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Pre-service teachers develop their mathematical knowledge for teaching using manipulative materials in mathematics

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Abstract

This manuscript aims to describe aspects of mathematical knowledge for teaching, MKT, identified in pre-service teachers (PSTs) when explaining an arithmetic property using manipulative materials. In particular, we are interested in the specialized mathematical knowledge, SCK, the pedagogical knowledge related to teaching, KCT, and the knowledge of content and curriculum, KCC. We proposed to record a video to a sample of 27 primary education students enrolled in their first mathematics education course. They had to explain an arithmetic property of natural numbers using manipulative materials. PSTs do not create contexts by the mere presence of manipulative material, but only rely on it for visual purposes; the meaning of these values are modified during the explanation. Evidence has been found of difficulties relating to the SCK such as the inadequate varying of the meanings given to the manipulative material, and to the KCC such as the selecting of an unsuitable material.

Keywords: pre-service teachers, manipulative materials, arithmetic properties, MKT

INTRODUCTION

The results of the 2008 teacher education and development study in mathematics (TEDS-M) for pre-service teachers (PSTs) in Spain regarding mathematical knowledge and mathematics didactics were not positive (Lacasa & Rodríguez, 2013). On the other hand, difficulties in the learning of PSTs are related to their lack of abstraction and generalization (Godino et al. 2003). Research works should seek explanations or solutions to the problems detected in teaching or learning (Popper, 1997); in our case we will try to approach the description of the difficulties associated with the skills, mathematical and didactic, shown by PST in relation to the teaching of arithmetic properties as a first step in the search for their solution.

The training in mathematics education of PST should contain scientific and pedagogical aspects; these aspects are clearly represented in the theoretical framework developed by Ball et al. (2008) based in the seminal work of Shulman (1986): MKT (mathematical knowledge for teaching) that includes the domains SMK (subject matter knowledge) and PCK (pedagogical content knowledge) whose subdomains we can see in **Figure 1**.

According to the studies that have worked with arithmetic properties (Ding et al., 2013) this training should include the learning of specialized mathematical content (SCK, subject content knowledge–subdomain of the SMK) that covers the mathematical content that a teacher needs from a teaching point of view (Ball et al., 2008), and aspects that future mathematics teachers would have to reinforce (Graciano-Barragán & Aké, 2021). In addition, pedagogical knowledge related to teaching (KCT, knowledge of content and teaching–subdomain of PCK) also appears as necessary in the design and implementation of activities for students (Butterfield & Chinnapan, 2011; Hill et al., 2008): teaching situations require the construction of statements, contexts, ... to set what Borasi (1986) called word-problems, which is an activity of some difficulty, since it implies being able to decompose the concept, property, ... to be explained into its basic components and to know how they are related to each other. This difficulty contributes to the often exclusively procedural treatment of the teaching of number sets and their properties (Montes et al., 2015), forgetting that primary school teachers need a better understanding of the properties of operations (Chapin et al., 2021).

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Contribution to the literature

- In this study we describe aspects of SMK and PCK developed by PSTs when explaining an arithmetic property using manipulative material.
- In our results we show how prospective teachers develop mathematical content (SCK), in particular the meaning of intermediate operations of the arithmetic property.
- PSTs showed difficulties in choosing appropriate numerical values and material to develop their teaching activity (KCT and KCC).

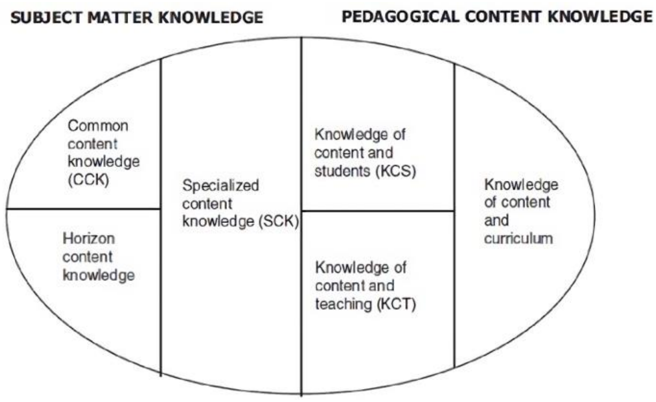


Figure 1. MKT model (Ball et al., 2008)

Studies by Skemp (1987) substantiate that early experiences and interactions with physical objects support later abstract learning leading to effective mathematics instruction (Pham, 2015). In addition, numerous studies support the use of manipulative materials in the formation of PST (Moyer, 2001; Pham, 2015). On the other hand, Maboya (2014) shows in her study that the use of manipulative materials entails a process of exploration by teachers that fosters the learning of specialized mathematical content (SCK). Several authors (Hodgen et al., 2018; Maboya, 2014) point out that the importance of the use of manipulative materials in mathematics also lies in establishing connections between mathematical ideas, concepts and procedures both in teaching and learning. Green et al. (2008) go further, stating that manipulatives can reverse old arithmetic misconceptions and facilitate increases in arithmetic knowledge before the PST reaches its future classroom.

