyle\int_{J_+} e^{-zt} f(z)\,dz, & t > 0, \end{cases} \tag{2.9}$$

and where J is any path of integration for the regularized functional calculus of bisectorial-like operators, $J_- := J \cap \{\mathfrak{Re}\, z < -a\}$ and $J_+ := J \cap \{\mathfrak{Re}\, z > a\}$.

Proof. The proof is the same as in the case of sectorial operators, see [Haa06, Lemma 3.3.1]. □

Remark 2.2.23. Let f be as above, and assume furthermore that $|f(z)| \lesssim |z|^{-(1+\varepsilon)}$ as $z \to \infty$ for some $\varepsilon > 0$. An easy application of Cauchy's theorem to (2.9) gives rise to

$$\psi_f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itu} f(iu)\, du, \quad t \in \mathbb{R}.$$

Proposition 2.2.24. *Let A be the generator of an exponentially bounded group $(T(t))$ on X satisfying $\|T(t)\|_{L(X)} \lesssim e^{a|t|}$ for some $a \geq 0$, so that $A \in \mathrm{BSect}(\pi/2, a)$. Let $\mu \in M_a(\mathbb{R})$ be such that $f(z) := \mathcal{F}\mu(-z) \in \mathcal{M}(A)$. Then,*

$$f(A) = \int_{-\infty}^{\infty} T(t)\, \mu(dt).$$

Proof. The proof follows as in the case of sectorial operators (see [Haa06, [Prop. 3.3.2]). □

Corollary 2.2.25. *Let A be the generator of an exponentially bounded group $(T(t))$ on a Banach space X satisfying that $\|T(t)\| \lesssim \exp(a|t|)$, $t \in \mathbb{R}$ for $a > 0$. Let $\mu \in M_a(\mathbb{R})$ such that $\mathcal{F}(\mu) \in \mathcal{M}(A)$ and such that $\mathcal{F}(\mu)$ is quasi-regular at $\{-a, a, \infty\}$. If $\mathcal{T} = \int_{-\infty}^{\infty} T(t)\, \mu(dt)$, then*

$$\begin{aligned}
\widetilde{\sigma}_i(\mathcal{T}) &= \mathcal{F}(\mu)(-\widetilde{\sigma}_i(\Delta)), \quad i \in \{0,1,2,3,4,5,8\}, \\
\mathcal{F}(\mu)(-\widetilde{\sigma}_6(\Delta)) &\subseteq \widetilde{\sigma}_6(\mathcal{T}), \\
\widetilde{\sigma}_7(\mathcal{T}) &\subseteq \mathcal{F}(\mu)(-\widetilde{\sigma}_7(\Delta)), \\
\mathcal{F}(\mu)(-\sigma_{point}(\Delta)) \subseteq \sigma_{point}(\mathcal{T}) &\subseteq \mathcal{F}(\mu)(-\sigma_{point}(\Delta) \cup -M_A).
\end{aligned}$$

*In addition, assume that $\mathcal{F}(\mu)(-\cdot)$ satisfies condition (**2.2.P1**). Then*

$$\sigma_{point}(\mathcal{T}) = \mathcal{F}(\mu)(-\sigma_{point}(\Delta)).$$

Proof. The result follows from Theorem 2.2.17 and Propositions 2.2.21 and 2.2.24. □

2.2.F Sectorial operators and semigroups

Lemmas 2.1.4, 2.1.5, 2.1.6 and 2.1.7 are the only properties of the regularized functional calculus of bisectorial-like operators that are used in the proofs given in Subsections 2.2.A and 2.2.B. Therefore, for any other regularized functional calculus satisfying such properties, one can prove spectral mapping theorems for essential spectra analogous to the ones given in Theorem 2.2.17. In particular, one has the following result for the regularized functional calculus of sectorial operators considered in [Haa05b].

Theorem 2.2.26. *Let A be a sectorial operator of angle $\phi \in [0, 2\pi)$, and let f be a function in the domain of the regularized functional calculus of A that is quasi-regular at $\{0, \infty\} \cap \widetilde{\sigma}(A)$. Then*

$$\begin{aligned} \widetilde{\sigma}_i(f(A)) &= f(\widetilde{\sigma}_i(A)), \qquad i \in \{0, 1, 2, 3, 4, 5, 8\}, \\ f(\widetilde{\sigma}_6(A)) &\subseteq \widetilde{\sigma}_6(f(A)), \\ \widetilde{\sigma}_7(f(A)) &\subseteq f(\widetilde{\sigma}_7(A)). \end{aligned}$$

Let us show an application of the above theorem to subordinated operators which will be helpful in Chapter 5. Let $(T(t))_{t\geq 0}$ be a uniformly bounded semigroup with generator A, so $-A$ is a sectorial operator of angle $\pi/2$. Recall that for a finite Borel measure μ on $[0, \infty)$, its Laplace transform $\mathcal{L}$, given by

$$\mathcal{L}\mu(z) = \int_0^\infty e^{-zt}\, \mu(dt), \qquad \mathfrak{Re}\, z \geq 0,$$

defines a holomorphic function on the half-right complex plane.

It is well known that the Laplace transform connects the subordinated operators of $(T(t))$ in terms of the functional calculus of $-A$. More precisely, one has the following result, which was given in [Haa06, Prop. 3.3.2].

Proposition 2.2.27. *Let $-A$ be the generator of a uniformly bounded semigroup $(T(t))_{t\geq 0}$ on a Banach space X. If μ is a finite Borel measure on $[0, \infty)$ such that $f := \mathcal{L}\mu \in \mathcal{M}(A)$, then $f(A) \in L(X)$ and*

$$f(A) = \int_0^\infty T(t)\, \mu(dt).$$

(**2.2.P2**) For each $d \in \widetilde{\sigma}(A) \cap \{0, \infty\}$ such that $f(d) \notin f(\sigma_p(A)) \cup \{\infty\}$, there is some $\beta > 0$ for which

- if $d = 0$, then $|f(z) - c_0| \gtrsim |z|^\beta$ as $z \to 0$, or
- if $d = \infty$, then $|f(z) - c_\infty| \gtrsim |z|^{-\beta}$ as $z \to \infty$,

where c_d denotes the limit of $f(z)$ as $z \to d$.

Corollary 2.2.28. *Let $-A$ be the generator of a uniformly bounded semigroup $(T(t))_{t\geq 0}$ on a Banach space X, and let μ is a finite Borel measure on $[0, \infty)$ such that $f := \mathcal{L}\mu \in \mathcal{M}(A)$. Assume that f is quasi-regular at $M_A = \widetilde{\sigma}(A) \cap \{0, \infty\}$. If $\mathcal{T} = \int_0^\infty T(t)\, \mu(dt)$, then*

$$\begin{aligned} \widetilde{\sigma}_i(\mathcal{T}) &= \mathcal{L}(\mu)(\widetilde{\sigma}_i(A)), \quad i \in \{0, 1, 2, 3, 4, 5, 8\}, \\ \mathcal{L}(\mu)(\widetilde{\sigma}_6(A)) &\subseteq \widetilde{\sigma}_6(\mathcal{T}), \\ \widetilde{\sigma}_7(\mathcal{T}) &\subseteq \mathcal{L}(\mu)(\widetilde{\sigma}_7(A)), \\ \mathcal{L}(\mu)(\sigma_{point}(A)) \subseteq \sigma_{point}(\mathcal{T}) &\subseteq \mathcal{L}(\mu)(\sigma_{point}(A) \cup M_A). \end{aligned}$$

*If, furthermore, $\mathcal{L}(\mu)$ satisfies the condition (**2.2.P2**), then $\sigma_{point}(\mathcal{T}) = \mathcal{L}(\mu)(\sigma_{point}(A))$.*

Proof. This is a direct consequence of Proposition 2.2.21 (adapted to sectorial operators), Theorem 2.2.26 and Proposition 2.2.27. □

Chapter 3

Scaling property extension and Black–Scholes equation

The present chapter is based on the article [OW23].

Sectorial operators and bisectorial operators are closely related. For instance, A^2 is sectorial whenever A is a bisectorial operator, see e.g. [AZ10, Prop. 5.1]. In the particular case that A generates a uniformly bounded group, then A^2 generates a uniformly bounded holomorphic semigroup (see for instance [Are+86, Th. 1.15] or [EN00, Cor. 4.9]). This result can be applied to study differential equations on the positive real line in terms of a possibly simpler equation. One may find a concrete example of this fact in [AP02], where the authors obtain properties of a version of the classical Black–Scholes equation

$$u_t = x^2 u_{xx} + x u_x, \qquad t, x > 0, \tag{BS}$$

through the simpler and elegant partial differential equation

$$u_t = -x u_x, \qquad t, x > 0.$$

The Black-Scholes equation is of particular importance since the seminal work [BS73], and has been an active topic of research in mathematical finance due to its importance in the modeling of pricing options contracts, see for instance [GMV97] and the references therein.

It sounds sensible to think of an extension of the generation result mentioned above for A^2, since, for instance, that extension could be a suitable tool to study a broader family of Black-Scholes equations. Note that this result resembles the scaling property of sectorial operators, i.e. A^α is a sectorial operator if A is sectorial and $\alpha > 0$ is small enough, see [Kat60, Th. 2]. As a particular case, such an extension result could lead to the study of a generalized version of the Black–Scholes equation (BS).

Following this direction, the main contribution of this chapter is to give a result for bisectorial-like operators, namely Theorem 3.1.9, which extends such a scaling property, and which is key to study fractional differential equations extending the classical equation (BS). More precisely, four new families of generalized Black–Scholes equations arise in a natural way as an application of Theorem 3.1.9. At this point, we wish to observe that part of our contribution is to show how the relations between bisectorial-like operators and sectorial operators that we develop here can be used successfully to solve several of those equations extending (BS). We are not dealing with mathematical finance in this memory.

Let us explain the method that we follow to extend (BS) to a fractional differential equation, and we will explain later on how the extension of the scaling property given in Theorem 3.1.9 is related to these differential equations. As said before, the classical Black–Scholes equation is studied in [AP02] by means of the following degenerate differential operator:

$$(Qf)(x) := -xf'(x), \qquad x > 0, \tag{3.1}$$

on $(L^1 - L^\infty)$ interpolation spaces, see Subsection 3.3. In [AP02], the authors use the connection between the operator Q and the classical Cesàro operator $\mathcal{C}$ given by

$$(\mathcal{C}f)(x) = \frac{1}{x}\int_0^x f(y)\,dy, \quad x > 0$$

and its adjoint Cesàro operator $\mathcal{C}^*$. This connection had been first pointed out in [Cow84] to study the Cesàro operator $\mathcal{C}$ on the half-plane. In addition, the differential operator Q was also related in [LMPS14] via subordination of the group $(E(t)f)(x) = f(e^{-t}x)$ to the generalized fractional version of the Cesàro operator $\mathcal{C}_\alpha$ and its adjoint operator $\mathcal{C}_\alpha^*$ on L^p-spaces, for real numbers $\alpha > 0$. Such fractional operators are related to the Riemann-Liouville $D^{-\alpha}$ and Weyl $W^{-\alpha}$ fractional integrals of order α, see Section 1.4.

Furthermore, the subordination formulae of $\mathcal{C}_\alpha$ and $\mathcal{C}_\alpha^*$ (1.9) yields a representation of these operators in the regularized functional calculus of Q, namely $\mathcal{C}_\alpha = \alpha\mathbb{B}(I-Q,\alpha)$ and $\mathcal{C}_\alpha^* = \alpha\mathbb{B}(Q,\alpha)$, where $\mathbb{B}$ denotes the Beta function. Since (BS) can be written in terms of the operators $\mathcal{C}$, $\mathcal{C}^*$, it seems natural to construct families of generalized Black–Scholes equations via $\mathcal{C}_\alpha$, $\mathcal{C}_\alpha^*$. This procedure gives rise to generalized Black–Scholes equations of three forms for x, $t > 0$ and $\alpha > 0$:

$$\begin{aligned} u_t &= \frac{1}{\Gamma(\alpha+1)^2} D^\alpha(x^\alpha D^\alpha(x^\alpha u)) - \frac{2}{\Gamma(\alpha+1)} D^\alpha(x^\alpha u) + u, \\ u_t &= \frac{1}{\Gamma(\alpha+1)^2} x^\alpha W^\alpha(x^\alpha W^\alpha u), \\ u_t &= -\frac{1}{\Gamma(\alpha+1)^2} D^\alpha(x^{2\alpha} W^\alpha u) + \frac{1}{\Gamma(\alpha+1)} x^\alpha W^\alpha u. \end{aligned} \tag{3.2}$$

(see Section 3.3 for more details and for a fourth family of generalized Black-Scholes equations). To deal with these equations, we apply the theory of bisectorial-like operators developed here, which yields the well-posedness and explicit integral expressions of the solutions of such equations, including initial and boundary conditions. Also, we recover all the classical known results at the limiting case $\alpha = 1$.

In order to obtain these results, Theorem 3.1.9 is of particular importance. This theorem proves that, if A is a bisectorial-like operator and g is a suitable function in the domain of the regularized functional calculus of A, whose range is contained in a sector, and which has a fractional-power behavior at the singularity points of the spectra of A and $g(A)$, then $g(A)$ is a sectorial operator. As one may expect, the setting for sectorial operators serves again as an inspiration for this result, and in particular it can be regarded as an extension of the scaling property for sectorial operators. In addition, we give supplementary results of special importance when $g(A)$ generates a semigroup, such as the characterization of the closure of its domain $\overline{\mathrm{Dom}(g(A))}$ or an integral expression for the semigroup that it generates in terms of the functional calculus of A.

Interestingly, as a consequence of our results one obtains that if A is bisectorial-like of angle $\frac{\pi}{2}$ and half-width $a \geq 0$ (in particular, if it generates an exponentially bounded group) then either $(A+a)^\alpha$ or $-(A+a)^\alpha$ generate a holomorphic semigroup for all $\alpha > 0$ with $\alpha \neq 1, 3, 5, \ldots$ This is a remarkable property that generalizes known results in the case $\alpha = 2$, see for example [Are+86, Th. 1.15] and [AZ10, Prop. 5.1], or in the case when A generates a bounded group, see [BHK09, Th. 4.6].

As a final remark, we mention that fractional versions of the Black–Scholes equation have been proposed and analyzed in a number of papers, see for instance [FNS19; Kum+12; SW13; ZLTY16]. In most of these references, authors only deal with time-fractional derivatives, that is, replacing the time derivative u_t in (BS) with a Riemann-Liouville or a Caputo type time fractional derivative. The well-posedness of such equations follows directly from the well-posedness of (3.3), see for example [LCL10, Th. 4.9(a)]. Spatial-fractional derivatives are indeed more difficult to deal with, since spatial terms are more complex than the time ones in the equation (BS). A spatial fractional Black–Scholes model can be found in [CXZ14] without giving further details. The fractional Black–Scholes equation we propose here also contains fractional powers acting as multiplication, yielding equations which are definitely difficult to solve by more classical methods such as the Laplace transform or the Fourier transform. Therefore, we notice that the fractional versions of (BS) proposed here seem to be notably difficult to be solved with classical methods.

In short, the contributions of the present chapter can be regarded as centered around two facts:

1. The introduction of new (generalized) fractional Black–Scholes equations arising from fractional Cesàro operators in a natural manner. As we have already noticed, such equations are difficult -maybe not possible- to solve by classical methods.

2. In order to overcome the quoted failure of usual methods, we establish a new connection, in an abstract setting, between bisectorial-like operators and sectorial operators. Such a connection extends notably previous results in the field. Actually our approach is based on the proof of the scaling property given in [AMN97, Prop. 5.2], but it requires quite more general functions to operate in functional calculi defined on the basis of more intricate integration paths, as well as nontrivial, more involved, approximation tools.

Section 3.1 is devoted to the extension of the scaling property to our setting. In Section 3.2 we give bounded holomorphic semigroups generation results for the sectorial operators that we have constructed in Section 3.1. The theory and applications of the generalized Black–Scholes equation are contained in Section 3.3.

3.1 Scaling property extension

In this section, we give the generalization of the scaling property namely mentioned above. The method to prove this generalization is based on the proof of the scaling property for sectorial operators given in [AMN97, Prop. 5.2], but its proof requires longer and more sophisticated techniques because of the more intricate setting.

We start with a definition referring to the limit behavior of a function at the singular points $\{-a, a, \infty\}$.

Definition 3.1.1. Let $A \in \mathrm{BSect}(\omega, a)$, $f \in \mathcal{M}[\Omega_A]$, $d \in \overline{\mathrm{Dom}(f)}$, and $c \in \mathbb{C}$.

1. For $d \in \mathbb{C}$, we say that $f(z) \to c$ exactly polynomially (of order α) as $z \to d$ if there exists $\alpha > 0$ such that $|f(z) - c| \sim |z - d|^{\alpha}$ as $z \to d$.
2. We say that $f(z) \to c$ exactly polynomially (of order α) as $z \to \infty$ if there exists $\alpha > 0$ such that $|f(z) - c| \sim |z|^{-\alpha}$ as $z \to \infty$.
3. We say that $f(z) \to \infty$ exactly polynomially (of order α) as $z \to d$ if $(1/f)(z) \to 0$ exactly polynomially as $z \to d$.

From now on, A denotes a bisectorial-like operator on a Banach space X of angle $\omega \in (0, \pi/2]$ and half-width $a \geq 0$, i.e. $A \in \mathrm{BSect}(\omega, a)$. Recall that $M_A = \{a, -a, \infty\} \cap \widetilde{\sigma}(A)$. For any $\lambda \in \mathbb{C}$, $f \in \mathcal{M}[\Omega_A]$, we let $R_f^{\lambda} \in \mathcal{M}[\Omega_A]$ be the meromorphic function given by

$$R_f^{\lambda}(z) := \frac{\lambda}{\lambda - f(z)}, \qquad z \in \mathrm{Dom}(f). \tag{3.3}$$

Remark 3.1.2. Through the following, we consider $\gamma \in [0, \pi)$ and a function $\widetilde{g} \in \mathcal{M}_A$ satisfying the following conditions:

1. $\mathrm{Ran}(\widetilde{g}) \subseteq \overline{S_\gamma} \cup \{\infty\}$.
2. $\widetilde{g}$ is quasi-regular at M_A. In particular, it has limits in M_A, which we denote by $c_d \in \mathbb{C}_\infty$ for $d \in M_A$.
3. $\widetilde{g}$ has exactly polynomial limits at $M_A \cap \widetilde{g}^{-1}(\{0, \infty\})$.

By the open mapping theorem, Property (a) implies that $\widetilde{g}$ does not have any zeros (unless $\widetilde{g} = 0$) or poles in $\mathrm{Dom}(\widetilde{g})$. In particular, if $\widetilde{g}$ is not the constant zero function, then both $\widetilde{g}$ and $\widetilde{g}^{-1}$ are holomorphic.

We present a family of functions which are crucial to prove our main result of this section.

Definition 3.1.3. Let $(f_\lambda)_{\lambda\notin\overline{S_\gamma}} \subset \mathcal{M}(A)$ and let $\widetilde{g} \in \mathcal{M}(A)$ be as in Remark 3.1.2. We say that $(f_\lambda)_{\lambda\notin\overline{S_\gamma}}$ makes $\left(R^\lambda_{\widetilde{g}}\right)_{\lambda\notin\overline{S_\gamma}}$ $\varepsilon-$uniformly bounded at $d \in M_A$ (with respect to the regularized functional calculus of A) if, for any $\varepsilon \in (0, \pi-\gamma)$, it satisfies the following properties:

1. $\sup\{|f_\lambda(z)| \,:\, z \in \mathrm{Dom}(f_\lambda),\ \lambda \notin \overline{S_{\gamma+\varepsilon}}\} < \infty$.
2. $(f_\lambda(A))_{\lambda\notin\overline{S_\gamma}} \subset L(X)$ and $\sup\{\|f_\lambda(A)\|_{L(X)} \,:\, \lambda \notin \overline{S_{\gamma+\varepsilon}}\} < \infty$.
3. Let Γ be an integration path for the regularized functional calculus of A (see Section 2.1). Then, for each $d' \in M_A \setminus \{d\}$, there exists a neighborhood $\Omega_{d'}$ of d' for which
$$\sup_{\lambda\notin\overline{S_{\gamma+\varepsilon}}} \int_{\Gamma\cap\Omega_{d'}} |f_\lambda(z)| \|(z-A)^{-1}\|_{L(X)}\,|dz| < \infty \quad \text{for each } d' \in M\backslash\{d\}.$$
4. Let Γ be as above. Then, there exists a neighborhood Ω_d of d for which
$$\sup_{\lambda\notin\overline{S_{\gamma+\varepsilon}}} \int_{\Gamma\cap\Omega_d} \left|R^\lambda_{\widetilde{g}}(z) - f_\lambda(z)\right| \|(z-A)^{-1}\|_{L(X)}\,|dz| < \infty.$$

The following lemmas are useful to find a family of functions that makes $\left(R^\lambda_{\widetilde{g}}\right)_{\lambda\notin\overline{S_\gamma}}$ $\varepsilon-$uniformly bounded. Let $d(z,\Omega)$ denote the distance between a point $z \in \mathbb{C}$ and a set $\Omega \subseteq \mathbb{C}$.

Lemma 3.1.4. *Let $\varepsilon > 0$. We have $d(z, \mathbb{C}\backslash\overline{S_{\gamma+\varepsilon}}) \gtrsim |z|$ for all $z \in \overline{S_\gamma}$ and $d(w, \overline{S_\gamma}) \gtrsim |w|$ for all $w \notin \overline{S_{\gamma+\varepsilon}}$. As a consequence, $|w/(w-z)|$ and $|z/(w-z)|$ are uniformly bounded for all $z \in \overline{S_\gamma}$ and $w \notin \overline{S_{\gamma+\varepsilon}}$.*

Proof. The first two inequalities follow from the fact that $d(z, \mathbb{C}\backslash\overline{S_{\gamma+\varepsilon}}) = |z|\sin(\gamma+\varepsilon-|\arg(z)|) \geq |z|\sin\varepsilon$ and $d(w, \overline{S_\gamma}) = |w|\sin(|\arg w|-\gamma) \geq |w|\sin\varepsilon$. The other inequalities follow from which we have already proven, and that $|z-w| \geq \max\{d(z, \mathbb{C}\backslash\overline{S_{\gamma+\varepsilon}}), d(w, \overline{S_\gamma})\}$ for all $z \in \overline{S_\gamma}$ and $w \notin \overline{S_{\gamma+\varepsilon}}$. ☐

As an immediate application of the lemma above we have the following result.

Lemma 3.1.5. *Let $c \in \overline{S_\gamma}\backslash\{0\}$, $\varepsilon \in (0, \pi-\gamma)$ and $f \in \mathcal{M}[\Omega_A]$ such that $\mathrm{Ran}(f) \subseteq \overline{S_\gamma} \cup \{\infty\}$. Then,*
$$\begin{aligned}\left|R^\lambda_f(z) - \frac{\lambda}{\lambda-c}\right| &\lesssim \min\{1, |f(z)-c|\},\\ \left|R^\lambda_f(z)\right| &\lesssim \min\{1, |\lambda||f(z)|^{-1}\},\\ \left|R^\lambda_f(z) - 1\right| &\lesssim \min\{1, |\lambda|^{-1}|f(z)|\},\end{aligned}$$
where all inequalities hold for all $z \in \mathrm{Dom}(f)$ and $\lambda \notin \overline{S_{\gamma+\varepsilon}}$.

Proof. It follows from Lemma 3.1.4 that all the above functions are uniformly bounded.
Now, let $c \in \overline{S_\gamma}$. One gets

$$\left|R_f^\lambda(z) - \frac{\lambda}{\lambda - c}\right| = \left|\frac{\lambda}{\lambda - f(z)} \frac{f(z) - c}{\lambda - c}\right| \lesssim \left|\frac{f(z) - c}{\lambda - c}\right| \lesssim |f(z) - c|,$$

for all $z \in \mathrm{Dom}(f)$, $\lambda \notin \overline{S_{\gamma+\varepsilon}}$. Likewise, one obtains

$$|R_f^\lambda(z)| = \left|\frac{\lambda}{\lambda - f(z)}\right| \lesssim |\lambda||f(z)|^{-1}, \qquad |R_f^\lambda(z) - 1| = \left|\frac{f(z)}{\lambda - f(z)}\right| \lesssim |\lambda|^{-1}|f(z)|,$$

for all $z \in \mathrm{Dom}(f)$, $\lambda \notin \overline{S_{\gamma+\varepsilon}}$, and the proof is finished. □

Lemma 3.1.6. *Let $I \subseteq (0, \infty)$ be a measurable subset and let $(f_\nu)_{\nu \in \mathcal{V}} : I \to \mathbb{C}$ be a family of complex-valued functions. Let $F_1, F_2 : I \to \mathbb{R}^+$ be some positive functions which are integrable with respect to the measure dx/x, and let $r_\nu > 0$ for all indices $\nu \in \mathcal{V}$, and $s_n, t_m > 0$ for $n = 1, ..., N$, $m = 1, ..., M$ for some $N, M \in \mathbb{N}$. Assume*

$$|f_\nu(x)| \lesssim \min\left\{F_1(x) + \sum_{n \le N} (r_\nu x)^{s_n}, F_2(x) + \sum_{m \le M} (r_\nu x)^{-t_m}\right\}, \tag{3.4}$$

for all $x \in I$ and all indices $\nu \in \mathcal{V}$. Then

$$\sup_\nu \int_I |f_\nu(x)| \frac{dx}{x} < \infty.$$

Proof. By adding terms of the type $(r_\nu x)^{t_m}$ to the first expression inside the brackets in (3.4), and terms of the type $(r_\nu x)^{-s_n}$ to the second one, one can assume $N = M$ and $s_n = t_n$ for all $n = 1, ..., N$. It follows that

$$\int_I |f_\nu(x)| \frac{dx}{x} \lesssim \int_I \Big(F_1(x) + F_2(x)\Big) \frac{dx}{x} + \int_I \min\left\{\sum_{n \le N} (r_\nu x)^{s_n}, \sum_{n \le N} (r_\nu x)^{-s_n}\right\} \frac{dx}{x}.$$

By the integrability hypothesis on F_1, F_2, it suffices to uniformly bound the second term for all indices $\nu \in \mathcal{V}$. Using the change of variable $r_\nu x \mapsto x$, one gets

$$\begin{aligned}\int_I \min\left\{\sum_{n \le N} (r_\nu x)^{s_n}, \sum_{n \le N} (r_\nu x)^{-s_n}\right\} \frac{dx}{x} &= \int_0^\infty \min\left\{\sum_{n \le N} x^{s_n}, \sum_{n \le N} x^{-s_n}\right\} \frac{dx}{x} \\ &= \sum_{n \le N} \left(\int_0^1 x^{s_n - 1}\, dx + \int_1^\infty x^{-s_n - 1}\, dx\right) < \infty.\end{aligned}$$

The proof is concluded. □

In order to prove the main result of this section, some integrals related to resolvent operators need to be bounded. The techniques to bound them vary from one case to another, as shows the proof of the proposition below.

Proposition 3.1.7. *Let $A \in \mathrm{BSect}(\omega, a)$ and let $\widetilde{g}$ be as in Remark 3.1.2. For each point $d \in M_A$, there exists a family of functions $(f_\lambda)_{\lambda \notin \overline{S_\gamma}}$ that makes $(R^\lambda_{\widetilde{g}})_{\lambda \notin \overline{S_\gamma}}$ $\varepsilon-$uniformly bounded at d.*

Proof. We proceed by examining all the possible cases. Throughout the proof ε is any appropriate number in $(0, \pi - \gamma)$ whenever it appears. Also, $b > a$ for the rest of the proof. We proceed in several steps.

Step 1: Let $d = a$ and $c_a \in \overline{S_\gamma} \backslash \{0, \infty\}$. We claim that the family of functions given by

$$f_\lambda(z) := \frac{\lambda}{\lambda - c_a} \frac{b^2 - a^2}{2a} \frac{a + z}{b^2 - z^2}, \quad z \in \mathrm{Dom}(\widetilde{g}),$$

makes $\left(R^\lambda_{\widetilde{g}}\right)_{\lambda \notin \overline{S_\gamma}}$ $\varepsilon-$uniformly bounded at a. Indeed, it follows from Lemma 3.1.4 that $|f_\lambda(z)|$ is uniformly bounded for all $z \in \mathrm{Dom}(f_\lambda)$ and $\lambda \notin \overline{S_{\gamma+\varepsilon}}$. Moreover, $f_\lambda(A) \in L(X)$ with

$$f_\lambda(A) = \frac{\lambda}{\lambda - c_a} \frac{b^2 - a^2}{2a} (a + A)(b^2 - A^2)^{-1},$$

so $\|f_\lambda(A)\|_{L(X)}$ is also uniformly bounded for all $\lambda \notin \overline{S_{\gamma+\varepsilon}}$ (note that $b^2 \notin \sigma(A^2) = \sigma(A)^2$). It is clear that the integrability property (c) in Definition 3.1.3 holds. Finally, we have

$$\begin{aligned} \left|R^\lambda_{\widetilde{g}}(z) - f_\lambda(z)\right| &\leq \left|R^\lambda_{\widetilde{g}}(z) - \frac{\lambda}{\lambda - c_a}\right| + \left|\frac{\lambda}{\lambda - c_a}\right| \left|\frac{b^2 - a^2}{2a} \frac{z + a}{b^2 - z^2} - 1\right| \\ &\lesssim |\widetilde{g}(z) - c_a| + |z - a|, \qquad z \to a. \end{aligned}$$

The estimate for the first summand above is a consequence of Lemma 3.1.5, and the second one is a consequence of Lemma 3.1.4 and Taylor's expansion of order 1. Since $\widetilde{g}$ is regular at a with limit c_a, it follows that $\left|R^\lambda_{\widetilde{g}}(z) - f_\lambda(z)\right|$ satisfies the integrability property (d) in Definition 3.1.3, and the claim is proven.

Step 2: Assume $d = a$ and $c_a = 0$. Since $a \in M_A \cap \widetilde{g}^{-1}(\{0, \infty\})$, by hypothesis, we have $|\widetilde{g}(z)| \sim |z - a|^\alpha$ as $ztoa$ for a real number $\alpha > 0$. Consider the family of functions given by

$$f_\lambda(z) := \frac{|\lambda|^{1/\alpha}}{|\lambda|^{1/\alpha} + a - z} \frac{b^2 - a^2}{2a} \frac{a + z}{b^2 - z^2}, \quad z \in \mathrm{Dom}(\widetilde{g}).$$

Let us show that $(f_\lambda)_{\lambda \notin \overline{S_{\gamma+\varepsilon}}}$ satisfies the desired properties. By Lemma 3.1.4, $|f_\lambda(z)|$ is uniformly bounded for all $z \in \mathrm{Dom}(f_\lambda)$ and $\lambda \notin \overline{S_{\gamma+\varepsilon}}$. Moreover, $f_\lambda(A) \in L(X)$ with

$$f_\lambda(A) = \frac{b^2 - a^2}{2a} \frac{a + A}{b^2 - A^2} (|\lambda|^{1/\alpha})(|\lambda|^{1/\alpha} - (A - a))^{-1}.$$

Since $A \in \mathrm{BSect}(\omega, a)$, one gets that $\|f_\lambda(A)\|_{L(X)}$ is uniformly bounded for all $\lambda \notin \overline{S_{\gamma+\varepsilon}}$. Also, $(f_\lambda)_{\lambda \notin \overline{S_\gamma}}$ satisfies Property (c) in Definition 3.1.3, since $|\lambda|^{1/\alpha}/(|\lambda|^{1/\alpha} + a - z)$ is uniformly bounded, see Lemma 3.1.4.

Let Ω_a be a (small enough) neighborhood of a. On the one hand, by the triangle inequality and various applications of Lemmas 3.1.4 and 3.1.5, we get

$$\begin{aligned}\left|R^\lambda_{\widetilde{g}}(z) - f_\lambda(z)\right| &\le \left|R^\lambda_{\widetilde{g}}(z)\right| + \frac{b^2-a^2}{2a}\left|\frac{a+z}{b^2-z^2}\right|\left|\frac{|\lambda|^{1/\alpha}}{|\lambda|^{1/\alpha}+a-z}\right| \\ &\lesssim |\lambda||\widetilde{g}(z)|^{-1} + |\lambda|^{1/\alpha}|z-a|^{-1} \\ &\lesssim |\lambda^{-1/\alpha}(z-a)|^{-\alpha} + |\lambda^{-1/\alpha}(z-a)|^{-1},\end{aligned}$$

for all $z \in \Omega_a \cap \mathrm{Dom}(\widetilde{g})$ and $\lambda \notin \overline{S_{\gamma+\varepsilon}}$. On the other hand, one has

$$\left|R^\lambda_{\widetilde{g}}(z) - f_\lambda(z)\right| \le |R^\lambda_{\widetilde{g}}(z) - 1| + |f_\lambda(z) - 1|, \quad z \in \Omega_a \cap \mathrm{Dom}(\widetilde{g}, \lambda \notin \overline{S_{\gamma+\varepsilon}}.$$

In addition, Lemma 3.1.5 yields $\left|R^\lambda_{\widetilde{g}}(z) - 1\right| \lesssim |\lambda|^{-1}|\widetilde{g}(z)| \lesssim |\lambda^{-1/\alpha}(z-a)|^\alpha$ for all $z \in \Omega_a \cap \mathrm{Dom}(\widetilde{g})$ and $\lambda \notin \overline{S_{\gamma+\varepsilon}}$. Moreover, one gets

$$\begin{aligned}|f_\lambda(z) - 1| &\le \left|f_\lambda(z) - \frac{b^2-a^2}{2a}\frac{a+z}{b^2-z^2}\right| + \left|\frac{b^2-a^2}{2a}\frac{a+z}{b^2-z^2} - 1\right| \\ &\lesssim \left|\frac{a-z}{|\lambda|^{1/\alpha}+a-z}\right| + |z-a| \lesssim |\lambda^{-1/\alpha}(z-a)| + |z-a|,\end{aligned}$$

for all $z \in \Omega_a \cap \mathrm{Dom}(\widetilde{g})$ and $\lambda \notin \overline{S_{\gamma+\varepsilon}}$. Summarizing, if we set $U_\lambda(z) := |\lambda|^{1/\alpha}|z-a|$, we obtain

$$|R^\lambda_{\widetilde{g}}(z) - f_\lambda(z)| \lesssim \min\left\{\sum_{j\in\{1,\alpha\}} U_\lambda(z)^{-j}, |z-a| + \sum_{j\in\{1,\alpha\}} U_\lambda(z)^j\right\},$$

for all $z \in \Omega_a \cap \mathrm{Dom}(\widetilde{g})$ and $\lambda \notin \overline{S_{\gamma+\varepsilon}}$. Lemma 3.1.6 together with the bound of the resolvent of a bisectorial-like operator, yield that $(f_\lambda)_{\lambda\notin\overline{S_\gamma}}$ satisfies Property (d) in Definition 3.1.3, so in fact $(f_\lambda)_{\lambda\notin\overline{S_\gamma}}$ makes $(R^\lambda_{\widetilde{g}})_{\lambda\notin\overline{S_\gamma}}$ $\varepsilon-$uniformly bounded at a.

Step 3: Now, let $d = a$ and $c_a = \infty$. By hypothesis, there exists $\alpha > 0$ and a neighborhood Ω_a of a such that $|\widetilde{g}(z)| \sim |z-a|^{-\alpha}$ for all $z \in \Omega_a \cap \mathrm{Dom}(\widetilde{g})$. Set

$$f_\lambda(z) := \frac{a-z}{|\lambda|^{-1/\alpha}+a-z}\frac{b^2-a^2}{2a}\frac{a+z}{b^2-z^2}, \quad z \in \mathrm{Dom}(\widetilde{g}).$$

Similar reasoning as in the above cases together with the identity

$$(a-A)(|\lambda|^{-1/\alpha} - (A-a))^{-1} = I - (|\lambda|^{-1/\alpha})(|\lambda|^{-1/\alpha} - (A-a))^{-1},$$

leads to the fact that the family $(f_\lambda)_{\lambda\notin\overline{S_\gamma}}$ satisfies Properties (a), (b) and (c) in Definition 3.1.3. By Lemma 3.1.4, it easily follows that $|f_\lambda(z)| \lesssim |\lambda|^{1/\alpha}|z-a|$. Therefore, the triangle inequality and an application of Lemma 3.1.5 yield

$$\left|R^\lambda_{\widetilde{g}}(z) - f_\lambda(z)\right| \le |\lambda^{1/\alpha}(z-a)|^\alpha + |\lambda^{1/\alpha}(z-a)|,$$

for all $z \in \Omega_a \cap \mathrm{Dom}(\widetilde{g})$ and $\lambda \notin \overline{S_{\gamma+\varepsilon}}$. This implies

$$\begin{aligned}|R^\lambda_{\widetilde{g}}(z) - f_\lambda(z)| &\le \left|R^\lambda_{\widetilde{g}}(z) - \frac{a-z}{|\lambda|^{-1/\alpha}+a-z}\right| + \left|\frac{a-z}{|\lambda|^{-1/\alpha}+a-z}\right|\left|\frac{b^2-a^2}{2a}\frac{a+z}{b^2-z^2}-1\right| \\ &\lesssim \left|R^\lambda_{\widetilde{g}}(z) - \frac{a-z}{|\lambda|^{-1/\alpha}+a-z}\right| + |z-a|,\end{aligned}$$

where we have used again the Taylor expansion of order 1 and the fact that $|(a-z)/(|\lambda|^{-1/\alpha}+a-z)|$ is uniformly bounded. In addition, one gets

$$\begin{aligned}\left|R^\lambda_{\widetilde{g}}(z) - \frac{a-z}{|\lambda|^{-1/\alpha}+a-z}\right| &= \left|\frac{\lambda|\lambda|^{-1/\alpha}+\widetilde{g}(z)(a-z)}{(\lambda-\widetilde{g}(z))(|\lambda|^{-1/\alpha}+a-z)}\right| \\ &\le \left|\frac{\lambda|\lambda|^{-1/\alpha}}{(\lambda-\widetilde{g}(z))(|\lambda|^{-1/\alpha}+a-z)}\right| + \left|\frac{\widetilde{g}(z)(a-z)}{(\lambda-\widetilde{g}(z))(|\lambda|^{-1/\alpha}+a-z)}\right| \\ &\lesssim |\lambda^{1/\alpha}(z-a)|^{-1} + |\lambda^{1/\alpha}(z-a)|^{-\alpha},\end{aligned}$$

for all $z \in \Omega_a \cap \mathrm{Dom}(\widetilde{g})$ and $\lambda \notin \overline{S_{\gamma+\varepsilon}}$, where we have used various applications of Lemmas 3.1.4 and 3.1.5 in the last step. Finally, reasoning as in step 2, one obtains that $(f_\lambda)_{\lambda\notin\overline{S_\gamma}}$ also satisfies Property (d) in Definition 3.1.3.

Step 4: Let $d \in \{-a,\infty\} \cap M_A$, and assume $d \in \overline{S_\gamma} \setminus 0,\infty$. Consider the family of functions $(f_\lambda)_{\lambda\notin\overline{S_\gamma}}$ given by

$$\begin{aligned}f_\lambda(z) &:= \frac{\lambda}{\lambda-c_{-a}}\frac{b^2-a^2}{2a}\frac{a-z}{b^2-z^2}, \quad z\in\mathrm{Dom}(\widetilde{g}), \quad \text{if } d=-a, \\ f_\lambda(z) &:= \frac{\lambda}{\lambda-c_\infty}\frac{a^2-z^2}{b^2-z^2}, \quad z\in\mathrm{Dom}(\widetilde{g}), \quad \text{if } d=\infty.\end{aligned}$$

Following a similar reasoning as in Step 1, one gets that $(f_\lambda)_{\lambda\notin\overline{S_\gamma}}$ makes $(R^\lambda_{\widetilde{g}})_{\lambda\notin\overline{S_\gamma}}$ $\varepsilon-$uniformly bounded at d.

Step 5: Let $d \in M_A \cap \{-a,\infty\}$ and assume $c_d = 0$. By hypothesis, there exists $\alpha > 0$ such that $\widetilde{g}(z) \to 0$ exactly polynomially of order α as $z \to d$. Proceeding as in Step 2, one has that the family of functions $(f_\lambda)_{\lambda\notin\overline{S_\gamma}}$ given by

$$\begin{aligned}f_\lambda(z) &:= \frac{|\lambda|^{1/\alpha}}{|\lambda|^{1/\alpha}+a+z}\frac{b^2-a^2}{2a}\frac{a-z}{b^2-z^2}, \quad z\in\mathrm{Dom}(\widetilde{g}), \quad \text{if } d=-a, \\ f_\lambda(z) &:= \frac{b-z}{|\lambda|^{-1/\alpha}+b-z}\frac{a^2-z^2}{b^2-z^2}, \quad z\in\mathrm{Dom}(\widetilde{g}), \quad \text{if } d=\infty,\end{aligned}$$

satisfies the desired properties.

Step 6: Finally, let $d \in M_A \cap \{-a, \infty\}$ with $c_d = \infty$. By hypothesis, there exists $\alpha > 0$ such that $\widetilde{g}(z) \to \infty$ exactly polynomially of order α as $z \to d$. An analogous reasoning as the one of Step 3 yields that the family of functions $(f_\lambda)_{\lambda \notin \overline{S_\gamma}}$ given by

$$f_\lambda(z) := \frac{a+z}{|\lambda|^{-1/\alpha} + a + z} \frac{b^2 - a^2}{2a} \frac{a-z}{b^2 - z^2}, \quad z \in \mathrm{Dom}(\widetilde{g}), \quad \text{if } d = -a,$$
$$f_\lambda(z) := \frac{|\lambda|^{1/\alpha}}{|\lambda|^{1/\alpha} + b - z} \frac{a^2 - z^2}{b^2 - z^2}, \quad z \in \mathrm{Dom}(\widetilde{g}), \quad \text{if } d = \infty,$$

makes $(R_{\widetilde{g}}^\lambda)_{\lambda \notin \overline{S_\gamma}}$ ε-uniformly bounded at d. The proof is complete. □

Remark 3.1.8. For each $d \in M_A$, let $(f_{d,\lambda})_{\lambda \notin \overline{S_\gamma}}$ be a family of functions that makes $\left(R_{\widetilde{g}}^\lambda\right)_{\lambda \notin \overline{S_\gamma}}$ $\varepsilon-$uniformly bound at d. From the bounds appearing in the proof of Proposition 3.1.7, one obtains $\left(R_{\widetilde{g}}^\lambda - \sum_{d \in M_A} f_{d,\lambda}\right) \in \mathcal{E}_0(A)$ for all $\lambda \notin \overline{S_\gamma}$.

We are now ready to give the main result of this section.

Theorem 3.1.9. *Let $(\omega, a) \in (0, \pi/2] \times [0, \infty)$ and $\beta \in [0, \pi)$. Let $A \in \mathrm{BSect}(\omega, a)$ in a Banach space X and $g \in \mathcal{M}(A)$. Assume the following:*

1. *For each $\gamma > \beta$, there exists $\theta \in (0, \omega)$ such that $g(BS_{\theta,a}) \subseteq \overline{S_\gamma} \cup \{\infty\}$.*
2. *g is quasi-regular at M_A.*
3. *g has exactly polynomial limits at $M_A \cap g^{-1}(\{0, \infty\})$.*

Then, $g(A)$ is a sectorial operator of angle β.

Proof. Our result follows once we prove that $g(A)$ is a sectorial operator of angle γ for all $\gamma > \beta$. Indeed, if this is true, one has

$$\beta \geq \inf_{\gamma \in [0,\pi)} \{\gamma : g(A) \text{ is sectorial of angle } \gamma\},$$

which implies that $g(A)$ is a sectorial operator of angle β, see [Haa06, Section 2.1].

Let $\gamma > \beta$ and set $\widetilde{g} := g|_{BS_{\theta,a}}$, where $\theta \in (0, \omega)$ is chosen such that $\mathrm{Ran}(\widetilde{g}) \subseteq \overline{S_\gamma} \cup \{\infty\}$. Notice that $\widetilde{g} \in \mathcal{M}(A)$ with $\widetilde{g}(A) = g(A)$. Now, the spectral inclusion $\widetilde{\sigma}(\widetilde{g}(A)) \subseteq \overline{S_\gamma} \cup \{\infty\}$ holds since $\widetilde{\sigma}(\widetilde{g}(A)) = \widetilde{g}(\widetilde{\sigma}(A)) \subseteq \overline{S_\gamma} \cup \infty$, see Theorem 2.2.17. It remains to prove the bound for the resolvent. More precisely, we have to show that for all $\varepsilon > 0$,

$$\sup_{\lambda \notin \overline{S_{\gamma+\varepsilon}}} \left\| \lambda (\lambda - \widetilde{g}(A))^{-1} \right\|_{L(X)} < \infty.$$

By Lemma 2.1.4, it follows that $\lambda(\lambda - \widetilde{g}(A))^{-1} = R^{\lambda}_{\widetilde{g}}(A)$ for all $\lambda \notin \overline{S_\gamma}$. Also, by Proposition 3.1.7, we have that for each $d \in M_A$, there exist some families of functions $(f_{d,\lambda})_{\lambda\notin\overline{S_\gamma}}$ which make $\left(R^{\lambda}_{\widetilde{g}}\right)_{\lambda\notin\overline{S_\gamma}}$ ε-uniformly bounded at d. Moreover,

$$\left\|\lambda(\lambda - \widetilde{g}(A))^{-1}\right\|_{L(X)} \leq \left\|\left(R^{\lambda}_{\widetilde{g}} - \sum_{d\in M_A} f_{d,\lambda}\right)(A)\right\|_{L(X)} + \sum_{d\in M_A} \|f_{d,\lambda}(A)\|_{L(X)}. \tag{3.5}$$

By Property (c) in Definition 3.1.3, one has that $\sup_{\lambda\notin\overline{S_{\gamma+\varepsilon}}} \|f_{d,\lambda}(A)\|_{L(X)} < \infty$ for each $d \in M_A$. It remains to uniformly bound the first term of the right hand side in (3.5).

Let Γ be an integration path for the (regularized) functional calculus of A, see Section 2.1, and let $(\Omega_d)_{d\in M_A}$ be some appropriate open sets for which $d \in \Omega_d$ and the uniform integral bounds of Definition 3.1.3 hold for each $(f_{d,\lambda})_{\lambda\notin\overline{S_\gamma}}$. Since $\left(R^{\lambda}_{\widetilde{g}} - \sum_{d\in M_A} f_{d,\lambda}\right) \in \mathcal{E}_0(A)$ (see Remark 3.1.8), one has

$$\left\|\left(R^{\lambda}_{\widetilde{g}} - \sum_{d\in M_A} f_{d,\lambda}\right)(A)\right\|_{L(X)} \leq \int_\Gamma \left|R^{\lambda}_{\widetilde{g}}(z) - \sum_{d\in M_A} f_{d,\lambda}(z)\right| \|(z-A)^{-1}\|_{L(X)}\, |dz|. \tag{3.6}$$

Now, we split the integral on Γ to the sum of integrals on $\Gamma \cap \Omega_d$ for each $d \in M_A$, and $\Gamma\setminus(\cup_{d\in M_A}\Omega_d)$. Notice that by Property (b) in Definition 3.1.3 and Lemma 3.1.5, $|f_{d,\lambda}(z)|$ and $\left|R^{\lambda}_{\widetilde{g}}(z)\right|$ are uniformly bounded in $\Gamma\setminus(\cup_{d\in M_A}\Omega_d)$. Thus,

$$\begin{aligned}
&\sup_{\lambda\notin\overline{S_{\gamma+\varepsilon}}} \int_{\Gamma\setminus(\cup_{d\in M_A}\Omega_d)} \left|R^{\lambda}_{\widetilde{g}}(z) - \sum_{d'\in M_A} f_{d',\lambda}(z)\right| \|(z-A)^{-1}\|_{L(X)}\, |dz| \\
&\lesssim \int_{\Gamma\setminus(\cup_{d\in M}\Omega_d)} \|(z-A)^{-1}\|_{L(X)}\, |dz| < \infty.
\end{aligned}$$

Finally, for each $d \in M_A$, one has

$$\begin{aligned}
&\sup_{\lambda\notin\overline{S_{\gamma+\varepsilon}}} \int_{\Gamma\cap\Omega_d} \left|R^{\lambda}_{\widetilde{g}}(z) - \sum_{d'\in M_A} f_{d',\lambda}(z)\right| \|(z-A)^{-1}\|_{L(X)}\, |dz| \\
&\leq \sup_{\lambda\notin\overline{S_{\gamma+\varepsilon}}} \sum_{d'\in M_A\setminus\{d\}} \int_{\Gamma\cap\Omega_d} |f_{d',\lambda}(z)| \|(z-A)^{-1}\|_{L(X)}\, |dz| \\
&\qquad + \sup_{\lambda\notin\overline{S_{\gamma+\varepsilon}}} \int_{\Gamma\cap\Omega_d} \left|R^{\lambda}_{\widetilde{g}}(z) - f_{d,\lambda}(z)\right| \|(z-A)^{-1}\|_{L(X)}\, |dz|.
\end{aligned}$$

The two supremums of the integrals above are finite by Properties (c) and (d) in Definition 3.1.3. Combining these estimates with (3.5)-(3.6), we get the bound for the resolvent, and as a consequence $g(A)$ is a sectorial operator of angle γ for all $\gamma > \beta$. Hence, the proof is finished. □

The following corollaries are immediate consequences of Theorem 3.1.9. The first one has been already shown in [BHK09, Th. 4.6] for the particular case where $-A$ generates a bounded group; that is, $A \in \mathrm{BSect}(\pi/2, 0)$.

Corollary 3.1.10. *Let $a \geq 0$, $A \in \mathrm{BSect}(\pi/2, a)$, and let $\alpha > 0$ with α not an odd number, so $\alpha \in (2n-1, 2n+1)$ for a unique $n \in \mathbb{N}$. Then, for any $\gamma > \pi \left|\frac{\alpha}{2} - n\right|$, there exists $\rho \geq 0$ such that $\rho I + (-1)^n (A+a)^\alpha$ is a sectorial operator of angle γ. Moreover, if $a = 0$ then we can take $\rho = 0$.*

Corollary 3.1.11. *Let $0 < \omega \leq \frac{\pi}{2}$ and $a \geq 0$. Let $A \in \mathrm{BSect}(\omega, a)$ in a Banach space X and $g \in \mathcal{M}(A)$. Assume there are $\beta \in [\pi/2, \pi)$ and $b \geq 0$ such that the following hold:*

1. *For each $\gamma \in (0, \beta)$, there exists $\theta \in (0, \omega)$ for which $g(BS_{\theta,a}) \subseteq \overline{BS}_{\gamma,b}$.*
2. *g is quasi-regular at M_A.*
3. *g has exactly polynomial limits at $M_A \cap g^{-1}(\{-b, b, \infty\})$.*

Then, $g(A) \in \mathrm{BSect}(\beta, b)$.

3.1.A Extension to sectorial operators

Similarly to Chapter 2, most proofs we have presented here are generic, and as a consequence, the results shown here hold for functional calculus analogous to the one presented in Section 2.1. In particular, in the setting of sectorial operators, one has the extension of the scaling property given below. We refer the reader to [Haa05a; Haa06] for the definition of the regularized functional calculus of sectorial operators.

Theorem 3.1.12. *Let $0 \leq \omega < \pi$, $\beta \in [0, \pi)$, A a sectorial operator of angle ω in a Banach space X, and g in the domain of the regularized functional calculus of A. Assume the following items hold:*

1. *For each $\gamma > \beta$, there exists $\theta \in (\omega, \pi)$ such that $g(S_\theta) \subseteq \overline{S_\gamma} \cup \{\infty\}$.*
2. *g is quasi-regular at $\{0, \infty\} \cap \widetilde{\sigma}(A)$.*
3. *g has exactly polynomial limits at $\{0, \infty\} \cap \widetilde{\sigma}(A) \cap g^{-1}(\{0, \infty\})$.*

Then, $g(A)$ is sectorial of angle β.

3.2 Generation results of holomorphic semigroups and their properties

Because of the bijection between generators of bounded holomorphic semigroups and sectorial operators (see Section 1.2) and the results obtained in Section 3.1, we obtain the following corollary.

Corollary 3.2.1. *Let A, β, g be as in Theorem 3.1.9. In addition, assume that $\beta \in [0, \pi/2)$. Then, $-g(A)$ generates a bounded holomorphic semigroup T_g of angle $\frac{\pi}{2} - \beta$.*

Proof. This is an immediate consequence of Theorem 3.1.9 and the fact that an operator B is sectorial of angle $\beta < \frac{\pi}{2}$ if and only if $-B$ is the generator of a bounded holomorphic semigroup of angle $\frac{\pi}{2} - \beta$, see for example [EN00, Th. 4.6] or [Haa06, Prop. 3.4.4]. □

Under the hypothesis of the corollary above, it seems natural to study the properties related to the holomorphic semigroup generated by $-g(A)$.

Recall that the space of strong continuity $\mathbb{D}_T$ of a (holomorphic) semigroup T generated by A is precisely $\overline{\mathrm{Dom}(A)}$. The following result characterizes the space $\mathbb{D}_T$ in our setting. Let us point out that the result holds even if the angle of sectoriality β of $g(A)$ is greater or equal than $\frac{\pi}{2}$.

Proposition 3.2.2. *Let A, g be as in Theorem 3.1.9. If $g^{-1}(\infty) \cap M_A = \emptyset$, then $\overline{\mathrm{Dom}(g(A))} = X$. Otherwise,*

$$\overline{\mathrm{Dom}(g(A))} = \bigcap_{d \in g^{-1}(\infty) \cap M_A} \overline{\mathrm{Ran}(d - A)},$$

where we set $\mathrm{Ran}(\infty I - A) := \mathrm{Dom}(A)$.

Proof. Notice that, if $g^{-1}(\infty) \cap M_A = \emptyset$, then $g^{-1}(\infty) = \emptyset$ (see Remark 3.1.2). Thus, by the spectral mapping theorem (Theorem 2.2.17), $g(A) \in L(X)$ and $\mathrm{Dom}(g(A)) = X$, so we are done in this case.

Assume now $g^{-1}(\infty) \neq \emptyset$. We can assume without loss of generality that $g^{-1}(\infty) \subseteq M_A$, see Remark 3.1.2. Let $d \in M_A$ with $g(d) \neq \infty$, so g is regular at d. For any $b > a$, consider

$$f_d(z) := \begin{cases} g(a)\dfrac{b^2 - a^2}{2a}\dfrac{a+z}{b^2 - z^2}, & \text{if } d = a, \\ g(-a)\dfrac{b^2 - a^2}{2a}\dfrac{a-z}{b^2 - z^2}, & \text{if } d = -a, \\ g(\infty)\dfrac{a^2 - z^2}{b^2 - z^2}, & \text{if } d = \infty. \end{cases}$$

Then, $g - f_d$ is regular at d with limit 0, and the behavior of $g - f_d$ at $M_A \backslash \{d\}$ remains the same as the behavior of g at these points. Moreover, since $f_d(A) \in L(X)$, one has

$$\mathrm{Dom}(g(A)) = \mathrm{Dom}(g_\bullet(A)) \quad \text{where} \quad g_\bullet(z) := g(z) - \sum_{d \notin g^{-1}(\infty) \cap M_A} f_d(z).$$

Thus, we can assume that g has regular limits equal to 0 at $M_A \backslash g^{-1}(\infty)$.

Now, we proceed by showing both inclusions $\subseteq, \supseteq$ of the statement, starting with the latter one. For all $t > 0$ small enough (for which $b \notin \sigma(tA)$), set

$$h_t(z) := \frac{(a-z)^{n_a}(a+z)^{n_{-a}} b^{n_\infty}}{(t+a-z)^{n_a}(t+a+z)^{n_{-a}}(b+tz)^{n_\infty}}, \quad z \in \mathrm{Dom}(g), \tag{3.7}$$

with $n_d = 0$ if $g(d) = 0$ and, if $g(d) = \infty$, take $n_d \in \mathbb{N}$ large enough so that $h_t g \in \mathcal{E}(A)$. Then $h_t g(A) \in L(X)$ with $\mathrm{Dom}((h_t g)(A)) = X$. Also, h_t is injective since $\sigma_p(A) \cap g^{-1}(\infty) = \emptyset$ (see Lemma 2.2.19), hence $h_t^{-1} \in \mathcal{M}(A)$. Therefore, $g(A) \supseteq (h_t g)(A) h_t^{-1}(A)$, which implies that $\mathrm{Dom}(g(A)) \supseteq \mathrm{Dom}(h_t^{-1}(A)) = \mathrm{Ran}(h_t(A))$ for all $t > 0$ small enough. In addition, since both $a + A$ and $a - A$ are sectorial operators, we have

$$\lim_{t \to 0} h_t(A)x = x, \quad \text{for all } x \in \bigcap_{d \in g^{-1}(\infty) \cap M_A} \overline{\mathrm{Ran}(d - A)},$$

see [Haa06, Prop. 2.1.1 (c)]. This implies

$$\bigcap_{d \in g^{-1}(\infty) \cap M_A} \overline{\mathrm{Ran}(d - A)} \subseteq \overline{\mathrm{Ran}(h_t(A))} \subseteq \overline{\mathrm{Dom}(g(A))}, \tag{3.8}$$

so we have proven the inclusion $\supseteq$ of the claim.

Let us prove the reverse inclusion $\subseteq$. If $\infty \in g^{-1}(\infty) \cap M_A$, then $|g(z)| \sim |z|^\alpha$ as $z \to \infty$ for some $\alpha > 0$. It follows that $(1+g(z))^{-1}$ regularizes $(z+a)^{\alpha'}$ for all $\alpha' \in (0, \alpha)$. This implies $\mathrm{Dom}(g(A)) \subseteq \mathrm{Dom}((A+a)^{\alpha'})$. Reasoning similarly with $-a, a$, one obtains, for all $\alpha' > 0$ small enough,

$$\mathrm{Dom}(g(A)) \subseteq \bigcap_{d \in g^{-1}(\infty) \cap M_A} \mathrm{Ran}((d - A)^{\alpha'}),$$

where $\overline{\mathrm{Ran}((\infty I - A)^{\alpha'})} := \mathrm{Dom}((a + A)^{\alpha'})$. Then, our proof is finished if we show $\overline{\mathrm{Ran}((d - A)^{\alpha'})} \subseteq \overline{\mathrm{Ran}(d - A)}$.

So assume $d = \infty$. It follows from Theorem 3.1.9 that $(a+A)^{\alpha'}$ is a sectorial operator for a small enough $\alpha' > 0$. Moreover, $a + A$ is also a sectorial operator, and $(a + A)^{\alpha'} = f_{\alpha'}(a+A)$, where $f_{\alpha'}(z) = z^{\alpha'}$, and where we consider the regularized functional calculus of sectorial operators (see [Haa06, Section 2.3]). Then, by the composition rule for sectorial operators (see e.g. [Haa06, Th. 2.4.2]), one has that $f_{1/\alpha'}((a + A)^{\alpha'}) = a + A$. Now, in the setting of sectorial operators, one can prove an analogous inclusion to (3.8) by mimicking the arguments given here. Applying this inclusion to $f_{\alpha'}$ and $(a + A)^{\alpha'}$, one has

$$\overline{\mathrm{Dom}(a + A)} = \overline{\mathrm{Dom}((f_{1/\alpha'})((a + A)^{\alpha'}))} \supseteq \overline{\mathrm{Dom}((a + A)^{\alpha'})},$$

as we wanted to prove. The cases $d \in \{-a, a\}$ are solved in an analogous way, by using the operators $(a + A)^{-\alpha'}, (a - A)^{-\alpha'}$, respectively. Thus, the proof is finished. □

As a consequence of the theorem above, we have

Corollary 3.2.3. *Let A, g be as in Theorem 3.1.9. If X is reflexive, then* $\overline{\mathrm{Dom}(g(A))} = X$.

Proof. By [Haa06, Prop. 2.1.1 (h)], one has that

$$X = \overline{\mathrm{Dom}(A)} = \mathcal{N}(a - A) \oplus \overline{\mathrm{Ran}(a - A)} = \mathcal{N}(a + A) \oplus \overline{\mathrm{Ran}(a + A)}$$

if X is reflexive. Since $\sigma_p(A) \cap g^{-1}(\infty) = \emptyset$ (see Lemma 2.2.19), the statement follows by Proposition 3.2.2. □

For each $w \in \mathbb{C}$, set the function $\exp_{-w}(z) := \exp(-wz)$, $z \in \mathbb{C}$. Under the hypothesis of Corollary 3.2.1, one has that the semigroup generated by $-g(A)$, T_g, is given by $T_g(w) = \exp_{-w}(g(A))$, see [Haa06, Prop. 3.4.4]. Thus, it seems natural to conjecture that $T_g(w) = (\exp_{-w} \circ g)(A)$. The theorem below answers this question positively. Its proof is inspired by the composition rule for sectorial operators given in [Haa05a], but carefully adapted to cover all our cases. Indeed, one could easily generalize the result below to a composition rule from bisectorial-like to sectorial operators, addressing a larger class of functions. However, this would require to introduce several new definitions and additional cumbersome notations. Thus, for the sake of clarity, we limit the result below to the exponential function.

