CONFORMAL YANGIAN AND TREE AMPLITUDES IN SCALAR AND GAUGE FIELD THEORIES

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ABSTRACT

Nikolaos Dokmetzoglou: Conformal Yangian and Tree Amplitudes
in Scalar and Gauge Field Theories
(Under the direction of Louise Dolan)

Scattering amplitudes are intimately related to both experimental and theoretical ongoing efforts of testing the Standard Model of particle physics and our understanding of quantum field theory at large, through their computation to higher orders of precision and the study of their often unexpected and fascinating mathematical properties. It is within this latter field of study that our research lies.

We investigate the infinite-dimensional Yangian extension of the conformal group $\mathrm{SO}(2,n)$, where n is the number of space-time dimensions, and its action on the tree-level scattering amplitudes of scalar $\lambda\,\phi^3$ theory and pure Yang-Mills theory. These two non-supersymmetric field theories are connected through the Cachazo-He-Yuan (CHY) scattering equations formalism.

We first establish the consistency of the conformal Yangian algebra, $Y[\mathrm{SO}(2,n)]$, for a differential operator representation of its generators in momentum-space. We prove that this representation satisfies the Serre relation, off-shell and in any number of space-time dimensions n for scalar fields, but only on-shell and in $n = 4$ space-time dimensions for spin-one gauge fields.

We then show that the conformal Yangian generators annihilate individual off-shell scalar $\lambda\,\phi^3$ Feynman tree graphs in $n = 6$ dimensions when the differential operator representation of $Y[\mathrm{SO}(2,n)]$ is extended by graph-specific so-called evaluation parameter terms. We further show that the action of the conformal Yangian generators on the on-shell three-point and four-point pure Yang-Mills theory gluon tree amplitudes has a compact, albeit non-vanishing, form in $n = 4$ dimensions. We conclude our investigation by exploring the action of the $Y[\mathrm{SO}(2,n)]$ generators on the off-shell scattering polynomials of the CHY formalism relating the two theories.

To my parents, Vasilis Dokmetzoglou and Popi Katsani.

For their endless love, encouragement and support.

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Finding myself at the end of my graduate school years at the University of North Carolina at Chapel Hill, having now completed the writing of this dissertation, I feel the need to take a moment to reflect on my journey so far, to take stock of the lessons I have gathered along the way and, most importantly, to acknowledge and thank the people in my life to whom I am deeply grateful.

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PREFACE

This dissertation is based on the research work conducted by the author at the University of North Carolina at Chapel Hill from August 2018 to May 2023 [1, 2]. Parts of this work have already been presented in the peer-reviewed publication

[1] N. Dokmetzoglou and L. Dolan, *Properties of the conformal Yangian in scalar and gauge field theories*, *JHEP* **02** (2023) 137 [arXiv:2207.14806 [hep-th]].

For the purpose of this dissertation, the content of the above publication has been significantly rewritten and expanded upon, though parts of the more technical discussions were adapted with minor changes only. The dissertation includes additional background material, several explicit examples and some analytic expressions for the results of calculations which were beyond the scope of the above publication, as well as an introduction to the computational tools developed by the author in the process of completing this work.

A detailed discussion of these computational tools, including a more thorough documentation, is in preparation to be submitted for publication, possibly to *Computer Physics Communications*, under the title

[2] N. Dokmetzoglou, *ConformalYangian: A Mathematica package for computations related to the action of the conformal Yangian Y[SO(2,n)]*, in preparation.

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LIST OF ABBREVIATIONS

AdS/CFT	Anti-de Sitter/Conformal Field Theory correspondence
BCFW	Britto-Cachazo-Feng-Witten recursion relations
BCJ	Bern-Carrasco-Johansson relations
CHY	Cachazo-He-Yuan scattering equations formalism
KK	Kleiss-Kuijf relations
KLT	Kawai-Lewellen-Tye relations
LHC	Large Hadron Collider
MHV	Maximally helicity violating
SYM	$\mathcal{N} = 4$ super Yang-Mills theory
YM	Pure Yang-Mills theory

LIST OF SYMBOLS

n	Number of space-time dimensions
d	Conformal dimension (canonical mass/scaling dimension)
N	Number of sites or number of particles
$\mathcal{N}$	Number of independent supersymmetries
$\mathrm{SO}(2,n)$	Conformal group in n space-time dimensions
$\mathrm{PSU}(2,2\|4)$	Superconformal group in $n=4$ space-time dimensions
$Y[\mathrm{SO}(2,n)]$	Yangian of the conformal group $\mathrm{SO}(2,n)$
$Y[\mathrm{PSU}(2,2\|4)]$	Yangian of the superconformal group $\mathrm{PSU}(2,2\|4)$
J^{AB}	Level-zero Yangian generators
$\widehat{J}^{AB}$	Level-one Yangian generators
$\widehat{J}'^{AB}$	Extended level-one Yangian generators
$\mathbb{P}^{\mu}$	Level-zero translation generators
$\mathbb{L}^{\mu\nu}$	Level-zero Lorentz transformation generators
$\mathbb{D}$	Level-zero dilatation generator
$\mathbb{K}^{\mu}$	Level-zero special conformal transformation generators
$\widehat{\mathbb{P}}^{\mu}$	Level-one translation generators
$\widehat{\mathbb{L}}^{\mu\nu}$	Level-one Lorentz transformation generators
$\widehat{\mathbb{D}}$	Level-one dilatation generator
$\widehat{\mathbb{K}}^{\mu}$	Level-one special conformal transformation generators
$k^{\mu}_{[I,J]}$	Sum of consecutive momenta
$k^{2}_{[I,J]}$	Off-shell kinematic invariant with consecutive momenta
h^{N}_{m}	Scattering polynomial
H_N	Product of scattering polynomials

CHAPTER 1

INTRODUCTION

From the perspective of a particle physicist in the first half of the 21$^{\text{st}}$ century, most of what we see around us can be understood, at a fundamental level, as the eternal dance of a set of quantum fields, permeating the entire universe, interacting with each other and continuously fluctuating, in a manner governed by the equations of their respective quantum field theories. The excitations of these fields correspond to the subatomic elementary particles, constituting all that we perceive as matter in the universe, as well as to the forces those particles exert on each other. There are reasons to suspect, however, that the perspective of a particle physicist in the second half of the 21$^{\text{st}}$ century could be somewhat different.

This latter statement is partially motivated by the several unexpected recent developments in the study of scattering amplitudes. Scattering amplitudes are arguably *the* fundamental physical observable within the field of particle physics. They are a measure of the probability of a certain set of incoming elementary particles colliding and producing a certain set of outgoing ones. As such, they provide a direct link between theory and experiment, with the theorists calculating predictions for those observables, and with the experimentalists testing those predictions at particle accelerators like the Large Hadron Collider (LHC) at CERN in Geneva, Switzerland. With the third run of the LHC having commenced in July 2022 and scheduled to continue for the next four years, delivering proton collisions at unprecedentedly high energies, theorists are feeling the pressure to push the state of the art in amplitude calculations, to higher orders of precision and to more involved collision processes.

However, this task is notoriously challenging. The traditional method, called the Feynman diagrams formalism, prescribes drawing several diagrams, with straight and wiggly lines, for a given collision event, one for each possible way that the set of incoming particles could interact and produce the outgoing ones, with each diagram corresponding to an often formidable integral one needs to compute. Some of those diagrams look like trees, and their sum corresponds to the (tree-level) zeroth-order approximation of the amplitude. The rest of the diagrams contain one, two or more loops, and correspond to the (loop-level) first-, second- or higher-order corrections to the amplitude. The more the particles in the collision and the higher the desired precision, the more the diagrams to draw and the harder the integrals to compute, making the calculation of the amplitudes for certain processes to high enough precision seemingly intractable.

Enter the recent developments in the field of scattering amplitudes within the last couple of decades. Theorists have, almost accidentally, stumbled upon some very intriguing mathematical properties of these fundamental physical observables, which have raised suspicions that there might exist some other, novel and perhaps deeper, ways of understanding the interactions of elementary particles [3]. Some of those unexpected discoveries include

- the Parke-Taylor formula for maximally helicity violating (MHV) gluon amplitudes,
- recursion relations used to decompose complicated amplitudes into the product of much simpler ones, such as the Britto-Cachazo-Feng-Witten (BCFW) recursion relations, the Kleiss-Kuijf (KK) and the Kawai-Lewellen-Tye (KLT) relations,
- dualities between amplitudes of supposedly unrelated processes, such as the result that graviton amplitudes are equal to the "square" of gluon ones, known as the double-copy Bern-Carrasco-Johansson (BCJ) relations or the color-kinematics duality [4–6],
- the duality between gluon scattering amplitudes and Wilson loops [7],
- the original and supersymmetric Ward identities,
- the Anti-de Sitter/Conformal Field Theory (AdS/CFT) correspondence and its connection to integrability [8–10],
- connections between amplitudes and higher-dimensional geometric structures, such as the amplituhedron and associahedron [11–16],

as well as, of more interest to us,

- the Cachazo-He-Yuan (CHY) scattering equations formalism [17–22], and
- the symmetry properties of certain scalar and gauge theories and their scattering amplitudes under the action of the conformal group $\mathrm{SO}(2,n)$, the superconformal group $\mathrm{PSU}(2,2|4)$, and their infinite-dimensional Yangian extensions, $Y[\mathrm{SO}(2,n)]$ and $Y[\mathrm{PSU}(2,2|4)]$ [23–31].

In Figure 1, we depict diagrammatically some of the fascinating interconnections between a subset of the above topics which are closely related to our work.

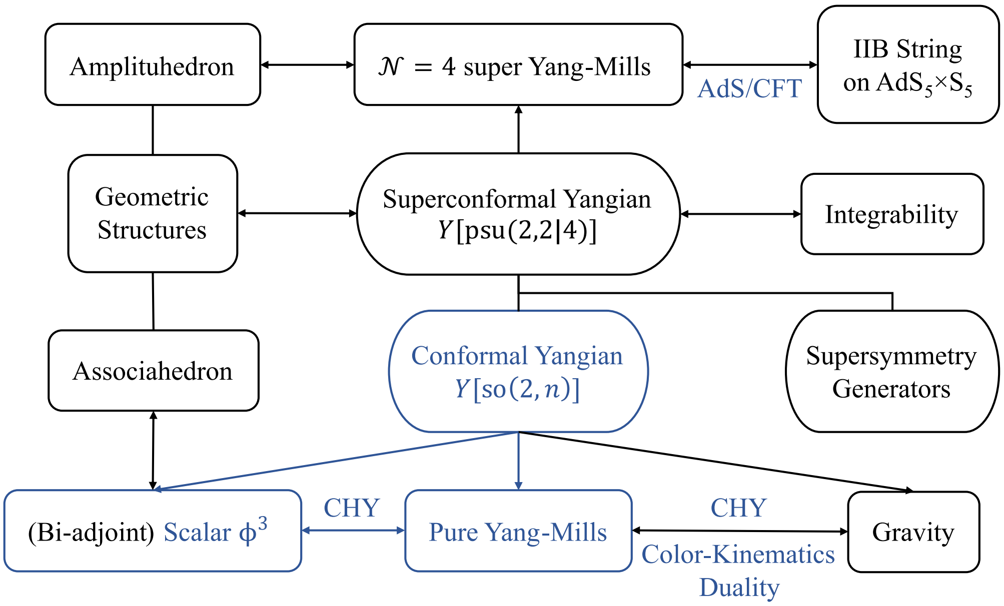

Figure 1: Diagram of interconnected topics of interest. Topics we have worked on to date are marked in blue.

We say that an amplitude has a symmetry under a given transformation if it remains invariant under it. For example, rotating all the particles of some collision event by the same arbitrary angle, leaves the probability of the event unchanged, and we can thus say that the corresponding scattering amplitude has a symmetry under rotations. Each symmetry corresponds to a conserved quantity and can be understood as an exploitable constraint on the amplitudes. To investigate the transformation properties of an amplitude under a given transformation, one can define a differential operator with respect to the particles' momenta, the so-called generator of the transformation, and then act with that generator on the amplitude of interest. If the action of a generator annihilates the amplitude, i.e. if it produces a zero, we say that the amplitude has a symmetry under the corresponding transformation. If it does not, such as is the case, for example, with the loop-level amplitudes of $\lambda\phi^3$ and pure Yang-Mills theory under the action of the generators of the conformal group [32], we say that the symmetry is broken, but it is still useful to understand how exactly that happens.

It is known that tree-level superamplitudes in $\mathcal{N} = 4$ super Yang-Mills (SYM) theory, i.e. in the maximally supersymmetric Yang-Mills theory, are invariant under the action of the superconformal group $\mathrm{PSU}(2,2|4)$ and the superconformal Yangian $Y[\mathrm{PSU}(2,2|4)]$ [26, 27, 33]. This infinite-dimensional symmetry of the $\mathcal{N} = 4$ SYM superamplitudes places some important constraints on these amplitudes at tree-level. The approach of inferring symmetry from the invariance of the superamplitudes under the action of the superconformal Yangian generators provides an alternative method for understanding the previously discovered PSU$(2, 2|4)$ Yangian symmetry of $\mathcal{N} = 4$ SYM theory, with SU(N) gauge group, in the planar limit (i.e. for large N), in $n = 4$ space-time dimensions [23–25]. This naturally raises the question [34]: *What happens if we remove all the (not yet experimentally observed) supersymmetry? Do the tree-level gluon amplitudes of the non-supersymmetric pure Yang-Mills theory also enjoy such an infinite-dimensional symmetry, perhaps under the action of the (non-supersymmetric) conformal Yangian $Y[SO(2, 4)]$?*

In this dissertation, we make progress towards answering this question. We first prove the algebraic consistency of the infinite-dimensional Yangian extension of the conformal group SO$(2, n)$, where n is the number of space-time dimensions, in the momentum-space differential operator representation, for both scalar and spin-one gauge fields. We then investigate the action of the generators of the conformal Yangian $Y[\mathrm{SO}(2, n)]$ on the tree-level scattering amplitudes of massless scalar $\lambda\,\phi^3$ theory and pure Yang-Mills (YM) theory, two non-supersymmetric theories which at tree-level are known to be conformally invariant in $n = 6$ and $n = 4$ space-time dimensions, respectively. Further motivated by the close connection of these two theories through the Cachazo-He-Yuan (CHY) scattering equations formalism [17–20], we conclude our investigation by exploring the action of the $Y[\mathrm{SO}(2, n)]$ generators on the off-shell scattering polynomials appearing in the so-called polynomial form of the CHY formalism [21, 22].

The CHY scattering equations formalism, developed by Cachazo, He and Yuan, is a novel, unified method for calculating N-point tree-level partial amplitudes of *several theories* and in *any number of space-time dimensions*, by the evaluation of a multi-variable contour integral encircling the simultaneous roots of a set of equations, the so-called *scattering equations*, which are the same for any theory [17–20]. In the polynomial form of the CHY scattering equations formalism, developed by Dolan and Goddard, a set of polynomials, h_m^N, with $1 \le m \le N - 3$, replace the original CHY scattering equations and extend the applicability of the CHY formalism to off-shell amplitudes [21, 22]. The *scattering polynomials* h_m^N, which appear as the product $H_N \equiv \prod_{m=1}^{N-3} h_m^N$ in the denominator of the N-point CHY integrand for any theory, reproduce the roots of the original scattering equations on shell, while at the same time allowing for more efficient computations of both the roots, i.e. the poles encircled by the CHY contour integral, and of the contour integral itself [35–37].

With the scattering polynomials h_m^N remaining the same for any theory, selecting the theory whose scattering amplitudes one wishes to calculate comes down to choosing an appropriate function Ψ_N, which is specific to the theory of interest and multiplies the inverse of the product of the scattering polynomials, $(H_N)^{-1}$, in the CHY integrand. The simplest choice is $\Psi_N|_{\lambda\,\phi^3\text{ theory}} = 1$, which corresponds to $\lambda\,\phi^3$ theory and further justifies our interest in this scalar theory's amplitudes. For pure YM theory, the appropriate function $\Psi_N|_{\text{Yang-Mills theory}}$ is the Pfaffian of an antisymmetric matrix, which is a function of the gluon momenta and polarization vectors, and of the complex variables of integration, discussed extensively in [18, 20] and summarized in Section E.3.

Since massless $\lambda\,\phi^3$ scalar theory is known to be invariant, as a classical field theory, under the conformal group SO(2, 6), in $n = 6$ space-time dimensions, and pure YM theory is known to be invariant, as a classical field theory, under the conformal group SO(2, 4), in $n = 4$ space-time dimensions, it can be shown that these symmetries also show up as invariances of the tree-level scattering amplitudes of the two theories, under the action of the generators of the conformal group SO$(2, n)$ in $n = 4$ and $n = 6$ space-time dimensions, respectively [38, 39]. Additionally, since the CHY integrands for the tree amplitudes of these two theories differ only by the function $\Psi_N|_{\text{Yang-Mills theory}}$, we expect to see the conformal symmetry of $\lambda\,\phi^3$ theory show up as the invariance of $(H_N)^{-1}$ under the action of the SO$(2, n)$ generators in $n = 6$ space-time dimensions, and the conformal symmetry of pure YM theory show up as the invariance of the product of the same $(H_N)^{-1}$ and the function $\Psi_N|_{\text{Yang-Mills theory}}$ under the action of the SO$(2, n)$ generators in $n = 4$ space-time dimensions, up to integration around a contour encircling the simultaneous roots of the scattering polynomials.

By the same logic, we should be able to see any further symmetry of the pure YM tree-level scattering amplitudes, e.g. under the conformal Yangian $Y[\text{SO}(2, n)]$, show up as an invariance of the product of $(H_N)^{-1}$ and the function $\Psi_N|_{\text{Yang-Mills theory}}$ under the action of the generators of the corresponding symmetry group, in this case the $Y[\text{SO}(2, n)]$ generators. And thus, as follow-up questions to our original question, we ask: *How do the generators of the conformal Yangian $Y[SO(2, n)]$ act on the scattering polynomials h_m^N? How do they act on the function $\Psi_N|_{\textit{Yang-Mills theory}}$? And do these results reproduce the action of the $Y[SO(2, n)]$ generators on the $\lambda\,\phi^3$ and pure YM theory tree amplitudes, as expected?*

To that end, in **Chapter 2**, we review the infinite-dimensional algebraic structure known as the Yangian $Y[G]$ of a semi-simple Lie group G. For the Yangian of the conformal group $Y[\text{SO}(2, n)]$, we define the *level-zero generators*, i.e. the generators of the conformal group SO$(2, n)$, in a differential operator representation in terms of (ordinary) momentum vectors, as opposed to the four-dimensional spinor-helicity variables used in most of the pertinent literature [27, 40, 41]. We choose to work in this momentum-space representation, firstly, because we want to carry out our calculations in arbitrary space-time dimensions, to the extent possible, and secondly, because, in the CHY formalism, the h_m^N scattering polynomials and the $\Psi_N|_{\text{Yang-Mills theory}}$ function are also expressed in terms of momentum vectors. We construct the *level-one generators* of $Y[\mathrm{SO}(2,n)]$ as bilocal combinations of the level-zero ones in a multi-site representation. To generate a consistent Yangian algebra, the level-zero and level-one generators need to be in a representation that satisfies the so-called *Serre relation*. We introduce a minimal condition that we call the *Single Site Serre Condition*, and which, if satisfied by our representation at a single site, guarantees the satisfaction of the multi-site Serre relation.

In **Chapter 3**, we prove that the momentum-space differential operator representation of the conformal Yangian algebra $Y[\mathrm{SO}(2,n)]$ satisfies the Serre relation, off-shell and in any number of space-time dimensions n for scalar fields, but only on-shell and in $n = 4$ space-time dimensions for spin-one gauge fields. To our knowledge, the consistency of the Yangian of the (non-supersymmetric) conformal group $Y[\mathrm{SO}(2,n)]$ had not been proven before in the literature.

In **Chapter 4**, we show that the level-zero generators annihilate individual Feynman tree graphs and tree-level partial amplitudes of off-shell $\lambda\,\phi^3$ theory in $n = 6$ space-time dimensions, and that the level-one generators, when extended by graph-dependent so-called evaluation parameter terms, annihilate *only individual* $\lambda\,\phi^3$ off-shell tree graphs in $n = 6$ space-time dimensions.

In **Chapter 5**, we show that the action of the (non-extended) level-one generators on the pure Yang-Mills theory 3-point and 4-point on-shell partial amplitudes has a compact, albeit non-vanishing, form in $n = 4$ space-time dimensions. These expressions can also be expressed in terms of traces of Dirac matrices, which we show originate from tree amplitudes with 2 fermions and $N - 2$ gluons. This result reflects the invariance of pure Yang-Mills gluon tree amplitudes under the PSU(2,2|4) Yangian, and may in the future lead to some interpretation of the role of supersymmetry in pure non-supersymmetric gauge theory.

In **Chapter 6**, we review the polynomial form of the CHY scattering equations formalism, and study the action of the level-zero and level-one generators on $(H_N)^{-1}$, the inverse of the product of the h_m^N scattering polynomials, which we show vanishes under certain conditions, in accordance with the results of the action of the generators on $\lambda\,\phi^3$ partial amplitudes and individual Feynman graphs. We then briefly comment on the action of the generators on the product of $(H_N)^{-1}$ and the pure YM-specific function $\Psi_N|_{\text{Yang-Mills theory}}$.

In **Chapter 7**, we introduce the computer algebra tools we used to complement our formal pen-and-paper calculations. Those are the MATHEMATICA packages **CONFORMALYANGIAN**, **TREEAMPLITUDESDATABASE** and **NUMERICALEVALUATIONFINITEFIELDS**, which we created for symbolically calculating the action of the conformal Yangian generators on the scalar and gauge theory tree amplitudes, and also for numerically evaluating those using finite fields methods, i.e. using modular arithmetic with large prime numbers.

Finally, in the appendices we present some additional background material and examples. In **Appendix A**, we present the explicit commutation relations of the generators of the conformal Yangian algebra $Y[\text{SO}(2,n)]$ and comment on the symmetry properties of the $\text{SO}(2,n)$ structure constants. In **Appendix B**, we show that the momentum-conserving Dirac delta function commutes with the level-one translations generator. In **Appendix C**, we give some explicit examples of tree-level partial and total amplitudes of scalar $\lambda\,\phi^3$ theory. In **Appendix D**, we present some additional explicit examples of the action of level-one $Y[\text{SO}(2,n)]$ generators on 4-, 5- and 6-point $\lambda\,\phi^3$ theory individual tree graphs. In **Appendix E**, we gather some additional definitions related to the CHY scattering equations formalism. And in **Appendix F**, we present some additional analytic expressions and explicit examples of the action of both level-zero and level-one $Y[\text{SO}(2,n)]$ generators on the $N = 4$, 5 and 6 scattering polynomials.

CHAPTER 2

CONFORMAL YANGIAN $Y[\mathbf{SO}(2,n)]$

In this chapter, we introduce the infinite-dimensional algebraic structure known as the *Yangian* of a semi-simple Lie group G, with its defining commutation relations expressed in terms of the first two levels of its generators. We focus on the Yangian of the conformal group $\mathrm{SO}(2,n)$, defined in terms of the structure constants of the conformal algebra in n space-time dimensions, and we discuss a multi-site representation of its so-called *level-zero* and *level-one generators*. To generate a consistent Yangian algebra, the level-zero and level-one generators need to be in a representation that satisfies the so-called *Serre relation*. For that to be the case, it is sufficient to show that the single-site level-zero generators satisfy a much simpler, though certainly non-trivial, constraint, which we call the *Single Site Serre Condition*. We conclude this chapter by defining a differential operator representation for the conformal Yangian $Y[\mathrm{SO}(2,n)]$ in momentum-space, which in the following chapter we prove satisfies this single-site constraint.

2.1 Yangian Algebra $Y[G]$ and its Generators in Single-Index Form

A Yangian algebra $Y[G]$ is an associative Hopf algebra generated by the elements J^A, with J^A taking values in the Lie algebra of an arbitrary semi-simple Lie group G with structure constants $f^{AB}{}_C$, and $\widehat{J}^A$ such that the following defining relations are satisfied [24, 25, 41–45]:

$$\left[J^A, J^B\right] = f^{AB}{}_C\, J^C \tag{2.1}$$

$$\left[J^A, \widehat{J}^B\right] = f^{AB}{}_C\, \widehat{J}^C \tag{2.2}$$

$$\left[\widehat{J}^A, \left[\widehat{J}^B, J^C\right]\right] + \left[\widehat{J}^B, \left[\widehat{J}^C, J^A\right]\right] + \left[\widehat{J}^C, \left[\widehat{J}^A, J^B\right]\right] = \frac{1}{24}\, f^{ADK} f^{BEL} f^{CFM} f_{KLM} \left\{J_D, J_E, J_F\right\} \tag{2.3}$$

$$\begin{aligned}&\left[\left[\widehat{J}^A, \widehat{J}^B\right], \left[J^C, \widehat{J}^D\right]\right] + \left[\left[\widehat{J}^C, \widehat{J}^D\right], \left[J^A, \widehat{J}^B\right]\right] \\ &\quad = \frac{1}{24}\Big(f^{AGL} f^{BEM} f^{KFN} f_{LMN} f^{CD}{}_K + f^{CGL} f^{DEM} f^{KFN} f_{LMN} f^{AB}{}_K\Big) \left\{J_G, J_E, \widehat{J}_F\right\}\end{aligned} \tag{2.4}$$

The latter two relations are called the Serre relations. Note that $\{J_D, J_E, J_F\}$ and $\{J_G, J_E, \widehat{J}_F\}$ are totally symmetrized triple products. Note further that the second Serre relation follows from the first one for $G \neq \mathrm{SU}(2)$ [24, 25, 41, 42]. Therefore, for all subsequent discussion of the conformal Yangian $Y[\mathrm{SO}(2,n)]$, we will be referring to (2.3) as the *Serre relation*. We call the elements J^A the *level-zero generators* and the elements $\widehat{J}^A$ the *level-one generators*.

For completeness we note that, as a Hopf algebra, $Y[G]$ is equipped with the coproduct Δ:

$$\Delta(J^A) = J^A \otimes \mathbb{1} + \mathbb{1} \otimes J^A, \quad \Delta(\widehat{J}^A) = \widehat{J}^A \otimes \mathbb{1} + \mathbb{1} \otimes \widehat{J}^A + f^A{}_{BC}\, J^B \otimes J^C\,, \tag{2.5}$$

which provides the prescription for lifting a single-site representation of the algebra to a multi-site one. In a multi-site representation, with single-site level-zero generators J_i^A satisfying the commutation relations

$$\left[J_i^A, J_j^B\right] = \delta_{ij}\, f^{AB}{}_C\, J_i^C \tag{2.6}$$

and N-site level-zero generators

$$J^A = \sum_{i=1}^{N} J_i^A \tag{2.7}$$

satisfying (2.1), a construction for the level-one generators that satisfies (2.2) is the following [25, 41, 45]:

$$\widehat{J}^A = f^A{}_{BC} \sum_{1\leq i<j\leq N} J_i^B J_j^C\,. \tag{2.8}$$

The infinite number of higher-level generators of the Yangian algebra, i.e. the level-two generators and above, call them $J^A_{(m)}$ for $m \geq 2$, are derived from commutators of level-one generators, and satisfy the following commutation relations with the level-zero generators [41, 44–46]:

$$\left[J^A, J^B_{(m)}\right] = f^{AB}{}_C\, J^C_{(m)}\,. \tag{2.9}$$

2.2 Conformal Yangian Algebra $Y[\mathbf{SO}(2,n)]$ and its Generators in Double-Index Form

For the Yangian of the conformal group $\mathrm{SO}(2,n)$, where n is the number of space-time dimensions, it is convenient to use a double-index form of the generators and defining relations. In this notation, the conformal group $\mathrm{SO}(2,n)$ generators J^{AB} satisfy the Lie algebra commutation relations

$$\left[J^{AB}, J^{CD}\right] = -\eta^{AC} J^{BD} - \eta^{BD} J^{AC} + \eta^{AD} J^{BC} + \eta^{BC} J^{AD} \ , \tag{2.10}$$

where

$$\begin{aligned} &J^{AB} = -J^{BA}, \qquad 0 \leq A, B \leq n+1, \qquad \eta^{AB} = \mathrm{diagonal}(1, -1, -1, -1, \ldots, -1, 1), \\ &g^{\mu\nu} = \eta^{\mu\nu} = \mathrm{diagonal}(1, -1, -1, -1, \ldots, -1), \qquad 0 \leq \mu, \nu \leq n-1 \ . \end{aligned} \tag{2.11}$$

To generate a consistent Yangian algebra, the level-zero generators J^{AB} and level-one generators $\widehat{J}^{AB}$ need to satisfy the following defining relations [25, 42–45]:

$$\left[J^{AB}, J^{CD}\right] = f^{ABCD}{}_{EF}\, J^{EF} \tag{2.12}$$

$$\left[J^{AB}, \widehat{J}^{CD}\right] = f^{ABCD}{}_{EF}\, \widehat{J}^{EF} \tag{2.13}$$

$$\begin{aligned} &\left[\widehat{J}^{AB}, \left[\widehat{J}^{CD}, J^{EF}\right]\right] + \left[\widehat{J}^{CD}, \left[\widehat{J}^{EF}, J^{AB}\right]\right] + \left[\widehat{J}^{EF}, \left[\widehat{J}^{AB}, J^{CD}\right]\right] \\ &= \frac{1}{24}\, f^{AB}{}_{GHMN}\, f^{CD}{}_{IJOP}\, f^{EF}{}_{KLQR}\, f^{MNOPQR}\ \left\{J^{GH}, J^{IJ}, J^{KL}\right\} \end{aligned} \tag{2.14}$$

where the totally symmetrized triple product is defined as

$$\begin{aligned} \left\{J^{GH}, J^{IJ}, J^{KL}\right\} \equiv\ & J^{GH} J^{IJ} J^{KL} + J^{IJ} J^{GH} J^{KL} + J^{KL} J^{IJ} J^{GH} \\ & + J^{GH} J^{KL} J^{IJ} + J^{IJ} J^{KL} J^{GH} + J^{KL} J^{GH} J^{IJ} \end{aligned} \tag{2.15}$$

and the structure constants of the conformal algebra (2.10) as

$$\begin{aligned} f^{ABCD}{}_{EF} \equiv \frac{1}{2}\Big(& -\eta^{AC}\, \delta^B_E\, \delta^D_F - \eta^{BD}\, \delta^A_E\, \delta^C_F + \eta^{AD}\, \delta^B_E\, \delta^C_F + \eta^{BC}\, \delta^A_E\, \delta^D_F \\ & + \eta^{AC}\, \delta^B_F\, \delta^D_E + \eta^{BD}\, \delta^A_F\, \delta^C_E - \eta^{AD}\, \delta^B_F\, \delta^C_E - \eta^{BC}\, \delta^A_F\, \delta^D_E \Big). \end{aligned} \tag{2.16}$$

The indices can be raised and lowered with the metric η^{AB}, e.g. $f^{AB}{}_{GHMN} = f^{ABG'H'}{}_{MN}\,\eta_{GG'}\eta_{HH'}$ and $f^{MNOPQR} = f^{MNOP}{}_{Q'R'}\,\eta^{QQ'}\eta^{RR'}$.

In a multi-site representation, with single-site level-zero generators J_i^{AB} satisfying the commutation relations

$$\left[J_i^{AB}, J_j^{CD}\right] = \delta_{ij}\left(-\eta^{AC} J_i^{BD} - \eta^{BD} J_i^{AC} + \eta^{AD} J_i^{BC} + \eta^{BC} J_i^{AD} \right) = \delta_{ij}\, f^{ABCD}{}_{EF}\, J_i^{EF} \tag{2.17}$$

and N-site level-zero generators

$$J^{AB} = \sum_{i=1}^{N} J_i^{AB} \tag{2.18}$$

satisfying (2.12), a construction for the level-one generators that satisfies (2.13) is the following:

$$\widehat{J}^{AB} = \frac{1}{2} f^{AB}{}_{CDEF} \sum_{1\le i<j\le N} J_i^{CD}\, J_j^{EF}\ . \tag{2.19}$$

In Chapter 4, we introduce an extension to this construction, in terms of so-called evaluation parameter terms, which also satisfies the defining relations (2.12)–(2.14).

2.3 Single Site Serre Condition

For a representation of level-zero generators J^{AB} and a construction of level-one generators $\widehat{J}^{AB}$ to generate a consistent Yangian algebra, it is not sufficient for the generators to satisfy the first two defining relations (2.12)–(2.13). They need to also satisfy the third one, the so-called Serre relation (2.14). From the definition of the N-site level-zero generators (2.18) and the construction of the level-one generators (2.19), it follows that for a single site, i.e. for $N = 1$, $J^{AB} = J_1^{AB}$ and $\widehat{J}^{AB} = 0$. Thus, for a single site, the Serre relation reduces to the following *Single Site Serre Condition*:

$$f^{AB}{}_{GHMN}\, f^{CD}{}_{IJOP}\, f^{EF}{}_{KLQR}\, f^{MNOPQR} \left\{ J_1^{GH}, J_1^{IJ}, J_1^{KL} \right\} = 0\ . \tag{2.20}$$

Once (2.20) is proved for a given representation of single-site level-zero generators, the full Serre relation (2.14) follows from the coproduct (2.5).

However, not all representations of the single-site level-zero generators satisfy (2.20). In Chapter 3 we prove that the momentum-space differential operator representation of the conformal Yangian $Y[\mathrm{SO}(2,n)]$, which is introduced in the following section (2.25), satisfies the Single Site Serre

Condition (2.20) for both scalar and spin-one gauge fields, in their respective domains of applicability.

In preparation for the proof, we rewrite the RHS of the Serre relation (2.14) as the sum of three cyclic terms:

$$
\begin{aligned}
& f^{AB}{}_{GHMN} f^{CD}{}_{IJOP} f^{EF}{}_{KLQR} f^{MNOPQR}\,\{J^{GH}, J^{IJ}, J^{KL}\} \\
=& \Big[4\,\eta_{WY} \\
& \cdot \Big(\eta^{BD} \big(\{J^{FA}, J^{EW}, J^{YC}\} + \{J^{EC}, J^{AW}, J^{YF}\} - \{J^{FC}, J^{EW}, J^{YA}\} - \{J^{EA}, J^{CW}, J^{YF}\} \big) \\
& - \eta^{AD} \big(\{J^{FB}, J^{EW}, J^{YC}\} + \{J^{EC}, J^{BW}, J^{YF}\} - \{J^{FC}, J^{EW}, J^{YB}\} - \{J^{EB}, J^{CW}, J^{YF}\} \big) \\
& - \eta^{BC} \big(\{J^{FA}, J^{EW}, J^{YD}\} + \{J^{ED}, J^{AW}, J^{YF}\} - \{J^{FD}, J^{EW}, J^{YA}\} - \{J^{EA}, J^{DW}, J^{YF}\} \big) \\
& + \eta^{AC} \big(\{J^{FB}, J^{EW}, J^{YD}\} + \{J^{ED}, J^{BW}, J^{YF}\} - \{J^{FD}, J^{EW}, J^{YB}\} - \{J^{EB}, J^{DW}, J^{YF}\} \big) \Big) \Big] \\
& + (ABCDEF \to CDEFAB) + (ABCDEF \to EFABCD)\,.
\end{aligned}
\tag{2.21}
$$

We then rewrite (2.21) in terms of anti-commutators as follows:

$$
\begin{aligned}
& f^{AB}{}_{GHMN} f^{CD}{}_{IJOP} f^{EF}{}_{KLQR} f^{MNOPQR}\,\{J^{GH}, J^{IJ}, J^{KL}\} \\
=& \Bigg[\Bigg(4\,\eta^{DB} \Big(3\,\eta_{WY}\, \big(J^{FA}\{J^{EW}, J^{YC}\} + J^{EC}\{J^{AW}, J^{YF}\} - J^{FC}\{J^{EW}, J^{YA}\} - J^{EA}\{J^{CW}, J^{YF}\} \big) \\
& \qquad - (\delta^W_W - 6)(\eta^{EC} J^{FA} + \eta^{FA} J^{EC} - \eta^{EA} J^{FC} - \eta^{FC} J^{EA}) \Big) - (A \leftrightarrow B) \Bigg) - (C \leftrightarrow D) \Bigg] \\
& + (ABCDEF \to CDEFAB) + (ABCDEF \to EFABCD)\,.
\end{aligned}
\tag{2.22}
$$

The terms proportional to $(\delta^W_W - 6)$ make zero contribution in (2.22) when all the cyclic permutations are performed, so we can drop them. Note that the identities (2.21) and (2.22) hold for any number of sites N, but we are interested to prove that they vanish on a single site. That is, the Single Site Serre Condition (2.20) becomes

$$
\begin{aligned}
& \Big[\Big(\eta^{DB} M^{EFCA} - (A \leftrightarrow B) \Big) \quad - (C \leftrightarrow D) \Big] \\
& \qquad + (ABCDEF \to CDEFAB) + (ABCDEF \to EFABCD) = 0\,,
\end{aligned}
\tag{2.23}
$$

where the single-site tensor M^{EFCA} is defined as

$$M^{EFCA} \equiv \big(- J_1^{EA}\{J_1^{CW}, J_1^{YF}\} - J_1^{FC}\{J_1^{EW}, J_1^{YA}\} + J_1^{EC}\{J_1^{AW}, J_1^{YF}\} + J_1^{FA}\{J_1^{EW}, J_1^{YC}\}\big)\, \eta_{WY} \, . \tag{2.24}$$

2.4 Momentum-Space Differential Operator Representation

We now introduce the momentum-space differential operator representation of the single-site level-zero generators of the conformal Yangian $Y[\mathrm{SO}(2,n)]$, i.e. the single-site conformal group $\mathrm{SO}(2,n)$ generators, for scalar and spin-one gauge fields [38,39,47,48]. In terms of the momenta k_i^μ and the partial derivatives $\partial_i^\mu \equiv \partial/\partial k_{i\,\mu}$, $\partial_{i\mu} \equiv \partial/\partial k_i^\mu$, $1 \le i \le N$, $0 \le \mu, \nu \le n-1$, we define the generators:

$$\begin{aligned} P_i^\mu &\equiv k_i^\mu & L_i^{\mu\nu} &\equiv k_i^\mu\, \partial_i^\nu - k_i^\nu\, \partial_i^\mu + \Sigma_i^{\mu\nu} \\ D_i &\equiv d + k_i^\nu\, \partial_{i\nu} & K_i^\mu &\equiv 2\, d\, \partial_i^\mu + 2\, k_i^\nu\, \partial_{i\nu}\, \partial_i^\mu - k_i^\mu\, \partial_{i\nu}\, \partial_i^\nu - 2\, \Sigma_i^{\mu\nu}\, \partial_{i\nu} \end{aligned} \tag{2.25}$$

where the conformal dimension $d \in \mathbb{R}$ (canonical mass/scaling dimension) takes the value $d = (n-2)/2$ for bosonic fields in n space-time dimensions. The spin operator $\Sigma_i^{\mu\nu}$ terms don't appear for scalar fields, i.e.

$$\Sigma_i^{\mu\nu}|_{\mathrm{scalar}} \equiv 0 \, , \tag{2.26}$$

while for gauge fields they are given by

$$\Sigma_i^{\mu\nu}|_{\mathrm{gauge}}\, A(k_j;\epsilon_j) \equiv \epsilon_i^\alpha\, \Sigma^{\mu\nu}_{(\alpha\beta)}\, \tilde{A}_i^\beta(k_j;\epsilon_{j\neq i}) = \epsilon_i^\alpha\, (\delta^\mu_\alpha\, \delta^\nu_\beta - \delta^\mu_\beta\, \delta^\nu_\alpha)\, \tilde{A}_i^\beta(k_j;\epsilon_{j\neq i}) \, , \tag{2.27}$$

where

$$\Sigma^{\mu\nu}_{(\alpha\beta)} \equiv \delta^\mu_\alpha\, \delta^\nu_\beta - \delta^\mu_\beta\, \delta^\nu_\alpha \, , \tag{2.28}$$

$A(k_j;\epsilon_j) \equiv \epsilon_i^\alpha\, g_{\alpha\beta}\, \tilde{A}_i^\beta(k_j;\epsilon_{j\neq i})$ is a scalar function of the momenta k_j^μ and polarization vectors ϵ_j^μ, linear in each one of the latter, and $\tilde{A}_i^\beta(k_j;\epsilon_{j\neq i})$ is the tensor we obtain by stripping off the i^{th} polarization vector. The above definition of $\Sigma_i^{\mu\nu}|_{\mathrm{gauge}}$ is equivalent to the following one appearing in [39]:

$$\Sigma_i^{\mu\nu}|_{\mathrm{gauge}} \equiv \epsilon_i^\mu\, \frac{\partial}{\partial \epsilon_{i\,\nu}} - \epsilon_i^\nu\, \frac{\partial}{\partial \epsilon_{i\,\mu}} \, . \tag{2.29}$$

These generators $\{P_i^\mu, L_i^{\mu\nu}, D_i, K_i^\mu\}$ are the momentum-space analogues of the generators of the conformal group $\mathrm{SO}(2,n)$ coordinate transformations in position-space, i.e. the generators of translations, Lorentz transformations, dilatation and special conformal transformations.