In this work a group of PST in their second year of the primary teaching degree faces the construction of situations for the teaching of basic but abstract questions, such as the arithmetic properties of natural numbers. In particular, we work on the following property: for any a, b and c belonging to the set of natural numbers, it is satisfied that: $a:(b:c) = (a:b)xc$. The result of the operations that appear on both sides of the equal symbol need not belong to this set, so we say that the property is not internal to the set of natural numbers, this is clearly observed in the tern $(a,b,c) = (6,4,2)$ $6:(4:2) = (6:4)x2$ i.e., $6:2 = 1{,}5x2$.

For the construction of such teaching situations, PSTs can use manipulative materials of their choice. From the above, we state our research question: What aspects of the teacher's mathematical knowledge (MKT) are employed when PSTs explain arithmetic properties using manipulative materials? We will try to approach the answer through two more specific objectives:

1. To describe aspects of subject matter knowledge (SMK), in particular specialized mathematical knowledge (SCK) displayed by PSTs when explaining arithmetic properties using manipulative material.
2. To describe the aspects of pedagogical knowledge (PCK), in particular pedagogical knowledge related to teaching (KCT) and curriculum knowledge (KCC) that PSTs show when explaining arithmetic properties using manipulative material.

THEORETICAL FRAMEWORK

MKT

SMK includes the subdomains common content knowledge (CCK), specialized content knowledge (SCK) and mathematical horizon knowledge (HCK). CCK includes mathematical knowledge used at different moments other than teaching. For example, knowledge of the arithmetic property. SCK encompasses mathematical knowledge specific to teaching (e.g., the different interpretations of the fraction). HCK incorporates how mathematical concepts are related in the framework of the mathematics included in the curriculum (e.g., the validity of a property in other numerical sets).

PCK includes the subdomains knowledge of content and students (KCS), knowledge of content and teaching (KCT), and knowledge of content and curriculum (KCC). KCS includes knowledge of how students interact with the discipline. For example, their interpretation of the equal sign. KCT encompasses knowledge related to instructional design. For example, the development of problem statements. KCC incorporates knowledge of the mathematics curriculum and the most appropriate resources to use. For example, the selection of specific manipulative materials.

Consider the property $a:(b:c) = (a:b)xc$. Knowing that $12:(6:2) = (12:6)x2$ is an example of the property can be considered CCK, a content that some people can have without being a teacher. But knowing what representations to show to teach this property in Primary Education requires SCK. On the other hand, within PCK we also find subdomains with a clear relationship with the teaching of the arithmetic property that concerns us. Knowing about students' comprehension problems is KCS, while developing strategies to help them overcome them would be part of KCT. To help us specify what is and what is not KCS, Ball et al. (2008) list a number of activities including, among others: developing examples to support a mathematical statement or connecting representations to the mathematical ideas that support them or to other representations.

Arithmetic Properties and Word Problems

Working with arithmetic expressions gives rise to relational thinking in which the algebraic part is connected with the generalization of patterns and relationships, being able to examine expressions in a global way and take advantage of them to solve a problem, decide or continue learning about a concept (Molina et al., 2006). This same line of reasoning is followed by Carmenates et al. (2005) who affirm that the search for relationships and, consequently, the development of relational thinking, are very useful for solving mathematical problems. Considering relational thinking implies considering arithmetic expressions and equations as a whole and not only as procedures to be related step by step, but that is also, using the fundamental properties of operations to, for example, relate expressions (Carpenter et al., 2003). Relational thinking implies understanding the equal sign as a balance between the right-hand and left-hand side, being able to focus on the relationships between arithmetic operations and their properties, rather than on their calculation (Fernández & Ivars, 2016). On the other hand, Castro and Molina (2007) consider that the understanding of the equal sign presents difficulties in primary school students, as well as other type of algebraic symbology (Cañadas et al., 2018), since they tend to consider it as a means to answer.