Theorem 3.2.4. *Let β, A, g be as in Corollary 3.2.1, so that $-g(A)$ generates a bounded holomorphic semigroup T_g of angle $\frac{\pi}{2} - \beta$. Then, for every $w \in S_{\pi/2-\beta}$, we have $\exp_{-w} \circ g \in \mathcal{M}(A)$ and*

$$T_g(w) = (\exp_{-w} \circ g)(A). \tag{3.9}$$

Proof. First of all, the claim is trivial if $g = 0$, so we can assume $g \neq 0$. Fix $w \in S_{\pi/2-\beta}$. Then, it is straightforward to check that $\exp_{-w} \circ g$ is regular at M_A, so Lemma 2.1.7 yields $\exp_{-w} \circ g \in \mathcal{M}(A)$.

Fix $w \in S_{\pi/2-\beta}$ for the rest of the proof, and set $f_w(z) := \exp_{-w}(z) - (1+z)^{-1}$. Then, f_w has regular limits equal to 0 in $\{0, \infty\}$. As $-1 \notin \sigma(g(A)) \subseteq \overline{S_\beta}$, Lemma 2.1.4 (f) yields $f_w \circ g \in \mathcal{M}(A)$ and $(I+g)^{-1}(A) = (I + g(A))^{-1}$. Therefore, our statement follows if we prove the identity $(f_w \circ g)(A) = f_w(g(A))$.

Recall that, for $\mu \in \mathbb{C}_\infty$, we denote by c_μ the limit of $g(z)$ as $z \to \mu$ whenever it exists. In particular, c_μ exists if $\mu \in \sigma_p(A)$, see Lemma 2.2.19. Fix $b > a$, and for each $\lambda \notin \overline{S_\beta}$, set

$$G_\lambda(z) := \frac{1}{\lambda - g(z)} - \sum_{d \in \sigma_p(A) \cap \{-a, a\}} \frac{1}{\lambda - c_d} \frac{z+d}{b-z} \frac{b-d}{2d}, \quad z \in \operatorname{Dom}(g).$$

Since $\lambda \in \rho(A)$, one has $G_\lambda \in \mathcal{M}(A)$. Moreover, $|G_\lambda|$ is uniformly bounded in $\operatorname{Dom}(g)$ with $G_\lambda(d) = 0$ for all $d \in \sigma_p(A) \cap \{-a, a\}$. Furthermore, it is readily seen that there exists a regularizer $e \in \mathcal{E}(A)$ for G_λ for every $\lambda \notin \overline{S_\beta}$. That is, $e(A)$ is bounded and injective and $eG_\lambda \in \mathcal{E}(A)$ for all $\lambda \notin \overline{S_\beta}$. Moreover, we can assume $eG_\lambda \in \mathcal{E}_0(A)$, $\lambda \notin \overline{S_\beta}$, i.e., eG_λ is regular with limits equal to 0 in M_A. To see this, note that the key point is the regularity of eG_λ at M_A. If $d' \in M_A$ with $d' \notin \sigma_{point(A)}$, one can add to e powers of the function $(z-d')/(z-b)^2$ if $d' \notin \sigma_p(A)$. If $d' \in M_A \cap \sigma_{point}(A)$, the regularity is obtained by the bounds in Lemma 3.1.5 (recall that in this case, $c_{d'} \neq \infty$ by Lemma 2.2.19).

Then, let Γ' be a path for the regularized functional calculus of the sectorial operator

$g(A)$ (see [Haa06, Section 2.5]. One gets

$$\begin{aligned} f_w(g(A)) &= e(A)^{-1}e(A)f_w(g(A)) \\ &= e(A)^{-1}\frac{1}{2\pi i}\int_{\Gamma'} f_w(\lambda)e(A)(\lambda - g(A))^{-1}\, d\lambda \\ &= e(A)^{-1}\frac{1}{2\pi i}\int_{\Gamma'} f_w(\lambda)(eG_\lambda)(A)\, d\lambda \\ &\quad + \sum_{d\in\sigma_p(A)\cap\{-a,a\}} \frac{b-d}{2d}(d+A)(b-A)^{-1}\ \frac{1}{2\pi i}\int_{\Gamma'} \frac{f_w(\lambda)}{\lambda - c_d}\, d\lambda. \end{aligned}$$

By Cauchy's integral theorem, one has that the last of the above terms is precisely

$$\sum_{d\in\sigma_p(A)\cap\{-a,a\}} \frac{b-d}{2d} f_w(c_d)(d+A)(b-A)^{-1}.$$

Now, let Γ be a path of the regularized functional calculus of the bisectorial-like operator A. Since $eG_\lambda \in \mathcal{E}_0(A)$, one has

$$\begin{aligned} &e(A)^{-1}\frac{1}{2\pi i}\int_{\Gamma'} f_w(\lambda)(e(z)G_\lambda(z))(A)\, d\lambda \\ =&e(A)^{-1}\frac{1}{(2\pi i)^2}\int_{\Gamma'} f_w(\lambda)\int_\Gamma e(z)G_\lambda(z)(z-A)^{-1}\, dz d\lambda \\ =&e(A)^{-1}\frac{1}{(2\pi i)^2}\int_{\Gamma} e(z)(z-A)^{-1}\int_{\Gamma'} f_w(\lambda)G_\lambda(z)\, d\lambda dz. \end{aligned}$$

Let us go on with the proof before checking the hypothesis for Fubini's theorem that we have applied in the last equality above. By Cauchy's theorem, it follows that

$$\frac{1}{2\pi i}\int_{\Gamma'} f_w(\lambda)G_\lambda(z)\, d\lambda = f_w(g(z)) - \sum_{d\in\sigma_p(A)\cap\{-a,a\}} f_w(c_d)\frac{z+d}{b-z}\frac{b-d}{2d}.$$

From this, we can conclude

$$\begin{aligned} &e(A)^{-1}\frac{1}{(2\pi i)^2}\int_{\Gamma} e(z)(z-A)^{-1}\int_{\Gamma} f_w(\lambda)G_\lambda(z)\, d\lambda dz \\ &\quad = (f_w\circ g)(A) - \sum_{d\in\sigma_p(A)\cap\{-a,a\}} \frac{b-d}{2d} f_w(c_d)(d+A)(b-A)^{-1}, \end{aligned}$$

and our assertion follows.

Let us check now that indeed Fubini's theorem can be applied. To do this, we have to check the integrability of the function

$$F(\lambda, z) := \frac{f_w(\lambda)}{\lambda}\lambda G_\lambda(z)\frac{e(z)}{\min\{|z-a|, |z+a|\}},$$

on $\Gamma' \times \Gamma$. First, $f_w(\lambda)/\lambda$ is clearly integrable on Γ' and, by Lemma 3.1.5, $\lambda G_\lambda(z)$ is uniformly bounded on $\Gamma' \times \Gamma$. Now, one can assume that $\frac{e(z)}{\min\{|z-a|,|z+a|\}}$ is integrable on Γ if $\{-a, a\} \cap \sigma_p(A) = \emptyset$. Otherwise, let $d \in \sigma_p(A) \cap \{-a, a\}$. Recall that in this case, $c_d \in \overline{S_\beta}$ with $c_d \neq \infty$. If $c_d \neq 0$, then λG_λ is of the same type as the functions (f_λ) appearing in Step 1 in the proof of Proposition 3.1.7. Using a similar argument as the one used there, one obtains that the function

$$\lambda G_\lambda \frac{e}{\min\{|(\cdot) - a|, |(\cdot) + a|\}}$$

is integrable on Γ for every $\lambda \in \Gamma'$, and that the norm of its integral value is uniformly bounded for all $\lambda \in \Gamma'$.

So assume $c_d = 0$. Thus, one has

$$|\lambda G_\lambda(z)| \lesssim \frac{|g(z)|}{|\lambda - g(z)|} + |z-d|, \qquad \text{as } z \to d,$$

where the $|z-d|$ term is the result of applying a Taylor expansion of order 1 in a similar way as in Step 2 in the proof of Proposition 3.1.7. It is readily seen that the $|z-d|$ term does not entangle the bound of $F(\lambda, z)$. Moreover, for any $\delta \in (0,1)$, one has that

$$\left| \frac{f_w(\lambda)}{\lambda} \frac{g(z)}{\lambda - g(z)} \frac{e(z)}{z-d} \right| = \left| \frac{f_w(\lambda)}{\lambda^{1+\delta}} \right| \left| \frac{\lambda^\delta g(z)^{1-\delta}}{\lambda - g(z)} \right| \left| \frac{(eg^\delta)(z)}{z-d} \right|.$$

It is easy to see that $f_w(\lambda)/\lambda^{1+\delta}$ is still integrable on Γ', and that the middle term is uniformly bounded. Moreover, since $c_d = 0$, we have by hypothesis that $|g(z)| \sim |z-d|^\alpha$ as $z \to d$ for some $\alpha > 0$. Thus $g^\delta(z) \lesssim |z-d|^{\alpha\delta}$, so the last term is integrable in Γ, and the proof is finished. □

3.3 Generalized Black-Scholes equations on interpolation spaces

Here, we apply the theory developed in the preceding sections to introduce and study generalized Black–Scholes equations on $(L^1 - L^\infty)$-interpolation spaces. Let us start with the definition of interpolation space.

3.3.A Fractional Cesàro operators on $(L^1 - L^\infty)$ interpolation spaces

Let X be a functional Banach space for which the inclusions $(L^1(0,\infty) \cap L^\infty(0,\infty)) \subseteq X \subseteq (L^1(0,\infty) + L^\infty(0,\infty))$ hold and are continuous. We say that X is a **$(L^1 - L^\infty)$-interpolation space** if, for every linear operator $S : (L^1(0,\infty) + L^\infty(0,\infty)) \to (L^1(0,\infty) + L^\infty(0,\infty))$ that restricts to bounded operators $S|_{L^1(0,\infty)} : L^1(0,\infty) \to L^1(0,\infty)$, $S|_{L^\infty(0,\infty)} : L^\infty(0,\infty) \to L^\infty(0,\infty)$, we have that its the restriction to X, $S|_X : X \to X$, is well defined and bounded. This class includes many of the classical function spaces (e.g. L^p-spaces, Orlicz spaces, Lorenz spaces, Marcinkiewiecz spaces).

Also, X is said to have an **order continuous norm** if $\|f_n\|_X \to 0$ for every sequence of functions $f_n \in X$ converging to 0 almost everywhere and for which $|f_n|$ is non-increasing. For more details about $(L^1 - L^\infty)$-interpolation spaces, we refer to the monograph [BS88].

Throughout the following, without any mention, X denotes a $(L^1 - L^\infty)$-interpolation space on $(0, \infty)$. We recall that

$$(E_X(t)f)(x) := f(e^{-t}x), \qquad x > 0,\ t \in \mathbb{R},\ f \in X,$$

defines a group of bounded operators $E_X = (E_X(t))_{t\in\mathbb{R}}$ on X with $\|E_X(t)\|_{L(X)} \le \max\{1, e^t\}$ for $t \in \mathbb{R}$, which is strongly continuous if and only X has order continuous norm. Then, the lower and upper Boyd indices $\underline{\eta}_X, \overline{\eta}_X$ are defined by

$$\underline{\eta}_X := -\lim_{t\to\infty} \frac{\log \|E_X(-t)\|_{L(X)}}{t}, \quad \overline{\eta}_X := \lim_{t\to\infty} \frac{\log \|E_X(t)\|_{L(X)}}{t},$$

and they satisfy $0 \le \underline{\eta}_X \le \overline{\eta}_X \le 1$. By [AP02, Th. 2.4], $(E_X(t))$ is strongly continuous if and only if X has order continuous norm.

Now, define the operator Q_X by

$$\begin{cases} \mathrm{Dom}(Q_X) = \Big\{ f \in X \, : f \in AC_{\mathrm{loc}}(0,\infty) \text{ and } -xf'(x) \in X \Big\}, \\ (Q_X f)(x) := -xf'(x), \quad x > 0,\ f \in \mathrm{Dom}(Q_X), \end{cases} \tag{3.10}$$

and the operators $\widetilde{E}_+(t), \widetilde{E}_-(t)$, $t \ge 0$, by

$$(\widetilde{E}_+(t)f)(x) := \int_0^t (E(s)f)(x)\, ds, \quad (\widetilde{E}_-(t)f)(x) := \int_0^t (E(-s)f)(x)\, ds, \quad x > 0,\ f \in X.$$

It was shown in [AP02] that $(\widetilde{E}_+(t))$ and $(\widetilde{E}_-(t))$ are integrated semigroups on X generated by A_X and $-A_X$ respectively. In particular, the resolvent identity holds pointwise, i.e., for $f \in X$ and a.e. $x > 0$,

$$((\lambda - Q_X)^{-1} f)(x) = \int_0^\infty e^{-\lambda t} (E_X(t)f)(x)\, dt, \quad \mathfrak{Re}\,\lambda > 1,$$

$$((\lambda - Q_X)^{-1} f)(x) = -\int_{-\infty}^0 e^{-\lambda t} (E_X(t)f)(x)\, dt, \quad \mathfrak{Re}\,\lambda < 0.$$

Also, the resolvent of Q_X satisfies the bounds $\|(\lambda - Q_X)^{-1}\|_{L(X)} \le (\mathfrak{Re}\,\lambda - 1)^{-1}$ if $\mathfrak{Re}\,\lambda > 1$ and $\|(\lambda - Q_X)^{-1}\|_{L(X)} \le (-\mathfrak{Re}\,\lambda)^{-1}$ if $\mathfrak{Re}\,\lambda < 0$. Moreover, they also obtained in [AP02, Th. 4.2] the spectrum of Q_X, given by

$$\sigma(Q_X) = \{\lambda \in \mathbb{C} \, : \, \underline{\eta}_X \le \mathfrak{Re}\,\lambda \le \overline{\eta}_X\}.$$

Hence, for each $\underline{\varepsilon}, \overline{\varepsilon} > 0$, both $\left(\underline{\eta}_X + \underline{\varepsilon}\right) I + Q_X$ and $(\overline{\eta}_X + \overline{\varepsilon})\, I - Q_X$ are sectorial operators of angle $\frac{\pi}{2}$. Notice that one may take $\underline{\varepsilon} = \overline{\varepsilon} = 0$ if $\underline{\eta}_X = 0$ and $\overline{\eta}_X = 1$, respectively. Therefore, $Q_X - \frac{\overline{\eta}_X + \underline{\eta}_X + \overline{\varepsilon} - \underline{\varepsilon}}{2} I$ is a bisectorial-like operator of angle $\pi/2$

and half-width $\frac{\overline{\eta}_X - \underline{\eta}_X + \overline{\varepsilon} + \varepsilon}{2}$. To avoid cumbersome notations we write $f(Q_X)$ to refer to $f_k(Q_X - k)$ for $k = \frac{\overline{\eta}_X + \underline{\eta}_X + \overline{\varepsilon} - \varepsilon}{2}$ and $f_k(z) = f(z+k)$.

In [AP02], the authors make use of the operator Q_X to study the classical Black-Scholes partial differential equation in (L^1, L^∞)-interpolation spaces. Recall that the classical Black-Scholes equation is the degenerate parabolic equation given by

$$u_t = x^2 u_{xx} + x u_x, \quad x, t > 0. \tag{3.11}$$

In fact, we can rewrite (3.11) as $u_t = Q_X^2 u$.

Next, we introduce the fractional operators that generalize the Black–Scholes equation (3.11). On the one hand, we consider fractional powers of the operator Q_X. If $\alpha \in (0, n)$, $n \in \mathbb{N}$, one has $\mathrm{Dom}(Q_X^n) \subseteq \mathrm{Dom}(Q_X^\alpha)$ (see [Haa06, Prop. 3.1.1]). If in addition $0 < \alpha < 1$, an application of Fubini's theorem to the Balakrishnan representation of $Q_X^\alpha f$ together with the resolvent identity yields, whenever $\underline{\eta}_X > 0$,

$$(Q_X^\alpha f)(x) = \frac{-1}{\Gamma(1-\alpha)} \int_x^\infty \left(\log \frac{s}{x}\right)^{-\alpha} f'(s)\, ds, \quad f \in \mathrm{Dom}(Q_X),\, x > 0.$$

If $\underline{\eta}_X = 0$, then one cannot apply Fubini's theorem to obtain the above expression. However, one can use the fact that $(Q_X + \varepsilon I)^\alpha f \to Q_X^\alpha f$ in X as $\varepsilon \downarrow 0$ (see [Haa06, Prop. 3.1.9]), together with

$$(Q_X + \varepsilon I)^\alpha f)(x) = \frac{-1}{\Gamma(1-\alpha)} \int_x^\infty \left(\log \frac{s}{x}\right)^{-\alpha} \left(\frac{x}{s}\right)^{\varepsilon} f'(s)\, ds, \tag{3.12}$$

for $f \in \mathrm{Dom}(Q_X)$ and $x, \varepsilon > 0$.

For a real number $\alpha > 0$, we define the fractional versions of the Cesàro operator are defined in an analogous way as in L^p-spaces. More precisely, for a $(L^1 - L^\infty)$−interpolation space X with $\overline{\eta}_X < 1$, set

$$(\mathcal{C}_{\alpha,X} f)(x) := \Gamma(\alpha+1) x^{-\alpha} (D^{-\alpha} f)(x), \qquad \text{a.e. } x > 0,\ f \in X,$$

and, for an interpolation space X with $\underline{\eta}_X > 0$,

$$(\mathcal{C}^*_{\alpha,X} f)(x) := \Gamma(\alpha+1) \left(W^{-\alpha}((\cdot)^{-\alpha} f)\right)(x), \qquad \text{a.e. } x > 0,\ f \in X.$$

It is readily seen that, as their L^p versions (see (1.7), (1.8)), these operators satisfy

$$\begin{aligned}
(\mathcal{C}_{\alpha,X} f)(x) &= \Gamma(\alpha+1) x^{-\alpha} (D^{-\alpha} f)(x), \qquad \text{a.e. } x > 0,\ f \in X,\\
(\mathcal{C}^*_{\alpha,X} f)(x) &= \Gamma(\alpha+1) (W^{-\alpha}((\cdot)^{-\alpha} f))(x), \qquad \text{a.e. } x > 0,\ f \in X.
\end{aligned}$$

Recall that we denote by $D^{-\alpha}$ the Riemann-Liouville fractional integral of order α, and by $W^{-\alpha}$ the Weyl fractional integral of order α, see (1.3) and (1.4), respectively. Recall also that these operators are injective, and that we denote their inverses $(\mathcal{C}_{\alpha,X})^{-1}$, $(\mathcal{C}^*_{\alpha,X})^{-1}$ by D_X^α, $\mathcal{W}_X^\alpha$ respectively.

Recall that, in an arbitrary $(L^1 - L^\infty)$–interpolation space (where the group $(E_X(t))$ may not be strongly continuous), Q_X, $-Q_X$ are the generators of the integrated semigroups $(\widetilde{E}_+(t))$, $\widetilde{E}_-(t))$ on X. The following subordination formula hold (in the pointwise sense)

$$(\mathcal{C}_{\alpha,X}f)(x) = \alpha \int_0^\infty e^{-s}(1-e^{-s})^{\alpha-1}(E_X(s)f)(x)\,ds, \qquad \text{a.e. } x>0,\ f\in X,$$
$$(\mathcal{C}^*_{\alpha,X}f)(x) = \alpha \int_{-\infty}^0 (1-e^{-s})^{\alpha-1}(E_X(s)f)(x)\,ds, \qquad \text{a.e. } x>0,\ f\in X.$$

Then, one can mimic the arguments of Proposition 2.2.24 and [Haa06, Prop. 3.3.2] to represent $\mathcal{C}_{\alpha,X}, \mathcal{C}^*_{\alpha,X}$ via the regularized functional calculus of Q_X. Namely,

$$\mathcal{C}_{\alpha,X} = \alpha\mathbb{B}(I-Q_X,\alpha), \qquad \text{if } \overline{\eta}_X < 1,$$
$$\mathcal{C}^*_{\alpha,X} = \alpha\mathbb{B}(Q_X,\alpha), \qquad \text{if } \underline{\eta}_X > 0,$$

where $\mathbb{B}$ denotes the usual Beta function.

3.3.B Generation results of fractional power operators

The identity $Q_X = \mathcal{W}^1_X = I - D^1_X$ holds whenever the operators are well defined on X (see e.g. [AP02]). In particular, we have

$$(Q_X)^2 = (I-\mathcal{D}^1_X)^2 = (\mathcal{W}^1_X)^2 = \mathcal{W}^1_X(I-\mathcal{D}^1_X). \tag{3.13}$$

This motivates us to study different fractional versions of the Black–Scholes equation (3.11), given by the fractional operators

$$(Q_X)^{2\alpha},\ \ (I-\mathcal{D}^\alpha_X)^2,\ \ (\mathcal{W}^\alpha_X)^2,\ \ \mathcal{W}^\alpha_X(I-\mathcal{D}^\alpha_X). \tag{3.14}$$

We show later on that such operators are generators of exponentially bounded holomorphic semigroups on X for suitable values of α.

We start with the operator $(Q_X)^{2\alpha}$.

Proposition 3.3.1. *Let X be a $(L^1 - L^\infty)$-interpolation space, $n \in \mathbb{N}$ and $\alpha > 0$ such that $\alpha \in \left(n-\frac{1}{2}, n+\frac{1}{2}\right)$. Then, the operator $(-1)^{n+1}(Q_X)^{2\alpha}$ generates an exponentially bounded holomorphic semigroup $T_{(-1)^{n+1}(Q_X)^{2\alpha}}$ of angle $\pi\left(\frac{1}{2}-|\alpha-n|\right)$, which is given by*

$$\left(T_{(-1)^{n+1}(Q_X)^{2\alpha}}(w)f\right)(x) = \frac{1}{2\pi}\int_0^\infty \frac{f(s)}{s}\int_{-\infty}^\infty \left(\frac{s}{x}\right)^{iu} \exp((-1)^{n+1}wu^{2\alpha})\,du\,ds, \quad x>0,$$

for $w \in S_{\pi\left(\frac{1}{2}-|\alpha-n|\right)}$ and $f \in X$. In addition, $\overline{\mathrm{Dom}((Q_X)^{2\alpha})} = \overline{\mathrm{Dom}(Q_X)}$.

Proof. That the operator $(-1)^{n+1}(J_X)^{2\alpha}$ generates an exponentially bounded holomorphic semigroup with the given angle follows from Corollary 3.1.10. The expression given for $T_{(-1)^{n+1}(J_X)^{2\alpha}}$ is an immediate consequence of Theorem 3.2.4 and Proposition 2.2.24. The assertion about $\overline{\mathrm{Dom}((J_X)^{2\alpha})}$ follows from Proposition 3.2.2. □

Next, we have the following result for the operator $(I-\mathcal{D}_X^\alpha)^2$.

Proposition 3.3.2. *Let X be a (L^1-L^∞)-interpolation space with $\overline{\eta}_X<1$, $n\in\mathbb{N}$ and $\alpha\in\left(n-\frac{1}{2},n+\frac{1}{2}\right)$. Then, the operator $(-1)^{n+1}(I-\mathcal{D}_X^\alpha)^2$ generates an exponentially bounded holomorphic semigroup $T_{(-1)^{n+1}(I-\mathcal{D}_X^\alpha)^2}$ of angle $\pi\left(\frac{1}{2}-|\alpha-n|\right)$, which is given by*

$$\begin{aligned}&\left(T_{(-1)^{n+1}(I-\mathcal{D}_X^\alpha)^2}(w)f\right)(x)\\ =&\frac{1}{2\pi}\int_0^\infty\frac{f(s)}{s}\int_{-\infty}^{\infty}\left(\frac{s}{x}\right)^{iu}\exp\left((-1)^{n+1}w\left(1-\frac{1}{\alpha\mathbb{B}(1-iu,\alpha)}\right)^2\right)duds,\end{aligned}$$

for $x>0$, $w\in S_{\pi\left(\frac{1}{2}-|\alpha-n|\right)}$ and $f\in X$. In addition, $\overline{\mathrm{Dom}((I-\mathcal{D}_X^\alpha)^2)}=\overline{\mathrm{Dom}(Q_X)}$.

Proof. First, recall that $\mathcal{D}_X^\alpha=(\alpha\mathbb{B}(I-Q_X,\alpha))^{-1}$, so $(I-\mathcal{D}_X^\alpha)^2=(I-\alpha\mathbb{B}(I-Q_X,\alpha)^{-1})^2$. It follows that

$$\left(1-\frac{1}{\alpha\mathbb{B}(1-z,\alpha)}\right)^2=\left(1-\frac{1}{\Gamma(\alpha+1)}\frac{\Gamma(1+\alpha-z)}{\Gamma(1-z)}\right)^2,$$

which is a holomorphic function in $\mathbb{C}\backslash\{1,2,3,...\}$. In addition, for $\lambda,z\in\mathbb{C}$, one has

$$\frac{\Gamma(z+\lambda)}{\Gamma(z)}=z^\lambda\left(1+O(|z|^{-1})\right),\qquad\text{as } |z|\to\infty,\tag{3.15}$$

whenever $z\neq 0,-1,-2,...$ and $z\neq-\lambda,-\lambda-1,-\lambda-2...$, (see e.g. [TE+51] for more details). As a consequence, one gets

$$\left(1-\frac{1}{\alpha\mathbb{B}(1-z,\alpha)}\right)^2=\frac{(-z)^{2\alpha}}{\alpha}\left(1+O(|z|^{-1})\right),\qquad\text{as } |z|\to\infty.$$

Thus, for each $\beta\in\left(0,\pi\left(\frac{1}{2}-|\alpha-n|\right)\right)$, there exists $\rho>0$ large enough such that the function $\rho+(-1)^{n+1}\left(1-\frac{1}{\alpha\mathbb{B}(1-z,\alpha)}\right)^2$ satisfies the hypothesis of Corollary 3.2.1, i.e. $(-1)^{n+1}(I-\mathcal{D}_X^\alpha)^2$ generates an exponentially bounded holomorphic semigroup of angle $\pi\left(\frac{1}{2}-|\alpha-n|\right)$. The rest of the statement follows by a similar reasoning as in the proof of Proposition 3.3.1. □

We have the following generation result for the operator $(\mathcal{W}_X^\alpha)^2$.

Proposition 3.3.3. *Let X be a (L^1-L^∞)-interpolation space with $\underline{\eta}_X>0$, $n\in\mathbb{N}$ and $\alpha\in\left(n-\frac{1}{2},n+\frac{1}{2}\right)$. Then, the operator $(-1)^{n+1}(\mathcal{W}_X^\alpha)^2$ generates an exponentially bounded holomorphic semigroup $T_{(-1)^{n+1}(\mathcal{W}_X^\alpha)^2}$ of angle $\pi\left(\frac{1}{2}-|\alpha-n|\right)$, which is given by*

$$\begin{aligned}&\left(T_{(-1)^{n+1}(\mathcal{W}_X^\alpha)^2}(w)f\right)(x)\\ =&\frac{1}{2\pi}\int_0^\infty\frac{f(s)}{s}\int_{-\infty}^{\infty}\left(\frac{s}{x}\right)^{iu+\delta}\exp\left((-1)^{n+1}w\left(\alpha\mathbb{B}(iu+\delta,\alpha)\right)^{-2}\right)duds,\end{aligned}$$

for $x > 0$*,* $w \in S_{\pi(\frac{1}{2}-|\alpha-n|)}$*, and* $f \in X$*, and* δ *is an arbitrary positive number. In addition,* $\overline{\mathrm{Dom}((\mathcal{W}_X^\alpha)^2)} = \overline{\mathrm{Dom}(Q_X)}$*.*

Proof. The proof is analogous to the proof of Proposition 3.3.2, using that $\mathcal{W}_X^\alpha = (\alpha\mathbb{B}(Q_X,\alpha))^{-1}$. The only difference comes out that one cannot apply Cauchy's Theorem and translate the inner integral path in u to make $\delta = 0$ since the Euler-Beta function $\mathbb{B}$ is not holomorphic on $(0,\alpha)$ for any non natural number α. □

Finally, we have the following generation result for the operator $\mathcal{W}_X^\alpha(I-\mathcal{D}_X^\alpha)$.

Proposition 3.3.4. *Let* X *be a* (L^1-L^∞)*-interpolation space with* $\underline{\eta}_X > 0$ *and* $\overline{\eta}_X < 1$*, and let* $\alpha > 0$*. Then,* $\mathcal{W}_X^\alpha(I-\mathcal{D}_X^\alpha)$ *generates an exponentially bounded holomorphic semigroup* $T_{\mathcal{W}_X^\alpha(I-\mathcal{D}_X^\alpha)}$ *of angle* $\frac{\pi}{2}$*, which is given by*

$$\begin{aligned}&\left(T_{\mathcal{W}_X^\alpha(I-\mathcal{D}_X^\alpha)}(w)f\right)(x)\\ =&\frac{1}{2\pi}\int_0^\infty \frac{f(s)}{s}\int_{-\infty}^\infty \left(\frac{s}{x}\right)^{iu+\delta} \exp\left(\frac{w}{\alpha\mathbb{B}(\delta+iu,\alpha)}\left(1-\frac{1}{\alpha\mathbb{B}(1-\delta-iu,\alpha)}\right)\right) duds,\end{aligned}$$

for $x > 0$*,* $w \in S_{\frac{\pi}{2}}$*, and* $f \in X$*, where* δ *is an arbitrary number in* $(0,1)$*. In addition,* $\overline{\mathrm{Dom}(\mathcal{W}_X^\alpha(I-\mathcal{D}_X^\alpha))} = \overline{\mathrm{Dom}(Q_X)}$*.*

Proof. The proof is analogous to the proof of Propositions3.3.2 and 3.3.3. Here, the statement is valid for any $\alpha > 0$ since, by (3.15), we have that

$$\frac{1}{\alpha\mathbb{B}(z,\alpha)}\left(1-\frac{1}{\alpha\mathbb{B}(1-z,\alpha)}\right) = \frac{z^\alpha(-z)^\alpha}{2\alpha}(1+O(|z|^{-1})), \qquad \text{as } |z|\to\infty,$$

and the proof is finished. □

3.3.C Generalized Black-Scholes partial differential equations

Let B_X be a closed linear operator on a Banach space X. We say that u is a solution of the abstract Cauchy problem associated with B_X, with initial condition $f \in X$, if u satisfies the following:

$$(ACP_0) \qquad \begin{cases} u \in C^1((0,\infty);X), & u(t)\in \mathrm{Dom}(B_X), \quad t>0,\\ u'(t) = B_X u(t), & t>0,\\ \lim_{t\downarrow 0} u(t) = f \in X. \end{cases}$$

We say that the Cauchy problem associated with B_X is well posed (or that (ACP_0) is well posed for short), if for each initial condition $f \in X$, there exists a unique solution u.

We are ready to state the following result concerning the well-posedness of the fractional Black-Scholes equation. Before that, let us state explicitly how these equations look like. Let $n \in \mathbb{N}$, $\alpha > 0$, and recall that D^α and W^α denote, respectively, the Riemann-Liouville and Weyl fractional derivatives of order α acting on the spatial domain.

(1) In the case $B_X = (-1)^{n+1}(Q_X)^{2\alpha}$ we have the following situation:

- If $\underline{\eta}_X > 0$, one can use the Balakrishnan representation, to obtain

$$(-1)^{n+1}u_t(x) = \frac{-1}{\Gamma(1-\alpha)} \int_x^{\infty} \left(\log \frac{s}{x}\right)^{-2\alpha+n} U_n'(s)\, ds, \quad t, x > 0.$$

- If $\underline{\eta}_X = 0$, one has to proceed as in (3.12) to obtain

$$(-1)^{n+1}u_t(x) = \lim_{\varepsilon \downarrow 0} \frac{-1}{\Gamma(1-\alpha)} \int_x^{\infty} \left(\log \frac{s}{x}\right)^{-2\alpha+n} \left(\frac{x}{s}\right)^{\varepsilon} U_n'(s)\, ds, \quad t, x > 0.$$

In both cases, $n \in \mathbb{N}$ is the whole part of 2α and $U_n := (Q_X)^n U$.

(2) If $B_X = (-1)^{n+1}(I - \mathcal{D}_X^{\alpha})^2$, one obtains the equation

$$(-1)^{n+1}u_t = \frac{1}{\Gamma(\alpha+1)^2} D^{\alpha}(x^{\alpha} D^{\alpha}(x^{\alpha}u)) - \frac{2}{\Gamma(\alpha+1)} D^{\alpha}(x^{\alpha}u) + u, \quad t, x > 0.$$

(3) If $B_X = (-1)^{n+1}(\mathcal{W}_X^{\alpha})^2$, one gets the equation

$$(-1)^{n+1}u_t = \frac{1}{\Gamma(\alpha+1)^2} x^{\alpha} W^{\alpha}(x^{\alpha} W^{\alpha} u), \quad t, x > 0.$$

(4) The case $B_X = \mathcal{W}_X^{\alpha}(I - \mathcal{D}_X^{\alpha})$ leads to the equation

$$u_t = \frac{1}{\Gamma(\alpha+1)} x^{\alpha} W^{\alpha} u - \frac{1}{\Gamma(\alpha+1)^2} D^{\alpha}(x^{2\alpha} W^{\alpha} u), \quad t, x > 0.$$

Theorem 3.3.5. *Let X be a $(L^1 - L^{\infty})$-interpolation space with order continuous norm, $n \in \mathbb{N}$, and $\alpha > 0$. Then, the following assertions hold.*

1. *If $\alpha \in \left(n - \frac{1}{2}, n + \frac{1}{2}\right)$, then ($ACP_0$) is well posed with $B_X = (-1)^{n+1}(Q_X)^{2\alpha}$.*
2. *If $\overline{\eta}_X < 1$ and $\alpha \in \left(n - \frac{1}{2}, n + \frac{1}{2}\right)$, then ($ACP_0$) is well posed with $B_X = (-1)^{n+1}(I - \mathcal{D}_X^{\alpha})^2$.*
3. *If $\underline{\eta}_X > 0$ and $\alpha \in \left(n - \frac{1}{2}, n + \frac{1}{2}\right)$, then ($ACP_0$) is well posed with $B_X = (-1)^{n+1} (\mathcal{W}_X^{\alpha})^2$.*
4. *If $\overline{\eta}_X < 1$ and $\underline{\eta}_X > 0$, then (ACP_0) is well posed with $B_X = \mathcal{W}_X^{\alpha}(I - \mathcal{D}_X^{\alpha})$.*

In any case, the solution u of (ACP_0) is given by $u(t) = T_{B_X}(t)f$ for $t > 0$. In addition, identifying $u(t,x) = u(t)(x)$, we obtain that $u \in C^{\infty}((0,\infty) \times (0,\infty))$.

Proof. In all cases, B_X is the generator of a holomorphic semigroup with $\overline{\mathrm{Dom}(B_X)} = \overline{\mathrm{Dom}(Q_X)}$ by Propositions 3.3.1, 3.3.2, 3.3.3, and 3.3.4. Moreover, T_{B_X} is strongly continuous since one has that $\mathrm{Dom}(Q_X)$ is dense in X if and only if X has order continuous norm (see e.g. [AP02, Remark 4.2]). Then, the assertions follow immediately by the relation between the well-posedness of a Cauchy problem, and the fact that B_X generates a strongly continuous semigroup (see for example [ABHN11, Prop. 3.1.2 and Theorem 3.1.12]).

Regarding the regularity result, one has that $u(t)$ is X-holomorphic in t in $(0, \infty)$ since T_{B_X} is a holomorphic semigroup. Even more, it satisfies $u(t) = T_X(t)f$, $u^{(k)}(t) = (B_X)^k u(t)$, and $u^{(k)}(t) \in \mathrm{Dom}((B_X)^n)$ for all $k, n \in \mathbb{N}$ and $t > 0$ (see [ABHN11, Chapter 3]). Now, reasoning as in the proof of Proposition 3.2.2 with B_X yields $\mathrm{Dom}(B_X) \subseteq \mathrm{Dom}((Q_X)^\varepsilon)$ for sufficiently small $\varepsilon > 0$. In addition, since $\mathrm{Dom}(Q_X) \subseteq AC_{\mathrm{loc}}(0, \infty)$, we have $\mathrm{Dom}((Q_X)^{j+1}) \subseteq C^j(0, \infty)$. As $u^{(k)}(t) \in \mathrm{Dom}((B_X)^n) \subseteq \mathrm{Dom}((Q_X)^{n\varepsilon})$ for all $k, n \in \mathbb{N}$, one obtains $u^{(k)}(t) \in C^\infty(0, \infty)$ for all $k \in \mathbb{N}$ and $t > 0$, and the proof is finished. □

Remark 3.3.6. T_{B_X} is strongly continuous if and only if X has order continuous norm, see [AP02, Remark 4.2]. Hence, Theorem 3.3.5 does not hold for a general $(L^1 - L^\infty)$-interpolation space. To address all interpolation spaces, we follow the ideas given in [AP02] and consider the Köthe dual $X^\star$ of X, given by

$$X^\star := \left\{ \theta : (0, \infty) \to \mathbb{C} \text{ measurable and } \int_0^\infty |f(x)\theta(x)|\, dx < \infty \quad \text{for all } f \in X \right\}.$$

Every $\theta \in X^\star$ defines a bounded linear functional L_θ on X, given by

$$L_\theta f := \langle f, \theta \rangle_{X, X^\star} := \int_0^\infty f(x)\theta(x)\, dx \ \text{ for all } \ f \in X.$$

In this way we can identify $X^\star$ with a subspace of the dual space X'.

It was proven in [AP02, Prop. 4.5] that $(E_X(t))$ is $\sigma(X, X^\star)$-continuous, that is the function from $\mathbb{R}$ to $\mathbb{C}$ given by $t \mapsto \langle E(t)f, \theta \rangle_{X, X^\star}$ is continuous for each $f \in X, \theta \in X^\star$. As a consequence, Fubini's theorem implies

$$\langle \widetilde{E}_\pm(t)f, \theta \rangle_{X, X^\star} = \int_0^t \langle E_X(\pm s)f, \theta \rangle_{X, X^\star}\, ds, \qquad f \in X, \theta \in X^\star.$$

From this, it is readily seen that Proposition 2.2.24 holds, understanding the integrals in the weak Köethe sense, even if X has not order continuous norm.

Now, we say that u is a solution in the Köethe dual sense of the abstract Cauchy problem associated with $B_X \in C(X)$, and with initial condition $f \in X$, if u satisfies the following:

$$(ACP_1) \qquad \begin{cases} u \in C^1((0, \infty); X), \quad u(t) \in \mathrm{Dom}(B_X), \quad t > 0, \\ u'(t) = B_X u(t), \qquad t > 0, \\ \lim\limits_{t \downarrow 0} \langle u(t), \theta \rangle_{X, X^\star} = \langle f, \theta \rangle_{X, X^\star}, \quad f \in X \text{ and for all } \theta \in X^\star. \end{cases}$$

Again, we say that the abstract Cauchy problem associated with B_X is well posed in the Köethe dual sense (or that (ACP_1) is well posed for short) if, for each initial condition $f \in X$, there exists a unique solution u of (ACP_1).

Theorem 3.3.7. *Let X be a $(L^1 - L^\infty)$-interpolation space, $n \in \mathbb{N}$, and $\alpha > 0$. Then, the following assertions hold.*

1. *If $\alpha \in \left(n - \frac{1}{2}, n + \frac{1}{2}\right)$, then ($ACP_1$) is well posed with $B_X = (-1)^{n+1}(Q_X)^{2\alpha}$.*

2. *If $\overline{\eta}_X < 1$ and if $\alpha \in \left(n - \frac{1}{2}, n + \frac{1}{2}\right)$, then ($ACP_1$) is well posed with $B_X = (-1)^{n+1}(I - \mathcal{D}_X^\alpha)^2$.*

3. *If $\underline{\eta}_X > 0$ and if $\alpha \in \left(n - \frac{1}{2}, n + \frac{1}{2}\right)$, then ($ACP_1$) is well posed with $B_X = (-1)^{n+1}(\mathcal{W}_X^\alpha)^2$.*

4. *If $\overline{\eta}_X < 1$ and $\underline{\eta}_X > 0$, then (ACP_1) is well posed with $B_X = \mathcal{W}_X^\alpha(I - \mathcal{D}_X^\alpha)$.*

In any case, the solution u of (ACP_1) is given by $(u(t))(x) = (T_{B_X}(t)f)(x)$ for $t > 0$. In addition, identifying $u(t,x) = u(t)(x)$, we obtain that $u \in C^\infty((0,\infty) \times (0,\infty))$.

To prove the theorem, we need the following lemma.

Lemma 3.3.8. *Let $a \geq 0$ and let $A \in \mathrm{BSect}(\pi/2, a)$ on X be such that A generates an exponentially bounded group $(T(t))_{t\in\mathbb{R}}$ for which $\|T(t)\| \lesssim e^{a|t|}$ for $t \in \mathbb{R}$. Let $g \in \mathcal{M}(A)$ satisfy all the hypothesis in Corollary 3.2.1. Assume furthermore that the following hold:*

1. *g is quasi-regular in $\{-a, a, \infty\}$ with $g(a), g(-a) \neq \infty$.*

2. *The group $(T(t))_{t\in\mathbb{R}}$ is $\sigma(X, X^\star)$-continuous.*

Then, the semigroup $(T_g(t))_{t\geq 0}$ generated by the operator $-g(A)$ is also $\sigma(X, X^\star)$-continuous.

Proof. We ask for the regularity conditions at $\{-a, a, \infty\}$ instead of just M_A in order to apply Proposition 2.2.24 in the weak Köethe sense, that is, $\langle T_g(t)f, \theta\rangle_{X,X^\star} = \int_{-\infty}^{\infty} \langle T(s)f, \theta\rangle_{X,X^\star}\, \mu_{h_t}(ds)$, for $f \in X, \theta \in X^\star$, where $h_t(z) := \exp(-tg(z))$, and $\mu_{h_t} \in M_a(\mathbb{R})$ is the Borel measure given in Lemma 2.2.22. By Lemma 2.2.22 again, one obtains

$$\int_{-\infty}^{\infty} \mu_{h_t}(ds) = \int_{-\infty}^{\infty} e^{i0s}\, \mu_{h_t}(ds) = h_t(0) = \exp(-tg(0)).$$

Then, for $f \in X$ and $\theta \in X^\star$, we have

(3.16)
$$\langle T_g(t)f, \theta\rangle_{X,X^\star} - \langle f, \theta\rangle_{X,X^\star} = \int_{-\infty}^{\infty} \langle T(s)f - f, \theta\rangle_{X,X^\star}\, \mu_{h_t}(ds) + (e^{-tg(0)} - 1)\langle f, \theta\rangle_{X,X^\star}.$$

We have to prove that the integral term in (3.16) tends to 0 as $t \downarrow 0$. Since by assumption $(T(t))_{t\in\mathbb{R}}$ is $\sigma(X, X^\star)$-continuous, we have $\lim_{t\downarrow 0}\langle T(t)f, \theta\rangle_{X,X^\star} = \langle f, \theta\rangle_{X,X^\star}$. Thus, for each $\varepsilon > 0$, there exists $\delta > 0$ such that $|\langle T(s)f - e^{irs}f, \theta\rangle_{X,X^\star}| < \varepsilon$ for all $|s| < \delta$. Hence, there exists $C > 0$ for which

$$\begin{aligned}&\limsup_{t\downarrow 0}\left|\int_{-\infty}^{\infty}\langle T(s)f - f, \theta\rangle_{X,X^\star}\,\mu_{h_t}(ds)\right|\\ \leq& C\varepsilon + \limsup_{t\downarrow 0}\left|\int_{|s|>\delta}\langle T(s)f - f, \theta\rangle_{X,X^\star}\,\mu_{h_t}(ds)\right|.\end{aligned}$$

Let us work with the above integral when $s > \delta$. The case $s < -\delta$ is completely analogous. By Lemma 2.2.22, one gets

$$\begin{aligned}&\int_{s>\delta}\langle T(s)f - f, \theta\rangle_{X,X^\star}\,\mu_{h_t}(ds)\\ =&\int_{s>\delta}\langle T(s)f - f, \theta\rangle_{X,X^\star}\frac{1}{2\pi i}\int_{\Gamma_+}e^{-zs}e^{-tg(z)}\,dzds\\ =&\int_{s>\delta}\langle T(s)f - f, \theta\rangle_{X,X^\star}\frac{1}{2\pi i}\int_{\Gamma_+}e^{-zs}\left(e^{-tg(z)} - e^{-tg(a)}\frac{b+a}{b+z}\right)\,dzds,\end{aligned}$$

where we have used, in the last equality, the identity

$$e^{-tg(a)}\int_{\Gamma_+}e^{-zs}\frac{b+a}{b+z}\,dz = 0, \qquad \text{for all } s,t > 0, \text{ and } b > a.$$

Now, applying the Lebesgue Dominated Convergence Theorem, we obtain

$$\begin{aligned}&\limsup_{t\downarrow 0}\left|\int_{s>\delta}\langle T(s)f - f, \theta\rangle_{X,X^\star}\,\mu_{h_t}(ds)\right|\\ =&\left|\int_{s>\delta}\langle T(s)f - f, \theta\rangle_{X,X^\star}\frac{1}{2\pi i}\int_{\Gamma_+}e^{-zs}\frac{z-a}{b+z}\,dzds\right| = 0,\end{aligned}$$

where we have used again Cauchy's theorem in the last equality. To check the hypothesis of the Dominated Convergence Theorem, one has to bound the following expression:

$$F_t(s,z) := e^{-s(\mathfrak{Re}\, z - a)}\left|e^{-tg(z)} - e^{-tg(a)}\frac{b+a}{b+z}\right|, \qquad s > \delta,\ z \in \Gamma_+,$$

by an integrable function for all $t \in (0, \varepsilon')$, where ε' is an arbitrary positive number. Also, $\mathfrak{Re}\, g(z) \geq 0$ implies $\sup_{t>0, z\in\Gamma_+}|e^{-tg(z)}| < \infty$. Thus, it is readily seen that

$$F_t(s,z) \lesssim e^{-s(\mathfrak{Re}\, z - a)}\min\left\{1, \frac{|z-a| + |g(z) - g(a)|}{|b+z|}\right\},$$

which is integrable since g is regular at a, and the proof is finished. □

Proof of Theorem 3.3.7. Once we have proven that $T_{B_X}(t)f$ is $\sigma(X, X^\star)$-continuous on t as $t \downarrow 0$ for all $f \in X$, the assertions follow by a similar reasoning as in the proofs of Theorem 3.3.5 and [AP02, Th. 5.8]. Then, we only have to check that the exponentially bound condition of Lemma 3.3.8 holds for $(E_X(t))$. But, except for the case $\underline{\eta}_X = 0$ and $B_X = (Q_X)^{2\alpha}$, we can always assume it is satisfied since the functions g_{B_X}, for which $B_X = g_{B_X}(Q_X)$, are holomorphic in $\underline{\eta}_X$ and $\overline{\eta}_X$. And regarding the case $\underline{\eta}_X = 0$, one has $\|E_X(t)\|_{L(X)} \lesssim 1$ for $t \leq 0$ (see e.g. [AP02]). Then, we can apply Lemma 3.3.8 to obtain that $T_{B_X}(t)f$ is $\sigma(X, X^\star)$-continuous. □

If $\alpha = 1$, all the different generalized Black–Scholes equations presented above yield the classical Black–Scholes equation given by (BS). In this case, the above results retrieve the ones obtained in [AP02, Section 5]. In particular, one gets the formula for the semigroup T_{B_X}, given by

$$\begin{aligned}\left(T_{B_X}(w)f\right)(x) &= \frac{1}{2\pi}\int_0^\infty \frac{f(s)}{s}\int_{-\infty}^\infty \left(\frac{s}{x}\right)^{iu} \exp\left(-wu^2\right) du ds \\ &= \frac{1}{\sqrt{4\pi w}}\int_0^\infty \exp\left(-\frac{(\log x - \log s)^2}{4w}\right)\frac{f(s)}{s}\, ds, \qquad x > 0,\ \mathfrak{Re}\, w > 0,\end{aligned}$$

where in the last equality we have made use of the integral identity [GR14, Formula 3.233(2)].

Remark 3.3.9. The above results do not cover (in general) the case $\alpha = 1/2, 3/2, 5/2, \ldots$. This is closely related to the odd powers of a generator of a group (see Corollary 3.1.10 and [BHK09, Th. 4.6]). Indeed, one can prove that when $\alpha = 1/2, 3/2, 5/2, \ldots$, the operators B_X considered there (except for $\mathcal{W}_X^\alpha(I - \mathcal{D}_X^\alpha)$), are bisectorial-like operators of angle $\frac{\pi}{2}$. Unfortunately, this is a necessary but not sufficient condition to determine that they generate semigroups.

Chapter 4

Weighted composition semigroups in the disc

Recall that $\mathcal{O}(\mathbb{D})$ denotes the Fréchet algebra of holomorphic functions on the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Let X be a functional Banach space continuously contained in $\mathcal{O}(\mathbb{D})$. Important examples of one-parameter (semi)groups in $L(X)$ are weighted composition (semi)groups. This type of (semi)groups play a central role in Chapters 5 and 6 of this monograph. In this chapter, we present the basic objects needed to work with such operators, such as (semi)flows and (semi)cocycles. Moreover, we give some new results regarding multivalued coboundaries in Section 4.2.

4.1 Weighted composition semigroups

A morphism of $\mathbb{D}$ is any function $\psi \in \mathcal{O}(\mathbb{D})$ such that $\psi(\mathbb{D}) \subseteq \mathbb{D}$. The semigroup of morphisms of $\mathbb{D}$ is denoted by $Mor(\mathbb{D})$. A family $(\varphi_t)_{t\geq 0}$ of non-trivial morphisms of $\mathbb{D}$ (i.e. φ_t is not the identity map for all $t \geq 0$) is said to be a one-parameter semigroup, or (holomorphic) semiflow, if

1. $\psi_0(z) = z$ for all $z \in \mathbb{D}$;
2. $\psi_{s+t} = \psi_s \circ \psi_t$ for all $s, t \geq 0$;
3. $\psi_t(z)$ is continuous in (t, z) on $[0, \infty) \times \mathbb{D}$.

When t runs over the whole real line in (ψ_t), and (2) and (3) hold for every $s, t \in \mathbb{R}$ the family $(\psi_t)_{t\in\mathbb{R}}$ is called one-parameter group or flow. Here we use preferably the term *semiflow* or *flow* to distinguish such families of morphisms from the so-called semigroups of operators (on Banach spaces).

The infinitesimal generator of a given semiflow (ψ_t) is the function Ψ defined by the limit $\Psi(z) := \lim_{t\to 0} t^{-1}(\psi_t(z) - z)$, $z \in \mathbb{D}$. Actually, the limit exists uniformly on compact subsets of $\mathbb{D}$, the mapping $t \mapsto \psi_t(z)$ is differentiable on $[0, \infty)$ for every $z \in \mathbb{D}$, and one has

$$\frac{\partial \psi_t(z)}{\partial t} = \Psi(\psi_t(z)) = \Psi(z)\frac{\partial \psi_t(z)}{\partial z}, \qquad z \in \mathbb{D},\ t \geq 0. \tag{4.1}$$

Furthermore, Ψ is an analytic function on $\mathbb{D}$ that has a unique representation

$$\Psi(z) = F(z)(\overline{a}z - 1)(z - a), \qquad z \in \mathbb{D}, \tag{4.2}$$

where F is an analytic function on $\mathbb{D}$ with $\mathfrak{Re}\, F \geq 0$, and the point $|a| \leq 1$ is called Denjoy-Wolff (DW) point of $(\psi_t)_{t\geq 0}$, see [BP78]. This notation is due to the Denjoy-Wolff theorem, which states that given a conformal mapping $\phi : \mathbb{D} \to \mathbb{D}$ which is not an elliptic Möbius automorphism of $\mathbb{D}$, there is a (unique) point $a \in \overline{\mathbb{D}}$ such that $\lim_{n\to\infty} \phi^n(z) \to a$ uniformly on compact subsets of $\mathbb{D}$. Note that $\Psi'(a) = -(1 - |a|^2)F(a)$.

It was proven in [Cow81; Sis85] that any semiflow (ψ_t) can be described as the conjugation of one of the two basic semigroups of the complex plane, i.e. $z \mapsto z + ct$ or $z \mapsto e^{-ct}$. More precisely,

1. If the DW point a of (ψ_t) is in $\mathbb{T}$, then there exists a unique univalent function $h : \mathbb{D} \to \mathbb{C}$ with $h(0) = 0$, $h'(0) = 1$ such that

$$h(\psi_t(z)) = h(z) + \Psi(0)t, \qquad z \in \mathbb{D},\ t \geq 0. \tag{4.3}$$

2. If the DW point a of (ψ_t) is in $\mathbb{D}$, then there exists a unique univalent function $h : \mathbb{D} \to \mathbb{C}$ with $h(0) = 0$, $h'(0) = 1$ such that

$$h(\phi_a(\psi_t)(z)) = e^{\Psi'(a)t} h(\phi_a(z)), \qquad z \in \mathbb{D},\ t \geq 0, \tag{4.4}$$

   where we set $\phi_a(z) = (z - a)/(1 - \overline{a}z)$. We denote by h_a the function $h \circ \phi_a$.

The function h in either (4.3) or (4.4) is called the *univalent function associated with* (ψ_t). For the above items and other details about semiflows and flows of self-analytic maps of $\mathbb{D}$, we refer the readers to [BKP74; BP78; CM95; Sis86; Sis98].

Let (ψ_t) be a semiflow. A family (u_t) of analytic functions $u_t \colon \mathbb{D} \to \mathbb{C}$ is called a (continuous) semicocycle for (ψ_t) if

1. $u_0(z) = 1$ for all $z \in \mathbb{D}$;
2. $u_{s+t} = u_t(u_s \circ \psi_t)$ for all $s, t \geq 0$;
3. the mapping $t \mapsto u_t(z)$ is continuous on $[0, \infty)$ for every $z \in \mathbb{D}$.

Suppose $(\psi_t)_{t\in\mathbb{R}}$ is a flow. If u_t is given for all $t \in \mathbb{R}$ and the above properties hold for every $t \in \mathbb{R}$ we say that $(u_t)_{t\in\mathbb{R}}$ is a cocycle for $(\psi)_{t\in\mathbb{R}}$.

If $t \mapsto u_t(z)$ above is differentiable on $[0, \infty)$ (on $\mathbb{R}$) for every $z \in \mathbb{D}$ the semicocycle (cocycle) (u_t) is called differentiable. The infinitesimal generator g of a differentiable semicocycle (cocycle) (u_t) is defined by $g(z) := \frac{\partial}{\partial t} u_t(z) \mid_{t=0}$.

We say that a subspace X of $\mathcal{O}(\mathbb{D})$ separates points in $\mathbb{D}$ if, for every $z \in \mathbb{D}$, there exists $f \in X$ such that $f(z) \neq 0$. We give below two results essentially contained, respectively, in [Kön90, Th. 1] and [Sis86, Th. 2] for H^p spaces. Their proofs run in our framework with minimal changes (cf. [Ber22, Th. 2.3] and [GSY22, Th. 2.1]).

Lemma 4.1.1. *Let X be a Banach space which separates point in $\mathbb{D}$ and which embeds continuously in $\mathcal{O}(\mathbb{D})$, and let $(u_t C_{\psi_t})$ be a C_0-semigroup of bounded operators on X. Then (u_t) is a differentiable cocycle such that its generator g is an analytic function in $\mathbb{D}$. Moreover,*

$$u_t(z) = \exp\left(\int_0^t g(\psi_s(z))\, ds\right), \qquad z \in \mathbb{D},\, t \geq 0.$$

Proposition 4.1.2. *Let $(u_t C_{\psi_t})$ be a strongly continuous weighted composition semigroup on a Banach space X which embeds continuously in $\mathcal{O}(\mathbb{D})$. Assume that (u_t) is a differentiable semicocycle such that its generator g is analytic in $\mathbb{D}$. Then, the infinitesimal generator Δ of $(u_t C_{\psi_t})$ is given by the differential operator*

$$\Delta(f) := \Psi f' + gf, \quad f \in \mathrm{Dom}(\Delta),$$

with $\mathrm{Dom}(\Delta) = \{f \in X : Gf' + gf \in X\}$.

4.2 Multivalued coboundaries

Let (ψ_t) be a semiflow and let ω be a holomorphic function on $\mathbb{D}$, non-vanishing except possibly at the DW point a (if $a \in \mathbb{D}$). It is readily seen that $\left(\frac{\omega\circ\psi_t}{\omega}\right)$ is a semicocycle for (ψ_t). Cocycles of this type are called coboundaries. If $a \in \mathbb{T}$, all cocycles (u_t) (for (ψ_t)) satisfying some mild assumptions are in fact coboundaries, see [Kön90, Lemma 2.2(b)].

In this section, we show that the natural setting for the case $a \in \mathbb{D}$ are multivalued functions. This fact was already noted in [Kön90], but it was not treated structurally.

So assume $a \in \mathbb{D}$ and let $p : E_a \to \mathbb{D} \setminus \{a\}$ denote the projection of the universal covering space E_a of the punctured disc $\mathbb{D}\setminus\{a\}$. Such covering space E_a can be realized as a two-dimensional surface on $\mathbb{C} \times \mathbb{R} \sim \mathbb{R}^3$ given by

$$E_a = \{(z, \theta) \in (\mathbb{D} \setminus \{a\}) \times \mathbb{R} \,:\, z = a + |z - a|e^{i\theta}\},$$

see Figure 4.1. As a matter of fact, E_a is a simply connected Riemann surface which is locally conformally equivalent to $\mathbb{D} \setminus \{a\}$ through the projection p, see for instance [Ahl79, Subsections 3.4.3 & 8.1.3]. E_a also satisfies that, for every path ν on $\mathbb{D} \setminus \{a\}$ (i.e., a continuous function $\nu : [0, 1] \to \mathbb{D} \setminus \{a\}$) and any $z' \in p^{-1}(z) \subset E_a$, there exists a unique lifting $\widetilde{\nu}$ of the path ν, i.e., a path $\widetilde{\nu} : [0, 1] \to E_a$ such that $p(\widetilde{\nu}(t)) = \nu(t)$, $t \in [0, 1]$.

Since E_a is simply connected, every non-vanishing holomorphic function f on E_a has holomorphic logarithms and fractional powers. In fact, one has

$$(\mathrm{Log}\, f)(z') := \mathrm{Log}(f(d')) + \int_{d'}^{z'} \frac{f'(\tau)}{f(\tau)}\, d\tau, \qquad z' \in E_a,$$

where d' is any point in E_a. Fixed $d' \in E_a$ above, each branch of $\mathrm{Log}(f(d'))$ induces a different branch of $\mathrm{Log}\, f$. This ambivalence in the definition of $\mathrm{Log}\, f$ is irrelevant for

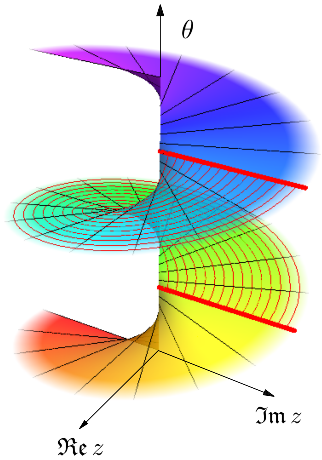


Figure 4.1: Universal covering space of $\mathbb{D}\setminus\{0\}$. Mathematica code from [Kan08].

most of our results. In such cases, we omit the branch of Log f we are using for the sake of briefness.

Regarding the fractional powers, for $\delta \in \mathbb{C}$ and a zero-free $f \in \mathcal{O}(E_a)$, set $f^\delta := \exp(\delta \operatorname{Log} f) \in \mathcal{O}(E_a)$. Also, given $f \in \mathcal{O}(\mathbb{D}\setminus\{a\})$ we denote by $\widetilde{f}$ the function in $\mathcal{O}(E_a)$ for which $\widetilde{f} = f \circ p$.

Remark 4.2.1. Given a semiflow (ψ_t) with *DW* point $a \in \mathbb{D}$, we define the lifted semiflow $(\widetilde{\psi}_t)$ by mappings $\widetilde{\psi}_t : E_a \to E_a$ as follows. For any $z' \in E_a$, let the ν be the path $\nu(s) = \psi_{st}(p(z'))$, $s \in [0,1]$. Then, we set $\widetilde{\psi}_t(z') := \widetilde{\nu}(1)$, where $\widetilde{\nu}$ is the lifting of ν with $\widetilde{\nu}(0) = z'$. It is readily seen that $\widetilde{\psi}_t$ is holomorphic on E_a for every $t \geq 0$, and that $\widetilde{\psi}_s \circ \widetilde{\psi}_t = \widetilde{\psi}_{s+t}$ for $s, t \geq 0$.

The universal covering space is key in the item ii) in the following result.