Note that, in the double-index form introduced earlier, the single-site level-zero $Y[\mathrm{SO}(2,n)]$ generators J_i^{AB}, $0 \leq A, B \leq n+1$ are given by:

$$J_i^{\mu\nu} \equiv L_i^{\mu\nu} \qquad J_i^{n,\mu} \equiv \tfrac{1}{2}\left(P_i^\mu - K_i^\mu\right) \qquad J_i^{n+1,\mu} \equiv \tfrac{1}{2}\left(P_i^\mu + K_i^\mu\right) \qquad J_i^{n,n+1} \equiv D_i \tag{2.30}$$

or more explicitly

$$\begin{aligned} J_i^{\mu\nu} &= k_i^\mu\, \partial_i^\nu - k_i^\nu\, \partial_i^\mu & J_i^{n,\mu} &= \tfrac{1}{2}\left(k_i^\mu - 2\,d\,\partial_i^\mu - 2\,k_i^\nu\,\partial_{i\nu}\,\partial_i^\mu + k_i^\mu\,\partial_{i\nu}\,\partial_i^\nu\right) \\ J_i^{n,n+1} &= d + k_i^\nu\,\partial_{i\nu} & J_i^{n+1,\mu} &= \tfrac{1}{2}\left(k_i^\mu + 2\,d\,\partial_i^\mu + 2\,k_i^\nu\,\partial_{i\nu}\,\partial_i^\mu - k_i^\mu\,\partial_{i\nu}\,\partial_i^\nu\right) \end{aligned} \tag{2.31}$$

Using (2.18) and (2.25), we can now define the N-site level-zero generators of $Y[\mathrm{SO}(2,n)]$

$$\mathbb{P}^\mu \equiv \sum_{i=1}^N P_i^\mu \qquad \mathbb{L}^{\mu\nu} \equiv \sum_{i=1}^N L_i^{\mu\nu} \qquad \mathbb{D} \equiv \sum_{i=1}^N D_i \qquad \mathbb{K}^\mu \equiv \sum_{i=1}^N K_i^\mu \tag{2.32}$$

in the momentum-space differential operator representation. And from (2.30), (2.19) and (2.16), we obtain the level-one $Y[\mathrm{SO}(2,n)]$ generators in this representation:

$$\begin{aligned} \widehat{\mathbb{P}}^\mu &\equiv -\sum_{1\leq i<j\leq N}\left[\left(P_i^\mu\,D_j + g_{\alpha\beta}\,P_i^\alpha\,L_j^{\mu\beta}\right) - (i \leftrightarrow j)\right] \\ \widehat{\mathbb{L}}^{\mu\nu} &\equiv -\frac{1}{2}\sum_{1\leq i<j\leq N}\left[\left(P_i^\mu\,K_j^\nu - P_i^\nu\,K_j^\mu + 2\,g_{\alpha\beta}\,L_i^{\mu\alpha}\,L_j^{\nu\beta}\right) - (i \leftrightarrow j)\right] \\ \widehat{\mathbb{D}} &\equiv -\frac{1}{2}\sum_{1\leq i<j\leq N}\left[\left(g_{\alpha\beta}\,P_i^\alpha\,K_j^\beta\right) - (i \leftrightarrow j)\right] \\ \widehat{\mathbb{K}}^\mu &\equiv \sum_{1\leq i<j\leq N}\left[\left(-D_i\,K_j^\mu + g_{\alpha\beta}\,L_i^{\mu\alpha}\,K_j^\beta\right) - (i \leftrightarrow j)\right] \end{aligned} \tag{2.33}$$

Finally, note that, in the double-index form, the level-one $Y[\mathrm{SO}(2,n)]$ generators $\widehat{J}^{AB}$, are given by:

$$\widehat{J}^{\mu\nu} \equiv \widehat{\mathbb{L}}^{\mu\nu} \qquad \widehat{J}^{n,\mu} \equiv \frac{1}{2}\left(\widehat{\mathbb{P}}^\mu - \widehat{\mathbb{K}}^\mu\right) \qquad \widehat{J}^{n+1,\mu} \equiv \frac{1}{2}\left(\widehat{\mathbb{P}}^\mu + \widehat{\mathbb{K}}^\mu\right) \qquad \widehat{J}^{n,n+1} \equiv \widehat{\mathbb{D}} \tag{2.34}$$

or more explicitly

$$
\begin{aligned}
\widehat{J}^{\mu\nu} &\equiv -\frac{1}{2} \sum_{1\leq i<j\leq N} \Big[\Big(P_i^{\mu}\,K_j^{\nu} - P_i^{\nu}\,K_j^{\mu} + 2\,g_{\alpha\beta}\,L_i^{\mu\alpha}\,L_j^{\nu\beta}\Big) - (i \leftrightarrow j)\Big] \\
\widehat{J}^{n,\mu} &\equiv \frac{1}{2} \sum_{1\leq i<j\leq N} \Big[\Big(D_i\,\Big(P_j^{\mu} + K_j^{\mu}\Big) + g_{\alpha\beta}\,L_i^{\mu\alpha}\,\Big(P_j^{\beta} - K_j^{\beta}\Big)\Big) - (i \leftrightarrow j)\Big] \\
\widehat{J}^{n+1,\mu} &\equiv \frac{1}{2} \sum_{1\leq i<j\leq N} \Big[\Big(D_i\,\Big(P_j^{\mu} - K_j^{\mu}\Big) + g_{\alpha\beta}\,L_i^{\mu\alpha}\,\Big(P_j^{\beta} + K_j^{\beta}\Big)\Big) - (i \leftrightarrow j)\Big] \\
\widehat{J}^{n,n+1} &\equiv -\frac{1}{2} \sum_{1\leq i<j\leq N} \Big[\Big(g_{\alpha\beta}\,P_i^{\alpha}\,K_j^{\beta}\Big) - (i \leftrightarrow j)\Big]
\end{aligned}
\tag{2.35}
$$

CHAPTER 3

PROOF OF THE SERRE RELATION

In this chapter[1], we prove that the Serre relation (2.14) is satisfied by the momentum-space differential operator representation of the conformal Yangian $Y[\mathrm{SO}(2,n)]$ for massless scalar and spin-one gauge fields. More specifically, we first prove that the representation (2.25) for scalar fields satisfies the Single Site Serre Condition (2.23), off-shell, for arbitrary conformal dimension d and in any number of space-time dimensions n. And then, we prove that for spin-one gauge fields this representation also satisfies (2.23), but only on-shell, for $d = 1$ and in $n = 4$ space-time dimensions.

To prove the Single Site Serre Condition (2.23) for the representation (2.25), we first write the single-site level-zero generators J_i^{AB} and their anti-commutators $S_i^{AD} \equiv \eta_{BC}\,\{J_i^{AB}, J_i^{CD}\}$ in terms of a smaller set of operators, κ_i^A, V_i^A and Σ_i^{AB}. For the remainder of this chapter all operators are to be still understood as acting on a single site, but we suppress the single-site notation for simplicity:

$$
\begin{aligned}
J^{AB} &= \kappa^A V^B - \kappa^B V^A + \Sigma^{AB} \\
S^{AD} &= -2\kappa^B \kappa_B V^A V^D - 2\,d\,\eta^{AD} - 2\,(d - \tfrac{n-2}{2})(\kappa^A V^D + \kappa^D V^A) \\
&\quad - 2\kappa^B V^A \Sigma^{CD}\eta_{BC} - 2\kappa^B V^D \Sigma^{CA}\eta_{BC} + \Sigma^{AB}\Sigma^{CD}\eta_{BC} + \Sigma^{CD}\Sigma^{AB}\eta_{BC}
\end{aligned}
\tag{3.1}
$$

[1]The content of this chapter was adapted with minor changes from Section 3 of the author's previously published paper in the Journal of High Energy Physics:
N. Dokmetzoglou and L. Dolan, *Properties of the conformal Yangian in scalar and gauge field theories*, *JHEP* **02** (2023) 137 [arXiv:2207.14806 [hep-th]].

where

$$\kappa^A = (\kappa^\mu, \kappa^n, \kappa^{n+1}), \qquad \kappa^\mu = k^\mu, \quad \kappa^n = -(d + k^\rho \partial_\rho) = -\kappa^{n+1}$$
$$V^A = (V^\mu, V^n, V^{n+1}), \qquad V^\mu = \partial^\mu, \qquad V^n = -\tfrac{1}{2}(1 + \partial^\rho \partial_\rho), \quad V^{n+1} = -\tfrac{1}{2}(1 - \partial^\rho \partial_\rho)$$
$$\Sigma^{AB} = \delta^A_\mu \delta^B_\nu \Sigma^{\mu\nu} + \delta^A_\mu \left(\eta^{Bn} + \eta^{B,n+1}\right) \Sigma^{\mu\rho} V_\rho - \delta^B_\mu \left(\eta^{An} + \eta^{A,n+1}\right) \Sigma^{\mu\rho} V_\rho \tag{3.2}$$

satisfy the simpler algebra

$$[\kappa^A, \kappa^B] = \tilde{c}^{AB}{}_D\, \kappa^D, \qquad [V^A, V^B] = 0, \qquad [\kappa^A, V^B] = -\eta^{AB} + c^{AB}{}_D\, V^D,$$
$$\tilde{c}^{AB}{}_D = -\delta^A_D(\eta^{Bn} + \eta^{B,n+1}) \ + \ \delta^B_D(\eta^{An} + \eta^{A,n+1}) = -\tilde{c}^{BA}{}_D$$
$$c^{AB}{}_D = -\delta^A_D(\eta^{Bn} + \eta^{B,n+1}) \ - \ \delta^B_D(\eta^{An} + \eta^{A,n+1}) = c^{BA}{}_D \tag{3.3}$$
$$[\Sigma^{\mu\nu}, \Sigma^{\rho\sigma}] = -\eta^{\mu\rho}\Sigma^{\nu\sigma} - \eta^{\nu\sigma}\Sigma^{\mu\rho} + \eta^{\mu\sigma}\Sigma^{\nu\rho} + \eta^{\nu\rho}\Sigma^{\mu\sigma}, \qquad [\Sigma^{\mu\nu}, \kappa^A] = [\Sigma^{\mu\nu}, V^A] = 0$$

To prove (2.23) we consider the scalar and gauge cases separately.

3.1 For Scalar Fields

From (2.26) we have that for scalar fields, $\Sigma^{AB} = \Sigma^{\mu\nu} = 0$, and thus, (3.1) becomes

$$J^{AB} = \kappa^A V^B - \kappa^B V^A,$$
$$S^{AD} = -2\kappa^B \kappa_B V^A V^D - 2\, d\, \eta^{AD} - 2\left(d - \tfrac{n-2}{2}\right)(\kappa^A V^D + \kappa^D V^A) \tag{3.4}$$

To construct the M^{EFCA} tensor (2.24),

$$M^{EFCA} = \left(-J^{EA} S^{CF} - (A \leftrightarrow C)\right) - (E \leftrightarrow F)\,, \tag{3.5}$$

we first compute

$$\begin{aligned}
-J^{EA}(\kappa^C V^F + \kappa^F V^C) &= -(\kappa^E V^A - \kappa^A V^E)\,(\kappa^C V^F + \kappa^F V^C) \\
&= -\kappa^E \kappa^C V^A V^F - \kappa^E \kappa^F V^A V^C + \kappa^A \kappa^C V^E V^F + \kappa^A \kappa^F V^E V^C \\
&\quad - \kappa^E [V^A, \kappa^C] V^F - \kappa^E [V^A, \kappa^F] V^C \\
&\quad + \kappa^A [V^E, \kappa^C] V^F + \kappa^A [V^E, \kappa^F] V^C
\end{aligned} \tag{3.6}$$

which, after performing the permutations as in the four tensor, becomes

$$
\begin{aligned}
&\big(-J^{EA}(\kappa^C V^F + \kappa^F V^C)\ - (A \leftrightarrow C)\big)\ -\ (E \leftrightarrow F) \\
&= -[\kappa^E, \kappa^C] V^A V^F + [\kappa^A, \kappa^F] V^E V^C + [\kappa^E, \kappa^A] V^C V^F - [\kappa^C, \kappa^F] V^E V^A \\
&\quad - J^{EC}[V^A, \kappa^F] + J^{AF}[V^E, \kappa^C] + J^{EA}[V^C, \kappa^F] - J^{CF}[V^E, \kappa^A] \\
&= -J^{EC}\eta^{AF} + J^{AF}\eta^{EC} + J^{EA}\eta^{CF} - J^{CF}\eta^{EA} \\
&\quad + (\eta^{En} + \eta^{E,n+1})\, 2J^{AC}\, V^F - (\eta^{Fn} + \eta^{F,n+1})\, 2J^{AC}\, V^E \\
&\quad - (\eta^{Cn} + \eta^{C,n+1})\, 2J^{FE}\, V^A + (\eta^{An} + \eta^{A,n+1})\, 2J^{FE}\, V^C
\end{aligned}
\tag{3.7}
$$

From the first term in S^{AD}, we have

$$
\begin{aligned}
-J^{EA}\, \kappa^2 V^C V^F &= -\kappa^2 J^{EA} V^C V^F - [J^{EA}, \kappa^2] V^C V^F \\
&= -\kappa^2 J^{EA} V^C V^F - (\eta^{En} + \eta^{E,n+1})\, \big(2\kappa^2 V^A + 2(d - \tfrac{n-2}{2})\, \kappa^A\big)\, V^C V^F \\
&\quad + (\eta^{An} + \eta^{A,n+1})\, \big(2\kappa^2 V^E + 2(d - \tfrac{n-2}{2})\, \kappa^E\big)\, V^C V^F
\end{aligned}
\tag{3.8}
$$

where $\kappa^2 = \eta_{AB}\, \kappa^A\, \kappa^B = k^2$, which after the permutations becomes

$$
\begin{aligned}
&\Big(-J^{EA}\, \kappa^2 V^C V^F - (A \leftrightarrow C)\Big) - (E \leftrightarrow F) = \Big(-[J^{EA}, \kappa^2] V^C V^F - (A \leftrightarrow C)\Big) - (E \leftrightarrow F) \\
&= -2\ \left(d - \tfrac{\delta^\rho_\rho - 2}{2}\right)\ \Big((\eta^{En} + \eta^{E,n+1})\, J^{AC}\, V^F - (\eta^{Fn} + \eta^{F,n+1})\, J^{AC}\, V^E \\
&\qquad - (\eta^{Cn} + \eta^{C,n+1})\, J^{FE}\, V^A + (\eta^{An} + \eta^{A,n+1})\, J^{FE}\, V^C\Big)
\end{aligned}
\tag{3.9}
$$

Then, combining (3.4), (3.5), (3.7) and (3.9), we find the tensor

$$
\begin{aligned}
M^{EFCA} &= -J^{EA} S^{CF} - J^{FC} S^{EA} + J^{EC} S^{AF} + J^{FA} S^{EC} \\
&= (n-2)\, (J^{EA}\eta^{CF} + J^{FC}\eta^{EA} - J^{EC}\eta^{AF} - J^{FA}\eta^{EC}) \\
&= (n-2)\, [J^{EF}, J^{CA}]
\end{aligned}
\tag{3.10}
$$

Finally, substituting this form of M^{EFCA} into the Single Site Serre Condition (2.23) gives a vanishing result due to the sum over the various permutations. So, we have proved the Serre relation for the momentum-space differential operator representation of the conformal Yangian $Y[\mathrm{SO}(2,n)]$ for scalar fields and for arbitrary k^2, d and n, i.e. off-shell, for any conformal dimension d and in any number of space-time dimensions n.

3.2 For Gauge Fields

For gauge fields, the proof of (2.23) is more complicated. The anti-commutator S^{AD} has additional terms, from (3.1) and (2.27), which can be simplified using the commutation relations (3.3),

$$\begin{aligned}&-2\kappa^B V^A \Sigma^{CD}\eta_{BC} - 2\kappa^B V^D \Sigma^{CA}\eta_{BC} + \Sigma^{AB}\Sigma^{CD}\eta_{BC} + \Sigma^{CD}\Sigma^{AB}\eta_{BC}\\ &= \left(-2V^A\, k_\mu\, \left(\delta^D_\rho + (\delta^D_{n+1} - \delta^D_n)V_\rho\right)\Sigma^{\mu\rho} + \Sigma^{A\mu}\Sigma^{\nu D}\eta_{\mu\nu}\right) + (A \leftrightarrow D)\end{aligned} \tag{3.11}$$

With the α, γ indices on $\Sigma^{\mu\nu}_{(\alpha\gamma)}$ displayed explicitly, (3.11) becomes

$$\begin{aligned}=\Big(&-2k_\alpha\, V^A\big(\delta^D_\gamma + (\delta^D_{n+1} - \delta^D_n)V_\gamma\big)\\ &+ (n-4)\big(\delta^A_\alpha + (\eta^{An} + \eta^{A,n+1})V_\alpha\big)\,\big(\delta^D_\gamma + (\delta^D_{n+1} - \delta^D_n)V_\gamma\big)\\ &- 2V^A\,(\delta^D_{n+1} - \delta^D_n)\eta_{\alpha\gamma} + 2V^A\big(\delta^D_\alpha + (\delta^D_{n+1} - \delta^D_n)V_\alpha\big)\,k_\gamma\\ &+ (\delta^A_\delta + (\delta^A_{n+1} - \delta^A_n)V_\delta)\,(\delta^D_{\delta'} + (\delta^D_{n+1} - \delta^D_n)V_{\delta'})\,\eta^{\delta\delta'}\eta_{\alpha\gamma}\Big) + (A \leftrightarrow D)\end{aligned} \tag{3.12}$$

where we used the anti-commutator

$$\begin{aligned}\Sigma^{A\mu}\Sigma^{\nu D}\eta_{\mu\nu} + (A \leftrightarrow D) = {}& 2\,(\delta^A_\delta + (\delta^A_{n+1} - \delta^A_n)V_\delta)\,(\delta^D_{\delta'} + (\delta^D_{n+1} - \delta^D_n)V_{\delta'})\,\eta^{\delta\delta'}\eta_{\alpha\gamma}\\ &+ (n-2)\,\Big(\big(\delta^A_\alpha + (\delta^A_{n+1} - \delta^A_n)V_\alpha\big)\,\big(\delta^D_\gamma + (\delta^D_{n+1} - \delta^D_n)V_\gamma\big)\\ &\qquad + \big(\delta^D_\alpha + (\delta^D_{n+1} - \delta^D_n)V_\alpha\big)\,\big(\delta^A_\gamma + (\delta^A_{n+1} - \delta^A_n)V_\gamma\big)\Big)\end{aligned} \tag{3.13}$$

We reduce (3.12) further using

$$\begin{aligned}&V^A\,(\delta^D_{n+1} - \delta^D_n) + V^D\,(\delta^A_{n+1} - \delta^A_n) - (\delta^A_\delta + (\delta^A_{n+1} - \delta^a_n)V_\delta)\,(\delta^D_{\delta'} + (\delta^D_{n+1} - \delta^D_n)V_{\delta'})\,\eta^{\delta\delta'}\\ &= -\delta^A_{n+1}\delta^D_{n+1} + \delta^A_n\delta^D_n - \delta^A_\delta\delta^D_{\delta'}\eta^{\delta\delta'} = -\eta^{AD}\end{aligned} \tag{3.14}$$

and combine it with the Σ-independent terms in (3.1) to evaluate the anti-commutator for the gauge field representation as

$$
\begin{aligned}
S^{AD} = \Big(& - 2k_\alpha\, V^A\big(\delta^D_\gamma + (\delta^D_{n+1} - \delta^D_n)V_\gamma\big) \\
& + (n-4)\big(\delta^A_\alpha + (\eta^{An} + \eta^{A,n+1})V_\alpha\big)\,\big(\delta^D_\gamma + (\delta^D_{n+1} - \delta^D_n)V_\gamma\big) \\
& + 2V^A(\delta^D_\alpha + (\delta^D_{n+1} - \delta^D_n)V_\alpha\big)\; k_\gamma \\
& + \eta_{\alpha\gamma}\big(- k^2 V^A V^D - \;(d-1)\,\eta^{AD} - 2\,(d - \tfrac{n-2}{2})\;\kappa^A V^D\big)\Big) \\
+ & (A \leftrightarrow D)
\end{aligned}
\tag{3.15}
$$

To construct the four tensor M^{EFCA} for the gauge field representation, we first compute from (3.15) the product

$$
\begin{aligned}
-J^{EA}S^{CF} &= -J^{EA}_{\alpha\beta'}\; S^{CF}_{\beta\gamma}\,\eta^{\beta\beta'} \\
&= \Big(2J^{EA}_{\alpha\beta}\; k^\beta V^C\big(\delta^F_\gamma + (\eta^{Fn} + \delta^{F,n+1})V_\gamma\big) + J^{EA}_{\alpha\gamma}\; k^2\; V^C V^F \\
&\quad - (n-4)J^{EA}_{\alpha\beta}\;\big(\eta^{C\beta} + (\eta^{Cn} + \eta^{C,n+1})V^\beta\big)\;\big(\delta^F_\gamma + (\eta^{Fn} + \eta^{F,n+1})V_\gamma\big) \\
&\quad - 2J^{EA}_{\alpha\beta}\; V^C\;(\eta^{F\beta} + (\eta^{Fn} + \eta^{F,n+1})V^\beta)\; k_\gamma \\
&\quad + (d-1)J^{EA}_{\alpha\gamma}\;\eta^{CF} + 2(d - \tfrac{n-2}{2})\; J^{EA}_{\alpha\gamma}\;\kappa^C V^F\Big)\; + (C \leftrightarrow F)
\end{aligned}
\tag{3.16}
$$

Finally, with the use of

$$
\begin{aligned}
J^{EA}_{\alpha\beta}\ k^{\beta} &= [J^{EA}_{\alpha\beta}, k^{\beta}] + k^{\beta}\ J^{EA}_{\alpha\beta} \\
&= k_{\alpha}\ \Big((\kappa^{E} + \eta^{En} + \eta^{E,n+1})\ V^{A} - (\kappa^{A} + \eta^{An} + \eta^{A,n+1})\ V^{E} \Big) \\
&\quad + (d+1-n)\ \Big(\delta^{A}_{\alpha}\ (\eta^{En} + \eta^{E,n+1}) - \delta^{E}_{\alpha}\ (\eta^{An} + \eta^{A,n+1}) \Big)
\end{aligned}
$$

$$
\begin{aligned}
J^{EA}_{\alpha\gamma}\ k^{2} &= [J^{EA}_{\alpha\gamma}, k^{2}] + k^{2}\ J^{EA}_{\alpha\gamma} \\
&= 2(\eta^{En} + \eta^{E,n+1})\ (k_{\alpha}\delta^{A}_{\gamma} - \delta^{A}_{\alpha}k_{\gamma}) - 2(\eta^{An} + \eta^{A,n+1})\ (k_{\alpha}\delta^{E}_{\gamma} - \delta^{E}_{\alpha}k_{\gamma}) \\
&\quad + \eta_{\alpha\gamma}\ \Big((\eta^{En} + \eta^{E,n+1})\ (2k^{2}V^{A} + 2(d - \tfrac{n-2}{2})\kappa^{A}) \\
&\qquad\qquad - (\eta^{An} + \eta^{A,n+1})\ (2k^{2}V^{E} + 2(d - \tfrac{n-2}{2})\kappa^{E})\ \Big) \\
&\quad + k^{2}\ J^{EA}_{\alpha\gamma}
\end{aligned}
$$

$$
\begin{aligned}
J^{EA}_{\alpha\gamma}\ k^{2}\ V^{C}V^{F} &= k_{\alpha}\ 2\ \big(\delta^{A}_{\gamma}\ (\eta^{En} + \eta^{E,n+1}) - \delta^{E}_{\gamma}\ (\eta^{An} + \eta^{A,n+1}) \big)\ V^{C}V^{F} \\
&\quad - 2\ \big(\delta^{A}_{\alpha}\ (\eta^{En} + \eta^{E,n+1}) - \delta^{E}_{\alpha}\ (\eta^{An} + \eta^{A,n+1}) \big)\ V^{C}V^{F}\ k_{\gamma} \\
&\quad + 2\ \big(\delta^{A}_{\alpha}\ (\eta^{En} + \eta^{E,n+1}) - \delta^{E}_{\alpha}\ (\eta^{An} + \eta^{A,n+1}) \big) \\
&\qquad \cdot (\delta^{C}_{\gamma} + (\eta^{Cn} + \eta^{C,n+1})V_{\gamma})\ V^{F} + V^{C}\ (\delta^{F}_{\gamma} + (\eta^{Fn} + \eta^{F,n+1})V_{\gamma}) \\
&\quad + \Big[\eta_{\alpha\gamma}\ \Big((\eta^{En} + \eta^{E,n+1})\ (2k^{2}V^{A} + 2(d - \tfrac{n-2}{2})\kappa^{A}) \\
&\qquad\qquad - (\eta^{An} + \eta^{A,n+1})\ (2k^{2}V^{E} + 2(d - \tfrac{n-2}{2})\kappa^{E}) \Big) \\
&\qquad + k^{2}\ J^{EA}_{\alpha\gamma} \Big]\ V^{C}V^{F}
\end{aligned}
\tag{3.17}
$$

the product becomes

$$
\begin{aligned}
&- J^{EA} S^{CF} \\
=& \Big[k_\alpha \, 2 \left((\kappa^E + \eta^{En} + \eta^{E,n+1}) \, V^A - (\kappa^A + \eta^{An} + \eta^{A,n+1}) \, V^E \right) \\
&+ (2(d+1-n)+4) \left(\delta^A_\alpha \, (\eta^{En} + \eta^{E,n+1}) - \delta^E_\alpha \, (\eta^{An} + \eta^{A,n+1}) \right) \Big] \\
&\qquad \cdot (\delta^C_\gamma + (\eta^{Cn} + \eta^{C,n+1}) V_\gamma) \, V^F + V^C \, (\delta^F_\gamma + (\eta^{Fn} + \eta^{F,n+1}) V_\gamma) \\
&+ k_\alpha \, 4 \left(\delta^A_\gamma \, (\eta^{En} + \eta^{E,n+1}) - \delta^E_\gamma \, (\eta^{An} + \eta^{A,n+1}) \right) V^C V^F \\
&- 4 \left(\delta^A_\alpha \, (\eta^{En} + \eta^{E,n+1}) - \delta^E_\alpha \, (\eta^{An} + \eta^{A,n+1}) \right) V^C V^F \, k_\gamma \\
&+ 2 \Big[\eta_{\alpha\gamma} \Big((\eta^{En} + \eta^{E,n+1})(2k^2 V^A + 2(d - \tfrac{n-2}{2}) \kappa^A) \\
&\qquad - (\eta^{An} + \eta^{A,n+1})(2k^2 V^E + 2(d - \tfrac{n-2}{2}) \kappa^E) \Big) + k^2 J^{EA}_{\alpha\gamma} \Big] V^C V^F \\
&+ \Big[\Big(- (n-4) J^{EA}_{\alpha\beta} \left(\eta^{C\beta} + (\eta^{Cn} + \eta^{C,n+1}) V^\beta \right) \left(\delta^F_\gamma + (\eta^{Fn} + \eta^{F,n+1}) V_\gamma \right) \\
&\quad - 2 J^{EA}_{\alpha\beta} \, V^C \, (\eta^{F\beta} + (\eta^{Fn} + \eta^{F,n+1}) V^\beta) \, k_\gamma + (d-1) J^{EA}_{\alpha\gamma} \, \eta^{CF} + 2(d - \tfrac{n-2}{2}) \, J^{EA}_{\alpha\gamma} \, \kappa^C V^F \Big) \\
&\quad + (C \leftrightarrow F) \Big]
\end{aligned} \tag{3.18}
$$

We see that (3.18) does not lead to a four tensor (3.5) that satisfies the Serre relation for arbitrary k^2, d, and n. But for $k^2 = 0$, $d = 1$, and $n = 4$, the product reduces to

$$
\begin{aligned}
-J^{EA} S^{CF} = 2k_\alpha & \left[(\kappa^E + \eta^{En} + \eta^{E,n+1}) \, V^A - (\kappa^A + \eta^{An} + \eta^{A,n+1}) \, V^E \right] \\
& \cdot \left[(\delta^C_\gamma + (\eta^{Cn} + \eta^{C,n+1}) V_\gamma) \, V^F + V^C \, (\delta^F_\gamma + (\eta^{Fn} + \eta^{F,n+1}) V_\gamma) \right] \\
&+ 4k_\alpha \left[\delta^A_\gamma \, (\eta^{En} + \eta^{E,n+1}) - \delta^E_\gamma \, (\eta^{An} + \eta^{A,n+1}) \right] V^C V^F \\
&- 4 \left[\delta^A_\alpha \, (\eta^{En} + \eta^{E,n+1}) - \delta^E_\alpha \, (\eta^{An} + \eta^{A,n+1}) \right] V^C V^F \;\; k_\gamma \\
&- 2 J^{EA}_{\alpha\beta} \left[V^C \, (\eta^{F\beta} + (\eta^{Fn} + \eta^{F,n+1}) V^\beta) \; + (C \leftrightarrow F) \right] k_\gamma
\end{aligned} \tag{3.19}
$$

which we recognize as a gauge transformation.

So $J^{EA} S^{CF} = 0$ when acting on on-shell gauge amplitudes since they are gauge invariant. Note that (3.19) is a single site expression, but it holds for any site i, $1 \le i \le N$. The gauge invariance of on-shell gauge amplitudes provides $(k_i \cdot \partial_{\epsilon_i}) A_N(k_1, \ldots, k_N; \epsilon_1, \ldots, \epsilon_N) = k_{i\gamma} \, \tilde{A}^\gamma_i(k_j; \epsilon_{j \neq i}) = 0$ for all i. For example, if we consider (3.19) at site one, $k_{1\gamma} \, \tilde{A}^\gamma_1(k_1, \ldots, k_N; \epsilon_2, \ldots, \epsilon_N) = 0$ will cause the terms in (3.19) proportional to k_γ to vanish. The terms proportional to k_α will vanish upon multiplication on the left of $J^{EA}S^{CF}$ with the polarization vector ϵ_1^α, due to the transversality condition $k_i \cdot \epsilon_i = 0$. That is to say

$$\epsilon_1^\alpha \left(J_1^{EA} S_1^{CF}\right)_{\alpha\gamma} \tilde{A}_1^\gamma(k_1, \ldots, k_N; \epsilon_2, \ldots, \epsilon_N) = 0 \ , \tag{3.20}$$

which implies M^{EFCA} vanishes on the amplitudes

$$\epsilon_1^\alpha \left(M^{EFCA}\right)_{\alpha\gamma} \tilde{A}_1^\gamma(k_1, \ldots, k_N; \epsilon_2, \ldots, \epsilon_N) = 0 \ , \tag{3.21}$$

and proves that the Single Site Serre Condition (2.23) for the gauge field representation (2.25) of the conformal Yangian $Y[\mathrm{SO}(2,n)]$ is satisfied, but only on-shell, for $d = 1$ and in $n = 4$ space-time dimensions.

This restriction to fields satisfying their free field equations of motion for representation theory is familiar from earlier discussions of the conformal group [49].

Proof of the Serre relation in the context of the $\mathrm{SU}(N)$ Yangian and the $PSU(2,2|4)$ Yangian was given in [24] using tensor operator methods. These methods also occur in [50]. In this work we emphasize that the $\mathrm{SO}(2,n)$ Yangian gauge field representation only has a consistent Serre relation for on-shell fields and for $n = 4$, $d = 1$. In contrast, the scalar field representation is consistent off shell, for arbitrary n and conformal dimension d.

CHAPTER 4

ACTION OF CONFORMAL YANGIAN $Y[\mathbf{SO}(2,n)]$ GENERATORS ON SCALAR $\lambda\,\phi^3$ THEORY

In this chapter, we investigate the action of the level-zero and level-one conformal Yangian $Y[\mathrm{SO}(2,n)]$ generators on scalar $\lambda\,\phi^3$ theory off-shell tree-level amplitudes and individual Feynman graphs.

We find that rewriting the momentum-space differential operator representation of the generators, discussed previously, in terms of derivatives with respect to kinematic invariants with consecutive momenta, significantly simplifies the calculation of their action on the off-shell $\lambda\,\phi^3$ theory tree-level amplitudes, which themselves can be written as functions of those invariants alone. We show that the level-zero generators ($\mathbb{P}^\mu$, $\mathbb{L}^{\mu\nu}$, $\mathbb{D}$, $\mathbb{K}^\mu$), i.e. the conformal group generators, annihilate all $\lambda\,\phi^3$ theory off-shell tree-level Feynman graphs in $n = 6$ space-time dimensions. This result can be easily extended to show that $\lambda\,\phi^3$ theory off-shell tree-level partial and total amplitudes are also annihilated by the level-zero generators in $n = 6$ space-time dimensions, as expected.

We find that the level-one generator $\widehat{\mathbb{P}}^\mu$ acts as a multiplicative factor of the form $\sum_{i=1}^{N} c_i k_i^\mu$ on individual $\lambda\,\phi^3$ theory off-shell tree-level Feynman graphs, where c_i are integer valued **graph-specific** parameters, which we call the **evaluation parameters**, adopting the language used in the literature [51]. Extending all the level-one generators $\left(\widehat{\mathbb{P}}^\mu,\, \widehat{\mathbb{L}}^{\mu\nu},\, \widehat{\mathbb{D}},\, \widehat{\mathbb{K}}^\mu\right)$ by terms that depend on these graph-specific evaluation parameters, as in

$$\begin{aligned} \widehat{\mathbb{P}}'^\mu &\equiv \widehat{\mathbb{P}}^\mu + \sum_{i=1}^{N} c_i\, P_i^\mu & \qquad \widehat{\mathbb{L}}'^{\mu\nu} &\equiv \widehat{\mathbb{L}}^{\mu\nu} + \sum_{i=1}^{N} c_i\, L_i^{\mu\nu} \\ \widehat{\mathbb{D}}' &\equiv \widehat{\mathbb{D}} + \sum_{i=1}^{N} c_i\, D_i & \qquad \widehat{\mathbb{K}}'^\mu &\equiv \widehat{\mathbb{K}}^\mu + \sum_{i=1}^{N} c_i\, K_i^\mu \end{aligned}\,, \tag{4.1}$$

we show that this extended construction of the level-one generators $\left(\widehat{\mathbb{P}}'^\mu,\, \widehat{\mathbb{L}}'^{\mu\nu},\, \widehat{\mathbb{D}}',\, \widehat{\mathbb{K}}'^\mu\right)$ annihilates **individual** $\lambda\,\phi^3$ theory off-shell tree-level Feynman graphs in $n = 6$ space-time dimensions. However, neither the original nor the extended construction of the level-one generators annihilates $\lambda\,\phi^3$ theory off-shell tree-level partial or total amplitudes, which are sums of individual Feynman graphs, each one with its own set of evaluation parameters.

4.1 Some Basic Definitions and Preliminary Comments

For all subsequent discussion, it is useful for us to define any N-point function $\mathcal{A}_N$ for which momentum conservation is assumed (e.g. an individual Feynman graph, partial or total amplitude) as follows:

$$\mathcal{A}_N(k_1, k_2, \ldots, k_N) \equiv \delta^n(k_1 + k_2 + \cdots + k_N)\; A_N(k_1, k_2, \ldots, k_N)\,, \tag{4.2}$$

where δ^n is the n-dimensional momentum-conserving Dirac delta function, A_N is the kinematic part of the N-point function, and where the caligraphic $\mathcal{A}_N$ and plain A_N have been chosen to distinguish between the full N-point function and its kinematic part with the momentum-conserving delta function stripped off. Further, we use the superscript Δ to denote an individual off-shell $\lambda\,\phi^3$ theory graph, e.g. $A_N^{\phi^3,\,\Delta}$. For a discussion of the distinction between individual graphs, partial amplitudes and total amplitudes, see Appendix C.