One of the options when trying to explain an arithmetic property is to pose a contextualized situation in which an element is unknown (word problem), although it does not necessarily meet all the conditions to be called a mathematical problem in the sense of Carrillo (1998, p. 87): "... the concept of problem must be associated with the meaningful (not mechanical) application of mathematical knowledge to unfamiliar situations, the awareness of such a situation, the existence of difficulty when facing it ..."

In fact, this type of problem has always been an important part of school mathematics all over the world (Verschaffel et al., 2020). Ding et al. (2013) propose word problem statements that can be solved in two ways as one of the strategies to present arithmetic properties. These authors study the associative property of multiplication and how PSTs design strategies for teaching it from a given statement: most of them proposed solving the problem in two different ways as a teaching strategy. Butterfield and Chinnapan (2011) propose problem posing as a way of developing the KCT sub-domain in the training of PSTs.

Use of Manipulative Materials

Since the early twentieth century, the use of manipulative materials has served as a tool for the development of mathematical and scientific knowledge. Working with them in the classroom facilitates the ability to compose numbers, to understand the structure of the number system and to be able to use arithmetic properties and the existing relationships between them (Bartolini & Martignone, 2020; NCTM, 2003). On the other hand, Boggan et al. (2010) determine that when manipulative materials are used in a practical way, students begin to construct their own mathematical understanding.

Moyer (2001), after a specific training to mathematics teachers of different educational levels, emphasizes the relevance of creating a context to foster learning when using manipulative materials and the possibility of showing abstract mathematical concepts from them. However, they verify that many of these teachers use these materials to change the pace of the subject, provide a more visual model or make it more fun, resisting the constructivist epistemology and misinterpreting the potentiality of the materials. Their use, following this author's assertions, is more complicated than it seems; the student's internal representation must be connected with the manipulative representation, and knowledge will be obtained with the relationship between the two.

The teacher's job is not only focused on how to teach mathematics from manipulative tasks, but to choose quality tasks and guide students towards a deep mathematical understanding (Maboya, 2014). Thus, the way in which manipulative materials are used in the classroom and the very selection of manipulatives will depend on the teacher's knowledge of the mathematical concepts involved and the interrelationships among them. Thus, the teacher's beliefs and pedagogy can only serve to support this knowledge. The teacher's selection of material can influence the understanding and thinking of his students (Hiebert, 1997), so much so that a proper selection of material allows word problems to be solved in two different ways (Borasi, 1986) and to justify their solutions on the basis of manipulation (Baroody, 1989).

Examples

According to Borasi (1986), examples play a central role in mathematics pedagogy and are frequently used in the teaching of elementary mathematics. In this sense,

Table 1. Variables & categories used in the analysis

Variable (subdomain)	Categories
Contextualization (KCT)	Includes explicit context/does not include (Borasi, 1986)
Variety of materials (KCC)	Single material/different materials
Meaning given to indeterminate in relation to material (SCK)	Representation only/variable/stable
Choice of numerical values (SCK)	No indication/some value is 1/all values are powers/quotient equal to third/different and not powers (Rowland, 2008)
Meaning of intermediate operations (SCK)	Division: Partitive/quotative (Fischbein et al., 1985) Multiplication: Repeated addition/meaningless
Property verification (SCK)	Not tested/only with material/only numerically/with material and numerically

a possible alternative approach to the statement of a problem would be the exposition of a series of concrete examples of application of the property in order to test its validity. We understand that an example is a particular case of a broader class of mathematical objects from which it is possible to generalize mathematical knowledge (Zodik & Zaslavsky, 2008), i.e., there must be a didactic intentionality in the choice of the example. According to Planas et al. (2018), to constitute a complete explanation of the property, these examples should at least meet the criteria of similarity and contrast. In this regard, they detail the criteria that a set of examples must meet to be usable for the development of students' mathematical learning: the elements that should remain invariant in the teaching object appear in the examples (similarity); the variation of elements that should remain invariant appear in them by showing them as non-examples or special cases (contrast). In order for teachers to create these sequences, they must have an adequate mathematical level (Zodik & Zaslavski, 2008), particularly in the SCK domain. In this sense, several studies (Rowland, 2008; Rowland et al. , 2003) identify three possible problems when formulating examples: that they hide the role of the variables in the example (two variables take the same value in the example), that the example designed to illustrate one procedure is more appropriate for another (proposing 302-299 to be solved by decomposition instead of counting from the subtrahend to the minuend) and that the examples are randomly generated (with a die for example).