Proposition 4.2.2. *Let (ψ_t) be a (non-trivial) semiflow with DW point $a \in \overline{\mathbb{D}}$, and let (u_t) be a differentiable cocycle for (ψ_t) such that its generator g is an analytic function on $\mathbb{D}$.*

i) If $a \in \mathbb{T}$, then there exists a non-vanishing holomorphic function $\omega : \mathbb{D} \to \mathbb{C}$ such that $u_t = (\omega \circ \psi_t)/\omega$ for all $t \geq 0$.

ii) If $a \in \mathbb{D}$, then there exits a non-vanishing holomorphic function $\omega : E_a \to \mathbb{C}$ such that, for $z \in \mathbb{D}\setminus\{a\}$ and $z' \in p^{-1}(z) \subset E_a$, we have

$$u_t(z) = \frac{\omega \circ \widetilde{\psi}_t(z')}{\omega(z')}, \qquad t \geq 0.$$

In any case, we say that such an holomorphic function ω as above is a *holomorphic function associated with* (u_t).

Proof. Item i) was already proven in [Kön90, Lemma 2.2]. For item ii), take $d' \in E_a$ and define

$$\omega(z') = \exp\left(\int_{d'}^{z'} \frac{\widetilde{g}(\tau)}{\widetilde{\Psi}(\tau)}\, d\tau\right) = \exp\left(\int_{\nu} \frac{g(\xi)}{\Psi(\xi)}\, d\xi\right), \qquad z' \in E_a,$$

where ν is the projection through $p : E_a \to \mathbb{D} \setminus \{a\}$ of the integration path taken in the integral with variable τ above. Note that the value of $\omega(z')$ is independent on the choice of the integration path since E_a is a simply connected manifold, and $\widetilde{\Psi}$ has no zeroes on E_a, see (4.2).

It is clear that ω is non-vanishing. Take $z_1', z_2' \in E_a$ such that $p(z_1') = p(z_2')$. It follows that

$$\frac{\omega(z_2')}{\omega(z_1')} = \exp\left(\int_{z_1'}^{z_2'} \frac{\widetilde{g}(\tau)}{\widetilde{\Psi}(\tau)}\, d\tau\right) = \exp\left(\int_{\nu} \frac{g(\xi)}{\Psi(\xi)}\, d\xi\right) = \exp\left(2\pi i \operatorname{Ind}_{\nu}(a)\frac{g(a)}{\Psi'(a)}\right),$$

where $\operatorname{Ind}_{\nu}(a)$ is the index of ν with respect to a. Note that ν is a closed path in $\mathbb{D} \setminus \{a\}$ since $p(z_1') = p(z_2')$. Hence, the quotient $\omega(z_2')/\omega(z_1')$ only depends on $\operatorname{Ind}_{\nu}(a)$. In addition, it is readily seen that $\operatorname{Ind}_{\nu}(a) = \operatorname{Ind}_{\mu}(a)$, where μ is the projection of a path on E_a from $\widetilde{\psi}_t(z_1')$ to $\widetilde{\psi}_t(z_2')$. Therefore, the quotient

$$v_t(z) := \frac{\omega(\widetilde{\psi}_t(z'))}{\omega(z')}, \qquad z \in \mathbb{D} \setminus \{a\},\ z' \in p^{-1}(z), \quad t \geq 0,$$

is independent of the choice of $z' \in p^{-1}(z)$, so v_t is a well-defined function on $\mathbb{D} \setminus \{a\}$.

Now, take $z \in \mathbb{D} \setminus \{a\}$, $z' \in p^{-1}(z)$, and consider the (differentiable) function $\lambda : [0, \infty) \to \mathbb{C}$ given by $\lambda(t) = v_t(z)$. We have $\lambda(0) = 1$ and

$$\begin{aligned}\lambda'(t) &= \frac{1}{\omega(z')} \frac{\partial \omega(\widetilde{\psi}_t(z'))}{\partial t} = \frac{1}{\omega(z')} \omega'(\widetilde{\psi}_t(z')) \frac{\partial \widetilde{\psi}_t(z')}{\partial t} \\ &= \frac{1}{\omega(z')} \omega(\widetilde{\psi}_t(z')) \frac{g(\psi_t(z))}{\Psi(\psi_t(z))} \Psi(\psi_t(z)) = \lambda(t) g(\psi_t(z)), \quad t \geq 0.\end{aligned}$$

By Lemma 4.1.1, the function $t \mapsto u_t(z)$ satisfies the same differential equation as above $\left(\text{i.e., } \frac{\partial u_t(z)}{\partial t} = u_t(z) g(\psi_t(z))\right)$ with the same initial condition. Thus, $v_t = u_t$ by the theory of differential equations, and the proof is finished. □

On the one hand, item ii) in Proposition 4.2.2 enables us to mimic the proof of [Sis86, Th. 3] (which is given for H^p and coboundaries) to obtain the point spectrum of the infinitesimal generator of $(u_t C_{\psi_t})$ in Proposition 4.2.4 below. We need the following remark first.

Remark 4.2.3. Using the same terminology as in the proof of Proposition 4.2.2ii), let $a \in \mathbb{D}$, set $\delta := g(a)/\Psi'(a) \in \mathbb{C}$, and let F be the function associated with Ψ as in (4.2). Set $e \in \mathcal{O}(\mathbb{D})$ by

$$e(z) := \frac{g(z) - \delta(\overline{a}z - 1)F(z)}{\Psi(z)}, \quad z \in \mathbb{D} \setminus \{a\},$$

and extend it by continuity to a. Then, the function $\rho(z) = \exp(\int_{p(d')}^{z} e(\xi)\,d\xi)$, $z \in \mathbb{D}$, is holomorphic in $\mathbb{D}$, with no zeroes, and we have

$$\omega(z') = \exp\left(\int_{d'}^{z'} \frac{\widetilde{g}(\tau)(\tau - a) - \delta\widetilde{\Psi}(\tau)}{\widetilde{\Psi}(\tau)(\tau - a)} + \frac{\delta}{\tau - a}\,d\tau\right)$$
$$= \exp\left(\int_{p(d')}^{p(z')} e(\xi)\,d\xi\right)\exp\left(\int_{d'}^{z'} \frac{\delta}{\tau - a}\,d\tau\right) = \rho(p(z'))\frac{(z'-a)^{\delta}}{(d'-a)^{\delta}}, \qquad z' \in E_a.$$

As a consequence, if $f \in \mathbb{D}$ with no zeroes on $\mathbb{D} \setminus \{a\}$ and a simple zero located at a, then $\widetilde{f}^{\delta}/\omega$ induces a holomorphic function in $\mathbb{D}$.

Proposition 4.2.4. *Let $(u_t C_{\psi_t})$ be a C_0-semigroup on a Banach space X continuously embedded in $\mathcal{O}(\mathbb{D})$, such that (u_t) is a differentiable semicocycle with an analytic generator g. Let h be the univalent function associated with (ψ_t), let Δ be the infinitesimal generator of $(u_t C_{\psi_t})$, and let ω be a holomorphic function associated with (u_t). Then,*

(a) if the DW point a of (ψ_t) lies in $\mathbb{T}$, then

$$\sigma_{point}(\Delta) = \{\lambda \in \mathbb{C} : f_\lambda \in X\},$$

where $f_\lambda(z) = 1/\omega(z)\exp(\lambda/\Psi(0)h(z))$, $z \in \mathbb{D}$. Moreover, each eigenspace is one-dimensional and is generated by f_λ.

(b) if the DW point a of (ψ_t) lies in $\mathbb{D}$, then

$$\sigma_{point}(\Delta) = \left\{ g(a) + \Psi'(a)k \,:\, k \in \mathbb{N}_0 \text{ such that } \frac{\widetilde{h_a}^{k+g(a)/\Psi'(a)}}{\omega} \in X \right\}.$$

Moreover, each eigenspace is one-dimensional and is generated by $\widetilde{h}^{k+g(a)/\Psi'(a)}/\omega$.

On the other hand, we use the lifted mappings $\widetilde{\psi}_t$ to construct semicocycles of fractional powers $((\psi_t')^{\delta})$ for $\delta \in \mathbb{C}$. (Note that (ψ_t') is a cocycle for (ψ_t), see (4.1).)

Remark 4.2.5. Let (ψ_t) be a semiflow with *DW* point $a \in \mathbb{D}$, and fix $\delta \in \mathbb{C}$. For $z \in \mathbb{D} \setminus \{a\}$ and $t \geq 0$, the quotient $(\widetilde{\psi}_t(z') - a)^{\delta}/(z' - a)^{\delta}$ is independent of the choice of $z' \in p^{-1}(z) \subset E_a$. To see this, it suffices to mimic the argument in the poof of Proposition 4.2.2 ii) involving the indexes with respect to a of certain integration paths.

Then, for $t \geq 0$, define

$$\left(\frac{\psi_t(z) - a}{z - a}\right)^{\delta} := \frac{(\widetilde{\psi}_t(z') - a)^{\delta}}{(z' - a)^{\delta}}, \qquad z \in \mathbb{D} \setminus \{a\}, \text{ and } z' \in p^{-1}(z). \tag{4.5}$$

It is readily seen that such a function is indeed a branch of the fractional power of order δ of $(\psi_t(\cdot) - a)/((\cdot) - a)$.

From now on, by $((\psi_t(\cdot) - a)/((\cdot) - a))^{\delta}$ we mean the branch of the fractional power of order δ of $(\psi_t(\cdot) - a)/((\cdot) - a)$ given by (4.5).

Definition 4.2.6. Let (ψ_t) be a semiflow with *DW* point $a \in \overline{\mathbb{D}}$ and let F be the function associated to the generator of (ψ_t) as in (4.2). Let $\delta \in \mathbb{C}$.

i) If $a \in \mathbb{T}$, set

$$(\psi_t')^\delta(z) := \frac{(F(\psi_t(z)))^\delta}{(F(z))^\delta} \frac{(1-\bar{a}\psi_t(z))^\delta}{(1-\bar{a}z)^\delta} \frac{(1-\psi_t(z)/a)^\delta}{(1-z/a)^\delta}, \qquad z \in \mathbb{D},\ t \geq 0,$$

where we take the principal branch of the logarithm in $\mathbb{C} \setminus (-\infty, 0]$ for all fractional powers above. (Note that all the bases of such powers lie in the right half-complex plane.)

ii) If $a \in \mathbb{D}$, we set

$$(\psi_t')^\delta(z) := \frac{(F(\psi_t(z)))^\delta}{(F(z))^\delta} \frac{(1-\bar{a}\psi_t(z))^\delta}{(1-\bar{a}z)^\delta} \left(\frac{\psi_t(z)-a}{z-a}\right)^\delta, \qquad z \in \mathbb{D},\ t \geq 0,$$

where the function $((\psi_t(\cdot)-a)/((\cdot)-a))^\delta$ is as defined in Remark 4.2.5, and the rest of fractional powers above are considered with the principal branch of the logarithm in $\mathbb{C} \setminus (-\infty, 0]$.

Lemma 4.2.7. *Let (ψ_t) be a semiflow and let $\delta \in \mathbb{C}$. The family $((\psi_t')^\delta)$ given in Definition 4.2.6 is a semicocycle for (ψ_t).*

Proof. If the *DW* point a of the semiflow (ψ_t) lies in the boundary $\mathbb{T}$, then the claim is obtained directly from the definition of $(\psi_t')^\delta$. If a lies in the disc $\mathbb{D}$, we have to prove the semicocycle property, i.e. that $(\psi_{s+t}')^\delta = (\psi_t')^\delta((\psi_s')^\delta \circ \psi_t)$ for all $s, t \geq 0$ (the continuity on t is trivial to check). To see that such an equality holds, note that

$$\begin{aligned}\left(\frac{\psi_{s+t}(z)-a}{z-a}\right)^\delta &= \exp\left(\int_{z'}^{\widetilde{\psi}_{s+t}(z')} \frac{\delta}{\tau-a}\,d\tau\right)\\ &= \exp\left(\int_{z'}^{\widetilde{\psi}_t(z')} \frac{\delta}{\tau-a}\,d\tau\right) \exp\left(\int_{\widetilde{\psi}_t(z')}^{\widetilde{\psi}_s(\widetilde{\psi}_t(z'))} \frac{\delta}{\tau-a}\,d\tau\right)\\ &= \left(\frac{\psi_t(z)-a}{z-a}\right)^\delta \left(\frac{\psi_s(\psi_t(z))-a}{\psi_t(z)-a}\right)^\delta, \quad z \in \mathbb{D}\setminus\{a\},\ s,t \geq 0,\end{aligned}$$

where z' is any point in $p^{-1}(z)$. By continuity, equality above is satisfied for all $z \in \mathbb{D}$. Now, it suffices to apply equality above in the definition of $(\psi_{s+t}')^\delta$ to obtain the claim. □

Chapter 5

Hausdorff matrices and weighted semigroups with DW point in $\mathbb{D}$

This chapter is based on the ongoing work [AO23]. We are aware that certain results presented here can be refined and/or improved. Moreover, we expect to obtain additional, deeper results (concerning the topic studied here) in the following months.

Let Δ be the forward difference operator acting on scalar sequences $a = (a_n)_{n=0}^{\infty}$, that is, $(\Delta a)_n = a_n - a_{n+1}$. The generalized Hausdorff matrix $H_a^{(\zeta)}$ generated by the sequence a and a real number ζ is the infinite lower triangular matrix given by

$$H_a^{(\zeta)}(i,j) = \begin{cases} 0, & i < j, \\ \binom{i+\zeta}{i-j}(\Delta^{i-j}a)_j, & i \geq j. \end{cases}$$

These matrices were defined independently in [End60; Jak59]. As a countably infinite matrix, each generalized Hausdorff matrix $H_a^{(\zeta)}$ induces an operator in sequence spaces on $\mathbb{N}_0$, denoted by $\mathcal{H}_a^{(\zeta)}$, determined by

$$(\mathcal{H}_a^{(\zeta)})a)_n := \sum_{k=0}^{n} H_a^{(\zeta)}(n,k)a_k, \qquad n \in \mathbb{N}_0,\ a = (a_n)_{n=0}^{\infty}.$$

Let μ be a finite Borel measure on $(0,1]$, and let (μ_n) be the sequence given by

$$\mu_n = \int_0^1 t^n d\mu(t), \quad n \in \mathbb{N}_0. \tag{5.1}$$

Then the Hausdorff matrix $H_{(\mu_n)}^{(0)}$ corresponds to the ordinary Hausdorff summability [Hau21]. In this case, it follows from the work of Hardy [Har43] that if $\int_0^1 t^{-1/p}d\mu(t) < \infty$, then the induced operator $\mathcal{H}_{(\mu_n)}^{(0)}$ is bounded on $\ell^p(\mathbb{N}_0)$ for $1 < p < \infty$.

Now, let $\zeta \in \mathbb{R}$, let μ be a finite Borel measure on $(0,1]$ and let (μ_n) be the sequence given by

$$\mu_n = \int_0^1 t^{n+\zeta} d\mu(t), \quad n \in \mathbb{N}_0. \tag{5.2}$$

An interesting family of generalized Hausdorff matrices, which contains the one associated with ordinary Hausdorff summability (5.1), is the one given by $H^{(\zeta)}_{(\mu_n)}$, for (μ_n) as in (5.2). Indeed, the behavior of the induced operators $\mathcal{H}^{(\zeta)}_{\mu}$ on sequence spaces has been object of study (or play a central role) in several papers, see for instance [JRT74; GRT77; Rho81; Rho89]. To avoid cumbersome notation, we denote such Hausdorff matrices by $H^{(\zeta)}_{\mu}$. Note that, in this case, the non-zero elements of $H^{(\zeta)}_{\mu}$ are given by

$$H^{(\zeta)}_{\mu}(i,j) = \binom{i+\zeta}{i-j} \int_0^1 t^{j+\zeta}(1-t)^{i-j}\, d\mu(t), \qquad 0 \le j \le i.$$

On the other hand, ordinary Hausdorff matrices $H^{(0)}_{\mu}$ (i.e., with sequences as in (5.1)) have been considered in [GS01; GP06] as operators on spaces of holomorphic functions (Hardy, Bergman, Dirichlet, Bloch and BMOA) on the disc via the coefficients of the power series of such functions. One of the crucial points in these studies is to represent such operators in terms of averages of weighted composition semigroups. We note here that such representation also holds for the Hausdorff matrices of type $H^{(\zeta)}_{\mu}$ for $\zeta \ge 0$. To see this, let $f \in \mathcal{O}(\mathbb{D})$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$, and let $\mathcal{H}^{(\zeta)}_{\mu}$ be the operator given by

$$\mathcal{H}^{(\zeta)}_{\mu} f(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} H^{(\zeta)}_{\mu}(n,k) a_k \right) z^n, \qquad z \in \mathbb{D}.$$

A few computations show that the series given above is absolutely convergent. Indeed, for each $z \in \mathbb{D}$,

$$\begin{aligned}\sum_{n=0}^{\infty} \left| \sum_{k=0}^{n} H^{(\zeta)}_{\mu}(n,k) a_k \right| |z|^n &\le \int_0^1 t^{\zeta} \sum_{k=0}^{\infty} |a_k| \frac{|z|^k t^k}{(1-|z|(1-t))^{k+\zeta+1}}\, d|\mu|(t) \\ &\le \frac{1}{1-|z|} |\mu|((0,1]) \sum_{k=0}^{\infty} |a_k||z|^k.\end{aligned}$$

Let $\psi(t) = \log(1/t)$, $t \in (0,1]$, and set $\nu = \psi(\mu)$, i.e., ν is the image measure (on $[0,\infty)$) of μ. Then, the absolute convergence of the series above gives

$$\begin{aligned}\mathcal{H}^{(\zeta)}_{\mu} f(z) &= \int_0^1 t^{\zeta} \sum_{k=0}^{\infty} a_k \frac{z^k t^k}{(1-z(1-t))^{k+\zeta+1}}\, d\mu(t) \\ &= \int_0^1 \frac{t^{\zeta}}{(1-z(1-t))^{\zeta+1}} f\left(\frac{zt}{1-z(1-t)} \right) d\mu(t) \\ &= \int_0^{\infty} u_t(z) C_{\phi_t} f(z)\, d\nu(t), \qquad z \in \mathbb{D},\ f \in \mathcal{O}(\mathbb{D}),\end{aligned}$$

where (ϕ_t) is the semiflow given by

$$\phi_t(z) = \frac{e^{-t} z}{(e^{-t}-1)z+1}, \qquad z \in \mathbb{D},\ t \ge 0, \tag{5.3}$$

and $u_t(z) = \left(\frac{\phi_t(z)}{z}\right)^{\zeta} \frac{1-\phi_t(z)}{1-z}$, $z \in \mathbb{D}, t \geq 0$, which is a semicocycle for (ϕ_t).

Then, it seems reasonable to study operators $\mathcal{H}_{\mu}^{(\zeta)}$ from the viewpoint of subordination in terms of weighted composition semigroups related to the semiflow (ϕ_t) and the semicocycle (u_t) as above. Even more, we consider operators $\mathcal{H}$ of the type

$$\mathcal{H}f = \int_0^{\infty} v_t C_{\psi_t} f \, d\nu(t), \qquad f \in \mathcal{O}(\mathbb{D}), \tag{5.4}$$

where (ψ_t) is a semiflow, (v_t) is a semicocycle for (ψ_t), and ν is a complex bounded Borel measure on $[0, +\infty)$. For practical reasons, we restrict our study to operators alike to the semiflow (5.3). In particular we ask (ψ_t) to consist of non-invertible morphism of $\mathbb{D}$ with DW point $a \in \mathbb{D}$ and which have a finite number (possibly none) of repulsive fixed points on $\mathbb{T}$, see Sections 5.2 and 5.3 for more details. One example of such a semiflow is, for $n \in \mathbb{N}$, the semiflow $(\phi_{t,n})$ given by

$$\phi_{t,n}(z) = \frac{e^{-t}z}{((e^{-nt}-1)z^n + 1)^{1/n}}, \qquad z \in \mathbb{D}, \, t \geq 0, \tag{5.5}$$

which appears in [Sis98]. Note that $(\phi_t) = (\phi_{t,1})$.

For such a semiflow (ψ_t), we use the representation of (ψ_t) in terms of its univalent function h_a (4.4), and the representation of the semicocycle (u_t) in terms of an holomorphic function (on E_a) ω (see Proposition 4.2.2). Then, the operator $\mathcal{H}$ can be written as

$$\mathcal{H}f(z) = \frac{1}{\omega(z)} \int_z^a \frac{\omega(\xi)}{\Psi(\xi)} f(\xi) \, d\nu\left(\frac{1}{\Psi'(a)} \log \frac{h_a(\xi)}{h_a(z)}\right), \qquad z \in \mathbb{D}, \tag{5.6}$$

where Ψ is the generator of (ψ_t). Operators above with $\omega(z) = z$ and ν equal to the Lebesgue measure, which are often labeled as Cesàro operators, have been object of study in different papers. Indeed, the boundedness on Hardy spaces of these Cesàro operators was analyzed in [Sis93] using semigroup theory techniques. A version (different from ours) of generalized Cesàro operators, which is not connected with semigroup theory, was treated in [AP10; Per08] to obtain their fine spectrum in some classical Banach spaces of analytic functions on $\mathbb{D}$.

Our study focuses on the boundedness and, mainly, the spectrum of operators (5.4) and (5.6). With such a purpose, the crucial point is the description of the spectrum of the infinitesimal generator Δ of the semigroup $(v_t C_{\psi_t})$. This spectrum is then transferred to the one of $\mathcal{H}$ via the functional calculus of sectorial operators and the spectral mapping theorems given in Section 2.2.

Also, recall that such generators are given by first order linear differential operators of the type

$$\Delta f = \Psi f' + gf, \qquad f \in \mathrm{Dom}(\Delta),$$

where g is the generator of (v_t), see Proposition 4.1.2. Then, the representation of the semicocycle (v_t) in terms of a non-vanishing holomorphic function ω on E_a, is convenient to obtain $\sigma(\Delta)$. Another key point result is that, if v_t is continuous at the repulsive points of (ψ_t), then ω has singularities of fractional type at these points.

We note that, in a different direction from the one taken here, spectral properties of weighted composition operators with DW point in $\mathbb{D}$ have been treated in several settings through different papers, see for instance [AL04; Bou12; GL18; GLW20; MS02]. In particular, spectral inclusions for weighted composition operators vC_ψ were obtained in [GLW20] under fairly general conditions for a long list of Banach spaces of holomorphic functions with domain the unit ball of a Banach space (for instance, Hardy, Bergman, Korenblum spaces on the polydisc).

5.1 Axiomatic spaces

Recall that by $Mul(X)$ we denote the space of multipliers of a (Banach space) $X \subseteq \mathcal{O}(\mathbb{D})$. On the other hand, by B_b (with $b \in \mathbb{D}$), we denote the backshift operator given by

$$(B_b f)(z) = \frac{f(z) - f(b)}{z - b}, \qquad f \in \mathcal{O}(\mathbb{D}).$$

For $\gamma \geq 0$, the Korenblum class $\mathcal{K}^{-\gamma}(\mathbb{D})$ is the Banach space of analytic functions f on $\mathbb{D}$ given by

$$\mathcal{K}^{-\gamma}(\mathbb{D}) := \{f \in \mathcal{O}(\mathbb{D}) : \|f\|_{\mathcal{K}^{-\gamma}} := \sup_{z\in\mathbb{D}}(1 - |z|^2)^{\gamma}|f(z)| < \infty\},$$

which is a Banach space when endowed with the norm $\|\cdot\|_{\mathcal{K}^{-\gamma}}$. Note that $\gamma = 0$ corresponds to $H^\infty(\mathbb{D})$.

In this chapter, we deal with Banach spaces $X \hookrightarrow \mathcal{K}^{-\gamma}(\mathbb{D})$ which contain the constant functions and satisfy the following conditions

(**Gam'1**) $\quad Mul(X) = H^\infty(\mathbb{D})$.

(**Gam'2**) $\quad$ For $b \in \mathbb{D}$, $B_b(X) \subseteq X$.

For $m \in \mathbb{N}_0$, set $Z_b^m = \{f \in X : f$ has a zero at b of order at least $m\}$ and let P_m be the space of polynomials of degree at most m. Since $P_m \subseteq X$ by (**Gam'1**), then $X = P_m \oplus Z_b^m$ whence Z_b^m has finite codimension in X. Moreover, the projection $X \to P_m$ is continuous since $X \hookrightarrow \mathcal{O}(\mathbb{D})$. This implies that the projection $X \to Z_b^m$ is continuous too. Thus, Z_b^m is a closed subspace of X.

Hence, if (**Gam'1**) and (**Gam'2**) hold, then Z_b^m is the range space of the multiplication operator by the function $z \mapsto z - b$. Also, $\|f\|_X \simeq \|B_b f\|_X$ for $f \in Z_b^m$ by the open mapping theorem.

Definition 5.1.1. Let X be a Banach space X of holomorphic functions in the disc containing the constant functions and satisfying properties (**Gam'1**) and (**Gam'2**). For $\gamma \geq 0$, we say that X is a γ^∞-space if the inclusion $X \hookrightarrow \mathcal{K}^{-\gamma}(\mathbb{D})$ holds.

We list below some examples of γ^∞ spaces. See Subsection 6.2.A for the proof that they fulfill all the properties required to be γ^∞-spaces.

1. *Little Korenblum classes.*

   If $\gamma > 0$, then the closure of polynomials in $\mathcal{K}^{-\gamma}(\mathbb{D})$ is the Little Korenblum growth class $\mathcal{K}_0^{-\gamma}(\mathbb{D})$ given by

$$\mathcal{K}_0^{-\gamma}(\mathbb{D}) := \{f \in \mathcal{K}^{-\gamma}(\mathbb{D}) \ : \ \lim_{|z|\to 1}(1-|z|^2)^{\gamma}|f(z)| = 0\},$$

   with norm $\|\cdot\|_{\mathcal{K}^{-\gamma}}$.

   It is clear that $H^{\infty}(\mathbb{D})$ and $\mathcal{K}^{-\gamma}(\mathbb{D}), \mathcal{K}_0^{-\gamma}(\mathbb{D})$ are γ^{∞} spaces for every $\gamma > 0$. However, we are only interested in $\mathcal{K}_0^{-\gamma}(\mathbb{D})$ as we explain in Section 5.4.

2. *Hardy spaces of integrable functions.* For $1 \le p < \infty$, let $H^p(\mathbb{D})$ be the Hardy space on $\mathbb{D}$ formed by all functions $f \in \mathcal{O}(\mathbb{D})$ such that

$$\|f\|_{H^p} = \sup_{0<r<1}\left(\int_0^{2\pi}|f(re^{i\theta})|^p\,\frac{d\theta}{2\pi}\right)^{1/p} < \infty,$$

   endowed with the norm $\|\cdot\|_{H^p}$. Then $H^p(\mathbb{D})$ is a γ^{∞} space for $p \ge 1$ with $\gamma = 1/p$.

3. *Weighted Bergman spaces.* Let $1 \le p < \infty$ and $\sigma > -1$. $\mathcal{A}^p_{\sigma}(\mathbb{D})$ denotes the weighted Bergman space formed by all holomorphic functions in $\mathbb{D}$ such that

$$\|f\|_{\mathcal{A}^p_{\sigma}} := \left(\int_{\mathbb{D}}|f(z))|^p d\mathcal{A}_{\sigma}(z)\right)^{1/p} < \infty,$$

   where $d\mathcal{A}_{\sigma}(z) = (1-|z|^2)^{\sigma}\,dA(z)$, and where dA is the Lebesgue measure of $\mathbb{D}$. The space $\mathcal{A}^p_{\sigma}(\mathbb{D})$, with norm $\|\cdot\|_{\mathcal{A}^p_{\sigma}}$, is a γ^{∞}-space with $\gamma = \frac{\sigma+2}{p}$.

5.2 Semiflows

One of the aims of this chapter is the spectral study of averaging operators and weighted composition operators related to semiflows with similar characteristics as $(\phi_{t,n})$ given in (5.5). We specify below the axiomatic properties of the semiflows (ψ_t) we deal with here.

(**SFlow1**) (ψ_t) is a semiflow of non-invertible (for $t > 0$) morphisms of $\mathbb{D}$ such that its *DW* point a lies in $\mathbb{D}$.

(**SFlow2**) There exists a finite (possibly empty) set $\{z_1, \dots, z_n\} \subseteq \mathbb{T}$ such that

   (a) For each $t \ge 0$ and $i = 1, \dots, n$, z_i is a fixed point of (ψ_t) in the sense that $\lim_{\mathbb{D}\ni z\to z_i}\psi_t(z) = z_i$.

   (b) For any neighborhoods Ω_i in $\mathbb{D}$ of the points z_i (i.e. $\Omega_i \supset U_i \cap \mathbb{D}$ for some open set $U_i \subseteq \mathbb{C}$ containing z_i), $i = 1, \dots, n$, and any open set Ω_a containing a, there exists $t \ge 0$ for which $\psi_t(\mathbb{D}\setminus(\cup_{i=1}^n\Omega_i)) \subseteq \Omega_a$.

   (c) For $t \ge 0$, $i = 1, \dots, n$, the limit $\lim_{z\to z_i}\psi'_t(z)$ exists in $\mathbb{C}$.

We label the points $z_1, \dots, z_n$ as above as *repulsive (fixed) points of the semiflow* (ψ_t).

From now on, p denote either the canonical projection of E_a, or the canonical projection of the universal covering space of $\mathbb{C} \setminus \{a\}$. Recall that, given a semiflow (ψ_t) with DW point $a \in \mathbb{D}$, we denote by $(\widetilde{\psi_t})$ to its lifting semigroup to E_a, see Remark 4.2.1. Also, given a holomorphic function f in $\mathbb{D} \setminus \{a\}$ (or in $\mathbb{D}$), we denote by $\widetilde{f}$ to the holomorphic function on E_a given by $\widetilde{f}(z') = f(p(z'))$, $z' \in E_a$. With this notation, $p^{-1}\{z_1, \dots, z_n\} \subset \overline{E_a}$. Moreover, we say that $\widetilde{\nu}$ is a path in E_a with starting (ending) point $z_i' \in p^{-1}(z_i)$ if $\widetilde{\nu}$ is a path in $\overline{E_a}$ with starting (ending) point z_i' such that $\widetilde{\nu}(0,1) \subset E_a$.

(**SFlow3**) For $t \geq 0$, one has $\psi_t', (\psi_t')^{-1} \in H^\infty(\mathbb{D})$.

Preceding axioms refer to intrinsic properties of the semiflow (ψ_t), and it is readily seen that the semiflows $(\phi_{t,n})$, $n \in \mathbb{N}$, satisfy all of them. Next condition connects (ψ_t) with the space of analytic functions. It is key for estimates of the essential norm of suitable weighted composition operators.

(**SFlow4**) Let X be a γ^∞-space with $\gamma \geq 0$, and let $((\psi_t')^\gamma)$ be as in Definition 4.2.6. Then, $(\psi_t')^\gamma C_{\psi_t} \in L(X)$ for $t \geq 0$, and

$$\sup_{t \geq 0} \|(\psi_t')^\gamma C_{\psi_t}\|_{L(X)} < \infty.$$

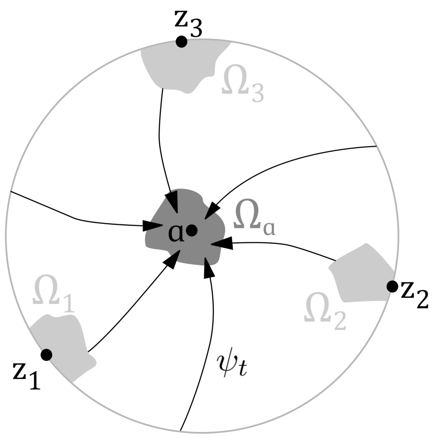


Figure 5.1: Graphical illustration of a semiflow (ψ_t) satisfying properties (**SFlow2**)(a) & (b), with limit fixed points z_1, z_2, z_3 lying on the boundary $\mathbb{T}$.

Let $X = \mathcal{A}^p_\sigma(\mathbb{D})$ with $\sigma > -1, p \geq 1$ (and $\gamma = (\sigma + 2)/p$) or $X = \mathcal{K}_0^{-\gamma}$ with $\gamma > 0$. An application of the Schwarz-Pick lemma shows that in fact $\|(\phi')C_\phi\|_L \leq 1$ for an arbitrary univalent $\phi \in Mor(\mathbb{D})$. Hence condition (**SFlow4**) is superfluous for these spaces. Proposition below shows that the semiflow $(\phi_{t,n})$, with $n \in \mathbb{N}$, satisfies (**SFlow4**) on $H^p(\mathbb{D})$.

Proposition 5.2.1. *Let* $(\phi_{t,n})$ *be the semiflow given in* (5.5)*. Then, for* $n \in \mathbb{N}$ *and* $p \geq 1$*, one has*

$$\sup_{t\geq 0} \|(\phi'_{t,n})^{1/p} C_{\phi_{t,n}}\|_{L(H^p)} < \infty.$$

Proof. By Littlewood's subordination theorem [Dur70, Th. 1.7], we have $\|C_{\phi_{n,t}}\|_{H^p} \leq 1$ for $t \geq 0$. Therefore, $\|(\phi'_{n,t})^{1/p} C_{\phi_{n,t}}\|_{L(H^p)} \leq \|(\phi'_{t,n})^{1/p}\|_\infty$, $t \geq 0$, and we get, for each $M > 0$,

$$\sup_{t\in[0,M]} \|(\phi'_{n,t})^{1/p} C_{\phi_{n,t}}\|_{L(H^p)} < \infty.$$

So let us prove that $\sup_{t\geq M} \|(\phi'_{t,n})^{1/p} C_{\phi_{n,t}}\|_{L(H^p)} < \infty$ for some $M > 0$. To do this, let P_z denote the Poisson kernel, i.e.

$$P_z(e^{i\theta}) = \mathfrak{Re}\left(\frac{1+ze^{i\theta}}{1-ze^{i\theta}}\right) = \frac{1-|z|^2}{1-2|z|\cos(\theta+\arg z)+|z|^2}, \qquad z \in \mathbb{D},\ \theta \in (-\pi,\pi].$$

Since $|f|^p$ is a subharmonic function, we have

$$\begin{aligned}\left\|(\phi'_{t,n})^{1/p} C_{\phi_{t,n}} f\right\|_{H^p}^p &= \int_{|z|=1} |\phi'_{t,n}||f\circ\phi_{t,n}|^p d\mu_0 = \int_{\phi_{t,n}\{|z|=1\}} |f|^p \, d\mu_t \\ &\leq \int_0^{2\pi} |f(e^{i\theta})|^p \int_{\phi_{t,n}\{|z|=1\}} P_z(e^{-i\theta})\, d\mu_t(z) d\theta, \qquad f \in H^p(\mathbb{D}),\ t \geq 0,\end{aligned}$$

where $d\mu_t$ is the arc-length measure of the closed curve $\phi_{t,n}(\{|z|=1\})$. Note that

$$\phi_{t,n}(z) = (\rho_{nt}\psi_{nt}(z^n) + \lambda_{nt})^{1/n}, \qquad z \in \overline{\mathbb{D}},\ t \geq 0,\ n \in \mathbb{N}, \tag{5.7}$$

where

$$\rho_s = \frac{1}{2-e^{-s}}, \qquad \lambda_s = \frac{1-e^{-s}}{2-e^{-s}} \quad \text{and} \quad \psi_s(z) = \frac{z-(1-e^{-s})}{1-(1-e^{-s})z}, \qquad s \geq 0,\ z \in \overline{\mathbb{D}}.$$

Since $\psi_s \in Aut(\mathbb{D})$ with $\psi_s(\mathbb{T}) = \mathbb{T}$, $s \geq 0$, it follows that $\phi_{t,n}(\{|z|=1\}) = \{z \in \overline{\mathbb{D}},\ |z^n - \lambda_{nt}| = \rho_{nt}\}$. Let $\gamma_{t,1}, \ldots, \gamma_{t,n}$ be the image of the circle $\{|z-\lambda_{nt}| = \rho_{nt}\}$ through the n different branches of the fractional power $(\cdot)^{1/n}$ (in $\mathbb{C}\setminus(-\infty,0]$), so that $\phi_{t,n}(\{|z|=1\}) = \cup_{i=1}^n \gamma_{t,i}$. Due to the symmetry between $\gamma_{t,1},\ldots,\gamma_{t,n}$, the proof is done if we show that

$$\sup_{t\geq M,\,\theta\in(-\pi,\pi]} \int_{\gamma_{t,1}} P_z(e^{-i\theta})\, d\mu_t(z) < \infty,$$

where $\gamma_{t,1}$ is the arc containing the point $z = 1$. Let $\gamma_{t,1}$ also denote the arc-length parametrization of the path $\gamma_{t,1}$ with $\gamma_{t,1}(0) = 1$. For $\ell > 0$, it is readily seen that

$$\sup_{t\geq M \text{ and } (|s|>\ell \text{ or } |\theta|>\ell)} P_{\gamma_{t,1}(s)}(e^{-i\theta}) < \infty.$$

One also has $\sup_{t\geq 0}(\text{length}(\phi_{n,t}\{|z|=1\})) < \infty$ by (5.7). Hence, we are done if we prove that, for some $\ell > 0$,

$$\sup_{t\geq M,\,|\theta|\leq\ell} \int_{|s|\leq\ell} P_{\gamma_{t,1}(s)}(e^{-i\theta})\, ds < \infty.$$

To do this, let ν_t be the arc-length parametrization of the circle $\{|z-\lambda_{nt}|=\rho_{nt}\}$ with $\nu_t(0)=1$. For sufficiently small $\ell>0$ one has, using the respective Taylor expansions,

$$|\gamma_{t,1}(s)-\nu_t(s)|\simeq |s|^2 \quad \text{and} \quad |\xi_{t,s}-e^{i\theta}|\gtrsim \max\{|\theta|^2,|s-\theta|\}, \quad \text{for } |s|,|\theta|\le \ell,\ t\ge M,$$

where $\xi_{t,s}$ is any point in the segment $[\nu_t(s),\gamma_{t,1}(s)]$. Hence, by the mean value theorem, we have

$$\begin{aligned}|P_{\gamma_{t,1}(s)}(e^{-i\theta})-P_{\nu_t(s)}(e^{-i\theta})| &\lesssim \frac{|\gamma_{t,1}(s)-\nu_t(s)|}{(\operatorname{dist}\{[\gamma_{t,1}(s),\nu_t(s)],e^{i\theta}\})^2}\\ &\lesssim \frac{s^2}{\theta^4+(s-\theta)^2}, \qquad |s|,|\theta|\le\ell,\ t\ge M,\end{aligned}$$

where dist denotes the distance between two subsets of $\mathbb{C}$.

On the other hand, by the mean value property of harmonic functions, we obtain $\int_{\nu_t} P_z(e^{-i\theta})\,d\mu(z) = P_{\lambda_{nt}}(e^{-i\theta})$, which is uniformly bounded for $t\ge 0, \theta\in(-\pi,\pi]$. Putting everything together one gets

$$\sup_{t\ge M,\,|\theta|\le\ell}\int_{|s|\le\ell} P_{\gamma_{t,1}(s)}(e^{-i\theta})\,ds \le \sup_{t\ge M,\,|\theta|\le\ell}\left(P_{\lambda_{nt}}(e^{-i\theta})+\int_{|s|\le\ell}\frac{s^2}{\theta^4+(s-\theta)^2}\,ds\right)<\infty.$$

To see this, note that a primitive of $s\mapsto \frac{s^2}{\theta^4+(s-\theta)^2}$ is given by

$$x\mapsto (1-\theta^2)\left(\arctan\left(\frac{x+\theta}{\theta^2}\right)+\theta\log(\theta^4+(x-\theta)^2)\right)+x.$$

Thus, the proof is finished. □

Next we give some technical results on semiflows (ψ_t) satisfying the axioms (**SFlow1**)-(**SFlow4**).

Lemma 5.2.2. *Let (ψ_t) be a semiflow satisfying* (**SFlow1**) *with DW point $a\in\mathbb{D}$, and let h be the univalent function associated with (ψ_t). A family of points $\{z_1,\dots,z_n\}\subseteq\mathbb{T}$ satisfies the conditions* (**SFlow2**)(a)&(b) *if and only if $\lim_{\mathbb{D}\ni z\to z_i}|h_a(z)|=\infty$ and, for every open neighborhoods Ω_i in $\mathbb{D}$ of z_i, one has $\sup_{z\in\mathbb{D}\setminus(\cup_i\Omega_i)}|h_a(z)|<\infty$.*

Proof. It is readily seen that (**SFlow2**)(b) is equivalent to $\sup_{z\in\mathbb{D}\setminus(\cup_i\Omega_i)}|h_a(z)|<\infty$ for every open neighborhoods Ω_i in $\mathbb{D}$ of z_i. So assume (**SFlow2**)(b) holds, and let us prove that (ψ_t) fulfills (**SFlow2**)(a) if and only if $\lim_{\mathbb{D}\ni z\to z_i}|h_a(z)|=\infty$.

First, fix $i\in\{1,\dots,n\}$ and assume $\lim_{\mathbb{D}\ni z\to z_i}\psi_t(z)=z_i$ for $t\ge0$. If there existed $M>0$ and a sequence $(w_k)\subseteq\mathbb{D}$ for which $\lim_{j\to\infty}w_k=z_i$ and $|h_a(w_k)|\le M$, then one would have $|h_a(\psi_t(w_k))|\le Me^{(\mathfrak{Re}\,\Psi'(a))t}$ for $k\in\mathbb{N}$, see (4.4). This implies that for $t>0$ big enough, all the points $(\psi_t(w_k))_{k=1}^{\infty}$ are contained in a neighborhood of a, reaching a contradiction. Thus $\lim_{z\to z_i}\psi_t(z)=z_i$ as claimed.

Assume now $\lim_{\mathbb{D}\ni z\to z_i}h_a(z)=\infty$ for $i=1,\dots,n$. If there existed $j\in\{1,\dots,n\}$ and $t>0$ for which $\lim_{\mathbb{D}\ni z\to z_j}\psi_t(z)\ne z_j$, then there would be an open neighborhood Ω_j

in $\mathbb{D}$ of z_j and a sequence $(w_k)_{k=1}^{\infty} \subset \mathbb{D}$ such that $\lim_{k\to\infty} w_k = z_j$ and $\psi_t(w_k) \notin \Omega_j$ for all $k \in \mathbb{N}$. Since $\lim_{\mathbb{D}\ni z\to z_i} h_a(z) = \infty$, we have $\lim_{k\to\infty} h_a(\psi_t(w_k)) = \infty$ by (4.4). Since we have assumed (**SFlow2**)(b), this would imply (by which we have already proven) that, for each $t \geq 0$, there is $i \neq j$ such that the sequence $(\psi_t(\omega_k))_{k=1}^{\infty}$ has z_i as accumulation point. As a consequence, there would be orbits of (ψ_t) from points arbitrary close to z_j to points arbitrary close to z_i. But this cannot hold since $|h_a|$ is decreasing through an orbit of (ψ_t), and we have proven above that h_a is bounded on subsets $A \subset \mathbb{D}$ with $z_1, \ldots, z_n \notin \overline{A}$. Thus, we have $\lim_{\mathbb{D}\ni z\to z_j} \psi_t(z) = z_j$ for $j = 1, \ldots, n$ and $t \geq 0$, and the proof is finished.

□

For $\lambda \in (-\pi/2, \pi/2)$, a holomorphic function $f \in \mathcal{O}(\mathbb{D})$ with $f(0) = 0$ is said to be λ-spirallike if, for each $z \in f(\mathbb{D})$, the spiral

$$\{z \exp(-e^{i\lambda}t) \,:\, t \geq 0\},$$

lies in $f(\mathbb{D})$, see [Dur83, Section 2.7]. This class of functions is closely related to semiflows (ψ_t) with *DW* point in $\mathbb{D}$, since the univalent function h associated with (ψ_t) is $\arg(-\Psi'(a))$-spirallike by (4.4).

For $\lambda \in (-\pi/2, \pi/2)$, the λ-argument, denoted by $\arg_\lambda$, is defined by $\theta = \arg_\lambda z$, $z \in \mathbb{C} \setminus \{0\}$, if there exists $t \in \mathbb{R}$ such that $z = e^{\lambda t + i\theta}$. Note that $\arg_0 = \arg$. Note also that one has freedom for the choice of $\arg_\lambda z$ up to an integer multiple of 2π as in the case of $\arg z$.

The following result was given in [KS12, Th. 3.2].

Lemma 5.2.3. *Let f be a λ-spirallike function with $\lambda \in (-\pi/2, \pi/2)$. Then the limit*

$$U(s) = \lim_{r\to 1^-} \arg_\lambda f(re^{is})$$

exists for every $s \in \mathbb{R}$, $U(s)$ is non-decreasing in s, $U(s + 2\pi) = U(s) + 2\pi$, and $U(s) = (\lim_{\varepsilon\to 0^+} U(s + \varepsilon) + U(s - \varepsilon))/2$.

Lemma 5.2.4. *Let (ψ_t) be a semiflow satisfying (**SFlow1**) with DW point a. Then*

$$(\forall t \geq 0) \qquad \sup_{z'\in E_a} |\arg(\widetilde{\psi}_t(z') - a) - \arg(z' - a)| < \infty.$$

Proof. Via composition with a suitable Möbius transform, we can assume $a = 0$ without loss of generality.

Since $\arg z' = \mathfrak{Im}\, \mathrm{Log}\, z'$, $z' \in E_0$, it is readily seen that, for each $t \geq 0$, the function $\arg \widetilde{\psi}_t(z)' - \arg z'$ induces an harmonic function V_t on $\mathbb{D}$ given by

$$V_t(z) = \mathfrak{Im}\left(\mathrm{Log}\, \frac{\psi_t(z)}{z}\right) = \mathfrak{Im}\left(-\int_0^t F(\psi_s(z))\, ds\right), \qquad z \in \mathbb{D}. \tag{5.8}$$

There, F is the holomorphic function associated with the generator Ψ by (4.2), and we take the branch of the above logarithm for which $0 \xmapsto{\mathrm{Log}(\cdots)} \Psi'(0)t$.

So assume $\sup_{z\in\mathbb{D}}|V_t(z)| = \infty$ for some $t > 0$, whence there exists a sequence $(z_n) \subset \mathbb{D}$ with $|z_n| \xrightarrow[n\to\infty]{} 1$ such that $|V_t(z_n)| \xrightarrow[n\to\infty]{} \infty$. Hence, for $M \in \mathbb{N}$ big enough, there exists a sequence of functions $s_n : [0, 2\pi) \to [0, t]$, for $n \geq M$, such that

$$\arg\psi_{s_n(\theta)}(z_n) = \theta \mod 2\pi, \quad \text{and} \quad |V_{s_n(\theta)}(z_n)| < 2\pi, \qquad \theta \in [0, 2\pi),\ n \geq M.$$

By the inequality above, we have

$$|V_{t-s_n(\theta)}(\psi_{s_n(\theta)}(z_n))| = |V_t(z_n) - V_{s_n(\theta)}(z_n)| \xrightarrow[n\to\infty]{} \infty, \quad \text{for all } \theta \in [0, 2\pi).$$

Thus $|\psi_{s_n(\theta)}(z_n)| \xrightarrow[n\to\infty]{} 1$ since $V_t(z) \xrightarrow[t\to 0^+]{} 0$ uniformly on compact subsets of $\mathbb{D}$ (which follows by the continuity of $\psi_t(z)$ on (t, z) and (5.8)). Hence,

$$\lim_{r\to 1^-} \arg_\lambda h(re^{i\theta}) = \lim_{n\to\infty} \arg_\lambda h(\psi_{s_n(\theta)}(z_n)) = \lim_{n\to\infty} h(z_n), \quad \theta \in [0, 2\pi),$$

where $\lambda = \arg(-\Psi'(0)) \in (-\pi/2, \pi/2)$ and where we have used the equality $\arg_\lambda(h(\psi_s(z))) = \arg_\lambda(h(z))$ for $z \in \mathbb{D}$, $s \geq 0$, see (4.4). Also, the existence of the limit above is guaranteed by Lemma 5.2.3. As a consequence, one gets that the function from $\mathbb{R}$ to $\mathbb{R}$ given by $s \mapsto \lim_{r\to 1^-} \arg_\lambda h(re^{is})$ is constant $\mod 2\pi$, which contradicts Lemma 5.2.3. Thus, our assumption $\sup_{z\in\mathbb{D}}|V_t(z)| = \infty$ was wrong, and the proof is finished. □

Lemma 5.2.5. *Let (ψ_t) be a semiflow satisfying axioms* (**SFlow1**) *and* (**SFlow2**) (a)&(b), *with repulsive points $z_1, \ldots, z_n \subseteq \mathbb{T}$, DW point a, and with univalent function h. Fix $i \in \{1, \ldots, n\}$ and take $z_i' \in p^{-1}(z_i) \subseteq \overline{E_a}$. Then, there exist $K \in \mathbb{R}$ and arbitrary small open neighborhoods B_1, B_2 in E_a of z_i' (i.e., $B_i = A_i \cap E_a$ for an open set $A_i \ni z_i'$) satisfying the following:*

- *$B_1 \subsetneq B_2$,*
- *$\widetilde{\psi}_{T(z')}(z') \in B_2 \setminus B_1$ for all $z' \in B_1$, where $T : B_1 \to (0, \infty)$ is given by*

$$T(z') = \frac{-1}{\mathfrak{Re}\,\Psi'(a)} \log|\widetilde{h}(z')| + K, \qquad z \in B_1.$$

Proof. Let $\theta_i \in \arg(z_i - a) + 2\pi\mathbb{Z}$ such that $z_i' = (z_i, \theta_i)$. Let $\mathcal{K} = \{(z, \theta) \in E_a : |\theta - \theta_i| \leq \pi\}$. Let Ω be an open subset of $\overline{E_a}$ satisfying the following properties

- $\overline{\mathcal{K}} \cap p^{-1}(\{z_1, \ldots, z_n\}) \subset \Omega$.
- If $a', b' \in \overline{\mathcal{K}} \cap p^{-1}(\{z_1, \ldots, z_n\})$ with $a' \neq b'$, then a', b' belong to different components of Ω.
- If $\widetilde{\nu}$ is a path in E_a from z_i' to a point $c' \in E_a \setminus \mathcal{K}$, then ν goes through $\mathcal{K} \setminus \Omega$.

It is readily seen that such a set Ω exits (it is enough to take small enough open neighborhoods of each point in $\overline{\mathcal{K}} \cap p^{-1}(\{z_1, \dots, z_n\})$).

One has $\sup_{z' \in \mathcal{K}\setminus\Omega} |\widetilde{h}(z')| =: M < \infty$ by Lemma 5.2.2. So let Ω_i be the component of Ω containing the point z_i', and let $U := \{z' \in E_a \cap \Omega_i : |\widetilde{h}(z')| > M+1\}$. Then U is an open neighborhood in E_a of z_i by Lemma 5.2.2. Set

$$T(z') := \frac{1}{\mathfrak{Re}\,\Psi'(a)} \log \frac{M+1/2}{|\widetilde{h}(z')|}, \qquad z' \in U.$$

Then $T(z') > 0$ for $z' \in U$ and

$$|\widetilde{h}(\widetilde{\psi}_{T(z')}(z'))| = |h(\psi_{T(z')}(p(z'))| = e^{\mathfrak{Re}\,\Psi'(a)T(z')}|h(p(z'))| = M+1/2, \qquad z' \in U.$$

In particular, $\widetilde{h}(\widetilde{\psi}_{T(z')}(z')) \notin U$. Since $|\widetilde{h}(\widetilde{\psi}_t(z'))|$ is decreasing on t, then $|\widetilde{h}(\widetilde{\psi}_t(z'))| \geq M+1/2$ for every $t \in [0, M+1/2]$. By the properties of Ω, it follows that $\widetilde{\psi}_t(z') \in \Omega_i$ for each $t \in [0, M+1/2]$, and we obtain the claim for the subsets $B_1 = U$, $B_2 = E_a \cap \Omega_i$. $\square$

Lemma 5.2.6. *Let (ψ_t) be a semiflow satisfying axioms* (**SFlow1**) *and* (**SFlow2**) (a)&(b)*, with DW point a, with repulsive points $z_1, \dots, z_n \subset \mathbb{T}$ and with univalent function h. Let $\mathcal{K}$ be a subset in E_a with $\sup_{z' \in \mathcal{K}} |\arg z'| < \infty$, and such that $\{a, z_1, \dots, z_n\} \cap \overline{p(\mathcal{K})} = \emptyset$. Then $\sup_{z' \in \mathcal{K}} |(\arg \widetilde{h})(z')| < \infty$.*

Proof. It follows by Lemma 5.2.2 that $\sup_{z' \in \mathcal{K}} |\widetilde{h}(z')| < \infty$. Moreover, since $a \notin \overline{p(\mathcal{K})}$, we also get $\inf_{z' \in \mathcal{K}} |\widetilde{h}(z')| > 0$. By the functional equation (4.4), there exists $s > 0$ such that the closed set $\overline{\Psi_s(p(\mathcal{K}))}$ is compact in $\mathbb{D}\setminus\{a\}$. In addition, $\sup_{z' \in \mathcal{K}} |\arg(\widetilde{\psi}_s(z')) - \arg z'| =: M < \infty$ by Lemma 5.2.4. Putting everything together, one obtains that the closed set $\overline{\widetilde{\psi}_s(\mathcal{K})}$ is compact in E_a. Thus $\sup_{z' \in \widetilde{\psi}_s(\mathcal{K})} |(\arg \widetilde{h})(z')| < \infty$ by the Weierstrass extreme value theorem.

On the other hand, it is readily seen from (4.4) that

$$(5.9) \qquad (\arg \widetilde{h})(\widetilde{\psi}_t(z')) = (\arg \widetilde{h})(z') + \mathfrak{Im}\,\Psi'(a)t, \qquad t \geq 0, z' \in E_a.$$

Therefore,

$$\sup_{z' \in \mathcal{K}} |(\arg \widetilde{h})(z')| \leq |\mathfrak{Im}\,\Psi'(a)|s + \sup_{z' \in \widetilde{\psi}_s(\mathcal{K})} |(\arg \widetilde{h})(z')| < \infty,$$

and the proof is finished. $\square$

Recall that, since h has no zeroes on $\mathbb{D} \setminus \{a\}$, the fractional power $\widetilde{h}^\lambda$ is well defined in E_a for arbitrary $\lambda \in \mathbb{C}$.

Corollary 5.2.7. *Let (ψ_t), h and $\mathcal{K}$ be as in Lemma 5.2.6, and let $\lambda \in \mathbb{C}$. Then $\sup_{z' \in \mathcal{K}} |\widetilde{h}^\lambda(z')| < \infty$ and $\inf_{z' \in \mathcal{K}} |\widetilde{h}^\lambda(z')| > 0$.*

Proof. Since $|\widetilde{h}^\lambda(z')| = |\widetilde{h}(z')|^{\mathfrak{Re}\,\lambda} e^{-(\mathfrak{Im}\,\lambda)(\arg \widetilde{h})(z')}$, $z' \in E_a$, the claim follows by Lemma 5.2.2 and Lemma 5.2.6. $\square$

Proposition 5.2.8. *Let (ψ_t) be a semiflow satisfying axioms* (**SFlow1**) *and* (**SFlow2**) (a)&(b) *with DW point a, and let h be the univalent function associated with (ψ_t). Let $z_i' \in p^{-1}(z_i)$, where z_i is a repulsive point of (ψ_t), and take $\lambda \in \mathbb{C}$. Then*

$$|\widetilde{h}^\lambda(z')| \simeq |\widetilde{h}(z')|^{\frac{\mathfrak{Re}\,(\lambda\Psi'(a))}{\mathfrak{Re}\,\Psi'(a)}}, \qquad \text{as } z' \to z_i'.$$

Proof. Let B_1, B_2 be open neighborhoods in E_a of z_i' as in Lemma 5.2.5, with associated function $T : B_1 \to (0, \infty)$, and with B_2 small enough such that $\mathcal{K} := B_2 \setminus B_1$ satisfies the hypothesis of Lemma 5.2.6. By (5.9), we have

$$\begin{aligned}(\arg\widetilde{h})(z') &= (\arg\widetilde{h})(\widetilde{\psi}_{T(z')}(z')) - \mathfrak{Im}\,\Psi'(a)T(z') \\ &= (\arg\widetilde{h})(\widetilde{\psi}_{T(z')}(z')) + \frac{\mathfrak{Im}\,\Psi'(a)}{\mathfrak{Re}\,\Psi'(a)} \log|\widetilde{h}(z')| + C \qquad z' \in B_1,\end{aligned}$$

where $C \in \mathbb{R}$ does not depend on z'. This implies

$$\exp\Big(-(\mathfrak{Im}\,\lambda)(\arg\widetilde{h})(z')\Big) = |\widetilde{h}(z')|^{-\frac{(\mathfrak{Im}\,\lambda)(\mathfrak{Im}\,\Psi'(a))}{\mathfrak{Re}\,\Psi'(a)}} \exp\big(\underbrace{-(\mathfrak{Im}\,\lambda)(\arg\widetilde{h})(\widetilde{\psi}_{T(z')}(z'))}_{①}\big),$$

for every $z' \in B_1$. Since $\widetilde{\psi}_{T(z')}(z')$ lies in $\Lambda = B_2 \setminus B_1$ and, by Lemma 5.2.6, $(\arg\widetilde{h})$ is bounded on Λ, it follows that the term ① above is uniformly bounded for all $z' \in B_1$. Thus, for every $z' \in B_1$,

$$|\widetilde{h}^\lambda(z')| = |\widetilde{h}(z')|^{\mathfrak{Re}\,\lambda} e^{-(\mathfrak{Im}\,\lambda)(\arg\widetilde{h})(z')} \simeq |\widetilde{h}(z')|^{\mathfrak{Re}\,\lambda - \frac{(\mathfrak{Im}\,\lambda)(\mathfrak{Im}\,\Psi'(a))}{\mathfrak{Re}\,\Psi'(a)}} = |\widetilde{h}(z')|^{\frac{\mathfrak{Re}\,(\lambda\Psi'(a))}{\mathfrak{Re}\,\Psi'(a)}},$$

and the claim is proved. □

5.3 Semicocycles

In this subsection, from now on, (v_t) is a semicocycle for a semiflow (ψ_t) which satisfies (at least) properties (**SFlow1**) and (**SFlow2**)(a)&(b), with *DW* point $a \in \mathbb{D}$ and repulsive points $z_1, \ldots, z_n \in \mathbb{T}$.

Now we turn to the axiomatic properties of the semicocycles we are concerned with.

(**SCo1**) The limit $v_t(z_i) := \lim_{z \to z_i} v_t(z)$ exists in $\mathbb{C} \cup \{\infty\}$ for any $t \geq 0$, $i \in \{1, \ldots, n\}$.

We refer the reader to [CGP15; ELM16; HLNS13] for the suitability of the condition above when dealing with spectra of invertible weighted composition operators.

If (v_t) is a semicocycle, the function v_t has no zeroes in $\mathbb{D}$ for any $t \geq 0$, see [Kön90, Lemma 2.1b)]. However, it may happen that $v_t(z_i) = 0$ or $v_t(z_i) = \infty$ for some $t \geq 0$, $i \in \{1, \ldots, n\}$. Following axiom concerns such cases.

(**SCo2**) Let Ω^0 (Ω^∞) be open neighborhoods in $\mathbb{D}$ containing the repulsive points $z_i \in \mathbb{T}$ of (ψ_t) for which $u_1(z_i) = 0$ ($u_1(z_i) = \infty$). Then

$$\sup_{z\in\mathbb{D}\setminus\Omega^\infty} |v_t(z)| < \infty, \qquad \inf_{z\in\mathbb{D}\setminus\Omega^0} |v_t(z)| > 0.$$

Similar conditions as (**SCo2**) arise naturally when studying the strong continuity of the semigroup $(v_t C_{\psi_t})$, see for instance [Sis86].

The remainder of this subsection is devoted to give technical results on semicocycles satisfying the properties above.

Remark 5.3.1. Let Ω^0, Ω^∞ be open sets as in (**SCo2**). We can assume, without loss of generality for all the arguments used here, that the subsets $\mathbb{D}\setminus\Omega^0, \mathbb{D}\setminus\Omega^\infty$ are C_{ψ_t}-invariant for all $t \geq 0$, see Lemma 5.2.2. That is, $\psi_t(\mathbb{D}\setminus\Omega^0) \subset \mathbb{D}\setminus\Omega^0$ and $\psi_t(\mathbb{D}\setminus\Omega^\infty) \subset \mathbb{D}\setminus\Omega^\infty$. In this case, it is readily seen that the functions given by $t \mapsto \sup_{z\in\mathbb{D}\setminus\Omega^\infty} \log|v_t(z)|$, $t \mapsto \sup_{z\in\mathbb{D}\setminus\Omega^0} \log(|v_t(z)|^{-1})$ are subadditive. Hence,

$$\exists \lim_{t\to\infty} \sup_{z\in\mathbb{D}\setminus\Omega^\infty} |v_t(z)|^{1/t} < \infty, \qquad \exists \lim_{t\to\infty} \inf_{z\in\mathbb{D}\setminus\Omega^0} |v_t(z)|^{1/t} > 0,$$

see for example [HP57, Th. 7.6.5]. As a consequence, for each $T > 0$, there exist $M, w > 0$ such that

$$\sup_{z\in\mathbb{D}\setminus\Omega^\infty} |v_t(z)| \leq Me^{wt}, \qquad \inf_{z\in\mathbb{D}\setminus\Omega^0} |v_t(z)| \geq Me^{-wt}, \qquad \text{for all } t \geq T. \tag{5.10}$$

We provide below two examples of cocycles satisfying the quoted properties.