Let S be any subset of the set $A \equiv \{1, 2, \ldots, N\}$. We define the corresponding kinematic invariant as $k_S^2 \equiv (\sum_{j\in S} k_j)^2$. Given the assumption of momentum conservation, any invariant k_S^2 can be rewritten as a linear combination of other invariants. Thus, one has some freedom in choosing the basis of independent kinematic invariants they wish to use. In [22] Dolan and Goddard show that a choice of basis of $\frac{1}{2}N(N-1)$ independent off-shell invariants with consecutive momenta, of the form $k^2_{[I,J]} \equiv (k_I + k_{I+1} + \ldots + k_{J-1} + k_J)^2 \equiv s_{I,I+1,\ldots,J-1,J}$, with $1 \le I \le J < N$, is convenient for expressing the polynomial form of the CHY scattering equations off-shell. We adopt this choice of basis of consecutive invariants here, as it appears to significantly simplify our calculations.

In this basis we can express all $\lambda\,\phi^3$ theory off-shell tree-level amplitudes as functions of $\frac{1}{2}N(N-3)$ of these consecutive off-shell invariants $k^2_{[I,J]}$, with $1 \le I < J < N$, but not $k^2_{[1,N-1]}$. Now, let us define the set of all consecutive subsets $\mathbb{A} \equiv \{[I,J] : 1 \le I \le J < N\}$, where $[I,J] = \{a \in \mathbb{N} : I \le a \le J\}$, such that $\{k_S^2 : S \in \mathbb{A}\}$ is our chosen set of all $\frac{1}{2}N(N-1)$ independent off-shell kinematic invariants. Let us further define for each individual off-shell $\lambda\,\phi^3$ theory Feynman graph, $A_N^{\phi^3,\,\Delta}$, a set $\Delta \subset \mathbb{A}$ which only includes the $N-3$ consecutive subsets $[I,J]$ which appear in the invariants of the denominator of the given graph, such that

$$A_N^{\phi^3,\,\Delta}(k_1,\ldots,k_N) \equiv (-1)^{N+1} \prod_{S\in\Delta} \frac{1}{k_S^2} = (-1)^{N+1} \prod_{[I,J]\in\Delta} \frac{1}{k_{[I,J]}^2}\,. \tag{4.3}$$

For example, for the graph $A_6^{\phi^3,\,\Delta} = -1/\left((k_1+k_2)^2(k_1+k_2+k_3)^2(k_1+k_2+k_3+k_4)^2\right)$, we define $\Delta \equiv \{\{1,2\},\{1,2,3\},\{1,2,3,4\}\} = \{[1,2],[1,3],[1,4]\}$, such that $A_6^{\phi^3,\,\Delta} = (-1)\prod_{S\in\Delta} 1/k_S^2$.

Similarly, for the two graphs that constitute the $N=4$ partial amplitude, $-\frac{1}{s_{12}}$ and $-\frac{1}{s_{23}}$, we define Δ as $\{[1,2]\}$ and $\{[2,3]\}$ respectively, and for the five graphs of the $N=5$ partial amplitude, $\frac{1}{s_{12}s_{123}}$, $\frac{1}{s_{12}s_{34}}$, $\frac{1}{s_{23}s_{123}}$, $\frac{1}{s_{23}s_{234}}$ and $\frac{1}{s_{34}s_{234}}$, we define Δ as $\{[1,2],[1,3]\}$, $\{[1,2],[3,4]\}$, $\{[2,3],[1,3]\}$, $\{[2,3],[2,4]\}$ and $\{[3,4],[2,4]\}$ respectively. We can find the $N>5$ graphs from the off-shell recurrence relation [22],

$$A_N^{\phi^3,\,\Delta}(k_1,\ldots,k_N) = -\frac{1}{s_{34}}\,A_{N-1}^{\phi^3,\,\Delta'}(k_1,k_2,k_3+k_4,k_5,\ldots,k_N) \tag{4.4}$$

where the momenta can be cycled to find all the subsets Δ.

Finally, we note that, for a given N-point function $\mathcal{A}_N$, showing that it is annihilated by all the level-zero $Y[\mathrm{SO}(2,n)]$ generators ($\mathbb{P}^\mu$, $\mathbb{L}^{\mu\nu}$, $\mathbb{D}$, $\mathbb{K}^\mu$) and by *at least one* of the level-one $Y[\mathrm{SO}(2,n)]$ generators $\left(\widehat{\mathbb{P}}^\mu,\, \widehat{\mathbb{L}}^{\mu\nu},\, \widehat{\mathbb{D}},\, \widehat{\mathbb{K}}^\mu\right)$ is sufficient for proving that $\mathcal{A}_N$ is annihilated by *all* the level-one generators, assuming that the level-zero and level-one generators satisfy the algebra (A.5). The same holds for the extended level-one generators $\left(\widehat{\mathbb{P}}'^\mu,\, \widehat{\mathbb{L}}'^{\mu\nu},\, \widehat{\mathbb{D}}',\, \widehat{\mathbb{K}}'^\mu\right)$.

For instance, assuming $\mathbb{P}^\mu\mathcal{A}_N = 0$, $\mathbb{L}^{\mu\nu}\mathcal{A}_N = 0$, $\mathbb{D}\mathcal{A}_N = 0$, $\mathbb{K}^\mu\mathcal{A}_N = 0$, and $\widehat{\mathbb{D}}\mathcal{A}_N = 0$, from (A.5) we have

$$\left[\mathbb{K}^\mu, \widehat{\mathbb{D}}\right]\mathcal{A}_N = \widehat{\mathbb{K}}^\mu\mathcal{A}_N \quad\Rightarrow\quad \mathbb{K}^\mu\cancelto{0}{\left(\widehat{\mathbb{D}}\mathcal{A}_N\right)} - \widehat{\mathbb{D}}\cancelto{0}{\left(\mathbb{K}^\mu\mathcal{A}_N\right)} = \widehat{\mathbb{K}}^\mu\mathcal{A}_N \quad\Rightarrow\quad \widehat{\mathbb{K}}^\mu\mathcal{A}_N = 0 \tag{4.5}$$

and

$$\left[\mathbb{P}^\mu, \widehat{\mathbb{D}}\right]\mathcal{A}_N = -\widehat{\mathbb{P}}^\mu\mathcal{A}_N \quad\Rightarrow\quad \mathbb{P}^\mu\cancelto{0}{\left(\widehat{\mathbb{D}}\mathcal{A}_N\right)} - \widehat{\mathbb{D}}\cancelto{0}{\left(\mathbb{P}^\mu\mathcal{A}_N\right)} = -\widehat{\mathbb{P}}^\mu\mathcal{A}_N \quad\Rightarrow\quad \widehat{\mathbb{P}}^\mu\mathcal{A}_N = 0, \tag{4.6}$$

which further implies that

$$\left[\mathbb{K}^\nu, \widehat{\mathbb{P}}^\mu\right] \mathcal{A}_N = 2\, g^{\mu\nu} \cancelto{0}{\left(\widehat{\mathbb{D}} \mathcal{A}_N\right)} + 2\, \widehat{\mathbb{L}}^{\mu\nu} \mathcal{A}_N \quad \Rightarrow \quad \mathbb{K}^\nu \cancelto{0}{\left(\widehat{\mathbb{P}}^\mu \mathcal{A}_N\right)} - \widehat{\mathbb{P}}^\mu \cancelto{0}{\left(\mathbb{K}^\nu \mathcal{A}_N\right)} = 2\, \widehat{\mathbb{L}}^{\mu\nu} \mathcal{A}_N$$
$$\Rightarrow \quad \widehat{\mathbb{L}}^{\mu\nu} \mathcal{A}_N = 0. \tag{4.7}$$

4.2 Level-Zero Generators on Scalar $\lambda\,\phi^3$ Theory Off-Shell Tree Graphs

Massless scalar $\lambda\,\phi^3$ theory is known to be conformally invariant as a classical field theory in $n = 6$ space-time dimensions. As such, we expect its tree-level amplitudes to be annihilated by the momentum-space generators of the conformal group SO(2, 6), as defined in (2.32), in $n = 6$ space-time dimensions. Here we show that that is indeed the case by acting on $\lambda\,\phi^3$ off-shell tree graphs with $\mathbb{P}^\mu$, $\mathbb{L}^{\mu\nu}$, $\mathbb{D}$, and $\mathbb{K}^\mu$. We show that $\mathbb{P}^\mu$ and $\mathbb{L}^{\mu\nu}$ annihilate any $\lambda\,\phi^3$ off-shell tree graph in any arbitrary number of space-time dimensions n, while $\mathbb{D}$ and $\mathbb{K}^\mu$ annihilate the graphs only in $n = 6$ space-time dimensions. We use the terms "conformal group SO(2, n) generators" and "level-zero conformal Yangian Y[SO(2, n)] generators" interchangeably.

4.2.1 Action of Level-Zero Translation Generators

The level-zero translation generators, by definition, act very simply on any N-point function. They simply multiply the function they are acting on by the sum of all the N external momenta, which by momentum-conservation vanishes in any number of space-time dimensions. Therefore, for a given $\lambda\,\phi^3$ off-shell tree graph, we have

$$\mathbb{P}^\mu\, \mathcal{A}_N^{\phi^3,\,\Delta} = \left(\sum_{i=1}^{N} k_i{}^\mu\right) \mathcal{A}_N^{\phi^3,\,\Delta} = \left(\sum_{i=1}^{N} k_i{}^\mu\right) \delta^n\left(\textstyle\sum_{j=1}^N k_j\right) A_N^{\phi^3,\,\Delta} = 0\,. \tag{4.8}$$

So, the level-zero translation generators, $\mathbb{P}^\mu$, annihilate all $\lambda\,\phi^3$ off-shell tree graphs in any number of space-time dimensions n.

Given that any $\lambda\,\phi^3$ off-shell tree-level partial or total amplitude is a sum of individual graphs, and that the level-zero translation generators act identically on all graphs, we can further conclude that $\mathbb{P}^\mu$ annihilates any $\lambda\,\phi^3$ off-shell tree-level partial or total amplitude in any number of space-time dimensions.

4.2.2 Action of Level-Zero Lorentz Transformation Generators

It can be shown that the level-zero Lorentz transformation generators simply pass through the momentum-conserving Dirac delta function. Since all $\lambda\,\phi^3$ theory off-shell graphs have kinematic parts that consist exclusively of scalar products of the external momenta, i.e. the off-shell kinematic invariants k_S^2, it can be easily shown that the level-zero Lorentz transformation generators, $\mathbb{L}^{\mu\nu}$, annihilate any $\lambda\,\phi^3$ theory off-shell graph in any number of space-time dimensions n, as in

$$\mathbb{L}^{\mu\nu}\,\mathcal{A}_N^{\phi^3,\,\Delta} = \delta^n\left(\textstyle\sum_{j=1}^N k_j\right)\mathbb{L}^{\mu\nu}\,A_N^{\phi^3,\,\Delta} = 0\,. \tag{4.9}$$

This result clearly extends to any $\lambda\,\phi^3$ off-shell tree-level partial or total amplitude, as was explained in the previous section for the level-zero translation generators.

4.2.3 Action of Level-Zero Dilatation Generator

The level-zero dilatation generator simply extracts the degree of homogeneity, or scaling dimension, of the function it is acting on. The momentum-conserving Dirac delta function has scaling dimension $-n$, while the kinematic part of any $\lambda\,\phi^3$ theory off-shell N-point graph is $-2(N-3)$. Therefore, for the kinematic part of a given $\lambda\,\phi^3$ theory off-shell graph, we have

$$\mathbb{D}\,A_N^{\phi^3,\,\Delta} = [N\,d - 2\,(N-3)]\,A_N^{\phi^3,\,\Delta}\,, \tag{4.10}$$

while for the full expression for the tree graph, we have

$$\mathbb{D}\,\mathcal{A}_N^{\phi^3,\,\Delta} = \mathbb{D}\left[\delta^n\left(\textstyle\sum_{j=1}^N k_j\right)A_N^{\phi^3,\,\Delta}\right] = [N\,(d-2)-(n-6)]\;\mathcal{A}_N^{\phi^3,\,\Delta}\,. \tag{4.11}$$

We conclude that the level-zero dilatation generator, $\mathbb{D}$, annihilates the full expression, i.e. including the momentum-conserving Dirac delta function, for any $\lambda\,\phi^3$ theory off-shell graph only for $d=2$ and in $n=6$ space-time dimensions. We note that, since the canonical mass dimension for bosonic fields is $d=(n-2)/2$, the conditions $d=2$ and $n=6$ are equivalent.

This annihilation result naturally extends to the action on the full expressions for any $\lambda\,\phi^3$ off-shell tree-level partial or total amplitude in $n=6$ space-time dimensions.

4.2.4 Action of Level-Zero Special Conformal Transformation Generators

It can be shown that the level-zero special conformal transformation generators simply pass through the momentum-conserving Dirac delta function [38]. However, the action of $\mathbb{K}^\mu$ on the kinematic parts of $\lambda\,\phi^3$ theory off-shell graphs is somewhat more involved compared to that of $\mathbb{P}^\mu$, $\mathbb{L}^{\mu\nu}$ and $\mathbb{D}$. To simplify our calculations, we rewrite all partial derivatives with respect to the momentum vectors, $\partial/\partial k_{j\,\mu}$, in terms of derivatives with respect to kinematic invariants, $\partial/\partial k_S^2$, using

$$\frac{\partial}{\partial k_{j\,\mu}} = \sum_{S\subset A} \frac{\partial k_S^2}{\partial k_{j\,\mu}} \frac{\partial}{\partial k_S^2} = \sum_{S\subset A} \left\{ \begin{array}{ll} 2\,k_S^\mu & \text{if } j\in S \\ 0 & \text{if } j\notin S \end{array} \right\} \frac{\partial}{\partial k_S^2} = \sum_{S\subset A} 2\,k_S^\mu\,[\![j\in S]\!]\,\frac{\partial}{\partial k_S^2}\,, \tag{4.12}$$

where in the last inequality we have re-expressed the conditional statement in terms of an Iverson bracket. Recall that, for a mathematical statement Q, the Iverson bracket is defined by

$$[\![Q]\!] \equiv \begin{cases} 1 & \text{if } Q \text{ is true} \\ 0 & \text{if } Q \text{ is false} \end{cases}\,. \tag{4.13}$$

These techniques, of converting all differential operators to derivatives with respect to kinematic invariants and of using Iverson brackets for the expression of conditional statements, prove to be even more useful for calculating the action of level-one generators later on.

Applying these methods to the action of $\mathbb{K}^\mu$ on a given $\lambda\,\phi^3$ theory off-shell graph, we find

$$\begin{aligned} \mathbb{K}^\mu\,\mathcal{A}_N^{\phi^3,\,\Delta} &= \mathbb{K}^\mu\,\left[\delta^n\Big(\textstyle\sum_{j=1}^N k_j\Big) A_N^{\phi^3,\,\Delta}\right] \\ &= 2\,\delta^n\Big(\textstyle\sum_{j=1}^N k_j\Big) \displaystyle\sum_{S\in\Delta} \left[2\,|S|\,(d-2)-(n-6)\right]\,k_S^\mu\,\left[\frac{\partial}{\partial k_S^2}\,A_N^{\phi^3,\,\Delta}\right] \\ &= -2\sum_{S\in\Delta}\left[2\,|S|\,(d-2)-(n-6)\right]\,\frac{k_S^\mu}{k_S^2}\,\mathcal{A}_N^{\phi^3,\,\Delta} \end{aligned} \tag{4.14}$$

where Δ is the set of consecutive subsets associated with the off-shell kinematic invariants appearing in the graph $A_N^{\phi^3,\,\Delta}$, as defined in (4.3), $k_S^\mu = \sum_{j\in S} k_j^\mu$ and $k_S^2 = (\sum_{j\in S} k_j)^2$. For the last equality we used the fact that for a subset $S \in \Delta$

$$\frac{\partial}{\partial k_S^2} A_N^{\phi^3,\Delta} = \frac{\partial}{\partial k_S^2} \left(\prod_{F\in\Delta} \frac{1}{k_F^2} \right) = -\frac{1}{k_S^2} A_N^{\phi^3,\Delta} . \tag{4.15}$$

We conclude that the level-zero special conformal transformation generators, $\mathbb{K}^\mu$, annihilate any $\lambda\,\phi^3$ theory off-shell graph only for $d = 2$ and in $n = 6$ space-time dimensions.

Alternatively, one can also prove that the annihilation $\mathbb{K}^\mu\, A_N^{\phi^3,\Delta}(k_1,\ldots,k_N) = 0$ results from the recurrence relation (4.4) as follows. If we can show

$$\begin{aligned}&(K_3^\mu + K_4^\mu) \left(-\frac{1}{s_{34}}\right) A_{N-1}^{\phi^3,\Delta'}(k_1,k_2,k_3+k_4,k_5,\ldots,k_N)\\ &= -\frac{1}{s_{34}}\, K_{3+4}^\mu\, A_{N-1}^{\phi^3,\Delta'}(k_1,k_2,k_3+k_4,k_5,\ldots,k_N)\end{aligned} \tag{4.16}$$

then

$$\begin{aligned}&\mathbb{K}^\mu \left(-\frac{1}{s_{34}}\right) A_{N-1}^{\phi^3,\Delta'}(k_1,k_2,k_3+k_4,k_5,\ldots,k_N)\\ &= -\frac{1}{s_{34}} (K_1+K_2+K_{3+4}+K_5+\ldots+K_N)^\mu\, A_{N-1}^{\phi^3,\Delta'}(k_1,k_2,k_3+k_4,k_5,\ldots,k_N)\end{aligned} \tag{4.17}$$

which implies

$$\begin{aligned}&\mathbb{K}^\mu\, A_N^{\phi^3,\Delta}(k_1,\ldots,k_N)\\ &= -\frac{1}{s_{34}}\, (K_1+K_2+K_{3+4}+K_5+\ldots+K_N)^\mu\; A_{N-1}^{\phi^3,\Delta'}(k_1,k_2,k_3+k_4,k_5,\ldots,k_N) = 0\end{aligned} \tag{4.18}$$

whenever $A_{N-1}^{\phi^3,\Delta'}$ is annihilated by its relevant special conformal generators. This is explicitly true for $N = 4$ and 5, so the annihilation for higher N follows iteratively.

To show (4.16), where $K_{3+4}^\mu \equiv (2d\partial_{3+4}^\mu + 2(k_3+k_4)\cdot\partial_{3+4}\partial_{3+4}^\mu - (k_3+k_4)^\mu\partial_{3+4}^\rho\partial_{3+4,\rho})$, we find, using $\partial_{3+4}^\mu \equiv \frac{\partial}{\partial(k_3+k_4)_\mu}$,

$$\begin{aligned}&(K_3^\mu + K_4^\mu) \left(-\frac{1}{s_{34}}\right) A_{N-1}^{\phi^3,\Delta'}(k_1,k_2,k_3+k_4,k_5,\ldots,k_N)\\ &= -\frac{1}{s_{34}}\Big[(4d-4)\partial_{3+4}^\mu - \frac{(8d-2n-4)(k_3+k_4)^\mu}{s_{34}} + 2(k_3+k_4)\cdot\partial_{3+4}\partial_{3+4}^\mu\\ &\qquad - (k_3+k_4)^\mu\partial_{3+4}^\rho\partial_{\rho,3+4}\Big]\, A_{N-1}^{\phi^3,\Delta'}(k_1,k_2,k_3+k_4,k_5,\ldots,k_N)\end{aligned} \tag{4.19}$$

which gives (4.16) for $n = 6$, $d = 2$.

This annihilation result for the individual tree graphs naturally extends to the action on any $\lambda\,\phi^3$ off-shell tree-level partial or total amplitude in $n = 6$ space-time dimensions.

4.3 Level-One Generators on Scalar $\lambda\,\phi^3$ Theory Off-Shell Tree Graphs

We now consider the action of the level-one conformal Yangian generators on individual $\lambda\,\phi^3$ off-shell tree-level Feynman graphs. We start by investigating the action of $\widehat{\mathbb{P}}^\mu$ on an individual tree graph, which does not by itself annihilate the graph. However, through the action of $\widehat{\mathbb{P}}^\mu$, we are able to identify the graph-specific evaluation parameters, which we subsequently use to extend the original construction of the level-one generators $\left(\widehat{\mathbb{P}}^\mu,\, \widehat{\mathbb{L}}^{\mu\nu},\, \widehat{\mathbb{D}},\, \widehat{\mathbb{K}}^\mu\right)$ by terms that depend on these parameters for each graph, as explained in (4.1). We conclude by showing that all the extended level-one generators $\left(\widehat{\mathbb{P}}'^\mu,\, \widehat{\mathbb{L}}'^{\mu\nu},\, \widehat{\mathbb{D}}',\, \widehat{\mathbb{K}}'^\mu\right)$ annihilate *individual* $\lambda\,\phi^3$ theory off-shell tree-level Feynman graphs in $n = 6$ space-time dimensions. However, as mentioned earlier, neither the original nor the extended construction of the level-one generators annihilates $\lambda\,\phi^3$ theory off-shell tree-level partial or total amplitudes, which are sums of individual Feynman graphs, each one with its own set of evaluation parameters, thus indicating that the conformal Yangian is not a symmetry of $\lambda\,\phi^3$ theory.

4.3.1 Action of Level-One Translation Generators

The level-one translation generators (2.33), in the momentum-space differential operator representation (2.25) for scalar fields, are defined as

$$\begin{aligned} \widehat{\mathbb{P}}^\mu &= -\sum_{1\le i<j\le N}\left[\left(P_i^\mu\, D_j + g_{\alpha\beta}\, P_i^\alpha\, L_j^{\mu\beta}\right) - (i \leftrightarrow j)\right] \\ &= -\sum_{1\le i<j\le N}\left[\left(k_i^\mu\,(d + k_j^\nu\,\partial_{j\nu}) + g_{\alpha\beta}\,k_i^\alpha\,(k_i^\mu\,\partial_i^\beta - k_i^\beta\,\partial_i^\mu)\right) - (i \leftrightarrow j)\right] \end{aligned} \tag{4.20}$$

For a given off-shell kinematic invariant, $k^2_{[I,J]}$, for $1 \le I < J < N$, but not $k^2_{[1,N-1]}$, we find that the level-one translation generators act as a multiplicative factor of the form

$$\widehat{\mathbb{P}}^\mu\, k^2_{[I,J]} = \left[-\,d\sum_{j=1}^N (N+1-2\,j)\,k_j^\mu + 2\,\left(-k^\mu_{[1,\,I-1]} + k^\mu_{[J+1,\,N]}\right)\right] k^2_{[I,J]}\;. \tag{4.21}$$

Note that the first sum simplifies to $\left(2\,d\,\sum_{j=1}^N j\,k_j^\mu\right)$ under the assumption of momentum conservation $\sum_{j=1}^N k_j^\mu = 0$. Apart from the terms proportional to d, which can be extracted straightforwardly, we prove this as follows.

$$\begin{aligned}
&\sum_{1\le i<j\le N} \left(P_i^\mu\, k_j\cdot\partial_j + P_{i\rho}L_j^{\mu\rho} - (i\leftrightarrow j)\right) k_{[I,J]}^2 \\
&= \sum_{i=I}^{J} \Big[(k_1+\ldots+k_{i-1}-k_{i+1}-\ldots-k_N)^\mu k_i\cdot\partial_i \\
&\qquad + (k_1+\ldots+k_{i-1}-k_{i+1}-\ldots-k_N)_\rho(k_i^\mu\partial_i^\rho - k_i^\rho\partial_i^\mu)\Big]\; k_{[I,J]}^2 \\
&= 2\sum_{i=I}^{J} \Big[(k_1+\ldots+k_{i-1}-k_{i+1}-\ldots-k_N)^\mu k_i\cdot k_{[I,J]} \\
&\qquad + k_i^\mu(k_1+\ldots+k_{i-1}-k_{i+1}-\ldots-k_N)\cdot k_{[I,J]} \\
&\qquad - k_{[I,J]}^\mu(k_1+\ldots+k_{i-1}-k_{i+1}-\ldots-k_N)\cdot k_i\Big]
\end{aligned} \tag{4.22}$$

We simplify each of the terms in (4.22), where the third term becomes

$$\begin{aligned}
&-2\sum_{i=I}^{J} k_{[I,J]}^\mu(k_1+\ldots+k_{i-1}-k_{i+1}-\ldots-k_N)\cdot k_i \\
&= -2k_{[I,J]}^\mu\,(k_1+k_2+\ldots+k_{I-1}-k_{J+1}-k_{J+2}-\ldots-k_N)\cdot k_{[I,J]}
\end{aligned} \tag{4.23}$$

and the first term contains the answer (4.21) plus a remainder,

$$\begin{aligned}
&2\sum_{i=I}^{J} (k_1+\ldots+k_{i-1}-k_{i+1}-\ldots-k_N)^\mu k_i\cdot k_{[I,J]} \\
&= 2(k_1+k_2+\ldots+k_{I-1}-k_{J+1}-k_{J+2}-\ldots-k_N)^\mu\, k_{[I,J]}^2 \\
&\quad + 2(-k_{I+1}-k_{I+2}-\ldots-k_J)^\mu\, k_I\cdot k_{[I,J]} \;+\ldots+\; 2(k_I+\ldots+k_{J-1})^\mu\, k_J\cdot k_{[I,J]}
\end{aligned} \tag{4.24}$$

Adding this remainder to (4.23) and the second term in (4.22), we find the cancellation

$$\begin{aligned}
&(-k_{I+1}-k_{I+2}-\ldots-k_J)^\mu\, k_I\cdot k_{[I,J]} + (k_I-k_{I+2}-k_{I+3}-\ldots-k_J)^\mu\, k_{I+1}\cdot k_{[I,J]} \\
&+\ldots+(k_I+\ldots+k_{J-1})^\mu\, k_J\cdot k_{[I,J]} \\
&+\sum_{i=I}^{J} k_i^\mu(k_1+\ldots+k_{i-1}-k_{i+1}-\ldots-k_N)\cdot k_{[I,J]} \\
&-k_{[I,J]}^\mu\,(k_1+k_2+\ldots+k_{I-1}-k_{J+1}-k_{J+2}-\ldots-k_N)\cdot k_{[I,J]} \;=\; 0
\end{aligned} \tag{4.25}$$

by identifying the coefficient of each k_i^μ, $I \leq i \leq J$ to be zero in (4.25). Here we have defined sums of consecutive momenta as $k_{[I,J]}^\mu \equiv (k_I + k_{I+1} + \cdots + k_{J-1} + k_J)^\mu$. This proves (4.21) which says that $\widehat{\mathbb{P}}^\mu$ acts on off-shell kinematic invariants as a simple multiplicative factor, similarly to the action of the level-zero translations generator $\mathbb{P}^\mu$.

Moreover, it can be shown that $\widehat{\mathbb{P}}^\mu$ simply passes through the momentum-conserving Dirac delta function (see Appendix B). Using these results, we find that $\widehat{\mathbb{P}}^\mu$ acts on the kinematic part of a given off-shell $\lambda\,\phi^3$ theory tree graph, $A_N^{\phi^3,\Delta}$, in a similarly simple manner:

$$\widehat{\mathbb{P}}^\mu\, A_N^{\phi^3,\Delta} = \left[-d \sum_{j=1}^{N} (N+1-2\,j)\, k_j^\mu - 2 \sum_{[I,J]\in\Delta} \left(-k^\mu_{[1,\,I-1]} + k^\mu_{[J+1,\,N]} \right) \right] A_N^{\phi^3,\Delta} \,. \tag{4.26}$$

As mentioned earlier, the first sum simplifies to $\left(2\,d \sum_{j=1}^{N} j\, k_j^\mu\right)$ under the assumption of momentum conservation. Clearly the level-one translations generator does not annihilate the graph $\mathcal{A}_N^{\phi^3,\Delta}$. Nonetheless, we can define a set of so-called evaluation parameters for each individual off-shell $\lambda\,\phi^3$ theory graph,

$$c_{N,j}^{\phi^3,\Delta} \equiv d\,(N+1-2\,j) + 2 \sum_{[I,J]\in\Delta} \begin{cases} -1 & \text{if } j \in [1,\, I-1] \\ +1 & \text{if } j \in [J+1,\, N] \end{cases} \,, \tag{4.27}$$

such that

$$\widehat{\mathbb{P}}'^\mu\, A_N^{\phi^3,\Delta} \equiv \left[\widehat{\mathbb{P}}^\mu + \sum_{i=1}^{N} c_{N,i}^{\phi^3,\Delta}\, P_i^\mu \right] A_N^{\phi^3,\Delta} = 0 \,. \tag{4.28}$$

That is, by definition of these evaluation parameters, we have:

$$\sum_{i=1}^{N} c_{N,i}^{\phi^3,\Delta}\, P_i^\mu\, A_N^{\phi^3,\Delta} = \sum_{i=1}^{N} c_{N,i}^{\phi^3,\Delta}\, k_i^\mu\, A_N^{\phi^3,\Delta} = -\widehat{\mathbb{P}}^\mu\, A_N^{\phi^3,\Delta} \,, \tag{4.29}$$

and thus, this graph-dependent extension of the level-one translations generator annihilates individual off-shell $\lambda\,\phi^3$ theory graphs

$$\widehat{\mathbb{P}}'^\mu\, \mathcal{A}_N^{\phi^3,\Delta} = \delta^n\left(\textstyle\sum_{j=1}^{N} k_j\right) \widehat{\mathbb{P}}'^\mu\, A_N^{\phi^3,\Delta} = 0 \tag{4.30}$$

for arbitrary n and d.

Example: $\widehat{\mathbb{P}}^{\mu}$ on $A_4^{\phi^3,\{[1,2]\}} = -1/s_{12} = -1/(k_1+k_2)^2$

$$\widehat{\mathbb{P}}^{\mu}\left(-\frac{1}{s_{12}}\right) = \left[-d\,(3k_1^{\mu}+k_2^{\mu}-k_3^{\mu}-3k_4^{\mu}) - 2\,(k_3^{\mu}+k_4^{\mu})\right]\left(-\frac{1}{s_{12}}\right)$$

Evaluation parameters : $\{c_1 = 3\,d, c_2 = d, c_3 = -d+2, c_4 = -3\,d+2\}$

$$\sum_{i=1}^{4} c_i P_i^{\mu}\left(-\frac{1}{s_{12}}\right) = \left[d\,(3k_1^{\mu}+k_2^{\mu}-k_3^{\mu}-3k_4^{\mu}) + 2\,(k_3^{\mu}+k_4^{\mu})\right]\left(-\frac{1}{s_{12}}\right) \tag{4.31}$$

$$\left[\widehat{\mathbb{P}}^{\mu} + \sum_{i=1}^{4} c_i P_i^{\mu}\right]\left(-\frac{1}{s_{12}}\right) = 0 \text{ , for arbitrary } n \text{ and } d.$$

4.3.2 Action of Remaining Level-One Generators

As discussed in Section 4.1, having shown that the level-zero generators annihilate individual $\lambda\,\phi^3$ theory off-shell graphs in $n = 6$ space-time dimensions,

$$J^{AB}\,\mathcal{A}_N^{\phi^3,\Delta} = J^{AB}\left[\delta^n\left(\textstyle\sum_{j=1}^{N} k_j\right) A_N^{\phi^3,\Delta}\right] = 0 \text{ ,} \tag{4.32}$$

and that $\widehat{\mathbb{P}}'^{\mu}$ does so too (4.30), we are guaranteed by the algebra (A.5), which is also satisfied by the level-zero and the extended level-one generators,

$$\widehat{J}'^{AB} \equiv \widehat{J}^{AB} + \sum_{i=1}^{N} c_{N,i}^{\phi^3,\Delta}\; J_i^{AB} \text{ ,} \tag{4.33}$$

that *all* the extended level-one generators annihilate *individual* $\lambda\,\phi^3$ theory off-shell graphs in $n = 6$ space-time dimensions,

$$\widehat{J}'^{AB}\,\mathcal{A}_N^{\phi^3,\Delta} = \widehat{J}'^{AB}\left[\delta^n\left(\textstyle\sum_{j=1}^{N} k_j\right) A_N^{\phi^3,\Delta}\right] = 0 \text{ .} \tag{4.34}$$

That is, we are guaranteed that the other three extended level-one generators also annihilate individual graphs in $n = 6$ space-time dimensions,

$$\begin{aligned}
\widehat{\mathbb{L}}'^{\mu\nu}\,\mathcal{A}_N^{\phi^3,\Delta} &= \widehat{\mathbb{L}}'^{\mu\nu}\left[\delta^n\left(\textstyle\sum_{j=1}^{N} k_j\right) A_N^{\phi^3,\Delta}\right] = 0\\
\widehat{\mathbb{D}}'\,\mathcal{A}_N^{\phi^3,\Delta} &= \widehat{\mathbb{D}}'\left[\delta^n\left(\textstyle\sum_{j=1}^{N} k_j\right) A_N^{\phi^3,\Delta}\right] = 0\\
\widehat{\mathbb{K}}'^{\mu}\,\mathcal{A}_N^{\phi^3,\Delta} &= \widehat{\mathbb{K}}'^{\mu}\left[\delta^n\left(\textstyle\sum_{j=1}^{N} k_j\right) A_N^{\phi^3,\Delta}\right] = 0 \text{ .}
\end{aligned} \tag{4.35}$$

However, since the evaluation parameters cannot be defined in a graph independent way, and since they are generally not equal for all the individual graphs which one has to sum over to obtain a $\lambda\,\phi^3$ theory partial or total amplitude, we conclude that neither the original $\widehat{J}^{AB}$ nor the extended $\widehat{J}'^{AB}$ level-one generators annihilate $\lambda\,\phi^3$ theory partial and total amplitudes. Thus, we are led to also conclude that the conformal Yangian $Y[\mathrm{SO}(2,n)]$ is not a symmetry of $\lambda\,\phi^3$ theory. Nevertheless, the invariance of the individual graphs is still an unexpected interesting structure worth investigating further.

In Appendix D, using the evaluation parameters $c_{N,\,i}^{\phi^3,\,\Delta}$, defined individually for each $\lambda\,\phi^3$ theory off-shell tree graph (4.27), we explicitly compute the action of *all* the extended level-one Yangian generators $\left(\widehat{\mathbb{P}}'^{\mu},\, \widehat{\mathbb{L}}'^{\mu\nu},\, \widehat{\mathbb{D}}',\, \widehat{\mathbb{K}}'^{\mu}\right)$ on the kinematic part $A_N^{\phi^3,\,\Delta}$ of several 4-, 5- and 6-point individual graphs, and show that it vanishes in $n=6$ space-time dimensions.

CHAPTER 5

ACTION OF CONFORMAL YANGIAN $Y[\mathbf{SO}(2,n)]$ GENERATORS ON PURE YANG-MILLS THEORY

In this chapter[1], we compute explicitly how the level one SO(2,4) Yangian generator $\widehat{\mathbb{P}}^\mu$ acts on gluon tree amplitudes in pure Yang-Mills theory, using the momentum space differential operator representation of $\widehat{\mathbb{P}}^\mu$. Although the generator does not annihilate the gluon amplitudes, we find a somewhat compact form. We show this for the 3-gluon amplitude and the 4-gluon partial amplitude, on shell. This could be useful to understand what role the SO(2,4) Yangian might play in pure Yang-Mills. Since it does not annihilate the amplitude, it is not a symmetry. But it may serve some function as a spectrum generating algebra would. We show that our answer can also be written in terms of traces of Dirac gamma matrices, which is motivated by the fact that the supersymmetic PSU(2,2|4) Yangian does annihilate the superamplitude. In particular, the PSU(2,2|4) Yangian level one generators annihilate the pure gluon amplitudes, and the SO(2,4) Yangian level one generator acting on the gluon amplitude changes it into an amplitude involving 2 fermions and 2 less gluons. This may help us to interpret a possible role of supersymmetry in non-supersymmetric gauge theory. But this is beyond the scope of this dissertation.

In the following sections, we find it useful to factor out the polarization vectors ϵ_j^μ from the pure Yang-Mills theory tree amplitudes,

$$A_N^{\text{YM}}(k;\epsilon) = \left(\epsilon_1^{\alpha_1}\epsilon_2^{\alpha_2}\dots\epsilon_N^{\alpha_N}\right)\left(\eta_{\alpha_1\gamma_1}\eta_{\alpha_2\gamma_2}\dots\eta_{\alpha_N\gamma_N}\right) A_N^{\gamma_1\gamma_2\cdots\gamma_N}(k;\not\epsilon)\ , \tag{5.1}$$

and act with the generators on the tensors $A_N^{\gamma_1\gamma_2\cdots\gamma_N}(k)$, which are independent of the polarization vectors.

[1] The content of this chapter was adapted with minor changes from Section 6 and Appendices C and D of the author's previously published paper in the Journal of High Energy Physics:
N. Dokmetzoglou and L. Dolan, *Properties of the conformal Yangian in scalar and gauge field theories*, JHEP **02** (2023) 137 [arXiv:2207.14806 [hep-th]].