Likewise, through set of examples, counterexamples and inferences it is possible to reach the proof or refutation of certain implications (Lee, 2016). Their choice may be unintentional, leading to naive conviction or persuasion, without arriving at inductive reasoning on relevant mathematical examples (Balacheff, 1988).

METHODOLOGY

The sample, taken on a purposive basis, consisted of 27 PSTs. All of them were in the 2nd year of the primary education degree. The subject in which they carried out the task included the properties of the natural number and its teaching and learning. They had not previously studied any subject that dealt with mathematical content or didactics of mathematics, nor any subject that dealt with the use of didactic materials from a more general point of view.

Each PST was given the task of making a video to explain, using a manipulate material of his/her choice, an arithmetic property according to the following statement:

Assignment: Make a video explaining, with a material you can manipulate, the property $a:(b:c) = (a:b)xc$ $(a, b, c\ are\ natural\ numbers)$.

The choice of this arithmetic property was motivated by the fact that other mathematical properties of the natural numbers, such as commutative and associative properties, among others, had been explained in the classroom. Likewise, before addressing the didactics of arithmetic for natural numbers, the subject had addressed three previous topics: legislative framework and curricular design in the area of mathematics, planning and design in the teaching-learning process of mathematics, and assessment in mathematics.

The PSTs had one week for the individual elaboration of the video, as well as the previous reflection of the mathematical and didactic aspects to be considered. Subsequently, it was sent through the Moodle of the subject. The study is of an exploratory nature with a fundamentally descriptive purpose (Elliot & Timulak, 2005). The approach is qualitative in which a frequency analysis of the variables considered is also carried out. These variables are contextualization, variety of materials, meaning given to the indeterminates in relation to the material, choice of numerical values, meaning of the intermediate operations and verification of the property (**Table 1**).

The activity proposed is consciously open and can be approached from different points of view, among them the proposal of a contextualized situation, giving rise to the *contextualization* variable. Following Borasi (1986), this variable gives rise to two possible categories: the first one if the PST does not provide the material to be chosen with a context, and the second one when it does. This variable is associated with the KCT subdomain since PSTs have to propose an appropriate situation for teaching the arithmetic property. Note that the existence or not of a context is not determinant in qualifying an activity as a problem. However, in the case that our PSTs

Figure 2. Use of materials without context (PST#4) (Source: Author's own elaboration)

pose a contextualized situation, we expect it to be a word-problem (Borasi, 1986): an explicit context, a unique solution, and a resolution by combining known algorithms.

As we have already mentioned above, the PSTs were able to choose the manipulative material to be used, giving rise to the *variety of materials* variable. In an inductive way, we have established the categories a single material or diverse materials. This selection can facilitate the creation of a context for demonstrating mathematical concepts (Moyer, 2001). In addition, this variable is related to the KCC subdomain (Ball et al., 2008).

The meaning taken by the material with which each indeterminate a, b and c is represented during the representation of the arithmetic property gave rise to the variable *meaning given to the material*. The categories considered, inductively, for this variable are representation only, only the meaning of cardinal is considered; variable, during the course of the explanation or operations the meaning of some indeterminate(s) varies; stable, each of the indeterminates is represented by a different manipulative material and does not modify its meaning during the whole explanation. This last value is the one that students are expected to reach in their training in didactics of mathematics. This variable is associated with the SCK subdomain given that the choice of material requires a certain mathematical understanding (Maboya, 2014).

As we indicated in the previous section, Planas et al. (2018) consider that the choice of the concrete numerical values that appear in the examples can facilitate or hinder the understanding of the explanations. We extend this statement to the examples proposed with the use of materials by assigning a concrete numerical value to each indeterminate, giving rise to the variable *choice of numerical values*. For the categories considered in this variable, we took as a starting point the studies of Rowland et al. (2003) with trainee teachers. Five categories were taken into account: if the PST does not indicate the numbers to be used; if one of the numerical values is the unit; if all of them are powers of the smallest; if the quotient of two of them is equal to the third without being all powers of the smallest; and finally, when all the numerical values are different from each other, are not powers of the first of them and the quotient of two of them are not equal to a third. In this last case we include the possibility of a/b=b/c since they are not on the same side of equality. We relate the first four categories to the first problem detected by Rowland (2008) in which the role of the variables in the example under consideration is hidden. This variable is associated with the SCK subdomain since, as mentioned by Zodik and Zaslavsky (2008), it is possible to generalize mathematical knowledge from an example.