Lemma 5.3.2. *Let (ψ_t) be a semiflow satisfying* (**SFlow1**)-(**SFlow3**). *Take $\delta \in \mathbb{C}$ and let $(\psi_t')^\delta$, $t \geq 0$, be as in Definition 4.2.6. Then $((\psi_t')^\delta)$ is a semicocycle for the (ψ_t) which satisfies* (**SCo1**) and (**SCo2**).

Proof. We know by Lemma 4.2.7 that $((\psi_t')^\delta)$ is a semicocycle for (ψ_t). On the one hand, $((\psi_t')^\delta)$ satisfies (**SCo1**) by (**SFlow2**)(c). On the other hand, it satisfies (**SCo2**) as a consequence of (**SFlow3**) and the fact that $\sup_{z\in\mathbb{D}} |\arg \psi_t'(z)| < \infty$. The boundedness of $\arg \psi_t'(z)$ in $\mathbb{D}$ follows by (4.1), (4.2) and Lemma 5.2.4. ∎

Lemma 5.3.3. *Let (ψ_t) be a semiflow satisfying* (**SFlow1**) *and* (**SFlow2**) *with repulsive points $z_1, \ldots, z_n \in \mathbb{T}$ and DW point $z_0 \in \mathbb{D}$. Take $\delta \in \mathbb{C}$ and $i \in \{0, 1, \ldots, n\}$. Let (v_t) be given by*

$$v_t(z) = \left(\frac{\psi_t(z) - z_i}{z - z_i}\right)^\delta, \qquad z \in \mathbb{D},\ t \geq 0,$$

where (for each $t \geq 0$) we consider the branch for which $\lim_{z\to z_i} v_t(z) = (\psi_t')^\delta(z_i)$, see Definition 4.2.6. Then (v_t) is a semicocycle for (ψ_t) fulfilling (**SCo1**) *and* (**SCo2**).

Proof. It is readily seen that (v_t) is a semicocycle for (ψ_t). Property (**SFlow2**)(c) and the mean value theorem imply that v_t is continuous at z_i for each $t \geq 0$, and property (**SFlow2**)(a) implies that v_t is continuous at z_j for $j \neq i$. Condition (**SCo2**) follows by which we have already proven and Lemma 5.2.2. □

We now study the asymptotic behavior of a semicocycle (v_t), which is crucial in the understanding of the spectrum of the infinitesimal generator of $(v_t C_{\psi_t})$.

Lemma 5.3.4. *Let (ψ_t) be a semiflow satisfying* (**SFlow1**) *and* (**SFlow2**)(a)&(b)*, and let (v_t) be a semicocycle satisfying* (**SCo1**) *and* (**SCo2**)*. Let $z_0 \in \mathbb{D}$, $z_1, \ldots, z_n \in \mathbb{T}$ be respectively the DW point and the repulsive points of the semiflow (ψ_t). We have*

$$\lim_{t\to\infty}\left(\sup_{z\in\mathbb{D}}|v_t(z)|\right)^{1/t} = \max_{i=0,1,\ldots,n}\{|u_1(z_i)|\},$$
$$\lim_{t\to\infty}\left(\inf_{z\in\mathbb{D}}|v_t(z)|\right)^{1/t} = \min_{i=0,1,\ldots,n}\{|u_1(z_i)|\}.$$

Proof. The proof runs analogously to [HLNS13, Lemma 4.4]. □

Remark 5.3.5. Let (ψ_t), (v_t) be as in the lemma above. For $t \geq 0$, take open neighborhoods $\Omega_{i,t}$ in $\mathbb{D}$ of z_i such that $z_j \notin \overline{\Omega_i}$ for $j \neq i$, and for which $\psi_s(\Omega_{i,t}) \subseteq \Omega_{i,0}$ for all $s \in [0,t]$. (Note that such sets exist by (**SFlow2**)(a).) Then, reasoning as in the proof of Lemma [HLNS13, Lemma 4.4], one gets

$$\lim_{t\to\infty}\left(\sup_{z\in\widetilde{\Omega}_{i,t}}|v_t(z)|\right)^{1/t} = |u_1(z_i)|, \quad \text{and} \quad \lim_{t\to\infty}\left(\inf_{z\in\widetilde{\Omega}_{i,t}}|v_t(z)|\right)^{1/t} = |u_1(z_i)|.$$

The elements of the extended real line α_i found in the following lemma will be called exponents of (v_t).

Lemma 5.3.6. *Let (ψ_t) be a semiflow satisfying* (**SFlow1**) *and* (**SFlow2**) *with DW point $z_0 \in \mathbb{D}$ and repulsive points $z_1, \ldots, z_n \in \mathbb{T}$. Let (v_t) be a semicocycle satisfying* (**SCo1**) *and* (**SCo2**)*. There exists $\alpha_i \in [-\infty, \infty]$ such that*

$$|v_t(z_i)| = e^{\alpha_i t}, \qquad t > 0,$$

where $e^{\infty} = \infty$ and $e^{-\infty} = 0$.

Proof. The mapping $t \mapsto |v_t(z_i)|$ is measurable since it is the limit of a countable family of continuous functions. Indeed, $|v_t(z_i)| = \lim_{k\to\infty}|v_t(w_k)|$ where $(w_k)_{k=1}^{\infty} \subset \mathbb{D}$ with $w_k \xrightarrow[k\to\infty]{} z_i$. Even more, one gets, by the semicocycle property, $u_{t+s}(z_i) = v_t(z_i)u_s(z_i)$ for $s, t \geq 0$. This together with (**SCo2**) implies that

i) either $|v_t(z_i)| \in (0, \infty)$ for all $t > 0$,

ii) or $|v_t(z_i)| = 0$ for all $t > 0$,

iii) or $|v_t(z_i)| = \infty$ for all $t > 0$.

If item ii) or item iii) holds, set $\alpha_i = -\infty$ or $\alpha_i = \infty$ respectively. If item i) holds, then $\mapsto |v_t(z_i)|$ is measurable and fulfills the Cauchy's exponential equation, so there exists $\alpha_i \in \mathbb{R}$ such that $|v_t(z_i)| = e^{\alpha_i t}$ for all $t \geq 0$, and the proof is done. □

Remark 5.3.7. Let (ψ_t) be a semiflow satisfying (**SFlow1**) and (**SFlow2**). An analogous reasoning as in Lemma 5.3.6 yields that there exist $\beta_0, \beta_1, \ldots, \beta_n \in [-\infty, \infty)$ for which $|\psi_t'(z_i)| = e^{\beta_i t}$, $t \geq 0$, $i = 0, 1, \ldots, n$. It is readily seen from (4.4) that $\beta_0 = \mathfrak{Re}\, \Psi'(a)$.

Such elements $\beta_0, \beta_1, \ldots, \beta_n$ will also be called the *exponents* of (ψ_t') even in the case the semicocycle (ψ_t') does not satisfy (**SCo2**).

Given a differentiable semicocycle (v_t) for the semiflow (ψ_t), it follows by Proposition 4.2.2 that there exists a holomorphic mapping $\omega : E_a \to \mathbb{C}$ without zeroes such that $v_t(z) = (\omega \circ \widetilde{\psi}_t(z'))/\omega(z')$ for all $t \geq 0$ and $z \in \mathbb{D}$, and where $z' \in p^{-1}(z)$.

Lemma 5.3.8. *Let (ψ_t) be a semiflow satisfying* (**SFlow1**) *and* (**SFlow2**) (a)&(b), *with DW point $z_0 \in \mathbb{D}$ and repulsive points $z_0, z_1, \ldots, z_n \in \mathbb{T}$. Let (v_t) be a differentiable semicocycle satisfying* (**SCo1**) *and* (**SCo2**)*, and let ω be an holomorphic function on E_a associated with (v_t). Let $\mathcal{K}$ be a subset in E_a with $\sup_{z' \in \mathcal{K}} |\arg z'| < \infty$, and for which there exist Ω_i open neighborhoods in $\mathbb{D}$ of z_i for each $i = 0, 1, \ldots, n$ with $p(\mathcal{K}) \subseteq \mathbb{D} \setminus (\cup_{i=0}^{n} \Omega_i)$. Then*

$$\sup_{z' \in \mathcal{K}} |\omega(z')| < \infty \quad \textit{and} \quad \inf_{z' \in \mathcal{K}} |\omega(z')| > 0.$$

Proof. The proof runs in a similar way to the proof of Lemma 5.2.6. □

Recall that, by Remark 4.2.3, ω behaves as a fractional power as $z \to a$. The next theorem gives some information about the behavior of ω near the repulsive points of (ψ_t).

Theorem 5.3.9. *Let (ψ_t) be a semiflow satisfying* (**SFlow1**) *and* (**SFlow2**)(a)&(b), *and let (v_t) be a differentiable semicocycle satisfying* (**SCo1**) *and* (**SCo2**)*. Let $a \in \mathbb{D}$ be the DW point of (ψ_t), let $z_1, \ldots, z_n \in \mathbb{T}$ be the repulsive points of (ψ_t), and let h be the univalent function associated with (ψ_t). Let also $\omega : E_a \to \mathbb{C}$ be an holomorphic function associated with (v_t). Then, for every $\varepsilon > 0$,*

$$|\omega(z')| \lesssim |\widetilde{h}(z')|^{\alpha_i/\mathfrak{Re}\,\Psi'(a)+\varepsilon}, \qquad \textit{as } z' \to z_i',$$
$$|\omega(z')| \gtrsim |\widetilde{h}(z')|^{\alpha_i/\mathfrak{Re}\,\Psi'(a)-\varepsilon}, \qquad \textit{as } z' \to z_i',$$

where z_i' is any point in $p^{-1}(z_i)$, $i = 1, \ldots, n$, and $\alpha_i \in [-\infty, \infty]$ are the exponents of (v_t).

If $\alpha_i = \infty$ ($\alpha_i = -\infty$), the above reads as, for each $\alpha > 0$, $|\omega(z')| \lesssim |\widetilde{h}(z)|^{-\alpha}$ ($|\omega(z')| \gtrsim |\widetilde{h}(z)|^{\alpha}$) as $z' \to z_i'$.

Proof. Fix $\varepsilon > 0$ and let Ω_i be a small enough open neighborhood in $\mathbb{D}$ of z_i for each $i = 1, \dots, n$. Let $M := \sup_{z\in\mathbb{D}\setminus\Omega_i} |h(z)|$, so $M < \infty$ by Lemma 5.2.2. Now fix $i \in \{1, \dots, n\}$ for the rest of the proof, and let B_1, B_2 be arbitrary small neighborhoods of $z_i' \in p^{-1}(z_i)$ taken as in Lemma 5.2.5, with function $T : B_1 \to (0, \infty)$. Set also $\mathcal{K} := B_2 \setminus B_1$.

Since $T(z') \to \infty$ as $z' \to z_i'$, Remark 5.3.5 implies

$$\begin{aligned} \left|\widetilde{u}_{T(z')}(z')\right| &\lesssim |u_1(z_i)|^{T(z')} e^{\varepsilon(T(z'))} = e^{(\alpha_i+\varepsilon)(s+T(z'))}, \qquad \text{as } z' \to z_i', \\ \left|\widetilde{u}_{T(z')}(z')\right| &\gtrsim |u_1(z_i)|^{T(z')} e^{-\varepsilon(T(z'))} = e^{(\alpha_i-\varepsilon)(T(z'))}, \qquad \text{as } z' \to z_i'. \end{aligned} \tag{5.11}$$

As $|h(\psi_t(z))/h(z)| = e^{\Psi'(a)t}$ for all $z \in \mathbb{D} \setminus \{a\}$ and $t \geq 0$, inequalities (5.11) yield

$$\begin{aligned} \left|\frac{\omega(\widetilde{\psi}_{T(z')}(z'))}{\omega(z')}\right| &= |\widetilde{u}_{T(z')}(z')| \lesssim \left|\frac{\widetilde{h}(\widetilde{\psi}_{T(z')}(z'))}{\widetilde{h}(z')}\right|^{(\alpha_i+\varepsilon)/\mathfrak{Re}\,\Psi'(a)}, \qquad \text{as } z' \to z_i', \\ \left|\frac{\omega(\widetilde{\psi}_{T(z')}(z'))}{\omega(z')}\right| &= |\widetilde{u}_{T(z')}(z')| \gtrsim \left|\frac{\widetilde{h}(\widetilde{\psi}_{T(z')}(z'))}{\widetilde{h}(z')}\right|^{(\alpha_i-\varepsilon)/\mathfrak{Re}\,\Psi'(a)}, \qquad \text{as } z' \to z_i'. \end{aligned}$$

The claim of the theorem follows from inequalities above since $\widetilde{\psi}_{T(z')}(z') \subseteq \mathcal{K}$, and we have $\sup_{z'\in\mathcal{K}} |\omega(z')|, \sup_{z'\in\mathcal{K}} |\widetilde{h}(z')| < \infty$ and $\inf_{z'\in\mathcal{K}} |\omega(z')|, \inf_{z'\in\mathcal{K}} |\widetilde{h}(z')| > 0$, see Lemma 5.3.8 and Corollary 5.2.7. □

Lemma 5.3.10. *Let (ψ_t) be a semiflow satisfying* (**SFlow1**) *and* (**SFlow2**), *DW point $a \in \mathbb{D}$ and with repulsive points $z_1, \dots, z_n \in \mathbb{T}$. Let $\beta_0, \beta_1, \dots, \beta_n$ be the exponents of (ψ_t'), and let h be the univalent function associated with (ψ_t). Then $\beta_i \geq |\Psi'(a)|/2$ for each $i = 1, \dots, n$ and, for every $\varepsilon > 0$,*

$$\begin{aligned} |h(z)| &\lesssim |z - z_i|^{(\mathfrak{Re}\,\Psi'(a))/\beta_i - \varepsilon}, \qquad \text{as } z \to z_i, \\ |h(z)| &\gtrsim |z - z_i|^{(\mathfrak{Re}\,\Psi'(a))/\beta_i + \varepsilon}, \qquad \text{as } z \to z_i. \end{aligned}$$

Proof. Fix $i = 1, \dots, n$ for the rest of the proof. Let $v_t(z) = (\psi_t(z) - z_i)/(z - z_i)$, $z \in \mathbb{D}$, $t \geq 0$. Then (v_t) satisfies axioms (**SCo1**) and (**SCo2**) by Lemma 5.3.3. Since the exponents of (v_t) are $\beta_0, \beta_1, \dots, \beta_n$, Theorem 5.3.9 implies, for each $\varepsilon > 0$,

$$\begin{aligned} |z - z_i| &\lesssim |h(z)|^{\beta_i/\mathfrak{Re}\,\Psi'(a)+\varepsilon}, \qquad \text{as } z \to z_i, \\ |z - z_i| &\gtrsim |h(z)|^{\beta_i/\mathfrak{Re}\,\Psi'(a)-\varepsilon}, \qquad \text{as } z \to z_i. \end{aligned} \tag{5.12}$$

As $\lim_{z\to z_i} |h(z)| = \infty$ by Lemma 5.2.2, one has $\beta_i \geq 0$ (otherwise, inequality $\gtrsim$ above would entail a contradiction). On the other hand, since h is λ-spirallike with $\lambda = \arg(-\Psi'(a)) \in (-\pi/2, \pi/2)$, we have $1 - |z| \leq |z/h(z)|^{1/(2\cos\lambda)}$, $z \in \mathbb{D}$, see [PB00, Th. 3.1(iii)]. Thus the inequality $\gtrsim$ in (5.12) yields $\beta_i \geq |\Psi'(a)|/2$ as claimed.

In particular, $\beta_i > 0$, and the inequalities asserted in the lemma follow from (5.12). □

As a consequence of the results above, we obtain the following.

Corollary 5.3.11. *Let (ψ_t) be a semiflow satisfying* (**SFlow1**) *and* (**SFlow2**)*, and let (v_t) be a differentiable semicocycle satisfying* (**SCo1**) *and* (**SCo2**)*. Using the notation of Theorem 5.3.9 and Lemma 5.3.10, we have, for each $\varepsilon > 0$ and $i \in \{1, \dots, n\}$,*

$$|\omega(z')| \lesssim |z' - z_i'|^{\frac{\alpha_i}{\beta_i} - \varepsilon}, \qquad |\omega(z')| \gtrsim |z' - z_i'|^{\frac{\alpha_i}{\beta_i} + \varepsilon}, \qquad \text{as } E_a \ni z' \to z_i',$$

where z_i' is any point in $p^{-1}(z_i)$.

If moreover (ψ_t) satisfies (**SFlow3**)*, then*

$$|\Psi(z)| \lesssim |z - z_i|^{1-\varepsilon}, \qquad |\Psi(z)| \gtrsim |z - z_i|^{1+\varepsilon}, \qquad \text{as } \mathbb{D} \ni z \to z_i,$$

where Ψ is the generator of the semiflow (ψ_t).

Proof. The first statement is a consequence of Theorem 5.3.9 and Lemma 5.3.10.

If (ψ_t) also satisfies (**SFlow3**), then the semicocycle (ψ_t') satisfies (**SCo1**) and (**SCo2**) with exponents $\alpha_i = \beta_i$ by Lemma 5.3.2. Since its associated holomorphic function is $\widetilde{\Psi}$ (recall that $\psi_t' = (\Psi \circ \psi_t)/\Psi$, see (4.1)), it is enough to apply which we have already proven. □

5.4 Spectrum of the infinitesimal generator

We deal in this section with the spectral properties of the generator Δ of $(v_t C_{\psi_t})$ on a γ^∞-space X. For Δ to be well defined, we assume that the semicocycles (v_t) (of a semiflow (ψ_t)) we are working with fulfill the following condition:

(**SCo3**) $(v_t C_{\psi_t})$ is a C_0-semigroup of bounded operators on a γ^∞-space X.

Unfortunately, (**SCo3**) rules out any Banach space X of holomorphic functions for which the inclusions $H^\infty(\mathbb{D}) \subseteq X \subseteq \mathcal{B}_1(\mathbb{D})$ hold, where $\mathcal{B}_1(\mathbb{D})$ denotes the Bloch space, since no weighted composition semigroup is strongly continuous (at 0) in such a space X, see [GSY22, Th. 4.1]. In particular, the results of this chapter do not cover spaces like $H^\infty(\mathbb{D})$ or the Korenblum classes $\mathcal{K}^{-\gamma}(\mathbb{D})$.

If $(v_t C_{\psi_t})$ satisfies (**SCo3**), it follows by Lemma 4.1.1 and Proposition 4.1.2 that the infinitesimal generator Δ of the C_0-semigroup $(v_t C_{\psi_t})$ is given by

$$\Delta f = \Psi f' + g f, \qquad f \in \operatorname{Dom}(\Delta), \tag{5.13}$$

with $\operatorname{Dom}(\Delta) = \{f \in X \,:\, \Psi f' + g f \in X\}$, and where g is the generator of (v_t), i.e. $g = \frac{\partial v_t}{\partial t}\big|_{t=0}$.

The following upper bound for the asymptotic behavior of the norm of $(v_t C_{\psi_t})$ yields the spectral inclusion given in corollary below. Recall that, for $b \in \mathbb{D}$, $m \in \mathbb{N}_0$, we denote by $\mathfrak{X}_b^m$ the subset of functions f in X which have a zero at b of order at least m, and by B_b we denote the backshift operator at b.

Proposition 5.4.1. *Let X be a γ^∞-space for $\gamma \geq 0$, let (ψ_t) be a semiflow satisfying* (**SFlow1**)-(**SFlow4**) *with DW point $a \in \mathbb{D}$, and let (v_t) be a semicocycle for (ψ_t) satisfying* (**SCo1**) *and* (**SCo2**)*. Let $m \in \mathbb{N}_0$. Then, $\mathfrak{X}_a^m$ is an invariant subspace of $(v_t C_{\psi_t})$, and*

$$\lim_{t\to\infty} \|v_t C_{\psi_t}\|_{L(\mathfrak{X}_a^m)}^{1/t} \leq \exp\left(\max\{\alpha_0 + (m-\gamma)\beta_0, \alpha_1 - \gamma\beta_1, \ldots, \alpha_n - \gamma\beta_n\}\right),$$

where $\alpha_0, \alpha_1, \ldots, \alpha_n$ and $\beta_0, \beta_1, \ldots, \beta_n$ are the exponents of (v_t) and (ψ_t') respectively.

Proof. Note that, since $t \mapsto \log \|v_t C_{\psi_t}\|_{L(\mathfrak{X}_a^m)}$ is a subadditive function, such a limit exists, see for example [HP57, Th. 7.6.5].

Now, the inclusion $(v_t C_{\psi_t})(\mathfrak{X}_a^m) \subseteq \mathfrak{X}_a^m$ follows from the fact ψ_t has a zero of order 1 at $z = a$. Hence, by (**Gam'1**) and (**Gam'2**) we have

$$\begin{aligned} \|v_t C_{\psi_t} f\|_X &\simeq \|B_a^m(v_t C_{\psi_t} f)\|_X = \|(B_a^m \psi_t)\, v_t C_{\psi_t}\,(B_a^m f)\|_X \\ &\lesssim \|(B_a^m \psi_t)\, v_t\,(\psi_t')^{-\gamma}\|_\infty \|(\psi_t')^\gamma C_{\psi_t}\,(B_a^m f)\|_X \\ &\lesssim \|(B_a^m \psi_t)\, v_t\,(\psi_t')^{-\gamma}\|_\infty \|f\|_X, \quad f \in \mathfrak{X}_a^m,\, t \geq 0, \end{aligned}$$

where we have used that $Mul(X) = H^\infty(\mathbb{D})$ by (**Gam'1**), and $\sup_{t\geq 0} \|(\psi_t')^\gamma C_{\psi_t}\|_{L(X)} < \infty$ by (**SFlow4**).

In addition, the semicocycle (w_t) given by $w_t := (B_a^m \psi_t)\, v_t\,(\psi_t')^{-\gamma}$, $t \in \mathbb{R}$, satisfies properties (**SCo1**) and (**SCo2**) since the cocycles (v_t), $((\psi_t')^{-\gamma})$, $(B_a^m \psi_t)$ do so, see Lemma 5.3.2 and Lemma 5.3.3. Hence, Lemma 5.3.4 yields

$$\begin{aligned} \lim_{t\to\infty} \left\|(B_a^m \psi_t)\, v_t\,(\psi_t')^{-\gamma}\right\|^{1/t} &= \max_{i=0,1,\ldots,n} \left\{\left((B_a^m \psi_t)\, v_t\,(\psi_t')^{-\gamma}\right)(z_i)\right\} \\ &= \exp\left(\max\{\alpha_0 + (m-\gamma)\beta_0, \alpha_1 - \gamma\beta_1, \ldots, \alpha_n - \gamma\beta_n\}\right), \end{aligned}$$

where $z_1, \ldots, z_n$ are the limit fixed points of (ψ_t) and z_0 is the DW point of (ψ_t). □

Corollary 5.4.2. *Let X be a γ^∞-space for $\gamma \geq 0$, let (ψ_t) be a semiflow satisfying* (**SFlow1**)-(**SFlow4**)*, and let (v_t) be a semicocycle for (ψ_t) satisfying* (**SCo1**)-(**SCo3**)*. Then*

$$\sigma(\Delta) \subseteq \{\lambda \in \mathbb{C} \,|\, \mathfrak{Re}\,(\lambda) \leq \max\{\alpha_1 - \gamma\beta_1, \ldots, \alpha_n - \gamma\beta_n\}\} \cup \sigma_{point}(\Delta),$$

where $\alpha_0, \alpha_1, \ldots, \alpha_n$ and $\beta_0, \beta_1, \ldots, \beta_n$ are the exponents of (v_t) and (ψ_t') respectively.

Proof. If $\alpha_i = \infty$ for some $i = 1, \ldots, n$, then the claim is trivial. So, assume $\alpha_i < \infty$, $i = 1, \ldots, n$ and set $B := \{\lambda \in \mathbb{C} : \mathfrak{Re}\,(\lambda) > \max\{\alpha_1 - \gamma\beta_1, \ldots, \alpha_n - \gamma\beta_n\}\}$.

Let $a \in \mathbb{D}$ be the DW point of (ψ_t). For $m \in \mathbb{N}$, $\Delta|_{\mathfrak{X}_a^m}$ is the generator of the C_0-semigroup $(v_t C_{\psi_t}|_{\mathfrak{X}_a^m})$, and if

$$\mathfrak{Re}\,(\lambda) > \max\{\alpha_0 + (m-\gamma)\beta_0, \alpha_1 - \gamma\beta_1, \ldots, \alpha_n - \gamma\beta_n\},$$

then $\lambda \in \rho(\Delta_m)$ (with $(\lambda - \Delta_m)^{-1} = \int_0^\infty e^{-\lambda t}(v_t C_{\psi_t})|_{\mathfrak{X}_a^m}\, dt$). To see this, it is enough to apply Proposition 4.1.2, Proposition 5.4.1 and [EN00, Th. II.1.10].

Since $\mathfrak{X}_a^m$ has finite codimension for all $m \in \mathbb{N}$ ($\operatorname{codim}(\mathfrak{X}_a^m) = m$), we have that B lies in the essential resolvent of Δ. Since B is a connected open set and $B \cap \rho(\Delta) \neq \emptyset$, the points in $\sigma(\Delta) \cap B$ are isolated eigenvalues, see for instance [EE87, Section I.4], and the claim follows. □

The point spectrum of Δ is given by

$$\sigma_{point}(\Delta) = \left\{ g(a) + \Psi'(a)k \,:\, k \in \mathbb{N}_0 \text{ and } \frac{\widetilde{h}^{k+g(a)/\Psi'(a)}}{\omega} \in X \right\},$$

see Proposition 4.2.4, where h is the univalent function associated with (ψ_t), and ω a holomorphic function (on E_a) associated with (v_t). The proposition below gives a little more information in the case that the γ^∞-space X satisfies the following condition:

(**Gam'3**) For every $\varepsilon > 0$ and $\theta_j \in [0, 2\pi)$, $j = 1, \ldots, m$ with $\theta_j \neq \theta_k$ if $j \neq k$, we have

$$\text{if } f \in \mathcal{O}(\mathbb{D}) \text{ with } |f(z)| \lesssim \prod_{j=1}^{m} |e^{i\theta_j} - z|^{-\gamma+\varepsilon}, \qquad \text{then} \qquad f \in X.$$

Note that the Hardy spaces, the (weighted) Bergman spaces and the little Korenblum classes satisfy such a property.

Proposition 5.4.3. *Let X be a γ^∞-space with $\gamma \geq 0$. Let (ψ_t) be a semiflow satisfying (**SFlow1**) and (**SFlow2**), and let (v_t) be a semicocycle for (ψ_t) satisfying (**SCo1**)-(**SCo3**). Let $\alpha_0, \alpha_1, \ldots, \alpha_n$ and $\beta_0, \beta_1, \ldots, \beta_n$ be the exponents of (v_t) and (ψ_t') respectively, and set $A = \frac{1}{\mathfrak{Re}\,\Psi'(a)} \left(\max\{\alpha_1 - \gamma\beta_1, \ldots, \alpha_n - \gamma\beta_n\} - \mathfrak{Re}\, g(a)\right)$. Then*

$$\sigma_{point}(\Delta) \subseteq \{g(a) + \Psi'(a)k \,:\, k \in \mathbb{N}_0 \text{ and } k \leq A\}.$$

*If in addition X fulfills (**Gam'3**), then*

$$\{g(a) + \Psi'(a)k \,:\, k \in \mathbb{N}_0 \text{ and } k < A\} \subseteq \sigma_{point}(\Delta).$$

Proof. Let $z_1, \ldots, z_n$ be the repulsive points of (ψ_t). Corollary 5.2.7, Proposition 5.2.8, Lemma 5.3.8, Theorem 5.3.9 and Lemma 5.3.10 imply, for each $\varepsilon > 0$,

$$\prod_{i=1}^{n} |z - z_i|^{\frac{\mathfrak{Re}\,(k\Psi'(a)+g(a))-\alpha_i}{\beta_i}+\varepsilon} \lesssim \left| \frac{\widetilde{h}^{k+g(a)/\Psi'(a)}}{\omega} \right| \lesssim \prod_{i=1}^{n} |z - z_i|^{\frac{\mathfrak{Re}\,(k\Psi'(a)+g(a))-\alpha_i}{\beta_i}-\varepsilon}, \quad z \in \mathbb{D}.$$

If $k > A$, then $\frac{\widetilde{h}^{k+g(a)/\Psi'(a)}}{\omega} \notin \mathcal{K}^{-\gamma}(\mathbb{D})$ by the first inequality above, whence it is not in X. If $k < A$ and X satisfies (**Gam'3**), then $\frac{\widetilde{h}^{k+g(a)/\Psi'(a)}}{\omega} \in X$ by the second inequality above. Then our claim follows by Proposition 4.2.4. □

As a consequence of the proposition above, one can improve the asymptotic bound given in Proposition 5.4.1. Recall that, for a bounded operator A on X, we denote by $r(A)$ the spectral radius of A.

Proposition 5.4.4. *Let X be a γ^∞-space for $\gamma \geq 0$, let (ψ_t) be a semiflow satisfying* (**SFlow1**)-(**SFlow4**) *with DW point $a \in \mathbb{D}$, and let (v_t) be a semicocycle for (ψ_t) satisfying* (**SCo1**)-(**SCo3**). *Then*

$$\lim_{t\to\infty} \|v_t C_{\psi_t}\|_{L(X)}^{1/t} \leq \exp\left(\max\{\alpha_0, \alpha_1 - \gamma\beta_1, \ldots, \alpha_n - \gamma\beta_n\}\right),$$

where $\alpha_0, \alpha_1, \ldots, \alpha_n$ and $\beta_0, \beta_1, \ldots, \beta_n$ are the exponents of (v_t) and (ψ_t') respectively.

Proof. By the spectral radius formula, we are done if we prove

$$r(v_1 C_{\psi_1}) \leq \exp\left(\max\{\alpha_0, \alpha_1 - \gamma\beta_1, \ldots, \alpha_n - \gamma\beta_n\}\right) =: C.$$

Moreover, by Proposition 5.4.1,

$$\sigma_{ess}(v_1 C_{\psi_1}) \subseteq \{\lambda \in \mathbb{C} \,:\, |\lambda| \leq \exp\left(\max\{\alpha_1 - \gamma\beta_1, \ldots, \alpha_n - \gamma\beta_n\}\right)\} =: K.$$

Thus, if $\lambda \in \sigma(v_1 C_{\psi_t})$ with $|\lambda| > K$, then λ is an isolated eigenvalue of $v_1 C_{\psi_t}$, see for instance [EE87, Section I.4]. In this case, $\lambda \in \exp(\sigma_{point}(\Delta))$, where Δ is the generator of $(v_t C_{\psi_t})$, see for example [EN00, Theorem IV.3.7]. Since $\mathfrak{Re}\,(g(a)) = \alpha_0$, it follows by Proposition 5.4.3 that $|\lambda| \leq C$, and the proof is finished. □

Remark 5.4.5. The natural framework to study the spectrum of the infinitesimal generator Δ is the universal covering E_a, where a is the DW point of (ψ_t). In fact, let $\widetilde{\Delta}$ denote the induced operator on E_a by Δ, i.e.,

$$\widetilde{\Delta} f = \widetilde{\Psi} f' + \widetilde{g} f = \widetilde{\Psi} f' + \widetilde{\Psi}\frac{\omega'}{\omega} f, \qquad f \in \mathcal{O}(E_a), \tag{5.14}$$

where ω is a holomorphic function associated with (v_t). Fix $d' \in E_a$. Recall that $\Psi = \Psi'(a)h/h'$ (see (4.4)), where h is the univalent function associated with (ψ_t). Then, for $f_0, f_1 \in \mathcal{O}(E_a)$ and $\lambda \in \mathbb{C}$, one obtains that $(\lambda - \widetilde{\Delta})f_0 = f_1$ holds if and only if there exist $A \in \mathbb{C}$ for which

$$f_0(z') = (\Lambda_A^\lambda f_1)(z') := \frac{\widetilde{h}^{\lambda/\Psi'(a)}(z')}{\omega(z')}\left(A - \int_{d'}^{z'} \frac{\omega(\tau)}{\widetilde{h}^{\lambda/\Psi'(a)}(\tau)}\frac{f_1(\tau)}{\widetilde{\Psi}(\tau)}\,d\tau\right), \quad z' \in E_a. \tag{5.15}$$

As a consequence, given $f \in X$ and $\lambda \in \mathbb{C}$, f belongs to $\mathrm{Ran}(\lambda - \Delta)$ if and only if there exists $A \in \mathbb{C}$ such that the function $\Lambda_A^\lambda \widetilde{f} \in \mathcal{O}(E_a)$ induces a holomorphic function on $\mathbb{D}$ which belongs to X.

The following functionals, which are inspired by the study of the spectra of Cesàro operators in [AP10; Per08], play a central role in the study of the spectrum of Δ. Let $z_i, i = 1, \ldots, n$ be the repulsive points of (ψ_t). For $\lambda \in \mathbb{C}, i \in \{1, \ldots, n\}$, set

$$L_{\nu_i}^\lambda f := \int_{\widetilde{\nu}_i} \frac{\omega(\tau)}{\widetilde{h}^{\lambda/\Psi'(a)}(\tau)}\frac{\widetilde{f}(\tau)}{\widetilde{\Psi}(\tau)}\,d\tau, \quad f \in \mathcal{O}(\mathbb{D}), \tag{5.16}$$

where $\widetilde{\nu}_i$ is a lifting in E_a of the path ν_i from z_i to 0 with range $\{\phi_a^{-1}(\phi_a(z_i)e^{\Psi'(a)t}) \in \mathbb{D} \setminus \{a\} : t > 0\}$ (recall that $\phi_a(z) = (z-a)/(1-\bar{a}z)$). Note that the path ν_i has finite length. It is readily seen that if $\widetilde{\nu}_{i,1}, \widetilde{\nu}_{i,2}$ are two such liftings, then there exits $K \in \mathbb{C} \setminus \{0\}$ for which $L^\lambda_{\widetilde{\nu}_{i,1}} = kL^\lambda_{\widetilde{\nu}_{i,2}}$. Such a constant k is is irrelevant for the results presented here, hence we denote by L_i^λ to any functional $L^\lambda_{\widetilde{\nu}_i}$ as (5.16).

Lemma 5.4.6. *Let X be a γ^∞-space for $\gamma \geq 0$, let (ψ_t) be a semiflow satisfying* (**SFlow1**)*-*(**SFlow3**) *with DW point $a \in \mathbb{D}$ and repulsive points $z_1, \dots, z_n \in \mathbb{T}$. Let (v_t) be a semicocycle for (ψ_t) satisfying* (**SCo1**)*-*(**SCo3**)*. Let $\lambda \in \mathbb{C}$ with $\mathfrak{Re}\,(\lambda) < \alpha_i - \gamma\beta_i$ for $i = 1, \dots, n$, where α_i, β_i are the exponents of $(v_t), (\psi_t')$ at z_i. Let $m \in \mathbb{N}_0$ be such that $m > \frac{\mathfrak{Re}\,\lambda - \mathfrak{Re}\,g(a)}{\mathfrak{Re}\,\Psi'(a)}$. Then L_i^λ is a continuous functional on $\mathfrak{X}_a^m$ for which $(\lambda - \Delta)(\mathfrak{X}_a^m) \subseteq \ker L_i^\lambda|_{\mathfrak{X}_a^m}$.*

Proof. Let $\nu_i : [0, \infty] \to \overline{\mathbb{D}}$ be parameterized as $\nu_i(t) = \phi_a^{-1}(e^{\Psi'(a)t}\phi_a(z_i))$ for $i = 1, \dots, n$. Then,

$$\nu_i(t) - a = e^{\Psi'(a)t}\frac{(1-|a|^2)\phi_a(z_i)}{1+\bar{a}e^{\Psi'(a)t}\phi_a(z_i)}, \qquad t \geq 0.$$

Hence, $\sup_{t\geq 0} |\operatorname{Log}\widetilde{\nu}_i(t) - \Psi'(a)t| < \infty$, where $\widetilde{\nu}_i$ is any integration path as in (5.16). Moreover, by Remark 4.2.3, there exists a zero-free function $\rho \in \mathcal{O}(\mathbb{D})$ such that $\omega(z') = (z'-a)^{g(a)/\Psi'(a)}\rho(p(z'))$, $z' \in E_a$. Recall also that both h, Ψ have a simple zero at a, and that $\|f/(z-a)^m\|_X \simeq \|f\|_X$, $f \in \mathfrak{X}_a^m$. Then, putting everything together,

$$\left|\frac{\omega(\widetilde{\nu}_i(t))}{\widetilde{h}^{\lambda/\Psi'(a)}(\widetilde{\nu}_i(t))}\frac{\widetilde{f}(\widetilde{\nu}_i(t))}{\widetilde{\Psi}(\widetilde{\nu}_i(t))}\right| |\widetilde{\nu}_i'(t)| \lesssim \exp\left(\mathfrak{Re}\,\left(g(a) - \lambda + \Psi'(a)m\right)t\right)\|f\|_X, \tag{5.17}$$

as $t \to \infty$. Therefore, the integral (5.16) is absolutely convergent as $t \to \infty$ through $\widetilde{\nu}_i$ if $m > \frac{\mathfrak{Re}\,\lambda - \mathfrak{Re}\,g(a)}{\mathfrak{Re}\,\Psi'(a)}$.

On the other hand, by the inclusion X $\hookrightarrow \mathcal{K}^{-\gamma}(\mathbb{D})$, Proposition 5.2.8, Lemma 5.3.10 and Corollary 5.3.11, we have, for all $\varepsilon > 0$,

$$\left|\frac{\omega(\tau)}{\widetilde{h}^{\lambda/\Psi'(a)}(\tau)}\frac{\widetilde{f}(\tau)}{\widetilde{\Psi}(\tau)}\right| \lesssim |\tau - z_i'|^{\frac{\alpha_i - \mathfrak{Re}\,\lambda}{\beta_i} - \gamma - 1 - \varepsilon}\|f\|_X, \quad f \in X, \tag{5.18}$$

as $\tau \to z_i'$ non-tangentially. As a consequence, the integral (5.16) is absolutely convergent as $\tau \to z_i'$ through $\widetilde{\nu}_i$ if $\mathfrak{Re}\,\lambda < \alpha_i - \gamma\beta_i$ (remember that $\beta_i > 0$ by Lemma 5.3.10). We have, by the preceding bounds, that L_i^λ is a bounded functional on $\mathfrak{X}_a^m$, as claimed.

Now, for every $z \in \operatorname{Ran}(\nu_i)$, let $\nu_{i,z}$ be the path obtained by restricting ν_i such that $\nu_{i,z}$ has starting point z_i and ending point z. Fix $f \in \mathfrak{X}_a^m$. By the bounds we have proven above, one obtains that the mapping from $\operatorname{Ran}(\nu_i)$ (including a and z_i) to $\mathbb{C}$ given by

$$z \mapsto \int_{\widetilde{\nu}_{i,z}} \frac{\omega(\tau)}{\widetilde{h}^{\lambda/\Psi'(a)}(\tau)}\frac{\widetilde{f}(\tau)}{\widetilde{\Psi}(\tau)}\,d\tau,$$

is continuous. (There, $\widetilde{\nu}_{i,z}$ is the lifting of $\nu_{i,z}$ to E_a through the same starting point as $\widetilde{\nu}_i$.) Therefore, for $A \in \mathbb{C}$ and $f \notin \ker L_i^\lambda$, either

$$(5.19)\qquad \begin{aligned} &|(\Lambda_A^\lambda \widetilde{f})(z')| \simeq \left|\frac{\widetilde{h}^{\lambda/\Psi'(a)}(z')}{\omega(z')}\right| \qquad \text{as } z' \to a \text{ through } \widetilde{\nu}_i, \\ \text{or} \qquad &|(\Lambda_A^\lambda \widetilde{f})(z')| \simeq \left|\frac{\widetilde{h}^{\lambda/\Psi'(a)}(z')}{\omega(z')}\right| \qquad \text{as } z' \to z_i' \text{ through } \widetilde{\nu}_i. \end{aligned}$$

So assume $f \in \mathrm{Ran}(\lambda - \Delta|_{\mathfrak{X}_a^m}) \setminus \ker L_i^\lambda$. Then $\Lambda_A^\lambda \widetilde{f}$ induces a holomorphic function on $\mathbb{D}$ which belongs to $\mathfrak{X}_a^m$ for some $A \in \mathbb{C}$, see Remark 5.4.5. However, in the first case of (5.19), one gets, by Remark 4.2.3,

$$|\Lambda_A^\lambda \widetilde{f}(z)| \simeq |z-a|^{\frac{\mathfrak{Re}\,\lambda - \mathfrak{Re}\,g(a)}{\mathfrak{Re}\,\Psi'(a)}} \qquad \text{as } z \to a \text{ through } \nu_i.$$

So in this case $\Lambda_A^\lambda \widetilde{f} \notin \mathfrak{X}_a^m$, obtaining a contradiction. Hence the second case of (5.19) holds. However, for any $\varepsilon > 0$, one has, by Proposition 5.2.8, Lemma 5.3.10 and Corollary 5.3.11,

$$|\Lambda_A^\lambda \widetilde{f}(z)| \gtrsim |z-a|^{\frac{\mathfrak{Re}\,\lambda - \alpha_i}{\beta_i} + \varepsilon} \qquad \text{as } z \to z_i \text{ through } \nu_i.$$

Hence, $\Lambda_A^\lambda \widetilde{f} \notin \mathcal{K}^{-\gamma}(X)$, so $\Lambda_A^\lambda \widetilde{f} \notin X$, reaching a contradiction again.

Therefore, we have $(\lambda - \Delta)(\mathfrak{X}_a^m) \subseteq \ker L_i^\lambda$, and the proof is finished. □

Remark 5.4.7. In the setting of the lemma above, L_i^λ is not the zero functional on $\mathfrak{X}_a^m$. To see this, set $f(z) = z(z - z_i)$, $z \in \mathbb{D}$, so $f \in X$ by (**Gam'1**), and set also

$$e = \left(\frac{\omega}{\widetilde{h}^{\lambda/\Psi'(a)}} \frac{\widetilde{f}}{\widetilde{\Psi}}\right)\Bigg|_{\widetilde{\nu}_i}.$$

Reasoning as in the proof of Lemma 5.4.6, it follows that e is a continuous function on $\widetilde{\nu}_i$. If L_i^λ were the zero functional, one would have $\int_{\widetilde{\nu}_i} ep = 0$ for any function p on $\widetilde{\nu}_i$ which induces a bounded holomorphic extension to $\mathbb{D}$. This would imply $e = 0$ by the Stone-Weierstrass theorem, reaching a contradiction. Hence, $L_i^\lambda \neq 0$.

The overall discussion carried out in this section leads to the following result.

Theorem 5.4.8. *Let X be a γ^∞-space for $\gamma \geq 0$, let (ψ_t) be a semiflow satisfying* (**SFlow1**)*-*(**SFlow4**)*, and let (v_t) be a semicocycle for (ψ_t) satisfying* (**SCo1**)*-*(**SCo3**)*. Let $z_1, \dots, z_n$ be the repulsive points of (ψ_t), and let $\alpha_0, \alpha_1, \dots, \alpha_n$ and $\beta_0, \beta_1, \dots, \beta_n$ be the exponents of (v_t) and (ψ_t') respectively. Then*

$$\sigma(\Delta) = \{\lambda \in \mathbb{C} \,|\, \mathfrak{Re}\,(\lambda) \leq \max\{\alpha_1 - \gamma\beta_1, \dots, \alpha_n - \gamma\beta_n\}\} \cup \sigma_{point}(\Delta).$$

Proof. We gave the inclusion $\subseteq$ in Corollary 5.4.2, so all that we need to prove now is the inclusion $\supseteq$. Let $\lambda \in \mathbb{C}$ with $\mathfrak{Re}\,(\lambda) < \alpha_i - \gamma\beta_i$ for some $i = 1, \dots, n$. Lemma 5.4.6 together with Remark 5.4.7 yield

$$(\lambda - \Delta)(Z^m) \subseteq \ker L_i^\lambda \subsetneq Z^m,$$

for some $m \in \mathbb{N}_0$ big enough. Therefore $\dim X/((\lambda-\Delta)(Z^m)) > \dim X/\mathfrak{X}_a^m = m$, which implies $\operatorname{codim}((\lambda-\Delta)(X)) \geq 1$. Thus $\lambda-\Delta$ is not surjective, so $\lambda \in \sigma(\Delta)$ and the proof is finished. □

5.5 Weighted Hausdorff matrices

Here, we apply the results obtained in the preceding section to study the boundedness and the spectrum, on a γ^∞-space, of an operator subordinated to a weighted composition semigroup.

Along this section, for each semigroup $(v_t C_{\psi_t})$ on a γ^∞-space X, such that (v_t) and (ψ_t) satisfy properties (**SCo1**)-(**SCo3**) and (**SFlow1**)-(**SFlow4**), we denote by c the real number $\max\{\alpha_0, \alpha_1 - \gamma\beta_1, \dots, \alpha_n - \gamma\beta_n\}$, where $\alpha_0, \alpha_1, \dots, \alpha_n$ and $\beta_0, \beta_1, \dots, \beta_n$ are the exponents of (v_t) and (ψ_t) respectively.

Theorem 5.5.1. *Let $(v_t C_{\psi_t})$ be a semigroup on a γ^∞-space X, such that (v_t) and (ψ_t) satisfy properties (**SCo1**)-(**SCo3**) and (**SFlow1**)-(**SFlow4**). Let ν be a complex Borel measure on $[0,+\infty)$, such that $\int_0^\infty e^{(c+\delta)t}|d\nu|(t) < \infty$ for some $\delta > 0$. Let the operator $\mathcal{H}$ be defined by*

$$\mathcal{H}f = \int_0^\infty v_t C_{\psi_t} f \, d\nu(t), \quad f \in X,$$

where the integral above is Bochner-convergent. Then $\mathcal{H}$ is a well-defined bounded operator on X.

Proof. This is a consequence of Proposition 5.4.4. □

Now we present a technical lemma. Assume that ν is an Borel measure on $[0,\infty)$ which is absolutely continuous with respect to the Lebesgue measure, so $d\nu(t) = \rho(t)\, dt$, for some $L^1[0,\infty)$ function ρ. So, if $\int_0^\infty |d\nu|(t) < \infty$, its Laplace transform $q(\lambda) := \mathcal{L}(\nu)(\lambda) = \int_0^\infty e^{-\lambda t}\rho(t)\, dt$ is well defined for $\mathfrak{Re}\,\lambda \geq 0$.

Lemma 5.5.2. *Suppose that ρ can be extended in an holomorphic way to a sector Σ_θ with $0 < \theta \leq \pi/2$, and that there exist $0 < \eta \leq 1, \xi \in (0,1)$ satisfying*

$$\sup_{z\in\Sigma_\varepsilon\cap\{|z|\leq 1\}} |z^{1-\eta}\rho(z)| < \infty \quad \textit{and} \quad \sup_{z\in\Sigma_\varepsilon\cap\{|z|\geq 1\}} |z^{1+\xi}\rho(z)| < \infty, \quad \textit{for all } 0<\varepsilon<\theta.$$

Then, its Laplace transform $q := \mathcal{L}(\rho)$ can be extended to $\Sigma_{\pi/2+\theta}$, and such extension satisfies

$$\sup_{\lambda\in\Sigma_{\pi/2+\varepsilon}\cap\{|\lambda|\geq 1\}} |\lambda^{\eta}q(\lambda)| < \infty \quad \textit{and} \quad \sup_{\lambda\in\Sigma_{\pi/2+\varepsilon}\cap\{|\lambda|\leq 1\}} |\lambda^{-\xi}(q(\lambda)-q(0))| < \infty,$$

for all $0 < \varepsilon < \theta$.

Proof. Let $0<\varepsilon<\theta$. Then there is $M>0$ such that $|\rho(z)|\le M|z|^{\eta-1}$ if $z\in\overline{\Sigma_\varepsilon}\cap\{|z|\le 1\}\setminus\{0\}$ and $|\rho(z)|\le \frac{M}{|z|^{\xi+1}}$ if $z\in\overline{\Sigma_\varepsilon}\cap\{|z|\ge 1\}$. Let $\Gamma_\pm$ the paths on the complex plane defined by $\Gamma_\pm:=\{se^{\pm\varepsilon}\,:\,0\le s<\infty\}$. Let $\lambda>0$, by Cauchy's theorem we get

$$q(\lambda)=\int_{\Gamma_\pm} e^{-\lambda z}\rho(z)\,dz=e^{\pm i\varepsilon}\int_0^\infty e^{-\lambda s e^{\pm i\varepsilon}}\rho(se^{\pm i\varepsilon})\,ds,$$

since

$$\int_0^{\pm\varepsilon}|e^{-\lambda Re^{i\theta}}\rho(Re^{i\theta})Rie^{i\theta}|\,d\theta\lesssim\frac{e^{-\lambda R\cos\varepsilon}}{R^\xi}\to 0,\quad R\to+\infty.$$

Let now $0<\tau<\pi/2-\varepsilon$, and $\lambda\in\mathbb{C}$ such that $-\pi/2-\varepsilon+\tau<\arg\lambda<\pi/2-\varepsilon-\tau$. Then $-\pi/2+\tau<\arg(e^{i\varepsilon}\lambda)<\pi/2-\tau$, and therefore $\mathfrak{Re}\,(e^{i\varepsilon}\lambda)\ge|\lambda|\sin\tau$. Then

$$|e^{-\lambda se^{i\varepsilon}}\rho(se^{i\varepsilon})|\le Me^{-|\lambda|s\sin\tau}s^{\eta-1},\quad s\in(0,1),$$

and

$$|e^{-\lambda se^{i\varepsilon}}\rho(se^{i\varepsilon})|\le M\frac{e^{-|\lambda|s\sin\tau}}{s^{\xi+1}},\quad s>1.$$

So, the integral

$$q_+(\lambda):=e^{i\varepsilon}\int_0^\infty e^{-\lambda se^{i\varepsilon}}\rho(se^{i\varepsilon})\,ds$$

is absolutely convergent and defines a holomorphic function in the region $-\pi/2-\varepsilon+\tau<\arg\lambda<\pi/2-\varepsilon-\tau$, satisfying

$$|\lambda^\eta q_+(\lambda)|\le M/(\sin\tau)^\eta.$$

In a similar way,

$$q_-(\lambda):=e^{-i\varepsilon}\int_0^\infty e^{-\lambda se^{-i\varepsilon}}\rho(se^{-i\varepsilon})\,ds$$

is absolutely convergent and defines a holomorphic function in the region $-\pi/2+\varepsilon+\tau<\arg\lambda<\pi/2+\varepsilon-\tau$, satisfying

$$|\lambda^\eta q_-(\lambda)|\le M/(\sin\tau)^\eta.$$

Then q_+ and q_- are holomorphic extensions of q, and they define a holomorphic extension to $\Sigma_{\pi/2+\varepsilon-\tau}$, satisfying $|\lambda^\eta q(\lambda)|\le M/(\sin\tau)^\eta$ in the sector. Since $\varepsilon<\theta$ and $0<\tau<\pi/2-\varepsilon$ are arbitrary, we have defined the extension of q in $\Sigma_{\pi/2+\theta}$ such that $\sup_{\lambda\in\Sigma_{\pi/2+\varepsilon}}|\lambda^\eta q(\lambda)|<\infty$ for all $0<\varepsilon<\theta$.

Now observe that by Cauchy's theorem we have $q(0)=e^{\pm i\varepsilon}\int_0^\infty\rho(se^{\pm i\varepsilon})\,ds$, since

$$\int_0^{\pm\varepsilon}|\rho(Re^{i\theta})Rie^{i\theta}|\,d\theta\le\frac{M\varepsilon}{R^\xi}\to 0,\quad R\to+\infty.$$

So, if $0<\tau<\pi/2-\varepsilon$, and $\lambda\in\mathbb{C}$ is such that $-\pi/2-\varepsilon+\tau<\arg\lambda<\pi/2-\varepsilon-\tau$, since $\mathfrak{Re}\,(e^{i\varepsilon}\lambda)\ge|\lambda|\sin\tau$, one has

$$|(e^{-\lambda se^{i\varepsilon}}-1)\rho(se^{i\varepsilon})|\le M|\lambda|s^{\eta-1}\int_0^s e^{-|\lambda|u\sin\tau}\,du\le M|\lambda|,\quad s\in(0,1),$$

and

$$|(e^{-\lambda s e^{i\varepsilon}}-1)\rho(se^{i\varepsilon})| \le M|\lambda|\frac{\int_0^s e^{-|\lambda|u\sin\tau}\,du}{s^{\xi+1}}, \quad s>1.$$

Then

$$|q(\lambda)-q(0)| \le M|\lambda|\Big(\int_0^1 ds + \int_1^\infty \frac{1}{s^{\xi+1}}\int_0^s e^{-|\lambda|u\sin\tau}\,du\,ds\Big).$$

Observe that

$$\begin{aligned}&\int_1^\infty \frac{1}{s^{\xi+1}}\int_0^s e^{-|\lambda|u\sin\tau}\,du\,ds\\ &=\int_0^1 e^{-|\lambda|u\sin\tau}\int_1^\infty \frac{1}{s^{\xi+1}}\,ds\,du + \int_1^\infty e^{-|\lambda|u\sin\tau}\int_u^\infty \frac{1}{s^{\xi+1}}\,ds\,du\\ &\lesssim 1+\int_1^\infty \frac{e^{-|\lambda|u\sin\tau}}{u^\xi}\,du = 1+(|\lambda|\sin\tau)^{\xi-1}\int_{|\lambda|\sin\tau}^\infty \frac{e^{-v}}{v^\xi}\,dv\\ &\lesssim 1+|\lambda|^{\xi-1}.\end{aligned}$$

Therefore $|q(\lambda)-q(0)| \lesssim |\lambda|^\xi$ with $-\pi/2-\varepsilon+\tau<\arg\lambda<\pi/2-\varepsilon-\tau$ and $|\lambda|\le 1$. Similarly, one gets that $|q(\lambda)-q(0)| \lesssim |\lambda|^\xi$ with $-\pi/2+\varepsilon+\tau<\arg\lambda<\pi/2+\varepsilon-\tau$ and $|\lambda|\le 1$. Since $\varepsilon<\theta$ and $0<\tau<\pi/2-\varepsilon$ are arbitrary, we have $\sup_{\lambda\in\Sigma_{\pi/2+\varepsilon}\cap\{|\lambda|\le 1\}}|\lambda^{-\xi}(q(\lambda)-q(0))|<\infty$ for all $0<\varepsilon<\theta$.

□

Given a sectorial operator A of angle $\pi/2$, recall that, in the context of the functional calculus of sectorial operators, an holomorphic function f on a sector $\sum_\theta$ (for some $\theta\in(\pi/2,\pi)$) belongs to the domain of the functional calculus of A, $\mathcal{E}(A)$, if f is regular at 0 and ∞, see [Haa05a; Haa05b; Haa06].

Note that, by Proposition 5.4.4, for every semigroup $(v_tC_{\psi_t})$ as above, one gets that $e^{-(c+\varepsilon)t}T(t)$ is a uniformly bounded semigroup for each $\varepsilon>0$, and therefore $c+\varepsilon-\Delta$ is sectorial of angle $\pi/2$, where Δ is the infinitesimal generator of $(v_tC_{\psi_t})$, which is given in (5.13). To avoid cumbersome notation, we write $f\in\mathcal{E}(-\Delta)$ if $f_{c+\varepsilon}\in\mathcal{E}(c+\varepsilon-\Delta)$, where $f_{c+\varepsilon}=f((\cdot)-c-\varepsilon)$. In this case, we set $f(-\Delta):=f_{c+\varepsilon}(c+\varepsilon-\Delta)$.