5.1 Level-One Translation Generators on the Three-Gluon On-Shell Tree Amplitude

The three-gluon tree amplitude is

$$\begin{aligned} A^{a\alpha_1,\, b\alpha_2,\, c\alpha_3} = & -g\ f^{abc}\ (2\pi)^4\delta^4(k_1+k_2+k_3) \\ & \cdot \Big(\eta^{\alpha_2\alpha_3}(k_2-k_3)^{\alpha_1} + \eta^{\alpha_3\alpha_1}(k_3-k_1)^{\alpha_2} + \eta^{\alpha_1\alpha_2}(k_1-k_2)^{\alpha_3}\Big) \end{aligned} \tag{5.2}$$

For $N=3$ the $\mathrm{SO}(2,4)$ Yangian level one generator for the gauge theory is

$$\begin{aligned} -\widehat{\mathbb{P}}^{\mu}{}_{\gamma_1\gamma_2\gamma_3}^{\ \alpha_1\alpha_2\alpha_3} = & \left(P_1^\mu(D_2+D_3) + P_2^\mu D_3 - (P_2+P_3)^\mu D_1 - P_3^\mu D_2\right)\ \delta^{\alpha_1}_{\gamma_1}\delta^{\alpha_2}_{\gamma_2}\delta^{\alpha_3}_{\gamma_3} \\ & + \left(P_{1\rho}(L_2+L_3)^{\mu\rho} + P_{2\rho}L_3^{\mu\rho} - (P_2+P_3)_\rho L_1^{\mu\rho} - P_{3\rho}L_2^{\mu\rho}\right)^{\alpha_1\alpha_2\alpha_3}_{\gamma_1\gamma_2\gamma_3} \\ = & \Big(2(k_1-k_3)^\mu - (k_2+k_3)^\mu k_1\cdot\partial_1 + (k_1-k_3)^\mu k_2\cdot\partial_2 + (k_1+k_2)^\mu k_3\cdot\partial_3 \\ & - (k_2+k_3)_\beta(k_1^\mu\partial_1^\beta - k_1^\beta\partial_1^\mu) + (k_1-k_3)_\beta(k_2^\mu\partial_2^\beta - k_2^\beta\partial_2^\mu) \\ & + (k_1+k_2)_\beta(k_3^\mu\partial_3^\beta - k_3^\beta\partial_3^\mu)\Big)\quad \delta^{\alpha_1}_{\gamma_1}\delta^{\alpha_2}_{\gamma_2}\delta^{\alpha_3}_{\gamma_3} \\ & - (k_2+k_3)_\beta(\eta^{\mu\alpha_1}\delta^\beta_{\gamma_1} - \eta^{\beta\alpha_1}\delta^\mu_{\gamma_1})\ \delta^{\alpha_2}_{\gamma_2}\delta^{\alpha_3}_{\gamma_3} + (k_1-k_3)_\beta(\eta^{\mu\alpha_2}\delta^\beta_{\gamma_2} - \eta^{\beta\alpha_2}\delta^\mu_{\gamma_2})\ \delta^{\alpha_1}_{\gamma_1}\delta^{\alpha_3}_{\gamma_3} \\ & + (k_1+k_2)_\beta(\eta^{\mu\alpha_3}\delta^\beta_{\gamma_3} - \eta^{\beta\alpha_3}\delta^\mu_{\gamma_3})\ \delta^{\alpha_1}_{\gamma_1}\delta^{\alpha_2}_{\gamma_2} \end{aligned} \tag{5.3}$$

This level one generator moves trivially through the delta function, as shown in Appendix B, and we find the compact formula

$$\begin{aligned} & -\widehat{\mathbb{P}}^{\mu\alpha_1\alpha_2\alpha_3}_{\gamma_1\gamma_2\gamma_3}\ A^{a\gamma_1,\, b\gamma_2,\, c\gamma_3} \\ & = -g\ f_{abc}\ (2\pi)^4\delta^4(k_1+k_2+k_3) \\ & \cdot\Big(\ 4k_1^\mu\ \left(\eta^{\alpha_2\alpha_3}k_2^{\alpha_1} + \eta^{\alpha_3\alpha_1}k_3^{\alpha_2} + 2\eta^{\alpha_1\alpha_2}k_1^{\alpha_3}\right) - 4k_3^\mu\ \left(2\eta^{\alpha_2\alpha_3}k_2^{\alpha_1} + \eta^{\alpha_3\alpha_1}k_3^{\alpha_2} + \eta^{\alpha_1\alpha_2}k_1^{\alpha_3}\right) \\ & \qquad + \eta^{\mu\alpha_1}\ (-4k_1^{\alpha_2}k_1^{\alpha_3}) + \eta^{\mu\alpha_2}\ (-4k_2^{\alpha_1}k_2^{\alpha_3}) + \eta^{\mu\alpha_3}\ (-4k_3^{\alpha_1}k_3^{\alpha_2})\ \Big) \end{aligned} \tag{5.4}$$

Since this expression is not zero, we know the $\mathrm{SO}(2,4)$ Yangian is not a symmetry of pure Yang-Mills theory. But it shows how this Yangian acts in a non-supersymmetric gauge theory. The expression is gauge invariant, *i.e.* it vanishes when multiplied by any $k_i^{\alpha_i}$ for $1 \leq i \leq 3$. It is not cyclic invariant since $\widehat{\mathbb{P}}^\mu$ is not cyclic. We assume transverse polarizations $k_i \cdot \epsilon_i(k_i) = 0$, and drop terms in (5.4) proportional to $k_i^{\alpha_i}$ since they correspond to gauge transformations. The on-shell conditions $k_i^2 = 0$ in four spacetime dimensions are required from the Serre relation. The presence of the $\eta^{\mu\alpha_i}$ is necessary for gauge invariance; and these fixed tensor terms cannot be removed by extending $\widehat{\mathbb{P}}^\mu$ by evaluation parameters, as they could in the case of scalar $\lambda\phi^3$ theory.

To pursue a simpler form of (5.4), we note it can be streamlined from the equivalence

$$\begin{aligned}
&\delta^4(k_1+k_2+k_3)\\
&\cdot\Big(\ 4k_1^\mu\left(\eta^{\alpha_2\alpha_3}k_2^{\alpha_1}+\eta^{\alpha_3\alpha_1}k_3^{\alpha_2}+2\eta^{\alpha_1\alpha_2}k_1^{\alpha_3}\right)-4k_3^\mu\left(2\eta^{\alpha_2\alpha_3}k_2^{\alpha_1}+\eta^{\alpha_3\alpha_1}k_3^{\alpha_2}+\eta^{\alpha_1\alpha_2}k_1^{\alpha_3}\right)\\
&\qquad+\eta^{\mu\alpha_1}\left(-4k_1^{\alpha_2}k_1^{\alpha_3}\right)+\eta^{\mu\alpha_2}\left(-4k_2^{\alpha_1}k_2^{\alpha_3}\right)+\eta^{\mu\alpha_3}\left(-4k_3^{\alpha_1}k_3^{\alpha_2}\right)\Big)\\
&=\delta^4(k_1+k_2+k_3)\ \big(tr(\gamma^{\alpha_2}\gamma^\zeta\gamma^{\alpha_3}\gamma^\omega\gamma^{\alpha_1}\gamma^\mu)k_{1\omega}k_{2\zeta}\\
&\qquad\qquad-tr(\gamma^{\alpha_3}\gamma^\zeta\gamma^{\alpha_2}\gamma^\omega\gamma^{\alpha_1}\gamma^\mu)k_{1\omega}k_{3\zeta}+tr(\gamma^{\alpha_3}\gamma^\zeta\gamma^{\alpha_1}\gamma^\omega\gamma^{\alpha_2}\gamma^\mu)k_{2\omega}k_{3\zeta}\big)
\end{aligned} \tag{5.5}$$

Here the Dirac γ matrices are in a Weyl representation,

$$\begin{aligned}
\gamma^\mu &= \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix},\ \sigma^\mu = (1, \sigma^i),\ \bar{\sigma}^\mu = (1, -\sigma^i),\\
\{\gamma^\mu, \gamma^\nu\} &= 2\eta^{\mu\nu},\ \eta^{\mu\nu} = \text{diag}\ (1, -1, -1, -1)
\end{aligned} \tag{5.6}$$

The equivalence can be checked using standard trace formulae,

$$\begin{aligned}
&tr(\gamma^\mu\gamma^\nu) = 4\eta^{\mu\nu}, \qquad tr(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\lambda) = 4\left(\eta^{\mu\nu}\eta^{\rho\lambda} - \eta^{\mu\rho}\eta^{\nu\lambda} + \eta^{\mu\lambda}\eta^{\nu\rho}\right)\\
&tr(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\lambda\gamma^\omega\gamma^\zeta) = \eta^{\mu\nu}tr(\gamma^\rho\gamma^\lambda\gamma^\omega\gamma^\zeta) - \eta^{\mu\rho}tr(\gamma^\nu\gamma^\lambda\gamma^\omega\gamma^\zeta) + \eta^{\mu\lambda}tr(\gamma^\nu\gamma^\rho\gamma^\omega\gamma^\zeta)\\
&\qquad\qquad - \eta^{\mu\omega}tr(\gamma^\nu\gamma^\rho\gamma^\lambda\gamma^\zeta) + \eta^{\mu\zeta}tr(\gamma^\nu\gamma^\rho\gamma^\lambda\gamma^\omega)
\end{aligned} \tag{5.7}$$

which can be extended to the trace of products of any number of γ matrices using the anticommutator.

Supercharge Contribution for the Three-Point Gauge Amplitude

We were motivated to find the identity (5.5) by extending the level one generator to include supercharges [26, 27]. In position space we would have

$$\langle 0|TA^{a\gamma_1}(x_1)A^{b\gamma_2}(x_2)A^{c\gamma_3}(x_3)|0\rangle = G^{a\gamma_1,b\gamma_2,c\gamma_3}(x_1x_2x_3), \tag{5.8}$$

and for PSU(2,2|4) Yangian invariance of the three-gluon tree amplitude we assume a level one generator of the form

$$\begin{aligned}
&-\widehat{\mathbb{P}}^{\mu\alpha_1\alpha_2\alpha_3}_{x,SS\ \gamma_1\gamma_2\gamma_3}\ G^{a\gamma_1,b\gamma_2,c\gamma_3}(x_1x_2x_3) = 0\\
&= \sum_{1\ \leq i<j\leq 3}\Big(P_i^\mu D_j + P_{i\rho}L_j^{\mu\rho} - \tfrac{1}{4}\bar{\sigma}^{\mu\dot{\alpha}\alpha}Q^A_{\alpha i}\tilde{Q}_{A\dot{\alpha}j} - (i\leftrightarrow j)\Big)^{\alpha_1\alpha_2\alpha_3}_{x\ \gamma_1\gamma_2\gamma_3}\ G^{a\gamma_1,b\gamma_2,c\gamma_3}(x_1x_2x_3)\\
&= \sum_{1\ \leq i<j\leq 3}\Big(P_i^\mu D_j + P_{i\rho}L_j^{\mu\rho} - (i\leftrightarrow j)\Big)^{\alpha_1\alpha_2\alpha_3}_{x\ \gamma_1\gamma_2\gamma_3}\ G^{a\gamma_1,b\gamma_2,c\gamma_3}(x_1x_2x_3)\\
&- \tfrac{1}{4}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\langle 0|TQ^A_{\alpha 1}A^{a\alpha_1}(x_1)\tilde{Q}_{A\dot{\alpha}2}A^{b\alpha_2}(x_2)A^{c\alpha_3}(x_3)|0\rangle\\
&- \tfrac{1}{4}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\langle 0|T\tilde{Q}_{A\dot{\alpha}1}A^{a\alpha_1}(x_1)Q^A_{\alpha 2}A^{b\alpha_2}(x_2)A^{c\alpha_3}(x_3)|0\rangle\\
&- \tfrac{1}{4}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\langle 0|TQ^A_{\alpha 1}A^{a\alpha_1}(x_1)A^{b\alpha_2}(x_2)\tilde{Q}_{A\dot{\alpha}3}A^{c\alpha_3}(x_3)|0\rangle\\
&- \tfrac{1}{4}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\langle 0|T\tilde{Q}_{A\dot{\alpha}1}A^{a\alpha_1}(x_1)A^{b\alpha_2}(x_2)Q^A_{\alpha 3}A^{c\alpha_3}(x_3)|0\rangle\\
&- \tfrac{1}{4}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\langle 0|TA^{a\alpha_1}(x_1)Q^A_{\alpha 2}A^{b\alpha_2}(x_2)\tilde{Q}_{A\dot{\alpha}3}A^{c\alpha_3}(x_3)|0\rangle\\
&- \tfrac{1}{4}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\langle 0|TA^{a\alpha_1}(x_1)\tilde{Q}_{A\dot{\alpha}2}A^{b\alpha_2}(x_2)Q^A_{\alpha 3}A^{c\alpha_3}(x_3)|0\rangle
\end{aligned} \tag{5.9}$$

Q^A_α, $\tilde{Q}_{A\dot{\alpha}}$ are the conformal supercharges appearing in the superconformal group $\mathrm{PSU}(2,2|4)$, where $1 \leq A \leq 4$. In this section we distinguish spinor indices $1 \leq \alpha, \dot{\alpha} \leq 2$ from the Lorentz indices $0 \leq \mu, \alpha_i, \gamma_i \leq 3$, and i denotes the site. The color indices a, b, c run over the dimension of the gauge group. The notation follows [52] where σ^i are the Pauli matrices,

$$\begin{aligned}
&\sigma^\mu_{\alpha\dot{\beta}} = (1, \sigma^i), \quad \bar{\sigma}^{\mu\ \dot{\alpha}\beta} = (1, -\sigma^i),\\
&\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}, \quad \epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha}, \quad \epsilon^{12} = \epsilon_{21} = 1 \quad \text{same for dotted indices}
\end{aligned} \tag{5.10}$$

The conformal supercharges rotate a gluon into a fermion in the adjoint representation,

$$Q^A_\alpha A^{a\mu} \sim \sigma^\mu_{\alpha\dot{\beta}} \epsilon^{\dot{\beta}\dot{\gamma}} \bar{\psi}^{Aa}_{\dot{\gamma}}, \quad \widetilde{Q}^A_{\dot{\beta}} A^{a\mu} \sim \epsilon_{\dot{\beta}\dot{\alpha}} \bar{\sigma}^{\mu\,\dot{\alpha}\gamma} \psi^{Aa}_\gamma \tag{5.11}$$

We start by computing the first $Q\tilde{Q}$ term in (5.9), using the Lagrangian coupling fermions and gluons

$$\begin{aligned}\mathcal{L} &= -\tfrac{1}{4} F^{\mu\nu a} F_{\mu\nu a} - ig\bar{\psi}^A \bar{\sigma}^\mu D_\mu \psi_A = \mathcal{L}_0 + \mathcal{L}_I \\ \mathcal{L}_I &= -g f_{abc} A^b_\mu A^c_\nu \partial^\mu A^{\nu a} - \frac{g^2}{4} f_{abc} f_{ade} A^b_\mu A^c_\nu A^{d\mu} A^{e\nu} - ig f_{def} \bar{\psi}^{Ad}_{\dot{\alpha}} \bar{\sigma}^{\mu\dot{\alpha}\alpha} A^e_\mu \psi^f_{A\alpha}\end{aligned} \tag{5.12}$$

Then moving from the Heisenberg picture to the interaction picture, and working to first order in the coupling g, and suppressing g,

$$\begin{aligned}&- \tfrac{1}{4} \bar{\sigma}^{\mu\dot{\alpha}\alpha} \langle 0| T Q^A_{\alpha 1} A^{a\alpha_1}(x_1) \tilde{Q}_{A\dot{\alpha}2} A^{b\alpha_2}(x_2) A^{c\alpha_3}(x_3)|0\rangle \\ &= -\tfrac{1}{4} \bar{\sigma}^{\mu\dot{\alpha}\alpha} \langle 0| T \sigma^{\alpha_1}_{\alpha\dot{\beta}} \epsilon^{\dot{\beta}\dot{\gamma}} \bar{\psi}^{Aa}_{\dot{\gamma}}(x_1) \epsilon_{\dot{\alpha}\dot{\kappa}} \bar{\sigma}^{\alpha_2\dot{\kappa}\gamma} \psi^b_{A\gamma}(x_2) A^{c\alpha_3}(x_3)|0\rangle \\ &= -\tfrac{1}{4} \bar{\sigma}^{\mu\dot{\alpha}\alpha}\ \sigma^{\alpha_1}_{\alpha\dot{\beta}} \epsilon^{\dot{\beta}\dot{\gamma}} \epsilon_{\dot{\alpha}\dot{\kappa}} \bar{\sigma}^{\alpha_2\dot{\kappa}\gamma} \langle 0| T \bar{\psi}^{Aa}_{\dot{\gamma}}(x_1) \psi^b_{A\gamma}(x_2) A^{c\alpha_3}(x_3) e^{i\int d^4 z \mathcal{L}_I}|0\rangle \\ &= -\tfrac{1}{4} \bar{\sigma}^{\mu\dot{\alpha}\alpha} \sigma^{\alpha_1}_{\alpha\dot{\beta}} \epsilon^{\dot{\beta}\dot{\gamma}} \epsilon_{\dot{\alpha}\dot{\kappa}} \bar{\sigma}^{\alpha_2\dot{\kappa}\gamma} \int d^4 z \langle 0| T \bar{\psi}^{Aa}_{\dot{\gamma}}(x_1) \psi^b_{A\gamma}(x_2) A^{c\alpha_3}(x_3) \bar{\psi}^{Bd}_{\dot{\delta}}(z) \bar{\sigma}^{\nu\dot{\delta}\delta} A^e_\nu(z) \psi^f_{B\delta}(z)|0\rangle f_{def} \\ &= \tfrac{1}{4} \bar{\sigma}^{\mu\dot{\alpha}\alpha}\ \sigma^{\alpha_1}_{\alpha\dot{\beta}} \epsilon^{\dot{\beta}\dot{\gamma}} \epsilon_{\dot{\alpha}\dot{\kappa}} \bar{\sigma}^{\alpha_2\dot{\kappa}\gamma} \bar{\sigma}^{\nu\dot{\delta}\delta} \int d^4 z D^{\alpha_3}_\nu(x_3 - z) S^F_{\delta\dot{\gamma}}(z - x_1) \delta^A_B S^F_{\gamma\dot{\delta}}(x_2 - z) \delta^B_A\ \delta^{ce} \delta^{af} \delta^{bd} f_{def}\end{aligned} \tag{5.13}$$

where the gauge and fermion propagators are found from

$$\begin{aligned}\delta^{ce} D^{\alpha_3}_\nu(x_3 - z) &= \langle 0| T A^{c\alpha_3}(x_3) A^e_\nu(z)|0\rangle = \delta^{ce} \int d^4 p_3 e^{-ip_3\cdot(x_3 - z)} \tilde{D}^{\alpha_3}_\nu(p_3), \quad \tilde{D}^{\alpha_3}_\nu(p) = -\frac{i}{p^2} \\ \delta^B_A \delta^{be} S^F_{\gamma\dot{\delta}}(x_2 - z) &= \langle 0| T \psi^b_{A\gamma}(x_2) \bar{\psi}^{Be}_{\dot{\delta}}(z)|0\rangle = \delta^B_A \delta^{be} \int d^4 p e^{-ip\cdot(x_2 - z)} \tilde{S}^F_{\gamma\dot{\delta}}(p),\ \tilde{S}^F_{\gamma\dot{\delta}}(p) = \frac{i\sigma^\omega_{\gamma\dot{\delta}} p_\omega}{p^2}\end{aligned} \tag{5.14}$$

Then the Fourier transform of (5.13) is

$$
\begin{aligned}
&\bar{\sigma}^{\mu\dot{\alpha}\alpha}\ \sigma^{\alpha_1}_{\alpha\dot{\beta}}\epsilon^{\dot{\beta}\dot{\gamma}}\epsilon_{\dot{\alpha}\dot{\kappa}}\bar{\sigma}^{\alpha_2\dot{\kappa}\gamma}\bar{\sigma}^{\nu\dot{\delta}\delta}\int d^4x_1d^4x_2d^4x_3e^{ik_1\cdot x_1}e^{ik_2\cdot x_2}e^{ik_3\cdot x_3}\\
&\qquad\qquad\cdot\int d^4zD^{\alpha_3}_{\nu}(x_3-z)S^F_{\delta\dot{\gamma}}(z-x_1)S^F_{\gamma\dot{\delta}}(x_2-z)f_{abc}\\
&=\bar{\sigma}^{\mu\dot{\alpha}\alpha}\ \sigma^{\alpha_1}_{\alpha\dot{\beta}}\epsilon^{\dot{\beta}\dot{\gamma}}\epsilon_{\dot{\alpha}\dot{\kappa}}\bar{\sigma}^{\alpha_2\dot{\kappa}\gamma}\bar{\sigma}^{\nu\dot{\delta}\delta}\ (2\pi)^4\delta^4(k_1+k_2+k_3)\tilde{S}^F_{\delta\dot{\gamma}}(-k_1)\tilde{S}^F_{\gamma\dot{\delta}}(k_2)\tilde{D}^{\alpha_3}_{\nu}(k_3)\ f_{abc}\\
&=\ (2\pi)^4\delta^4(k_1+k_2+k_3)\ \bar{\sigma}^{\mu\dot{\alpha}\alpha}\ \sigma^{\alpha_1}_{\alpha\dot{\beta}}\epsilon^{\dot{\beta}\dot{\gamma}}\epsilon_{\dot{\alpha}\dot{\kappa}}\bar{\sigma}^{\alpha_2\dot{\kappa}\gamma}\bar{\sigma}^{\nu\dot{\delta}\delta}\ \frac{(-i\sigma^{\omega}_{\delta\dot{\gamma}}k_{1\omega})}{k_1^2}\frac{(i\sigma^{\zeta}_{\gamma\dot{\delta}}k_{2\zeta})}{k_2^2}\tilde{D}^{\alpha_3}_{\nu}(k_3)\ f_{abc}
\end{aligned}
\tag{5.15}
$$

Since the three-gluon amplitude in (5.2) has had the three external propagators truncated, we now truncate (5.15) by multiplying it by the inverse propagators, $\tilde{D}^{-1\ \tilde{\alpha}_1\alpha_1}(k_1)\tilde{D}^{-1\ \tilde{\alpha}_2\alpha_2}(k_2)\tilde{D}^{-1\ \tilde{\alpha}_3\alpha_3}(k_3) = -\eta^{\tilde{\alpha}_1\alpha_1}k_1^2\eta^{\tilde{\alpha}_2\alpha_2}k_2^2\tilde{D}^{-1\ \tilde{\alpha}_3\alpha_3}(k_3)$, where $\tilde{D}^{-1\ \tilde{\alpha}\alpha}(k)\tilde{D}_{\alpha\nu}(k) = \delta^{\tilde{\alpha}}_{\nu}$. Then (5.15) truncated becomes

$$
\begin{aligned}
&-\ (2\pi)^4\delta^4(k_1+k_2+k_3)\ \bar{\sigma}^{\mu\dot{\alpha}\alpha}\ \sigma^{\tilde{\alpha}_1}_{\alpha\dot{\beta}}\epsilon^{\dot{\beta}\dot{\gamma}}\epsilon_{\dot{\alpha}\dot{\kappa}}\bar{\sigma}^{\tilde{\alpha}_2\dot{\kappa}\gamma}\bar{\sigma}^{\tilde{\alpha}_3\dot{\delta}\delta}\ \sigma^{\omega}_{\delta\dot{\gamma}}\sigma^{\zeta}_{\gamma\dot{\delta}}k_{1\omega}k_{2\zeta}\ f_{abc}\\
&=-\ (2\pi)^4\delta^4(k_1+k_2+k_3)\ (-(\bar{\sigma}^{\tilde{\alpha}_2}\sigma^{\zeta}\bar{\sigma}^{\tilde{\alpha}_3}\sigma^{\omega}\bar{\sigma}^{\tilde{\alpha}_1}\sigma^{\mu})^{\dot{\kappa}}{}_{\dot{\kappa}})\ k_{1\omega}k_{2\zeta}\ f_{abc}
\end{aligned}
\tag{5.16}
$$

Here we have used properties of the $\sigma, \bar{\sigma}$ matrices

$$
\epsilon^{\tau\kappa}\sigma^{\mu}_{\kappa\dot{\beta}}\epsilon^{\dot{\beta}\dot{\gamma}} = -(\sigma^2\sigma^{\mu}\sigma^2)^{\tau\dot{\gamma}} = -\bar{\sigma}^{\mu\dot{\gamma}\tau}
\tag{5.17}
$$

since $\sigma^2\sigma^{\mu}\sigma^2 = \bar{\sigma}^{\mu T}$. And similarly $\epsilon_{\dot{\kappa}\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\epsilon_{\alpha\tau} = -(\sigma^2\bar{\sigma}\sigma^2)_{\dot{\kappa}\tau} = -\sigma^{\mu}_{\tau\dot{\kappa}}$, since $\sigma^2\bar{\sigma}^{\mu}\sigma^2 = \sigma^{\mu T}$. The second $Q\tilde{Q}$ term in (5.9) leads to

$$
\begin{aligned}
&-\tfrac{1}{4}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\langle 0|T\tilde{Q}_{A\dot{\alpha}1}A^{a\alpha_1}(x_1)Q^A_{\alpha 2}A^{b\alpha_2}(x_2)A^{c\alpha_3}(x_3)|0\rangle\\
&\to -\ (2\pi)^4\delta^4(k_1+k_2+k_3)\ (\sigma^{\mu}\bar{\sigma}^{\tilde{\alpha}_1}\sigma^{\omega}\bar{\sigma}^{\tilde{\alpha}_3}\sigma^{\zeta}\bar{\sigma}^{\tilde{\alpha}_2})_{\kappa}{}^{\kappa}\ k_{1\omega}k_{2\zeta}\ f_{acb}
\end{aligned}
\tag{5.18}
$$

On combining (5.16) and (5.18), the truncated Fourier transform sums to

$$
\begin{aligned}
& -\tfrac{1}{4}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\langle 0|TQ^{A}_{\alpha 1}A^{a\alpha_1}(x_1)\tilde{Q}_{A\dot{\alpha}2}A^{b\alpha_2}(x_2)A^{c\alpha_3}(x_3)|0\rangle \\
& -\tfrac{1}{4}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\langle 0|T\tilde{Q}_{A\dot{\alpha}1}A^{a\alpha_1}(x_1)Q^{A}_{\alpha 2}A^{b\alpha_2}(x_2)A^{c\alpha_3}(x_3)|0\rangle \\
& \to\ (2\pi)^4\delta^4(k_1+k_2+k_3)\left((\bar{\sigma}^{\tilde{\alpha}_2}\sigma^{\zeta}\bar{\sigma}^{\tilde{\alpha}_3}\sigma^{\omega}\bar{\sigma}^{\tilde{\alpha}_1}\sigma^{\mu})^{\dot{\kappa}}{}_{\dot{\kappa}}+(\sigma^{\mu}\bar{\sigma}^{\tilde{\alpha}_1}\sigma^{\omega}\bar{\sigma}^{\tilde{\alpha}_3}\sigma^{\zeta}\bar{\sigma}^{\tilde{\alpha}_2})_{\kappa}{}^{\kappa}\right) k_{1\omega}k_{2\zeta}\ f_{abc} \\
& =\ (2\pi)^4\delta^4(k_1+k_2+k_3)\ tr(\gamma^{\tilde{\alpha}_2}\gamma^{\zeta}\gamma^{\tilde{\alpha}_3}\gamma^{\omega}\gamma^{\tilde{\alpha}_1}\gamma^{\mu})\ k_{1\omega}k_{2\zeta}\ f_{abc}
\end{aligned} \tag{5.19}
$$

Here we use further properties of the γ matrices in the Weyl representation (5.6)

$$
\begin{aligned}
& \gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\lambda}\gamma^{\omega}\gamma^{\zeta}=\begin{pmatrix}(\sigma^{\mu}\bar{\sigma}^{\nu}\sigma^{\rho}\bar{\sigma}^{\lambda}\sigma^{\omega}\bar{\sigma}^{\zeta})_{\kappa}{}^{\tau} & 0\\ 0 & (\bar{\sigma}^{\mu}\sigma^{\nu}\bar{\sigma}^{\rho}\sigma^{\lambda}\bar{\sigma}^{\omega}\sigma^{\zeta})^{\dot{\kappa}}{}_{\dot{\tau}}\end{pmatrix}, \\
& tr(\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\lambda}\gamma^{\omega}\gamma^{\zeta}) \\
& =(\sigma^{\mu}\bar{\sigma}^{\nu}\sigma^{\rho}\bar{\sigma}^{\lambda}\sigma^{\omega}\bar{\sigma}^{\zeta})_{\kappa}{}^{\kappa}+(\bar{\sigma}^{\mu}\sigma^{\nu}\bar{\sigma}^{\rho}\sigma^{\lambda}\bar{\sigma}^{\omega}\sigma^{\zeta})^{\dot{\kappa}}{}_{\dot{\kappa}}=(\sigma^{\zeta}\bar{\sigma}^{\omega}\sigma^{\lambda}\bar{\sigma}^{\rho}\sigma^{\nu}\bar{\sigma}^{\mu})_{\kappa}{}^{\kappa}+(\bar{\sigma}^{\mu}\sigma^{\nu}\bar{\sigma}^{\rho}\sigma^{\lambda}\bar{\sigma}^{\omega}\sigma^{\zeta})^{\dot{\kappa}}{}_{\dot{\kappa}}
\end{aligned} \tag{5.20}
$$

The contribution of the last four terms of (5.9) can be found by exchanging $b, \alpha_2, k_2 \to c, \alpha_3, k_3$ and from there $a, \alpha_1, k_1 \to b, \alpha_2, k_2$ by inspection, to yield that the truncated Fourier transform of the last six terms of (5.9) is

$$
\begin{aligned}
& -\tfrac{1}{4}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\langle 0|TQ^{A}_{\alpha 1}A^{a\alpha_1}(x_1)\tilde{Q}_{A\dot{\alpha}2}A^{b\alpha_2}(x_2)A^{c\alpha_3}(x_3)|0\rangle \\
& -\tfrac{1}{4}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\langle 0|T\tilde{Q}_{A\dot{\alpha}1}A^{a\alpha_1}(x_1)Q^{A}_{\alpha 2}A^{b\alpha_2}(x_2)A^{c\alpha_3}(x_3)|0\rangle \\
& -\tfrac{1}{4}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\langle 0|TQ^{A}_{\alpha 1}A^{a\alpha_1}(x_1)A^{b\alpha_2}(x_2)\tilde{Q}_{A\dot{\alpha}3}A^{c\alpha_3}(x_3)|0\rangle \\
& -\tfrac{1}{4}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\langle 0|T\tilde{Q}_{A\dot{\alpha}1}A^{a\alpha_1}(x_1)A^{b\alpha_2}(x_2)Q^{A}_{\alpha 3}A^{c\alpha_3}(x_3)|0\rangle \\
& -\tfrac{1}{4}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\langle 0|TA^{a\alpha_1}(x_1)Q^{A}_{\alpha 2}A^{b\alpha_2}(x_2)\tilde{Q}_{A\dot{\alpha}3}A^{c\alpha_3}(x_3)|0\rangle \\
& -\tfrac{1}{4}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\langle 0|TA^{a\alpha_1}(x_1)\tilde{Q}_{A\dot{\alpha}2}A^{b\alpha_2}(x_2)Q^{A}_{\alpha 3}A^{c\alpha_3}(x_3)|0\rangle
\end{aligned} \tag{5.21}
$$

$$
\begin{aligned}
& \to\ (2\pi)^4\delta^4(k_1+k_2+k_3)\Big(tr(\gamma^{\tilde{\alpha}_2}\gamma^{\zeta}\gamma^{\tilde{\alpha}_3}\gamma^{\omega}\gamma^{\tilde{\alpha}_1}\gamma^{\mu})\ k_{1\omega}k_{2\zeta} \\
& \qquad -tr(\gamma^{\tilde{\alpha}_3}\gamma^{\zeta}\gamma^{\tilde{\alpha}_2}\gamma^{\omega}\gamma^{\tilde{\alpha}_1}\gamma^{\mu})\ k_{1\omega}k_{3\zeta}\ +tr(\gamma^{\tilde{\alpha}_3}\gamma^{\zeta}\gamma^{\tilde{\alpha}_1}\gamma^{\omega}\gamma^{\tilde{\alpha}_2}\gamma^{\mu})\ k_{2\omega}k_{3\zeta}\Big)f_{abc}
\end{aligned} \tag{5.22}
$$

which with the coupling g reinserted, cancels (5.4). This motivates the identity (5.5).

In summary, we have observed that the action of the SO(2,4) Yangian on a pure gluon amplitude is equivalent to an appropriately truncated amplitude with 2 fermions and two less gluons, as expected from the PSU(2,2|4) Yangian symmetry of the pure gluon amplitude. If we could interpret the fermion amplitude as the effect of some process in pure Yang-Mills theory, we might realize a role for the level one generators of the SO(2,4) Yangian in pure gluon theory. Such an analysis could lead to a deeper understanding of the dynamics of non-supersymmetric non-abelian gauge theory, and help to interpret a non-supersymmetric extension of [12, 16, 30]. The physical meaning of our results is they may be evidence for some fermionic structure in Yang-Mills theory.

5.2 Level-One Translation Generators on the Four-Gluon On-Shell Tree Amplitude

We give an analogous identity for the four-point partial gluon amplitude, whose derivation illuminates further how to extend these identities for all N. We expect the relation between the action of the Yangian on the N-gluon tree, and the $(N-2)$-gluon with 2 fermion graphs, to hold for all N due to the known PSU(2,2|4) Yangian invariance of the planar $\mathcal{N}=4$ super Yang-Mills theory. We remark that the $(N-2)$ gluon - 2 fermion trees we find differ slightly from standard expressions [53] due to our procedure for truncating the external legs, as explained below (5.22).