The arithmetic property involves the arithmetic operations of multiplication and division, so in this study we consider the variable *meaning of the intermediate operations*. In the case of multiplication, we contemplate repeated addition and mere calculation without any reasoning. For division we establish two possible meanings: partitioning, partitive division; and grouping, quotative division (Fischbein et al., 1985). In addition, Simon (1993) detected a certain weakness, on the part of teachers, in the knowledge of the relationship between both meanings, real context problems and the identification of units. This variable is associated with the subdomain SCK since it gives information about the students' knowledge of the different meanings that multiplication and division can adopt.

Finally, we examined the variable *property testing,* which will take the values yes or no, both for testing with manipulative material and numerically. Its analysis is descriptive in nature. This variable is associated with the subdomain SCK since it gives information on the knowledge of the PSTs about the level of certainty obtained from the work developed.

RESULTS

We show below the results obtained in the study in which all the PSTs explain the property taking as a starting point a specific choice of numerical values to the indeterminates, which is a specific example according to Zodik and Zaslavsky (2008).

Contextualization

We find that most of the PSTs, 81.5%, although they use manipulative materials, do not provide a context that relates them globally. Only 18.5% pose a problem that serves as a context relating the materials present. In these contexts, we find distribution of candies among students seated in pairs, large and small bowls to store caps, cups and groups of cups to store pens, among others.

As an example of absence of context, we show PST#4 (**Figure 2**) who proposes the use of matches to represent the values of a, b, and c on both sides of the equality and manipulate them in parallel. However, these matches only serve to represent the numerical values assigned to the indeterminates initially and the results of the intermediate operations. At no point has any contextualized problem been posed, but only formal operations with material support have been presented.

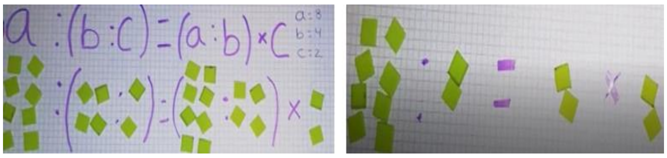
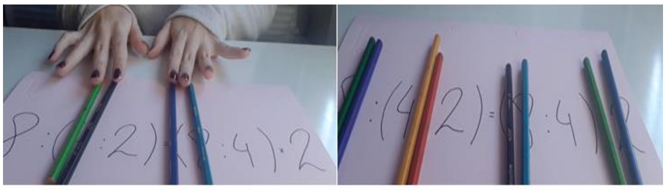


Figure 3. Materials used by PST#22 (Source: Author's own elaboration)



Left side Right side

Figure 4. Meanings of material on each side of equality (PST#25) (Source: Author's own elaboration)

During the video, PST#4 represents the cardinal of each set of matches 8, 4, and 2, twice, once for each side of equality and describes the following:

> PST#4: "We are going to operate, as follows: 8 divided by 4 divided by 2, these two [pointing to 4 matches and 2 matches] would go in parentheses, equal to 8 divided by 4 that would go in parentheses, by 2. As the first parenthesis we have 4 divided by 2, we substitute these matches for 2 matches."

Subsequently, PST#4 continues to replace the existing matches for each of the operations.

As we can see in the transcript, in spite of using a manipulative material, such as matches, a contextualized problem is not included. The matches are only the cardinal presentation of the number.

Variety of Materials

Our PSTs are clearly divided into two groups with respect to the variety of the materials: 63% of them use only one material throughout the explanation to represent the three variables present in the property; on the other hand, 37.0% of the PSTs select two or three different types of material to represent the three variables. Among the most used materials are markers, matches, candies, and glasses.

Meaning Given to Indeterminates with Respect to the Material

Some of the PSTs use the selected material only to represent the numerical values assigned to the indeterminates and the results of the intermediate operations (**Figure 3**), placing as many objects as indicated by each number that appears (25.9% of the total). All these PSTs use the same material for the representation of all the indeterminates present in the property. For example, PST#22 uses only some pieces of

Left side Right side

Figure 5. Meanings of material on each side of equality (PST#2) (Source: Author's own elaboration)

Figure 6. Material used by PST #16 (Source: Author's own elaboration)

paper, which he moves to represent the numbers 8, 4, and 2 (**Figure 3**–left), as well as the results obtained in the partial operations (**Figure 3**–right). PSTs who use the material in this way only represent the numbers that appear and solve the operations mentally.