Corollary 5.5.3. *Let $(v_tC_{\psi_t})$ be a semigroup on a γ^∞-space X such that the semicocycle (v_t) and the semiflow (ψ_t) satisfy properties* (**SCo1**)-(**SCo3**) *and* (**SFlow1**)-(**SFlow4**) *respectively. Let $\alpha_0,\alpha_1,\dots,\alpha_n$, and $\beta_0,\beta_1,\dots,\beta_n$ be the exponents of (v_t) and (ψ_t') respectively. Let ν be a complex Borel measure on $[0,+\infty)$, such that $\int_0^\infty e^{(c+\delta)t}|d\nu|(t)<\infty$ for some $\delta>0$. Assume $d\nu(t)=\rho(t)\,dt$, and ρ can be extended in an holomorphic way to a sector Σ_θ with $0<\theta\le\pi/2$, and there exist $0<\eta\le 1,\xi\in(0,1)$ satisfying*

$$\sup_{z\in\Sigma_\varepsilon\cap\{|z|\le 1\}}|z^{1-\eta}\rho(z)|<\infty \quad \textit{and} \quad \sup_{z\in\Sigma_\varepsilon\cap\{|z|\ge 1\}}|z^{1+\xi}e^{ct}\rho(z)|<\infty, \quad \textit{for all } 0<\varepsilon<\theta.$$

Then

$$\sigma(\mathcal{H})=\{0\}\cup\mathcal{L}(\nu)(-\tilde{\sigma}(\Delta))=\mathcal{L}(\nu)((d+\Sigma_{\pi/2})\cup-\sigma_{point}(\Delta)),$$

where $d:=\min\{\gamma\beta_1-\alpha_1,\dots,\gamma\beta_n-\alpha_n\}$. Also

$$\mathcal{L}(\nu)(-\sigma_{point}(\Delta))\subseteq\sigma_{point}(\mathcal{H})\subseteq\{0\}\cup\mathcal{L}(\nu)(-\sigma_{point}(\Delta)).$$

Proof. By Lemma 5.5.2, the function $\mathcal{L}(\nu)$ belongs to $\mathcal{E}(-\Delta)$ with $\mathcal{L}(\nu)(\infty) = 0$. Hence, the claim follows by the spectral mapping theorem given in Corollary 2.2.28 and Theorem 5.4.8. □

5.6 Examples

Here we apply our results to some generalized Hausdorff operators $\mathcal{H}_\mu^{(\zeta)}$ on a γ^∞-space X (for some $\gamma \geq 0$) for complex measures μ on $(0, 1]$ and $\zeta \in \mathbb{C}$. Let $\nu = \kappa(\mu)$ be the Borel image measure on $[0, \infty)$ by the function $\kappa : (0, 1] \to [0, +\infty)$ given by $\kappa(t) = \log(1/t)$. Then we have

$$\mathcal{H}_\mu^{(\zeta)} f(z) = \int_0^\infty \left(\frac{\phi_t(z)}{z}\right)^\zeta \frac{1 - \phi_t(z)}{1 - z} C_{\phi_t} f(z)\, d\nu(t), \quad z \in \mathbb{D},\ f \in X,$$

where ϕ_t is given by (5.3). In this case, the semigroup $(v_t C_{\phi_t})$ given by $v_t(z) = \left(\frac{\phi_t}{z}\right)^\zeta \frac{1-\phi_t(z)}{1-z}, z \in \mathbb{D}, t \geq 0$, has infinitesimal generator $\Delta f(z) = \Psi(z) f'(z) + g(z) f(z)$, with $\Psi(z) = -z(1 - z)$ and $g(z) = -z(\zeta + 1)$. It is readily seen that the DW point of (ϕ_t) is $z_0 = 0$ and that (ϕ_t) has one repulsive point in $z_1 = 1$. Moreover, the exponents of the semicocycle (v_t) are given by $\alpha_0 = -\mathfrak{Re}\,\zeta$ and $\alpha_1 = 1$, and the exponents of (ϕ_t') are given by $\beta_0 = -1$ and $\beta_1 = 1$. Thus, following the notation of Section 5.5, $c = \max\{-\mathfrak{Re}\,\zeta, 1 - \gamma\}$. Then, by Theorem 5.4.8,

$$\sigma(\Delta) = \{\lambda \in \mathbb{C} \ :\ \mathfrak{Re}\,\lambda \leq 1 - \gamma\} \cup \sigma_{point}(\Delta). \tag{5.20}$$

Remark 5.6.1. Let X be any of the examples of γ^∞-spaces listed in Section 5.1 (i.e., Hardy spaces, weighted Bergman spaces and little Korenblum classes). Then:

a) The growth bound of the semigroup $(v_t C_{\phi_t})$ described above is $c = \max\{-\mathfrak{Re}\,\zeta, 1-\gamma\}$, that is, there is $M > 0$ such that $\|T(t)\| \leq M e^{ct}$, for $t \geq 0$. So, it is enough to assume that $\int_0^\infty e^{ct} |d\nu|(t) < \infty$ to get that $\mathcal{H}_\mu^{(\zeta)}$ is a bounded operator on X.

b) By Proposition 4.2.4, we obtain

$$\begin{aligned}\sigma_{point}(\Delta) &= \left\{-\zeta - k \ :\ k \in \mathbb{N}_0 \text{ such that } \frac{z^k}{(1 - z)^{k+\zeta+1}} \in X\right\} \\ &= \{-\zeta - k \ :\ k \in \mathbb{N}_0 \text{ with } k < \gamma - \mathfrak{Re}\,\zeta - 1\}.\end{aligned}$$

5.6.A Generalized Cesàro operators

Let $\alpha \in \mathbb{C}$ with $\mathfrak{Re}\,\alpha > 0$. Let μ_α the Borel measure on $(0, 1]$ such that $d\mu_\alpha(t) = \alpha(1 - t)^{\alpha-1}\, dt$. Then $d\nu_\alpha(t) = \alpha(1 - e^{-t})^{\alpha-1} e^{-t}\, dt$. Note that $(\mu_\alpha)_n = \frac{\Gamma(\alpha+1)\Gamma(n+\zeta+1)}{\Gamma(n+\zeta+\alpha+1)}$, for $n \in \mathbb{N}_0$. Then, for $\mathfrak{Re}\,\zeta > -1$, the generalized Cesàro operator $\mathcal{C}_\alpha^\zeta$ is defined as the associated Hausdorff operator to μ_α and ζ, that is, $\mathcal{C}_\alpha^\zeta = \mathcal{H}_{\mu_\alpha}^{(\zeta)}$. It is readily seen that

$$(\mathcal{C}_\alpha^\zeta f)(z) = \frac{\alpha}{z^{\zeta+\alpha}} \int_0^z \frac{w^\zeta (z - w)^{\alpha-1}}{(1 - w)^\alpha} f(w)\, dw, \quad z \in \mathbb{D},\ f \in \mathcal{O}(\mathbb{D}).$$

Now, let X be a γ^∞-space for some $\gamma > 0$. Then, for each $\delta \in (0, \min\{\mathfrak{Re}\,\zeta + 1, \gamma\})$, one has

$$\int_0^\infty e^{(c+\delta)t} |d\nu_\alpha|(t) = |\alpha| \int_0^\infty e^{\delta t} e^{\max\{-(\zeta+1), -\gamma\}t} (1 - e^{-t})^{\mathfrak{Re}\,\alpha - 1}\, dt < \infty.$$

As a consequence, $\mathcal{C}_\alpha^\zeta$ is a well-defined bounded operator on X, see Theorem 5.5.1. In addition, it is readily seen that

$$\mathcal{L}(\nu_\alpha)(z) = \int_0^\infty e^{-zt}\, d\nu_\alpha(t) = \alpha \mathbb{B}(z+1, \alpha), \quad \mathfrak{Re}\, z > -1.$$

Hence, the hypothesis of Corollary 2.2.28 are satisfied, see the proof of Proposition 3.3.2 for more details. As a consequence, we obtain the following

Theorem 5.6.2. *Let $X = H^p(\mathbb{D}), \mathcal{A}^p_\sigma(\mathbb{D}), \mathcal{K}_0^{-\widetilde{\gamma}}(\mathbb{D})$ for $p \geq 1, \sigma > -1, \widetilde{\gamma} > 0$, so X is a γ^∞ space for $\gamma = 1/p, (\sigma+2)/p, \widetilde{\gamma}$ respectively. Let $\mathfrak{Re}\,\alpha > 0$ and $\mathfrak{Re}\,\zeta > -1$. Then $\mathcal{C}_\alpha^\zeta$ is a bounded operator on X such that*

$$\sigma(\mathcal{C}_\alpha^\zeta) = \{\alpha \mathbb{B}(z, \alpha) \,:\, \mathfrak{Re}\, z \geq \gamma, \text{ or } z = \zeta + k \text{ with } k \in \mathbb{N} \text{ and } k < \gamma - \mathfrak{Re}\,\zeta\},$$

and

$$\sigma_{point}(\mathcal{C}_\alpha^\zeta) = \{\alpha \mathbb{B}(z, \alpha) \,:\, z = \zeta + k \text{ with } k \in \mathbb{N} \text{ and } k < \gamma - \mathfrak{Re}\,\zeta\}.$$

Proof. The statement follows from the comments above together with (5.20) and Remark 5.6.1b). □

5.6.B Hölder operators

Let $\alpha \in \mathbb{C}$ with $\mathfrak{Re}\,\alpha > 0$. Let μ_α the Borel measure on $(0, 1]$ such that $d\mu_\alpha(t) = \frac{1}{\Gamma(\alpha)} \Big(\log(1/t)\Big)^{\alpha-1} dt$. Then $d\nu_\alpha(t) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1} e^{-t}\, dt$. Note that $\mu_n = \frac{1}{(n+\zeta+1)^\alpha}$, for $n \in \mathbb{N}_0$. Then, for $\mathfrak{Re}\,\zeta > -1$, the generalized Hölder operator $\mathfrak{H}_\alpha^\zeta$ is defined as the associated Hausdorff operator to μ_α and ζ, that is, $\mathfrak{H}_\alpha^\zeta = \mathcal{H}_{\mu_\alpha}^{(\zeta)}$. It is readily seen that

$$(\mathfrak{H}_\alpha^\zeta f)(z) = \frac{1}{\Gamma(\alpha)} \frac{1}{z^{\zeta+1}} \int_0^z \frac{w^\zeta}{1-w} \left(\log \frac{z(1-w)}{w(1-z)} \right)^{\alpha-1} f(w)\, dw, \quad z \in \mathbb{D},\, f \in \mathcal{O}(\mathbb{D}).$$

Now, let X be a γ^∞-space for some $\gamma > 0$. Then, for each $\delta \in (0, \min\{\mathfrak{Re}\,\zeta + 1, \gamma\})$, one has

$$\int_0^\infty e^{(c+\delta)t} |d\nu_\alpha|(t) = \int_0^\infty e^{\delta t} e^{-\min\{\mathfrak{Re}\,\zeta+1, \gamma\}t} t^{\mathfrak{Re}\,\alpha - 1}\, dt < \infty.$$

As a consequence, $\mathfrak{H}_\alpha^\zeta$ is a well-defined bounded operator on X, see Theorem 5.5.1. In addition, it is readily seen that

$$\mathcal{L}(\nu_\alpha)(z) = \int_0^\infty e^{-zt}\, d\nu_\alpha(t) = \frac{1}{(z+1)^\alpha}, \quad \mathfrak{Re}\, z > -1.$$

Hence, the hypothesis of Corollary 2.2.28 are satisfied, see the proof of Proposition 3.3.2 for more details. As a consequence, we obtain the following

Theorem 5.6.3. *Let* $X = H^p(\mathbb{D}), \mathcal{A}^p_\sigma(\mathbb{D}), \mathcal{K}_0^{-\widetilde{\gamma}}(\mathbb{D})$ *for* $p \geq 1, \sigma > -1, \widetilde{\gamma} > 0$*, so* X *is a* γ^∞ *space for* $\gamma = 1/p, (\sigma+2)/p, \widetilde{\gamma}$ *respectively. Let* $\mathfrak{Re}\,\alpha > 0$ *and* $\mathfrak{Re}\,\zeta > -1$*. Then* $\mathfrak{H}^\zeta_\alpha$ *is a bounded operator on* X *such that*

$$\sigma(\mathfrak{H}^\zeta_\alpha) = \{z^{-\alpha} \,:\, \mathfrak{Re}\,z \geq \gamma, \text{ or } z = \zeta + k \text{ with } k \in \mathbb{N} \text{ and } k < \gamma - \mathfrak{Re}\,\zeta\},$$

and

$$\sigma_{point}(\mathfrak{H}^\zeta_\alpha) = \{z^{-\alpha} \,:\, z = \zeta + k \text{ with } k \in \mathbb{N} \text{ and } k < \gamma - \mathfrak{Re}\,\zeta\}.$$

Proof. The statement follows from the comments above together with (5.20) and Remark 5.6.1b). □

Chapter 6

Weighted hyperbolic groups

The purpose of this chapter is twofold. In one way, we search for providing a spectral picture of weighted hyperbolic composition groups on $\mathbb{D}$. On the other hand, we look for giving spectral descriptions of integral operators subordinated to the quoted groups.

Our interest in the above operators and groups has been motivated by several issues arising in different, though connected, ways. There is a vast literature dealing with properties (norm, compactness, spectrum, ...) of families of averaging integral operators acting on Banach spaces X of holomorphic functions in $\mathbb{D}$. Recall, the Cesàro integral operator $\mathcal{C}$ and its equivalent formulation $\mathfrak{C}$ on sequences are defined respectively by

$$(\mathcal{C}f)(z) := \frac{1}{z}\int_0^z \frac{f(w)}{1-w}\,dw\ ; \quad (\mathfrak{C}\widehat{f})(n) := \frac{1}{n+1}\sum_{j=0}^{n} \widehat{f}(j),$$

for $z \in \mathbb{D}$, $n \in \mathbb{N}\cup\{0\}$, $f \in X$, where $\widehat{f} = (\widehat{f}(n))$ denotes the coefficient Taylor sequence of the analytic function f. The corresponding adjoint operators of $\mathcal{C}$ and $\mathfrak{C}$ are given by

$$(\mathcal{C}^*f)(z) := \frac{1}{z-1}\int_1^z f(\xi)\ d\xi\,; \quad (\mathfrak{C}^*\widehat{f})(n) := \sum_{j=n}^{\infty} \frac{\widehat{f}(j)}{j+1} \quad (z \in \mathbb{D}, n \in \mathbb{N}\cup\{0\}).$$

Let $\mathcal{J}$ denote the operator defined by

$$(\mathcal{J}f)(z) := \frac{1}{1-z}\int_1^z \frac{f(\xi)}{1+\xi}\ d\xi,\ z \in \mathbb{D},$$

which was introduced in [Sis86], where its norm, spectrum and point spectrum in Hardy spaces $H^p(\mathbb{D})$, $p \geq 1$, were studied. Here, we call $\mathcal{J}$ *Siskakis' operator*. Even though it formally looks a weighted version of $\mathcal{C}^*$ (in fact, $\mathcal{J}f = -\mathcal{C}^*\left((1+(\cdot))^{-1}f)\right)$ they behave different from a spectral viewpoint. A reason for this is seen below, via certain one-parameter operator families.

Likewise, there are also the so-called Hilbert matrix operator $\mathfrak{H}$ and the *reduced* Hilbert matrix operator $\mathcal{H}$ defined respectively by

$$(\mathfrak{H}f)(z) := \int_0^1 \frac{f(\xi)}{1-z\xi}d\xi, \quad (\mathcal{H}f)(z) := \int_{-1}^1 \frac{f(\xi)}{1-z\xi}d\xi, \qquad z \in \mathbb{D},$$

see [DS00] for $\mathfrak{H}$. While working on the present chapter, the authors have been aware of the fact that A. Aleman, A. Siskakis and D. Vukotic have recently approached the study of the operator $\mathfrak{H}$ using its reduced version $\mathcal{H}$ as a key tool. We are not following this idea here.

In recent times, a line of research has emerged that takes families of (multi-parameterized) generalizations of Cesàro operators as study objects. An interesting representative of one of such families is $\mathcal{T}_{\mu,\nu}$, $\mu,\nu\in\mathbb{R}$, given by the formula

$$(\mathcal{T}_{\mu,\nu}f)(z) := z^{\mu-1}(1-z)^{-\nu}\int_0^z \xi^{-\mu}(1-\xi)^{\nu-1}f(\xi)\,d\xi, \qquad z\in\mathbb{D}.$$

The operator $\mathcal{T}_{\mu,\nu}$ generalizes $\mathcal{C}$ (note, $\mathcal{T}_{0,0}=\mathcal{C}$) as well as other operators related with $\mathcal{C}$, see [AP10; BMM14] and references therein. There are other generalizations of Cesàro operators in the literature, see [AP10; Bla+13; Per08; Sis93; Ste94; Xia97]; in particular averaging operators of the form $\frac{1}{z}\int_0^z f(\xi)g'(\xi)d\xi$ for generic functions g' of essentially rational type.

In a similar way, it sounds sensible to consider parameterized averaging operators generalizing $\mathcal{J}$ and to investigate their spectral properties. Here we approach the study of the family of operators $\mathcal{J}_\delta^{\mu,\nu}$ given by

$$(\mathcal{J}_\delta^{\mu,\nu}f)(z) := \frac{1}{(1+z)^{\nu+\delta}(1-z)^{\mu+\delta}}\int_z^1 (1+\xi)^{\nu}(1-\xi)^{\mu}(\xi-z)^{\delta-1}f(\xi)\,d\xi, \qquad z\in\mathbb{D},$$

for $z\in\mathbb{D}$, $f\in X$ and suitable values of parameters $\mu,\nu,\delta\in\mathbb{C}$.

This family generalizes the Siskakis' operator since $\mathcal{J}=-\mathcal{J}_1^{0,-1}$. For other particular values of μ, ν and δ, operators $\mathcal{J}_\delta^{\mu,\nu}$ are isometric, up to constants, to certain parameterized operators, defined on fractional subspaces of $L^2(0,\infty)$ and $H^2(\mathbb{C}^+)$, considered in [GMS21; LMPS14]. The extension of the above operators to arbitrary parameters μ, ν, δ (whenever there is convergence of the integrals) seems to be natural. Weights $(1\pm z)^\alpha$, $\alpha\in\mathbb{R}$, also arise in a natural way if we think of the action of composition operators (see below in this introduction) on spaces like $H^p(\mathbb{D})$ with weights of the same type; see for example [CMW92, Section 4].

As regards generalizations of the reduced Hilbert matrix operator, we will deal with the family $\mathcal{H}_\delta^{\mu,\nu}$, for suitable $\mu,\nu,\delta\in\mathbb{C}$, given by

$$(\mathcal{H}_\delta^{\mu,\nu}f)(z) := \frac{1}{(1+z)^{\nu-\delta+1}(1-z)^{\mu-\delta+1}}\int_{-1}^1 (1+\xi)^{\nu}(1-\xi)^{\mu}\frac{f(\xi)}{(1-z\xi)^{\delta}}\,d\xi,$$

for $z\in\mathbb{D}$, $f\in X$. Clearly, $\mathcal{H}=\mathcal{H}_1^{0,0}$. On the other hand, operators $\mathcal{H}_\delta^{\mu,\nu}$ are also a generalization of other operators isometric to the Stieltjes transform or Poisson-like integrals; see [MO21].

Operators $\mathcal{J}_\delta^{\mu,\nu}$ and $\mathcal{H}_\delta^{\mu,\nu}$ are closely related to groups of automorphisms on the unit disc, in particular with the hyperbolic one, as we explain later on.

When acting on a function Banach space X, all families of integral operators quoted above share the property that their elements, say $\mathcal{T}$, can be expressed on appropriate X

by subordination to suitable vector-valued functions $V\colon \mathbb{R} \to L(X)$; that is, $\mathcal{T}$ can be written in the form

$$\mathcal{T}f = \int_{-\infty}^{\infty} g(t)V(t)f\,dt, \quad f \in X, \tag{6.1}$$

where g is locally integrable on $\mathbb{R}$ and $V(t)$ is related with semigroups of composition operators or it is a semigroup itself. We put $V(t) = S(t)$ in this case, and write the semigroup (or group) often as $(S(t))$. The above representation (6.1) is relevant for the study of boundedness and norms, spectra and other properties like subnormality, compactness and so on. The idea to exploit subordination, as in (6.1), in the study of properties of $\mathcal{T}$ dates back to [Cow84] at least. A systematic approach to classical averaging operators $\mathcal{T}$ based upon the analysis of the infinitesimal generators of semigroups $S(t)$ was undertaken by A. Siskakis in several papers [DS00; Sis86; Sis87]. In these works, subordination is mostly restricted to give integral expressions of inverses of generators and, more generally, of resolvent functions. Families $\{\mathcal{J}_\delta^{\mu,\nu}\}$ and $\{\mathcal{H}_\delta^{\mu,\nu}\}$ lie in the framework yield around (6.1). To see this, we need to say some words about composition groups of automorphisms.

Assume that X is a function Banach space continuously contained in the Fréchet space $\mathcal{O}(\mathbb{D})$ of all holomorphic functions on $\mathbb{D}$. In this chapter, we are interested in weighted composition groups $(S(t))$ where (ψ_t) is a flow of *hyperbolic* automorphisms. Up to isomorphism, the class of groups of hyperbolic automorphisms of $\mathbb{D}$ is reduced to the hyperbolic flow (φ_t) where

$$\varphi_t(z) := \frac{(e^t+1)z + e^t - 1}{(e^t-1)z + e^t + 1}, \quad z \in \mathbb{D},\ t \in \mathbb{R}. \tag{6.2}$$

The operator $\mathcal{T}_{\mu,\nu}$ as well as other generalizations of Cesàro's operator admit to be represented by subordination, as in (6.1), to semigroups of weighted composition operators, see [Sis98]. In turn, operators $\mathcal{J}_\delta^{\mu,\nu}$ and $\mathcal{H}_\delta^{\mu,\nu}$ can be represented by subordination to a weighted composition group $(u_t C_{\varphi_t})$; namely

$$\mathcal{J}_\delta^{\mu,\nu} = \int_{-\infty}^{\infty} g_\delta(t)\, u_t C_{\varphi_t}\, dt, \qquad \mathcal{H}_\delta^{\mu,\nu} = \int_{-\infty}^{\infty} h_\delta(t)\, u_t C_{\varphi_t}\, dt, \tag{6.3}$$

where, for $t \in \mathbb{R}$ and $\mathfrak{Re}\,\delta > 0$, $g_\delta(t) = 2^{-\delta}(1-e^{-t})^{\delta-1}\chi_{(0,\infty)}(t)$ and $h_\delta(t) = 2^{\delta-1}(1+e^t)^{-\delta}$, see Section 6.8. Notice that the functions g_δ, h_δ appear on the other hand as subordinating functions in [AM18; GMS21; LMPS14; MO21]. This fact also suggested considering operators $\mathcal{J}_\delta^{\mu,\nu}$, $\mathcal{H}_\delta^{\mu,\nu}$.

One of the aims in this chapter is to describe the fine structure of the spectrum of the operators $\mathcal{J}_\delta^{\mu,\nu}$ and $\mathcal{H}_\delta^{\mu,\nu}$. To do so in a unified way, we connect this question with the regularized functional calculus of the generator of the group $(u_t C_{\varphi_t})$ and suitable operating functions. More precisely, we adopt the Siskakis' view, and therefore we undertake a detailed study of the infinitesimal generator Δ of $(u_t C_{\varphi_t})$. Such a generator is a bisectorial-like operator, so that we apply results of Chapter 2 on spectral mappings to transfer the information on the spectrum of Δ to the one of $\mathcal{J}_\delta^{\mu,\nu}$ and $\mathcal{H}_\delta^{\mu,\nu}$.

We wish to establish our results here for a class of Banach spaces as larger as possible, following a unified approach. Thus we introduce the notion of Banach γ-*space*, depending on a non-negative parameter γ, which includes classical Banach spaces usually considered in the subject. Among these spaces, one has for instance Hardy spaces, (weighted) Bergman spaces, little Korenblum spaces and the disc algebra, (weighted) Dirichlet spaces and little Bloch spaces.

On the other hand, the study of weighted hyperbolic groups $(u_tC_{\varphi_t})$ has interest in its own. This was another of our aims in the beginning of this work, as well as finding out applications to weighted hyperbolic composition operators, say vC_ψ. Let ψ denote a hyperbolic automorphism and let v denote a weight or multiplier. It is still an open question, in general, whether or not the spectrum $\sigma(vC_\psi)$ is an annulus and, in such a case, which are its radii. Just citing the most recent papers on that question, one has in [CGP15] that, for the classical Dirichlet space ($\mathcal{D}_0^2$ in our notation), v continuous at the DW points a and b of ψ, and vC_ψ invertible,

$$\sigma(vC_\psi) \subseteq \{\lambda \in \mathbb{C} : \min\{|v(a)|, |v(b)|\}\psi'(a) \leq |\lambda| \leq \max\{|v(a)|, |v(b)|\}\psi'(b)\}.$$

The above inclusion is improved in [ELM16], where it is shown that

$$\sigma(vC_\psi) \subseteq \{\lambda \in \mathbb{C} : \min\{|v(a)|, |v(b)|\} \leq |\lambda| \leq \max\{|v(a)|, |v(b)|\}\},$$

whenever v is in the disc algebra. It is also conjectured that

$$\sigma(vC_\psi) = \{\lambda \in \mathbb{C} : \min\{|v(a)|, |v(b)|\} \leq |\lambda| \leq \max\{|v(a)|, |v(b)|\}\}, \tag{6.4}$$

for the Dirichlet space and Bloch space.

Furthermore, for the spaces $H^p(\mathbb{D})$, $\mathcal{A}^p_\sigma(\mathbb{D})$, $\mathcal{K}_0^{-\gamma}(\mathbb{D})$, $p \geq 1$, and vC_ψ invertible, it is proved in [HLNS13] that the spectrum of vC_ψ is contained in the annulus of radii $\min\{|v(a)|\psi'(a)^{-\gamma}, |v(b)|\psi'(b)^{-\gamma}\}$ and $\max\{|v(a)|\psi'(a)^{-\gamma}, |v(b)|\psi'(b)^{-\gamma}\}$ and that, provided $|v(b)|\psi'(b)^{-\gamma} \leq |v(a)|\psi'(a)^{-\gamma}$,

$$\sigma(vC_\psi) = \{\lambda \in \mathbb{C} : |v(b)|\psi'(b)^{-\gamma} \leq |\lambda| \leq |v(a)|\psi'(a)^{-\gamma}\}, \tag{6.5}$$

as well as that $\mathrm{Int}(\sigma(vC_\psi)) \subseteq \sigma_{point}(vC_\psi)$. The question of whether or not the corresponding equality is true in the case $|v(b)|\psi'(b)^{-\gamma} > |v(a)|\psi'(a)^{-\gamma}$ is left open in [HLNS13] as a conjecture in the positive.

Every hyperbolic automorphism ψ can be embedded in a hyperbolic flow (ψ_t), in the sense that $\psi = \psi_1$. If the weight v can also be embedded in a cocycle (v_t) for (ψ_t), then the spectrum of the infinitesimal generator Δ of $(v_tC_{\psi_t})$ provides substantial information about the one of $v_1C_{\psi_1} = vC_\psi$. With this method, we prove that conjectures (6.4) and (6.5) are true if the operator vC_ψ can be embedded in a C_0-group $(v_tC_{\psi_t})_{t\in\mathbb{R}}$, and for all the spaces quoted above, see Theorem 6.7.2. Moreover, the theorem provides information about subspectra of vC_ψ which seems to be of interest, in particular for Dirichlet spaces. The ideas considered in the chapter could be helpful to study arbitrary invertible weighted hyperbolic operators uC_{ψ_1} by means of quasi-nilpotent perturbations

$uC_{\psi_1} - v_1 C_{\psi_1}$, since $uC_{\psi_1} - v_1 C_{\psi_1}$ is a quasi-nilpotent operator for a suitable cocycle (v_t) for (ψ_t).

In view of the above, the description of spectra of the infinitesimal generator Δ turns out to be the key point of the chapter. Thus another question of importance is to find families of cocycles (u_t) for which the spectral picture of Δ is available. In this respect, it is useful the representation of (u_t) as a coboundary, i.e.,

$$u_t = \frac{\omega \circ \varphi_t}{\omega},\ t \in \mathbb{R},$$

for some non-vanishing holomorphic function $\omega : \mathbb{D} \to \mathbb{C}$, see Proposition 4.2.2. We obtain the notable property that, under fairly mild conditions on (u_t) (namely, that (u_t) is a DW-continuous cocycle, see Section 6.1), ω presents zeroes or singularities of polynomial type at the Denjoy-Wolf points of (φ_t). This property is crucial (and enough) to give a detailed spectral picture of Δ for Hardy spaces, Bergman spaces, little Korenblum classes and the disc algebra. The case of Dirichlet spaces and little Bloch spaces require an extra condition on ω which does not seem to be strong.

We now outline how the chapter is organized.

In Section 6.1, we define DW-continuous cocycles and explain that, in most of the chapter, we will focus on the hyperbolic flow (φ_t) of DW points 1 and -1. Conditions or properties defining Banach γ-spaces are given in Section 6.2, together with some lemmas which provide us with a number of such spaces, including the examples quoted above. In particular, condition (**Gam5**) is introduced to place Dirichlet spaces and little Bloch spaces into the setting. For the other examples it is sufficient to recall the well known fact that (**Gam5**) hods for $\varepsilon = 0$. The notion of γ-space covers a range of spaces a bit larger than other systems of axioms do.

Section 6.3 is devoted to prove that the holomorphic function ω associated with a cocycle (u_t) for the flow (φ_t) is tempered at the DW points -1, 1. The overall argument to prove that is rather involved and culminates with Theorem 6.3.11. In order to establish our results on spectra in a general form, we also introduce spectrally DW-contractive cocycles, and hyperbolically DW-contractive spaces accordingly (see definitions there), and show that the examples of γ-spaces of Subsection 6.2.A are hyperbolically DW-contractive.

In Section 6.4 estimates on the group $(u_t C_{\varphi_t})$ of asymptotic type related to the spectral radius are given. In Section 6.5, properties of two helpful integrals related to the resolvent operator are presented, as preparation to Section 6.6 where the fine structure of the spectrum of Δ is exposed, see Theorem 6.6.6. This theorem widely extends results of [Sis86]. At this point, it must be said that the ideas behind the results of this chapter, in particular in Section 6.5 and Section 6.6, have been mainly inspired by papers [AP10; CGP15; HLNS13; Per08; Sis86]. The level of generality that such ideas present in this chapter, in the direction considered here, has been very much facilitated by the quoted Theorem 6.3.11.

Features of spectra of the generator Δ are transferred, first to the weighted hyperbolic group $u_t C_{\varphi_t} = e^{t\Delta}$ (Theorem 6.7.1), and then to arbitrary weighted hyperbolic groups $(v_t C_{\psi_t})$ (under corresponding assumptions on (v_t)) by composition with suitable automorphisms, in Section 6.7, Theorem 6.7.2. It is to be noticed that Theorem 6.7.2 gives us information on the full spectrum, essential spectrum, point spectrum and residual spectrum of $v_tC_{\psi_t}$, $t \in \mathbb{R}$. In Remark 6.7.3, we point out that Theorem 6.7.2 provides partial solutions, even for Dirichlet and little Bloch spaces, to the conjectures discussed around (6.4) and (6.5).

Finally, in Section 6.8 the results obtained in preceding sections are applied to the aforementioned integral averaging operators which generalize the Siskakis' operator and the reduced Hilbert matrix operator.

6.1 DW-continuous flows

Recall that $\mathcal{O}(\mathbb{D})$ denotes the Fréchet algebra of holomorphic functions on the unit disc $\mathbb{D}$, and that $Aut(\mathbb{D})$ is the group of automorphisms of the disc, that is, $\phi \in Aut(\mathbb{D})$ if and only if $\phi \in \mathcal{O}(\mathbb{D})$ and it is of the form $\phi(z) := e^{i\theta}\phi_\xi(z)$ for all $z \in \mathbb{D}$, where $\xi \in \mathbb{D}$ and $\theta \in [0, 2\pi)$, and where $\phi_\xi(z) = (1 - \overline{\xi}z)^{-1}(z - \xi)$.

Flows of automorphisms are classified according to their fixed points. Namely, one says that a flow of automorphisms (ψ_t) is: 1) elliptic, if it has a unique fixed point in $\mathbb{D}$; 2) parabolic, if it has a unique fixed point in $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$; 3) hyperbolic, if it has two distinct fixed points in $\mathbb{T}$.

Here we deal with *flows* of hyperbolic automorphisms. For such a given flow (ψ_t) the well-known Denjoy-Wolff theorem states that its fixed points in $\mathbb{T}$ are obtained as

$$a := \lim_{t\to+\infty} \psi_t(z), \; b := \lim_{t\to-\infty} \psi_t(z), \quad z \in \mathbb{D}.$$

Points a and b are called attractive and repulsive DW points, respectively. There always exists an automorphism ϕ of $\mathbb{D}$ such $\phi(a) = 1$ and $\phi(b) = -1$, so that there exists $c > 0$ for which $\varphi_{ct} := \phi\circ\psi_t\circ\phi^{-1}$, $t \in \mathbb{R}$, where (φ_t) is the hyperbolic flow (6.2) with DW points 1 (attractive) and -1 (repulsive). The generator G of (φ_t) is given by $G(z) = \frac{1}{2}(1 - z^2)$, $z \in \mathbb{D}$, and one also has

$$\frac{\partial\varphi_t(z)}{\partial t} = G(\varphi_t(z)) = \frac{\partial\varphi_t(z)}{\partial z}G(z), \quad z \in \mathbb{D}, \, t \in \mathbb{R}, \tag{6.6}$$

see Chapter 4.

In this chapter, we will consider cocycles (v_t) enjoying the following property:

(Co1) $(\forall t \in \mathbb{R})$ There exist $v_t(b) := \lim_{\mathbb{D}\ni z\to b} v_t(z) \in \mathbb{C}$, $v_t(a) := \lim_{\mathbb{D}\ni z\to a} v_t(z) \in \mathbb{C}$;

see [CGP15; HLNS13] for the suitability of this condition when dealing with the spectrum of weighted composition operators on Banach spaces.

Let X be a Banach function space continuously contained in $\mathcal{O}(\mathbb{D})$ (that is, $X \hookrightarrow \mathcal{O}(\mathbb{D})$ for short). The function spaces X which we are dealing with in this chapter satisfy that composition operators $C_\phi : X \to X$ ($C_\phi f = f \circ \phi$), $\phi \in Aut(\mathbb{D})$, are bounded isomorphisms of X, see Remark 6.2.2. Since multiplication by v_t is decomposed as

$$f \xrightarrow{C_{\psi_t^{-1}}} f \circ \psi_t^{-1} \xrightarrow{v_t C_{\psi_t}} v_t f,$$

we have that $v_t C_{\psi_t}$ is bounded on X if and only if the multiplication operator $f \mapsto v_t f$ is bounded on X which is to say v_t is a multiplier of X. Recall that we denote the space of multipliers of X by $Mul(X)$. In view of above, it sounds sensible to consider the following property for a cocycle (v_t):

(**Co2**) The mapping $t \mapsto v_t$ is Bochner-measurable from $\mathbb{R}$ to $Mul(X)$.

Definition 6.1.1. Let (v_t) be a continuous cocycle for a hyperbolic flow (ψ_t). We say that (v_t) is a *DW-continuous cocycle* (for the flow (ψ_t)) on X if it satisfies conditions (**Co1**) and (**Co2**).

We are interested in groups $(v_t C_{\psi_t})$ where (ψ_t) is a hyperbolic flow and v_t is a DW-continuous cocycle. We have seen before that composition (on the left and on the right) of (ψ_t) with suitable $\phi \in Aut(\mathbb{D})$ turns (ψ_t) into the standard hyperbolic group (φ_t) of generator $G(z) = (1 - z^2)/2$. Let us now see how the action of ϕ affects weighted composition operators, under mild assumptions.

So let (ψ_t) be a hyperbolic flow of $\mathbb{D}$ with DW points $a, b \in \mathbb{T}$ and let (v_t) be a DW-continuous cocycle for (ψ_t) so that $(v_t C_{\psi_t})$ is a one-parameter group in $L(X)$. Take $\phi \in Aut(\mathbb{D})$ such that $\phi(a) = 1$, $\phi(b) = -1$. Hence there exists $c > 0$ for which $\varphi_{ct} = \phi \circ \psi_t \circ \phi^{-1}$ for all $t \in \mathbb{R}$, see [BP78]. Now set $u_t := v_{c^{-1}t} \circ \phi^{-1}$, thus $u_{ct} C_{\varphi_{ct}} = C_{\phi^{-1}} \circ (v_t C_{\psi_t}) \circ C_\phi$. It is readily seen that $t \mapsto u_t$ is measurable if and only if $t \mapsto v_t$ is measurable, hence (u_t) satisfies (**Co2**). Moreover, if there exist $v_t(a) := \lim_{\mathbb{D} \ni z \to a} v_t(z)$ and $v_t(b) := \lim_{\mathbb{D} \ni z \to b} v_t(z)$ in $\mathbb{C}$, then there exist $u_t(-1) := \lim_{\mathbb{D} \ni z \to -1} u_t(z)$, $u_t(1) := \lim_{\mathbb{D} \ni z \to 1} u_t(z)$ in $\mathbb{C}$, for all t, so (u_t) also satisfies (**Co1**), i.e. (u_t) is a DW-continuous cocycle for (φ_t). Since the operators C_ϕ and $C_{\phi^{-1}}$ are isomorphisms, the spectra of $v_t C_{\psi_t}$ and $u_{ct} C_{\varphi_t ct}$ are the same. Thus, from now on, we concentrate our study of spectra of weighted hyperbolic groups on families $(u_t C_{\varphi_t})$ of bounded operators on X where (φ_t) is the hyperbolic flow of (6.2) and (u_t) is a DW-continuous cocycle for (φ_t).

6.2 γ-conformal spaces

One of the aims of this chapter is to study spectra of weighted composition groups $(u_t C_{\psi_t})$ acting on Banach spaces $X \hookrightarrow \mathcal{O}(\mathbb{D})$. In this section, we put up the setting where to work by introducing a number of conditions on X. We also show that most classical function spaces satisfy such conditions. The two first of these conditions, namely (**Gam1**) and (**Gam2**), concern *multipliers*. For every open subset $U \subseteq \mathbb{C}$, let $H^\infty(U)$ be the Banach algebra of bounded analytic functions on U endowed with the sup-norm

$\|f\|_{H^\infty(U)} := \sup_{z\in U} |f(z)|$, $f \in H^\infty(U)$. If $U = \mathbb{D}$ we write $\|\cdot\|_{H^\infty(\mathbb{D})} = \|\cdot\|_\infty$. Then, set

(**Gam1**) $$\bigcup_{\overline{\mathbb{D}}\subseteq U open} H^\infty(U) \hookrightarrow Mul(X),$$

where the "hook" arrow on the right means that $\|F\|_{Mul(X)} \leq K_U \|F\|_{H^\infty(U)}$, if $F \in H^\infty(U)$, $\overline{\mathbb{D}} \subseteq U$ open, and K_U is a constant depending on U. By [DRS69, Lemma 11], we have $Mul(X) \hookrightarrow H^\infty(\mathbb{D})$.

Let $\mathcal{P}$ denote the set of functions $f \in \mathcal{O}(\mathbb{D})$ of the form $f(z) = (\lambda z + \mu)^\delta$, $z \in \mathbb{D}$, with $\delta > 0$ and $\lambda, \mu \in \mathbb{C}$ such that $|\mu| \geq |\lambda|, \mu \neq 0$. Then, set

(**Gam2**) $$\mathcal{P} \subseteq Mul(X).$$

The next property is a kind of splitting condition on X related, as we will see, with concentration on DW points. For the rest of the chapter, let ι denote the number -1 or 1. Let $\mathbb{D}_1 := \mathbb{D} \cap \{z : 0 < \mathfrak{Re}\, z\}$ and $\mathbb{D}_{-1} := \mathbb{D} \cap \{z : \mathfrak{Re}\, z < 0\}$.

(**Gam3**) There are two Banach spaces $X_1 \hookrightarrow \mathcal{O}(\mathbb{D}_1)$, $X_{-1} \hookrightarrow \mathcal{O}(\mathbb{D}_{-1})$ such that the following holds true

- $X = \{f \in \mathcal{O}(\mathbb{D}) : f|_{\mathbb{D}_\iota} \in X_\iota, \iota = -1, 1\}$ (note the mappings $f \mapsto f|_{\mathbb{D}_\iota}$ are continuous by the closed graph theorem).
- If U is an open set containing $\overline{\mathbb{D}_\iota}$, then $\mathcal{O}(U) \subseteq Mul(X_\iota)$.

In order to take advantage of the theory of C_0-groups, we also assume that

(**Gam4**) The one-parameter group of operators $(C_{\varphi_t})_{t\in\mathbb{R}}$ is strongly continuous on X.

The latter property is a mild assumption since every strongly measurable group of operators is strongly continuous on $\mathbb{R}$ as a consequence of [HP57, Th. 10.2.3].

Moreover, since (φ_t) is holomorphic in $\overline{\mathbb{D}}$, (**Gam4**) holds if $\mathfrak{A}(\mathbb{D}) \hookrightarrow X$ [Sis98, Section 4]. Here, $\mathfrak{A}(\mathbb{D})$ is the disc algebra; that is, the Banach algebra of functions in $\mathcal{O}(\mathbb{D})$ with continuous extension to the closure $\overline{\mathbb{D}}$, endowed with the sup-norm.

Let us set some notation before introducing the two last properties. For $\rho \in \mathbb{R}$ and $\phi \in Aut(\mathbb{D})$ let $C_{\phi,\rho}$ denote the operator on $\mathcal{O}(\mathbb{D})$ given by $C_{\phi,\rho} := (\phi')^\rho C_\phi$, where ϕ' is the derivative of ϕ.

Definition 6.2.1. Let $\gamma \geq 0$ and let X be a Banach space such that $X \hookrightarrow \mathcal{O}(\mathbb{D})$, which separates points of $\mathbb{D}$, and such that it satisfies properties (**Gam1**)-(**Gam4**). We say that the space X is conformally invariant of index γ and tempered type, or just γ-space for short, if $C_{\phi,\gamma} \in L(X)$ for all $\phi \in Aut(\mathbb{D})$ and

(**Gam5**) $$(\forall \varepsilon > 0) \qquad \sup_{\phi\in Aut(\mathbb{D})} (1 - |\phi(0)|)^\varepsilon \|C_{\phi,\gamma}\|_{L(X)} < \infty.$$

Let $\mathfrak{S}$ be a subset of $\mathcal{O}(\mathbb{D})$ which is invariant for multiplication by functions $z \mapsto (1-z)^\lambda(1+z)^\mu$ for any $\lambda, \mu \in \mathbb{C}$. We say that the pair $(X, \mathfrak{S})$ is a *DW*-conditioned pair of index γ, or $\gamma-$ *pair* for short, if X is a γ-space and

(Gam6)

$f \in \mathfrak{S}$ such that $|f(z)| \lesssim |(1-z)(1+z)|^{-\gamma+\varepsilon}, z \in \mathbb{D},$ for some $\varepsilon > 0 \implies f \in X.$

Remark 6.2.2. (1) Since $\phi \in Aut(\mathbb{D})$ and $C_{\phi,\gamma} \in L(X)$, it follows from $C_\phi = (\phi')^{-\gamma} C_{\phi,\gamma}$ that C_ϕ is a bounded isomorphism of X.

(2) One obtains from (**Gam5**) that $\sigma(C_{\phi,\gamma}) \subseteq \overline{\mathbb{D}}$. Indeed, if $\phi = \varphi_t$ for some $t \in \mathbb{R} \setminus \{0\}$ (the claim is trivial if $t = 0$), a straightforward calculation gives us

$$\|C^n_{\varphi_t,\gamma}\|_{L(X)} = \|C_{\varphi_{nt},\gamma}\|_{L(X)} \lesssim (1 - |\varphi_{nt}(0)|)^{-\varepsilon} \lesssim (1 + e^{n|t|})^\varepsilon, \tag{6.7}$$

for every $\varepsilon > 0$. Then, the spectral radius formula yields $\sigma(C_{\varphi_t,\gamma}) \subseteq \overline{\mathbb{D}}$, and our claim follows. If now ϕ is an arbitrary hyperbolic automorphism one can show, via some $\tilde{\phi} \in Aut(\mathbb{D})$, that the operator $C_{\phi,\gamma}$ is similar to $C_{\varphi_t,\gamma}$ for some $t \in \mathbb{R}$, thus $\sigma(C_{\phi,\gamma}) = \sigma(C_{\varphi_t,\gamma}) \subseteq \overline{\mathbb{D}}$.

Remark 6.2.3. The definition of γ-pair explicitly involves the canonical hyperbolic flow (φ_t) with *DW* points -1 and 1. It must be noticed that such a definition could be also given in terms of an arbitrary hyperbolic flow (ψ_t) with *DW* points $a, b \in \mathbb{T}$ instead. Since γ-spaces are C_ϕ-invariant ($\phi \in Aut(\mathbb{D})$), see Remark 6.2.2(1), all these definitions are indeed equivalent.

6.2.A Examples

Here we list several classical Banach spaces which provide examples of γ-pairs.

1. *Little Korenblum classes and the disc algebra.* For $\gamma \geq 0$, recall that $\mathcal{K}^{-\gamma}(\mathbb{D})$ is the weighted Korenblum growth class of order γ given by

$$\mathcal{K}^{-\gamma}(\mathbb{D}) = \{f \in \mathcal{O}(\mathbb{D}) : \|f\|_{\mathcal{K}^{-\gamma}} = \sup_{z\in\mathbb{D}} (1-|z|^2)^\gamma |f(z)| < \infty\},$$

which is a Banach space endowed with the norm $\|\cdot\|_{\mathcal{K}^{-\gamma}}$. These spaces fulfill all conditions (**Gam1**)-(**Gam6**), except for the strong continuity condition (**Gam4**). Indeed, for $f(z) = (i-z)^{-\gamma}$ if $\gamma > 0$, and $f(z) = (i-z)^i$ if $\gamma = 0$, one can check that the mapping $t \mapsto C_{\varphi_t} f$ is not norm continuous. However, as we pointed out above, the closure of $\mathfrak{A}(\mathbb{D})$ in these spaces satisfies (**Gam4**).

If $\gamma > 0$, recall that the closure of $\mathfrak{A}(\mathbb{D})$ in $\mathcal{K}^{-\gamma}(\mathbb{D})$ is the Little Korenblum growth class $\mathcal{K}_0^{-\gamma}(\mathbb{D})$ given by

$$\mathcal{K}_0^{-\gamma}(\mathbb{D}) = \{f \in \mathcal{K}^{-\gamma}(\mathbb{D}) : \lim_{|z|\to 1} (1-|z|^2)^\gamma |f(z)| = 0\},$$

with norm $\|\cdot\|_{\mathcal{K}^{-\gamma}}$. Then $(\mathcal{K}_0^{-\gamma}(\mathbb{D}), \mathcal{O}(\mathbb{D}))$ is a γ-pair for every $\gamma > 0$ which satisfies properties (**Gam1**)-(**Gam6**) as we check next.

(**Gam1**) and (**Gam2**): These are clear since $H^\infty(\mathbb{D}) \hookrightarrow Mul(\mathcal{K}_0^{-\gamma}(\mathbb{D}))$.

(**Gam3**): Let $C_0(\mathbb{D}_\iota, (1-|z|^2)^\gamma)$ be the Banach weighted space of continuous functions f on $\mathbb{D}_\iota$ such that

$$\lim_{|z|\to 1, z\in\mathbb{D}_\iota} (1-|z|^2)^\gamma |f(z)| = 0 \quad \text{and} \quad \|f\|_{\mathcal{K}_\iota^{-\gamma}} := \sup_{z\in\mathbb{D}_\iota} (1-|z|^2)^\gamma |f(z)| < \infty.$$

Define

$$\mathcal{K}_0^{-\gamma}(\mathbb{D})_\iota := \mathcal{O}(\mathbb{D}_\iota) \cap C_0(\mathbb{D}_\iota, (1-|z|^2)^\gamma),$$

endowed with the norm $\|\cdot\|_{\mathcal{K}_\iota^{-\gamma}}$, for $i = 1, -1$. Since convergence in the norm $\|\cdot\|_{\mathcal{K}_\iota^{-\gamma}}$ implies uniform convergence on compact subsets of $\mathbb{D}_\iota$, it follows that $\mathcal{K}_0^{-\gamma}(\mathbb{D})_\iota$ is closed in the space $C_0(\mathbb{D}_\iota, (1-|z|^2)^\gamma)$. So $\mathcal{K}_0^{-\gamma}(\mathbb{D})_\iota$ is complete. It is also clear that $\mathcal{O}(U) \subseteq Mul(\mathcal{K}_0^{-\gamma}(\mathbb{D})_\iota)$ for all open subset $U \subseteq \mathbb{C}$ containing $\mathbb{D}_\iota$. Then the spaces $\mathcal{K}_0^{-\gamma}(\mathbb{D})_\iota$ satisfy (**Gam3**).

(**Gam4**): This holds since the disc algebra $\mathfrak{A}(\mathbb{D})$ is a subspace dense in $\mathcal{K}_0^{-\gamma}(\mathbb{D})$.

(**Gam5**) and (**Gam6**): In fact, we have $\sup_{\phi\in Aut(\mathbb{D})} \|C_{\phi,\gamma}\|_{L(\mathcal{K}_0^{-\gamma})} = 1$, as it was noted in [AP10; HLNS13]. Also, it is clear that (**Gam6**) holds for every $\gamma > 0$ and $f \in \mathcal{O}(\mathbb{D})$. So $(\mathcal{K}_0^{-\gamma}(\mathbb{D}), \mathcal{O}(\mathbb{D}))$ is a γ-pair for every $\gamma > 0$.

If $\gamma = 0$, when $\mathcal{K}^{-\gamma}(\mathbb{D})$ is $H^\infty(\mathbb{D})$, we have that the closure of the disc algebra $\mathfrak{A}(\mathbb{D})$ in $H^\infty(\mathbb{D})$ is $\mathfrak{A}(\mathbb{D})$ itself. Take $\mathfrak{S}(\mathfrak{A}) := \{f \in \mathcal{O}(\mathbb{D}) : f$ extends continuously to $\overline{\mathbb{D}} \setminus \{1, -1\}\}$. Then one can easily check that $(\mathfrak{A}(\mathbb{D}), \mathfrak{S}(\mathfrak{A}))$ is a 0-pair. For instance, condition (**Gam3**) is satisfied if we consider the Banach spaces of continuous functions $\mathfrak{A}(\mathbb{D})_\iota := \mathcal{O}(\mathbb{D}_\iota) \cap C(\overline{\mathbb{D}_\iota})$ with the sup-norm on $\mathbb{D}_\iota$.

Remark 6.2.4. Spaces $\mathcal{K}^{-\gamma}(\mathbb{D})$, $\gamma \geq 0$, enjoy the property that, for each $\gamma \geq 0$ and $\varepsilon > 0$, $\mathcal{K}^{-\gamma-\varepsilon}(\mathbb{D})$ contains every Banach space X satisfying (**Gam5**). In effect, in this case, for $f \in X$ and one has

$$\begin{aligned}
\sup_{z\in\mathbb{D}} (1-|z|^2)^{\gamma+\varepsilon} |f(z)| &= \sup_{\phi\in Aut(\mathbb{D})} (1-|\phi(0)|^2)^{\gamma+\varepsilon} |f(\phi(0))| \\
&= \sup_{\phi\in Aut(\mathbb{D})} (1-|\phi(0)|^2)^{\varepsilon} |(C_{\phi,\gamma} f)(0)| \\
&\lesssim \sup_{\phi\in Aut(\mathbb{D})} (1-|\phi(0)|^2)^{\varepsilon} \|C_{\phi,\gamma} f\|_X \lesssim \|f\|_X,
\end{aligned}$$

where Schwarz-Pick's Lemma has been used in the second equality. This bound obviously implies $X \hookrightarrow \mathcal{K}^{-\gamma-\varepsilon}(\mathbb{D})$ as claimed.

Notice that if (**Gam5**) holds for $\varepsilon = 0$, then mimicking the above argument we have $X \hookrightarrow \mathcal{K}^{-\gamma}(\mathbb{D})$.

2. *Hardy spaces of integrable functions.* For $1 \le p < \infty$, recall that $H^p(\mathbb{D})$ is the Hardy space on $\mathbb{D}$ formed by all functions $f \in \mathcal{O}(\mathbb{D})$ such that

$$\|f\|_{H^p} = \sup_{0<r<1} \left(\int_0^{2\pi} |f(re^{i\theta})|^p \, \frac{d\theta}{2\pi} \right)^{1/p} < \infty,$$

endowed with the norm $\|\cdot\|_{H^p}$.

We claim that $(H^p(\mathbb{D}), \mathcal{O}(\mathbb{D}))$ if a γ-pair for $\gamma = 1/p$. First, $H^\infty(\mathbb{D}) = Mul(H^p(\mathbb{D}))$ and therefore (**Gam1**), (**Gam2**) are fulfilled. (**Gam4**) holds since the disc algebra $\mathfrak{A}(\mathbb{D})$ is dense in $H^p(\mathbb{D})$. It is well known that they satisfy (**Gam5**) even for $\varepsilon = 0$; in fact, operators $C_{\phi,\gamma}$ are isometries in this case, see [For64, Th. 2]. And (**Gam6**) is clear. Checking property (**Gam3**) requires a bit more of work:

Given a Banach space Z with norm $\|\cdot\|_Z$ and a set J, let $B(J;Z)$ denote the Banach space of $\|\cdot\|_Z$-bounded Z-valued functions on J, with norm $\|F\|_{Z,\infty} := \sup_{j\in J} \|F(j)\|_Z$. Put $\mathbb{T}_1 := \{z \in \mathbb{T} : \mathfrak{Re}\, z > 0\}$ and $\mathbb{T}_{-1} := \{z \in \mathbb{T} : \mathfrak{Re}\, z < 0\}$, and consider the Banach spaces

$$L^p(\mathbb{T}_{-1}) := \left\{ f\colon \mathbb{T}_{-1} \to \mathbb{C} : \|f\|_{p,-1} = \left(\int_{\pi/2}^{3\pi/2} |f(e^{i\theta})|^p \, \frac{d\theta}{2\pi} \right)^{1/p} < \infty \right\},$$

$$L^p(\mathbb{T}_1) := \left\{ f\colon \mathbb{T}_1 \to \mathbb{C} : \|f\|_{p,1} = \left(\int_{-\pi/2}^{\pi/2} |f(e^{i\theta})|^p \, \frac{d\theta}{2\pi} \right)^{1/p} < \infty \right\}.$$

Take the interval $J = (0,1)$ in $\mathbb{R}$ and $Z = L^p(\mathbb{T}_\iota)$, $\iota = -1, 1$. Define

$$H^p(\mathbb{D})_\iota := \mathcal{K}^{-\gamma}(\mathbb{D}_\iota) \cap B((0,1); L^p(\mathbb{T}_\iota)),$$

where $\mathcal{K}^{-\gamma}(\mathbb{D}_\iota) = \{f \in \mathcal{O}(\mathbb{D}_\iota) : \|f\|_{\mathcal{K}_\iota^{-\gamma}} < \infty\}$. In such an intersection, an element $F \in \mathcal{K}^{-\gamma}(\mathbb{D}_\iota)$ is regarded as the family $(F_r)_{0<r<1}$ of functions on $\mathbb{T}$ where $F_r(z) := F(rz)$ for $r \in (0,1)$, $z \in \mathbb{T}$. Thus $F \in H^p(\mathbb{D})_\iota$ means that $F \in \mathcal{K}^{-\gamma}(\mathbb{D}_\iota)$ and $\widetilde{F}\colon (0,1) \to L^p(\mathbb{T}_\iota)$ given by $\widetilde{F}(r) := F_r$ satisfies $\sup_{0<r<1} \|\widetilde{F}(r)\|_{p,\iota} < \infty$. Then the space $H^p(\mathbb{D})_\iota$, provided with the norm

$$\|F\|_{H^p_\iota} := \|F\|_{\mathcal{K}_\iota^{-\gamma}} + \sup_{0<r<1} \|\widetilde{F}(r)\|_{p,\iota},$$

is a Banach space. Since $H^p(\mathbb{D}) \hookrightarrow \mathcal{K}^{-\gamma}(\mathbb{D})$, see the end of Remark 6.2.4, it is readily seen that $H^p(\mathbb{D})_\iota$ satisfies (**Gam3**).

3. *Weighted Bergman spaces.* Let $1 \le p < \infty$ and $\sigma > -1$. Recall that by $\mathcal{A}^p_\sigma(\mathbb{D})$ we denote the weighted Bergman space formed by all holomorphic functions in $\mathbb{D}$ such that

$$\|f\|_{\mathcal{A}^p_\sigma} := \left(\int_{\mathbb{D}} |f(z))|^p d\mathcal{A}_\sigma(z) \right)^{1/p} < \infty,$$

where $d\mathcal{A}_\sigma(z) = (1-|z|^2)^\sigma\, dA(z)$, and where dA is the Lebesgue measure on $\mathbb{D}$. The space $\mathcal{A}^p_\sigma(\mathbb{D})$, with norm $\|\cdot\|_{\mathcal{A}^p_\sigma}$, is a Banach space such that the pair $(\mathcal{A}^p_\sigma(\mathbb{D}), \mathcal{O}(\mathbb{D}))$ is a γ-pair with for $\gamma = \frac{\sigma+2}{p}$.

Indeed, as in the above examples, $H^\infty(\mathbb{D}) = Mul(\mathcal{A}^p_\sigma(\mathbb{D}))$, so (**Gam1**), (**Gam2**) hold. Define $\mathcal{A}^p_\sigma(\mathbb{D})_\iota := \mathcal{O}(\mathbb{D}_\iota) \cap L^p(\mathbb{D}_\iota, (1-|z|^2)^\sigma)$. Clearly, $\mathcal{A}^p_\sigma(\mathbb{D})_\iota$ endowed with the usual norm of $L^p(\mathbb{D}_\iota, (1-|z|^2)^\sigma)$ satisfies (**Gam3**). Moreover, $\mathcal{A}^p_\sigma(\mathbb{D})$ satisfies (**Gam4**) since $\mathfrak{A}(\mathbb{D})$ is dense in $\mathcal{A}^p_\sigma(\mathbb{D})$. It is well known that $\mathcal{A}^p_\sigma(\mathbb{D})$ satisfies (**Gam5**); see for instance Section 5.2.

Finally, (**Gam6**) is also satisfied. To see this, set $h_\varepsilon(z) := (1-z^2)^{-\gamma+\varepsilon}$, $z \in \mathbb{D}$, for $\varepsilon > 0$. Let us check that h_ε belongs to $\mathcal{A}^p_\sigma(\mathbb{D})$. Note that $h_\varepsilon \in \mathcal{A}^p_\sigma(\mathbb{D})$ if and only if $\int_{\mathbb{D}} |1-z^2|^{-\sigma-2+p\varepsilon} d\mathcal{A}_\sigma(z) < \infty$. Then the finiteness of the integral readily follows by decomposing it in three (finite, eventually) terms corresponding to the (integration) domains $\mathbb{D} \cap D(-1;1/2)$, $\mathbb{D} \setminus (D(-1;1/2) \cup D(1;1/2))$ and $\mathbb{D} \cap D(1;1/2)$ where $D(w;r) := \{z : |z-w| < r\}$, $w \in \mathbb{C}$, $r > 0$.

The two following examples are provided by Dirichlet spaces and Bloch spaces. To deal with them, we introduce the set $\mathfrak{S}_{log}$ of all functions $f \in \mathcal{O}(\mathbb{D})$, zero-free on $\mathbb{D}$, such that

$$(\forall \varepsilon > 0) \qquad \sup_{z\in\mathbb{D}} |(1-z^2)|^{1+\varepsilon} \left|\frac{f'(z)}{f(z)}\right| < \infty.$$

4. *Weighted Dirichlet spaces.* For $p \geq 1$ and $\sigma > -1$, let $\mathcal{D}^p_\sigma(\mathbb{D})$ denote the weighted Dirichlet space of all functions $f \in \mathcal{O}(\mathbb{D})$ such that $f' \in \mathcal{A}^p_\sigma(\mathbb{D})$ and

$$\|f\|_{\mathcal{D}^p_\sigma} := \left(|f(0)|^p + \|f'\|^p_{\mathcal{A}^p_\sigma}\right)^{1/p} < \infty.$$

Then $\mathcal{D}^p_\sigma(\mathbb{D})$ is a Banach space with norm given by $\|\cdot\|_{\mathcal{D}^p_\sigma}$. When $\sigma > p-1$ one has that $\mathcal{D}^p_\sigma(\mathbb{D}) = \mathcal{A}^p_{\sigma-p}(\mathbb{D})$ with equivalent norms, see e.g. [Fle72, Th. 6]. Hence $(\mathcal{D}^p_\sigma(\mathbb{D}), \mathcal{O}(\mathbb{D}))$ is a γ-pair for $\gamma = \frac{\sigma+2}{p} - 1$.

In the case $p-2 \leq \sigma \leq p-1$, we prove that the pair $(\mathcal{D}^p_\sigma(\mathbb{D}), \mathfrak{S}_{log})$ is a γ-pair for $\gamma = \frac{\sigma+2}{p} - 1$. The following lemma concerns multipliers and shows that $\mathcal{D}^p_\sigma(\mathbb{D})$ satisfies properties (**Gam1**) and (**Gam2**).

Lemma 6.2.5. *Let $\sigma > -1, p \geq 1$ be such that $p-2 \leq \sigma \leq p-1$. Then $H^\infty(U) \hookrightarrow Mul(\mathcal{D}^p_\sigma(\mathbb{D}))$ for every open subset U of $\mathbb{C}$ such that $\overline{\mathbb{D}} \subseteq U$, and also $\mathcal{P} \subseteq Mul(\mathcal{D}^p_\sigma(\mathbb{D}))$.*

Proof. (1) The inclusion $H^\infty(U) \hookrightarrow Mul(\mathcal{D}^p_\sigma(\mathbb{D}))$ is well known. We include here a proof for the sake of completeness. Let U be an open subset of $\mathbb{C}$ such that $\overline{\mathbb{D}} \subseteq U$.

Let $h \in H^\infty(U)$. For every $f \in \mathcal{D}^p_\sigma(\mathbb{D})$, one has $\|hf\|^p_{\mathcal{D}^p_\sigma} = |h(0)f(0)|^p + \|(hf)'\|^p_{\mathcal{A}^p_\sigma}$ with

$$\begin{aligned}\|(hf)'\|_{\mathcal{A}^p_\sigma} &\leq \|hf'\|_{\mathcal{A}^p_\sigma} + \|h'f\|_{\mathcal{A}^p_\sigma} \leq \|h\|_\infty \|f'\|_{\mathcal{A}^p_\sigma} + \|h'\|_\infty \|f\|_{\mathcal{A}^p_\sigma} \\ &\lesssim (\|h\|_\infty + \|h'\|_\infty)\|f'\|_{\mathcal{A}^p_\sigma},\end{aligned}$$

where we have used that $\|f\|_{\mathcal{A}^p_\sigma} \lesssim \|f'\|_{\mathcal{A}^p_{\sigma+p}} \leq \|f'\|_{\mathcal{A}^p_\sigma}$ for all $f \in \mathcal{D}^p_\sigma(\mathbb{D})$, see for instance [Fle72, Th. 6]. Now, using Cauchy's estimate for the derivative, one has $\|h'\|_\infty \lesssim \|h\|_{H^\infty(U)}$, and we are done.

Let now $g(z) = cz + d$, $z \in \mathbb{D}$, with $c, d \in \mathbb{C}$ such that $|c| \leq |d|$ and take $\delta > 0$. If $|c| < |d|$ the function g^δ is a holomorphic function in an open set containing $\overline{\mathbb{D}}$ and therefore it is a multiplier of $\mathcal{D}^p_\sigma(\mathbb{D})$ as seen before. If $|c| = |d|$ one can assume that $g(z) = 1 - z$ since rotations are isometries of $\mathcal{D}^p_\sigma(\mathbb{D})$. Then, for every $f \in \mathcal{D}^p_\sigma(\mathbb{D})$, one has $\|g^\delta f\|^p_{\mathcal{D}^p_\sigma} = |g^\delta(0)f(0)|^p + \|(g^\delta f)'\|^p_{\mathcal{A}^p_\sigma}$ with

$$\begin{aligned}\|(g^\delta f)'\|_{\mathcal{A}^p_\sigma} &\leq \|g^\delta\|_\infty \|f'\|_{\mathcal{A}^p_\sigma} + \delta \|g^\delta\|_\infty \|g^{-1} f\|_{\mathcal{A}^p_\sigma} \\ &\leq 2^\delta \|f'\|_{\mathcal{A}^p_\sigma} + \delta 2^\delta \left(\int_{\mathbb{D}} |f(z)|^p \rho(z) dA(z) \right)^{1/p},\end{aligned}$$

where $\rho(z) := (1 - |z|^2)^\sigma |1 - z|^{-p}$, $z \in \mathbb{D}$.

Assume first $\sigma > p - 2$. Then, using [HKZ00, Th. 1.7], one has

$$\int_{\mathbb{D}} \frac{\rho(\zeta)}{|1 - \bar{\zeta} z|^{\eta+2}} dA(\zeta) = \int_{\mathbb{D}} \frac{(1 - |\zeta|^2)^\sigma}{|1 - \zeta|^p |1 - \bar{\zeta} z|^{\eta+2}} dA(\zeta) \lesssim \frac{\rho(z)}{(1 - |z|)^\eta}, \quad z \in \mathbb{D}.$$

In the terminology of [AC09], the above inequality implies that $\rho/(1 - |\cdot|)^\eta \in B_1^*(\eta)$, $\eta > \sigma$. Moreover, a few computations show

$$\|\nabla \rho(z)\|_{\mathbb{R}^2} \leq 2\sqrt{2}(|\sigma| + p) \frac{\rho(z)}{1 - |z|^2}, \qquad z \in \mathbb{D},$$

where $\nabla \rho$ denotes the gradient of the differentiable function ρ. In short, ρ satisfies condition (3.21) of [AC09]. Hence, we can apply [AC09, Th. 3.2(iv)] in the inequality "$\lesssim$" coming in to obtain

$$\begin{aligned}\int_{\mathbb{D}} |f(z)|^p \rho(z) dA(z) &\lesssim |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^p \rho(z) dA(z) \\ &\leq |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^\sigma dA(z) = \|f\|^p_{\mathcal{D}^p_\sigma},\end{aligned}$$

(see also [AP10, Prop. 3.1]).