The on-shell four-gluon total tree amplitude [4] is

$$A^{a\alpha_1,\, b\alpha_2,\, c\alpha_3,\, d\alpha_4} = g^2\,(2\pi)^4\,\delta^4(k_1+k_2+k_3+k_4) \cdot \left(\frac{c_s n_s^{\alpha_1\alpha_2\alpha_3\alpha_4}}{s} + \frac{c_t n_t^{\alpha_1\alpha_2\alpha_3\alpha_4}}{t} + \frac{c_u n_u^{\alpha_1\alpha_2\alpha_3\alpha_4}}{u} \right) \tag{5.23}$$

$$c_s \equiv f^{abe} f^{ecd}, \quad c_t \equiv f^{bce} f^{ead}, \quad c_u \equiv f^{cae} f^{ebd}, \quad c_s + c_u + c_t = 0, \quad n_s + n_u + n_t = 0$$

where $s = 2\,k_1 \cdot k_2$, $t = 2\,k_2 \cdot k_3$ and $u = 2\,k_1 \cdot k_3$. We focus on the gauge invariant partial amplitude

$A(1234)$, where the polarization vectors have been removed,

$$A(1234) = g^2\,(2\pi)^4\,\delta^4(k_1+k_2+k_3+k_4)\left(i\,\frac{n_s}{s} - i\,\frac{n_t}{t}\right),$$

$$\begin{aligned}
&i\,n_s^{\alpha_1\alpha_2\alpha_3\alpha_4}\\
&= \left(\eta^{\alpha_1\alpha_2}(k_1-k_2)_\sigma + 2k_2^{\alpha_1}\delta_\sigma^{\alpha_2} - 2k_1^{\alpha_2}\delta_\sigma^{\alpha_1}\right)\left(\eta^{\alpha_3\alpha_4}(k_3-k_4)^\sigma + 2k_4^{\alpha_3}\eta^{\sigma\alpha_4} - 2k_3^{\alpha_4}\eta^{\sigma\alpha_3}\right)\\
&\quad + \left(\eta^{\alpha_1\alpha_3}\eta^{\alpha_2\alpha_4} - \eta^{\alpha_1\alpha_4}\eta^{\alpha_2\alpha_3}\right)\,s\\
&i\,n_t^{\alpha_1\alpha_2\alpha_3\alpha_4}\\
&= \left(\eta^{\alpha_2\alpha_3}(k_2-k_3)_\sigma + 2k_2^{\alpha_3}\delta_\sigma^{\alpha_3} - 2k_2^{\alpha_3}\delta_\sigma^{\alpha_2}\right)\left(\eta^{\alpha_4\alpha_1}(k_1-k_4)^\sigma - 2k_1^{\alpha_4}\eta^{\sigma\alpha_1} + 2k_4^{\alpha_1}\eta^{\sigma\alpha_4}\right)\\
&\quad + \left(-\,\eta^{\alpha_2\alpha_4}\eta^{\alpha_3\alpha_1} + \eta^{\alpha_2\alpha_1}\eta^{\alpha_3\alpha_4}\right)\,t
\end{aligned} \tag{5.24}$$

We simplify the expression for $\widehat{\mathbb{P}}^\mu$ using momentum conservation and $k_i^2=0$. For $N=4$, $d=1$,

$$\begin{aligned}
-\widehat{\mathbb{P}}^{\mu}{}_{\gamma_1\gamma_2\gamma_3\gamma_4}^{\;\alpha_1\alpha_2\alpha_3\alpha_4} &= \sum_{1\,\leq i<j\leq 4}\left(P_i^\mu D_j + P_{i\rho}L_j^{\mu\rho} - (i\leftrightarrow j)\right)_{\gamma_1\gamma_2\gamma_3\gamma_4}^{\alpha_1\alpha_2\alpha_3\alpha_4}\\
&= \Big(\;(3k_1+k_2-k_3-3k_4)^\mu\\
&\qquad + 2k_1^\mu k_1\cdot\partial_1 + 2(k_1+k_2)^\mu k_2\cdot\partial_2 - 2(k_3+k_4)^\mu k_3\cdot\partial_3 - 2k_4^\mu k_4\cdot\partial_4\\
&\qquad + 2k_2^\mu k_1\cdot\partial_2 - 2k_3^\mu k_4\cdot\partial_3 - 2k_1\cdot k_2\partial_2^\mu + 2k_1\cdot k_2\partial_3^\mu\Big)\;\delta_{\gamma_1}^{\alpha_1}\delta_{\gamma_2}^{\alpha_2}\delta_{\gamma_3}^{\alpha_3}\delta_{\gamma_4}^{\alpha_4}\\
&\qquad + \eta^{\mu\alpha_1}k_{1\gamma_1}\;\delta_{\gamma_2}^{\alpha_2}\delta_{\gamma_3}^{\alpha_3}\delta_{\gamma_4}^{\alpha_4} + \eta^{\mu\alpha_2}(2k_1+k_2)_{\gamma_2}\;\delta_{\gamma_1}^{\alpha_1}\delta_{\gamma_3}^{\alpha_3}\delta_{\gamma_4}^{\alpha_4}\\
&\qquad - \eta^{\mu\alpha_3}(k_3+2k_4)_{\gamma_3}\;\delta_{\gamma_1}^{\alpha_1}\delta_{\gamma_2}^{\alpha_2}\delta_{\gamma_4}^{\alpha_4} - \eta^{\mu\alpha_4}k_{4\gamma_4}\;\delta_{\gamma_1}^{\alpha_1}\delta_{\gamma_2}^{\alpha_2}\delta_{\gamma_3}^{\alpha_3}\\
&\qquad - \delta_{\gamma_2}^\mu 2k_1^{\alpha_2}\delta_{\gamma_1}^{\alpha_1}\delta_{\gamma_3}^{\alpha_3}\delta_{\gamma_4}^{\alpha_4} + \delta_{\gamma_3}^\mu 2k_4^{\alpha_3}\delta_{\gamma_1}^{\alpha_1}\delta_{\gamma_2}^{\alpha_2}\delta_{\gamma_4}^{\alpha_4}
\end{aligned} \tag{5.25}$$

Then from (5.24) and (4.21),

$$
\begin{aligned}
&-\widehat{\mathbb{P}}^{\mu}\left(i\,\frac{n_s}{s}\right)\\
&=\sum_{1\,\leq i<j\leq 4}\Big(P_i^{\mu}D_j+P_{i\rho}L_j^{\mu\rho}-(i\leftrightarrow j)\Big)^{\alpha_1\alpha_2\alpha_3\alpha_4}_{\gamma_1\gamma_2\gamma_3\gamma_4}\;\left(\frac{i\,n_s^{\gamma_1\gamma_2\gamma_3\gamma_4}}{s}\right)\\
&=\sum_{1\,\leq i<j\leq 4}\Big(P_i^{\mu}D_j+P_{i\rho}L_j^{\mu\rho}-(i\leftrightarrow j)\Big)^{\alpha_1\alpha_2\alpha_3\alpha_4}_{\gamma_1\gamma_2\gamma_3\gamma_4}\;\left(\eta^{\gamma_1\gamma_3}\eta^{\gamma_2\gamma_4}-\eta^{\gamma_1\gamma_4}\eta^{\gamma_2\gamma_3}\right)\\
&+2(k_3+k_4)^{\mu}\;\frac{1}{s}\left(\eta^{\alpha_1\alpha_2}(k_1-k_2)_{\sigma}+2k_2^{\alpha_1}\delta_{\sigma}^{\alpha_2}-2k_1^{\alpha_2}\delta_{\sigma}^{\alpha_1}\right)\\
&\qquad\qquad\cdot\left(\eta^{\alpha_3\alpha_4}(k_3-k_4)^{\sigma}+2k_4^{\alpha_3}\eta^{\sigma\alpha_4}-2k_3^{\alpha_4}\eta^{\sigma\alpha_3}\right)\\
&+\frac{1}{s}\sum_{1\,\leq i<j\leq 4}\Big(P_i^{\mu}D_j+P_{i\rho}L_j^{\mu\rho}-(i\leftrightarrow j)\Big)^{\alpha_1\alpha_2\alpha_3\alpha_4}_{\gamma_1\gamma_2\gamma_3\gamma_4}\\
&\qquad\cdot\left(\eta^{\gamma_1\gamma_2}(k_1-k_2)_{\sigma}+2k_2^{\gamma_1}\delta_{\sigma}^{\gamma_2}-2k_1^{\gamma_2}\delta_{\sigma}^{\gamma_1}\right)\left(\eta^{\gamma_3\gamma_4}(k_3-k_4)^{\sigma}+2k_4^{\gamma_3}\eta^{\sigma\gamma_4}-2k_3^{\gamma_4}\eta^{\sigma\gamma_3}\right)
\end{aligned}
\tag{5.26}
$$

$$
\begin{aligned}
&-\widehat{\mathbb{P}}^{\mu}\left(i\,\frac{n_t}{t}\right)\\
&=\sum_{1\,\leq i<j\leq 4}\Big(P_i^{\mu}D_j+P_{i\rho}L_j^{\mu\rho}-(i\leftrightarrow j)\Big)^{\alpha_1\alpha_2\alpha_3\alpha_4}_{\gamma_1\gamma_2\gamma_3\gamma_4}\;\left(\frac{i\,n_t^{\gamma_1\gamma_2\gamma_3\gamma_4}}{t}\right)\\
&=\sum_{1\,\leq i<j\leq 4}\Big(P_i^{\mu}D_j+P_{i\rho}L_j^{\mu\rho}-(i\leftrightarrow j)\Big)^{\alpha_1\alpha_2\alpha_3\alpha_4}_{\gamma_1\gamma_2\gamma_3\gamma_4}\;\left(-\eta^{\gamma_1\gamma_3}\eta^{\gamma_2\gamma_4}+\eta^{\gamma_1\gamma_2}\eta^{\gamma_3\gamma_4}\right)\\
&+2(-k_1+k_4)^{\mu}\;\frac{1}{t}\left(\eta^{\alpha_2\alpha_3}(k_2-k_3)_{\sigma}+2k_3^{\alpha_2}\delta_{\sigma}^{\alpha_3}-2k_2^{\alpha_3}\delta_{\sigma}^{\alpha_2}\right)\\
&\qquad\qquad\cdot\left(\eta^{\alpha_4\alpha_1}(k_1-k_4)^{\sigma}-2k_1^{\alpha_4}\eta^{\sigma\alpha_1}+2k_4^{\alpha_1}\eta^{\sigma\alpha_4}\right)\\
&+\frac{1}{t}\sum_{1\,\leq i<j\leq 4}\Big(P_i^{\mu}D_j+P_{i\rho}L_j^{\mu\rho}-(i\leftrightarrow j)\Big)^{\alpha_1\alpha_2\alpha_3\alpha_4}_{\gamma_1\gamma_2\gamma_3\gamma_4}\\
&\qquad\cdot\left(\eta^{\gamma_2\gamma_3}(k_2-k_3)_{\sigma}+2k_3^{\gamma_2}\delta_{\sigma}^{\gamma_3}-2k_2^{\gamma_3}\delta_{\sigma}^{\gamma_2}\right)\left(\eta^{\gamma_4\gamma_1}(k_1-k_4)^{\sigma}-2k_1^{\gamma_4}\eta^{\sigma\gamma_1}+2k_4^{\gamma_1}\eta^{\sigma\gamma_4}\right)
\end{aligned}
\tag{5.27}
$$

Evaluating (5.26), (5.27) using momentum conservation, dropping terms that are gauge transformations *i.e.* proportional to $k_i^{\alpha_i}$, and using the on-shell conditions $k_i^2 = 0$, yields

$$
\begin{aligned}
&-\widehat{\mathbb{P}}^\mu \left(i\,\frac{n_s}{s} - i\,\frac{n_t}{t}\right) \\
&= \Bigg[\eta^{\mu\alpha_1}\Big(\eta^{\alpha_2\alpha_3}\big(4k_3^{\alpha_4} - \frac{8k_1^{\alpha_4}k_1\cdot k_2}{t}\big) - \eta^{\alpha_3\alpha_4}(4k_3^{\alpha_2} - \frac{8k_1^{\alpha_2}k_1\cdot k_4}{s}) \\
&\qquad - \frac{8k_1^{\alpha_2}\,(k_1^{\alpha_4}k_4^{\alpha_3} + k_3^{\alpha_4}k_2^{\alpha_3})}{s} + \frac{8k_1^{\alpha_4}\,(k_1^{\alpha_2}k_2^{\alpha_3} + k_3^{\alpha_2}k_4^{\alpha_3})}{t}\Big) \\
&\quad + \eta^{\mu\alpha_2}\Big(\eta^{\alpha_1\alpha_4}\big(-4k_4^{\alpha_3} + \frac{8k_2^{\alpha_3}k_1\cdot k_2}{t}\big) + \eta^{\alpha_3\alpha_4}(4k_4^{\alpha_1} - \frac{8k_2^{\alpha_1}k_1\cdot k_4}{s}) \\
&\qquad + \frac{8k_2^{\alpha_1}\,(k_2^{\alpha_3}k_3^{\alpha_4} + k_4^{\alpha_3}k_1^{\alpha_4})}{s} - \frac{8k_2^{\alpha_3}\,(k_2^{\alpha_1}k_1^{\alpha_4} + k_4^{\alpha_1}k_3^{\alpha_4})}{t}\Big) \\
&\quad + k_1^\mu\,\Big(6\,\eta^{\alpha_1\alpha_3}\eta^{\alpha_2\alpha_4} - \eta^{\alpha_1\alpha_4}\eta^{\alpha_2\alpha_3}\,(6 + \frac{8k_1\cdot k_2}{t}) - \eta^{\alpha_1\alpha_2}\eta^{\alpha_3\alpha_4}\,(6 + \frac{16k_1\cdot k_4}{s}) \\
&\qquad + \eta^{\alpha_1\alpha_2}\,\,(\frac{16k_4^{\alpha_3}k_1^{\alpha_4} - 16k_3^{\alpha_4}k_1^{\alpha_3} - 4k_4^{\alpha_3}k_3^{\alpha_4}}{s} - \frac{12k_1^{\alpha_4}k_2^{\alpha_3}}{t}) \\
&\qquad + \eta^{\alpha_3\alpha_4}\,\,(\frac{8k_1^{\alpha_2}k_4^{\alpha_1} - 16k_2^{\alpha_1}k_4^{\alpha_2} - 4k_1^{\alpha_2}k_2^{\alpha_1}}{s} - \frac{12k_3^{\alpha_2}k_4^{\alpha_1}}{t}) \\
&\qquad + \eta^{\alpha_1\alpha_3}\,\,(\frac{8k_1^{\alpha_2}k_3^{\alpha_4}}{s} + \frac{12k_1^{\alpha_4}k_3^{\alpha_2}}{t}) + \eta^{\alpha_2\alpha_4}\,\,(\frac{16k_2^{\alpha_1}k_4^{\alpha_3}}{s} + \frac{12k_2^{\alpha_3}k_4^{\alpha_1}}{t}) \\
&\qquad + \eta^{\alpha_1\alpha_4}\,\,(-\frac{8k_1^{\alpha_2}k_4^{\alpha_3}}{s} + \frac{-8k_2^{\alpha_3}k_4^{\alpha_2} + 8k_3^{\alpha_2}k_4^{\alpha_3} + 4k_2^{\alpha_3}k_3^{\alpha_2}}{t}) \\
&\qquad + \eta^{\alpha_2\alpha_3}\,\,(-\frac{16k_2^{\alpha_1}k_3^{\alpha_4}}{s} + \frac{-12k_1^{\alpha_4}k_3^{\alpha_1} + 12k_4^{\alpha_1}k_3^{\alpha_4}}{t})\Big) \\
&\quad + k_2^\mu\,\Big(2\,\eta^{\alpha_1\alpha_3}\eta^{\alpha_2\alpha_4} - \eta^{\alpha_1\alpha_4}\eta^{\alpha_2\alpha_3}\,(2 + \frac{8k_1\cdot k_2}{t}) - 2\eta^{\alpha_1\alpha_2}\eta^{\alpha_3\alpha_4} \\
&\qquad + \eta^{\alpha_1\alpha_2}\,\,(\frac{4k_4^{\alpha_3}k_3^{\alpha_4}}{s} + \frac{4k_1^{\alpha_4}k_2^{\alpha_3}}{t}) + \eta^{\alpha_3\alpha_4}\,\,(\frac{4k_2^{\alpha_1}(k_3-k_4)^{\alpha_2}}{s} - \frac{12k_3^{\alpha_2}k_4^{\alpha_1}}{t}) \\
&\qquad + \eta^{\alpha_1\alpha_3}\,\,(\frac{4k_1^{\alpha_4}k_3^{\alpha_2}}{t}) + \eta^{\alpha_2\alpha_4}\,\,(\frac{8k_2^{\alpha_1}k_4^{\alpha_3}}{s} + \frac{4k_2^{\alpha_3}k_4^{\alpha_1}}{t}) + \eta^{\alpha_1\alpha_4}\,\,(\frac{4k_3^{\alpha_2}(k_4-k_1)^{\alpha_3}}{t}) \\
&\qquad + \eta^{\alpha_2\alpha_3}\,\,(\frac{-8k_2^{\alpha_1}k_3^{\alpha_4}}{s} + \frac{4k_1^{\alpha_4}k_2^{\alpha_1} - 4k_4^{\alpha_1}k_2^{\alpha_4} + 8k_3^{\alpha_4}k_4^{\alpha_1}}{t})\Big) \\
&\quad - (1\leftrightarrow 4),(2\leftrightarrow 3)\text{ simultaneous exchange}\Bigg]
\end{aligned}
\tag{5.28}
$$

which can also be expressed in terms of Dirac matrices

$$
\begin{aligned}
&= \frac{(k_1+k_2)_\tau}{(k_1+k_2)^2}\Big[-tr(\gamma^{\tilde{\alpha}_1}\gamma^{\mu}\gamma^{\tilde{\alpha}_3}\gamma^{\zeta}\gamma^{\tilde{\alpha}_4}\gamma^{\tau}\gamma^{\tilde{\alpha}_2}\gamma^{\omega})k_{1\omega}k_{3\zeta} + tr(\gamma^{\tilde{\alpha}_2}\gamma^{\mu}\gamma^{\tilde{\alpha}_3}\gamma^{\zeta}\gamma^{\tilde{\alpha}_4}\gamma^{\tau}\gamma^{\tilde{\alpha}_1}\gamma^{\omega})k_{2\omega}k_{3\zeta} \\
&\qquad + tr(\gamma^{\tilde{\alpha}_1}\gamma^{\mu}\gamma^{\tilde{\alpha}_4}\gamma^{\zeta}\gamma^{\tilde{\alpha}_3}\gamma^{\tau}\gamma^{\tilde{\alpha}_2}\gamma^{\omega})k_{1\omega}k_{4\zeta} - tr(\gamma^{\tilde{\alpha}_2}\gamma^{\mu}\gamma^{\tilde{\alpha}_4}\gamma^{\zeta}\gamma^{\tilde{\alpha}_3}\gamma^{\tau}\gamma^{\tilde{\alpha}_1}\gamma^{\omega})k_{2\omega}k_{4\zeta}\Big] \\
&- \frac{(k_2+k_3)_\tau}{(k_2+k_3)^2}\Big[tr(\gamma^{\tilde{\alpha}_1}\gamma^{\mu}\gamma^{\tilde{\alpha}_2}\gamma^{\zeta}\gamma^{\tilde{\alpha}_3}\gamma^{\tau}\gamma^{\tilde{\alpha}_4}\gamma^{\omega})k_{1\omega}k_{2\zeta} - tr(\gamma^{\tilde{\alpha}_1}\gamma^{\mu}\gamma^{\tilde{\alpha}_3}\gamma^{\zeta}\gamma^{\tilde{\alpha}_2}\gamma^{\tau}\gamma^{\tilde{\alpha}_4}\gamma^{\omega})k_{1\omega}k_{3\zeta} \\
&\qquad + tr(\gamma^{\tilde{\alpha}_2}\gamma^{\mu}\gamma^{\tilde{\alpha}_4}\gamma^{\zeta}\gamma^{\tilde{\alpha}_1}\gamma^{\tau}\gamma^{\tilde{\alpha}_3}\gamma^{\omega})k_{2\omega}k_{4\zeta} - tr(\gamma^{\tilde{\alpha}_3}\gamma^{\mu}\gamma^{\tilde{\alpha}_4}\gamma^{\zeta}\gamma^{\tilde{\alpha}_1}\gamma^{\tau}\gamma^{\tilde{\alpha}_2}\gamma^{\omega})k_{3\omega}k_{4\zeta}\Big] \\
&- \Bigg[\frac{k_{1\omega}k_{2\tau}}{(k_3+k_4)^2}\Big[-2k_4^{\tilde{\alpha}_3}\,tr(\gamma^{\tilde{\alpha}_1}\gamma^{\mu}\gamma^{\tilde{\alpha}_2}\gamma^{\tau}\gamma^{\tilde{\alpha}_4}\gamma^{\omega}) + 2k_3^{\tilde{\alpha}_4}tr(\gamma^{\tilde{\alpha}_1}\gamma^{\mu}\gamma^{\tilde{\alpha}_2}\gamma^{\tau}\gamma^{\tilde{\alpha}_3}\gamma^{\omega}) \\
&\qquad + \eta^{\tilde{\alpha}_3\tilde{\alpha}_4}\Big(-k_3+k_4\Big)_\nu\,tr(\gamma^{\tilde{\alpha}_1}\gamma^{\mu}\gamma^{\tilde{\alpha}_2}\gamma^{\tau}\gamma^{\nu}\gamma^{\omega})\Big] \\
&\qquad + \frac{k_{3\omega}k_{4\tau}}{(k_3+k_4)^2}\Big[-2k_2^{\tilde{\alpha}_1}\,tr(\gamma^{\tilde{\alpha}_3}\gamma^{\mu}\gamma^{\tilde{\alpha}_4}\gamma^{\tau}\gamma^{\tilde{\alpha}_2}\gamma^{\omega}) + 2k_1^{\tilde{\alpha}_2}tr(\gamma^{\tilde{\alpha}_3}\gamma^{\mu}\gamma^{\tilde{\alpha}_4}\gamma^{\tau}\gamma^{\tilde{\alpha}_1}\gamma^{\omega}) \\
&\qquad + \eta^{\tilde{\alpha}_1\tilde{\alpha}_2}\Big(-k_1+k_2\Big)_\nu\,tr(\gamma^{\tilde{\alpha}_3}\gamma^{\mu}\gamma^{\tilde{\alpha}_4}\gamma^{\tau}\gamma^{\nu}\gamma^{\omega})\Big]\Bigg] \\
&+ \Bigg[\frac{k_{2\omega}k_{3\tau}}{(k_1+k_4)^2}\Big[-2k_4^{\tilde{\alpha}_1}\,tr(\gamma^{\tilde{\alpha}_2}\gamma^{\mu}\gamma^{\tilde{\alpha}_3}\gamma^{\tau}\gamma^{\tilde{\alpha}_4}\gamma^{\omega}) + 2k_1^{\tilde{\alpha}_4}\,tr(\gamma^{\tilde{\alpha}_2}\gamma^{\mu}\gamma^{\tilde{\alpha}_3}\gamma^{\tau}\gamma^{\tilde{\alpha}_1}\gamma^{\omega}) \\
&\qquad + \eta^{\tilde{\alpha}_1\tilde{\alpha}_4}\Big(-k_1+k_4\Big)_\nu\,tr(\gamma^{\tilde{\alpha}_2}\gamma^{\mu}\gamma^{\tilde{\alpha}_3}\gamma^{\tau}\gamma^{\nu}\gamma^{\omega})\Big] \\
&\qquad + \frac{k_{1\omega}k_{4\tau}}{(k_2+k_3)^2}\Big[-2k_3^{\tilde{\alpha}_2}\,tr(\gamma^{\tilde{\alpha}_1}\gamma^{\mu}\gamma^{\tilde{\alpha}_4}\gamma^{\tau}\gamma^{\tilde{\alpha}_3}\gamma^{\omega}) + 2k_2^{\tilde{\alpha}_3}tr(\gamma^{\tilde{\alpha}_1}\gamma^{\mu}\gamma^{\tilde{\alpha}_4}\gamma^{\tau}\gamma^{\tilde{\alpha}_2}\gamma^{\omega}) \\
&\qquad + \eta^{\tilde{\alpha}_2\tilde{\alpha}_3}\Big(-k_2+k_3\Big)_\nu\,tr(\gamma^{\tilde{\alpha}_1}\gamma^{\mu}\gamma^{\tilde{\alpha}_4}\gamma^{\tau}\gamma^{\nu}\gamma^{\omega})\Big]\Bigg]
\end{aligned} \tag{5.29}
$$

The equivalence between (5.28) and (5.29) is motivated below, and can be checked manually via the trace relations (5.7), or by using computer algebra. Here we have isolated a supercharge contribution (5.29) for a Yangian level one generator without using the spinor-helicity and superspace formalism. We display explicitly that the SO(2,4) Yangian level one generator acts on the four-point pure gluon amplitude by transforming it to another amplitude which could be interpreted to involve fermions, fields not in the pure gauge theory. Also the differential operator representation provides more versatility when comparing how the Yangian acts on scalar and gauge theories.

Supercharge Contribution for the Four-Point Gauge Amplitude

For the four-point function we motivate the identity between (5.28) and (5.29) by adding supercharges to the level one generator for $N = 4$,

$$\langle 0|TA^{\gamma_1}(x_1)A^{\gamma_2}(x_2)A^{\gamma_3}(x_3)A^{\gamma_4}(x_4)|0\rangle = G^{\gamma_1\gamma_2\gamma_3\gamma_4}(x_1x_2x_3x_4) \tag{5.30}$$

For superconformal PSU(2,2|4) Yangian invariance,

$$\begin{aligned}
&-\widehat{\mathbb{P}}^{\mu\alpha_1\alpha_2\alpha_3\alpha_4}_{x,SS\ \gamma_1\gamma_2\gamma_3\gamma_4}\ G^{\gamma_1\gamma_2\gamma_3\gamma_4}(x_1x_2x_3x_4) = 0\\
&= \sum_{1\ \leq i<j\leq 4}\Big(P_i^{\mu}D_j + P_{i\rho}L_j^{\mu\rho} - (i\leftrightarrow j)\\
&\qquad\qquad - \tfrac{1}{4}\bar{\sigma}^{\mu\dot{\alpha}\alpha}Q^A_{\alpha i}\tilde{Q}_{A\dot{\alpha}j} - (i\leftrightarrow j)\Big)^{\alpha_1\alpha_2\alpha_3\alpha_4}_{x\ \gamma_1\gamma_2\gamma_3\gamma_4} G^{\gamma_1\gamma_2\gamma_3\gamma_4}(x_1x_2x_3x_4)\\
&= \sum_{1\ \leq i<j\leq 4}\Big(P_i^{\mu}D_j + P_{i\rho}L_j^{\mu\rho} - (i\leftrightarrow j)\Big)^{\alpha_1\alpha_2\alpha_3\alpha_4}_{x\ \gamma_1\gamma_2\gamma_3\gamma_4} G^{\gamma_1\gamma_2\gamma_3\gamma_4}(x_1x_2x_3x_4)\\
&- \tfrac{1}{4}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\langle 0|TQ^A_{\alpha 1}A^{\alpha_1}(x_1)\tilde{Q}_{A\dot{\alpha}2}A^{\alpha_2}(x_2)A^{\alpha_3}(x_3)A^{\alpha_4}(x_4)|0\rangle\\
&- \tfrac{1}{4}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\langle 0|T\tilde{Q}_{A\dot{\alpha}1}A^{\alpha_1}(x_1)Q^A_{\alpha 2}A^{\alpha_2}(x_2)A^{\alpha_3}(x_3)A^{\alpha_4}(x_4)|0\rangle\\
&- \tfrac{1}{4}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\langle 0|TQ^A_{\alpha 1}A^{\alpha_1}(x_1)A^{\alpha_2}(x_2)\tilde{Q}_{A\dot{\alpha}3}A^{\alpha_3}(x_3)A^{\alpha_4}(x_4)|0\rangle\\
&- \tfrac{1}{4}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\langle 0|T\tilde{Q}_{A\dot{\alpha}1}A^{\alpha_1}(x_1)A^{\alpha_2}(x_2)Q^A_{\alpha 3}A^{\alpha_3}(x_3)A^{\alpha_4}(x_4)|0\rangle\\
&- \tfrac{1}{4}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\langle 0|TA^{\alpha_1}(x_1)Q^A_{\alpha 2}A^{\alpha_2}(x_2)\tilde{Q}_{A\dot{\alpha}3}A^{\alpha_3}(x_3)A^{\alpha_4}(x_4)|0\rangle\\
&- \tfrac{1}{4}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\langle 0|TA^{\alpha_1}(x_1)\tilde{Q}_{A\dot{\alpha}2}A^{\alpha_2}(x_2)Q^A_{\alpha 3}A^{\alpha_3}(x_3)A^{\alpha_4}(x_4)|0\rangle\\
&\\
&- \tfrac{1}{4}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\langle 0|TQ^A_{\alpha 1}A^{\alpha_1}(x_1)A^{\alpha_2}(x_2)A^{\alpha_3}(x_3)\tilde{Q}_{A\dot{\alpha}4}A^{\alpha_4}(x_4)|0\rangle\\
&- \tfrac{1}{4}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\langle 0|T\tilde{Q}_{A\dot{\alpha}1}A^{\alpha_1}(x_1)A^{\alpha_2}(x_2)A^{\alpha_3}(x_3)Q^A_{\alpha 4}A^{\alpha_4}(x_4)|0\rangle\\
&- \tfrac{1}{4}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\langle 0|TA^{\alpha_1}(x_1)Q^A_{\alpha 2}A^{\alpha_2}(x_2)A^{\alpha_3}(x_3)\tilde{Q}_{A\dot{\alpha}4}A^{\alpha_4}(x_4)|0\rangle\\
&- \tfrac{1}{4}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\langle 0|TA^{\alpha_1}(x_1)\tilde{Q}_{A\dot{\alpha}2}A^{\alpha_2}(x_2)A^{\alpha_3}(x_3)Q^A_{\alpha 4}A^{\alpha_4}(x_4)|0\rangle\\
&- \tfrac{1}{4}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\langle 0|TA^{\alpha_1}(x_1)A^{\alpha_2}(x_2)Q^A_{\alpha 3}A^{\alpha_3}(x_3)\tilde{Q}_{A\dot{\alpha}4}A^{\alpha_4}(x_4)|0\rangle\\
&- \tfrac{1}{4}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\langle 0|TA^{\alpha_1}(x_1)A^{\alpha_2}(x_2)\tilde{Q}_{A\dot{\alpha}3}A^{\alpha_3}(x_3)Q^A_{\alpha 4}A^{\alpha_4}(x_4)|0\rangle
\end{aligned} \tag{5.31}$$

Then working to second order in the coupling g,

$$
\begin{aligned}
&-\tfrac{1}{4}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\langle 0|TQ^{A}_{\alpha 1}A^{a\alpha_1}(x_1)\tilde{Q}_{A\dot{\alpha}2}A^{b\alpha_2}(x_2)A^{c\alpha_3}(x_3)A^{d\alpha_4}(x_4)|0\rangle \\
&= -\tfrac{1}{4}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\ \sigma^{\alpha_1}_{\alpha\dot{\beta}}\epsilon^{\dot{\beta}\dot{\gamma}}\epsilon_{\dot{\alpha}\dot{\kappa}}\bar{\sigma}^{\alpha_2\dot{\kappa}\gamma}\langle 0|T\bar{\psi}^{Aa}_{\dot{\gamma}}(x_1)\psi^{b}_{A\gamma}(x_2)A^{c\alpha_3}(x_3)A^{d\alpha_4}(x_4)e^{i\int d^4 z\mathcal{L}_I}|0\rangle \qquad (5.32)\\
&= -g^2\tfrac{1}{8}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\ \sigma^{\alpha_1}_{\alpha\dot{\beta}}\epsilon^{\dot{\beta}\dot{\gamma}}\epsilon_{\dot{\alpha}\dot{\kappa}}\bar{\sigma}^{\alpha_2\dot{\kappa}\gamma}\int d^4z_1 d^4z_2\langle 0|T\bar{\psi}^{Aa}_{\dot{\gamma}}(x_1)\psi^{b}_{A\gamma}(x_2)A^{c\alpha_3}(x_3)A^{d\alpha_4}(x_4) \\
&\qquad\qquad \cdot\bar{\psi}^{Bm}_{\dot{\delta}}(z_1)\bar{\sigma}^{\nu\dot{\delta}\delta}A^{e}_{\nu}(z_1)\psi^{f}_{B\delta}(z_1)\ \bar{\psi}^{Ch}_{\dot{\epsilon}}(z_2)\bar{\sigma}^{\rho\dot{\epsilon}\epsilon}A^{j}_{\rho}(z_2)\psi^{\ell}_{C\epsilon}(z_2)|0\rangle\ f_{mef}f_{hj\ell} \\
&= \tfrac{1}{2}g^2\bar{\sigma}^{\mu\dot{\alpha}\alpha}\ \sigma^{\alpha_1}_{\alpha\dot{\beta}}\epsilon^{\dot{\beta}\dot{\gamma}}\epsilon_{\dot{\alpha}\dot{\kappa}}\bar{\sigma}^{\alpha_2\dot{\kappa}\gamma}\bar{\sigma}^{\nu\dot{\delta}\delta}\bar{\sigma}^{\rho\dot{\epsilon}\epsilon}\int d^4z_1 d^4z_2 \\
&\ \cdot\big[D^{\alpha_3}_{\nu}(x_3-z_1)D^{\alpha_4}_{\rho}(x_4-z_2)S^{F}_{\epsilon\dot{\gamma}}(z_2-x_1)S^{F}_{\gamma\dot{\delta}}(x_2-z_1)S^{F}_{\delta\dot{\epsilon}}(z_1-z_2)\ f_{bch}f_{hda} \\
&\quad + D^{\alpha_3}_{\nu}(x_3-z_1)D^{\alpha_4}_{\rho}(x_4-z_2)S^{F}_{\delta\dot{\gamma}}(z_1-x_1)S^{F}_{\gamma\dot{\epsilon}}(x_2-z_2)S^{F}_{\epsilon\dot{\delta}}(z_2-z_1)\ f_{\ell ca}f_{bd\ell} \\
&\quad + D^{\alpha_3}_{\rho}(x_3-z_2)D^{\alpha_4}_{\nu}(x_4-z_1)S^{F}_{\epsilon\dot{\gamma}}(z_2-x_1)S^{F}_{\gamma\dot{\delta}}(x_2-z_1)S^{F}_{\delta\dot{\epsilon}}(z_1-z_2)\ f_{bdh}f_{hca} \\
&\quad + D^{\alpha_3}_{\rho}(x_3-z_2)D^{\alpha_4}_{\nu}(x_4-z_1)S^{F}_{\delta\dot{\gamma}}(z_1-x_1)S^{F}_{\gamma\dot{\epsilon}}(x_2-z_2)S^{F}_{\epsilon\dot{\delta}}(z_2-z_1)\ f_{\ell da}f_{bc\ell}\big] \qquad (5.33)
\end{aligned}
$$

with $\mathcal{L}_I$ given in (5.12). As in Section 5.1, in this section $1 \le \alpha, \dot{\alpha} \le 2$ and $0 \le \alpha_i \le 3$ for site i. With multiplication by four inverse gluon propagators, the truncated Fourier transform of (5.33) is

$$
\begin{aligned}
&-i\,g^2\,(2\pi)^4\,\delta^4(k_1+k_2+k_3+k_4) \\
&\ \cdot\Big((-\bar{\sigma}^{\tilde{\alpha}_1}\sigma^{\mu}\bar{\sigma}^{\tilde{\alpha}_2}\sigma^{\zeta}\bar{\sigma}^{\tilde{\alpha}_3}\sigma^{\tau}\bar{\sigma}^{\tilde{\alpha}_4}\sigma^{\omega})^{\dot{\gamma}}{}_{\dot{\gamma}}\ k_{1\omega}k_{2\zeta}(k_2+k_3)_{\tau}\ \frac{1}{(k_2+k_3)^2}\ f_{bce}f_{eda} \\
&\quad + (-\bar{\sigma}^{\tilde{\alpha}_1}\sigma^{\mu}\bar{\sigma}^{\tilde{\alpha}_2}\sigma^{\zeta}\bar{\sigma}^{\tilde{\alpha}_4}\sigma^{\tau}\bar{\sigma}^{\tilde{\alpha}_3}\sigma^{\omega})^{\dot{\gamma}}{}_{\dot{\gamma}}\ k_{1\omega}k_{2\zeta}(k_2+k_4)_{\tau}\ \frac{1}{(k_2+k_4)^2}\ \ f_{cae}f_{ebd}\Big) \qquad (5.34)
\end{aligned}
$$

There is also a contribution to (5.32) from the interaction Lagrangian given by

$$
\begin{aligned}
& -\tfrac{1}{4}\,\bar\sigma^{\mu\dot\alpha\alpha}\langle 0|TQ^A_{\alpha 1}A^{a\alpha_1}(x_1)\tilde Q_{A\dot\alpha 2}A^{b\alpha_2}(x_2)A^{c\alpha_3}(x_3)A^{d\alpha_4}(x_4)|0\rangle \\
& = -\tfrac{1}{4}\,\bar\sigma^{\mu\dot\alpha\alpha}\;\sigma^{\alpha_1}_{\alpha\dot\beta}\epsilon^{\dot\beta\dot\gamma}\epsilon_{\dot\alpha\dot\kappa}\bar\sigma^{\alpha_2\dot\kappa\gamma}\langle 0|T\bar\psi^{Aa}_{\dot\gamma}(x_1)\psi^b_{A\gamma}(x_2)A^{c\alpha_3}(x_3)A^{d\alpha_4}(x_4)e^{i\int d^4z\mathcal{L}_I}|0\rangle \\
& = i\,g^2\,\tfrac{1}{4}\bar\sigma^{\mu\dot\alpha\alpha}\;\sigma^{\alpha_1}_{\alpha\dot\beta}\epsilon^{\dot\beta\dot\gamma}\epsilon_{\dot\alpha\dot\kappa}\bar\sigma^{\alpha_2\dot\kappa\gamma}\int d^4z_1d^4z_2\langle 0|T\bar\psi^{Aa}_{\dot\gamma}(x_1)\psi^b_{A\gamma}(x_2)A^{c\alpha_3}(x_3)A^{d\alpha_3}(x_4) \\
& \qquad\qquad \cdot\bar\psi^{Bm}_{\dot\delta}(z_1)\bar\sigma^{\nu\dot\delta\delta}A^e_\nu(z_1)\psi^f_{B\delta}(z_1)\;A^j_\rho(z_2)A^\ell_\sigma(z_2)\partial^\rho A^{\sigma h}(z_2)|0\rangle\;f_{mef}f_{hj\ell} \\
& = i\,g^2\,\bar\sigma^{\mu\dot\alpha\alpha}\;\sigma^{\alpha_1}_{\alpha\dot\beta}\epsilon^{\dot\beta\dot\gamma}\epsilon_{\dot\alpha\dot\kappa}\bar\sigma^{\alpha_2\dot\kappa\gamma}\bar\sigma^{\nu\dot\delta\delta}\int d^4z_1d^4z_2\;S^F_{\delta\dot\gamma}(z_1-x_1)S^F_{\gamma\dot\delta}(x_2-z_1)\;f_{bea} \\
& \quad\cdot\big[D^{\alpha_3}_\rho(x_3-z_2)D^{\alpha_4}_\sigma(x_4-z_2)\partial^\rho_{z_2}D^\sigma_\nu(z_1-z_2) \\
& \qquad -D^{\alpha_3}_\rho(x_3-z_2)\partial^\rho_{z_2}D^{\alpha_4\sigma}(x_4-z_2)D_{\nu\sigma}(z_1-z_2) \\
& \qquad -D^{\alpha_3}_\sigma(x_3-z_2)D^{\alpha_4}_\rho(x_4-z_2)\partial^\rho_{z_2}D^\sigma_\nu(z_1-z_2) \\
& \qquad +D^{\alpha_3}_\sigma(x_3-z_2)\partial^\rho_{z_2}D^{\alpha_4\sigma}(x_4-z_2)D_{\nu\rho}(z_1-z_2) \\
& \qquad +\partial^\rho_{z_2}D^{\alpha_3\sigma}(x_3-z_2)D^{\alpha_4}_\rho(x_4-z_2)D_{\nu\sigma}(z_1-z_2) \\
& \qquad -\partial^\rho_{z_2}D^{\alpha_3\sigma}(x_3-z_2)D^{\alpha_4}_\sigma(x_4-z_2)D_{\nu\rho}(z_1-z_2)\Big]f_{ecd}
\end{aligned}
\tag{5.35}
$$

The truncated Fourier transform of (5.35) is

$$
\begin{aligned}
& -i\,g^2\,(2\pi)^4\,\delta^4(k_1+k_2+k_3+k_4)\;f_{bea}f_{ecd} \\
& \quad\cdot\Big[(\bar\sigma^{\tilde\alpha_1}\sigma^\mu\bar\sigma^{\tilde\alpha_2}\sigma^\tau\bar\sigma^{\tilde\alpha_4}\sigma^\omega)^{\dot\gamma}_{\dot\gamma}\;\frac{k_{1\omega}k_{2\tau}}{(k_3+k_4)^2}\;\Big(-k_3-2k_4\Big)^{\tilde\alpha_3} \\
& \qquad +(\bar\sigma^{\tilde\alpha_1}\sigma^\mu\bar\sigma^{\tilde\alpha_2}\sigma^\tau\bar\sigma^{\tilde\alpha_3}\sigma^\omega)^{\dot\gamma}_{\dot\gamma}\;\frac{k_{1\omega}k_{2\tau}}{(k_3+k_4)^2}\;\Big(2k_3+k_4\Big)^{\tilde\alpha_4} \\
& \qquad +(\bar\sigma^{\tilde\alpha_1}\sigma^\mu\bar\sigma^{\tilde\alpha_2}\sigma^\tau\bar\sigma^{\nu}\sigma^\omega)^{\dot\gamma}_{\dot\gamma}\;\frac{k_{1\omega}k_{2\tau}}{(k_3+k_4)^2}\;\eta^{\tilde\alpha_3\tilde\alpha_4}\Big(-k_3+k_4\Big)_\nu\Big]
\end{aligned}
\tag{5.36}
$$

The second supercharge term in (5.31) is evaluated in a similar way, which when added to (5.34) and

(5.36) promotes the sigma matrices to Dirac matrices (5.20). The truncated Fourier transform of

$$
\begin{aligned}
&-\tfrac{1}{4}\bar\sigma^{\mu\dot\alpha\alpha}\langle 0|TQ^A_{\alpha 1}A^{a\alpha_1}(x_1)\tilde Q_{A\dot\alpha 2}A^{b\alpha_2}(x_2)A^{c\alpha_3}(x_3)A^{d\alpha_4}(x_4)|0\rangle\\
&-\tfrac{1}{4}\bar\sigma^{\mu\dot\alpha\alpha}\langle 0|T\tilde Q_{A\dot\alpha 1}A^{a\alpha_1}(x_1)Q^A_{\alpha 2}A^{b\alpha_2}(x_2)A^{c\alpha_3}(x_3)A^{d\alpha_4}(x_4)|0\rangle
\end{aligned}
\tag{5.37}
$$

is

$$
\begin{aligned}
&i\,g^2(2\pi)^4\delta^4(k_1+k_2+k_3+k_4)\;tr(\gamma^{\tilde\alpha_1}\gamma^\mu\gamma^{\tilde\alpha_2}\gamma^\zeta\gamma^{\tilde\alpha_3}\gamma^\tau\gamma^{\tilde\alpha_4}\gamma^\omega)\;k_{1\omega}k_{2\zeta}\frac{(k_2+k_3)_\tau}{(k_2+k_3)^2}\;f_{bce}f_{eda}\\
&+ig^2(2\pi)^4\delta^4(k_1+k_2+k_3+k_4)\;tr(\gamma^{\tilde\alpha_1}\gamma^\mu\gamma^{\tilde\alpha_2}\gamma^\zeta\gamma^{\tilde\alpha_4}\gamma^\tau\gamma^{\tilde\alpha_3}\gamma^\omega)\;k_{1\omega}k_{2\zeta}\frac{(k_2+k_4)_\tau}{(k_2+k_4)^2}\;f_{cae}f_{ebd}\\
&-ig^2(2\pi)^4\delta^4(k_1+k_2+k_3+k_4)\;f_{bea}f_{ecd}\left[tr(\gamma^{\tilde\alpha_1}\gamma^\mu\gamma^{\tilde\alpha_2}\gamma^\tau\gamma^{\tilde\alpha_4}\gamma^\omega)\;\frac{k_{1\omega}k_{2\tau}}{(k_3+k_4)^2}\;\Big(-2k_4^{\tilde\alpha_3}\Big)\right.\\
&\qquad\qquad+tr(\gamma^{\tilde\alpha_1}\gamma^\mu\gamma^{\tilde\alpha_2}\gamma^\tau\gamma^{\tilde\alpha_3}\gamma^\omega)\;\frac{k_{1\omega}k_{2\tau}}{(k_3+k_4)^2}\;\Big(2k_3^{\tilde\alpha_4}\Big)\\
&\qquad\qquad\left.+tr(\gamma^{\tilde\alpha_1}\gamma^\mu\gamma^{\tilde\alpha_2}\gamma^\tau\gamma^\nu\gamma^\omega)\;\frac{k_{1\omega}k_{2\tau}}{(k_3+k_4)^2}\;\eta^{\tilde\alpha_3\tilde\alpha_4}\Big(-k_3+k_4\Big)_\nu\right]
\end{aligned}
\tag{5.38}
$$

where also we could have used the antisymmetry under $a \leftrightarrow b$, $x_1 \leftrightarrow x_2$, $\tilde\alpha_1 \leftrightarrow \tilde\alpha_2$, to generate the second term from the first term in (5.37). We have dropped gauge transformations, *i.e.* those terms proportional to $k_i^{\tilde\alpha_i}$.