Within this large group we find two well differentiated cases: those who, as in the previous case, use the same material (37.1% of the total) and those who use two different materials (25.9% of the total). We present two examples of this situation, PST#25 and PST#2. PST#25 uses a single material, pencils (**Figure 4**). The values of the tern (a, b, and c) are (8, 4, and 2). The indeterminate b, which is assigned the value 4, acquires two meanings, on the left side of the equality it represents 4 pencils while on the right side it represents 4 groups of pencils.

PST#2 uses different materials and modifies the meaning of the numbers (**Figure 5**). In the left part of the equality, the PST uses 8 markers, 4 glasses and 2 groups of glasses. In the second one, the PST keeps the 8 markers and 4 cups, but the last number changes its meaning becoming an operator that doubles the number of markers per cup.

Finally, we find some PSTs that maintain the meaning of each of the indeterminates throughout the explanation (11.1% of the total), needing to use two or three different materials. PST#16 uses 16 bottle caps, 4 large bowls and 2 small bowls, in both sides of the equality (**Figure 6**).

In the video, we note that for the first part of equality he states the following:

> PST#16: "We have 2 large bowls inside each small bowl [...] We are going to distribute the bottle caps one by one inside each small bowl. The final result is the number of caps in each small bowl."

Left side Right side

Figure 7. Final representations on each side of equality of PST#8 (Source: Author's own elaboration)

Subsequently, for the second part of equality, he says:

> PST#16: "We are going to divide our 16 caps in our four bowls [indicates the large ones] [...] We place two large bowls into each small bowl. The result is the number of caps we have in each small bowl."

In this transcription, PST#16 gives a cardinal meaning to the number 16 (caps) and a group meaning to the numbers 4 and 2 (groups formed in the large and small bowls).

Choice of Numerical Values

The PSTs were not restricted in their choice of numerical values, even so, 40.8% chose two of the numbers to be powers of the smallest. In most of these cases, the values selected were 2, 4, and 8. In addition, 25.9% of the sample components used numbers whose quotient was equal to the third one, such as 12, 6, and 2, or 16, 8, and 4. 3.7% did not indicate the numbers to be used, making any manipulation option impossible. On the other hand, 14.8% took the unit as the value of c. Only 14.8% of the PSTs took values different from the three numbers without the particular characteristics mentioned. It must be said that, although this property is valid for any values of the tern (a, b, and c) with a, b, c belonging to the set of natural numbers, all the PSTs imposed to the chosen numerical values that a:b and b:c were natural numbers.

The work of PST#8 shows the relevance of this variable, PST#8 chooses (8, 4, and 2) as values from the tern (a, b, and c). The chosen numbers lead to the appearance of the number 2 with four different functions in the explanation: At the beginning of the video, he works on the left side of the equality (**Figure 4**) assigning the value 2 to c (the number of groups in which he is going to distribute the 4 straws). After doing the distribution and the previous operation, a second 2 appears, which is the number of straws in each of the 2 groups. He says: "finally, we have 2 groups with 4 straws each, which is what we wanted to check", without specifying the result of the left side of the equality. Later, working on the right side of the equality, he divides 8 straws into 4 groups and says: "thus, we have 4 groups with 2 straws each". Finally, he notes that "we want each group to have twice as many straws as we have," he says pointing to the 2 on the right side of the equality. "Since we have 2 straws, the double is 4 and we keep distributing". At this point he takes 8 more straws and hands out 2 extra straws to each of the 4 groups. We can observe in **Figure 7** the presence of 16 straws to represent the right side of equality.

Meaning of Intermediate Operations

In each of the intermediate operations performed, we have distinguished whether the PSTs give a meaning or not. Moreover, in the case of divisions, we established two possible meanings (**Table 2**): distribution (partitive division) and grouping (quotative division). In addition, we find the non-representation of the result and the realization of the division in a formal way without the use of manipulative material.

77.8% of the PSTs give a meaning to the operation b:c, 51.9% of them perform a division with the material, while 25.9% form groups with it. On the other hand, 22.2% of the PST represent only the result, carrying out the division without the need of the manipulative material. An example of division is found in the work of PST#17 in which b=6, paper clips, and c=2, groups, are considered.

> PST#17: "We have 6 orange paper clips divided into 2 groups. We make 2 groups, distributing the clips equally, 3 clips in each group."