Assume now $\sigma = p - 2$ and take $\varepsilon \in (0, \delta)$. One gets

$$\begin{aligned}\|(g^\delta f)'\|_{\mathcal{A}^p_\sigma} &\leq \|g^\delta\|_\infty \|f'\|_{\mathcal{A}^p_\sigma} + \delta \|g^{\delta-\varepsilon}\|_\infty \|g^{-(1-\varepsilon)} f\|_{\mathcal{A}^p_\sigma} \\ &\leq 2^\delta \|f'\|_{\mathcal{A}^p_\sigma} + \delta 2^{\delta-\varepsilon} \left(\int_{\mathbb{D}} |f(z)|^p \rho_\varepsilon(z) dA(z) \right)^{1/p}\end{aligned}$$

with $\rho_\varepsilon(z) := (1-|z|^2)^\sigma |1-z|^{-p(1-\varepsilon)}$, $z \in \mathbb{D}$. The remainder of the argument goes along the same lines as in the case $\sigma > p-2$, where the weight ρ should be replaced by the weight ρ_ε.

All in all, one has $g^\delta \in Mul(\mathcal{D}^p_\sigma(\mathbb{D}))$ for every $\delta > 0$ and therefore $\mathcal{P} \subseteq Mul(\mathcal{D}^p_\sigma(\mathbb{D}))$. □

Let $\mathcal{D}^p_\sigma(\mathbb{D})_\iota := \{f \in \mathcal{O}(\mathbb{D}_\iota) : f' \in L^p(\mathbb{D}_\iota, (1-|z|^2)^\sigma)\}$ equipped with the norm

$$\|f\|_{\mathcal{D}^p_\sigma,\iota} := \left(|f(\iota/2)|^p + \int_{\mathbb{D}_\iota} |f'(z)|^p \, d\mathcal{A}_\sigma(z) \right)^{1/p},$$

which satisfies (**Gam3**). Note that if (f_n) is a Cauchy sequence in $\mathcal{D}^p_\sigma(\mathbb{D})_\iota$ then there exists $g \in \mathcal{A}^p_\sigma(\mathbb{D})_\iota$ such that $\lim_n f_n' = g$ in $\mathcal{A}^p_\sigma(\mathbb{D})_\iota$. Since $\mathbb{D}_\iota$ is simply connected there exists a primitive function f of g, which we take such that $f(\iota/2) = \lim_n f_n(\iota/2)$. Thus we have that $\lim_n f_n = f$ in $\mathcal{D}^p_\sigma(\mathbb{D})_\iota$ and it follows that this space is complete. Moreover, (**Gam4**) is also satisfied since polynomials are dense in $\mathcal{D}^p_\sigma(\mathbb{D})$, and it is readily seen that the mapping $t \mapsto C_{\varphi_t} Q$ is norm continuous for every polynomial Q. The fact that the Dirichlet space satisfies (**Gam5**) and (**Gam6**) is proved in the following lemma.

Lemma 6.2.6. *Let $p \geq 1$ and $\sigma > -1$ be such that $p-2 \leq \sigma \leq p-1$. Then $(\mathcal{D}^p_\sigma(\mathbb{D}), \mathfrak{S}_{\log})$ is a γ-pair with $\gamma = \frac{\sigma+2}{p} - 1$.*

Proof. As noticed above, all that is left to prove is that the pair $(\mathcal{D}^p_\sigma(\mathbb{D}), \mathfrak{S}_{\log})$ satisfies properties (**Gam5**) and (**Gam6**). It is known that $\sup_{\phi \in Aut(\mathbb{D})} \|C_{\phi,\gamma}\|_{L(\mathcal{D}^p_\sigma)} < \infty$ if and only if $\sigma > p-2$ with $\gamma = (\sigma+2)/p - 1$ [AP10, Prop. 3.1]. Thus $\mathcal{D}^p_\sigma(\mathbb{D})$ satisfies (**Gam5**) when $\sigma > p-2$. For $\sigma = p-2$, whence $\gamma = 0$, we show that $\mathcal{D}^p_{p-2}$ is a 0-space as follows.

Let $f \in \mathcal{D}^p_{p-2}(\mathbb{D})$ so that $f' \in \mathcal{A}^p_{p-2}(\mathbb{D}) \hookrightarrow \mathcal{K}^{-1}(\mathbb{D})$, see the end of Remark 6.2.4. Then, since $f(z) = f(0) + \int_0^z f'(\xi) d\xi$ for all $z \in \mathbb{D}$, we have

$$\begin{aligned} |f(z)| &\leq |f(0)| + \int_{[0,z]} \|f'\|_{\mathcal{A}^p_{p-2}} (1-|\xi|)^{-1} |d\xi| \\ &\leq |f(0)| - \|f'\|_{\mathcal{A}^p_{p-2}} \log(1-|z|) \leq \|f\|_{\mathcal{D}^p_{p-2}} (1 - \log(1-|z|)), \end{aligned}$$

for all $z \in \mathbb{D}$ and $f \in \mathcal{D}^p_{p-2}(\mathbb{D})$. Hence, for every $\phi \in Aut(\mathbb{D})$,

$$\begin{aligned} \|f \circ \phi\|_{\mathcal{D}^p_{p-2}} &= \left(|f(\phi(0))|^p + \|\phi'(f' \circ \phi)\|^p_{\mathcal{A}^p_{p-2}} \right)^{1/p} \\ &= \left(|f(\phi(0))|^p + \|f'\|^p_{\mathcal{A}^p_{p-2}} \right)^{1/p} \\ &\leq \|f\|_{\mathcal{D}^p_{p-2}} (1 - \log(1 - |\phi(0)|)), \end{aligned} \tag{6.8}$$

where we have used that $C_{\phi,1}$ is an isometric isomorphism in $\mathcal{A}^p_{p-2}$ and the previous estimate for $|f(z)|$, $z \in \mathbb{D}$. Thus $\mathcal{D}^p_{p-2}(\mathbb{D})$ satisfies (**Gam5**) with $\gamma = 0$.

As for condition (**Gam6**), let $\gamma = \frac{\sigma+2}{p} - 1$ and $f \in \mathfrak{S}_{\log}$ such that, for some $\varepsilon > 0$, we have $|f(z)| \lesssim |1-z^2|^{-\gamma+\varepsilon}$ for all $z \in \mathbb{D}$. Then

$$|f'(z)| = |f(z)|\frac{|f'(z)|}{|f(z)|} \lesssim |1-z^2|^{-\gamma+\varepsilon}|1-z^2|^{-1-\varepsilon/2} = |1-z^2|^{-\gamma-1+\varepsilon/2},$$

for every $z \in \mathbb{D}$. Since $\gamma + 1 = (\sigma+2)/p$ one gets $f' \in \mathcal{A}^p_\sigma(\mathbb{D})$; that is, $f \in \mathcal{D}^p_\sigma(\mathbb{D})$, which implies that $(\mathcal{D}^p_\sigma(\mathbb{D}), \mathfrak{S}_{\log})$ is a γ-pair with $\gamma = \frac{\sigma+2}{p} - 1$. □

5. *Bloch spaces.* For $\delta > 0$, let $B_\delta(\mathbb{D})$ denote the Bloch space, that is, the space of holomorphic functions on $\mathbb{D}$ such that

$$\|f\|_{B_\delta} := |f(0)| + \sup_{z\in\mathbb{D}} (1-|z|^2)^\delta |f'(z)| < \infty,$$

endowed with the norm $\|\cdot\|_{B_\delta}$. Let $B_{\delta,0}(\mathbb{D})$ denote the little Bloch space, consisting of the closure of polynomials in $B_\delta(\mathbb{D})$. One has indeed

$$B_{\delta,0}(\mathbb{D}) = \{f \in B_\delta(\mathbb{D}) : \lim_{|z|\to 1} (1-|z|^2)^\delta |f'(z)| = 0\},$$

see [Zhu93, Prop. 2]. For $\delta > 1$ these spaces are Korenblum classes; i. e.,

$$B_\delta(\mathbb{D}) = \mathcal{K}^{-(\delta-1)}(\mathbb{D}) \quad \text{and} \quad B_{\delta,0}(\mathbb{D}) = \mathcal{K}_0^{-(\delta-1)}(\mathbb{D})$$

with corresponding equivalent norms, see [Zhu93, Prop. 7].

For $\delta = 1$, $B_1(\mathbb{D})$ fails to satisfy condition (**Gam4**). In fact, the mapping $t \mapsto C_{\varphi_t} f$, where $f(z) = \log(i-z)$, $z \in \mathbb{D}$, is not norm continuous. On the other hand, $B_{1,0}(\mathbb{D})$ satisfies (**Gam4**) since the mapping $t \in \mathbb{R} \mapsto C_{\varphi_t} Q \in B_{1,0}(\mathbb{D})$ is continuous for every analytic polynomial Q and the space of analytic polynomials is dense in $B_{1,0}(\mathbb{D})$.

Let us show that the little Bloch space $B_{1,0}(\mathbb{D})$ is a 0-space and that $(B_{1,0}(\mathbb{D}), \mathfrak{S}_{log})$ is a 0-pair. We know that (**Gam4**) holds. As regards multipliers, we have

$$\begin{aligned} Mul(B_1(\mathbb{D}) &= Mul(B_{1,0}(\mathbb{D})) \\ &= \{f \in H^\infty(\mathbb{D}) : (1-|\cdot|^2)\log(1-|\cdot|^2) f' \in H^\infty(\mathbb{D})\}, \end{aligned}$$

see [Zhu93, Th. 27], from which (**Gam1**), (**Gam2**) follow.

Define $B_1(\mathbb{D})_\iota := \{f \in \mathcal{O}(\mathbb{D}_\iota) : \sup_{z\in\mathbb{D}_\iota} (1-|z|^2)|f'(z)| < \infty\}$, with norm $\|f\|_{B_{\sigma,\iota}} := |f(\iota/2)| + \sup_{z\in\mathbb{D}_\iota}(1-|z|^2)|f'(z)|$, and let $B_{1,0}(\mathbb{D})_\iota$ denote the closure of the polynomials in $B_1(\mathbb{D})_\iota$. Then, if (f_n) is a Cauchy sequence in $B_{1,0}(\mathbb{D})_\iota$ it is convergent to g in $\mathcal{K}^{-1}(\mathbb{D})_\iota$. Taking $f \in \mathcal{O}(\mathbb{D}_\iota)$ with $f' = g$ and $f(\iota/2) = \lim_n f_n(\iota/2)$

we get $\lim_n f_n = f$ in $B_{1,0}(\mathbb{D})_\iota$. In short, $B_{1,0}(\mathbb{D})_\iota$ is complete, and it is readily seen that that $B_{1,0}(\mathbb{D})_\iota$ satisfies (**Gam3**) for $B_{1,0}(\mathbb{D})$.

Now, for every $\phi \in Aut(\mathbb{D})$,

$$\|f \circ \phi\|_{B_{1,0}} = |f(\phi(0))| + \sup_{z\in\mathbb{D}}(1-|z|^2)|\phi'(z)f'(\phi(z))|,$$

with

$$\begin{aligned} |f(\phi(0))| &\leq |f(0)| + \int_0^{\phi(0)} |f'(\xi)||d\xi| \\ &\lesssim \|f\|_{B_{1,0}}\left(1 + \int_0^{\phi(0)} (1-|\xi|)^{-1} d\xi\right) = \|f\|_{B_{1,0}}(1-\log(1-|\phi(0)|)). \end{aligned}$$

On the other hand, using the Schwarz-Pick lemma one has

$$\sup_{z\in\mathbb{D}}(1-|z|^2)|\phi'(z)f'(\phi(z))| \leq \sup_{z\in\mathbb{D}}(1-|\phi(z)|^2)|f'(\phi(z))| \leq \|f\|_{B_{1,0}}.$$

Thus (**Gam5**) holds. Finally, by an argument like in the case of Dirichlet spaces, it can be seen that $(B_{1,0}(\mathbb{D}), \mathfrak{S}_{\log})$ satisfies (**Gam6**).

6.3 Cocycles for the hyperbolic group on γ-spaces

Let X be a γ-space for some $\gamma \geq 0$ and let (u_t) be a DW-continuous cocycle for the hyperbolic flow (φ_t) on X. Condition (**Co2**) together with (**Gam4**) imply that the mapping $t \mapsto u_t C_{\varphi_t}$ is strongly measurable, hence $(u_t C_{\varphi_t})$ is a C_0-group of bounded operators on X, see [HP57, Th. 10.2.3]. Hence, there exists a non-vanishing holomorphic function $\omega : \mathbb{D} \to \mathbb{C}$ such that $u_t = (\omega \circ \varphi_t)/\omega$ for all $t \in \mathbb{R}$, see Theorem 4.2.2 i).

The first part of this section is devoted to show that the functions ω associated to DW-continuous cocycles $(u_t) \subseteq Mul(X)$, present zeroes or singularities of polynomial type at -1 and 1. In the second part, further additional properties of γ-spaces, regarding DW-continuous cocycles, are introduced.

Every measurable subadditive function on $(0, \infty)$ is locally bounded [DS58, p. 618]. Inspired by this result, we obtain the lemma which follows.

Lemma 6.3.1. *Let $g \colon (0, \infty) \to \mathbb{R}$ be a measurable function such that*

$$g(s+t) \leq g(s) + g(t) + H(s,t) \quad s,t > 0,$$

where H is non-decreasing if s, t increase simultaneously. Then g is locally bounded on $(0, \infty)$.

Proof. Take $a > 0$ and put $F := \{t \in (0,a) : g(t) \geq (g(a) - H(a,a))/2\}$. For a given $t \in (0,a)$ with $t \notin F$ one has $g(t) < g(a)/2 - H(t, a-t)/2$. Also, $g(a) \leq$

$g(t) + g(a-t) + H(t, a-t)$. All in all,

$$\begin{aligned} g(a-t) &\geq g(a) - g(t) - H(t, a-t) \\ &> g(a) - \frac{g(a) - H(t, a-t)}{2} - H(t, a-t) \geq \frac{g(a)}{2} - \frac{H(a,a)}{2}, \end{aligned}$$

since H is non-decreasing. Hence $t \in a - F$; that is, $(0, a) = F \cup (a - F)$ and so $\mu(F) \geq a/2$.

Suppose now, if possible, that g is unbounded on $[c, d]$ for some $c, d > 0$. Take a sequence (s_n) in $[c, d]$ such that $g(s_n) \geq 2n$ for each $n \in \mathbb{N}$. Put $B_n := \{0 < t < d : g(t) \geq n - H(d, d)\}$, $n \geq 1$. Applying the above argument to $F_n := \{0 < t < s : g(t) \geq (g(s_n) - H(s_n, s_n))/2\}$ we get $\mu(B_n) \geq c/2$ since $F_n \subseteq B_n$, for all $n \geq 1$. Then, taking $t \in \bigcap_{n=1}^{\infty} B_n$ one gets $g(t) = \infty$, which is a contradiction.

In conclusion, g is locally bounded, as we claimed. □

Lemma 6.3.2. *For (u_t) as above, the mapping $t \mapsto \|u_t\|_{Mul(X)}$ is locally bounded on $\mathbb{R}$.*

Proof. First, we prove that for every $\varepsilon > 0$ there is $K_\varepsilon > 0$ such that

$$\|u_{s+t}\|_{Mul(X)} \leq \|u_s\|_{Mul(X)} \|u_t\|_{Mul(X)} \left(K_\varepsilon e^{\varepsilon \min\{|s|,|t|\}}\right)^2, \qquad s, t \in \mathbb{R}. \tag{6.9}$$

Note that $(u_s \circ \varphi_t) f = C_{\varphi_t,\gamma}(u_s C_{\varphi_{-t},\gamma} f)$ for any $f \in X$, thus $u_s \circ \varphi_t \in Mul(X)$ for every $s, t \in \mathbb{R}$. Moreover, by the cocycle property $u_{s+t} = u_s(u_t \circ \varphi_s) = u_t(u_s \circ \varphi_t)$, hence

$$\|u_{s+t}\|_{Mul(X)} \leq \min\left\{\|u_s\|_{Mul(X)} \|u_t \circ \varphi_s\|_{Mul(X)}, \|u_t\|_{Mul(X)} \|u_s \circ \varphi_t\|_{Mul(X)}\right\}, \quad s, t \in \mathbb{R}.$$

In addition, $\|u_s \circ \varphi_t\|_{Mul(X)} \leq \|C_{\varphi_t,\gamma}\|_{L(X)} \|u_s\|_{Mul(X)} \|C_{\varphi_{-t},\gamma}\|_{L(X)}$. Since $\|C_{\varphi_t,\gamma}\|_{L(X)} \leq K_\varepsilon e^{\varepsilon|t|}$ for $t \in \mathbb{R}$ (see (6.7)), the inequality (6.9) follows. Hence, for $s, t \in \mathbb{R}$,

$$\log \|u_{s+t}\|_{Mul(X)} \leq \log \|u_s\|_{Mul(X)} + \log \|u_t\|_{Mul(X)} + 2(\varepsilon \min\{|t|, |s|\} + \log K_\varepsilon). \tag{6.10}$$

Thus applying Lemma 6.3.1 to $g(t) := \log \|u_t\|_{Mul(X)}$ and $H(s,t) := 2(\varepsilon \min\{|t|, |s|\} + \log K_\varepsilon)$, $s, t > 0$, we obtain that $t \mapsto \|u_t\|_{Mul(X)}$ is bounded on $[c, d]$ if $cd > 0$. So it remains to prove the result for $[c, d]$ with $c < 0$ and $d > 0$.

Fix s big enough so that $s >> |c|$ and $s >> d$. By (6.9)

$$\|u_t\|_{Mul(X)} \leq \|u_s\|_{Mul(X)} \|u_{t-s}\|_{Mul(X)} \left(K_\varepsilon e^{\varepsilon \min\{|s|,|t-s|\}}\right)^2, \qquad t \in [c, d],$$

which is uniformly bounded since $s, t - s$ are bounded away from zero. □

Lemma 6.3.3. *Let (u_t) be a cocycle as above. Then, u_t has no zero for any $t \in \mathbb{R}$, and the family (u_t^{-1}) is a DW-continuous cocycle for the flow (φ_t) on X.*

Proof. First, for each $t \in \mathbb{R}$, u_t has no zero on $\mathbb{D}$, see [Kön90, Lemma 2.1], so u_t^{-1} is well defined. Moreover, by the cocycle property of (u_t) it follows that $u_t^{-1} = u_{-t} \circ \varphi_t$, $t \in \mathbb{R}$, and then it is readily seen that (u_t^{-1}) is a continuous cocycle for (φ_t).

Now, note that $(u_{-t} \circ \varphi_t)f = C_{\varphi_t}(u_{-t}C_{\varphi_{-t}}f)$, $f \in X$, so that $u_t^{-1} = u_{-t} \circ \varphi_t$ is a multiplier in X since $C_{\varphi_t}, C_{\varphi_{-t}}$ are isomorphisms on X, see Remark 6.2.2(1). In fact, u_t^{-1} is the inverse multiplier of u_t.

Recall that $Mul(X) \hookrightarrow H^\infty(\mathbb{D})$ as we pointed out in Section 6.2. This implies that u_t^{-1} is bounded, hence $u_t(1), u_t(-1) \neq 0$ for any $t \in \mathbb{R}$, and as a consequence u_t^{-1} is continuous at the DW points $-1, 1$, that is, it satisfies (**Co1**). Finally, the mapping $t \mapsto u_t^{-1}$ is measurable since is is the composition of the measurable mapping $t \mapsto u_t$ and the (continuous) inversion map in the group of invertible multipliers of X. Hence, (u_t^{-1}) fulfills (**Co2**). □

Lemma 6.3.4. *Let (u_t) be a cocycle as above. Then there are $K, w > 0$ such that, for every $t \in \mathbb{R}$,*

$$\sup \left\{ \|u_t\|_{Mul(X)}, \|u_t^{-1}\|_{Mul(X)} \right\} \leq K e^{w|t|},$$
$$\sup \left\{ \|u_t\|_\infty, \|u_t^{-1}\|_\infty \right\} \leq K e^{w|t|}.$$

Proof. By Lemma 6.3.2 there exists $M > 0$ for which $\sup_{-1 \leq t \leq 1} \log \|u_t\|_{Mul(X)} \leq M$. We will show by induction that $\log \|u_t\|_{Mul(X)} \leq M + m|t|$ for every $t \in \mathbb{R}$, where $m = 2(\varepsilon + \log K_\varepsilon)$, where $K_\varepsilon, \varepsilon$ are taken as in (6.10). The claim is trivial if $|t| \leq 1$, so assume it holds for all $|t| \leq n$ for some $n \in \mathbb{N}$. Then, for $t \in [n, n+1]$, the inequality (6.10) implies

$$\begin{aligned} \log \|u_t\|_{Mul(X)} &\leq \log \|u_{t-1}\|_{Mul(X)} + \log \|u_1\|_{Mul(X)} + m \\ &\leq M + m|t-1| + m = M + m|t|. \end{aligned}$$

The above inequality is proven analogously for $t \in [-n-1, -n]$, thus the induction holds true and the bound of the lemma follows for $\|u_t\|_{Mul(X)}$.

As regards the inequality for $\|u_t^{-1}\|_{Mul(X)}$, Lemma 6.3.3 implies that (u_t^{-1}) is a well-defined DW-continuous cocycle for the flow (φ_t), hence the claim follows by what we have already proven for u_t.

To finish the proof, recall that by [DRS69, Lemma 11], the continuous inclusion $Mul(X) \hookrightarrow H^\infty(\mathbb{D})$ holds, so the inequalities of the claim for $\|u_t\|_\infty, \|u_t^{-1}\|_\infty$ follow from the ones we have already proven. □

The real numbers α_u, β_u found in the following lemma will be called *exponents* of (u_t). They play a central role in our spectral discussion in this chapter. Recall that $u_t(1) := \lim_{\mathbb{D} \ni z \to 1} u_t(z)$ and $u_t(-1) := \lim_{\mathbb{D} \ni z \to -1} u_t(z)$.

Lemma 6.3.5. *There exists some $\alpha_u, \beta_u \in \mathbb{R}$ such that*

$$|u_t(1)| = e^{\alpha_u t}, \quad |u_t(-1)| = e^{\beta_u t}, \qquad t \in \mathbb{R}.$$

Proof. The mapping $t \mapsto |u_t(\iota)|$ is a group homomorphism for $\iota = -1, 1$ since

$$u_{s+t}(\iota) = \lim_{\mathbb{D} \ni z \to \iota} u_{s+t}(z) = \left(\lim_{\mathbb{D} \ni z \to \iota} u_s(z) \right) \left(\lim_{\mathbb{D} \ni z \to \iota} u_t(\varphi_s(z)) \right) = u_s(\iota) u_t(\iota), \quad s, t \in \mathbb{R},$$

where we have used that $\lim_{\mathbb{D}\ni z\to\iota}\varphi_t(z)=\iota$ through $\mathbb{D}$ for all $t\in\mathbb{R}$. It follows from Lemma (6.3.4) that $t\mapsto|u_t(\iota)|$ is a locally bounded homomorphism from $\mathbb{R}$ to $(0,\infty)$, so it satisfies Cauchy's exponential functional equation. Hence there exists $c_\iota\in\mathbb{R}$ such that $u_t(\iota)=e^{c_\iota t}$, and the claim follows. □

One has $\lim_{\mathbb{N}\ni n\to\infty}\|u_n\|_\infty^{1/n}=\max\{|u_1(1)|,|u_1(-1)|\}$ for every DW-continuous cocycle (u_t), see [HLNS13, Lemma 4.4]. We need extensions of this property, which are pointed out in the following lemma.

Lemma 6.3.6. *Let $t\in\mathbb{R}\setminus\{0\}$. Then*

$$\lim_{x\to\infty}\|u_{xt}\|_\infty^{1/x}=\max\{|u_t(1)|,|u_t(-1)|\}.$$

In addition, for $t>0$ it holds that

$$\lim_{x\to\infty}\|u_{xt}\|_{H^\infty(\mathbb{D}_1)}^{1/x}=|u_t(1)|,\qquad \lim_{x\to-\infty}\|u_{xt}\|_{H^\infty(\mathbb{D}_{-1})}^{-1/x}=|u_{-t}(-1)|.$$

Proof. The existence of $\lim_{x\to\infty}\|u_{xt}\|_\infty^{1/x}$, as well as the first equality, is a consequence the fact that $t\mapsto\log\|u_t\|_\infty$ is a subadditive function of [HLNS13, Lemma 4.4].

The other claims in the statement regarding the limits are obtained similarly to the above, and reasoning as in the proof of [HLNS13, Lemma 4.4]. □

We show in Theorem 6.3.11 that a holomorphic function ω associated with (u_t), see Proposition 4.2.2, has tempered zeroes or singularities at the DW points. This property is one of the key facts through our discussion in this chapter.

Remark 6.3.7. In terms of the function ω, Lemma 6.3.6, second half, reads

$$\lim_{s\to\infty}\left\|\frac{\omega\circ\varphi_s}{\omega}\right\|_{H^\infty(\mathbb{D}_1)}^{1/s}=e^{\alpha_u},\qquad \lim_{s\to-\infty}\left\|\frac{\omega\circ\varphi_s}{\omega}\right\|_{H^\infty(\mathbb{D}_{-1})}^{-1/s}=e^{-\beta_u}.$$

Lemma 6.3.8. *Let ω be as above, and let $\lambda,\nu\in\mathbb{C}$ and set $\rho(z)=\omega(z)(1-z)^\lambda(1+z)^\nu$ for $z\in\mathbb{D}$. Then the cocycle (v_t) given by $v_t=(\rho\circ\varphi_t)/\rho$ is a DW-continuous cocycle for (φ_t) on X with exponents $\alpha_v=\alpha_u-\mathfrak{Re}\,\lambda$ and $\beta_v=\beta_u+\mathfrak{Re}\,\nu$.*

Proof. Given a bounded interval $I\subseteq\mathbb{R}$ and $t\in\mathbb{R}$ there exists an open subset U containing the closed disc $\overline{\mathbb{D}}$ such that the function h_t given by

$$h_t(z)=\left(\frac{1-\varphi_t(z)}{1-z}\right)^\lambda\left(\frac{1+\varphi_t(z)}{1+z}\right)^\nu=\left(\frac{2}{(e^t-1)z+e^t+1}\right)^{\lambda+\nu}e^{\nu t},\quad z\in U,$$

is holomorphic in U for all $t\in I$. Then we have that $v_t=u_th_t$ is a continuous cocycle which is continuous at the DW points $-1,1$. Thus it satisfies (**Co1**).

Moreover, U can be chosen for the mapping $t\in I\mapsto h_t\in H^\infty(U)$ to be continuous. Since $H^\infty(U)\hookrightarrow Mul(X)$ by (**Gam1**), it follows that the mapping $t\in I\mapsto v_t\in Mul(X)$ is measurable, so that (v_t) satisfies (**Co2**), that is, (v_t) is a DW-continuous cocycle.

Regarding the exponents of (v_t), a few computations show that $\lim_{z\to 1}(1-\varphi_t(z))/(1-z) = e^{-t}$ and $\lim_{z\to -1}(1+\varphi_t(z))/(1+z) = e^t$, $t \in \mathbb{R}$. In addition, $\lim_{z\to -1}(1-\varphi_t(z))/(1-z) = \lim_{z\to 1}(1+\varphi_t(z))/(1+z) = 1$ since both $-1, 1$ are fixed points of φ_t. Hence we conclude $\lim_{z\to 1}|v_t(z)| = \lim_{z\to 1}|u_t(z)||h_t(z)| = e^{\alpha_u t}|e^{-\lambda t}| = e^{(\alpha_u - \mathfrak{Re}\,\lambda)t}$, i.e. $\alpha_v = \alpha_u - \mathfrak{Re}\,\lambda$. Similarly we obtain $\beta_v = \beta_u + \mathfrak{Re}\,\nu$ and the proof is finished. □

Remark 6.3.9. According to (6.6), the following equality holds

$$(\varphi_t')^\delta = \frac{G^\delta \circ \varphi_t}{G^\delta}; \quad t \in \mathbb{R},\ \delta \in \mathbb{R},$$

where G is the generator of the flow (φ_t) given by $G(z) = (1-z^2)/2$, $z \in \mathbb{D}$. Whence, it follows by Lemma 6.3.8 that, for every $\delta \in \mathbb{R}$ and an arbitrary DW-continuous cocycle (u_t) for the flow (φ_t) on X, the family $(u_t(\varphi_t')^\delta)$ is a DW-continuous cocycle for the flow (φ_t) on X. In particular, taking $u_t = \mathbf{1}$ (i.e. the constant function equal to 1) we have that $((\varphi_t')^\delta)$ is a DW-continuous cocycle for the flow (φ_t) on X.

Lemma 6.3.10. *Let $A \subseteq \mathbb{D}$ be such that $\{-1,1\} \cap \overline{A} = \emptyset$. For ω as above, $\sup_{z\in A}|\omega(z)| < \infty$ and $\inf_{z\in A}|\omega(z)| > 0$.*

Proof. The claim is trivial if ω is a constant function, so let us assume that ω is not constant.

As neither -1 nor 1 belong to $\overline{A}$, it is readily seen that there exists $R > 0$ such that for any $z \in \overline{A}$, there are (unique) $x \in (-1,1)$ and $t \in [-R, R]$ such that $z = \varphi_t(ix)$.

Then, we prove that $\sup_{x\in(-1,1)}|\omega(ix)| < \infty$ by reaching a contradiction. So assume $\sup_{x\in(-1,1)}|\omega(ix)| = \infty$. In this case, for some $d \in \{-1,1\}$, there exists a sequence $(-1,1) \ni x_n \to d$ such that $\lim_{n\to\infty}|\omega(ix_n)| = \infty$. As a consequence, if the limit $\lim_{(-1,1)\ni x\to d}|\omega(ix)|$ existed, it would be equal to ∞. Assume this is the case. Now, for $\theta \in (0,\pi)$, let t_θ denote the unique real number for which $\varphi_{t_\theta}(i) = e^{i\theta}$. A few computations show that

$$t_\theta = 2\tanh^{-1}\left(\frac{-\cos\theta}{1+\sin\theta}\right), \quad \theta \in (0,\pi).$$

Therefore, the mapping $\Phi : [0,1] \times (0,\pi) \to \mathbb{C}$ given by $\Phi(x,\theta) = \varphi_{t_\theta}(ix)$ is continuous. Even more, $\Phi([0,1) \times (0,\pi)) \subseteq \mathbb{D}$ and $\Phi(1,\theta) = e^{i\theta}$, so Φ is a continuous family of paths in the sense of [DT85, pp. 83]. Since there exist $K, w > 0$ such that the bound $\|u_t^{-1}\|_\infty \le Ke^{w|t|}$ holds for all $t \in \mathbb{R}$ (see Lemma 6.3.4), it follows that

$$\lim_{x\to 1^-}|\omega(\Phi(x,\theta))| = \lim_{x\to 1^-}|\omega(\varphi_{t_\theta}(ix))| = \lim_{x\to 1^-}|u_{t_\theta}(ix)||\omega(ix)| = \infty,$$

for all $\theta \in (-\pi,\pi)$, which is absurd by the uniqueness of limits along the family of continuous path Φ, see [DT85, pp. 83].

Before continuing with the proof, we assume furthermore that $\alpha_u < 0$ and $\beta_u > 0$. Then, Remark 6.3.7 implies that there exists $M > 0$ such that

$$|\omega(\varphi_s(ix))| < |\omega(ix)|, \qquad \text{for all } |s| \ge M,\ x \in (-1,1). \tag{6.11}$$

We now continue with the proof of the lemma. As $\lim_{(-1,1)\ni x\to d}|\omega(ix)|$ does not exist and in particular is not equal to ∞ for neither $d=-1$ nor $d=-1$, there exist $K>0$ and a sequence $(y_n)_{n\in\mathbb{N}}\subseteq(-1,1)$ with accumulation points $-1,1$ and such that $|\omega(iy_n)|\le K$ for all $n\in\mathbb{N}$. One has that $\mu:=\sup_{t\in[-M,M]}\|u_t\|_\infty<\infty$ by Lemma 6.3.4, where $M>0$ is as in (6.11). Take C such that $C>\max\{\mu,1\}$ and $\widetilde{x}:=x_{N_1}$, $\widetilde{y}:=y_{N_2}$, $\widetilde{z}=y_{N_3}$ for $N_1,N_2,N_3\in\mathbb{N}$ such that $|\omega(i\widetilde{x})|>CK$ and $\widetilde{z}<\widetilde{x}<\widetilde{y}$. Let $B\subseteq\mathbb{D}$ be the compact subset

$$B:=\{\varphi_s(ix)\,|\,(x,s)\in[\widetilde{z},\widetilde{y}]\times[-M,M]\}\,.$$

We now prove that $|\omega|$ reaches its maximum in B in its interior, which contradicts the maximum modulus principle. Let $L=\max_{x\in[\widetilde{z},\widetilde{y}]}|\omega(ix)|$, which is attained in $(\widetilde{z},\widetilde{y})$ since $|\omega(i\widetilde{x})|>|\omega(i\widetilde{y})|,|\omega(i\widetilde{z})|$. Now, notice that

$$\max\{|\omega(\varphi_s(i\widetilde{z}))|,\,|\omega(\varphi_s(i\widetilde{y}))|\}\le C\max\{|\omega(i\widetilde{z})|,\,|\omega(i\widetilde{y})|\}\le CK<|\omega(i\widetilde{x})|\le L,$$

for all $s\in[-M,M]$. Also, by (6.11),

$$\max\{|\omega(\varphi_{-M}(ix))|,\,|\omega(\varphi_M(ix))|\}<|\omega(ix)|\le L,\quad x\in[\widetilde{z},\widetilde{y}].$$

Hence the maximum of $|\omega|$ in B is not attained in its boundary, reaching a contradiction. Therefore, $\sup_{x\in(-1,1)}|\omega(ix)|<\infty$.

If $\alpha_u\ge 0$ or $\beta_u\le 0$, we consider the weight $\rho(z):=\omega(z)(1-z)^{-N}(1+z)^M$ and its associated cocycle $v_t:=(\rho\circ\varphi_t)/\rho$, where $N>|\alpha_u|$, $M>|\beta_u|$. It follows by Lemma 6.3.8 that (v_t) is a DW-continuous cocycle with $\alpha_v=\alpha_u-N<0$ and $\beta_v=\beta_u+M>0$, so by what we have already proven, $\sup_{x\in(-1,1)}|\rho(ix)|<\infty$, and as a consequence, $\sup_{x\in(-1,1)}|\omega(ix)|\le 2^{N/2}\sup_{x\in(-1,1)}|\rho(ix)|<\infty$, as we wanted to show.

Finally, consider the DW-continuous cocycle given by (u_t^{-1}), see Lemma 6.3.3, and let A be a subset as in the statement. Then the weight associated with (u_t^{-1}) is ω^{-1}, whence it follows from the above that $\sup_{z\in A}|\omega(z)^{-1}|<\infty$, that is, $\inf_{z\in A}|\omega(z)|=(\sup_{z\in A}|\omega^{-1}(z)|)^{-1}>0$. □

Theorem 6.3.11. *Lt ω be a holomorphic function associated with a DW-continuous cocycle (u_t). Let α_u,β_u be the exponents of (u_t). Then, for every $\varepsilon>0$, one has*

$$|\omega(z)|\lesssim|1-z|^{-\alpha_u-\varepsilon}|1+z|^{\beta_u-\varepsilon},\quad z\in\mathbb{D},$$
$$|\omega(z)|\gtrsim|1-z|^{-\alpha_u+\varepsilon}|1+z|^{\beta_u+\varepsilon},\quad z\in\mathbb{D}.$$

Proof. By Lemma 6.3.10, we only have to prove the inequalities of the claim for some arbitrary neighborhoods $\mathcal{U}_{-1},\mathcal{U}_1$ of -1, 1 respectively. We will prove it for $\mathcal{U}_1$ of 1, being the other one analogous. One has

$$\frac{1-\varphi_s(z)}{1-z}\to e^{-s},\ \text{as } z\to 1,$$

uniformly on $s > 0$. On the other hand, by Remark 6.3.7, for any $\varepsilon' > 0$, there exists some $M > 0$ such that

$$|\omega(\varphi_s(z))| \leq |\omega(z)|e^{s(\alpha_u+\varepsilon')}, \qquad \text{for all } s \geq M,\, z \in \mathbb{D}_1.$$

Hence, for every $\varepsilon > 0$, $C > 1$, there exists a neighborhood $\mathcal{U}$ of 1, and $M > 0$ such that

$$|\omega(\varphi_s(z))| \leq C|\omega(z)| \left|\frac{1-z}{1-\varphi_s(z)}\right|^{\alpha_u+\varepsilon}, \qquad \text{for all } s \geq M,\, z \in \mathcal{U} \cap \mathbb{D}. \tag{6.12}$$

Since φ_{-M} is analytic at 1 and $\varphi_{-M}(1) = 1$ there is an open subset $\mathcal{V}$ such that $1 \in \mathcal{V} \subseteq \mathcal{U}$ and $\varphi_{-M}(\mathcal{V}) \subseteq \mathcal{U}$. It follows by Lemma 6.3.10 that ω is bounded on $\mathbb{D}_1 \setminus \mathcal{V}$. Moreover, taking $\mathcal{V}$ such that $\mathcal{D} \setminus \mathcal{U}, \varphi_{-M}(\mathcal{V})$ are two disjoint connected sets, it is easy to see that for all $v \in \mathcal{V} \cap \mathbb{D}$ there is $s(v) \geq M$ such that $\varphi_{-s(v)}(v) \in \mathbb{D} \cap (\mathcal{U} \setminus \mathcal{V})$. But then, (6.12) applied to $z = \varphi_{-s(v)}(v)$ implies, for any $\varepsilon > 0$,

$$|\omega(v)| \leq C|\omega(\varphi_{-s(v)}(v)| \left|\frac{1-\varphi_{-s(v)}(v)}{1-v}\right|^{\alpha_u+\varepsilon} \lesssim |1-v|^{-\alpha_u-\varepsilon}, \quad v \in \mathcal{V},$$

where, in the second inequality, we have used Lemma 6.3.10 for $|\omega|$, and that the terms $|1-\varphi_{-s(v)}(v)|, |\omega(\varphi_{-s_v}(v))|$ are bounded away from zero, since $\varphi_{-s(v)}(v) \notin \mathcal{V}$. As said above, one can analogously obtain that there exists a neighborhood $\mathcal{U}_{-1} \subseteq \mathbb{D}$ of -1 such that $|\omega(z)| \lesssim |1+z|^{\beta_u-\varepsilon}$, $z \in \mathcal{U}_{-1} \cap \mathbb{D}$. Altogether, one gets $|\omega(z)| \lesssim |1-z|^{-\alpha_u-\varepsilon}|1+z|^{\beta_u-\varepsilon}$, $z \in \mathbb{D}$.

Finally, the inequality $\gtrsim$ of the claim follows by an application of what we have already proven to the DW-continuous cocycle $(v_t) := (u_t^{-1})$ with weight $\rho = \omega^{-1}$, see Lemma 6.3.3. Indeed, since $\alpha_v = -\alpha_u$ and $\beta_v = -\beta_u$, one has that for any $\varepsilon > 0$, $|\omega(z)^{-1}| = |\rho(z)| \lesssim |1-z|^{\alpha_u-\varepsilon}|1+z|^{-\beta_u-\varepsilon}$ for all $z \in \mathbb{D}$. Thus the proof is concluded. □

Theorem 6.3.11 is a significant step in our discussion since it shows that, under mild conditions on a cocycle, its associated weight ω must be tempered at DW points. Besides such a property we next introduce two other conditions of asymptotic type that are needed for the unified approach we carry out in Section 6.5 and Section 6.6. Recall that by ι, we denote either the number -1 or 1.

Definition 6.3.12. Let X be a γ-space and, for $\iota \in \{-1, 1\}$, let X_ι be Banach spaces for which property (**Gam3**) holds. A DW-continuous cocycle (u_t) for the hyperbolic flow (φ_t) is said to be spectrally DW-contractive (DW-contractive for short) if it satisfies the following conditions:

(SpC1)
$$\limsup_{t\to\infty} \|u_{\iota t}\|_{Mul(X)}^{1/t} \leq \max\{|u_\iota(-1)|, |u_\iota(1)|\};$$

and

(SpC2)
$$\limsup_{t\to\infty} \|u_{\iota t} f_t\|_{X_\iota}^{1/t} \leq |u_\iota(\iota)|,$$

for every family $(f_t) \subseteq X$ such that $\limsup_{t\to\infty} \|f_t\|_X^{1/t} \le 1$.

We say that a γ-space is hyperbolically DW-contractive if every DW-continuous cocycle is spectrally DW-contractive.

Remark 6.3.13. Similarly to the definition of γ-pair, hyperbolically DW-contractivity can be equivalently formulated in terms of cocycles (v_t) associated to hyperbolic flows (ψ_t) with arbitrary DW points $a, b \in \mathbb{T}$. This fact and Remark 6.2.3 mean that cocycles (v_t) as above satisfy analogous properties to (**SpC1**) and (**SpC2**) when acting on a hyperbolically DW-contractive γ-space X.

Let X be any of the examples of γ-spaces given in Section 6.2. Next proposition proves that X is hyperbolically DW-contractive. The cases of Hardy spaces, Bergman spaces, little Korenblum classes and the disc algebra are covered by item (1) below.

Proposition 6.3.14. *1. Let X be a γ-space, for $\gamma \ge 0$, such that the continuous inclusions $Mul(X) \hookrightarrow H^\infty(\mathbb{D})$, $Mul(X_{-1}) \hookrightarrow H^\infty(\mathbb{D}_{-1})$, $Mul(X_1) \hookrightarrow H^\infty(\mathbb{D}_1)$ are bounded below mappings. Then X is hyperbolically DW-contractive.*

2. Let either $X = \mathcal{D}^p_\sigma(\mathbb{D})$ for $\sigma > -1, p \ge 1$, and $p-2 \le \sigma \le p-1$ or $X = B_{1,0}(\mathbb{D})$. Then X is hyperbolically DW-contractive.

Proof. (1) By hypothesis, $\|u\|_{Mul(X)} \lesssim \|u\|_\infty$, $\|v\|_{Mul(X_\iota)} \lesssim \|v\|_{H^\infty(\mathbb{D}_\iota)}$ for every $u \in Mul(X)$, $v \in Mul(X_\iota)$ respectively (recall that the embedding $Mul(Y) \hookrightarrow H^\infty(E)$ is continuous for any space Y such that $Y \hookrightarrow \mathcal{O}(E)$, where E is an open subset of $\mathbb{C}$, see [DRS69, Lemma 11]). Let (u_t) be a DW-continuous cocycle for (φ_t). It follows by Lemma 6.3.6 that

$$\limsup_{t\to\infty} \|u_{\iota t}\|^{1/t}_{Mul(X)} \le \lim_{t\to\infty} \|u_{\iota t}\|^{1/t}_{H^\infty(\mathbb{D})} = \max\{|u_\iota(1)|, |u_\iota(-1)|\},$$

so that condition (**SpC1**) is fulfilled. Let now $(f_t) \subseteq X$ be such that $\limsup_{t\to\infty} \|f_t\|_X^{1/t} \le 1$, thus $\limsup_{t\to\infty} \|f_t\|_{X_\iota}^{1/t} \le 1$ since $X \hookrightarrow X_\iota$. Another application of Lemma 6.3.6 yields that

$$\limsup_{t\to\infty} \|u_{\iota t} f_t\|^{1/t}_{X_\iota} \le \limsup_{t\to\infty} \|u_{\iota t}\|^{1/t}_{Mul(X_\iota)} \|f_t\|^{1/t}_{X_\iota} \le \lim_{t\to\infty} \|u_{\iota t}\|^{1/t}_{H^\infty(\mathbb{D}_\iota)} = |u_\iota(\iota)|,$$

so X satisfies (**SpC2**) and our claim is proven.

(2) Property (**SpC1**) is essentially proved in [ELM16, Th. 5.2] for $\mathcal{D}^2_0(\mathbb{D})$. The proof for arbitrary σ, p as in the statement, as well as for $B_{1,0}(\mathbb{D})$, runs similarly.

□

6.4 Estimates of hyperbolic composition groups

Let X be a γ-space with $\gamma \geq 0$ and let (u_t) be a DW-continuous cocycle for the hyperbolic flow (φ_t) on X given by (6.2). Let ω be the non-vanishing holomorphic function associated with (u_t), so $u_t = (\omega \circ \varphi_t)/\omega$, $t \in \mathbb{R}$, see Proposition 4.2.2. Define

$$S_\omega(t) := u_t C_{\varphi_t} \quad t \in \mathbb{R}.$$

Proposition 6.4.1. *For (u_t) and ω as above, the family $(S_\omega(t))$ is a C_0-group in $B(X)$.*

Proof. It follows that $(S_\omega(t))$ is strongly measurable since (u_t) is strongly measurable by (**Co2**), and C_{φ_t} is strongly continuous on X by (**Gam4**). Hence, $(S_\omega(t))$ is strongly continuous since every strongly measurable group is strongly continuous [HP57, Th. 10.2.3]. □

Here we deal with asymptotic estimates of the norm of operators $S_\omega(t)$, $t \in \mathbb{R}$. For the sake of convenience we set $\alpha := \alpha_u$, $\beta := \beta_u$ where α_u, β_u are the exponents of the cocycle (u_t) obtained in Lemma 6.3.5; that is, $|u_t(1)| = e^{\alpha t}$, $|u_t(-1)| = e^{\beta t}$ for $t \in \mathbb{R}$.

Proposition 6.4.2. *Let X be a hyperbolically DW-contractive γ-space for some $\gamma \geq 0$. For $(S_\omega(t))$ as above,*

$$\lim_{t\to\infty} \|S_\omega(t)\|_{L(X)}^{1/t} \leq \max\{e^{\beta-\gamma}, e^{\alpha+\gamma}\},$$

and

$$\lim_{t\to\infty} \|S_\omega(-t)\|_{L(X)}^{1/t} \leq \max\{e^{-\beta+\gamma}, e^{-\alpha-\gamma}\}.$$

Proof. Let $\varepsilon > 0$ and let $\iota = -1, 1$. Since X is a γ-space we have $\|C_{\varphi_{\iota t},\gamma}\|_{L(X)} \leq K_\varepsilon e^{\varepsilon t}$, for $t > 0$; see (6.7). On the other hand, $S_\omega(t) = u_t(\varphi_t')^{-\gamma} C_{\varphi_t,\gamma}$, $t \in \mathbb{R}$, where $(u_t(\varphi_t')^\gamma)$ is a DW-continuous cocycle for the flow (φ_t) with exponents $\alpha_{u(\varphi')^{-\gamma}} = \alpha + \gamma$ and $\beta_{u(\varphi')^{-\gamma}} = \beta - \gamma$, see Lemma 6.3.8 and Remark 6.3.9. As a consequence,

$$\|S_\omega(\iota t)\|_{L(X)} \leq \|u_{\iota t}(\varphi_{\iota t}')^{-\gamma}\|_{Mul(X)} \|C_{\varphi_{\iota t},\gamma}\|_{L(X)} \leq \|u_{\iota t}(\varphi_{\iota t}')^{-\gamma}\|_{Mul(X)} K_\varepsilon e^{\varepsilon t}, \quad t > 0.$$

Since X satisfies (**SpC1**), it follows that

$$(\forall \varepsilon > 0) \qquad \lim_{t\to\infty} \|S_\omega(\iota t)\|_{L(X)}^{1/t} \leq \max\{e^{\iota(\beta-\gamma)}, e^{\iota(\alpha+\gamma)}\} e^\varepsilon.$$

Then, making $\varepsilon \to 0$ one obtains the result. □

The following result is about localization at the DW points of the norm of the hyperbolic group. For $\delta < 0$, set $\mathfrak{X}_{-1}^\delta := \{f \in X : G_{-1}^\delta f \in X\}$ and $\mathfrak{X}_1^\delta := \{f \in X : G_1^\delta f \in X\}$, where $G_{-1}(z) := (1+z)$, $G_1(z) := (1-z)$ for $z \in \mathbb{D}$.

Proposition 6.4.3. *For X, (u_t), ω, α and β as above, assume $\beta - \alpha < 2\gamma$. Then* *(i)* $\lim_{t\to\infty} \|S_\omega(t)f\|_X^{1/t} \le e^{\beta-\gamma}$ *for all* $f \in \mathfrak{X}_1^{\beta-\alpha-2\gamma}$.

(ii) $\lim_{t\to\infty} \|S_\omega(-t)f\|_X^{1/t} \le e^{-\alpha-\gamma}$ *for all* $f \in \mathfrak{X}_{-1}^{\beta-\alpha-2\gamma}$.

Proof. (i) For $\delta < 0$ and $f \in \mathfrak{X}_1^\delta$, put $f_{\delta,1} := G_1^\delta f$. Then, for $t > 0$,

$$\begin{aligned} S_\omega(t)f &= \frac{(\omega\circ\varphi_t)(G_1^{-\delta}\circ\varphi_t)}{\omega}(f_{\delta,1}\circ\varphi_t) \\ &= G_1^{-\delta}\frac{(G_1^{-\delta}\omega)\circ\varphi_t}{G_1^{-\delta}\omega}(f_{\delta,1}\circ\varphi_t) = G_1^{-\delta}(S_{G_1^{-\delta}\omega}(t)f_{\delta,1}). \end{aligned}$$

The cocycle (v_t) given by $v_t = ((G_1^{-\delta}\omega)\circ\varphi_t)/(G_1^{-\delta}\omega)$ is a DW-continuous cocycle with exponents $\alpha+\delta$ and β (associated to the DW points $1, -1$ respectively) by Lemma 6.3.8. Moreover, $G_1^{-\delta} \in Mul(X)$ by (**Gam2**). Hence, an application of Proposition 6.4.2 to the group $(S_{G_1^{-\delta}\omega}(t))$ yields that

$$\lim_{t\to\infty} \|S_{G_1^{-\delta}\omega}(t)\|_{L(X)}^{1/t} \le \max\{e^{\alpha+\delta+\gamma}, e^{\beta-\gamma}\},$$

and then

$$\lim_{t\to\infty} \|S_\omega(t)f\|_X^{1/t} \le \lim_{t\to\infty}\left(\|G_1^{-\delta}\|_{Mul(X)}^{1/t}\|S_{G_1^{-\delta}\omega}(t)\|_{L(X)}^{1/t}\|f_{\delta,1}\|_X^{1/t}\right) \le \max\{e^{\alpha+\delta+\gamma}, e^{\beta-\gamma}\}.$$

Taking now $\delta = \beta - \alpha - 2\gamma$ one obtains $\lim_{t\to\infty} \|S_\omega(t)f\|_X^{1/t} \le e^{\beta-\gamma}$ for every $f \in \mathfrak{X}_1^{\beta-\alpha-2\gamma}$, as we wanted to show.

(ii) The argument to prove this part is similar to the preceding one. We leave it to the reader.

□

6.5 Two useful integrals

Through this section, let X be a hyperbolically DW-contractive γ-space and let $(S_\omega(t))$ be a weighted composition group as in Section 6.4, with α, β the exponents of $((\omega\circ\varphi_t)/\omega)$. Inspired by some ideas exposed within [Per08], which were further developed in [AP10], we introduce two integral operators which play a key role in the study of the spectrum of $(S_\omega(t))$ in Section 6.6.

For $z \in \mathbb{D}$, $f \in \mathcal{O}(\mathbb{D})$ and $\lambda \in \mathbb{C}$ (and $\iota = -1, 1$), set

$$(\Lambda_\omega^{\lambda,\iota} f)(z) := \frac{-2}{\omega(z)}\left(\frac{1+z}{1-z}\right)^\lambda \int_\iota^z \frac{(1-\xi)^{\lambda-1}}{(1+\xi)^{\lambda+1}}\,\omega(\xi)f(\xi)\,d\xi, \quad z \in \mathbb{D}, \tag{6.13}$$

where the integration path is to be understood as any simple path in $\mathbb{D}\cup\{\iota\}$ going from ι to z and leaving ι non-tangentially (it will be seen next that the value of the integral is independent of the chosen path), and

$$L_\omega^\lambda f := \int_{-1}^1 \frac{(1-\xi)^{\lambda-1}}{(1+\xi)^{\lambda+1}}\,\omega(\xi)f(\xi)\,d\xi, \tag{6.14}$$

where the integral is understood on any path in $\mathbb{D}$ between -1 and 1 touching $-1, 1$ non-tangentially.

The convergence of the above integrals is considered right now.

Lemma 6.5.1. *Let $f \in X$, $z \in \mathbb{D}$, $\lambda \in \mathbb{C}$. Then, the following holds.*

- *$(\Lambda_\omega^{\lambda,-1} f)(z)$ converges (absolutely) if $\mathfrak{Re}\,\lambda < \beta - \gamma$.*
- *$(\Lambda_\omega^{\lambda,1} f)(z)$ converges (absolutely) if $\mathfrak{Re}\,\lambda > \alpha + \gamma$.*

In any of the above cases, the value of $(\Lambda_\omega^{\lambda,\iota} f)(z)$ is independent on the integration path taken, whenever it is a simple path in $\mathbb{D} \cup \{\iota\}$ leaving ι non-tangentially. Also, the function $\Lambda_\omega^{\lambda,\iota} f$ is holomorphic in the disc.

Proof. Let us show the claims for $\Lambda_\omega^{\lambda,-1}$. Let θ_0 be a fixed angle such that $|\theta_0| < (\pi/2)$. Then, for $\xi \in \mathbb{D}$ such that $1 + \xi = |1 + \xi| e^{i\theta}$ with $|\theta| \leq |\theta_0|$ and $|1 + \xi| < \cos\theta_0$, one has $|\xi|^2 = |1 + \xi|^2 + 1 - 2|1 + \xi| \cos\theta$, whence $1 - |\xi|^2 = |1 + \xi|(2\cos\theta - |1 + \xi|) \geq |1 + \xi|(2\cos\theta_0 - |1 + \xi|) > (\cos\theta_0)|1 + \xi|$. In short,

$$1 - |\xi|^2 > (\cos\theta_0)|1 + \xi|, \tag{6.15}$$

for every ξ in the sector $-1 + \sum_{\theta_0}$ of angle θ_0, with vertex at -1 and symmetric with respect to $(-1, \infty)$, such that $|1 + \xi| < \cos\theta_0$.

Let $f \in X$ and $\varepsilon > 0$. Since $X \subseteq \mathcal{K}^{-\gamma-\varepsilon}(\mathbb{D})$ by Remark 6.2.4, one has $|f(\xi)| \lesssim (1 - |\xi|^2)^{-\gamma-\varepsilon} \|f\|_X$. Hence $|f(\xi)| \lesssim (\cos\theta_0)|1 + \xi|^{-\gamma-\varepsilon} \|f\|_X$ for all ξ as in (6.15). Also, ω has exponent β at -1 and so $|\omega(\xi)| \lesssim |1 + \xi|^{\beta-\varepsilon}$ for ξ as before, see Theorem 6.3.11.

Altogether,

$$|\omega(\xi) f(\xi)||1 - \xi|^{-\mathfrak{Re}\,\lambda+1} \lesssim |1 + \xi|^{\beta-\gamma-\mathfrak{Re}\,\lambda-1-2\varepsilon},$$

for every $\xi \in (-1 + \sum_{\theta_0})$ such that $|1 + \xi| < \cos\theta_0$, which readily implies the convergence of $\Lambda_\omega^{\lambda,-1} f$ on any path touching -1 non-tangentially, provided $\mathfrak{Re}\,\lambda < \beta - \gamma$.

The statement for $\Lambda_\omega^{\lambda,1} f$, that is, $\Lambda_\omega^{\lambda,1} f$ converges provided $\mathfrak{Re}\,\lambda > \alpha + \gamma$, is proven using analogous argument to the above one . It is left to the reader.

Let us now assume that $\mathfrak{Re}\,\lambda < \beta - \gamma$ and let τ be a closed path in $\mathbb{D}$ joining -1 and a fixed $z \in \mathbb{D}$, and being non-tangential (to $\mathbb{T}$) at -1. For $\delta > 0$ small enough, we can assume that the circle $\{\xi \in \mathbb{C} : |1 + \xi| < \delta\}$ intersects τ exactly twice. So let C_δ be the arc in $\mathbb{D}$ of such circle joining these two intersection points. Let τ_1, τ_{-1} be paths defined by $\tau_1 := (\tau \cap \{\xi \in \mathbb{D} : |1 + \xi| \geq \delta\}) \cup C_{\delta,-}$ and $\tau_{-1} := (\tau \cap \{\xi \in \mathbb{D} : |1 + \xi| < \delta\}) \cup C_{\delta,+}$, where $C_{\delta,-}$ (respectively $C_{\delta,+}$) is C_δ negatively (positively) orientated. Then we have $\int_{\tau_1} \frac{(1 - \xi)^{\lambda-1}}{(1 + \xi)^{\lambda+1}} \omega(\xi) f(\xi)\, d\xi = 0$ by Cauchy's theorem and therefore

$$\begin{aligned}
&\int_\tau \frac{(1 - \xi)^{\lambda-1}}{(1 + \xi)^{\lambda+1}} \omega(\xi) f(\xi)\, d\xi = \int_{\tau_{-1}} \frac{(1 - \xi)^{\lambda-1}}{(1 + \xi)^{\lambda+1}} \omega(\xi) f(\xi)\, d\xi \\
&= \int_{\tau_{-1}} \chi_{(\tau_{-1} \setminus C_{\delta,+})}(\xi) \frac{(1 - \xi)^{\lambda-1}}{(1 + \xi)^{\lambda+1}} \omega(\xi) f(\xi)\, d\xi + \int_{C_{\delta,+}} \frac{(1 - \xi)^{\lambda-1}}{(1 + \xi)^{\lambda+1}} \omega(\xi) f(\xi)\, d\xi
\end{aligned}$$

where $\chi_{(\tau_{-1}\setminus C_{\delta,+})}$ is the characteristic function of $\tau_{-1}\setminus C_{\delta,+}$. The first term of the two latter integrals tends to zero as $\delta \to 0$ by the dominated convergence theorem. As regards the second one, it is bounded up to a constant by $\int_{C_{\delta,+}} |1+\xi|^{\beta-\gamma-\mathfrak{Re}\,\lambda-1-2\varepsilon}|d\xi|$, which in turn, using the parametrization $1+\xi=\delta e^{i\theta}$, $\theta_1\le\theta\le\theta_2$, where θ_1 and θ_2 are the arguments of the extreme points of the arc $C_{\delta,+}$, equals

$$\int_{\theta_1}^{\theta_2} \delta^{\beta-\gamma-\mathfrak{Re}\,\lambda-1-2\varepsilon}\delta\, d\theta \le \pi\delta^{\beta-\gamma-\mathfrak{Re}\,\lambda-2\varepsilon}$$

(with ε small enough).

In conclusion, one has $\int_\tau \frac{(1-\xi)^{\lambda-1}}{(1+\xi)^{\lambda+1}}\omega(\xi)f(\xi)\,d\xi=0$ and so the integral which defines $(\Lambda_\omega^{\lambda,-1}f)(z)$ is independent of paths in $\mathbb{D}$ joining -1 and $z\in\mathbb{D}$ non-tangentially at -1. The case $\Lambda_\omega^{\lambda,1}f$ is proven in the same way.

Finally, it is readily seen that, under the above hypothesis, the functions $\Lambda_\omega^{\lambda,\iota}$, $\iota=-1,1$, are holomorphic in $\mathbb{D}$. □

In the following corollary, we extend the values of λ for which $\Lambda_\omega^{\lambda,\iota}f$ is well defined in the case that f belongs to the subspaces $\mathfrak{X}_\iota^\delta$ introduced prior to Proposition 6.4.3.

Corollary 6.5.2. *Assume that $\beta-\alpha<2\gamma$. Let $f\in X$ and $z\in\mathbb{D}$. Then, on every path as in Lemma 6.5.1, $(\Lambda_\omega^{\lambda,-1}f)(z)$ converges (absolutely) if $\mathfrak{Re}\,\lambda<\gamma+\alpha$ and $f\in\mathfrak{X}_{-1}^{\beta-\alpha-2\gamma}$; and $(\Lambda_\omega^{\lambda,1}f)(z)$ converges (absolutely) if $\mathfrak{Re}\,\lambda>\beta-\gamma$ and $f\in\mathfrak{X}_1^{\beta-\alpha-2\gamma}$.*

Moreover, the value of $\Lambda_\omega^{\lambda,\iota}f$ is independent on the integration taken, whenever it is a simple path in $\mathbb{D}\cup\{\iota\}$ leaving ι non-tangentially. Also, the function $\Lambda_\omega^{\lambda,\iota}f$ is holomorphic in $\mathbb{D}$.

Proof. The statement is an immediate consequence of Lemma 6.5.1 applied to the function $(1+\cdot)^{\beta-\alpha-2\gamma}f$ if $f\in\mathfrak{X}_{-1}^{\beta-\alpha-2\gamma}$, and to the function $(1-\cdot)^{\beta-\alpha-2\gamma}f$ if $f\in\mathfrak{X}_1^{\beta-\alpha-2\gamma}$. □

We show now the relationship between the integrals of (6.13) and the group $(S_\omega(t))$.