Analogous symmetries can be used to generate the remaining ten supercharge terms in (5.31) from the first two (5.38), to find that total supercharge contribution to (5.31) is

$$
\begin{aligned}
& i\, g^2\, (2\pi)^4 \delta^4(k_1+k_2+k_3+k_4) \\
& \cdot \Bigg(f_{bce} f_{ead}\, \frac{(k_2+k_3)_\tau}{(k_2+k_3)^2} \Big[tr(\gamma^{\tilde\alpha_1}\gamma^\mu\gamma^{\tilde\alpha_2}\gamma^\zeta\gamma^{\tilde\alpha_3}\gamma^\tau\gamma^{\tilde\alpha_4}\gamma^\omega) k_{1\omega} k_{2\zeta} - tr(\gamma^{\tilde\alpha_1}\gamma^\mu\gamma^{\tilde\alpha_3}\gamma^\zeta\gamma^{\tilde\alpha_2}\gamma^\tau\gamma^{\tilde\alpha_4}\gamma^\omega) k_{1\omega} k_{3\zeta} \\
& \qquad + tr(\gamma^{\tilde\alpha_2}\gamma^\mu\gamma^{\tilde\alpha_4}\gamma^\zeta\gamma^{\tilde\alpha_1}\gamma^\tau\gamma^{\tilde\alpha_3}\gamma^\omega) k_{2\omega} k_{4\zeta} - tr(\gamma^{\tilde\alpha_3}\gamma^\mu\gamma^{\tilde\alpha_4}\gamma^\zeta\gamma^{\tilde\alpha_1}\gamma^\tau\gamma^{\tilde\alpha_2}\gamma^\omega) k_{3\omega} k_{4\zeta} \Big] \\
& + f_{abe} f_{ecd}\, \frac{(k_1+k_2)_\tau}{(k_1+k_2)^2} \Big[- tr(\gamma^{\tilde\alpha_1}\gamma^\mu\gamma^{\tilde\alpha_3}\gamma^\zeta\gamma^{\tilde\alpha_4}\gamma^\tau\gamma^{\tilde\alpha_2}\gamma^\omega) k_{1\omega} k_{3\zeta} + tr(\gamma^{\tilde\alpha_2}\gamma^\mu\gamma^{\tilde\alpha_3}\gamma^\zeta\gamma^{\tilde\alpha_4}\gamma^\tau\gamma^{\tilde\alpha_1}\gamma^\omega) k_{2\omega} k_{3\zeta} \\
& \qquad + tr(\gamma^{\tilde\alpha_1}\gamma^\mu\gamma^{\tilde\alpha_4}\gamma^\zeta\gamma^{\tilde\alpha_3}\gamma^\tau\gamma^{\tilde\alpha_2}\gamma^\omega) k_{1\omega} k_{4\zeta} - tr(\gamma^{\tilde\alpha_2}\gamma^\mu\gamma^{\tilde\alpha_4}\gamma^\zeta\gamma^{\tilde\alpha_3}\gamma^\tau\gamma^{\tilde\alpha_1}\gamma^\omega) k_{2\omega} k_{4\zeta} \Big] \\
& + f_{cae} f_{ebd}\, \frac{(k_1+k_3)_\tau}{(k_1+k_3)^2} \Big[tr(\gamma^{\tilde\alpha_1}\gamma^\mu\gamma^{\tilde\alpha_2}\gamma^\zeta\gamma^{\tilde\alpha_4}\gamma^\tau\gamma^{\tilde\alpha_3}\gamma^\omega) k_{1\omega} k_{2\zeta} + tr(\gamma^{\tilde\alpha_2}\gamma^\mu\gamma^{\tilde\alpha_3}\gamma^\zeta\gamma^{\tilde\alpha_1}\gamma^\tau\gamma^{\tilde\alpha_4}\gamma^\omega) k_{2\omega} k_{3\zeta} \\
& \qquad - tr(\gamma^{\tilde\alpha_1}\gamma^\mu\gamma^{\tilde\alpha_4}\gamma^\zeta\gamma^{\tilde\alpha_2}\gamma^\tau\gamma^{\tilde\alpha_3}\gamma^\omega) k_{1\omega} k_{4\zeta} + tr(\gamma^{\tilde\alpha_3}\gamma^\mu\gamma^{\tilde\alpha_4}\gamma^\zeta\gamma^{\tilde\alpha_2}\gamma^\tau\gamma^{\tilde\alpha_1}\gamma^\omega) k_{3\omega} k_{4\zeta} \Big] \\
& - f_{abe} f_{ecd} \Bigg[\frac{k_{1\omega} k_{2\tau}}{(k_3+k_4)^2} \Big[-2k_4^{\tilde\alpha_3}\, tr(\gamma^{\tilde\alpha_1}\gamma^\mu\gamma^{\tilde\alpha_2}\gamma^\tau\gamma^{\tilde\alpha_4}\gamma^\omega) + 2k_3^{\tilde\alpha_4} tr(\gamma^{\tilde\alpha_1}\gamma^\mu\gamma^{\tilde\alpha_2}\gamma^\tau\gamma^{\tilde\alpha_3}\gamma^\omega) \\
& \qquad\qquad + \eta^{\tilde\alpha_3\tilde\alpha_4}\Big(-k_3+k_4\Big)_\nu\, tr(\gamma^{\tilde\alpha_1}\gamma^\mu\gamma^{\tilde\alpha_2}\gamma^\tau\gamma^\nu\gamma^\omega)\Big] \\
& \qquad + \frac{k_{3\omega} k_{4\tau}}{(k_3+k_4)^2} \Big[-2k_2^{\tilde\alpha_1}\, tr(\gamma^{\tilde\alpha_3}\gamma^\mu\gamma^{\tilde\alpha_4}\gamma^\tau\gamma^{\tilde\alpha_2}\gamma^\omega) + 2k_1^{\tilde\alpha_2} tr(\gamma^{\tilde\alpha_3}\gamma^\mu\gamma^{\tilde\alpha_4}\gamma^\tau\gamma^{\tilde\alpha_1}\gamma^\omega) \\
& \qquad\qquad + \eta^{\tilde\alpha_1\tilde\alpha_2}\Big(-k_1+k_2\Big)_\nu\, tr(\gamma^{\tilde\alpha_3}\gamma^\mu\gamma^{\tilde\alpha_4}\gamma^\tau\gamma^\nu\gamma^\omega)\Big]\Bigg] \\
& - f_{bce} f_{ead} \Bigg[\frac{k_{2\omega} k_{3\tau}}{(k_1+k_4)^2} \Big[-2k_4^{\tilde\alpha_1}\, tr(\gamma^{\tilde\alpha_2}\gamma^\mu\gamma^{\tilde\alpha_3}\gamma^\tau\gamma^{\tilde\alpha_4}\gamma^\omega) + 2k_1^{\tilde\alpha_4}\, tr(\gamma^{\tilde\alpha_2}\gamma^\mu\gamma^{\tilde\alpha_3}\gamma^\tau\gamma^{\tilde\alpha_1}\gamma^\omega) \\
& \qquad\qquad + \eta^{\tilde\alpha_1\tilde\alpha_4}\Big(-k_1+k_4\Big)_\nu\, tr(\gamma^{\tilde\alpha_2}\gamma^\mu\gamma^{\tilde\alpha_3}\gamma^\tau\gamma^\nu\gamma^\omega)\Big] \\
& \qquad + \frac{k_{1\omega} k_{4\tau}}{(k_2+k_3)^2} \Big[-2k_3^{\tilde\alpha_2}\, tr(\gamma^{\tilde\alpha_1}\gamma^\mu\gamma^{\tilde\alpha_4}\gamma^\tau\gamma^{\tilde\alpha_3}\gamma^\omega) + 2k_2^{\tilde\alpha_3} tr(\gamma^{\tilde\alpha_1}\gamma^\mu\gamma^{\tilde\alpha_4}\gamma^\tau\gamma^{\tilde\alpha_2}\gamma^\omega) \\
& \qquad\qquad + \eta^{\tilde\alpha_2\tilde\alpha_3}\Big(-k_2+k_3\Big)_\nu\, tr(\gamma^{\tilde\alpha_1}\gamma^\mu\gamma^{\tilde\alpha_4}\gamma^\tau\gamma^\nu\gamma^\omega)\Big]\Bigg] \\
& - f_{cae} f_{ebd} \Bigg[\frac{k_{1\omega} k_{3\tau}}{(k_2+k_4)^2} \Big[2k_4^{\tilde\alpha_2}\, tr(\gamma^{\tilde\alpha_1}\gamma^\mu\gamma^{\tilde\alpha_3}\gamma^\tau\gamma^{\tilde\alpha_4}\gamma^\omega) - 2k_2^{\tilde\alpha_4}\, tr(\gamma^{\tilde\alpha_1}\gamma^\mu\gamma^{\tilde\alpha_3}\gamma^\tau\gamma^{\tilde\alpha_2}\gamma^\omega) \\
& \qquad\qquad + \eta^{\tilde\alpha_2\tilde\alpha_4}\Big(k_2-k_4\Big)_\nu\, tr(\gamma^{\tilde\alpha_1}\gamma^\mu\gamma^{\tilde\alpha_3}\gamma^\tau\gamma^\nu\gamma^\omega)\Big] \\
& \qquad + \frac{k_{2\omega} k_{4\tau}}{(k_1+k_3)^2} \Big[2k_3^{\tilde\alpha_1}\, tr(\gamma^{\tilde\alpha_2}\gamma^\mu\gamma^{\tilde\alpha_4}\gamma^\tau\gamma^{\tilde\alpha_3}\gamma^\omega) - 2k_1^{\tilde\alpha_3} tr(\gamma^{\tilde\alpha_2}\gamma^\mu\gamma^{\tilde\alpha_4}\gamma^\tau\gamma^{\tilde\alpha_1}\gamma^\omega) \\
& \qquad\qquad + \eta^{\tilde\alpha_1\tilde\alpha_3}\Big(k_1-k_3\Big)_\nu\, tr(\gamma^{\tilde\alpha_2}\gamma^\mu\gamma^{\tilde\alpha_4}\gamma^\tau\gamma^\nu\gamma^\omega)\Big]\Bigg]\Bigg) \\
& = g^2 (2\pi)^4 \delta^4(k_1+k_2+k_3+k_4)\ \cdot \Big(f_{abe} f_{ecd}\, \widehat{\mathbb{P}}^\mu\, \frac{n_s}{s} + f_{bce} f_{ead}\, \widehat{\mathbb{P}}^\mu\, \frac{n_t}{t} + f_{cae} f_{ebd}\, \widehat{\mathbb{P}}^\mu\, \frac{n_u}{u} \Big)
\end{aligned}
\tag{5.39}
$$

Then the terms relating to $\widehat{\mathbb{P}}^\mu \, A(1234) \equiv \widehat{\mathbb{P}}^\mu \left(i\,\frac{n_s}{s} - i\,\frac{n_t}{t}\right)$ can be read off from (5.39), by taking the coefficient of $f_{abe}f_{ecd}$ and subtracting from it the coefficient of $f_{bce}f_{ead}$. That results in i times the expression (5.29) given in Section 5.1. This motivates the equivalence of (5.28) and (5.29).

CHAPTER 6

ACTION OF CONFORMAL YANGIAN $Y[\mathbf{SO}(2,n)]$ GENERATORS ON THE SCATTERING EQUATIONS FORMALISM

6.1 CHY Scattering Equations Formalism

From [21, 22], we have that the kinematic part of the N-point partial amplitude can be computed by evaluating the multi-variable contour integral

$$A_N^{\text{partial}} = \oint_{\mathcal{O}} \frac{\Psi_N}{H_N} \frac{1}{z_{N-1}} \prod_{2\leq a<b\leq N-1} (z_a - z_b) \prod_{a=2}^{N-2} \frac{z_a \, dz_{a+1}}{(z_a - z_{a+1})^2} \equiv \oint_{\mathcal{O}} \mathcal{I}_N \,, \tag{6.1}$$

where $z_2 = 1$, $z_N = 0$, H_N is the product of $N-3$ polynomials, called the h_m^N *scattering polynomials*, which we define below,

$$H_N \equiv \prod_{m=1}^{N-3} h_m^N \,, \tag{6.2}$$

and $\mathcal{O}$ signifies the closed contour surrounding only the simultaneous zeroes of those $N-3$ polynomials, also known as the solutions of the scattering equations, of which there are $(N-3)!$. The scattering polynomials were initially defined on-shell by Dolan and Goddard in [21], and later generalized to their off-shell form in [22]. See Appendix E for more details.

The function Ψ_N is the piece of the integrand $\mathcal{I}_N$ which specifies the theory whose partial amplitudes we want to compute. Taking $\Psi_N = 1$ yields massless scalar $\lambda\,\phi^3$ theory amplitudes. The function Ψ_N corresponding to pure Yang-Mills theory is discussed in Section E.3.

The off-shell scattering polynomials $h_m^{N,\text{ off-shell}}$ are defined in [22] as:

$$\begin{aligned} h_m^{N,\text{ off-shell}} = & \sum_{J=2}^{N-2} k_{[1,J]}^2 \, (z_J - z_{J+1}) \, \Pi_{[1,J]^o}^{m-1} - \sum_{J=3}^{N-1} k_{[2,J]}^2 \, (z_J - z_{J+1}) \, \Pi_{[2,J]^o}^{m-1} \\ & + \sum_{3\leq I<J<N} k_{[I,J]}^2 \, (z_I - z_{I-1}) \, (z_J - z_{J+1}) \, \Pi_{[I,J]'^o}^{m-2} \end{aligned} \,, \quad 1 \leq m \leq N-3 \tag{6.3}$$

where $z_2 = 1$, $z_N = 0$, $[I,J]'^o = [I,J]^o \cap A'$, $[I,J] = \{a : I \leq a \leq J\}$, $1 \leq I \leq J < N$, $[I,J]^o$ is the complement of $\{I-1, I, J, J+1\}$ in $A = \{1, 2, \ldots, N\}$, $A' = \{2, 3, \ldots, N\}$, and Π_V^n is the symmetric function,

$$\Pi_V^n = \sum_{\substack{i_1 < i_2 < \cdots < i_n \\ i_a \in V}} z_{i_1} z_{i_2} \cdots z_{i_n} \tag{6.4}$$

where $V \subset A$ and $n \leq |V|$. Note that $\Pi_V^0 = 1$ and $\Pi_V^{|V|} = \prod_{i \in V} z_i \equiv z_V$.

Example: The $N = 4,\, 5,\, 6$ off-shell scattering polynomials

$$\begin{aligned}
h_1^{4,\,\text{off-shell}} &= (1-z_3)(k_1+k_2)^2 - z_3(k_2+k_3)^2 \\
h_1^{5,\,\text{off-shell}} &= (1-z_3)(k_1+k_2)^2 - (z_3-z_4)(k_2+k_3)^2 \\
&\quad + (z_3-z_4)(k_1+k_2+k_3)^2 - z_4(k_2+k_3+k_4)^2 \\
h_2^{5,\,\text{off-shell}} &= z_4(1-z_3)(k_1+k_2)^2 - z_4(1-z_3)(k_3+k_4)^2 \\
&\quad + (z_3-z_4)(k_1+k_2+k_3)^2 - z_3z_4(k_2+k_3+k_4)^2 \\
h_1^{6,\,\text{off-shell}} &= (z_4-z_5)(k_1+k_2+k_3+k_4)^2 + (z_3-z_4)(k_1+k_2+k_3)^2 + (1-z_3)(k_1+k_2)^2 \\
&\quad - z_5(k_2+k_3+k_4+k_5)^2 - (z_4-z_5)(k_2+k_3+k_4)^2 - (z_3-z_4)(k_2+k_3)^2 \\
h_2^{6,\,\text{off-shell}} &= (z_3+1)(z_4-z_5)(k_1+k_2+k_3+k_4)^2 + (z_5+1)(z_3-z_4)(k_1+k_2+k_3)^2 \\
&\quad + (1-z_3)(z_4+z_5)(k_1+k_2)^2 - z_5(z_3+z_4)(k_2+k_3+k_4+k_5)^2 \\
&\quad - z_3(z_4-z_5)(k_2+k_3+k_4)^2 - z_5(z_3-z_4)(k_2+k_3)^2 + z_5(z_4-z_3)(k_4+k_5)^2 \\
&\quad + (z_3-1)z_5(k_3+k_4+k_5)^2 + (z_3-1)(z_4-z_5)(k_3+k_4)^2 \\
h_3^{6,\,\text{off-shell}} &= z_3(z_4-z_5)(k_1+k_2+k_3+k_4)^2 + z_5(z_3-z_4)(k_1+k_2+k_3)^2 \\
&\quad + (1-z_3)z_4z_5(k_1+k_2)^2 - z_3z_4z_5(k_2+k_3+k_4+k_5)^2 \\
&\quad + (z_3-1)z_4z_5(k_3+k_4+k_5)^2 + z_5(z_4-z_3)(k_4+k_5)^2
\end{aligned} \tag{6.5}$$

6.2 Scalar $\lambda\,\phi^3$ Theory Off-Shell Amplitudes from CHY Formalism

Partial amplitudes

As discussed above, from [22] we have that an off-shell $\lambda\,\phi^3$ theory partial amplitude, $A_N^{\phi^3,\,\text{partial}}$, i.e. the sum of $\lambda\,\phi^3$ theory planar tree graphs, can be obtained by evaluating the multi-variable contour integral

$$A_N^{\phi^3,\,\text{partial}} = \oint_{\mathcal{O}} \frac{1}{H_N^{\text{off-shell}}} \frac{1}{z_{N-1}} \prod_{2\leq a<b\leq N-1} (z_a - z_b) \prod_{a=2}^{N-2} \frac{z_a\, dz_{a+1}}{(z_a - z_{a+1})^2} \,. \tag{6.6}$$

Using the off-shell scattering polynomials (6.3) and methods of multivariate contour integration, such as the *Global Residue Theorem* [54, 55], which is a multi-variable generalization of Cauchy's theorem, one can show that the $\lambda\,\phi^3$ theory off-shell 4-point partial amplitude is given by

Example: The $\lambda\,\phi^3$ theory off-shell 4-point partial amplitude

$$A_4^{\phi^3,\,\text{partial}} = \oint_{\mathcal{O}} \frac{dz_3}{h_1^{4,\,\text{off-shell}}} \frac{1}{z_3\,(1-z_3)} = -\frac{1}{s_{12}} - \frac{1}{s_{23}} \,, \tag{6.7}$$

and the 5-point partial amplitude by

Example: The $\lambda\,\phi^3$ theory off-shell 5-point partial amplitude

$$\begin{aligned} A_5^{\phi^3,\,\text{partial}} &= \oint_{\mathcal{O}} \frac{dz_3\, dz_4}{h_1^{5,\,\text{off-shell}}\, h_2^{5,\,\text{off-shell}}} \frac{z_3\,(1-z_4)}{z_4\,(1-z_3)\,(z_3-z_4)} \\ &= \frac{1}{s_{12}s_{123}} + \frac{1}{s_{23}s_{234}} + \frac{1}{s_{12}s_{34}} + \frac{1}{s_{123}s_{23}} + \frac{1}{s_{234}s_{34}} \,. \end{aligned} \tag{6.8}$$

Individual Feynman graphs

From [22] we have that an individual off-shell $\lambda\,\phi^3$ theory Feynman graph, $A_N^{\phi^3,\,\Delta}$, can be obtained by evaluating the multi-variable contour integral

$$A_N^{\phi^3,\,\Delta} = \oint_{\mathcal{O}} \frac{1}{H_N^{\text{off-shell}}} \frac{1}{u_\Delta} \prod_{\substack{2\leq a<b\leq N \\ d(a,b)>2}} (z_a - z_b) \prod_{a=3}^{N-1} dz_a \tag{6.9}$$

where $d(a,b) \equiv \min(|a-b|, N-|a-b|)$, with $1 \leq a < b \leq N$, and u_Δ is the product of $N-3$ cross-ratios, one for each of the $N-3$ propagators of the given graph,

$$u_\Delta \equiv \prod_{[I,J]\in\Delta} u_{I,J} \tag{6.10}$$

of the form

$$u_{I,J} = \frac{(z_I - z_J)(z_{I-1} - z_{J+1})}{(z_I - z_{J+1})(z_{I-1} - z_J)}\,, \tag{6.11}$$

as introduced originally by Koba and Nielsen. Here we have once again used the definition (4.3) of the set of $N-3$ consecutive subsets, Δ, associated with the kinematic invariants appearing in the denominator of the given tree graph, $A_N^{\phi^3,\Delta}$. The product $1/u_\Delta$ has only simple poles in the variables z_a, and these only occur at locations of the form $z_a = z_b$ where $|a-b| \neq 2$.

Example: The two 4-point $\lambda\,\phi^3$ theory tree graphs

$$\begin{aligned} A_4^{\phi^3,\{[1,2]\}} &= -\frac{1}{s_{12}} = -\frac{1}{k_{[1,2]}^2} = \oint_{\mathcal{O}} \frac{dz_3}{h_1^{4,\text{ off-shell}}}\,\frac{1}{z_3} \\ A_4^{\phi^3,\{[2,3]\}} &= -\frac{1}{s_{23}} = -\frac{1}{k_{[2,3]}^2} = \oint_{\mathcal{O}} \frac{dz_3}{h_1^{4,\text{ off-shell}}}\,\frac{1}{1-z_3} \end{aligned} \tag{6.12}$$

Example: The five 5-point $\lambda\,\phi^3$ theory tree graphs

$$\begin{aligned} A_5^{\phi^3,\{[1,2],[1,3]\}} &= \frac{1}{s_{12}s_{123}} = \frac{1}{k_{[1,2]}^2 k_{[1,3]}^2} = \oint_{\mathcal{O}} \frac{dz_3 dz_4}{h_1^{5,\text{ off-shell}}\,h_2^{5,\text{ off-shell}}}\,\frac{1}{z_4} \\ A_5^{\phi^3,\{[2,3],[2,4]\}} &= \frac{1}{s_{23}s_{234}} = \frac{1}{k_{[2,3]}^2 k_{[2,4]}^2} = \oint_{\mathcal{O}} \frac{dz_3 dz_4}{h_1^{5,\text{ off-shell}}\,h_2^{5,\text{ off-shell}}}\,\frac{1}{1-z_3} \\ A_5^{\phi^3,\{[1,2],[3,4]\}} &= \frac{1}{s_{12}s_{34}} = \frac{1}{k_{[1,2]}^2 k_{[3,4]}^2} = \oint_{\mathcal{O}} \frac{dz_3 dz_4}{h_1^{5,\text{ off-shell}}\,h_2^{5,\text{ off-shell}}}\,\frac{1-z_4}{z_3-z_4} \\ A_5^{\phi^3,\{[1,3],[2,3]\}} &= \frac{1}{s_{123}s_{23}} = \frac{1}{k_{[1,3]}^2 k_{[2,3]}^2} = \oint_{\mathcal{O}} \frac{dz_3 dz_4}{h_1^{5,\text{ off-shell}}\,h_2^{5,\text{ off-shell}}}\,\frac{z_3(1-z_4)}{(1-z_3)z_4} \\ A_5^{\phi^3,\{[2,4],[3,4]\}} &= \frac{1}{s_{234}s_{34}} = \frac{1}{k_{[2,4]}^2 k_{[3,4]}^2} = \oint_{\mathcal{O}} \frac{dz_3 dz_4}{h_1^{5,\text{ off-shell}}\,h_2^{5,\text{ off-shell}}}\,\frac{z_3}{z_3-z_4} \end{aligned} \tag{6.13}$$

6.3 Level-Zero Generators on Off-Shell Scattering Polynomials

It is straightforward to show that the level-zero generators $\mathbb{P}^\mu$, $\mathbb{L}^{\mu\nu}$ and $\mathbb{D}$ annihilate the inverse of the product of the $N-3$ off-shell scattering polynomials, $\left(H_N^{\text{off-shell}}\right)^{-1}$, as expected from the conformal invariance of the off-shell $\lambda\,\phi^3$ theory partial amplitudes (6.6) and individual graphs (6.9), established earlier in (4.8), (4.9) and (4.11),

$$\begin{aligned} \mathbb{P}^\mu \left[\delta^n\!\left(\textstyle\sum_{j=1}^N k_j\right)\left[H_N^{\text{off-shell}}\right]^{-1}\right] &= \left(\sum_{i=1}^N k_i{}^\mu\right)\delta^n\!\left(\textstyle\sum_{j=1}^N k_j\right)\left[H_N^{\text{off-shell}}\right]^{-1} = 0 \\ \mathbb{L}^{\mu\nu}\left[\delta^n\!\left(\textstyle\sum_{j=1}^N k_j\right)\left[H_N^{\text{off-shell}}\right]^{-1}\right] &= 0 \\ \mathbb{D}\left[\delta^n\!\left(\textstyle\sum_{j=1}^N k_j\right)\left[H_N^{\text{off-shell}}\right]^{-1}\right] &= \left[N\,(d-2)-(n-6)\right]\;\delta^n\!\left(\textstyle\sum_{j=1}^N k_j\right)\left[H_N^{\text{off-shell}}\right]^{-1} = 0 \end{aligned} \tag{6.14}$$

for $d=2$ and in $n=6$ space-time dimensions.

The special conformal generator $\mathbb{K}^\mu$ is not first order in the derivative operators, and thus acts on the scattering polynomials in a more complicated way. Here, we show how it acts on the contour integral for the two Feynman graphs of the $\lambda\,\phi^3$ $N=4$ partial amplitude (6.12). Its action on higher N involve multivariable contour integrals, but they must also vanish for $n=6$, $d=2$, in accordance with (4.14). See Appendix F for some additional discussion and examples.

Example: $\mathbb{K}^\mu$ on $\left[H_4^{\text{off-shell}}\right]^{-1}$

$$\begin{aligned} \mathbb{K}^\mu\left[h_1^{4,\text{ off-shell}}\right]^{-1} &= \mathbb{K}^\mu\left[(1-z_3)(k_1+k_2)^2 - z_3(k_2+k_3)^2\right]^{-1} \\ &= -\frac{2\,(4\,d-n-2)\left[(1-z_3)\left(k_1^\mu+k_2^\mu\right)-z_3\left(k_2^\mu+k_3^\mu\right)\right]}{\left(h_1^{4,\text{ off-shell}}\right)^2} \\ &\quad -\frac{8(z_3-1)z_3\left(\left(k_3{}^2-k_2{}^2\right)k_1^\mu+\left(k_4{}^2-k_2{}^2\right)k_2^\mu+\left(k_1{}^2-k_2{}^2\right)k_3^\mu\right)}{\left(h_1^{4,\text{ off-shell}}\right)^3} \end{aligned} \tag{6.15}$$

Let $h \equiv h_1^{4,\text{ off-shell}} = (1-z)\,s_{12} - z\,s_{23}$, with $z=z_3$. The $N=4$ individual graphs are

$$-\frac{1}{s_{12}} = \oint_{h=0}\frac{dz}{h}\,\frac{1}{z}, \qquad -\frac{1}{s_{23}} = \oint_{h=0}\frac{dz}{h}\,\frac{1}{(1-z)} \tag{6.16}$$

From (6.15) the surviving integrals are

$$\begin{aligned}
\mathbb{K}^\mu \oint_{h=0} \frac{dz}{h}\,\frac{1}{z} &= (-8d+2n+4)(k_1+k_2)^\mu \oint_{h=0} \frac{dz}{(h)^2}\,\frac{1}{z} \\
&= (8d-2n-4)(k_1+k_2)^\mu \oint_{z=0} \frac{dz}{z}\,\frac{1}{(h)^2} = \frac{(8d-2n-4)(k_1+k_2)^\mu}{s_{12}^2}, \\
\mathbb{K}^\mu \oint_{h=0} \frac{dz}{h}\,\frac{1}{(1-z)} &= (8d-2n-4)(k_2+k_3)^\mu \oint_{h=0} \frac{dz}{(h)^2}\,\frac{z}{(1-z)} \\
&= (-8d+2n+4)(k_2+k_3)^\mu \oint_{z=1} \frac{dz}{(1-z)}\,\frac{1}{(h)^2} = \frac{(8d-2n-4)(k_2+k_3)^\mu}{s_{23}^2}
\end{aligned} \tag{6.17}$$

In the integrands with $\frac{1}{(h)^3}$ the contour encircles all the zeros of the denominator, so those integrals vanish. In (6.17) we can swap the contour around $h=0$ to minus the contour around $z=0$, etc. since there is no residue at infinity.

For $N=5$, acting with $\mathbb{K}^\mu$ results in terms like $\oint_{h_1=h_2=0} \frac{dz_3 dz_4}{h_1^2 h_2 z_4} \equiv \text{Res}(h_1^2, h_2)$, where $h_1 \equiv h_1^{5,\text{ off-shell}}$ and $h_2 \equiv h_2^{5,\text{ off-shell}}$, which can be evaluated with the Global Residue Theorem, by dividing the factors in the denominator into two disjoint sets, $\{h_1^2, z_4\}$ and $\{h_2\}$, so that $\text{Res}(h_1^2, h_2) = -\text{Res}(z_4, h_2)$.

6.4 Level-One Generators on Off-Shell Scattering Polynomials

The action of the level-one generators on the scattering polynomials is similarly complicated. As far as the level-one generator $\widehat{\mathbb{P}}^\mu$ is concerned, we find that:

$$\begin{aligned}
\widehat{\mathbb{P}}^\mu \left[H_N^{\text{off-shell}}\right]^{-1} &= -d \left[\sum_{i=1}^{N} (N+1-2\,i)\, k_i^\mu\right] \left[H_N^{\text{off-shell}}\right]^{-1} \\
&- 2\left[H_N^{\text{off-shell}}\right]^{-1} \sum_{m=1}^{N-3} \left[h_m^{N,\text{ off-shell}}\right]^{-1} \sum_{[I,J]\in\mathbb{A}} \left(-k^\mu_{[1,\,I-1]} + k^\mu_{[J+1,\,N]}\right) k^2_{[I,J]} \left(\frac{\partial h_m^{N,\text{ off-shell}}}{\partial k^2_{[I,J]}}\right),
\end{aligned} \tag{6.18}$$

where $\mathbb{A} \equiv \{[I,J] : 1 \le I \le J < N\}$ is the set of all consecutive subsets of the form $[I,J] = \{a \in \mathbb{N} : I \le a \le J\}$. It is clear that the action of the level-one generator $\widehat{\mathbb{P}}^\mu$ does not appear to vanish in any number of space-time dimensions. It also does not act as a multiplicative factor of the form $\sum_{i=1}^{N} c_i k_i^\mu$, as it does when it acts on individual $\lambda\,\phi^3$ theory off-shell tree-level Feynman graphs. Therefore, there is no obvious way of extending $\widehat{\mathbb{P}}^\mu$ by some evaluation parameters term, so that it annihilates $\left(H_N^{\text{off-shell}}\right)^{-1}$.

For the $N = 4$ polynomial $h \equiv h_1^{4,\text{ off-shell}}$, we have:

Example: $\widehat{\mathbb{P}}^\mu$ on $\left[H_4^{\text{off-shell}}\right]^{-1}$

$$\begin{aligned}\widehat{\mathbb{P}}^\mu \left[h_1^{4,\text{ off-shell}}\right]^{-1} &= -\frac{d\,(3k_1^\mu + k_2^\mu - k_3^\mu - 3k_4^\mu)}{\left(h_1^{4,\text{ off-shell}}\right)} \\ &\quad - \frac{2\left((1-z_3)(k_1+k_2)^2\,(k_3^\mu + k_4^\mu) - z_3\,(k_2+k_3)^2\,(-k_1^\mu + k_4^\mu)\right)}{\left(h_1^{4,\text{ off-shell}}\right)^2}\end{aligned} \tag{6.19}$$

The polynomial does not transform simply. However, the partial amplitude (6.7) transforms as expected after contour integration, in accordance with (4.26),

$$\begin{aligned}&-\widehat{\mathbb{P}}^\mu \oint \frac{dz}{h}\left(\frac{1}{z} + \frac{1}{1-z}\right) \\ &= 2k_3^\mu s_{12} \oint \frac{dz}{h^2}\frac{1}{z} + 2k_1^\mu s_{23} \oint \frac{dz}{h^2}\frac{1}{1-z} + 2k_4^\mu \oint \frac{dz}{h}\left(\frac{1}{z} + \frac{1}{1-z}\right) + \mathcal{O}(d) \\ &= 2(k_3+k_4)^\mu \left(-\frac{1}{s_{12}}\right) + 2(-k_1+k_4)^\mu \left(-\frac{1}{s_{23}}\right) + \mathcal{O}(d)\end{aligned} \tag{6.20}$$

For a more general theory beyond $\lambda\,\phi^3$, the amplitude (6.6) acquires a further numerator which may depend on the momenta. In particular for Yang-Mills theory, the CHY integral has also a Pfaffian in the numerator that depends on the momenta (see Section E.3). For these conformally invariant amplitudes, including the Pfaffian Ψ_N^{YM} has the effect of dropping the space-time dimensions in which the amplitudes are invariant from six to four. This further reflects that the Yangian generators must now act on both the polynomials and the Pfaffian. For instance, the level-one $\widehat{\mathbb{P}}^\mu$ generator, as a first-order differential operator, acts on the pure YM theory CHY integrand as follows:

$$\begin{aligned}\widehat{\mathbb{P}}^\mu \left[\frac{\Psi_N^{\text{YM}}}{H_N^{\text{on-shell}}}\right] &= -\,d\left[\sum_{i=1}^{N}(N+1-2\,i)\,k_i^\mu\right]\left[\frac{\Psi_N^{\text{YM}}}{H_N^{\text{on-shell}}}\right] + \Psi_N^{\text{YM}}\left[\widehat{\mathbb{P}}^\mu\Big|_{d=0}\ \frac{1}{H_N^{\text{on-shell}}}\right] \\ &\quad + \frac{1}{H_N^{\text{on-shell}}}\left[\widehat{\mathbb{P}}^\mu\Big|_{d=0}\ \Psi_N^{\text{YM}}\right].\end{aligned} \tag{6.21}$$

CHAPTER 7

COMPUTATIONAL METHODS

Packages Loaded:

AmplitudesGlobal (last updated: April 7, 2023)
TreeAmplitudesDatabase (last updated: March 17, 2022)
LoopAmplitudesDatabase (last updated: March 19, 2023)
YangianGenerators (last updated: February 10, 2022)
YangianGeneratorsOnTreeAmplitudesDatabase (last updated: April 29, 2021)
NumericalEvaluationFiniteFields (last updated: February 21, 2023)

Author: Nikos Dokmetzoglou

Institution: University of North Carolina at Chapel Hill

Please report any bugs to: nidokme@live.unc.edu

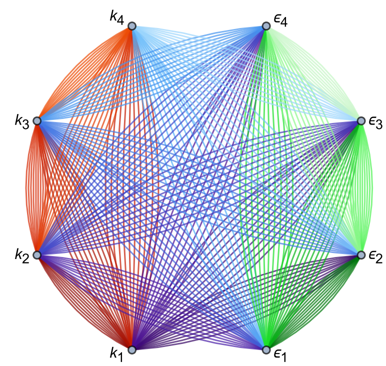

Figure 2: MATHEMATICA packages startup message

In addition to the pen-and-paper calculations we carried out for the formal proofs in this work, we have created and made use of computer algebra tools which have proven very useful. More specifically, we have created three MATHEMATICA packages, which we plan on making publicly available on GitHub:

- **CONFORMALYANGIAN**: to be submitted for publication in the journal Computer Physics Communications [2]. This package consists of the subpackages:
  - YANGIANGENERATORS: Definitions of the level-zero and level-one $Y[\mathrm{SO}(2,n)]$ generators in the momentum-space differential operator representation for scalar and spin-one gauge fields, in terms of the momenta and polarization vectors, with which one can perform efficient symbolic computations of their action on any expression with kinematic dependence. For the definition of these differential operators we have made use of the MATHEMATICA package FEYNCALC [56–58].
  - YANGIANGENERATORSONTREEAMPLITUDESDATABASE: Database of the action of level-zero and level-one $Y[\mathrm{SO}(2,n)]$ generators on scalar and gauge theory amplitudes.