While an example of grouping can be seen in PST#27, considering b=6, pencils, and c=3, number of pencils in each group.

> PST#27 "We have 6 pens, and we want to divide into groups of 3. We have 2 groups of 3."

Although the final representation of the division is the same in both cases, the values taken at the beginning are different as we have observed in their speeches.

Regarding the operation a:(), 74.1% of the PSTs give a meaning to the operation: 55.6% distributing and 18.5% grouping. However, 25.9% do not give meaning to this operation: 18.5% only represent the final result while 7.4% do not even represent the final result. In the case of the a:b division we find the same percentages as for the operation a:() of those who give a meaning.

In the case of multiplication, it is usually shown as a repeated addition (59.3%) or as a representation of the final result without meaning (40.7%). In this second case, multiplication is done in a formal way without using manipulative material and without explanation of the followed process.

Property Verification

The last of the variables studied is property verification. A total of 51.9% of the PSTs checked the correctness of the property by means of the material and arithmetically. In addition, another 22.2% checked only

Table 2. Meaning of intermediate operations

	Meaningless		Meaningful	
	Non-representation of result	Representation of result only	Distribution	Grouping
b:c	0 (0.0%)	6 (22.2%)	14 (51.9%)	7 (25.9%)
a:()	2 (7.4%)	5 (18.5%)	15 (55.6%)	5 (18.5%)
a:b	1 (3.7%)	6 (22.2%)	15 (55.6%)	5 (18.5%)

Figure 8. Material used by PST#1 (Source: Author's own elaboration)

arithmetically and 14.8% only by means of the material. Only 11.1% of the PSTs did not check the property at all.

PST#1 checks both through the material and arithmetically the property. He poses a problem to perform such a check:

> "In my class there are six students sitting in pairs. I want to distribute 12 pieces of candy among pairs. How many pieces of candy will get each pair?"

In addition, in each of the cases the PST verbalizes what kind of verification is being performed: "now we are going to verify numerically", "the same as in the previous case", "the numerical result agrees with the one we have obtained with the figurines". The problem is solved in two different ways, corresponding to the two sides of the property, leading to the same result. Both processes start and end in the same way (**Figure 8**), but in the first one she distributes the candies by pair, while in the second she distributes the candies to each student (figurine) and group them in pairs after that.

DISCUSSION AND CONCLUSIONS

We have analyzed how PSTs use materials to explain an arithmetic property of natural numbers. The variables we have employed inform on three subdomains of MKT, one concerning mathematical knowledge (SCK) and two about pedagogical knowledge (KCT and KCC). Given that this task was carried out in a mathematics education course of a teaching degree, it could have happened that other subdomains would have been present in the explanations. For example, the HCK, in the case that a PST had commented on the validity of the property in other sets such as the rational ones. We consider the proposed task to be appropriate for MKT development as it creates opportunities for flexible understanding of mathematical ideas, leads to the use of multiple representations and solving methods, and gives opportunities to carry out mathematical practices important for teaching such as explaining, representing, or asking questions (Suzuka et al., 2009). Next, we contrast the results obtained in each *variable* of analysis with previous research.

Regarding *contextualization*, we can state that the mere presence of materials has not induced, in many cases, the creation of contexts to explain the property globally. On the contrary, they were mostly used during the solving of the partial operations on each side of the equality to verify that the same result was obtained on both sides. These operations were fundamentally formal, the material playing only a role of visual support. A global context in which a question involving the use of this property will arise could be considered a problem. In this sense, our results agree with Ding et al. (2013) who showed how working with problem statements is difficult for PSTs when studying the associative property of multiplication. The low percentage of explanations that include contextualization (18.5%) could be improved with an intense practice of *problem-posing* involving different operations with natural numbers.

Regarding the variety of materials used and the meaning given to them, Charalambous and Hill (2012) affirm that they are a determining factor in the quality of teaching, especially when they help the teacher's practice, allowing to support the construction of meanings, among other aspects. Consequently, the variables *variety of materials* and *meaning given to the indeterminate in relation to material* are closely related to the discussion of the results obtained. One of the concerns of the current study was the meaning of the indeterminates when they were represented with the materials chosen by the PST. Aspects related to the selection of the materials, as already established by Hiebert (1997), can influence the understanding that students reach and their way of thinking about mathematics. In the first place, we find the PSTs that select a single material. Among these we distinguish two cases: those who use the material without giving them any meaning and those who realize as the explanation progresses the need to give them different meanings. In both cases, the choice of a single material prevents a deep understanding of the task (Maboya, 2014) when trying to combine the different meanings. Secondly, some PSTs choose two or more manipulative materials to work with the property. In this case we must also distinguish those that alter the meaning of the numerical values during the course of the testing, from those in which the meaning is maintained throughout the manipulation. Only in the latter case can we claim that the PST has a deep mathematical understanding of the representation of the operations involved (Boggan et al., 2010). When the meaning of the variables in the expression changes from one side of the equality to the other it makes a correct interpretation of the formula difficult and leads, at best, to a merely formal learning of the formula.