Proposition 6.5.3. *Let $f\in X$. Then*

(i) $\Lambda_\omega^{\lambda,1}f=\int_0^\infty e^{-\lambda t}S_\omega(t)f\,dt$, *in X, provided $\mathfrak{Re}\,\lambda>\max\{\beta-\gamma,\alpha+\gamma\}$.*

(ii) $\Lambda_\omega^{\lambda,-1}f=-\int_0^\infty e^{\lambda t}S_\omega(-t)f\,dt$, *in X, provided $\mathfrak{Re}\,\lambda<\min\{\beta-\gamma,\alpha+\gamma\}$.*

Assume furthermore that $\beta-\alpha<2\gamma$. Then

(iii) $\Lambda_\omega^{\lambda,1}f=\int_0^\infty e^{-\lambda t}S_\omega(t)f\,dt$, *in X, provided $\mathfrak{Re}\,\lambda>\beta-\gamma$ and $f\in\mathfrak{X}_1^{\beta-\alpha-2\gamma}$.*

(iv) $\Lambda_\omega^{\lambda,-1}f=-\int_0^\infty e^{\lambda t}S_\omega(-t)f\,dt$, *in X, provided $\mathfrak{Re}\,\lambda<\alpha+\gamma$ and $f\in\mathfrak{X}_{-1}^{\beta-\alpha-2\gamma}$.*

Proof. (i) Let $f \in X$. The map $t \in [0,\infty) \mapsto S_\omega(t)f \in X$ is norm continuous and

$$\|S_\omega(t)f\|_X \le K_\varepsilon \max\{e^{(\beta-\gamma+\varepsilon)t}, e^{(\alpha+\gamma+\varepsilon)t}\},$$

for $\varepsilon > 0$, by Proposition 6.4.2. Hence, choosing ε small enough, one obtains that the integral $\int_0^\infty e^{-\lambda t} S_\omega(t)f\, dt$ is Bochner-convergent in X for $\mathfrak{Re}\,\lambda > \max\{\beta-\gamma, \alpha+\gamma\}$.

Now, for $z \in \mathbb{D}$, we apply Lemma 6.5.1 with the path $\xi = \dfrac{z+r}{1+rz}$, $0 \le r \le 1$, and make the variable change $r = \tanh(t/2)$, to obtain

$$\begin{aligned}
&\frac{2}{\omega(z)}\left(\frac{1+z}{1-z}\right)^{\lambda}\int_z^1 \frac{(1-\xi)^{\lambda-1}}{(1+\xi)^{\lambda+1}}\omega(\xi)f(\xi)\,d\xi\\
&= \frac{2}{\omega(z)}\int_0^1 \frac{(1-r)^{\lambda-1}}{(1+r)^{\lambda+1}}\omega\left(\frac{z+r}{1+zr}\right) f\left(\frac{z+r}{1+zr}\right) d\xi \qquad (6.16)\\
&= \int_0^\infty e^{-\lambda t}\frac{(\omega(\varphi_t(z))}{\omega(z)} f(\varphi_t(z))dt = \int_0^\infty e^{-\lambda t}(S_\omega(t)f)(z)\,dt,
\end{aligned}$$

for every $\lambda \in \mathbb{C}$ such that $\mathfrak{Re}\,\lambda > \alpha+\gamma$. Since the latter integral, regarded as a vector-valued integral, is Bochner convergent for $\mathfrak{Re}\,\lambda > \max\{\beta-\gamma, \alpha+\gamma\}$ we get the wished-for result.

(ii) This part follows along the same lines as before, by applying Proposition 6.4.2 to the semigroup $(S_\omega(-t))_{t\ge 0}$.

Items (iii) and (iv) are obtained with an analogous argument. Corollary 6.5.2 states that $\Lambda_\omega^{\lambda,1} f$, $\Lambda_\omega^{\lambda,-1} f$ are well-defined in these cases, and the sharper asymptotic bounds for $\|S_\omega(t)f\|_X$ as $t \to \iota\infty$ given in Proposition 6.4.3 imply that the integrals of the statement are convergent in the Bochner sense. □

The following lemma is significant to study the residual spectrum of the infinitesimal generator of the C_0-group $(S_\omega(t))$.

Lemma 6.5.4. *Assume $\beta - \alpha > 2\gamma$ and $\gamma + \alpha < \lambda < \beta - \gamma$. Then the mapping $L_\omega^\lambda : X \to \mathbb{C}$ given by (6.14) is a continuous linear functional on X.*

Moreover, if $f \in \ker L_\omega^\lambda$, then $\Lambda_\omega^{\lambda,1} f = \Lambda_\omega^{\lambda,-1} f \in X$.

Proof. Let $\varepsilon > 0$. By Remark 6.2.4, we have $\sup_{z\in\mathbb{D}}(1-|z|)^{\gamma+\varepsilon}|f(z)| \lesssim \|f\|_X$ for all $f \in X$. Moreover, $|\omega(z)| \lesssim |1-z|^{-\alpha-\varepsilon}|1+z|^{\beta-\varepsilon}$ for all $z \in \mathbb{D}$. Therefore,

$$\begin{aligned}
|L_\omega^\lambda f| &\le \int_{-1}^1 \left|\frac{(1-\xi)^{\lambda-1}}{(1+\xi)^{\lambda+1}}\omega(\xi)f(\xi)\right| d\xi\\
&\lesssim \|f\|_X \int_{-1}^1 (1-\xi)^{\mathfrak{Re}\,\lambda-\alpha-\gamma-2\varepsilon-1}(1+\xi)^{-\mathfrak{Re}\,\lambda+\beta-\gamma-2\varepsilon-1}\,d\xi.
\end{aligned}$$

The last integral is finite for $\varepsilon > 0$ small enough, hence L_ω^λ is a well-defined bounded functional on X.

Now, it follows by Lemma 6.5.1 that $\Lambda_\omega^{\lambda,1} f, \Lambda_\omega^{\lambda,-1} f \in \mathcal{O}(\mathbb{D})$ for each $f \in X$. Moreover, a simple computation shows that

$$(\Lambda_\omega^{\lambda,-1} f)(z) = (\Lambda_\omega^{\lambda,1} f)(z) - \frac{2}{\omega(z)} \left(\frac{1+z}{1-z}\right)^\lambda L_\omega^\lambda f, \quad z \in \mathbb{D},\ f \in X.$$

Hence $\Lambda_\omega^{\lambda,1} f = \Lambda_\omega^{\lambda,-1} f$ if $f \in \ker L_\gamma^\lambda$ as claimed.

Now we prove that $\Lambda_\omega^{\lambda,1} f \in X_1$, where X_1 is the subspace of $\mathcal{O}(\mathbb{D}_1)$ associated to X through (**Gam3**). Note that the equality (6.16) holds whenever $\mathfrak{Re}\,\lambda > \alpha+\gamma$. Moreover

$$(\Lambda_\omega^{\lambda,1} f)(z) = \int_0^\infty e^{-\lambda t} (S_\omega(t) f)(z)\, dt = \int_0^\infty e^{-\lambda t} u_t(z) (\varphi_t'(z))^{-\gamma} (C_{\varphi_t,\gamma} f)(z)\, dt, \quad z \in \mathbb{D},$$

with $\lim_{t\to\infty} \|C_{\varphi_t,\gamma} f\|_X^{1/t} \le 1$ by (**Gam5**). Since X is hyperbolically DW-contractive and $(u_t(\varphi_t')^{-\gamma})$ is a DW-continuous cocycle with exponents $\alpha+\gamma$, $\beta-\gamma$ (see Lemma 6.3.8 and Remark 6.3.9), it follows by condition (**SpC2**) that ,for $\varepsilon > 0$,

$$\|e^{-\lambda t} u_t(\varphi_t')^{-\gamma} C_{\varphi_t,\gamma} f\|_{X_1} \lesssim e^{-\mathfrak{Re}\,\lambda t} e^{\varepsilon t} |u_1(1)(\varphi_1'(1))^{-\gamma}|^t = e^{(-\mathfrak{Re}\,\lambda+\gamma+\alpha+\varepsilon)t}, \qquad t \ge 0.$$

Therefore, the integral $\int_0^\infty e^{-\lambda t} S_\omega(t) f\, dt$ is Bochner-convergent in the Banach space X_1, the equality $\Lambda_\omega^{\lambda,1} f = \int_0^\infty e^{-\lambda t} S_\omega(t) f\, dt \in X_1$ holds, and in particular $\Lambda_\omega^{\lambda,1} f \in X_1$.

Reasoning along similar lines, one obtains that $\Lambda_\omega^{\lambda,-1} f \in X_{-1}$. Hence $\Lambda_\omega^{\lambda,1} f \in X$ since $X = \mathcal{O}(\mathbb{D}) \cap X_{-1} \cap X_1$ (see condition (**Gam3**)), and the proof is finished. ☐

Remark 6.5.5. Under the conditions of Lemma 6.5.4, the kernel of the functional L_ω^λ is not the whole space X, i.e. $L_\omega^\lambda \neq 0$. Indeed, assume that $L_\omega^\lambda = 0$, and we will reach a contradiction.

Take a non-zero $f \in X$. Since $|\omega(z)| \lesssim |1-z|^{-\alpha+\varepsilon}|1+z|^{\beta+\varepsilon}$ (Theorem 6.3.11) and $f \in \mathcal{K}^{-\gamma-\varepsilon}(\mathbb{D})$ (Remark 6.2.4), one has that the function $(1-\cdot)^\lambda(1+\cdot)^{-\lambda}\omega f$ is a continuous function when restricted to the real interval $[-1,1]$. By the density of polynomials in $C([-1,1])$ (the set of continuous complex-valued functions on $[-1,1]$), it follows that the functional $L : C([-1,1]) \to \mathbb{C}$ given by $g \mapsto \int_{-1}^1 g(1-\cdot)^\lambda(1+\cdot)^{-\lambda}\omega f = L_\omega^\lambda(p(1-(\cdot)^2)f)$ is the zero functional, hence the function $(1-\cdot)^\lambda(1+\cdot)^{-\lambda}\omega f$ is the zero function, which is nonsense.

Remark 6.5.6. Under the conditions of Lemma 6.5.4, fix $f \in X$. Using a similar reasoning as in the beginning of the proof of Lemma 6.5.4, one obtains that the mapping from $\mathbb{D} \cup \{-1,1\}$ to $\mathbb{C}$ given by

$$z \mapsto \int_0^z \frac{(1-\xi)^{\lambda-1}}{(1+\xi)^{\lambda+1}}\, \omega(\xi) f(\xi)\, d\xi,$$

is continuous, whenever z approaches $-1, 1$ via non-tangential paths.

6.6 Spectra of the generator

Let X be a hyperbolically DW-contractive γ-space, $\gamma \geq 0$, and let $\mathfrak{S}$ be a subset of $\mathcal{O}(\mathbb{D})$ such that $(X, \mathfrak{S})$ is a γ-pair. Let $\omega \in \mathcal{O}(\mathbb{D})$ be non-vanishing, let $(S_\omega(t))$ be the weighted composition group defined as in Section 6.4 and let Δ_ω denote its infinitesimal generator. The aim of this section is to describe the fine structure of the spectrum of Δ_ω. For $c, d \in \mathbb{R}$, we set $|c, d| = \{z \in \mathbb{C} : \min\{c, d\} \leq z \leq \max\{c, d\}\}$.

Recall that, by Proposition 4.1.2, we have

$$\Delta_\omega(f) := \frac{\omega'}{\omega} Gf + Gf', \quad f \in \mathrm{Dom}(\Delta_\omega),$$

with $\mathrm{Dom}(\Delta_\omega) = \{f \in X : (\omega'/\omega)Gf + Gf' \in X\}$, and where G is the generator of the hyperbolic flow (φ_t).

The following lemma is a consequence of Propositions 6.4.2 and 6.5.3.

Lemma 6.6.1. *The spectrum $\sigma(\Delta_\omega)$ of the infinitesimal generator Δ_ω satisfies*

$$\sigma(\Delta_\omega) \subseteq |\beta - \gamma, \gamma + \alpha|.$$

Moreover,

$$(\lambda - \Delta_\omega)^{-1} f = \Lambda_\omega^{\lambda, \iota} f, \quad f \in X, \tag{6.17}$$

for $\iota = 1$ if $\lambda > \max\{\beta - \gamma, \gamma + \alpha\}$ and for $\iota = -1$ if $\mathfrak{Re}\,\lambda < \min\{\beta - \gamma, \gamma + \alpha\}$.

Proof. By the spectral mapping inclusion for C_0-semigroups (see e.g. [EN00, Th. IV.3.6]) we have $e^{t\sigma(\Delta_\omega)} \subseteq \sigma(S_\omega(t))$ for $t \in \mathbb{R}$. Also, $r(S_\omega(t)) \leq e^{\max\{(\beta-\gamma)t, (\gamma+\alpha)t\}}$ and $r(S_\omega(t)^{-1}) = r(S_\omega(-t)) \leq e^{\max\{-(\beta-\gamma)t, -(\alpha+\gamma)t\}}$ by Proposition 6.4.2. Hence we obtain $\sigma(\Delta_\omega) \subseteq |\beta - \gamma, \gamma + \alpha|$ as claimed.

Let now $\mathfrak{Re}\,\lambda > \max\{\beta - \gamma, \gamma + \alpha\}$. Using the integral representation of the resolvent operator of Δ_ω in terms of the semigroup $(S_\omega(t))_{t \geq 0}$ (see e.g. [EN00, Th. II.1.10]) and Proposition 6.5.3(i), one has

$$(\lambda - \Delta_\omega)^{-1} f = \int_0^\infty e^{-\lambda t} S_\omega(t) f \, dt = \Lambda_\omega^{\lambda, 1} f, \qquad f \in X, \, \mathfrak{Re}\,\lambda > \max\{\alpha + \gamma, \beta - \gamma\}$$

If $\mathfrak{Re}\,\lambda < \min\{\beta - \gamma, \gamma + \alpha\}$, it suffices to apply the integral representation of the resolvent of $-\Delta_\omega$ in terms of the semigroup $(S_\omega(-t))_{t \geq 0}$ and Proposition 6.5.3(ii) to obtain the result. $\square$

In the remainder of the section, we describe several spectral sets of Δ_ω. For a suitable understanding of the arguments we divide the overall proof in a series of results and remarks.

Proposition 6.6.2. *The point spectrum of the infinitesimal generator* Δ_ω *is given by*

$$\sigma_{point}(\Delta_\omega) = \{\lambda \in \mathbb{C} : g_\lambda \in X\}, \quad g_\lambda(z) := \frac{1}{\omega(z)} \left(\frac{1+z}{1-z}\right)^\lambda, \; z \in \mathbb{D}.$$

The eigenspace of each $\lambda \in \sigma_{point}(\Delta_\omega)$ *is one-dimensional and generated by* g_λ. *If in addition* $\omega^{-1} \in \mathfrak{S}$, *then* $\sigma_{point}(\Delta_\omega)$ *satisfies the following inclusions:*

$$\{\lambda \in \mathbb{C} : \beta - \gamma < \mathfrak{Re}\,\lambda < \alpha + \gamma\} \subseteq \sigma_{point}(\Delta_\omega) \subseteq \{\lambda \in \mathbb{C} : \beta - \gamma \leq \mathfrak{Re}\,\lambda \leq \alpha + \gamma\},$$

if $\beta - \alpha \leq 2\gamma$; *and* $\sigma_{point}(\Delta_\omega) = \emptyset$ *if* $\beta - \alpha > 2\gamma$.

Proof. The equality $\sigma_{point}(\Delta_\omega) = \{\lambda \in \mathbb{C} : g_\lambda \in X\}$ was given in Proposition 4.2.4 (a).

By Theorem 6.3.11, for every $\varepsilon > 0$ we have

$$|1-z|^{\alpha+\varepsilon}|1+z|^{-\beta+\varepsilon} \lesssim |\omega(z)|^{-1} \lesssim |1-z|^{\alpha-\varepsilon}|1+z|^{-\beta-\varepsilon}, \quad z \in \mathbb{D}.$$

Thus, for $\gamma' > \gamma$,

$$|1-z^2|^{\gamma'}|g_\lambda(z)| \gtrsim |1-z|^{\gamma'+\alpha+\varepsilon-\mathfrak{Re}\,\lambda}|1+z|^{\gamma'-\beta+\varepsilon+\mathfrak{Re}\,\lambda}, \quad z \in \mathbb{D}.$$

Hence $\sup_{z\in\mathbb{D}} |1-z^2|^{\gamma'}|g_\lambda(z)| = \infty$ for some $\gamma' > \gamma$, provided $\mathfrak{Re}\,\lambda < \beta - \gamma$ or $\mathfrak{Re}\,\lambda > \alpha + \gamma$. It follows that $g_\lambda \notin \mathcal{K}^{-\gamma'}(\mathbb{D})$ and therefore $g_\lambda \notin X$, see Remark 6.2.4. This implies the inclusion $\sigma_{point}(\Delta_\omega) \subseteq \{\lambda \in \mathbb{C} : \beta - \gamma \leq \mathfrak{Re}\,\lambda \leq \alpha + \gamma\}$.

Now, fix $\lambda \in \mathbb{C}$ with $\beta - \gamma < \mathfrak{Re}\,\lambda < \alpha + \gamma$. Then $g_\lambda \in \mathfrak{S}$ since, for any $\lambda \in \mathbb{C}$, $\mathfrak{S}$ is invariant by multiplication with the function $z \mapsto (1+z)^\lambda(1-z)^{-\lambda}$. Then Theorem 6.3.11 implies, for $\delta > 0$ small enough, that $|g_\lambda(z)| \lesssim |1-z^2|^{-\gamma+\delta}$, $z \in \mathbb{D}$. Therefore, $g_\lambda \in X$ by property (**Gam6**), so that $\lambda \in \sigma_{point}(\Delta_\omega)$. Thus $\{\lambda \in \mathbb{C} : \beta - \gamma < \mathfrak{Re}\,\lambda < \alpha + \gamma\} \subseteq \sigma_{point}(\Delta_\omega)$ as we wanted to prove. □

The assumption $\omega^{-1} \in \mathfrak{S}$ in Proposition 6.6.2 is superfluous when $X = H^p(\mathbb{D})$, $\mathcal{A}^p_\sigma(\mathbb{D})$, $\mathcal{K}_0^{-\gamma}(\mathbb{D})$ for $1 \leq p < \infty$, $\sigma > -1$ and $\gamma > 0$, since in any of these examples $\mathfrak{S}$ is the set $\mathcal{O}(\mathbb{D})$ of all holomorphic functions in the disc $\mathbb{D}$. The next result shows that such an assumption is also redundant for the disc algebra $\mathfrak{A}(\mathbb{D})$. We conjecture that there exist subsets $\mathfrak{S}(\mathcal{D}^p_\sigma)$, $\mathfrak{S}(B_{1,0})$ such that $(\mathcal{D}^p_\sigma(\mathbb{D}), \mathfrak{S}(\mathcal{D}^p_\sigma))$ and $(B_{1,0}(\mathbb{D}), \mathfrak{S}(B_{1,0}))$ are γ-pairs and the assumptions $\omega^{-1} \in \mathfrak{S}(\mathcal{D}^p_\sigma), \mathfrak{S}(B_{1,0})$ are redundant as well.

Proposition 6.6.3. *Let* (u_t) *be a DW-continuous cocycle for the flow* (φ_t) *on the disc algebra* $\mathfrak{A}(\mathbb{D})$ *with weight* ω, *i.e.* $u_t = (\omega \circ \varphi_t)/\omega$. *Then* $\omega^{-1} \in \mathfrak{S}(\mathfrak{A})$.

Proof. Recall that $\mathfrak{S}(\mathfrak{A})$ is the subset of functions of $\mathcal{O}(\mathbb{D})$ which can be continuously extended to $\overline{\mathbb{D}} \setminus \{-1, 1\}$.

First note that ω can be extended to almost every point of $\mathbb{T} \setminus \{-1, 1\}$ via non-tangential limits. Indeed, the holomorphic function $z \mapsto (1-z^2)^\lambda\omega(z)$ lies in $H^\infty(\mathbb{D})$ for $\lambda > 0$ big enough, thus $(1-(\cdot)^2)^\lambda\omega$ can be extended a.e. via non-tangential limits to $\mathbb{T}$ (see for instance [Hof62, p.38]), whence the same holds true for ω in $\mathbb{T} \setminus \{-1, 1\}$. Moreover, such non-tangential limits are never equal to 0 by Theorem 6.3.11.

We claim that such limits exist for every point in $\mathbb{T} \setminus \{-1, 1\}$. To see this, fix $v \in \mathbb{T} \setminus \{-1, 1\}$ with $\mathfrak{Im}\, v > 0$ such that the (non-tangential) limit $\lim_{z \to v} \omega(z)$ exists. Notice that $u_t = (\omega \circ \varphi_t)/\omega \in Mul(\mathfrak{A}(\mathbb{D})) = \mathfrak{A}(\mathbb{D})$ for each $t \in \mathbb{R}$. Since $\varphi_t \in Aut(\mathbb{D})$, it follows from $\omega \circ \varphi_t = u_t \omega$ that the limit $\lim_{z \to \varphi_t(v)} \omega(z)$ exists, that is, ω has non-tangential limits at $\{\varphi_t(v) : t \in \mathbb{R}\} = \{z \in \mathbb{T} : \mathfrak{Im}\, z > 0\}$. After repeating the argument with $v \in \mathbb{T}$ such that $\mathfrak{Im}\, v < 0$, we obtain that ω has non-tangential limits at every point in $\mathbb{T} \setminus \{-1, 1\}$.

Now we show that the extension of ω to $\mathbb{D} \setminus \{-1, 1\}$ via non-tangential limits is continuous when restricted to $\mathbb{T} \setminus \{-1, 1\}$. Note that the mapping $t \in \mathbb{R} \mapsto u_t = S_\omega(t)\mathbf{1} \in \mathfrak{A}(\mathbb{D})$ is continuous, where $\mathbf{1}$ denotes the constant function $\mathbf{1}(z) = 1$. As a consequence, the mapping $t \mapsto u_t(v)$ is continuous for any $v \in \overline{\mathbb{D}}$. Hence the mapping $t \mapsto \omega(\varphi_t(v)) = u_t(v)\omega(v)$ is also continuous. Note also that $t \mapsto \varphi_t(v)$ is a homeomorphism from $\mathbb{R}$ to $\{z \in \mathbb{T} : \operatorname{sgn} \mathfrak{Im}\, z = \operatorname{sgn} \mathfrak{Im}\, v\}$ for every $v \in \mathbb{T} \setminus \{-1, 1\}$. Thus ω is continuous on $\mathbb{T} \setminus \{-1, 1\}$.

Taking λ as at the beginning of the proof, we obtain that the function $(1 - (\cdot)^2)^\lambda \omega$ is holomorphic and bounded on $\mathbb{D}$, and that it can be extended to every point in $\overline{\mathbb{D}}$ via non-tangential limits, being such an extension continuous when restricted to the boundary $\mathbb{T}$. Using the Poisson kernel, one gets $(1 - (\cdot)^2)^\lambda \omega \in \mathfrak{A}(\mathbb{D})$. Since ω has no zeros in $\overline{\mathbb{D}} \setminus \{-1, 1\}$, we conclude that $\omega^{-1} \in \mathfrak{S}(\mathfrak{A})$ and the proof is finished. □

Remark 6.6.4. We now study the range space of the operator $\lambda - \Delta_\omega : \operatorname{Dom}(\Delta_\omega) \to X$ for a fixed $\lambda \in \mathbb{C}$. To begin with, a few computations show that all the solutions $(g_{f,K})_{K \in \mathbb{C}} \in \mathcal{O}(\mathbb{D})$ of the differential equation $(\lambda - G\omega'/\omega)g - Gg' = f$, $f \in \mathcal{O}(\mathbb{D})$, are given by

$$g_{f,K}(z) = \frac{1}{\omega(z)} \frac{(1+z)^\lambda}{(1-z)^\lambda} \left(K - 2 \int_0^z \frac{(1-\xi)^{\lambda-1}}{(1+\xi)^{\lambda+1}} \omega(\xi) f(\xi)\, d\xi \right), \quad z \in \mathbb{D}, K \in \mathbb{C}. \tag{6.18}$$

Thus, we have by Proposition 4.1.2 that a function $f \in X$ lies in the range of $\lambda - \Delta_\omega$ if and only if there exists some $K \in \mathbb{C}$ such that the function $g_{f,K}$ given in (6.18) belongs to X. Indeed, if this is the case, then $g_{f,K} \in \operatorname{Dom}(\Delta_\omega)$ and $(\lambda - \Delta_\omega) g_{f,K} = f$.

The lemma below gives the range space $\operatorname{Ran}(\lambda - \Delta_\omega)$ when $\beta - \alpha \neq 2\gamma$. Notice that, by Lemma 6.6.1, $\lambda - \Delta_\omega$ is a surjective (moreover, invertible) operator whenever $\lambda \notin |\beta - \gamma, \alpha + \gamma|$.

Lemma 6.6.5. *Let $\lambda \in \mathbb{C}$. We have*

$$\operatorname{Ran}(\lambda - \Delta_\omega) = \begin{cases} X, & \text{if } \beta - \alpha < 2\gamma, \text{ and } \beta - \gamma < \mathfrak{Re}\, \lambda < \alpha + \gamma, \\ \ker L_\omega^\lambda \subsetneq X, & \text{if } \beta - \alpha > 2\gamma, \text{ and } \alpha + \gamma < \mathfrak{Re}\, \lambda < \beta - \gamma. \end{cases}$$

Proof. Assume first $\beta - \alpha < 2\gamma$ and $\beta - \gamma < \mathfrak{Re}\, \lambda < \alpha + \gamma$. Let $m \in \mathbb{N}$ be such that $m \geq 2(2\gamma + \alpha - \beta)$. For $f \in X$, set

$$f_j(z) := 2^{-m} \binom{m}{j} (1-z)^j (1+z)^{m-j} f(z), \quad z \in \mathbb{D}, 0 \leq j \leq m. \tag{6.19}$$

Notice that $(1+\imath z)^{\delta} \in \mathcal{P} \subseteq Mul(X)$ for all $\delta \geq 0$ by (**Gam2**), so $f_j \in X$ for all $0 \leq j \leq m$. Moreover, $f_j \in \mathfrak{X}_{-1}^{\beta-\alpha-2\gamma}$ if $j \leq m/2$, and $f_j \in \mathfrak{X}_{1}^{\beta-\alpha-2\gamma}$ otherwise. It follows from Proposition 6.5.3(iii) and (iv) that $\Lambda_{\omega}^{\lambda,c_j} f_j \in X$ for all j, where $c_j = -1$ if $j \leq m/2$ and $c_j = 1$ otherwise. Set

$$K_j := -2\int_{c_j}^{0} \frac{(1-\xi)^{\lambda-1}}{(1+\xi)^{\lambda+1}}\omega(\xi) f_j(\xi)\,d\xi, \qquad 0 \leq j \leq m.$$

Corollary 6.5.2 shows that the complex numbers K_j, $0 \leq j \leq m$, are well defined and that $g_{f_j,K_j} = \Lambda_{\omega}^{\lambda,c_j} f_j \in X$ for all $0 \leq j \leq m$. Hence, by Remark 6.6.4 we have $g_{f_j,K_j} = \Lambda_{\omega}^{\lambda,c_j} f_j \in \mathrm{Dom}(\Delta_\omega)$ and $(\lambda-\Delta_\omega) g_{f_j,K_j} = f_j$, that is $f_j \in \mathrm{Ran}(\lambda-\Delta_\omega)$ for all $0 \leq j \leq m$. Since $f = \sum_{j=0}^{m} f_j$, it follows that $f \in \mathrm{Ran}(\lambda-\Delta_\omega)$ and we conclude that $\mathrm{Ran}(\lambda-\Delta_\omega) = X$.

Assume now $\beta-\alpha > 2\gamma$ and $\alpha+\gamma < \lambda < \beta-\gamma$. By Lemma 6.5.4, L_{ω}^{λ} is a continuous functional on X, and $\Lambda_{\omega}^{\lambda,1} f = \Lambda_{\omega}^{\lambda,-1} f \in X$ if $f \in \ker L_{\omega}^{\lambda}$. By Lemma 6.5.1,

$$K := -2\int_{-1}^{0} \frac{(1-\xi)^{\lambda-1}}{(1+\xi)^{\lambda+1}}\omega(\xi) f(\xi)\,d\xi$$

is well defined and $g_{f,K} = \Lambda_{\omega}^{\lambda,-1} f \in X$. By Remark 6.6.4, $g_{f,K} \in \mathrm{Dom}(\Delta_\omega)$ and $(\lambda-\Delta_\omega) g_{f,K} = f$, so that $f \in \mathrm{Ran}(\lambda-\Delta_\omega)$. Thus, $\ker L_{\omega}^{\lambda} \subseteq \mathrm{Ran}(\lambda-\Delta_\omega)$.

Let now $f \in X \setminus \ker L_{\omega}^{\lambda}$. The mapping $z \mapsto \int_0^z (1-\xi)^{\lambda-1}(1+\xi)^{-\lambda-1}\omega(\xi) f(\xi)\,d\xi$ is continuous from $\mathbb{D} \cup \{-1,1\}$ to $\mathbb{C}$, see Remark 6.5.6. Hence, for all $K \in \mathbb{C}$,

$$\begin{aligned} &K - 2\int_{0}^{z} \frac{(1-\xi)^{\lambda-1}}{(1-\xi)^{\lambda+1}}\omega(\xi) f(\xi)\,d\xi \xrightarrow[z\to 1]{} c_{K,+} \in \mathbb{C}, \\ &K - 2\int_{0}^{z} \frac{(1-\xi)^{\lambda-1}}{(1-\xi)^{\lambda+1}}\omega(\xi) f(\xi)\,d\xi \xrightarrow[z\to -1]{} c_{K,-} \in \mathbb{C}, \end{aligned} \tag{6.20}$$

whenever $z \to -1, 1$ non-tangentially. Since $c_{K,+} - c_{K,-} = -2L_{\omega}^{\lambda} f \neq 0$, either $c_{K,+} \neq 0$ or $c_{K,-} \neq 0$. By Theorem 6.3.11, we have $|\omega(z)^{-1}| \gtrsim |1-z|^{\alpha+\varepsilon}|1+z|^{-\beta+\varepsilon}$, for all $z \in \mathbb{D}$, $\varepsilon > 0$. Applying this bound in (6.18) one gets that either $|g_{f,K}(x)| \gtrsim |1-x|^{-\gamma'}$ as $x \to 1$ through $(0,1)$ or $|g_{f,K}(x)| \gtrsim |1+x|^{-\gamma'}$, as $x \to -1$ through $(-1,0)$, for some $\gamma' > \gamma$ and some $K \in \mathbb{C}$. In any case, there exists $\delta > 0$ such that $g_{f,K} \notin \mathcal{K}^{-\gamma-\delta}(\mathbb{D})$, and therefore $g_{f,K} \notin X$, see Remark 6.2.4. As a consequence, $f \notin \mathrm{Ran}(\lambda-\Delta_\omega)$ by Remark 6.6.4. Thus $\mathrm{Ran}(\lambda-\Delta_\omega) \subseteq \ker L_{\omega}^{\lambda}$, and the proof is finished. □

The following theorem gives the spectrum of the generator Δ_ω.

Theorem 6.6.6. *Let X be a hyperbolically DW-contractive γ-space. Let ω be a weight, with exponents α and β, such that $\omega^{-1} \in \mathfrak{S}$. Then*

$$\sigma(\Delta_\omega) = |\beta-\gamma, \alpha+\gamma|.$$

Proof. First assume $\beta - \alpha \neq 2\gamma$. The inclusion $\sigma(\Delta_\omega) \subseteq |\beta-\gamma, \alpha+\gamma|$ is in Lemma 6.6.1. On the other hand, $\mathrm{Int}(|\beta-\gamma,\alpha+\gamma|) \subseteq \sigma(\Delta_\omega)$ by Proposition 6.6.2 in the case $\beta-\alpha<2\gamma$, and by Lemma 6.6.5 if $\beta-\alpha>2\gamma$. Therefore $\sigma(\Delta_\omega) = |\beta-\gamma,\alpha+\gamma|$ since the spectrum of a closed operator is a closed subset of $\mathbb{C}$.

In the case $\beta-\alpha = 2\gamma$ one cannot directly use the results obtained in Section 6.5. Instead, we use the invariance of ω in the sense of Lemma 6.3.8 to slightly modify the exponents α, β and then take advantage of what has been already proved for $\beta-\alpha\neq 2\gamma$.

As above, one has $\sigma(\Delta_\omega) \subseteq |\beta-\gamma,\alpha+\gamma|$ by Lemma 6.6.1. To prove the reverse inclusion, take $\lambda \in \mathbb{C}$ in $|\alpha+\gamma,\beta-\gamma|$, which is to say $\mathfrak{Re}\,\lambda = \alpha+\gamma = \beta-\gamma$. Recall

$$g_\lambda(z) = \omega(z)^{-1}(1+z)^{\lambda}(1-z)^{-\lambda}, \quad z \in \mathbb{D}.$$

If $g_\lambda \in X$ then $\lambda \in \sigma_{point}(\Delta_\omega)$ by Proposition 6.6.2, and we are done. Thus we assume $g_\lambda \notin X$ and $\lambda - \Delta_\omega$ injective. Under this assumption we show next, by contradiction, that $\lambda-\Delta_\omega$ is not surjective, whence $\lambda \in \sigma(\Delta_\omega)$ and the proof will be finished.

Thus suppose that $\lambda - \Delta_\omega$ is a surjective operator. As noticed in Remark 6.6.4, this implies that, for every $f \in X$, there exists $K\in\mathbb{C}$ for which the function $g_{f,K}$ in (6.18) lies in X. Since $g_\lambda \in \mathcal{O}(\mathbb{D})\setminus X$, we have that either $g_\lambda \notin X_1$ or $g_\lambda \notin X_{-1}$ (meaning that the restriction of g_λ to $\mathbb{D}_1$ or $\mathbb{D}_{-1}$ is not in X_1 or X_{-1} respectively), where X_{-1}, X_1 are the Banach spaces given in (**Gam3**).

Suppose $g_\lambda \notin X_{-1}$ without loss of generality. For $c>0$, set $\omega_c(z) := \omega(z)/(1-z)^c$, $z\in\mathbb{D}$, and $v_t := (\omega_c \circ \varphi_t)/\omega_c$, $t \in \mathbb{R}$. Then (v_t) is a DW-continuous cocycle for the flow (φ_t) on X with exponents $\alpha_c = \alpha + c > \alpha$ and $\beta_c = \beta$, see Lemma 6.3.8. In particular $\beta_c - \alpha_c = 2\gamma - c < 2\gamma$, so we conclude that $\sigma(\Delta_{\omega_c}) = |\beta_c-\gamma,\alpha_c+\gamma| = |\beta-\gamma,\alpha+\gamma+c|$ by the first part of this proof. In particular, $\lambda\in\sigma(\Delta_{\omega_c})$ and so $\lambda-\Delta_{\omega_c}$ is either not injective or not surjective.

If $\lambda - \Delta_{\omega_c}$ is not injective Lemma 6.6.2 implies that the holomorphic function $g_\lambda(1-(\cdot))^c$ is in X, and therefore its restriction to $\mathbb{D}_{-1}$ is in X_{-1}. However, the function $(1-(\cdot))^{-c}$ is in $Mul(X_{-1})$ since it is holomorphic in an open set containing $\overline{\mathbb{D}_{-1}}$, see (**Gam3**). Hence we have $g_\lambda \in X_{-1}$, which is a contradiction since we have assumed the opposite. Therefore $\lambda-\Delta_{\omega_c}$ must be an injective operator, which implies in turn that $\lambda-\Delta_{\omega_c}$ is not surjective by the preceding paragraph. However, we shall show next that $\lambda-\Delta_{\omega_c}$ is also surjective, reaching again a contradiction:

By a similar trick as after (6.19), one gets $X = \mathfrak{X}_{-1}^{-c} + \mathfrak{X}_{1}^{-c}$, and then it is enough to show that $\mathfrak{X}_{-1}^{-c}$ and $\mathfrak{X}_{1}^{-c}$ are subspaces of $\mathrm{Ran}(\lambda-\Delta_{\omega_c})$. Take $f\in\mathfrak{X}_{-1}^{-c} = \mathfrak{X}_{-1}^{\beta_c-\alpha_c-2\gamma}$. By Proposition 6.5.3(iv), $\Lambda_{\omega_c}^{\lambda,-1} f \in X$ and, as in the proof of Lemma 6.6.5, case $\beta-\alpha<2\gamma$, one obtains $(\lambda-\Delta_{\omega_c})\Lambda_{\omega_c}^{\lambda,-1} f = f$. Thus $f\in\mathrm{Ran}(\lambda-\Delta_{\omega_c})$ and then it follows that $\mathfrak{X}_{-1}^{-c}\subseteq\mathrm{Ran}(\lambda-\Delta_{\omega_c})$. Take now $f\in\mathfrak{X}_{1}^{-c}$ and define $f_c \in X$ by $f_c(z) = (1-z)^{-c}f(z)$ for $z\in\mathbb{D}$. There exists $K\in\mathbb{C}$ such that $g_{f_c,K}\in X$, see Remark 6.6.4. Since $(1-(\cdot))^c \in \mathcal{P}\subseteq Mul(X)$ by (**Gam2**) one has $(1-(\cdot))^c g_{f_c,K}\in X$. Using again Lemma Remark 6.6.4 with the weight ω_c instead ω one gets $f\in\mathrm{Ran}(\lambda-\Delta_{\omega_c})$. So $\mathfrak{X}_1^c\subseteq\mathrm{Ran}(\lambda-\Delta_{\omega_c})$.

Therefore, $\lambda-\Delta_{\omega_c}$ is surjective, hence invertible, reaching the forecasted contradiction since $\lambda\in\sigma(\Delta_{\omega_c})$. We finally conclude that our assumption $\lambda\notin\sigma(\Delta_\omega)$ is incorrect, and the proof is finished. □

The overall discussion carried out in preceding places of this chapter leads to the following detailed description of $\sigma(\Delta_\omega)$. Recall that the approximate spectrum and residual spectrum of a closed operator A are denoted by $\sigma_{ap}(A)$ and $\sigma_{res}(A)$ respectively.

Theorem 6.6.7. *Let $\gamma \geq 0$ and let X be a γ-space which is hyperbolically DW-contractive, and let $\mathfrak{S}$ be such that $(X, \mathfrak{S})$ is a γ-pair. Let (u_t) be a hyperbolically DW-continuous cocycle for (φ_t), so that $(u_t C_{\varphi_t})$ is a C_0-group in $L(X)$. Let α, β be the exponents of (u_t), and let ω be a non-vanishing holomorphic function associated with (u_t). Let Δ_ω be the infinitesimal generator of $(S_\omega(t)) := (u_t C_{\varphi_t})$. Assume $\omega^{-1} \in \mathfrak{S}$ Then one has the following.*

i) The full spectrum $\sigma(\Delta_\omega)$ of Δ_ω is the strip $|\alpha+\gamma, \beta-\gamma|$.

ii) The essential spectrum of Δ_ω is the boundary of $\sigma(\Delta_\omega)$, that is,

$$\sigma_{ess}(\Delta_\omega) = \partial(|\alpha+\gamma, \beta-\gamma|).$$

iii) The approximate spectrum of Δ_ω is given by

$$\sigma_{ap}(\Delta_\omega) = \begin{cases} |\alpha+\gamma, \beta-\gamma|, & \text{if} \quad \beta-\alpha \leq 2\gamma; \\ \partial(|\alpha+\gamma, \beta-\gamma|), & \text{if} \quad \beta-\alpha > 2\gamma. \end{cases}$$

iv) The point spectrum $\sigma_{point}(\Delta_\omega)$ of Δ_ω satisfies

$$\{\lambda \in \mathbb{C} : \beta-\gamma < \mathfrak{Re}\,\lambda < \alpha+\gamma\} \subseteq \sigma_{point}(\Delta_\omega) \subseteq \{\lambda \in \mathbb{C} : \beta-\gamma \leq \mathfrak{Re}\,\lambda \leq \alpha+\gamma\}.$$

The eigenspace of $\lambda \in \sigma_{point}(\Delta_\omega)$ is the one-dimensional subspace $\mathbb{C}g_\lambda$.

v) The residual spectrum $\sigma_{res}(\Delta_\omega)$ of Δ_ω on X satisfies

$$\{\lambda \in \mathbb{C} : \alpha+\gamma < \mathfrak{Re}\,\lambda < \beta-\gamma\} \subseteq \sigma_{res}(\Delta_\omega), \quad \beta-\alpha > 2\gamma;$$
$$\sigma_{res}(\Delta_\omega) \subseteq \{\lambda \in \mathbb{C} : \mathfrak{Re}\,\lambda = \alpha+\gamma \text{ or } \mathfrak{Re}\,\lambda = \beta-\gamma\}, \quad \beta-\alpha \leq 2\gamma.$$

Proof. i) This is Theorem 6.6.6.

ii) Let $\lambda \in \sigma(\Delta_\omega) = |\alpha+\gamma, \beta-\gamma|$. The kernel of $\lambda - \Delta_\omega$ is at most one-dimensional by Proposition 6.6.2, so $\dim(\ker(\lambda-\Delta_\omega)) < \infty$. In addition, if $\lambda \in \operatorname{Int}(|\alpha+\gamma, \beta-\gamma|)$, then $\dim(X/\operatorname{Ran}(\lambda-\Delta_\omega)) \leq 1 < \infty$ by Lemma 6.6.5, so we conclude that $\operatorname{Int}(|\alpha+\gamma, \beta-\gamma|) \cap \sigma_{ess}(\Delta_\omega) = \emptyset$.

Now let $\lambda \in \partial(|\alpha+\gamma, \beta-\gamma|)$. By item i), λ is an accumulation point of both the resolvent set $\rho(\Delta_\omega)$ and the spectrum $\sigma(\Delta_\omega)$. As a consequence, $\lambda \in \sigma_{ess}(\Delta_\omega)$, see for example [EE87, Th. I.3.25].

iii) First, the inclusion $\partial\sigma(A) \subseteq \sigma_{ap}(A)$ holds for any closed operator A, see for example [EN00, p. IV.1.10]. Now take an arbitrary $\lambda \in \operatorname{Int}(\sigma(\Delta_\omega)) = \operatorname{Int}(|\beta-\gamma, \alpha+\gamma|)$. Then $\operatorname{Ran}(\lambda-\Delta_\omega)$ is a closed subspace by Lemma 6.6.5, and $\lambda-\Delta_\omega$ is not injective if and only if $\beta-\alpha < 2\gamma$, see Proposition 6.6.2.

iv) This is Proposition 6.6.2.

v) This is a direct consequence of Lemma 6.6.5.

□

Remark 6.6.8. (1) From item i) in the theorem above, (6.17) gives the resolvent $(\lambda - \Delta_\omega)^{-1}$ for all $\lambda \in \rho(\Delta_\omega)$.

(2) For Hardy spaces, weighted Bergman spaces, Little Korenblum spaces, and the disc algebra, condition $\omega^{-1} \in \mathfrak{S}$ in Theorem 6.6.7 is superfluous, in view of Proposition 6.6.3 and the comment prior to Proposition 6.6.3.

6.7 Spectra of weighted hyperbolic composition groups

Let $\omega, (S_\omega(t))$ be as in Section 6.4. The spectral analysis of the infinitesimal generator Δ_ω of $(S_\omega(t))$ given in Theorem 6.6.7 is here transferred to the group $(S_\omega(t))$.

Theorem 6.7.1. *Let X, $\mathfrak{S}$ and $S_\omega(t)$ be as in Theorem 6.6.7. Let $t \in \mathbb{R}$. Then*

i) The full spectrum of $S_\omega(t)$ is the annulus

$$\sigma(S_\omega(t)) = \{\lambda \in \mathbb{C} : e^{\min\{(\alpha+\gamma)t,(\beta-\gamma)t\}} \le |\lambda| \le e^{\max\{(\alpha+\gamma)t,(\beta-\gamma)t\}}\}.$$

ii) The essential spectrum of $S_\omega(t)$ coincides with the full spectrum, i.e.

$$\sigma_{ess}(S_\omega(t)) = \sigma(S_\omega(t)).$$

iii) The point spectrum $\sigma_{point}(S_\omega(t))$ of $S_\omega(t)$ satisfies

$$\{\lambda : e^{(\beta-\gamma)t} < |\lambda| < e^{(\alpha+\gamma)t}\} \subseteq \sigma_{point}(S_\omega(t)) \subseteq \{\lambda : e^{(\beta-\gamma)t} \le |\lambda| \le e^{(\alpha+\gamma)t}\}.$$

Moreover, the eigenspace of λ is:

$$\overline{\mathrm{span}}\{g_\mu : \mu \in W_\lambda\}, \text{ if } \lambda \in \mathrm{Int}(\sigma_{point}(S_\omega(t)))$$

and

$$\overline{\mathrm{span}}\{g_\mu : \mu \in W_\lambda \text{ and } g_\mu \in X\} \text{ if } \lambda \in \partial(\sigma_{point}(S_\omega(t))),$$

where $W_\lambda = \{\mu \in \mathbb{C} : e^{\mu t} = \lambda\}$.

iv) The residual spectrum $\sigma_{res}(S_\omega(t))$ of $S_\omega(t)$ on X satisfies

$$\{\lambda : e^{(\alpha+\gamma)t} < |\lambda| < e^{(\beta-\gamma)t}\} \subseteq \sigma_{res}(S_\omega(t)), \quad \text{if } \beta - \alpha > 2\gamma;$$
$$\sigma_{res}(S_\omega(t)) \subseteq \{\lambda : |\lambda| = e^{(\alpha+\gamma)t} \text{ or } |\lambda| = e^{(\beta-\gamma)t}\}, \quad \text{if } \beta - \alpha \le 2\gamma.$$

If $\lambda \in \mathrm{Int}(\sigma_{\mathrm{res}}(\mathrm{S}_\omega(\mathrm{t})))$ then $\mathrm{Ran}(\lambda - S_\omega(t)) \subseteq \bigcap_{\mu \in W_\lambda} \ker L_\omega^\mu$.

Proof. i) We have $e^{t\sigma(\Delta_\omega)} \subseteq \sigma(S_\omega(t))$ for any $t \in \mathbb{R}$ by the spectral mapping inclusion for C_0-semigroups, see [EN00, p. IV.3.6]. Thus the inclusion $\supseteq$ of the statement follows from Theorem 6.6.7. The reverse inclusion $\subseteq$ follows from the spectral radius theorem together with the asymptotic bounds for $\|S_\omega(t)\|_{L(X)}$ given in Proposition 6.4.2.

ii) By item i), we have to prove $\sigma_{ess}(S_\omega(t)) = \sigma(S_\omega(t))$. If $\lambda \in \partial(\sigma(S_\omega(t)))$, then item i) shows that λ is an accumulation point of both the resolvent set $\rho(S_\omega(t))$ and the spectrum $\sigma(S_\omega(t))$. As a consequence, $\lambda \in \sigma_{ess}(S_\omega(t))$, see [EE87, Th. I.3.25].

Now let $\lambda \in \operatorname{Int}(\sigma(S_\omega(t)))$. One can assume $\beta - \alpha \neq 2\gamma$ since otherwise one has $\operatorname{Int}(\sigma(S_\omega(t))) = \emptyset$ by item i). If $\beta - \alpha < 2\gamma$ then $\dim(\ker(\lambda - S_\omega(t))) = \infty$, as we see in the proof of item iii), so $\lambda \in \sigma_{ess}(S_\omega(t))$. On the other hand, if $\beta - \alpha > 2\gamma$, then

$$(6.21) \qquad \operatorname{Ran}(\lambda - S_\omega(t)) \subseteq \cap_{\mu \in W_\lambda} \operatorname{Ran}(\mu - \Delta_\omega) = \cap_{\mu \in W_\lambda} \ker L_\omega^\mu,$$

by [EN00, Equation (IV.3.14)] and Lemma 6.6.5.

Moreover, $\{L_\omega^\mu\}$ is linearly independent in the dual space of X since L_ω^μ is an eigenvector associated to the eigenvalue μ of the adjoint operator of Δ_ω, see Lemma 6.6.5. Therefore the subspace $\cap_{\mu \in W_\lambda} \ker L_\omega^\mu$ has infinite codimension [Rud91, Lemma 3.9], and we conclude that $\lambda \in \sigma_{ess}(S_\omega(t))$, as we wanted to prove.

This proves the claim made at iv) about $\operatorname{Ran}(\lambda - S_\omega(t))$ since $\operatorname{Ran}(\mu - \Delta_\omega) = \ker L_\omega^\mu$ for all $\mu \in W_\lambda$ by Lemma 6.6.5.

iii) & iv) We have $\sigma_{point}(S_\omega(t)) = e^{t\sigma_{point}(\Delta_\omega)}$ and $\sigma_{res}(S_\omega(t)) = e^{t\sigma_{res}(\Delta_\omega)}$, $t \in \mathbb{R}$, see for instance [EN00, Th. IV.3.7]. Thus the given inclusions for the respective spectra are immediate consequences of Theorem 6.6.7. The claim about the eigenspaces follows from the fact that the kernel of $\lambda - S_\omega(t)$ is the closure of the linear span of the eigenspaces of $\mu - \Delta_\omega$, where $\mu \in W_\lambda$, see e.g. [EN00, Cor. IV.3.8]. The claim made about $\operatorname{Ran}(\lambda - S_\omega(t))$ follows from (6.21).

□

As a consequence of Theorem 6.7.1, one obtains the fine spectrum of weighted composition groups of the form $(v_t C_{\psi_t})$ where (ψ_t) is an arbitrary hyperbolic flow.

Theorem 6.7.2. *Let $(X, \mathfrak{S})$ be a γ-pair with $\gamma \geq 0$ such that X is hyperbolically DW-contractive. Let (ψ_t) be a hyperbolic flow with DW points a (attractive), b (repulsive) $\in \mathbb{T}$, and let (v_t) be a DW-continuous cocycle for (ψ_t) on X. Let ϖ be a non-vanishing holomorphic function associated with (v_t) and assume $\varpi^{-1} \in C_\phi(\mathfrak{S})$, where $\phi \in Aut(\mathbb{D})$ is such that $\phi(a) = 1$, $\phi(b) = -1$. Then, for $t \in \mathbb{R}$,*

i) The full spectrum of $v_t C_{\psi_t}$ is the set

$$\sigma(v_t C_{\psi_t}) = \left\{ \lambda \in \mathbb{C} \,:\, \min\left\{ \frac{|v_t(a)|}{\psi_t'(a)^\gamma}, \frac{|v_t(b)|}{\psi_t'(b)^\gamma} \right\} \leq |\lambda| \leq \max\left\{ \frac{|v_t(a)|}{\psi_t'(a)^\gamma}, \frac{|v_t(b)|}{\psi_t'(b)^\gamma} \right\} \right\}.$$

ii) The essential spectrum of $v_tC_{\psi_t}$ coincides with its full spectrum, that is,

$$\sigma_{ess}(v_tC_{\psi_t}) = \sigma(v_tC_{\psi_t}).$$

iii) The point spectrum of $v_tC_{\psi_t}$ satisfies

$$\left\{\lambda \in \mathbb{C} \,:\, \frac{|v_t(b)|}{\psi_t'(b)^\gamma} < |\lambda| < \frac{|v_t(a)|}{\psi_t'(a)^\gamma}\right\} \subseteq \sigma_{point}(v_tC_{\psi_t}) \subseteq \left\{\lambda \in \mathbb{C} \,:\, \frac{|v_t(b)|}{\psi_t'(b)^\gamma} \le |\lambda| \le \frac{|v_t(a)|}{\psi_t'(a)^\gamma}\right\}.$$

Moreover, the eigenspace of λ is:

$$\overline{\operatorname{span}}\{\widetilde{g}_\mu \,:\, \mu \in \widetilde{W}_\lambda\} \text{ if } \lambda \in \operatorname{Int}(\sigma_{point}(v_tC_{\psi_t})),$$

where $\widetilde{g}_\mu(z) := \frac{1}{\varpi(z)}\frac{(b-z)^\mu}{(a-z)^\mu}$, $z \in \mathbb{D}$, and

$$\overline{\operatorname{span}}\{\widetilde{g}_\mu \,:\, \mu \in \widetilde{W}_\lambda \text{ and } \widetilde{g}_\mu \in X\} \text{ if } \lambda \in \partial(\sigma_{point}(v_tC_{\psi_t})),$$

where $\widetilde{W}_\lambda = \{\mu \in \mathbb{C} \,:\, \psi_t'(a)^\mu = \lambda^{-1}\}$.

iv) The residual spectrum of $v_tC_{\psi_t}$ satisfies

$$\left\{\lambda \in \mathbb{C} \,:\, \frac{|v_t(a)|}{\psi_t'(a)^\gamma} < |\lambda| < \frac{|v_t(b)|}{\psi_t'(b)^\gamma}\right\} \subseteq \sigma_{res}(v_tC_{\psi_t}), \quad \text{if } \frac{|v_t(a)|}{\psi_t'(a)^\gamma} < \frac{|v_t(b)|}{\psi_t'(b)^\gamma};$$
$$\sigma_{res}(v_tC_{\psi_t}) \subseteq \left\{\lambda \in \mathbb{C} \,:\, |\lambda| = \frac{|v_t(a)|}{\psi_t'(a)^\gamma} \text{ or } |\lambda| \le \frac{|v_t(b)|}{\psi_t'(b)^\gamma}\right\}, \quad \text{if } \frac{|v_t(a)|}{\psi_t'(a)^\gamma} \le \frac{|v_t(b)|}{\psi_t'(b)^\gamma}.$$

If $\lambda \in \operatorname{Int}(\sigma_{res}(v_tC_{\psi_t}))$, then $\operatorname{Ran}(\lambda - v_tC_{\psi_t}) \subseteq \cap_{\mu\in\widetilde{W}_\lambda} \ker \tilde{L}^\mu_\varpi$, where $\tilde{L}^\mu_\varpi : X \to \mathbb{C}$ is the continuous functional on X given by

$$(6.22) \qquad \tilde{L}^\mu_\varpi f = \int_b^a \frac{(a-\xi)^{\mu-1}}{(b-\xi)^{\mu+1}}\varpi(\xi)f(\xi)\,d\xi, \qquad f \in X.$$

Here, we can take any simple integration path in $\mathbb{D}$ from b to a such that approaches both b, a non-tangentially.

Proof. There is $c > 0$ such that $v_tC_{\psi_t} = C_\phi(u_{ct}C_{\varphi_{ct}})C_{\phi^{-1}}$, where $(u_t) := (v_{c^{-1}t} \circ \phi^{-1})$, $t \in \mathbb{R}$ is a DW-continuous cocycle for (φ_t), see the end of Section 6.1. Therefore, it is enough to obtain the spectral sets for the operator $u_{ct}C_{\varphi_{ct}}$.

It is readily seen that $u_t = ((\varpi \circ \phi^{-1}) \circ \varphi_t)/(\varpi \circ \phi^{-1})$. Hence $u_tC_{\varphi_t} = S_\omega(t)$, $t \in \mathbb{R}$, in the notation of Section 6.6, where $\omega := \varpi \circ \phi^{-1}$. Thus $\omega^{-1} \in \mathfrak{S}$ and we have that the hypotheses of Theorem 6.7.1 are satisfied.

Therefore we can apply Theorem 6.7.1 to $S_\omega(ct)$. So all all we have to prove is $e^{(\alpha+\gamma)ct} = |v_t(a)|\psi_t'(a)^{-\gamma}$ and $e^{(\beta-\gamma)ct} = |v_t(b)|\psi_t'(b)^{-\gamma}$, where α, β are the exponents of the DW-continuous cocycle (u_t), see Lemma 6.3.5. From here, our claims regarding the spectra of $u_{ct}S_\omega(ct)$ follow immediately. Let us see.

On the one hand, $e^{\alpha ct} = e^{\alpha(ct)} = \lim_{z\to 1}|u_{ct}(z)| = \lim_{z\to a}|v_t(z)| = |v_t(a)|$. On the other hand, $\psi_t'(a) = (\phi^{-1}\circ\varphi_{ct}\circ\phi)'(a) = \varphi_{ct}'(1) = e^{-ct}$, $t\in\mathbb{R}$, and then $e^{c\gamma t} = \psi_t'(a)^{-\gamma}$. Thus $e^{(\alpha+\gamma)ct} = |v_t(a)||\psi_t'(a)^{-\gamma}$. The identity $e^{(\beta-\gamma)ct} = |v_t(b)||\psi_t'(b)^{-\gamma}$ can be obtained analogously.

Now we prove the claim made on the eigenspaces of $v_tC_{\psi_t}$. Let $\lambda\in\operatorname{Int}(\sigma_{point}(v_tC_{\psi_t})) = \operatorname{Int}(\sigma_{point}(S_\omega(ct)))$. By Theorem 6.7.1, the eigenspace of $S_\omega(ct)$ associated with the eigenvalue λ is $\overline{\operatorname{span}}\{g_\nu : (\psi_t'(a))^\nu = \lambda^{-1}\} = \overline{\operatorname{span}}\{g_\nu : \nu\in\widetilde{W_\lambda}\}$. Therefore the eigenspace of $v_tC_{\psi_t}$ associated to the eigenvalue λ is $\overline{\operatorname{span}}\{g_\nu\circ\phi : \nu\in\widetilde{W_\lambda}\}$. It is readily seen that the linear fractional mapping $(1+\phi)/(1-\phi)$ has one zero at $z=b$ and one pole at $z=a$, so that it is equal to $(b-(\cdot))/(a-(\cdot))$ up to a constant. Thus $\mathbb{C}\widetilde{g}_\nu = \mathbb{C}(g_\nu\circ\phi)$, that is, the eigenspaces of $v_tC_{\psi_t}$ are as claimed in the statement. The case $\lambda\in\partial(\sigma_{point}(v_tC_{\psi_t}))$ runs similarly.

It only remains to prove the claim made about the range space $\operatorname{Ran}(\lambda - v_tC_{\psi_t})$. Take any $\lambda\in\operatorname{Int}(\sigma_{res}(v_tC_{\psi_t}))$. By Theorem 6.7.1, $\operatorname{Ran}(\lambda - v_tC_{\psi_t}) = C_\phi(\operatorname{Ran}(\lambda - S_\omega(ct)))\subseteq C_\phi(\ker L_\omega^\mu) = \ker(L_\omega^\mu C_{\phi^{-1}})$ for all $\mu\in\widetilde{W_\lambda}$, where L_ω^μ is a continuous functional on X, see Lemma 6.5.4. Now, we are going to prove that $\tilde{L}_\varpi^\mu = kL_\omega^\mu C_{\phi^{-1}}$ for some $k\in\mathbb{C}\setminus\{0\}$, and the proof will be done.

Recall that Ψ denotes the generator (ψ_t). One has $\Psi(z) = \frac{c}{a-b}(a-z)(b-z) = G(\phi(z))/\phi'(z)$ for $z\in\mathbb{D}$, see [BP78, Th. 1.6]. As a consequence, the change of variable $z=\phi^{-1}(\xi)$ in the integral below yields

$$\begin{aligned} L_\omega^\mu C_{\phi^{-1}}f &= \int_{-1}^{1}\frac{(1-\xi)^{\mu-1}}{(1+\xi)^{\mu+1}}\omega(\xi)(f\circ\phi^{-1}(\xi))\,d\xi \\ &= k\int_b^a\frac{(a-z)^{\mu-1}}{(b-z)^{\mu+1}}\varpi(z)f(z)\,dz = k\,\tilde{L}_\varpi^\mu f, \qquad f\in X, \end{aligned}$$

as we wanted to prove. □

Remark 6.7.3. (1) As it has been shown in Section 6.2, Section 6.3 and Section 6.6, spaces $H^p(\mathbb{D})$, $\mathcal{A}_\sigma^p(\mathbb{D})$, $\mathcal{K}_0^{-\gamma}(\mathbb{D})$, $\mathfrak{A}(\mathbb{D})$, $\mathcal{D}_\sigma^p(\mathbb{D})$ and $B_{1,0}(\mathbb{D})$, for $p\geq 1$, $\sigma>-1$, $\gamma>0$, satisfy the conditions assumed on X in Theorem 6.7.2. Furthermore, for $H^p(\mathbb{D})$, $\mathcal{A}_\sigma^p(\mathbb{D})$, $\mathcal{K}_0^{-\gamma}(\mathbb{D})$ and $\mathfrak{A}(\mathbb{D})$ the hypothesis $\varpi^{-1}\in C_\phi(\mathfrak{S})$ is superfluous, see Remark 6.6.8(2). For $\mathcal{D}_\sigma^p(\mathbb{D})$, we conjecture that there exist a subset $\mathfrak{S}(\mathcal{D}_\sigma^p)$ defined in terms of Carleson measures such that $(\mathcal{D}_\sigma^p(\mathbb{D}),\mathfrak{S}(\mathcal{D}_\sigma^p))$ is a γ-pair and that the assumption $\varpi^{-1}\in C_\phi(\mathfrak{S}(\mathcal{D}_\sigma^p))$ is redundant as well.