- **TreeAmplitudesDatabase**: Database of all N-point tree-level amplitudes of scalar $\lambda\,\phi^3$ and pure Yang-Mills theory, $3 \leq N \leq 7$, along with their dual description through the CHY scattering equations formalism.

- **NumericalEvaluationFiniteFields**: Package with functions for the numerical evaluation of expressions invovling momenta and polarization vectors using finite fields, i.e. using modular arithmetic with modulus a large prime number, for the purposes of quick numerical checks.

APPENDIX A

$Y[\mathbf{SO}(2,n)]$ ALGEBRA : ADDITIONAL INFORMATION

A.1 Explicit Checks of the $Y[\mathbf{SO}(2,n)]$ Yangian Algebra

We have explicitly checked the algebra of the level-zero $Y[\mathrm{SO}(2,n)]$ generators, in the momentum-space differential operator representation, for both the scalar and gauge field cases:

$$[J^{AB}, J^{CD}] = f^{ABCD}{}_{EF}\, J^{EF} = -\eta^{AC}J^{BD} - \eta^{BD}J^{AC} + \eta^{AD}J^{BC} + \eta^{BC}J^{AD} \tag{A.1}$$

In $\{\mathbb{P}^\mu, \mathbb{L}^{\mu\nu}, \mathbb{D}, \mathbb{K}^\mu\}$ form:

$$\begin{aligned}
&\left[\mathbb{L}^{\mu\nu}, \mathbb{L}^{\lambda\rho}\right] = g^{\nu\lambda}\mathbb{L}^{\mu\rho} - g^{\mu\lambda}\mathbb{L}^{\nu\rho} + g^{\mu\rho}\mathbb{L}^{\nu\lambda} - g^{\nu\rho}\mathbb{L}^{\mu\lambda}\\
&\left[\mathbb{L}^{\mu\nu}, \mathbb{P}^{\lambda}\right] = g^{\nu\lambda}\mathbb{P}^{\mu} - g^{\mu\lambda}\mathbb{P}^{\nu}\\
&\left[\mathbb{L}^{\mu\nu}, \mathbb{K}^{\lambda}\right] = g^{\nu\lambda}\mathbb{K}^{\mu} - g^{\mu\lambda}\mathbb{K}^{\nu}\\
&[\mathbb{L}^{\mu\nu}, \mathbb{D}] = 0\\
&[\mathbb{D}, \mathbb{P}^{\mu}] = \mathbb{P}^{\mu}\\
&[\mathbb{D}, \mathbb{K}^{\mu}] = -\mathbb{K}^{\mu}\\
&[\mathbb{P}^{\mu}, \mathbb{K}^{\nu}] = -2g^{\mu\nu}\mathbb{D} - 2\mathbb{L}^{\mu\nu}\\
&[\mathbb{P}^{\mu}, \mathbb{P}^{\nu}] = 0 \qquad [\mathbb{K}^{\mu}, \mathbb{K}^{\nu}] = 0 \qquad [\mathbb{D}, \mathbb{D}] = 0
\end{aligned} \tag{A.2}$$

In J^{AB} form:

$$\begin{aligned}
&\left[J^{\mu\nu}, J^{\lambda\rho}\right] = g^{\nu\lambda}J^{\mu\rho} - g^{\mu\lambda}J^{\nu\rho} + g^{\mu\rho}J^{\nu\lambda} - g^{\nu\rho}J^{\mu\lambda}\\
&\left[J^{\mu\nu}, J^{n,\lambda}\right] = g^{\nu\lambda}J^{n,\mu} - g^{\mu\lambda}J^{n,\nu}\\
&\left[J^{\mu\nu}, J^{n+1,\lambda}\right] = g^{\nu\lambda}J^{n+1,\mu} - g^{\mu\lambda}J^{n+1,\nu}\\
&\left[J^{\mu\nu}, J^{n,n+1}\right] = 0\\
&\left[J^{n,n+1}, J^{n,\mu}\right] = J^{n+1,\mu} \qquad \left[J^{n,n+1}, J^{n+1,\mu}\right] = J^{n,\mu}\\
&\left[J^{n,\mu}, J^{n+1,\nu}\right] = -g^{\mu\nu}J^{n,n+1}\\
&[J^{n,\mu}, J^{n,\nu}] = J^{\mu\nu} \qquad \left[J^{n+1,\mu}, J^{n+1,\nu}\right] = -J^{\mu\nu}\\
&\left[J^{n,n+1}, J^{n,n+1}\right] = 0
\end{aligned} \tag{A.3}$$

We have also checked the algebra between the level-zero and level-one $Y[\mathrm{SO}(2,n)]$ generators:

$$\left[J^{AB}, \widehat{J}^{CD}\right] = f^{ABCD}{}_{EF}\, \widehat{J}^{EF} \tag{A.4}$$

In $\{\widehat{\mathbb{P}}^\mu, \widehat{\mathbb{L}}^{\mu\nu}, \widehat{\mathbb{D}}, \widehat{\mathbb{K}}^\mu\}$ form:

$$\begin{aligned}
&\left[\mathbb{L}^{\mu\nu}, \widehat{\mathbb{L}}^{\lambda\rho}\right] = g^{\nu\lambda}\widehat{\mathbb{L}}^{\mu\rho} - g^{\mu\lambda}\widehat{\mathbb{L}}^{\nu\rho} + g^{\mu\rho}\widehat{\mathbb{L}}^{\nu\lambda} - g^{\nu\rho}\widehat{\mathbb{L}}^{\mu\lambda} \\
&\left[\mathbb{L}^{\mu\nu}, \widehat{\mathbb{P}}^{\lambda}\right] = g^{\nu\lambda}\widehat{\mathbb{P}}^{\mu} - g^{\mu\lambda}\widehat{\mathbb{P}}^{\nu} \qquad \left[\mathbb{P}^{\lambda}, \widehat{\mathbb{L}}^{\mu\nu}\right] = -g^{\nu\lambda}\widehat{\mathbb{P}}^{\mu} + g^{\mu\lambda}\widehat{\mathbb{P}}^{\nu} \\
&\left[\mathbb{L}^{\mu\nu}, \widehat{\mathbb{K}}^{\lambda}\right] = g^{\nu\lambda}\widehat{\mathbb{K}}^{\mu} - g^{\mu\lambda}\widehat{\mathbb{K}}^{\nu} \qquad \left[\mathbb{K}^{\lambda}, \widehat{\mathbb{L}}^{\mu\nu}\right] = -g^{\nu\lambda}\widehat{\mathbb{K}}^{\mu} + g^{\mu\lambda}\widehat{\mathbb{K}}^{\nu} \\
&\left[\mathbb{L}^{\mu\nu}, \widehat{\mathbb{D}}\right] = 0 \qquad \left[\mathbb{D}, \widehat{\mathbb{L}}^{\mu\nu}\right] = 0 \\
&\left[\mathbb{D}, \widehat{\mathbb{P}}^{\mu}\right] = \widehat{\mathbb{P}}^{\mu} \qquad \left[\mathbb{P}^{\mu}, \widehat{\mathbb{D}}\right] = -\widehat{\mathbb{P}}^{\mu} \\
&\left[\mathbb{D}, \widehat{\mathbb{K}}^{\mu}\right] = -\widehat{\mathbb{K}}^{\mu} \qquad \left[\mathbb{K}^{\mu}, \widehat{\mathbb{D}}\right] = \widehat{\mathbb{K}}^{\mu} \\
&\left[\mathbb{P}^{\mu}, \widehat{\mathbb{K}}^{\nu}\right] = -2g^{\mu\nu}\widehat{\mathbb{D}} - 2\widehat{\mathbb{L}}^{\mu\nu} \qquad \left[\mathbb{K}^{\nu}, \widehat{\mathbb{P}}^{\mu}\right] = 2g^{\mu\nu}\widehat{\mathbb{D}} + 2\widehat{\mathbb{L}}^{\mu\nu} \\
&\left[\mathbb{P}^{\mu}, \widehat{\mathbb{P}}^{\nu}\right] = 0 \qquad \left[\mathbb{K}^{\mu}, \widehat{\mathbb{K}}^{\nu}\right] = 0 \qquad \left[\mathbb{D}, \widehat{\mathbb{D}}\right] = 0
\end{aligned} \tag{A.5}$$

A.2 Symmetry Properties of the $Y[\mathbf{SO}(2,n)]$ Structure Constants

We have checked and confirmed the following symmetry properties of the $Y[\mathrm{SO}(2,n)]$ structure constants. For

$$\begin{aligned}
f^{ABCD}{}_{EF} \equiv \frac{1}{2}\Big(&- \eta^{AC}\,\delta^B_E\,\delta^D_F - \eta^{BD}\,\delta^A_E\,\delta^C_F + \eta^{AD}\,\delta^B_E\,\delta^C_F + \eta^{BC}\,\delta^A_E\,\delta^D_F \\
&+ \eta^{AC}\,\delta^B_F\,\delta^D_E + \eta^{BD}\,\delta^A_F\,\delta^C_E - \eta^{AD}\,\delta^B_F\,\delta^C_E - \eta^{BC}\,\delta^A_F\,\delta^D_E\Big) ,
\end{aligned} \tag{A.6}$$

we have

$$f^{ABCD}{}_{EF} = -f^{BACD}{}_{EF} = -f^{ABDC}{}_{EF} = -f^{CDAB}{}_{EF} = -f^{ABCD}{}_{FE} , \tag{A.7}$$

and for

$$\begin{aligned}
f^{AB}{}_{CDEF} \equiv \frac{1}{2}\Big(&- \delta^A_C\,\delta^B_E\,\eta_{DF} - \delta^A_E\,\delta^B_D\,\eta_{CF} + \delta^A_D\,\delta^B_E\,\eta_{CF} + \delta^A_E\,\delta^B_C\,\eta_{DF} \\
&+ \delta^A_C\,\delta^B_F\,\eta_{DE} + \delta^A_F\,\delta^B_D\,\eta_{CE} - \delta^A_D\,\delta^B_F\,\eta_{CE} - \delta^A_F\,\delta^B_C\,\eta_{DE}\Big) ,
\end{aligned} \tag{A.8}$$

we have

$$f^{AB}{}_{CDEF} = -f^{BA}{}_{CDEF} = -f^{AB}{}_{DCEF} = -f^{AB}{}_{EFCD} = -f^{AB}{}_{CDFE} . \tag{A.9}$$

APPENDIX B

COMMUTATION WITH MOMENTUM-CONSERVING DELTA FUNCTION

In this appendix[1], we show that $\widehat{\mathbb{P}}^\mu$ commutes with the momentum-conserving Dirac delta function

$$\widehat{\mathbb{P}}^\mu\, \delta^n\Big(\textstyle\sum_{j=1}^N k_j\Big)\, M(k) = \delta^n\Big(\textstyle\sum_{j=1}^N k_j\Big)\, \widehat{\mathbb{P}}^\mu\, M(k) \tag{B.1}$$

for any function of the momenta $M(k) = M(k_1, \dots, k_N)$.

Note that, for any function f of the sum of the momenta $\sum_{j=1}^N k_j^\mu$, we have

$$\frac{\partial}{\partial {k_1}^\nu} f\left(k_1^\mu + k_2^\mu + \ldots + k_N^\mu\right) = \frac{\partial}{\partial {k_2}^\nu} f\left(k_1^\mu + k_2^\mu + \ldots + k_N^\mu\right) = \frac{\partial}{\partial {k_i}^\nu} f\left(k_1^\mu + k_2^\mu + \ldots + k_N^\mu\right) \tag{B.2}$$

for any $1 \le i \le N$. We prove (B.1) as follows:

$$\begin{aligned}
&\widehat{\mathbb{P}}^\mu\, \delta^n\Big(\textstyle\sum_{j=1}^N k_j\Big)\, M(k) \\
&= -\sum_{1\le i<j\le N} \Big[P_i^\mu D_j + P_{\rho i} L_j^{\mu\rho} - (i \leftrightarrow j)\Big]\, \delta^n\Big(\textstyle\sum_{j=1}^N k_j\Big)\, M(k) \\
&= -\sum_{1\le i<j\le N} \Big[k_i^\mu (d + k_j \cdot \partial_j) + k_{i\rho}(k_j^\mu \partial_j^\rho - k_j^\rho \partial_j^\mu + \Sigma_j^{\mu\rho}) - (i \leftrightarrow j)\Big]\, \delta^n\Big(\textstyle\sum_{j=1}^N k_j\Big)\, M(k) \\
&= \delta^n\Big(\textstyle\sum_{j=1}^N k_j\Big)\, \widehat{\mathbb{P}}^\mu\, M(k) \\
&\quad - \left(\sum_{1\le i<j\le N} \Big[k_i^\mu k_j \cdot \partial_j + k_{i\rho}(k_j^\mu \partial_j^\rho - k_j^\rho \partial_j^\mu) - (i \leftrightarrow j)\Big]\, \delta^n\Big(\textstyle\sum_{j=1}^N k_j\Big)\right) M(k)
\end{aligned} \tag{B.3}$$

The last term is zero because the momentum-conserving Dirac delta function is a function of the sum of the N momenta, which from (B.2) implies that all the partial derivatives ∂_i act on it in the

[1] The content of this appendix was adapted with minor changes from Appendix B of the author's previously published paper in the Journal of High Energy Physics:
N. Dokmetzoglou and L. Dolan, *Properties of the conformal Yangian in scalar and gauge field theories*, *JHEP* **02** (2023) 137 [arXiv:2207.14806 [hep-th]].

same way, as in $\partial_i = \frac{\partial}{\partial \sum_{\ell=1}^{N} k_\ell}$,

$$\begin{aligned}
&\sum_{1\leq i<j\leq N} \Big[k_i^\mu k_j \cdot \partial_j + k_{i\rho}(k_j^\mu \partial_j^\rho - k_j^\rho \partial_j^\mu) - (i \leftrightarrow j)\Big]\, \delta^n\Big(\textstyle\sum_{j=1}^{N} k_j\Big) \\
&= \sum_{1\leq i<j\leq N} \Big[(k_i^\mu k_{j\rho} + k_j^\mu k_{i\rho})\, \partial_j^\rho - k_{i\rho} k_j^\rho \partial_j^\mu - (i \leftrightarrow j)\Big]\, \delta^n\Big(\textstyle\sum_{j=1}^{N} k_j\Big) \\
&= \sum_{1\leq i<j\leq N} \Big[(k_i^\mu k_{j\rho} + k_j^\mu k_{i\rho})\, \frac{\partial}{\partial \sum_{\ell=1}^{N} k_{\ell\rho}} - k_{i\rho} k_j^\rho\, \frac{\partial}{\partial \sum_{\ell=1}^{N} k_{\ell\mu}} - (i \leftrightarrow j)\Big]\, \delta^n\Big(\textstyle\sum_{j=1}^{N} k_j\Big) = 0 \, ,
\end{aligned} \tag{B.4}$$

since the coefficients of $\frac{\partial}{\partial \sum_{\ell=1}^{N} k_{\ell\rho}}$ and $\frac{\partial}{\partial \sum_{\ell=1}^{N} k_{\ell\mu}}$, after the inclusion of the interchange $(i \leftrightarrow j)$, sum to zero.

APPENDIX C

TREE AMPLITUDES DATABASE

C.1 Scalar $\lambda\,\phi^3$ Theory

C.1.1 Partial Amplitudes

The tree-level partial amplitudes of $\lambda\,\phi^3$ theory are defined as the sum of cyclic permutations of individual Feynman graphs of the form (4.3).

4-point

$$A_4^{\text{partial}} = -\frac{1}{s_{12}} - \frac{1}{s_{23}} = -\frac{1}{s_{12}} + \text{cyclic} \qquad \text{2 terms}$$

5-point

$$A_5^{\text{partial}} = \frac{1}{s_{12}s_{123}} + \frac{1}{s_{23}s_{234}} + \frac{1}{s_{34}s_{345}} + \frac{1}{s_{45}s_{451}} + \frac{1}{s_{51}s_{512}} = \frac{1}{s_{12}s_{123}} + \text{cyclic} \qquad \text{5 terms}$$

6-point

$$\begin{aligned} A_6^{\text{partial}} = & -\frac{1}{s_{12}s_{123}s_{1234}} + \text{cyclic} \qquad \text{6 terms} \\ & -\frac{1}{s_{12}s_{345}s_{6123}} + \text{cyclic} \qquad \text{3 terms} \\ & -\frac{1}{s_{34}s_{345}s_{2345}} + \text{cyclic} \qquad \text{3 terms} \\ & -\frac{1}{s_{12}s_{34}s_{1234}} + \text{cyclic} \qquad \text{2 terms} \end{aligned}$$

7-point

$$\begin{aligned} A_7^{\text{partial}} = & \frac{1}{s_{12}s_{123}s_{1234}s_{12345}} + \text{cyclic} \qquad \text{7 terms} \\ & + \frac{1}{s_{23}s_{123}s_{567}s_{56}} + \text{cyclic} \qquad \text{7 terms} \\ & + \frac{1}{s_{12}s_{123}s_{567}s_{56}} + \text{cyclic} \qquad \text{7 terms} \\ & + \frac{1}{s_{23}s_{123}s_{567}s_{67}} + \text{cyclic} \qquad \text{7 terms} \\ & + \frac{1}{s_{12}s_{34}s_{567}s_{67}} + \text{cyclic} \qquad \text{7 terms} \\ & + \frac{1}{s_{12}s_{45}s_{123}s_{67}} + \text{cyclic} \qquad \text{7 terms} \end{aligned}$$

C.1.2 Total Amplitudes

The tree-level N-point total amplitudes are related to the N-point partial amplitudes by:

$$\mathcal{A}_N^{\text{total}}(k_1, k_2, \ldots, k_N) = i\,\frac{g^{N-2}}{2^{N-2}\,N}\,(2\pi)^n\,\delta^n(k_1 + k_2 + \cdots + k_N)\sum_{\rho\in\mathfrak{S}_N} A_N^{\text{partial}}(k_{\rho(1)}, k_{\rho(2)}, \ldots, k_{\rho(N)})\,, \tag{C.1}$$

where $\mathfrak{S}_N$ is the permutation group on N objects (here the N momenta), and g is the coupling constant. For $\lambda\,\phi^3$ theory, we set $g = \lambda$.

Therefore, the $\lambda\,\phi^3$ theory tree-level 4-point total amplitude is given by:

$$\mathcal{A}_4^{\text{total}}(k_1, k_2, k_3, k_4) = -i\,\lambda^2\,(2\pi)^n\,\delta^n(k_1 + k_2 + k_3 + k_4)\left(\frac{1}{s_{12}} + \frac{1}{s_{23}} + \frac{1}{s_{13}}\right)\quad, \tag{C.2}$$

where $s_{12} \equiv (k_1 + k_2)^2$, $s_{23} \equiv (k_2 + k_3)^2$ and $s_{13} \equiv (k_1 + k_3)^2$. Likewise, the 5-point total amplitude is given by:

$$\begin{aligned}\mathcal{A}_5^{\text{total}}(k_1, k_2, k_3, k_4, k_5) = i\,\lambda^3\,(2\pi)^n\,\delta^n(k_1 + k_2 + k_3 + k_4 + k_5)&\\ \times\Bigg(\frac{1}{s_{12}}\frac{1}{s_{34}} + \frac{1}{s_{12}}\frac{1}{s_{35}} + \frac{1}{s_{12}}\frac{1}{s_{45}} + \frac{1}{s_{13}}\frac{1}{s_{24}} + \frac{1}{s_{13}}\frac{1}{s_{25}}&\\ + \frac{1}{s_{13}}\frac{1}{s_{45}} + \frac{1}{s_{14}}\frac{1}{s_{23}} + \frac{1}{s_{15}}\frac{1}{s_{23}} + \frac{1}{s_{45}}\frac{1}{s_{23}} + \frac{1}{s_{14}}\frac{1}{s_{25}}&\\ + \frac{1}{s_{14}}\frac{1}{s_{35}} + \frac{1}{s_{15}}\frac{1}{s_{24}} + \frac{1}{s_{35}}\frac{1}{s_{24}} + \frac{1}{s_{15}}\frac{1}{s_{34}} + \frac{1}{s_{25}}\frac{1}{s_{34}}\Bigg)&\quad,\end{aligned} \tag{C.3}$$

where $s_{a_1,a_2\ldots,a_m} \equiv (k_{a_1} + k_{a_2} + \ldots + k_{a_m})^2$, and the 3-point total amplitude is simply:

$$\mathcal{A}_3^{\text{total}}(k_1, k_2, k_3) = i\,\lambda\,(2\pi)^n\,\delta^n(k_1 + k_2 + k_3)\quad. \tag{C.4}$$

C.2 Pure Yang-Mills Theory

For pure YM theory in four dimensions, with gauge group $SU(m)$, the total amplitude is given by:

$$\mathcal{A}_N^{\text{total}}(k;\epsilon) = 2^{N/2} g^{N-2} (2\pi)^4 \, \delta^4 \left(\sum_{i=1}^{N} k_i \right) \sum_{\rho \in S_N/Z_N} \text{tr}\left[T^{a_{\rho(1)}} \dots T^{a_{\rho(N)}} \right] \times A_N^{\text{partial}}(k_{\rho(1)}, \dots, k_{\rho(N)}; \epsilon_{\rho(1)}, \dots, \epsilon_{\rho(N)}) \, , \tag{C.5}$$

where the sum runs over all possible non-cyclic permutations ρ of the set $\{1, \dots, N\}$ and T^a are the generators of the gauge group [26].

APPENDIX D

ACTION OF CONFORMAL YANGIAN $Y[\mathbf{SO}(2,n)]$ GENERATORS ON SCALAR $\lambda\,\phi^3$ THEORY : ADDITIONAL EXAMPLES

In this appendix, we present some additional examples of the action of level-one $Y[\mathrm{SO}(2,n)]$ generators on individual off-shell Feynman tree graphs of scalar $\lambda\,\phi^3$ theory. These results have been checked computationally using the Mathematica packages we developed.

D.1 Level-One Generators on Scalar $\lambda\,\phi^3$ Theory Off-Shell Tree Graphs

Given our earlier discussion in Section 4.3, we expect the level-one $\mathrm{SO}(2,n)$ Yangian generators $\left(\widehat{\mathbb{P}}^{\mu},\, \widehat{\mathbb{L}}^{\mu\nu},\, \widehat{\mathbb{D}},\, \widehat{\mathbb{K}}^{\mu}\right)$ (2.33), when extended appropriately $\left(\widehat{\mathbb{P}}'^{\mu},\, \widehat{\mathbb{L}}'^{\mu\nu},\, \widehat{\mathbb{D}}',\, \widehat{\mathbb{K}}'^{\mu}\right)$ (4.1), by the graph-dependent evaluation parameters $c^{\phi^3,\,\Delta}_{N,\,i}$ (4.27), to annihilate any individual off-shell scalar $\lambda\,\phi^3$ theory tree graph $\mathcal{A}^{\phi^3,\,\Delta}_{N} = \delta^n\left(\sum_{j=1}^{N} k_j\right) A^{\phi^3,\,\Delta}_{N}$, in $n = 6$ space-time dimensions (4.34). Here, we show explicitly that in $n = 6$ space-time dimensions the extended level-one generators annihilate the kinematic part of several off-shell scalar $\lambda\,\phi^3$ theory tree graphs $A^{\phi^3,\,\Delta}_{N}$ (4.3), where the momentum-conserving Dirac delta function has been stripped off. These results imply that the extended level-one generators commute with the momentum-conserving Dirac delta function for these examples. We express the scalar theory tree graphs in terms of the off-shell kinematic invariants $s_{I,I+1,\ldots,J-1,J} \equiv k^2_{[I,J]} \equiv (k_I + k_{I+1} + \ldots + k_{J-1} + k_J)^2$.

D.1.1 Action of Level-One Translation Generators

$N=4$ Examples

Example: $\widehat{\mathbb{P}}^\mu$ on $A_4^{\phi^3,\{[1,2]\}} = -1/s_{12} = -1/(k_1+k_2)^2$

$$\widehat{\mathbb{P}}^\mu\left(-\frac{1}{s_{12}}\right) = \left[-d\left(3k_1^\mu + k_2^\mu - k_3^\mu - 3k_4^\mu\right) - 2\left(k_3^\mu + k_4^\mu\right)\right]\left(-\frac{1}{s_{12}}\right)$$

$$\text{Evaluation parameters}:\ \{c_1 = 3\,d, c_2 = d, c_3 = -d+2, c_4 = -3\,d+2\}$$

$$\sum_{i=1}^{4} c_i P_i^\mu\left(-\frac{1}{s_{12}}\right) = \left[d\left(3k_1^\mu + k_2^\mu - k_3^\mu - 3k_4^\mu\right) + 2\left(k_3^\mu + k_4^\mu\right)\right]\left(-\frac{1}{s_{12}}\right) \tag{D.1}$$

$$\left[\widehat{\mathbb{P}}^\mu + \sum_{i=1}^{4} c_i P_i^\mu\right]\left(-\frac{1}{s_{12}}\right) = 0\ ,\ \text{for arbitrary } n \text{ and } d.$$

Example: $\widehat{\mathbb{P}}^\mu$ on $A_4^{\phi^3,\{[2,3]\}} = -1/s_{23} = -1/(k_2+k_3)^2$

$$\widehat{\mathbb{P}}^\mu\left(-\frac{1}{s_{23}}\right) = \left[-d\left(3k_1^\mu + k_2^\mu - k_3^\mu - 3k_4^\mu\right) - 2\left(-k_1^\mu + k_4^\mu\right)\right]\left(-\frac{1}{s_{23}}\right)$$

$$\text{Evaluation parameters}:\ \{c_1 = 3\,d-2, c_2 = d, c_3 = -d, c_4 = -3\,d+2\}$$

$$\sum_{i=1}^{4} c_i P_i^\mu\left(-\frac{1}{s_{23}}\right) = \left[d\left(3k_1^\mu + k_2^\mu - k_3^\mu - 3k_4^\mu\right) + 2\left(-k_1^\mu + k_4^\mu\right)\right]\left(-\frac{1}{s_{23}}\right) \tag{D.2}$$

$$\left[\widehat{\mathbb{P}}^\mu + \sum_{i=1}^{4} c_i P_i^\mu\right]\left(-\frac{1}{s_{23}}\right) = 0\ ,\ \text{for arbitrary } n \text{ and } d.$$

$N = 5$ Examples

Example: $\widehat{\mathbb{P}}^\mu$ on $A_5^{\phi^3, \{[1,2],[1,3]\}} = 1/(s_{12}s_{123}) = 1/\left((k_1+k_2)^2(k_1+k_2+k_3)^2\right)$

$$\widehat{\mathbb{P}}^\mu\left(\frac{1}{s_{12}s_{123}}\right) = \left[-d\left(4k_1^\mu + 2k_2^\mu - 2k_4^\mu - 4k_5^\mu\right) - 2\left(k_3^\mu + k_4^\mu + k_5^\mu\right) - 2\left(k_4^\mu + k_5^\mu\right)\right]\left(\frac{1}{s_{12}s_{123}}\right)$$

$$\text{Evaluation parameters}: \{c_1 = 4\,d, c_2 = 2\,d, c_3 = 2, c_4 = -2\,d+4, c_5 = -4\,d+4\}$$

$$\sum_{i=1}^{5} c_i P_i^\mu\left(\frac{1}{s_{12}s_{123}}\right) = \left[d\left(4k_1^\mu + 2k_2^\mu - 2k_4^\mu - 4k_5^\mu\right) + 2\,k_3^\mu + 4\left(k_4^\mu + k_5^\mu\right)\right]\left(\frac{1}{s_{12}s_{123}}\right)$$

$$\left[\widehat{\mathbb{P}}^\mu + \sum_{i=1}^{5} c_i P_i^\mu\right]\left(\frac{1}{s_{12}s_{123}}\right) = 0 \text{ , for arbitrary } n \text{ and } d. \tag{D.3}$$

Example: $\widehat{\mathbb{P}}^\mu$ on $A_5^{\phi^3, \{[1,2],[3,4]\}} = 1/(s_{12}s_{34}) = 1/\left((k_1+k_2)^2(k_3+k_4)^2\right)$

$$\widehat{\mathbb{P}}^\mu\left(\frac{1}{s_{12}s_{34}}\right) = \left[-d\left(4k_1^\mu + 2k_2^\mu - 2k_4^\mu - 4k_5^\mu\right)\right.$$
$$\left. -2\left(k_3^\mu + k_4^\mu + k_5^\mu\right) - 2\left(-k_1^\mu - k_2^\mu + k_5^\mu\right)\right]\left(\frac{1}{s_{12}s_{34}}\right)$$

$$\text{Evaluation parameters}: \{c_1 = 4\,d-2, c_2 = 2\,d-2, c_3 = 2, c_4 = -2\,d+2, c_5 = -4\,d+4\}$$

$$\sum_{i=1}^{5} c_i P_i^\mu\left(\frac{1}{s_{12}s_{34}}\right) = \left[d\left(4k_1^\mu + 2k_2^\mu - 2k_4^\mu - 4k_5^\mu\right)\right.$$
$$\left. +2\left(k_3^\mu + k_4^\mu + k_5^\mu\right) + 2\left(-k_1^\mu - k_2^\mu + k_5^\mu\right)\right]\left(\frac{1}{s_{12}s_{34}}\right)$$

$$\left[\widehat{\mathbb{P}}^\mu + \sum_{i=1}^{5} c_i P_i^\mu\right]\left(\frac{1}{s_{12}s_{34}}\right) = 0 \text{ , for arbitrary } n \text{ and } d. \tag{D.4}$$

Example: $\widehat{\mathbb{P}}^\mu$ on $A_5^{\phi^3,\{[2,3],[1,3]\}} = 1/(s_{23}s_{123}) = 1/\left((k_2+k_3)^2(k_1+k_2+k_3)^2\right)$

$$\widehat{\mathbb{P}}^\mu\left(\frac{1}{s_{23}s_{123}}\right) = \left[-d\left(4k_1^\mu+2k_2^\mu-2k_4^\mu-4k_5^\mu\right)-2\left(-k_1^\mu+k_4^\mu+k_5^\mu\right)-2\left(k_4^\mu+k_5^\mu\right)\right]\left(\frac{1}{s_{23}s_{123}}\right)$$

Evaluation parameters : $\{c_1 = 4\,d-2, c_2 = 2\,d, c_3 = 0, c_4 = -2\,d+4, c_5 = -4\,d+4\}$

$$\sum_{i=1}^{5} c_i P_i^\mu\left(\frac{1}{s_{23}s_{123}}\right) = \left[d\left(4k_1^\mu+2k_2^\mu-2k_4^\mu-4k_5^\mu\right)-2\,k_1^\mu+4\left(k_4^\mu+k_5^\mu\right)\right]\left(\frac{1}{s_{23}s_{123}}\right)$$

$$\left[\widehat{\mathbb{P}}^\mu+\sum_{i=1}^{5} c_i P_i^\mu\right]\left(\frac{1}{s_{23}s_{123}}\right) = 0\ ,\ \text{for arbitrary } n \text{ and } d. \tag{D.5}$$

Example: $\widehat{\mathbb{P}}^\mu$ on $A_5^{\phi^3,\{[2,3],[2,4]\}} = 1/(s_{23}s_{234}) = 1/\left((k_2+k_3)^2(k_2+k_3+k_4)^2\right)$

$$\widehat{\mathbb{P}}^\mu\left(\frac{1}{s_{23}s_{234}}\right) = \left[-d\left(4k_1^\mu+2k_2^\mu-2k_4^\mu-4k_5^\mu\right)-2\left(-k_1^\mu+k_4^\mu+k_5^\mu\right)-2\left(-k_1^\mu+k_5^\mu\right)\right]\left(\frac{1}{s_{23}s_{234}}\right)$$

Evaluation parameters : $\{c_1 = 4\,d-4, c_2 = 2\,d, c_3 = 0, c_4 = -2\,d+2, c_5 = -4\,d+4\}$

$$\sum_{i=1}^{5} c_i P_i^\mu\left(\frac{1}{s_{23}s_{234}}\right) = \left[d\left(4k_1^\mu+2k_2^\mu-2k_4^\mu-4k_5^\mu\right)+2\,k_4^\mu+4\left(-k_1^\mu+k_5^\mu\right)\right]\left(\frac{1}{s_{23}s_{234}}\right)$$

$$\left[\widehat{\mathbb{P}}^\mu+\sum_{i=1}^{5} c_i P_i^\mu\right]\left(\frac{1}{s_{23}s_{234}}\right) = 0\ ,\ \text{for arbitrary } n \text{ and } d. \tag{D.6}$$

Example: $\widehat{\mathbb{P}}^\mu$ on $A_5^{\phi^3,\{[3,4],[2,4]\}} = 1/(s_{34}s_{234}) = 1/\left((k_3+k_4)^2(k_2+k_3+k_4)^2\right)$

$$\widehat{\mathbb{P}}^\mu\left(\frac{1}{s_{34}s_{234}}\right) = \left[-d\left(4k_1^\mu+2k_2^\mu-2k_4^\mu-4k_5^\mu\right)-2\left(-k_1^\mu-k_2^\mu+k_5^\mu\right)-2\left(-k_1^\mu+k_5^\mu\right)\right]\left(\frac{1}{s_{34}s_{234}}\right)$$

Evaluation parameters : $\{c_1 = 4\,d-4, c_2 = 2\,d-2, c_3 = 0, c_4 = -2\,d, c_5 = -4\,d+4\}$

$$\sum_{i=1}^{5} c_i P_i^\mu\left(\frac{1}{s_{34}s_{234}}\right) = \left[d\left(4k_1^\mu+2k_2^\mu-2k_4^\mu-4k_5^\mu\right)-2\,k_2^\mu+4\left(-k_1^\mu+k_5^\mu\right)\right]\left(\frac{1}{s_{34}s_{234}}\right)$$

$$\left[\widehat{\mathbb{P}}^\mu+\sum_{i=1}^{5} c_i P_i^\mu\right]\left(\frac{1}{s_{34}s_{234}}\right) = 0\ ,\ \text{for arbitrary } n \text{ and } d. \tag{D.7}$$

$N = 6$ Examples

Example: $\widehat{\mathbb{P}}^\mu$ on $A_6^{\phi^3,\{[1,2],[1,3],[1,4]\}} = -1/(s_{12}s_{123}s_{1234}) = -1/\left(k_{[1,2]}^2 k_{[1,3]}^2 k_{[1,4]}^2\right)$

$$\widehat{\mathbb{P}}^\mu\left(-\frac{1}{s_{12}s_{123}s_{1234}}\right) = [-d\,(5k_1^\mu + 3k_2^\mu + k_3^\mu - k_4^\mu - 3k_5^\mu - 5k_6^\mu) - 2\,(k_3^\mu + k_4^\mu + k_5^\mu + k_6^\mu) \\ -2\,(k_4^\mu + k_5^\mu + k_6^\mu) - 2\,(k_5^\mu + k_6^\mu)]\left(-\frac{1}{s_{12}s_{123}s_{1234}}\right)$$

Evaluation parameters :

$$\{c_1 = 5\,d, c_2 = 3\,d, c_3 = d+2, c_4 = -d+4, c_5 = -3\,d+6, c_6 = -5\,d+6\}$$

$$\sum_{i=1}^{6} c_i P_i^\mu\left(-\frac{1}{s_{12}s_{123}s_{1234}}\right) = [d\,(5k_1^\mu + 3k_2^\mu + k_3^\mu - k_4^\mu - 3k_5^\mu - 5k_6^\mu) \\ +2\,k_3^\mu + 4\,k_4^\mu + 6\,(k_5^\mu + k_6^\mu)]\left(-\frac{1}{s_{12}s_{123}s_{1234}}\right)$$

$$\left[\widehat{\mathbb{P}}^\mu + \sum_{i=1}^{6} c_i P_i^\mu\right]\left(-\frac{1}{s_{12}s_{123}s_{1234}}\right) = 0 \text{ , for arbitrary } n \text{ and } d. \tag{D.8}$$

Example: $\widehat{\mathbb{P}}^\mu$ on $A_6^{\phi^3,\{[2,3],[4,5],[2,5]\}} = -1/(s_{23}s_{45}s_{2345}) = -1/\left(k_{[2,3]}^2 k_{[4,5]}^2 k_{[2,5]}^2\right)$

$$\widehat{\mathbb{P}}^\mu\left(-\frac{1}{s_{23}s_{45}s_{2345}}\right) = [-d\,(5k_1^\mu + 3k_2^\mu + k_3^\mu - k_4^\mu - 3k_5^\mu - 5k_6^\mu) - 2\,(-k_1^\mu + k_4^\mu + k_5^\mu + k_6^\mu) \\ -2\,(-k_1^\mu - k_2^\mu - k_3^\mu + k_6^\mu) - 2\,(-k_1^\mu + k_6^\mu)]\left(-\frac{1}{s_{23}s_{45}s_{2345}}\right)$$

Evaluation parameters :

$$\{c_1 = 5\,d-6, c_2 = 3\,d-2, c_3 = d-2, c_4 = -d+2, c_5 = -3\,d+2, c_6 = -5\,d+6\}$$

$$\sum_{i=1}^{6} c_i P_i^\mu\left(-\frac{1}{s_{23}s_{45}s_{2345}}\right) = [d\,(5k_1^\mu + 3k_2^\mu + k_3^\mu - k_4^\mu - 3k_5^\mu - 5k_6^\mu) + 6\,(-k_1^\mu + k_6^\mu) \\ +2\,(k_4^\mu + k_5^\mu) + 2\,(-k_2^\mu - k_3^\mu)]\left(-\frac{1}{s_{23}s_{45}s_{2345}}\right)$$

$$\left[\widehat{\mathbb{P}}^\mu + \sum_{i=1}^{6} c_i P_i^\mu\right]\left(-\frac{1}{s_{23}s_{45}s_{2345}}\right) = 0 \text{ , for arbitrary } n \text{ and } d. \tag{D.9}$$