Concerning the variable on *the choice of numerical values* assigned to the tern (*a, b,* and *c*), we can say that in most cases the numbers were smaller than 15 and their choice was made in order to facilitate the solving of intermediate operations, rather than searching a certain generality. Rowland (2008) identified the challenges that arise when examples consistently have very specific characteristics. In particular, all the PSTs chose the values of *a*, *b* and *c* such that the intermediate ratios *a:b* and *b:c* were also natural numbers and some of them chose the same numerical value for different indeterminates. Although this could have to do with the PSTs considering the presented property to be internal, we understand that it may be motivated by the type of material they chose, which was always a discrete set of objects and in no case fractionable. No PST considered other possibilities, for example, taking an object of a certain length and working with it. This fact can be justified because students perform better on partitioning tasks in discrete contexts (Llinares & Sanchez, 1988). The choice of numerical values in the approach of didactic situations has been long studied, especially in relation to the construction of examples (Planas et al., 2018). Considering that the criteria of similarity and contrast must be met, a series of chained examples must be posed rather than a single one as all our PSTs posed. Given the problems they showed, we affirm that they either had difficulties with the mathematical content or with the translation of such content to the teaching activity (Zodik & Zaslavsky, 2008).

Most of the PSTs give *meaning to the intermediate operations* they must face to verify the property. Thus, they understand the concepts of division and multiplication, and not only know how to operate with algorithms. Concept and algorithm are two aspects related to each arithmetic operation that are often confused assuming that the correct use of the algorithm is associated with the understanding of the concept (Fuentes & Olmos, 2019). Those who understand division, and consequently grant meaning to each of the divisions that appear in the property, employ one of the two interpretations offered by Fischbein et al. (1985): partitive division or quotative division. In many cases, the appearance of multiplication in the arithmetic property does not generate any kind of reasoning nor is it given any meaning outside the formal one; they simply need to solve the multiplication so that the numbers obtained on both sides of the equality coincide. Probably, if the left and right sides of the property had been interchanged, the multiplication would have been interpreted according to the materials and the interpretation of the division *a:()* would have been postponed. There is a computational tendency to proceed from left to right, conceiving the equal sign as a command that makes it possible to give an answer (Castro & Molina, 2007).

Regarding the *property verification*, we note that most of the PSTs give an example either with materials, numerically or both. The rest, since they do not verify it in any way, do not realize the equality between the obtained results. In this sense, it is interesting to point out that 33.3% of the PST do not use the material to carry out the verification. Note that they had selected it by themselves, and the perception or physical manipulation of the materials could increase the certainty and persuasion (Ibañez & Ortega, 2001).

The first objective of the work, related to describing the aspects of the SCK, has been achieved in greater depth than the second one, related to describing the aspects of the KCT and KCC. This is not surprising, since it is a classroom task and not a real explanation, where the Primary school students' reactions or their questions would have determined the subsequent responses of the PST and their abilities would have been more evident: regarding KCT, such as the ability to reformulate an example, or HCC, such as the ability to consider the appropriateness of the chosen materials. We consider it interesting to analyze how they would put into practice in a real situation the explanation of this property and review the role of each subdomain of the MKT framework in this new situation.

In short, according to research question formulated at the beginning of this paper, we can state that the aspects of the teacher's mathematical knowledge (MKT) used by the PSTs to explain the arithmetic property at hand with manipulative material are both the SCK, pertaining to content knowledge (SMK), and the KCT and KCC, pertaining to pedagogical knowledge (PCK).

As a limitation of this study, and pretending to be a future perspective of it, we could point out the convenience of sharing some of the videos with the PSTs. This moment would also be used to know the reasons for choosing certain numbers, as well as the noncontextualization of the statements. All this would make it possible to adapt the teaching of this arithmetic property, and others of a similar nature, carried out by these PSTs with their future students.

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