(2) Theorem 6.7.2 answers in the positive the conjectures established in [CGP15; ELM16; HLNS13] about the spectrum of a weighted hyperbolic invertible operator vC_ψ on γ-spaces in the case that v can be embedded in a cocycle for (ψ_t), where $\psi_1=\psi$ (see the beginning of this chapter).

Remark 6.7.4. Non-separable Korenblum spaces, H^∞ in particular, and Bloch spaces are not under the scope of the chapter since weighted composition groups are not strongly continuous on them. These cases will be specifically approached in a forthcoming paper.

6.8 Weighted averaging operators

Here, we make use of the theory developed in the preceding sections to study the boundedness and spectral sets of two families of weighted averaging operators acting on γ-spaces. Throughout all this section, $(X, \mathfrak{S})$ will denote a γ-pair for some $\gamma \geq 0$ such that X is hyperbolically DW-contractive and such that the constant function $\mathbf{1}$ lies in $\mathfrak{S}$. In particular, it applies to any of the γ-spaces listed in the examples of Subsection 6.2.A.

Recall that we denote by $B(\cdot,\cdot)$, $\Gamma(\cdot)$ the Beta function and the Gamma function respectively. The following estimate for the Gamma function will be used in the sequel.

For $\lambda \in \mathbb{C}$, one has

(6.23)
$$\frac{\Gamma(z+\lambda)}{\Gamma(z)} = z^{\lambda}\left(1+\frac{\lambda(\lambda+1)}{2z}+O(|z|^{-2})\right) = z^{\lambda}\left(1+O(|z|^{-1})\right), \quad z\in\mathbb{C},\ |z|\to\infty,$$

whenever $z \neq 0, -1, -2, \ldots$ and $z \neq -\lambda, -\lambda-1, -\lambda-2 \ldots$, see [TE+51] for more details.

6.8.A Siskakis type operators

Let $\mu, \nu, \delta \in \mathbb{C}$. Here we analyze the weighted averaging operators given by

$$(\mathcal{J}_{\delta}^{\mu,\nu} f)(z) = \frac{1}{(1+z)^{\nu+\delta}(1-z)^{\mu+\delta}} \int_z^1 (1+\xi)^{\nu}(1-\xi)^{\mu}(\xi-z)^{\delta-1} f(\xi)\, d\xi, \qquad z\in\mathbb{D}.$$

Proposition 6.8.1. *Let $\mathfrak{Re}\,\mu-\gamma+1$, $\gamma-\mathfrak{Re}\,(\nu+\delta)$, $\mathfrak{Re}\,\delta>0$. Let $\omega(z)=(1+z)^{\nu+\delta}(1-z)^{\mu+1}$ for $z\in\mathbb{D}$. Then,*

$$\mathcal{J}_{\delta}^{\mu,\nu} f = 2^{-\delta}\int_0^{\infty}(1-e^{-t})^{\delta-1} S_{\omega}(t) f\, dt, \qquad f\in X, \tag{6.24}$$

where the integral is Bochner-convergent. In particular, $\mathcal{J}_{\delta}^{\mu,\nu}$ is a bounded operator on X.

Proof. Set $(u_t) = ((\omega\circ\varphi_t)/\omega)$, so (u_t) is a DW-continuous cocycle for the hyperbolic flow (φ_t) on X with exponents $\alpha = -\mathfrak{Re}\,\mu - 1$, $\beta = \mathfrak{Re}\,(\nu+\delta)$, see Lemma 6.3.8. By Proposition 6.4.2, for every $\varepsilon \in (0, \min\{\mathfrak{Re}\,\mu-\gamma+1,\ \gamma-\mathfrak{Re}\,(\nu+\delta)\})$, there exists $K_\varepsilon > 0$ such that

$$\|S_\omega(t)\|_{L(X)} \leq K_\varepsilon e^{-t\min\{\gamma-\mathfrak{Re}\,(\nu+\delta),\,\mathfrak{Re}\,\mu-\gamma+1\}+\varepsilon t}, \qquad t\geq 0.$$

Hence,

$$\begin{aligned}\left\|\int_0^{\infty}(1-e^{-t})^{\delta-1}S_\omega(t)\,dt\right\|_{L(X)} &\leq K_\varepsilon\int_0^{\infty}(1-e^{-t})^{\mathfrak{Re}\,\delta-1}e^{t(\varepsilon-\min\{\gamma-\mathfrak{Re}\,(\nu+\delta),\,\mathfrak{Re}\,\mu-\gamma+1\})}\,dt\\ &= K_\varepsilon B(\mathfrak{Re}\,\delta, \min\{\gamma-\mathfrak{Re}\,(\nu+\delta),\,\mathfrak{Re}\,\mu-\gamma+1\}-\varepsilon) < \infty.\end{aligned}$$

As a consequence, the integral $\int_0^\infty (1-e^{-t})^{\delta-1} S_\omega(t)dt$ is strongly convergent in the Bochner sense and it defines a bounded operator on X. Moreover, for $f \in X$ and $z \in \mathbb{D}$,

$$\begin{aligned}
&\int_0^\infty (1-e^{-t})^{\delta-1}(S_\omega(t)f)(z)\,dt \\
&= \int_0^\infty (1-e^{-t})^{\delta-1}\left(\frac{1+\varphi_t(z)}{1+z}\right)^{\nu+\delta}\left(\frac{1-\varphi_t(z)}{1-z}\right)^{\mu+1} f(\varphi_t(z))\,dt \\
&= \frac{2^\delta}{(1+z)^{\nu+\delta}(1-z)^{\mu+\delta}}\int_z^1 (1+\xi)^\nu(1-\xi)^\mu(\xi-z)^{\delta-1}f(\xi)\,d\xi = 2^\delta(\mathcal{J}_\delta^{\mu,\nu}f)(z),
\end{aligned}$$

where we used the change of variable $\xi = \varphi_t(z)$, and the proof is done. □

Theorem 6.8.2. *Let* $\mathfrak{Re}\,\mu-\gamma+1$, $\gamma-\mathfrak{Re}\,(\nu+\delta)$, $\mathfrak{Re}\,\delta>0$. *Then the spectrum, essential spectrum and point spectrum of* $\mathcal{J}_\delta^{\mu,\nu}$ *on* X *are*

$$\begin{aligned}
\sigma(\mathcal{J}_\delta^{\mu,\nu}) &= \left\{2^{-\delta}B(\delta,\lambda)\;:\;\lambda\in|\gamma-\mathfrak{Re}\,(\nu+\delta),\,\mathfrak{Re}\,\mu-\gamma+1|\right\}\cup\{0\},\\
\sigma_{ess}(\mathcal{J}_\delta^{\mu,\nu}) &= \left\{2^{-\delta}B(\delta,\lambda)\;:\;\mathfrak{Re}\,\lambda=\gamma-\mathfrak{Re}\,(\nu+\delta)\ \text{ or }\ \mathfrak{Re}\,\lambda=\mathfrak{Re}\,\mu-\gamma+1\right\}\cup\{0\},\\
\sigma_{point}(\mathcal{J}_\delta^{\mu,\nu}) &= \left\{2^{-\delta}B(\delta,\lambda)\;:\;\lambda\in\mathbb{C}\ \textit{such that}\ \left[\xi\mapsto(1+\xi)^{\lambda-\nu-\delta}(1-\xi)^{\mu-\lambda+1}\right]\in X\right\}.
\end{aligned}$$

In particular,

$$\{2^{-\delta}B(\delta,\lambda)\;:\;\mathfrak{Re}\,\mu-\gamma+1<\mathfrak{Re}\,\lambda<\gamma-\mathfrak{Re}\,(\nu+\delta)\}\subseteq\sigma_{point}(\mathcal{J}_\delta^{\mu,\nu}),$$

if $\mathfrak{Re}\,(\mu+\nu+\delta)<2\gamma-1$, *and*

$$\sigma_{point}(\mathcal{J}_\delta^{\mu,\nu})=\emptyset,\qquad \textit{if}\ \mathfrak{Re}\,(\mu+\nu+\delta)>2\gamma-1.$$

Proof. Set $\rho=\mathfrak{Re}\,(\nu+\delta-\mu-1)/2$ and $\omega(z)=(1+z)^{\nu+\delta}(1-z)^{\mu+1}$ for $z\in\mathbb{D}$. By Proposition 6.8.1, one has

$$\mathcal{J}_\delta^{\mu,\nu}=2^{-\delta}\int_0^\infty(1-e^{-t})^{\delta-1}S_\omega(t)\,dt=\int_{-\infty}^\infty e^{-\rho t}S_\omega(t)\,d\tilde\mu(t),$$

where $d\tilde\mu(t)=e^{\rho t}2^{-\delta}(1-e^{-t})^{\delta-1}\chi_{(0,\infty)}(t)\,dt$.

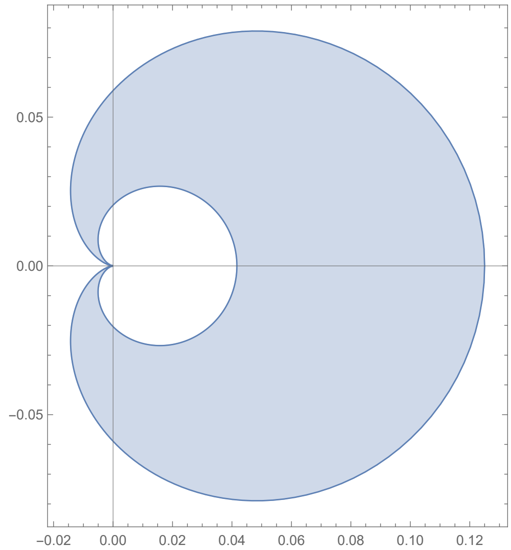
By Proposition 6.4.2 and Proposition 4.1.2, the infinitesimal generator $\Delta_\omega-\rho$ of the C_0-group $(e^{-\rho t}S_\omega(t))$ is bisectorial-like of angle $\pi/2$ and half-width c, for any $c>|\mathfrak{Re}\,(\mu+\nu+\delta)-2\gamma+1|/2$; see for instance [Haa06, Subsection 2.1.1]. Moreover, c can be taken such that $\int_{-\infty}^\infty e^{c|t|}\,|d\tilde\mu|(t)<\infty$ (see the proof of Proposition 6.8.1).

Define $f\in\mathcal{O}(\mathcal{D})$ by

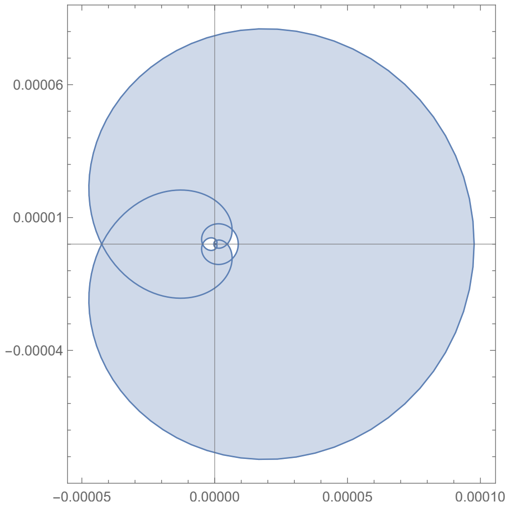
$$f(z)=\mathcal{F}(\tilde\mu)(-z)=\int_{-\infty}^\infty e^{zt}d\tilde\mu(t)=2^{-\delta}\int_0^\infty(1-e^{-t})^{\delta-1}e^{(z+\rho)t}\,dt=2^{-\delta}B(\delta,-z-\rho),$$

for all $z\in\mathbb{C}$ with $|\mathfrak{Re}\,z|<c$. Note that f can be analytically extended to the bisector $BS_{\theta,c}$ for any $\theta\in(0,\pi/2)$. Also, by (6.23),

$$f(z)=2^{-\delta}\frac{\Gamma(\delta)\Gamma(-\rho-z)}{\Gamma(\delta-\rho-z)}=2^{-\delta}\Gamma(\delta)(-\rho-z)^{-\delta}(1+O(|z+\rho|^{-1}))^{-1},\qquad |z|\to\infty\ (z\in BS_{\theta,c}).$$



(a) Spectrum of $\mathcal{J}_2^{2,-2}$ on 1-spaces



(b) Spectrum of $\mathcal{J}_{10}^{2,-10}$ on 2-spaces

Figure 6.1: Spectral pictures for two Siskakis type operators. The bold lines depict the essential spectrum

Thus, f has regular limit (equal to 0) at ∞, so $f \in \mathcal{E}(\Delta_\omega - \rho)$ satisfying condition (**2.2.P2**). Hence, we can apply Corollary 2.2.28 to get $\tilde{\sigma}(\mathcal{J}_\delta^{\mu,\nu}) = f(\tilde{\sigma}(\Delta_\omega - \rho))$, $\tilde{\sigma}_{ess}(\mathcal{J}_\delta^{\mu,\nu}) = f(\tilde{\sigma}_{ess}(\Delta_\omega - \rho))$ and $\sigma_{point}(\mathcal{J}_\delta^{\mu,\nu}) = f(\sigma_{point}(\Delta_\omega - \rho))$. Now, it suffices to apply Proposition 6.6.2 and Theorem 6.6.7 to obtain the claim. (Note that $\infty \in \tilde{\sigma}_{ess}(\Delta_\omega - \rho))$ since $\tilde{\sigma}_{ess}(A)$ is a closed subset of the Riemann sphere $\mathbb{C} \cup \{\infty\}$ for any closed operator A with non-empty resolvent, see for instance Section 2.2.) □

Corollary 6.8.3. *Let $0 < \gamma < 1$. The Siskakis operator $\mathcal{J}$ is a bounded operator on X, and the following holds true.*

- *$\sigma(\mathcal{J})$ is the region between the circles $C_1 := \{z \in \mathbb{C} : |z + 1/\gamma| = 1/\gamma\}$ and $C_2 := \{z \in \mathbb{C} : |z + 1/(1-\gamma)| = 1/(1-\gamma)\}$.*
- *$\sigma_{ess}(\mathcal{J}) = C_1 \cup C_2$.*
- *If $\gamma > 1/2$, then $\mathrm{Int}(\sigma(\mathcal{J})) \subseteq \sigma_{point}(\mathcal{J})$. If $\gamma < 1/2$, then $\sigma_{point}(\mathcal{J}) = \emptyset$.*

6.8.B Reduced Hilbert type operators

Let $\mu, \nu, \delta \in \mathbb{C}$. In this subsection, we study the spectrum of the multiparameter family of operators $(\mathfrak{H}_\delta^{\mu,\nu})$, with

(6.25)
$$(\mathfrak{H}_\delta^{\mu,\nu} f)(z) = \frac{1}{(1+z)^{\nu-\delta+1}(1-z)^{\mu-\delta+1}} \int_{-1}^{1} (1+\xi)^\nu (1-\xi)^\mu \frac{f(\xi)}{(1-z\xi)^\delta}\, d\xi, \quad z \in \mathbb{D}.$$

The next result gives sufficient conditions on μ, ν, δ for the boundedness of $\mathfrak{H}_\delta^{\mu,\nu}$ on X. We recall the following representation of the Gaussian hypergeometric function ${}_2F_1$. For $a \in \mathbb{C}$ and $c > b > 0$,

$$ {}_2F_1(a,b;c;z) := \frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)} \int_0^1 s^{b-1}(1-s)^{c-b-1}(1-zs)^{-a}\, ds, \quad z \in \mathbb{C}\backslash[1,+\infty), $$

see [GR14, Formula 9.111].

Proposition 6.8.4. *Assume* $\mathfrak{Re}\,\mu$, $\mathfrak{Re}\,\nu > \gamma - 1$ *and* $\mathfrak{Re}\,(\delta-\mu), \mathfrak{Re}\,(\delta-\nu) > 1-\gamma$. *Set* $\omega(z) = (1+z)^{\nu+1}(1-z)^{\mu-\delta+1}$ *for* $z \in \mathbb{D}$. *Then*

$$ \mathfrak{H}_\delta^{\mu,\nu} f = \int_{-\infty}^{\infty} \frac{2^{\delta-1}}{(1+e^t)^\delta} S_\omega(t) f\, dt, \qquad f \in X, $$

where the integral is Bochner-convergent. In particular, $\mathfrak{H}_\delta^{\mu,\nu}$ *is a bounded operator on* X.

Proof. The proof is similar to the proof of Proposition 6.8.1.

Here, the DW-continuous cocycle $((\omega\circ\varphi_t)/\omega)$ has exponents $\alpha = \mathfrak{Re}\,(\delta-\mu)-1$, $\beta = \mathfrak{Re}\,\nu+1$. Fix $\varepsilon > 0$ small enough, and set $\rho := \varepsilon + \max\{\mathfrak{Re}\,\nu - \gamma+1, \mathfrak{Re}\,(\delta-\mu)+\gamma-1\}$ and $\widetilde{\rho} := \varepsilon + \max\{\mathfrak{Re}\,(\delta-\nu)+\gamma-1, \mathfrak{Re}\,\mu-\gamma+1\}$. Then, there exists $K_\varepsilon > 0$ such that

$$ \begin{aligned} &\left\| \int_{-\infty}^{\infty} \frac{1}{(1+e^t)^\delta} S_\omega(t)\, dt \right\|_{L(X)} \\ \le& K_\varepsilon \left(\int_0^\infty \frac{e^{t(\varepsilon+\max\{\beta-\gamma,\alpha+\gamma\}}}{(1+e^t)^{\mathfrak{Re}\,\delta}}\, dt + \int_{-\infty}^0 \frac{e^{-t(\varepsilon+\max\{\gamma-\beta,-\alpha-\gamma\})}}{(1+e^t)^{\mathfrak{Re}\,\delta}}\, dt \right) \\ =& K_\varepsilon \left(\int_1^\infty \frac{x^{\rho-1}}{(1+x)^{\mathfrak{Re}\,\delta}}\, dx + \int_1^\infty \frac{x^{\widetilde{\rho}-1}}{(1+x)^{\mathfrak{Re}\,\delta}}\, dx \right) \\ =& K_\varepsilon 2^{1-\mathfrak{Re}\,\delta} \left(\frac{{}_2F_1(1-\rho,1;-\rho;-1)}{\mathfrak{Re}\,\delta-\rho} + \frac{{}_2F_1(1-\widetilde{\rho},1;-\widetilde{\rho};-1)}{\mathfrak{Re}\,\delta-\widetilde{\rho}} \right) < \infty, \end{aligned} $$

where we have applied [GR14, 3.197, (2)] in the last equality, and we have used the change of variables $e^t = x$ and $e^{-t} = x$, respectively in each integral sign, in the second-to-last equality. We conclude that $\int_{-\infty}^{\infty}(1+e^t)^{-\delta} S_\omega(t)\, dt$ is Bochner-strongly convergent, whence it defines a bounded operator. Similar computations as in the proof of Proposition 6.8.1 give us

$$ \begin{aligned} &\int_{-\infty}^{\infty} \frac{2^{\delta-1}}{(1+e^t)^\delta} (S_\omega(t)f)(z)\, dt \\ =& \int_{-\infty}^{\infty} \frac{2^{\delta-1}}{(1+e^t)^\delta} \left(\frac{1-\varphi_t(z)}{1-z}\right)^{\mu-\delta+1} \left(\frac{1+\varphi_t(z)}{1+z}\right)^{\nu+1} f(\varphi_t(z))\, dt \\ =& \frac{1}{(1+z)^{\nu-\delta+1}(1-z)^{\mu-\delta+1}} \int_{-1}^1 (1+\xi)^\nu (1-\xi)^\mu \frac{f(\xi)}{(1-z\xi)^\delta}\, dw \\ =& (\mathfrak{H}_\delta^{\mu,\nu} f)(z), \quad z \in \mathbb{D},\ f \in X, \end{aligned} $$

and the proof is finished.

□

Now, we obtain the spectra of operators $\mathfrak{H}_\delta^{\mu,\nu}$. First, we prove the following lemma.

Lemma 6.8.5. *Assume $\mathfrak{Re}\,\mu$, $\mathfrak{Re}\,\nu > \gamma - 1$ and $\mathfrak{Re}\,(\delta-\mu), \mathfrak{Re}\,(\delta-\nu) > 1-\gamma$. Then $\mathfrak{H}_\delta^{\mu,\nu}$ is an injective operator on X.*

Proof. Let $f \in X$, put $g(\xi) := (1+\xi)^\nu (1-\xi)^\mu f(\xi)$ for $\xi \in (-1,1)$, and fix any $\varepsilon > 0$ small enough. Then $|f(\xi)| \lesssim (1-\xi^2)^{-\gamma-\varepsilon}$ for all $\xi \in (-1,1)$ by Remark 6.2.4. Hence

$$\int_{-1}^1 |g(\xi)|\,dt \lesssim \int_{-1}^1 (1+\xi)^{\mathfrak{Re}\,\nu-\gamma}(1-\xi)^{\mathfrak{Re}\,\mu-\gamma}\,dt < \infty,$$

that is, $g \in L^1(-1,1)$.

Now, assume furthermore $f \in \ker \mathfrak{H}_\delta^{\mu,\nu}$, and let $K^\delta(n)$, $n \in \mathbb{N}_0$ be such that $(1-z)^{-\delta} = \sum_{n=0}^\infty K^\delta(n) z^n$, $z \in \mathbb{D}$. One has

$$\begin{aligned}(\mathfrak{H}_\delta^{\mu,\nu} f)(z) &= \int_{-1}^1 (1+\xi)^\nu (1-\xi)^\mu \frac{f(\xi)}{(1-z\xi)^\delta}\,d\xi = \int_{-1}^1 \frac{g(\xi)}{(1-z\xi)^\delta}\,d\xi \\ &= \int_{-1}^1 g(\xi) \sum_{n=0}^\infty K^\delta(n)(z\xi)^n\,d\xi = \sum_{n=0}^\infty z^n K^\delta(n) \int_{-1}^1 \xi^n g(\xi)\,d\xi = 0, \quad z \in \mathbb{D},\end{aligned}$$

where have used Fubini's theorem since

$$\sum_{n=0}^\infty \int_{-1}^1 \left| K^\delta(n) z^n \xi^n g(\xi)\,d\xi \right| \le \|g\|_{L^1(-1,1)} (1-|z|)^\delta < \infty.$$

As a consequence, $K^\delta(n)\int_{-1}^1 \xi^n g(\xi)\,d\xi = 0$, $n \in \mathbb{N}_0$, which implies $\int_{-1}^1 \xi^n g(\xi)\,d\xi = 0$, $n \in \mathbb{N}_0$ (note that $\mathfrak{Re}\,\delta > 0$ by the hypotheses assumed and so $K^\delta(n) \neq 0$, $n \in \mathbb{N}_0$). In short, $g = 0$, thus $f = 0$ and our claim follows. □

Theorem 6.8.6. *Assume $\mathfrak{Re}\,\mu$, $\mathfrak{Re}\,\nu > \gamma - 1$ and $\mathfrak{Re}\,(\delta-\mu), \mathfrak{Re}\,(\delta-\nu) > 1-\gamma$. Then the spectrum, essential spectrum and point spectrum of $\mathfrak{H}_\delta^{\mu,\nu}$ are*

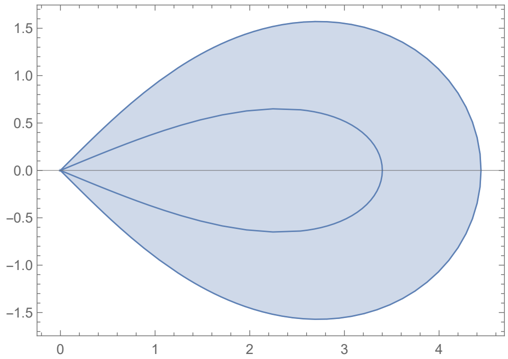
$$\begin{aligned}\sigma(\mathfrak{H}_\delta^{\mu,\nu}) &= \{2^{\delta-1}B(z,\delta-z) \ : \ z \in |\mathfrak{Re}\,\nu - \gamma + 1, \mathfrak{Re}\,(\delta-\mu)+\gamma-1|\} \cup \{0\}, \\ \sigma_{ess}(\mathfrak{H}_\delta^{\mu,\nu}) &= \{2^{\delta-1}B(z,\delta-z) \ : \ \mathfrak{Re}\,z = \mathfrak{Re}\,\nu-\gamma+1 \text{ or } \mathfrak{Re}\,z = \mathfrak{Re}\,(\delta-\mu)+\gamma-1\} \cup \{0\}, \\ \sigma_{point}(\mathfrak{H}_\delta^{\mu,\nu}) &= \{2^{\delta-1}B(z,\delta-z) \ : \ z \in \mathbb{C} \text{ such that } \left[\xi \mapsto (1+\xi)^{z-\nu-1}(1-\xi)^{\mu-\delta-z+1}\right] \in X\}.\end{aligned}$$

In particular,

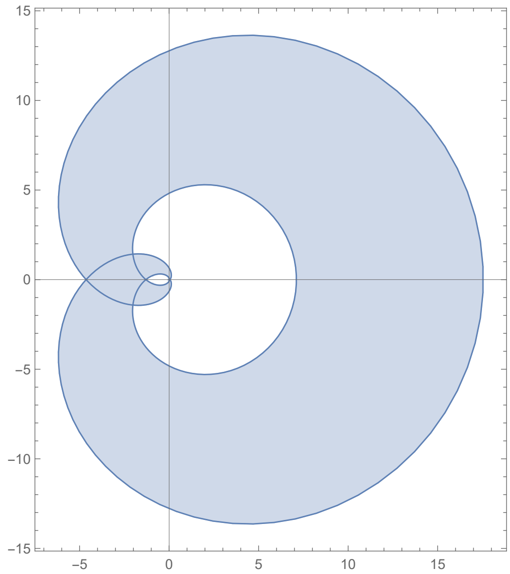
$$\{2^{\delta-1}B(z,\delta-z) \ : \ \mathfrak{Re}\,\nu-\gamma+1 < \mathfrak{Re}\,z < \mathfrak{Re}\,(\delta-\mu)+\gamma-1\} \subseteq \sigma_{point}(\mathfrak{H}_\delta^{\mu,\nu}),$$

if $\mathfrak{Re}\,(\mu+\nu-\delta) < 2(\gamma-1)$, and

$$\sigma_{point}(\mathfrak{H}_\delta^{\mu,\nu}) = \emptyset, \qquad \text{if } \mathfrak{Re}\,(\mu+\nu-\delta) > 2(\gamma-1).$$



(a) Spectrum of $\mathcal{H}_1^{1/4,1/8}$ on 1/2-spaces



(b) Spectrum of $\mathcal{H}_{10}^{8,3/2}$ on 1-spaces

Figure 6.2: Spectral pictures for two reduced Hilbert type operators. The bold lines depict the essential spectrum

Proof. The proof runs along similar lines as Theorem 6.8.2.

For $\rho = \mathfrak{Re}\,(\nu + \delta - \mu)/2$, we have $\mathfrak{H}_\delta^{\mu,\nu} = \int_{-\infty}^{\infty} e^{-\rho t} S_\omega(t)\, d\tilde{\mu}(t)$, where $d\tilde{\mu}(t) = 2^{\delta-1} e^{\rho t}(1+e^t)^{-\delta}\, dt$ for $t \in \mathbb{R}$, see Proposition 6.8.4. On the other hand, it follows by Proposition 6.4.2 and Proposition 4.1.2 that, for all $c > |\mathfrak{Re}\,(\delta - \mu - \nu) + 2(\gamma - 1)|/2$, the infinitesimal generator $\Delta_\omega - \rho$ of $(e^{-\rho t} S_\omega(t))$ is sectorial of angle $\pi/2$ and half-width c, see [Haa06, Subsection 2.1.1].

Define $f \in \mathcal{O}(\mathbb{D})$ by

$$f(z) = (\mathcal{F}\tilde{\mu})(-z) = \int_{-\infty}^{\infty} e^{zt} d\tilde{\mu}(t) = 2^{\delta-1} \int_{-\infty}^{\infty} \frac{e^{(z+\rho)t}}{(1+e^t)^\delta}\, dt = 2^{\delta-1} B(z+\rho, \delta - z - \rho),$$

for all $|\mathfrak{Re}\, z| < c$. Note that f can be analytically extended to a bisector $BS_{\theta,c}$ for any $\theta \in (0, \pi/2)$. We claim that there exists $K > 0$ for which $|f(z)| \lesssim e^{-K|z|}$ as $z \to \infty$ through $BS_{\theta,c}$. This is true if $\delta = 1$ since in this case $f(z) = \frac{\pi}{\sin \pi(z+\rho)}$ for all $z \in \mathbb{C} \setminus \{-\rho, -\rho-1, -\rho-2, ...; \rho-1, \rho-2, ...\}$, and $|\sin \pi(z+\rho)| \gtrsim e^{\pi \sin\theta |z|}$ as $z \to \infty$ through $BS_{\theta,c}$. If $\delta \neq 1$, note that

$$f(z) = B(z+\rho, \delta - z - \rho) = \frac{\Gamma(\delta - z - \rho)}{\Gamma(\delta-1)\Gamma(1-z-\rho)} \frac{\pi}{(\delta-1)\sin \pi(1-z-\rho)},$$

for all $z \in \mathbb{C} \setminus \{-\rho, -\rho-1, -\rho-2, ...; \rho-1, \rho-2, ...\}$. Thus, it follows by (6.23) that

$$f(z) = \frac{(-z-\rho)^{\delta-1}}{\Gamma(\delta-1)} \frac{\pi}{(\delta-1)\sin \pi(1-z-\rho)} (1 + O(|z+\rho|^{-1}))^{-1}, \quad z \in BS_{\theta,c},$$

obtaining the fore-mentioned inequality. Thus f is regular at ∞ with $f(\infty) = 0$, $f \in \mathcal{E}(\Delta_\omega - \rho)$ and the hypotheses of Corollary 2.2.28 are satisfied. As a consequence, $\widetilde{\sigma}(\mathfrak{H}_\delta^{\mu,\nu}) = f(\widetilde{\sigma}(\Delta_\omega - \rho))$, $\widetilde{\sigma}_{ess}(\mathfrak{H}_\delta^{\mu,\nu}) = f(\widetilde{\sigma}_{ess}(\Delta_\omega - \rho))$ and $f(\sigma_{point}(\Delta_\omega - \rho)) \subseteq \sigma_{point}(\mathfrak{H}_\delta^{\mu,\nu}) \subseteq f(\sigma_{point}(\Delta_\omega - \rho)) \cup \{0\}$. The statement follows since $\mathfrak{H}_\delta^{\mu,\nu}$ is injective by Lemma 6.8.5, and the different spectra of Δ_ω were given in Theorem 6.6.7. $\square$

Addendum A
Hardy operators

Let $1 \leq p < \infty$ and let $H : (0,\infty) \times (0,\infty) \to \mathbb{C}$ be a measurable map. H is said to be a Hardy kernel of index p if the following conditions hold.

(i) H is homogeneous of degree -1; that is, for all $\lambda > 0$, $H(\lambda r, \lambda s) = \lambda^{-1} H(r,s)$ for all $r, s > 0$.

(ii) $\int_0^\infty |H(1,s)| s^{-1/p} ds < \infty$.

Then, the Hardy operator T_H associated with H is defined by

$$(T_H f)(y) := \int_0^\infty H(x,y) f(x)\, dx, \qquad a.e.\ y > 0.$$

By Hardy's inequality [HLP34, Th. 319], T_H induces a well-defined bounded operator on $L^p(0,\infty)$ if H is a Hardy kernel of index p. As said in the Introduction, T_H can be represented as

$$T_H = \int_{-\infty}^{\infty} g_H(t) E(t)\, dt, \tag{A.1}$$

where $E(t)f = f(e^{-t}(\cdot))$ and $g_H(t) = e^{-t} H(e^{-t}, 1)$, $t \in \mathbb{R}$. The generator Δ_E of $(E(t))$, acting on $L^p(0,\infty)$, is given by the differential operator

$$(\Delta_E f)(x) = -x f'(x), \qquad a.e.\ x > 0,$$

with domain $\mathrm{Dom}(\Delta_E) = \{f \in L^p : f \in AC_{\mathrm{loc}}(0,\infty) \text{ and } -xf'(x) \in X\}$ and spectrum $\sigma(\Delta_E) = i\mathbb{R} + 1/p$, see [AS13, Prop. 2.3].

Proposition A.0.1. *Let H be a Hardy kernel of index $p \in [1,\infty)$. Then, the spectrum of T_H on $L^p(0,\infty)$ is given by*

$$\sigma(T_H) = \left\{ \widehat{H}\left(\frac{1}{q} + i\xi\right) : \xi \in \mathbb{R} \right\} \cup \{0\}$$

where q is such that $1/p + 1/q = 1$ ($q = \infty$ if $p = 1$) and $\widehat{H}(\zeta) = \int_0^\infty s^{\zeta - 1} H(s,1)\, ds$, that is, the Mellin transform of $H(\cdot, 1)$.

Proof. First, notice that $g_H \in L^1(\mathbb{R})$ and

$$(\mathcal{F}g_H)(\xi) = \int_{-\infty}^{\infty} H(e^{-r},1)e^{-r-ir\xi}dr = \int_0^{\infty} H(s,1)s^{i\xi}ds = \widehat{H}\,(1+i\xi)\,, \quad \xi \in \mathbb{R}.$$

Then, our claim follows by the spectral mapping theorem for group isometries (see for example [Sef06, Th. 3.1], we have

$$\sigma(T_H) = \overline{(\mathcal{F}g_H)\,(\sigma(i\Delta_E))} = (\mathcal{F}g_H)\left(\mathbb{R}+\frac{i}{p}\right)\cup\{0\} = \widehat{H}\left(\frac{1}{q}+i\mathbb{R}\right)\cup\{0\}.$$

□

Remark A.0.2. (1) Proposition A.0.1 was proven in [Boy73; FJL76] with techniques of convolution products. Here we have given the above proof to illustrate once again subordination to groups.

(2) On Hilbertian spaces $L^2(0,\infty)$ and $H^2(\mathbb{C}^+)$, Hardy kernels are approached in [Oli22a] from the viewpoint of reproducing kernels in the context of operator ranges. In this way, part of results of [GMS21] on range spaces associated with fractional Cesàro operators are extended, and on the other hand, several results of [GMS21] are given with simpler proofs.

Generalized Stieltjes operators

A particular case of Hardy operator is the classical Stieltjes operator $\mathcal{S}$ given by

$$(\mathcal{S}f)(t) := \int_0^{\infty} \frac{f(s)}{s+t}ds, \qquad a.e.\ t>0.$$

The operator is the origin of diverse theories in many areas of mathematical analysis and differential equations in real and complex variable, see [Sti18, p. 473]. One generalization of the Stieltjes operator is the following one:

$$\mathcal{S}_{\beta,\mu}f(t) := t^{\mu-\beta}\int_0^{\infty} \frac{s^{\beta-1}}{(t+s)^{\mu}}f(s)\,ds, \qquad a.e.\ t>0,$$

for $\beta,\mu \in \mathbb{R}$.

For $\alpha > 0$ and $p \in [1,\infty)$, let $\mathcal{T}_p^{(\alpha)}(t^\alpha)$ be the Banach space consisting of functions $f \in L^p(0,\infty)$ such that

$$||f||_{\alpha,p} := \frac{1}{\Gamma(\alpha+1)}\left(\int_0^{\infty} |W^\alpha f(t)|^p t^{\alpha p}dt\right)^{\frac{1}{p}} < \infty,$$

where W^α denotes the Weyl derivative of order α, see Section 1.4. These spaces are defined and studied in [Roy08].

Proposition A.0.3. *Let $1 \leq p < \infty$ and $0 < \beta - 1/p < \mu$. Then, $\mathcal{S}_{\beta,\mu}$ is a bounded operator on $\mathcal{T}_p^{(\alpha)}(t^\alpha)$ with spectrum given by*

$$\sigma(\mathcal{S}_{\beta,\mu}) = \left\{ B\left(\beta - \frac{1}{p} + i\xi, \mu - \beta + \frac{1}{p} - i\xi\right) : \xi \in \mathbb{R} \right\} \cup \{0\}.$$

Proof. The proof is analogous to the one given in Proposition A.0.1. □

Remark A.0.4. The above proposition, together with other results which are not of spectral nature, are given in [MO21]. For instance,

(1) a Hölder inequality-type for the elements of spaces $\mathcal{T}_p^{(\alpha)}(t^\alpha)$;

(2) the factorization $\gamma B(\gamma, \mu - \gamma)\mathcal{S}_{1,\mu-\gamma} = \mathcal{S}_{\gamma+1,\mu}\mathcal{C}_\gamma$ for $\gamma > 0$, $\mu > \gamma + 1 - \frac{1}{p}$;

(3) the functional convolution equation

$$\begin{aligned} \mathcal{S}_{n,m}(f \otimes g) = &\sum_{i=n}^{m} \binom{m}{i} \sum_{j=0}^{i-n} \mathcal{S}_{i-j,m} f \cdot \mathcal{S}_{n+j,m} g \\ &- \sum_{i=0}^{n-2} \binom{m}{i} \sum_{j=0}^{n-2-i} \mathcal{S}_{n-j-1,m} f \cdot \mathcal{S}_{i+j+1,m} g, \end{aligned}$$

where $p, q \in (1, \infty)$, $r \geq 1$ are such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, $n, m \in \mathbb{N}$ are such that $0 < n - \frac{1}{r} < m$, and $f \in L^p(0, \infty)$, $g \in L^q(0, \infty)$. In the above equation, the convolution $f \otimes g$ is defined by

$$f \otimes g := f \cdot \mathcal{H}_+ g + g \cdot \mathcal{H}_+ f, \qquad f \in L^p(\mathbb{R}^+),\ g \in L^q(\mathbb{R}^+),$$

where $\mathcal{H}_+$ is the one-sided Hilbert transform in $(0, \infty)$. The above equality is an extension of the identity $\mathcal{S}(f \otimes g) = (\mathcal{S}f)(\mathcal{S}g)$ proved in [ST95].

Bibliography

[AGMO22] L. Abadías, J.E. Galé, P.J. Miana, and J. Oliva-Maza. "Weighted hyperbolic composition groups on the disc and subordinated integral operators". Preprint. 2022.

[AM18] L. Abadías and P.J. Miana. "Generalized Cesàro operators, fractional finite differences and Gamma functions". In: *J. Funct. Anal.* 274.5 (2018), pp. 1424–1465.

[AO23] L. Abadías and J. Oliva-Maza. *Spectra of generalized weighted Hausdorff operators*. Ongoing work. 2023.

[Ahl79] L.V. Ahlfors. *Complex analysis: an introduction to the theory of analytic functions of one complex variable, 3ed.* McGraw-Hill, 1979.

[AC09] A. Aleman and O. Constantin. "Spectra of integration operators on weighted Bergman spaces". In: *J. Anal. Math.* 109.1 (2009), pp. 199–231.

[AP10] A. Aleman and A.M. Persson. "Resolvent estimates and decomposable extensions of generalized Cesàro operators". In: *J. Funct. Anal.* 258.1 (2010), pp. 67–98.

[ABHN11] W. Arendt, C.J.K. Batty, M. Hieber, and F. Neubrander. *Vector-valued Laplace transforms and Cauchy problems*. Berlin: Springer, 2011.

[AD06] W. Arendt and M. Duelli. "Maximal L^p-regularity for parabolic and elliptic equations on the line". In: *J. Evol. Equ.* 6.4 (2006), p. 773.

[AP02] W. Arendt and B. de Pagter. "Spectrum and asymptotics of the Black–Scholes partial differential equation in (L^1, L^∞)-interpolation spaces". In: *Pacific J. Math.* 202.1 (2002), pp. 1–36.

[AZ10] W. Arendt and A. Zamboni. "Decomposing and twisting bisectorial operators". In: *Studia Math.* 3.197 (2010), pp. 205–227.

[Are+86] W. Arendt et al. *One-parameter semigroups of positive operators*. Vol. 1184. Springer, 1986.

[ACG09] M.L. Arias, G. Corach, and M.C. Gonzalez. "Lifting properties in operator ranges". In: *Acta Sci. Math. (Szeged)* 75.3–4 (2009), pp. 635–653.

[ACG13] M.L. Arias, G. Corach, and M.C. Gonzalez. "Additivity properties of operator ranges". In: *Linear Algebra Appl.* 439.11 (2013), pp. 3581–3590.

[AL04] R. Aron and M. Lindström. “Spectra of weighted composition operators on weighted Banach spaces of analytic functions”. In: *Israel J. Math.* 141.1 (2004), pp. 263–276.

[AS13] A.G. Arvanitidis and A.G. Siskakis. “Cesàro operators on the Hardy spaces of the half-plane”. In: *Canad. Math. Bull.* 56.2 (2013), pp. 229–240.

[AMN97] P. Auscher, A. McIntosh, and A. Nahmod. “Holomorphic functional calculi of operators, quadratic estimates and interpolation”. In: *Indiana Univ. Math. J.* (1997), pp. 375–403.

[Bad53] W. Bade. “An operational calculus for operators with spectrum in a strip”. In: *Pac. J. Math.* 3.2 (1953), pp. 257–290.

[BHK09] B. Baeumer, M. Haase, and M. Kovács. “Unbounded functional calculus for bounded groups with applications”. In: *J. Evol. Equ.* 9.1 (2009), pp. 171–195.

[BMM14] S. Ballamoole, T.L. Miller, and V.G. Miller. “A class of integral operators on spaces of analytic functions”. In: *J. Math. Anal. Appl.* 414.1 (2014), pp. 188–210.

[BG14] D. Beltiţă and J.E. Galé. “Linear connections for reproducing kernels on vector bundles”. In: *Math. Z.* 277.1-2 (2014), pp. 29–62.

[BS88] C. Bennett and R.C. Sharpley. *Interpolation of operators.* London: Academic press, 1988.

[BKP74] E. Berkson, R. Kaufman, and H. Porta. “Möbius transformations of the disc and one-parameter groups of isometries of H^p”. In: *Trans. Amer. Math. Soc.* 199 (1974), pp. 223–239.

[BP78] E. Berkson and H. Porta. “Semigroups of analytic functions and composition operators”. In: *Mich. Math. J.* 25.1 (1978), pp. 101–115.

[Ber22] E. Bernard. “Weighted composition semigroups on Banach spaces of holomorphic functions”. In: *Arch. Math.* (2022), pp. 1–12.

[BS73] F. Black and M. Scholes. “The pricing of options and corporate liabilities”. In: *J. Polit. Econ.* 81 (1973), pp. 637–654.

[Bla+13] O. Blasco et al. “Semigroups of composition operators and integral operators in spaces of analytic functions”. In: *Ann. Acad. Sci. Fenn. Math.* 38 (2013), pp. 67–90.

[Bou12] P.S. Bourdon. “Spectra of some composition operators and associated weighted composition operators”. In: *J. Oper. Theory* (2012), pp. 537–560.

[Boy73] D.W. Boyd. “Spectra of convolution operators”. In: *Acta Sci. Math.* 35 (1973), pp. 31–37.

[Bro61] F.E. Browder. “On the spectral theory of elliptic differential operators. I”. In: *Math. Ann.* 142.1 (1961), pp. 22–130.

[CGP15] I. Chalendar, E.A. Gallardo-Gutiérrez, and J.R. Partington. “Weighted composition operators on the Dirichlet space: boundedness and spectral properties”. In: *Math. Ann.* 363.3 (2015), pp. 1265–1279.

[CXZ14] W. Chen, X. Xu, and S. Zhu. “Analytically pricing European-style options under the modified Black-Scholes equation with a spatial-fractional derivative”. In: *Quart. Appl. Math.* 72.3 (2014), pp. 597–611.

[Cow81] C.C. Cowen. “Iteration and the solution of functional equations for functions analytic in the unit disk”. In: *Trans. Amer. Math. Soc.* 265.1 (1981), pp. 69–95.

[Cow84] C.C. Cowen. “Subnormality of the Cesàro operator and a semigroup of composition operators”. In: *Indiana Univ. Math. J.* 33.2 (1984), pp. 305–318.

[CM95] C.C. Cowen and B.D. MacCluer. *Composition operators on spaces of analytic functions*. Boca Raton: CRC Press, 1995.

[CDMY96] M. Cowling, I. Doust, A. Mcintosh, and A. Yagi. “Banach space operators with a bounded H^∞ functional calculus”. In: *J. Aust. Math. Soc.* 60.1 (1996), pp. 51–89.

[CMW92] M. Cwikel, J.E. McCarthy, and T.H. Wolff. “Interpolation between weighted Hardy spaces”. In: *P. Am. Math. Soc.* 116.2 (1992), pp. 381–388.

[DS00] E. Diamantopoulos and A.G. Siskakis. “Composition operators and the Hilbert matrix”. In: *Studia Math.* 140.2 (2000), pp. 191–198.

[Dix49] J. Dixmier. “Étude sur les variétés et les opérateurs de Julia, avec quelques applications”. In: *Bull. Soc. Math. France* 77 (1949), pp. 11–101.

[DT85] E.P. Dolzhenko and G.T. Tumarkin. “NN Luzin and the theory of boundary properties of analytic functions”. In: *Russ. Math. Surv.* 40.3 (1985), p. 79.

[DS58] N. Dunford and J.T. Schwartz. *Linear operators I: general theory (Vol. VII)*. New York: John Wiley & Sons, 1958.

[DS63] N. Dunford and J.T. Schwartz. *Linear operators. II*. New York: Interscience Publishers, 1963.

[Dur70] P.L. Duren. *Theory of H^p Spaces*. New York: Academic Press, 1970.

[Dur83] P.L. Duren. *Univalent functions*. Springer-Verlag, 1983.

[DRS69] P.L. Duren, B.W. Romberg, and A.L. Shields. “Linear functionals on H^p spaces with $0 < p < 1$”. In: *J. Reine Angew. Math.* 238 (1969), pp. 32–60.

[EE87] D.E. Edmunds and W.D. Evans. *Spectral theory and differential operators*. Oxford/New York: Oxford University Press, 1987.

[ELM16] T. Eklund, M. Lindström, and P. Mleczko. “Spectral properties of weighted composition operators on the Bloch and Dirichlet spaces”. In: *Studia Math.* 232 (2016), pp. 1–18.

[End60] K. Endl. “Untersuchungen tiber Momentprobleme bei Verfahren vom Hausdorffschen Typus”. In: *Math. Ann.* 139 (1960), pp. 403–432.

[EN00] K.J. Engel and R. Nagel. *One-parameter semigroups for linear evolution equations.* Vol. 194. New York: Graduate texts in Mathematics. Springer, 2000.

[FJL76] E.B. Fabes, M. Jodeit Jr, and J.E. Lewis. “On the spectra of a Hardy kernel”. In: *J. Funct. Anal.* 21.2 (1976), pp. 187–194.

[FNS19] A.N. Fall, S.N. Ndiaye, and N. Sene. “Black–Scholes option pricing equations described by the Caputo generalized fractional derivative”. In: *Chaos Solitons Fractals* 125 (2019), pp. 108–118.

[FW71] P.A. Fillmore and J.P. Williams. “On operator ranges”. In: *Adv. Math.* 7.3 (1971), pp. 254–281.

[Fle72] T.M. Flett. “The dual of an inequality of Hardy and Littlewood and some related inequalities”. In: *J. Math. Anal. Appl.* 38.3 (1972), pp. 746–765.

[Foi72] C. Foiaş. “Invariant para-closed subspaces”. In: *Indiana Univ. Math. J.* 21.10 (1972), pp. 887–906.

[For64] F. Forelli. “The isometries of H^p”. In: *Canad. J. Math.* 16 (1964), pp. 721–728.

[GP06] P. Galanopoulos and M. Papadimitrakis. “Hausdorff and quasi-Hausdorff matrices on spaces of analytic functions”. In: *Canad. J. Math.* 58.3 (2006), pp. 548–579.

[GS01] P. Galanopoulos and A.G. Siskakis. “Hausdorff matrices and composition operators”. In: *Illinois J. Math.* 45.3 (2001), pp. 757–773.

[GMS21] J.E. Galé, P.J. Miana, and L. Sánchez–Lajusticia. “RKH spaces of Brownian type defined by Cesàro–Hardy operators”. In: *Anal. Math. Phys.* 11.3 (2021), pp. 1–34.

[GL18] P. Galindo and M. Lindström. “Spectra of some weighted composition operators on dual Banach spaces of analytic functions”. In: *Integr. Equ. Oper. Theory* 90.3 (2018), pp. 1–12.

[GLW20] P. Galindo, M. Lindström, and N. Wikman. “Spectra of weighted composition operators on analytic function spaces”. In: *Mediterr. J. Math.* 17.1 (2020), pp. 1–22.

[GSY22] E.A. Gallardo-Gutiérrez, A.G. Siskakis, and D. Yakubovich. “Generators of C_0-semigroups of weighted composition operators”. In: *Israel J. Math.* (2022), pp. 1–18.

[GRT77] B.K. Ghosh, B.E. Rhoades, and D. Trutt. “Subnormal generalized Hausdorff operators”. In: *Proc. Amer. Math. Soc.* 66.2 (1977), pp. 261–265.

[GO85] M. González and V.M. Onieva. “On the spectral mapping theorem for essential spectra”. In: *Publ. Sec. Mat. Univ. Autònoma Barcelona* (1985), pp. 105–110.

[GMV97] F. Gozzi, R. Monte, and V. Vespri. “Generation of analytic semigroups for degenerate elliptic operators arising in financial mathematics.” In: *Seminare* 2 (1997), pp. 155–196.

[GR14] I.S. Gradshteyn and I.M. Ryzhik. *Table of integrals, series, and products.* Amsterdam: Academic press, 2014.

[GL71] B. Gramsch and D. Lay. “Spectral mapping theorems for essential spectra”. In: *Math. Ann.* 192.1 (1971), pp. 17–32.

[GW69] K. Gustafson and J. Weidmann. “On the essential spectrum”. In: *J. Math. Anal. Appl.* 25.1 (1969), pp. 121–127.

[Haa05a] M. Haase. “A general framework for holomorphic functional calculi”. In: *Proc. Edinb. Math. Soc.* 48.2 (2005), pp. 423–444.

[Haa05b] M. Haase. “Spectral mapping theorems for holomorphic functional calculi”. In: *J. London Math. Soc.* 71.3 (2005), pp. 723–739.

[Haa06] M. Haase. *The functional calculus for Sectorial operators.* Vol. 169. Birkhäuser, Basel: Oper. Theory Adv. Appl., 2006.

[Har43] G.H. Hardy. “An inequality for Hausdorff means”. In: *J. London Math. Soc.* 18 (1943), pp. 46–50.

[HLP34] G.H. Hardy, J.E. Littlewood, and G. Pólya. *Inequalities.* London/New York: Cambridge U. P., 1934.

[Hau21] F. Hausdorff. “Summationmethoden und Momentfolgen I”. In: *Math. Z.* 9 (1921), pp. 74–109.

[HKZ00] H. Hedenmalm, B. Korenblum, and K. Zhu. *Theory of Bergman spaces.* Vol. 199. New York: Springer-Verlag, 2000.

[HP57] E. Hille and R.S. Phillips. *Functional analysis and semi-groups.* Vol. 31. Providence: American Mathematical Soc., 1957.

[Hof62] K. Hoffman. *Banach spaces of analytic functions.* Englewood Cliffs: Prentice-Hall, 1962.

[HLNS13] O. Hyvärinen, M. Lindström, I. Nieminen, and E. Saukko. “Spectra of weighted composition operators with automorphic symbols”. In: *J. Funct. Anal.* 265.8 (2013), pp. 1749–1777.

[Jak59] A. Jakimovski. *The product of summability methods.* Tech. rep. part 2. Technical Report 8, Jerusalem, 1959.

[JRT74] A. Jakimovski, B.E. Rhoades, and J. Tzimbalario. “Hausdorff matrices as bounded operators over ℓ^p”. In: *Mat. Z.* 138.2 (1974), pp. 173–181.

[Kan08] Kan8eDie. *Principle branch of Arg on Riemann.png*. [Online; accessed October 31, 2022]. 2008. URL: https://upload.wikimedia.org/wikipedia/commons/7/79/Principle_branch_of_arg_on_Riemann.png.

[Kat60] T. Kato. "Note on fractional powers of linear operators". In: *Proc. Japan Academy* 36.3 (1960), pp. 94–96.

[Kat66] T. Kato. *Perturbation theory for linear operators*. Berlin: Springer-Verlag, 1966.

[KS12] Y.C. Kim and T. Sugawa. "Correspondence between spirallike functions and starlike functions". In: *Math. Nachr.* 285.2-3 (2012), pp. 322–331.

[KT11] D. Kitson and R.M. Timoney. "Operator ranges and spaceability". In: *J. Math. Anal. Appl.* 378.2 (2011), pp. 680–686.

[Kön90] W. König. "Semicocycles and weighted composition semigroups on H^p". In: *Michigan Math. J.* 37 (1990), pp. 469–476.

[Kum+12] S. Kumar et al. "Analytical solution of fractional Black-Scholes European option pricing equation by using Laplace transform". In: *J. Fract. Calc. Appl.* 2.8 (2012), pp. 1–9.

[Lay68] D. Lay. "Characterizations of the essential spectrum of FE Browder". In: *Bull. Amer. Math. Soc.* 74.2 (1968), pp. 246–248.

[LCL10] M. Li, C. Chen, and F.B. Li. "On fractional powers of generators of fractional resolvent families". In: *J. Funct. Anal.* 259.10 (2010), pp. 2702–2726.

[LMPS14] C. Lizama, P.J. Miana, R. Ponce, and L. Sánchez-Lajusticia. "On the boundedness of generalized Cesàro operators on Sobolev spaces". In: *J. Math. Anal. Appl.* 419.1 (2014), pp. 373–394.

[MS02] B. MacCluer and K. Saxe. "Spectra of composition operators on the Bloch and Bergman spaces". In: *Israel J. Math.* 128.1 (2002), pp. 325–354.

[McI86] A. McIntosh. "Operators which have an H_∞ functional calculus". In: *Miniconference on operator theory and partial differential equations*. Vol. 14. Canberra: Austral. Nat. Univ., Mathematical Sciences Institute, 1986, pp. 210–232.

[MO20] P.J. Miana and J. Oliva-Maza. "Spectra of generalized Poisson integral operators on $L^p(\mathbb{R}^+)$". In: *Operator Theory*. Ed. by R.M. Aron et al. Berlin, Boston: De Gruyter, 2020. Chap. 18, pp. 281–296.

[MO21] P.J. Miana and J. Oliva-Maza. "Integral operators on Sobolev–Lebesgue spaces". In: *Banach J. Math. Anal.* 15.3 (2021), pp. 1–30.

[Mie87] A. Mielke. "Über maximale L^p-Regularität für Differentialgleichungen in Banach- und Hilbert-Räumen". In: *Math. Ann.* 277.1 (1987), pp. 121–133.

[MR93] K.S. Miller and B. Ross. *An introduction to the fractional calculus and fractional differential equations*. New York: Wiley, 1993.

[Mor10] A.J. Morris. "Local quadratic estimates and the holomorphic functional calculi". In: *The AMSI–ANU Workshop on Spectral Theory and Harmonic Analysis*. Vol. 44. Proc. Centre Math. Appl. Austral. Nat. Univ. 2010, pp. 211–231.

[NRRR79] E. Nordgren, M. Radjabalipour, H. Radjavi, and P. Rosenthal. "On invariant operator ranges". In: *Trans. Amer. Math. Soc.* 251 (1979), pp. 389–398.

[NRR+76] E. Nordgren, H. Radjavi, P. Rosenthal, et al. "Operator algebras leaving compact operator ranges invariant". In: *Michigan Math. J.* 23.4 (1976), pp. 375–377.

[Oli21a] J. Oliva-Maza. "On Lie group representations and operator ranges". In: *Proc. Amer. Math. Soc.* 149.10 (2021), pp. 4317–4329.

[Oli21b] J. Oliva-Maza. "On the connection between Hardy kernels and reproducing kernels". In: *TEMat monográficos* 2 (2021), pp. 187–190.

[Oli22a] J. Oliva-Maza. "On Hardy kernels as reproducing kernels". In: *Canad. Math. Bull.* (2022), pp. 1–16.

[Oli22b] J. Oliva-Maza. "Spectral mapping theorems for essential spectra and regularized functional calculi". In: *arXiv:2206.13955v2* (2022).

[OW23] J. Oliva-Maza and M. Warma. "Introducing and solving generalized Black-Scholes PDEs through the use of functional calculus". In: *J. Evol. Equ.* 23.1 (2023), pp. 1–40.

[Per08] A.M. Persson. "On the spectrum of the Cesàro operator on spaces of analytic functions". In: *J. Math. Anal. Appl.* 340.2 (2008), pp. 1180–1203.

[Pie60] A. Pietsch. "Zur Theorie der σ-Transformationen in lokalkonvexen Vektorräumen". In: *Math. Nach.* 21.6 (1960), pp. 347–369.

[PB00] Y. Polatoglu and M. Bolcal. "Some results on the Janowski's starlike functions of complex order". In: *arXiv preprint math/0007133* (2000).

[Rho81] B.E. Rhoades. "Generalized Hausdorff matrices bounded on ℓ^p and c". In: *Acta Sci. Math.* 43 (1981), pp. 333–345.

[Rho89] B.E. Rhoades. "The point spectra for generalized Hausdorff operators". In: *Acta Sci. Math.* 53 (1989), pp. 111–118.

[Roy08] J.J. Royo. "Convolution algebras and modules on $\mathbb{R}^+$ defined by fractional derivative,(in spanish)". PhD thesis. Universidad de Zaragoza, 2008.

[Rud91] W. Rudin. *Functional analysis*. 2nd ed. Singapore: McGraw-Hill, 1991.

[SKM93] S.G. Samko, A.A. Kilbas, and O.I. Marichev. *Fractional integrals and derivatives. Theory and applicatons*. New York: Gordon-Breach, 1993.

[Sch66] M. Schechter. "On the essential spectrum of an arbitrary operator. I". In: *J. Math. Anal. Appl.* 13.2 (1966), pp. 205–215.

[Sch00] S. Schweiker. “Asymptotics, regularity and well-posedness of first-and second-order differential equations on the line”. PhD thesis. Verlag nicht ermittelbar, 2000.

[Sef06] H. Seferoğlu. “A spectral mapping theorem for representations of one-parameter groups”. In: *Proc. Amer. Math. Soc.* 134.8 (2006), pp. 2457–2463.

[Sis85] A.G. Siskakis. “Semigroups of composition opeators and the Cesàro operator on H^p”. PhD thesis. Univ. of Illinois, 1985.

[Sis86] A.G. Siskakis. “Weighted composition semigroups on Hardy spaces”. In: *Linear Algebra Appl.* 84 (1986), pp. 359–371.

[Sis87] A.G. Siskakis. “Composition semigroups and the Cesàro operator on H^p”. In: *J. London Math. Soc.(2)* 36.1 (1987), pp. 153–164.

[Sis93] A.G. Siskakis. “The Koebe semigroup and a class of averaging operators on $H^p(D)$”. In: *Trans. Amer. Math. Soc.* 339.1 (1993), pp. 337–350.

[Sis98] A.G. Siskakis. “Semigroups of composition operators on spaces of analytic functions, a review”. In: *Contemp. Math.* 213 (1998), pp. 229–252.

[SW13] L. Song and W. Wang. “Solution of the fractional Black-Scholes option pricing model by finite difference method”. In: *Abstr. Appl. Anal.* Vol. 2013. Hindawi. 2013.

[ST95] H.M. Srivastava and V.K. Tuan. “A new convolution theorem for the Stieltjes transform and its application to a class of singular integral equations”. In: *Arch. Math. (Basel)* 64.2 (1995), pp. 144–149.

[Ste94] K. Stempak. “Cesàro averaging operators”. In: *Proc. Roy. Soc. Edinburgh Sect. A* 124.1 (1994), pp. 121–126.

[Sti18] T.J. Stieltjes. *Œvres Complètes II.* Groningen: Noordhoff, 1918.

[TL58] A.E. Taylor and D.C. Lay. *Introduction to Functional Analysis.* New York: Wiley & sons, 1958.

[TE+51] F.G. Tricomi, A. Erdélyi, et al. “The asymptotic expansion of a ratio of gamma functions.” In: *Pac. J. Math.* 1.1 (1951), pp. 133–142.

[Wol59] F. Wolf. “On the essential spectrum of partial differential boundary problems”. In: *Commun. Pure Appl. Math.* 12.2 (1959), pp. 211–228.

[Xia97] J. Xiao. “Cesàro-type operators on Hardy, BMOA and Bloch spaces”. In: *Arch. Math.* 68.5 (1997), pp. 398–406.

[Yoo51] B. Yood. “Properties of linear transformations preserved under addition of a completely continuous transformation”. In: *Duke Math. J.* 18.3 (1951), pp. 599–612.

[ZLTY16] H. Zhang, F. Liu, I. Turner, and Q. Yang. “Numerical solution of the time fractional Black–Scholes model governing European options”. In: *Comput. Math. Appl.* 71.9 (2016), pp. 1772–1783.

[Zhu93] K. Zhu. “Bloch type spaces of analytic functions”. In: *Rocky Mountain J. Math.* 23.3 (1993), pp. 1143–1177.