D.1.2 Action of Level-One Lorentz Transformation Generators

$N = 4$ Examples

Example: $\widehat{\mathbb{L}}^{\mu\nu}$ on $A_4^{\phi^3,\{[1,2]\}} = -1/s_{12} = -1/(k_1+k_2)^2$

$$\widehat{\mathbb{L}}^{\mu\nu}\left(-\frac{1}{s_{12}}\right) = -\left[4(d-2)-(n-6)\right]\frac{(k_1^\mu+k_2^\mu)(k_3^\nu+k_4^\nu)-(k_1^\nu+k_2^\nu)(k_3^\mu+k_4^\mu)}{{s_{12}}^2} - 4\,d\,\frac{k_1^\mu\,k_2^\nu - k_1^\nu\,k_2^\mu}{{s_{12}}^2}$$

Evaluation parameters : $\{c_1 = 3\,d, c_2 = d, c_3 = -d+2, c_4 = -3\,d+2\}$

$$\sum_{i=1}^{4} c_i L_i^{\mu\nu}\left(-\frac{1}{s_{12}}\right) = 4\,d\,\frac{k_1^\mu\,k_2^\nu - k_1^\nu\,k_2^\mu}{{s_{12}}^2} \tag{D.10}$$

$$\left[\widehat{\mathbb{L}}^{\mu\nu} + \sum_{i=1}^{4} c_i L_i^{\mu\nu}\right]\left(-\frac{1}{s_{12}}\right) = -\left[4(d-2)-(n-6)\right]\frac{(k_1^\mu+k_2^\mu)(k_3^\nu+k_4^\nu)-(k_1^\nu+k_2^\nu)(k_3^\mu+k_4^\mu)}{{s_{12}}^2}$$

$$\left[\widehat{\mathbb{L}}^{\mu\nu} + \sum_{i=1}^{4} c_i L_i^{\mu\nu}\right]\left(-\frac{1}{s_{12}}\right) = 0\ ,\ \text{when } n=6 \text{ and } d=2.$$

Example: $\widehat{\mathbb{L}}^{\mu\nu}$ on $A_4^{\phi^3,\{[2,3]\}} = -1/s_{23} = -1/(k_2+k_3)^2$

$$\widehat{\mathbb{L}}^{\mu\nu}\left(-\frac{1}{s_{23}}\right) = -\left[4(d-2)-(n-6)\right]\frac{(k_2^\mu+k_3^\mu)(-k_1^\nu+k_4^\nu)-(k_2^\nu+k_3^\nu)(-k_1^\mu+k_4^\mu)}{{s_{23}}^2} - 4\,d\,\frac{k_2^\mu\,k_3^\nu - k_2^\nu\,k_3^\mu}{{s_{23}}^2}$$

Evaluation parameters : $\{c_1 = 3\,d-2, c_2 = d, c_3 = -d, c_4 = -3\,d+2\}$

$$\sum_{i=1}^{4} c_i L_i^{\mu\nu}\left(-\frac{1}{s_{23}}\right) = 4\,d\,\frac{k_2^\mu\,k_3^\nu - k_2^\nu\,k_3^\mu}{{s_{23}}^2} \tag{D.11}$$

$$\left[\widehat{\mathbb{L}}^{\mu\nu} + \sum_{i=1}^{4} c_i L_i^{\mu\nu}\right]\left(-\frac{1}{s_{23}}\right) = -\left[4(d-2)-(n-6)\right]\frac{(k_2^\mu+k_3^\mu)(-k_1^\nu+k_4^\nu)-(k_2^\nu+k_3^\nu)(-k_1^\mu+k_4^\mu)}{{s_{23}}^2}$$

$$\left[\widehat{\mathbb{L}}^{\mu\nu} + \sum_{i=1}^{4} c_i L_i^{\mu\nu}\right]\left(-\frac{1}{s_{23}}\right) = 0\ ,\ \text{when } n=6 \text{ and } d=2.$$

$N=5$ Examples

Example: $\widehat{\mathbb{L}}^{\mu\nu}$ on $A_5^{\phi^3,\{[1,2],[1,3]\}} = 1/(s_{12}s_{123}) = 1/\left((k_1+k_2)^2(k_1+k_2+k_3)^2\right)$

$$\widehat{\mathbb{L}}^{\mu\nu}\left(\frac{1}{s_{12}s_{123}}\right) \neq 0$$

$$\text{Evaluation parameters : } \{c_1 = 4\,d, c_2 = 2\,d, c_3 = 2, c_4 = -2\,d+4, c_5 = -4\,d+4\} \tag{D.12}$$

$$\left[\widehat{\mathbb{L}}^{\mu\nu} + \sum_{i=1}^{5} c_i L_i^{\mu\nu}\right]\left(\frac{1}{s_{12}s_{123}}\right) = 0 \text{ , when } n=6 \text{ and } d=2.$$

Example: $\widehat{\mathbb{L}}^{\mu\nu}$ on $A_5^{\phi^3,\{[2,3],[2,4]\}} = 1/(s_{23}s_{234}) = 1/\left((k_2+k_3)^2(k_2+k_3+k_4)^2\right)$

$$\widehat{\mathbb{L}}^{\mu\nu}\left(\frac{1}{s_{23}s_{234}}\right) \neq 0$$

$$\text{Evaluation parameters : } \{c_1 = 4\,d-4, c_2 = 2\,d, c_3 = 0, c_4 = -2\,d+2, c_5 = -4\,d+4\}$$

$$\left[\widehat{\mathbb{L}}^{\mu\nu} + \sum_{i=1}^{5} c_i L_i^{\mu\nu}\right]\left(\frac{1}{s_{23}s_{234}}\right) = 0 \text{ , when } n=6 \text{ and } d=2. \tag{D.13}$$

$N=6$ Example

Example: $\widehat{\mathbb{L}}^{\mu\nu}$ on $A_6^{\phi^3,\{[1,2],[1,3],[1,4]\}} = -1/(s_{12}s_{123}s_{1234}) = -1/\left(k_{[1,2]}^2 k_{[1,3]}^2 k_{[1,4]}^2\right)$

$$\widehat{\mathbb{L}}^{\mu\nu}\left(-\frac{1}{s_{12}s_{123}s_{1234}}\right) \neq 0$$

Evaluation parameters :

$$\{c_1 = 5\,d, c_2 = 3\,d, c_3 = d+2, c_4 = -d+4, c_5 = -3\,d+6, c_6 = -5\,d+6\} \tag{D.14}$$

$$\left[\widehat{\mathbb{L}}^{\mu\nu} + \sum_{i=1}^{6} c_i L_i^{\mu\nu}\right]\left(-\frac{1}{s_{12}s_{123}s_{1234}}\right) = 0 \text{ , when } n=6 \text{ and } d=2.$$

D.1.3 Action of Level-One Dilatation Generator

$N = 4$ Examples

Example: $\widehat{\mathbb{D}}$ on $A_4^{\phi^3,\{[1,2]\}} = -1/s_{12} = -1/(k_1+k_2)^2$

$$
\begin{aligned}
&\widehat{\mathbb{D}}\left(-\frac{1}{s_{12}}\right) = \frac{1}{2}\left[4(d-2)-(n-6)\right]\left(\frac{-s_{12}-s_{34}+s_{1234}}{{s_{12}}^2}\right) - 2d\left(\frac{k_1^2-k_2^2}{{s_{12}}^2}\right)\\
&\text{Evaluation parameters}:\ \{c_1 = 3\,d, c_2 = d, c_3 = -d+2, c_4 = -3\,d+2\}\\
&\sum_{i=1}^{4} c_i D_i\left(-\frac{1}{s_{12}}\right) = 2d\left(\frac{k_1^2-k_2^2}{{s_{12}}^2}\right)\\
&\left[\widehat{\mathbb{D}} + \sum_{i=1}^{4} c_i D_i\right]\left(-\frac{1}{s_{12}}\right) = \frac{1}{2}\left[4(d-2)-(n-6)\right]\left(\frac{-s_{12}-s_{34}+s_{1234}}{{s_{12}}^2}\right)\\
&\left[\widehat{\mathbb{D}} + \sum_{i=1}^{4} c_i D_i\right]\left(-\frac{1}{s_{12}}\right) = 0\ ,\ \text{when } n=6 \text{ and } d=2.
\end{aligned}
\tag{D.15}
$$

Example: $\widehat{\mathbb{D}}$ on $A_4^{\phi^3,\{[2,3]\}} = -1/s_{23} = -1/(k_2+k_3)^2$

$$
\begin{aligned}
&\widehat{\mathbb{D}}\left(-\frac{1}{s_{23}}\right) = \frac{1}{2}\left[4(d-2)-(n-6)\right]\left(\frac{-s_{123}+s_{234}+k_1^2-k_4^2}{{s_{23}}^2}\right) - 2d\left(\frac{k_2^2-k_3^2}{{s_{23}}^2}\right)\\
&\text{Evaluation parameters}:\ \{c_1 = 3\,d-2, c_2 = d, c_3 = -d, c_4 = -3\,d+2\}\\
&\sum_{i=1}^{4} c_i D_i\left(-\frac{1}{s_{23}}\right) = 2d\left(\frac{k_2^2-k_3^2}{{s_{23}}^2}\right)\\
&\left[\widehat{\mathbb{D}} + \sum_{i=1}^{4} c_i D_i\right]\left(-\frac{1}{s_{23}}\right) = \frac{1}{2}\left[4(d-2)-(n-6)\right]\left(\frac{-s_{123}+s_{234}+k_1^2-k_4^2}{{s_{23}}^2}\right)\\
&\left[\widehat{\mathbb{D}} + \sum_{i=1}^{4} c_i D_i\right]\left(-\frac{1}{s_{23}}\right) = 0\ ,\ \text{when } n=6 \text{ and } d=2.
\end{aligned}
\tag{D.16}
$$

$N=5$ Examples

Example: $\widehat{\mathbb{D}}$ on $A_5^{\phi^3,\{[1,2],[1,3]\}} = 1/(s_{12}s_{123}) = 1/\left((k_1+k_2)^2(k_1+k_2+k_3)^2\right)$

$$\begin{aligned}
\widehat{\mathbb{D}}\left(\frac{1}{s_{12}s_{123}}\right) = &-\frac{1}{2}\left[4(d-2)-(n-6)\right]\left(\frac{s_{12345}-s_{12}-s_{345}}{{s_{12}}^2 s_{123}}\right)\\
&-\frac{1}{2}\left[6(d-2)-(n-6)\right]\left(\frac{s_{12345}-s_{123}-s_{45}}{s_{12}{s_{123}}^2}\right)\\
&-2d\left(-\frac{k_1^2-k_2^2}{{s_{12}}^2 s_{123}}-\frac{s_{12}-s_{23}+k_1^2-k_3^2}{s_{12}{s_{123}}^2}\right)-2\left(\frac{-s_{123}+s_{12}-k_3^2}{s_{12}{s_{123}}^2}\right)
\end{aligned}$$

Evaluation parameters : $\{c_1 = 4\,d, c_2 = 2\,d, c_3 = 2, c_4 = -2\,d+4, c_5 = -4\,d+4\}$

$$\sum_{i=1}^{5} c_i D_i\left(\frac{1}{s_{12}s_{123}}\right) = 2d\left(-\frac{k_1^2-k_2^2}{{s_{12}}^2 s_{123}}-\frac{s_{12}-s_{23}+k_1^2-k_3^2}{s_{12}{s_{123}}^2}\right)+2\left(\frac{-s_{123}+s_{12}-k_3^2}{s_{12}{s_{123}}^2}\right)$$

$$\begin{aligned}
\left[\widehat{\mathbb{D}}+\sum_{i=1}^{5} c_i D_i\right]\left(\frac{1}{s_{12}s_{123}}\right) = &-\frac{1}{2}\left[4(d-2)-(n-6)\right]\left(\frac{s_{12345}-s_{12}-s_{345}}{{s_{12}}^2 s_{123}}\right)\\
&-\frac{1}{2}\left[6(d-2)-(n-6)\right]\left(\frac{s_{12345}-s_{123}-s_{45}}{s_{12}{s_{123}}^2}\right)
\end{aligned}$$

$$\left[\widehat{\mathbb{D}}+\sum_{i=1}^{5} c_i D_i\right]\left(\frac{1}{s_{12}s_{123}}\right) = 0\ ,\ \text{when } n=6 \text{ and } d=2. \tag{D.17}$$

Example: $\widehat{\mathbb{D}}$ on $A_5^{\phi^3,\{[2,3],[2,4]\}} = 1/(s_{23}s_{234}) = 1/\big((k_2+k_3)^2(k_2+k_3+k_4)^2\big)$

$$\begin{aligned}
\widehat{\mathbb{D}}\left(\frac{1}{s_{23}s_{234}}\right) = & -\frac{1}{2}\left[4(d-2)-(n-6)\right]\left(\frac{-s_{123}+s_{2345}-s_{45}+k_1^2}{{s_{23}}^2 s_{234}}\right) \\
& -\frac{1}{2}\left[6(d-2)-(n-6)\right]\left(\frac{-s_{1234}+s_{2345}+k_1^2-k_5^2}{s_{23}{s_{234}}^2}\right) \\
& -2d\left(-\frac{k_2^2-k_3^2}{{s_{23}}^2 s_{234}}-\frac{s_{23}-s_{34}+k_2^2-k_4^2}{s_{23}{s_{234}}^2}\right)-2\left(\frac{-s_{234}+s_{23}-k_4^2}{s_{23}{s_{234}}^2}\right)
\end{aligned}$$

Evaluation parameters : $\{c_1 = 4\,d-4, c_2 = 2\,d, c_3 = 0, c_4 = -2\,d+2, c_5 = -4\,d+4\}$

$$\sum_{i=1}^{5} c_i D_i\left(\frac{1}{s_{23}s_{234}}\right) = 2d\left(-\frac{k_2^2-k_3^2}{{s_{23}}^2 s_{234}}-\frac{s_{23}-s_{34}+k_2^2-k_4^2}{s_{23}{s_{234}}^2}\right)+2\left(\frac{-s_{234}+s_{23}-k_4^2}{s_{23}{s_{234}}^2}\right)$$

$$\begin{aligned}
\left[\widehat{\mathbb{D}}+\sum_{i=1}^{5} c_i D_i\right]\left(\frac{1}{s_{23}s_{234}}\right) = & -\frac{1}{2}\left[4(d-2)-(n-6)\right]\left(\frac{-s_{123}+s_{2345}-s_{45}+k_1^2}{{s_{23}}^2 s_{234}}\right) \\
& -\frac{1}{2}\left[6(d-2)-(n-6)\right]\left(\frac{-s_{1234}+s_{2345}+k_1^2-k_5^2}{s_{23}{s_{234}}^2}\right)
\end{aligned}$$

$$\left[\widehat{\mathbb{D}}+\sum_{i=1}^{5} c_i D_i\right]\left(\frac{1}{s_{23}s_{234}}\right) = 0\ ,\ \text{when } n=6 \text{ and } d=2. \tag{D.18}$$

Example: $\widehat{\mathbb{D}}$ on $A_5^{\phi^3,\{[3,4],[2,4]\}} = 1/(s_{34}s_{234}) = 1/\big((k_3+k_4)^2(k_2+k_3+k_4)^2\big)$

$$\begin{aligned}
\widehat{\mathbb{D}}\left(\frac{1}{s_{34}s_{234}}\right) = & -\frac{1}{2}\left[4(d-2)-(n-6)\right]\left(\frac{-s_{1234}+s_{12}+s_{345}-k_5^2}{{s_{34}}^2 s_{234}}\right) \\
& -\frac{1}{2}\left[6(d-2)-(n-6)\right]\left(\frac{-s_{1234}+s_{2345}+k_1^2-k_5^2}{s_{34}{s_{234}}^2}\right) \\
& -2d\left(-\frac{k_3^2-k_4^2}{{s_{34}}^2 s_{234}}-\frac{s_{23}-s_{34}+k_2^2-k_4^2}{s_{34}{s_{234}}^2}\right)+2\left(\frac{-s_{234}+s_{34}-k_2^2}{s_{34}{s_{234}}^2}\right)
\end{aligned}$$

Evaluation parameters : $\{c_1 = 4\,d-4, c_2 = 2\,d-2, c_3 = 0, c_4 = -2\,d, c_5 = -4\,d+4\}$

$$\sum_{i=1}^{5} c_i D_i\left(\frac{1}{s_{34}s_{234}}\right) = 2d\left(-\frac{k_3^2-k_4^2}{{s_{34}}^2 s_{234}}-\frac{s_{23}-s_{34}+k_2^2-k_4^2}{s_{34}{s_{234}}^2}\right)-2\left(\frac{-s_{234}+s_{34}-k_2^2}{s_{34}{s_{234}}^2}\right)$$

$$\begin{aligned}
\left[\widehat{\mathbb{D}}+\sum_{i=1}^{5} c_i D_i\right]\left(\frac{1}{s_{34}s_{234}}\right) = & -\frac{1}{2}\left[4(d-2)-(n-6)\right]\left(\frac{-s_{1234}+s_{12}+s_{345}-k_5^2}{{s_{34}}^2 s_{234}}\right) \\
& -\frac{1}{2}\left[6(d-2)-(n-6)\right]\left(\frac{-s_{1234}+s_{2345}+k_1^2-k_5^2}{s_{34}{s_{234}}^2}\right)
\end{aligned}$$

$$\left[\widehat{\mathbb{D}}+\sum_{i=1}^{5} c_i D_i\right]\left(\frac{1}{s_{34}s_{234}}\right) = 0\ ,\ \text{when } n=6 \text{ and } d=2. \tag{D.19}$$

$N = 6$ Example

Example: $\widehat{\mathbb{D}}$ on $A_6^{\phi^3,\{[1,2],[1,3],[1,4]\}} = -1/(s_{12}s_{123}s_{1234}) = -1/\left(k^2_{[1,2]}k^2_{[1,3]}k^2_{[1,4]}\right)$

$$\widehat{\mathbb{D}}\left(-\frac{1}{s_{12}s_{123}s_{1234}}\right) = \frac{1}{2}\left[4(d-2)-(n-6)\right]\left(\frac{s_{123456}-s_{12}-s_{3456}}{{s_{12}}^2 s_{123}s_{1234}}\right)$$
$$+\frac{1}{2}\left[6(d-2)-(n-6)\right]\left(\frac{s_{123456}-s_{123}-s_{456}}{s_{12}{s_{123}}^2 s_{1234}}\right)$$
$$+\frac{1}{2}\left[8(d-2)-(n-6)\right]\left(\frac{s_{123456}-s_{1234}-s_{56}}{s_{12}s_{123}{s_{1234}}^2}\right)$$
$$+2d\left(-\frac{k_1^2-k_2^2}{{s_{12}}^2 s_{123}s_{1234}}-\frac{s_{12}-s_{23}+k_1^2-k_3^2}{s_{12}{s_{123}}^2 s_{1234}}\right.$$
$$\left.-\frac{s_{123}+s_{12}-s_{234}-s_{34}+k_1^2-k_4^2}{s_{12}s_{123}{s_{1234}}^2}\right)$$
$$+2\left(\frac{-s_{123}+s_{12}-k_3^2}{s_{12}{s_{123}}^2 s_{1234}}+\frac{-s_{1234}+s_{123}-k_4^2}{s_{12}s_{123}{s_{1234}}^2}+\frac{-s_{1234}+s_{12}-s_{34}}{s_{12}s_{123}{s_{1234}}^2}\right)$$

Evaluation parameters :

$$\{c_1 = 5\,d, c_2 = 3\,d, c_3 = d+2, c_4 = -d+4, c_5 = -3\,d+6, c_6 = -5\,d+6\}$$

$$\sum_{i=1}^{6} c_i D_i\left(-\frac{1}{s_{12}s_{123}s_{1234}}\right) =$$
$$= -\,2d\left(-\frac{k_1^2-k_2^2}{{s_{12}}^2 s_{123}s_{1234}}-\frac{s_{12}-s_{23}+k_1^2-k_3^2}{s_{12}{s_{123}}^2 s_{1234}}-\frac{s_{123}+s_{12}-s_{234}-s_{34}+k_1^2-k_4^2}{s_{12}s_{123}{s_{1234}}^2}\right)$$
$$-\,2\left(\frac{-s_{123}+s_{12}-k_3^2}{s_{12}{s_{123}}^2 s_{1234}}+\frac{-s_{1234}+s_{123}-k_4^2}{s_{12}s_{123}{s_{1234}}^2}+\frac{-s_{1234}+s_{12}-s_{34}}{s_{12}s_{123}{s_{1234}}^2}\right)$$

$$\left[\widehat{\mathbb{D}}+\sum_{i=1}^{6} c_i D_i\right]\left(-\frac{1}{s_{12}s_{123}s_{1234}}\right) = \frac{1}{2}\left[4(d-2)-(n-6)\right]\left(\frac{s_{123456}-s_{12}-s_{3456}}{{s_{12}}^2 s_{123}s_{1234}}\right)$$
$$+\frac{1}{2}\left[6(d-2)-(n-6)\right]\left(\frac{s_{123456}-s_{123}-s_{456}}{s_{12}{s_{123}}^2 s_{1234}}\right)$$
$$+\frac{1}{2}\left[8(d-2)-(n-6)\right]\left(\frac{s_{123456}-s_{1234}-s_{56}}{s_{12}s_{123}{s_{1234}}^2}\right)$$

$$\left[\widehat{\mathbb{D}}+\sum_{i=1}^{6} c_i D_i\right]\left(-\frac{1}{s_{12}s_{123}s_{1234}}\right) = 0 \text{ , when } n=6 \text{ and } d=2. \tag{D.20}$$

D.1.4 Action of Level-One Special Conformal Transformation Generators

$N = 4$ Examples

Example: $\widehat{\mathbb{K}}^\mu$ on $A_4^{\phi^3,\{[1,2]\}} = -1/s_{12} = -1/(k_1+k_2)^2$

$$\widehat{\mathbb{K}}^\mu\left(-\frac{1}{s_{12}}\right) = 4\,d\,[4(d-2)-(n-6)]\,\frac{k_1^\mu+k_2^\mu}{{s_{12}}^2} + 2\,d\,(n-6)\,\frac{k_1^\mu-k_2^\mu}{{s_{12}}^2} + 16\,d\,\frac{(k_1^2-k_2^2)(k_1^\mu+k_2^\mu)}{{s_{12}}^3}$$

Evaluation parameters : $\{c_1 = 3\,d, c_2 = d, c_3 = -d+2, c_4 = -3\,d+2\}$

$$\sum_{i=1}^{4} c_i K_i^\mu\left(-\frac{1}{s_{12}}\right) = 4\,d\,[4(d-2)-(n-6)]\,\frac{k_1^\mu+k_2^\mu}{{s_{12}}^2} - 2\,d\,(n-6)\,\frac{k_1^\mu-k_2^\mu}{{s_{12}}^2} - 16\,d\,\frac{(k_1^2-k_2^2)(k_1^\mu+k_2^\mu)}{{s_{12}}^3} \tag{D.21}$$

$$\left[\widehat{\mathbb{K}}^\mu + \sum_{i=1}^{4} c_i K_i^\mu\right]\left(-\frac{1}{s_{12}}\right) = 8\,d\,[4(d-2)-(n-6)]\,\frac{k_1^\mu+k_2^\mu}{{s_{12}}^2}$$

$$\left[\widehat{\mathbb{K}}^\mu + \sum_{i=1}^{4} c_i K_i^\mu\right]\left(-\frac{1}{s_{12}}\right) = 0\ ,\ \text{when } n=6 \text{ and } d=2.$$

Example: $\widehat{\mathbb{K}}^\mu$ on $A_4^{\phi^3,\{[2,3]\}} = -1/s_{23} = -1/(k_2+k_3)^2$

$$\widehat{\mathbb{K}}^\mu\left(-\frac{1}{s_{23}}\right) = 2\,d\,(n-6)\,\frac{k_2^\mu-k_3^\mu}{{s_{23}}^2} + 16\,d\,\frac{(k_2^2-k_3^2)(k_2^\mu+k_3^\mu)}{{s_{23}}^3}$$

Evaluation parameters : $\{c_1 = 3\,d-2, c_2 = d, c_3 = -d, c_4 = -3\,d+2\}$

$$\sum_{i=1}^{4} c_i K_i^\mu\left(-\frac{1}{s_{23}}\right) = -2\,d\,(n-6)\,\frac{k_2^\mu-k_3^\mu}{{s_{23}}^2} - 16\,d\,\frac{(k_2^2-k_3^2)(k_2^\mu+k_3^\mu)}{{s_{23}}^3} \tag{D.22}$$

$$\left[\widehat{\mathbb{K}}^\mu + \sum_{i=1}^{4} c_i K_i^\mu\right]\left(-\frac{1}{s_{23}}\right) = 0\ ,\ \text{for arbitrary } n \text{ and } d.$$

$N = 5$ Examples

Example: $\widehat{\mathbb{K}}^\mu$ on $A_5^{\phi^3,\{[1,2],[1,3]\}} = 1/(s_{12}s_{123}) = 1/\left((k_1+k_2)^2(k_1+k_2+k_3)^2\right)$

$$\widehat{\mathbb{K}}^\mu\left(\frac{1}{s_{12}s_{123}}\right) \neq 0$$

$$\text{Evaluation parameters}: \{c_1 = 4\,d, c_2 = 2\,d, c_3 = 2, c_4 = -2\,d+4, c_5 = -4\,d+4\} \tag{D.23}$$

$$\left[\widehat{\mathbb{K}}^\mu + \sum_{i=1}^{5} c_i K_i^\mu\right]\left(\frac{1}{s_{12}s_{123}}\right) = 0 \text{ , when } n = 6 \text{ and } d = 2.$$

Example: $\widehat{\mathbb{K}}^\mu$ on $A_5^{\phi^3,\{[1,2],[3,4]\}} = 1/(s_{12}s_{34}) = 1/\left((k_1+k_2)^2(k_3+k_4)^2\right)$

$$\widehat{\mathbb{K}}^\mu\left(\frac{1}{s_{12}s_{34}}\right) \neq 0$$

$$\text{Evaluation parameters}: \{c_1 = 4\,d-2, c_2 = 2\,d-2, c_3 = 2, c_4 = -2\,d+2, c_5 = -4\,d+4\}$$

$$\left[\widehat{\mathbb{K}}^\mu + \sum_{i=1}^{5} c_i K_i^\mu\right]\left(\frac{1}{s_{12}s_{34}}\right) = 0 \text{ , when } n = 6 \text{ and } d = 2. \tag{D.24}$$

Example: $\widehat{\mathbb{K}}^\mu$ on $A_5^{\phi^3,\{[2,3],[1,3]\}} = 1/(s_{23}s_{123}) = 1/\left((k_2+k_3)^2(k_1+k_2+k_3)^2\right)$

$$\widehat{\mathbb{K}}^\mu\left(\frac{1}{s_{23}s_{123}}\right) \neq 0$$

$$\text{Evaluation parameters}: \{c_1 = 4\,d-2, c_2 = 2\,d, c_3 = 0, c_4 = -2\,d+4, c_5 = -4\,d+4\}$$

$$\left[\widehat{\mathbb{K}}^\mu + \sum_{i=1}^{5} c_i K_i^\mu\right]\left(\frac{1}{s_{23}s_{123}}\right) = 0 \text{ , when } n = 6 \text{ and } d = 2. \tag{D.25}$$

Example: $\widehat{\mathbb{K}}^\mu$ on $A_5^{\phi^3,\{[2,3],[2,4]\}} = 1/(s_{23}s_{234}) = 1/\left((k_2+k_3)^2(k_2+k_3+k_4)^2\right)$

$$\widehat{\mathbb{K}}^\mu\left(\frac{1}{s_{23}s_{234}}\right) \neq 0$$

$$\text{Evaluation parameters}: \{c_1 = 4\,d-4, c_2 = 2\,d, c_3 = 0, c_4 = -2\,d+2, c_5 = -4\,d+4\}$$

$$\left[\widehat{\mathbb{K}}^\mu + \sum_{i=1}^{5} c_i K_i^\mu\right]\left(\frac{1}{s_{23}s_{234}}\right) = 0 \text{ , when } n = 6 \text{ and } d = 2. \tag{D.26}$$

Example: $\widehat{\mathbb{K}}^\mu$ on $A_5^{\phi^3,\{[3,4],[2,4]\}} = 1/(s_{34}s_{234}) = 1/\left((k_3+k_4)^2(k_2+k_3+k_4)^2\right)$

$$\widehat{\mathbb{K}}^\mu\left(\frac{1}{s_{34}s_{234}}\right) \neq 0$$

Evaluation parameters : $\{c_1 = 4\,d-4, c_2 = 2\,d-2, c_3 = 0, c_4 = -2\,d, c_5 = -4\,d+4\}$

$$\left[\widehat{\mathbb{K}}^\mu + \sum_{i=1}^{5} c_i K_i^\mu\right]\left(\frac{1}{s_{34}s_{234}}\right) = 0 \text{ , when } n = 6 \text{ and } d = 2. \tag{D.27}$$

<u>$N = 6$ Example</u>

Example: $\widehat{\mathbb{K}}^\mu$ on $A_6^{\phi^3,\{[1,2],[1,3],[1,4]\}} = -1/(s_{12}s_{123}s_{1234}) = -1/\left(k_{[1,2]}^2 k_{[1,3]}^2 k_{[1,4]}^2\right)$

$$\widehat{\mathbb{K}}^\mu\left(-\frac{1}{s_{12}s_{123}s_{1234}}\right) \neq 0$$

Evaluation parameters :

$$\{c_1 = 5\,d, c_2 = 3\,d, c_3 = d+2, c_4 = -d+4, c_5 = -3\,d+6, c_6 = -5\,d+6\} \tag{D.28}$$

$$\left[\widehat{\mathbb{K}}^\mu + \sum_{i=1}^{6} c_i K_i^\mu\right]\left(-\frac{1}{s_{12}s_{123}s_{1234}}\right) = 0 \text{ , when } n = 6 \text{ and } d = 2.$$

APPENDIX E

CHY SCATTERING EQUATIONS FORMALISM

E.1 Off-Shell Scattering Polynomials

E.1.1 Möbius Covariant Off-Shell Scattering Polynomials

The off-shell scattering polynomials can equivalently be defined as follows:

$$h_m^{N,\text{ off-shell}}(z,k) = \lim_{z_1\to\infty} \tilde{h}_{m+1}/z_1, \quad 1 \le m \le N-3\,, \tag{E.1}$$

where

$$\tilde{h}_m(z,k) = \sum_{\substack{1\le I<J<N \\ (I,J)\neq(1,N-1)}} k^2_{[I,J]}\,(z_I - z_{I-1})\,(z_J - z_{J+1})\,\Pi^{m-2}_{[I,J]^o}\,, \quad 2 \le m \le N-2 \tag{E.2}$$

are the Möbius covariant off-shell scattering polynomials.

E.1.2 Scattering Equations Solutions

Using the off-shell polynomials (6.3) as defined in [22]:

<u>4-point</u>

$$h_1^{4,\text{ off-shell}}(z,k) = (1-z_3)\,s_{12} - z_3\,s_{23} \tag{E.3}$$

The solution of the scattering equation $h_1^{4,\text{ off-shell}}(z,k) = 0$ is:

$$z_3 = \frac{s_{12}}{s_{12}+s_{23}} \tag{E.4}$$

<u>5-point</u>

$$\begin{aligned} h_1^{5,\text{ off-shell}}(z,k) &= (1-z_3)\,s_{12} + z_3 s_{123} - z_4 s_{123} - z_4 s_{234} + (z_4 - z_3)\,s_{23} \\ h_2^{5,\text{ off-shell}}(z,k) &= z_3 s_{123} + z_3 z_4 s_{34} - z_3 z_4 s_{234} - z_4 s_{34} - z_4 s_{123} + z_4\,(1-z_3)\,s_{12} \end{aligned} \tag{E.5}$$

The two simultaneous solutions of the scattering equations $h_1^{5,\text{ off-shell}}(z,k)=0$ and $h_2^{5,\text{ off-shell}}(z,k)=0$ are:

$$\begin{aligned}
z_3^{\pm} = & \frac{2s_{12}\left(s_{12}-s_{34}\right)+s_{12}\left(s_{23}-2s_{123}+s_{234}\right)-s_{23}s_{34}+s_{34}s_{123}-s_{123}s_{234}}{2\left(s_{12}+s_{23}-s_{123}\right)\left(s_{12}-s_{34}+s_{234}\right)} \\
& \pm \frac{\sqrt{\left(-s_{23}s_{34}+s_{123}\left(s_{34}-s_{234}\right)+s_{12}\left(s_{23}-2s_{123}-s_{234}\right)\right)^2-4s_{12}s_{123}\left(s_{12}-s_{34}+s_{234}\right)\left(-s_{23}+s_{123}+s_{234}\right)}}{2\left(s_{12}+s_{23}-s_{123}\right)\left(s_{12}-s_{34}+s_{234}\right)} \\
z_4^{\pm} = & \frac{s_{12}\left(s_{23}-2s_{123}-s_{234}\right)-s_{23}s_{34}+s_{34}s_{123}-s_{123}s_{234}}{2\left(s_{23}-s_{123}-s_{234}\right)\left(s_{12}-s_{34}+s_{234}\right)} \\
& \pm \frac{\sqrt{\left(-s_{23}s_{34}+s_{123}\left(s_{34}-s_{234}\right)+s_{12}\left(s_{23}-2s_{123}-s_{234}\right)\right)^2-4s_{12}s_{123}\left(s_{12}-s_{34}+s_{234}\right)\left(-s_{23}+s_{123}+s_{234}\right)}}{2\left(s_{23}-s_{123}-s_{234}\right)\left(s_{12}-s_{34}+s_{234}\right)}
\end{aligned} \tag{E.6}$$

E.2 On-Shell Scattering Polynomials

$$h_m^{N,\text{ on-shell}}(z,k)=\frac{1}{m!}\sum_{\substack{a_1,a_2,\ldots,a_m\in A' \\ a_i \text{ uneq.}}}\sigma_{a_1a_2\ldots a_m}z_{a_1}z_{a_2}\ldots z_{a_m}, \quad 1\le m\le N-3 \tag{E.7}$$

where $\sigma_{a_1a_2\ldots a_m}=k^2_{1a_1a_2\ldots a_m}=(k_1+k_{a_1}+k_{a_2}+\cdots+k_{a_m})^2\equiv s_{1a_1a_2\ldots a_m}$, $A'=\{a\in A : a\neq 1,N\}$, and $A=\{1,2,\ldots,N\}$.

E.2.1 Möbius Covariant On-Shell Scattering Polynomials

The on-shell scattering polynomials can equivalently be defined as follows:

$$h_m^{N,\text{ on-shell}}(z,k)=\lim_{z_1\to\infty}\hat{h}_{m+1}/z_1, \quad 1\le m\le N-3\;, \tag{E.8}$$

where

$$\hat{h}_m(z,k)=\sum_{\substack{S\subset A \\ |S|=m}}k_S^2 z_S, \quad 2\le m\le N-2 \tag{E.9}$$

are the Möbius covariant on-shell scattering polynomials and

$$k_S=\sum_{b\in S}k_b, \quad z_S=\prod_{a\in S}z_a, \quad S\subset A. \tag{E.10}$$

E.2.2 Scattering Equations Solutions

Using the original on-shell polynomials (E.7) as defined in [21]:

4-point

$$h_1^{4,\text{ on-shell}}(z,k) = s_{12} + s_{13}\, z_3 \tag{E.11}$$

The solution of the scattering equation $h_1^{4,\text{ on-shell}}(z,k) = 0$ is:

$$z_3 = -\frac{s_{12}}{s_{13}} \tag{E.12}$$

5-point

$$h_1^{5,\text{ on-shell}}(z,k) = s_{12} + s_{13}\, z_3 + s_{14}\, z_4 \tag{E.13}$$

$$h_2^{5,\text{ on-shell}}(z,k) = s_{123}\, z_3 + s_{124}\, z_4 + s_{134}\, z_3\, z_4 \tag{E.14}$$

The two simultaneous solutions of the scattering equations $h_1^{5,\text{ on-shell}}(z,k) = 0$ and $h_2^{5,\text{ on-shell}}(z,k) = 0$ are:

$$\begin{aligned} z_3^{\pm} &= -\frac{(s_{12}\, s_{134} + s_{13}\, s_{124} - s_{14}\, s_{123}) \pm \sqrt{(s_{12}\, s_{134} + s_{13}\, s_{124} - s_{14}\, s_{123})^2 - 4\, s_{12}\, s_{13}\, s_{124}\, s_{134}}}{2\, s_{13}\, s_{134}} \\ z_4^{\pm} &= \frac{-(s_{12}\, s_{134} - s_{13}\, s_{124} + s_{14}\, s_{123}) \pm \sqrt{(s_{12}\, s_{134} + s_{13}\, s_{124} - s_{14}\, s_{123})^2 - 4\, s_{12}\, s_{13}\, s_{124}\, s_{134}}}{2\, s_{14}\, s_{134}} \end{aligned} \tag{E.15}$$

E.3 Pure Yang-Mills Theory Partial Amplitudes from CHY Formalism

N-point

$$A_N^{\text{YM, partial}} = \oint_{\mathcal{O}} \frac{\Psi_N^{\text{YM}}}{H_N^{\text{on-shell}}} \frac{1}{z_{N-1}} \prod_{2 \le a < b \le N-1} (z_a - z_b) \prod_{a=2}^{N-2} \frac{z_a \, dz_{a+1}}{(z_a - z_{a+1})^2} \, . \tag{E.16}$$

From [18, 20], the $\Psi_N^{\text{YM}}(z;k;\epsilon)$ function, appearing in the integrand of the CHY formalism, for pure Yang-Mills theory is given by:

$$\Psi_N^{\text{YM}}(z;k;\epsilon) = \lim_{z_1 \to \infty} \left(z_1^2 \, \text{Pf}' \, \widetilde{\Psi}_N(z;k;\epsilon) \right) \prod_{a=2}^{N-1} (z_a - z_{a+1}) \ , \tag{E.17}$$

where

$$\text{Pf}' \, \widetilde{\Psi}_N(z;k;\epsilon) = 2 \frac{(-1)^{a+b}}{z_a - z_b} \, \text{Pf} \, \widetilde{\Psi}_N^{(a,b)}(z;k;\epsilon) \tag{E.18}$$

is the reduced Pfaffian of the matrix

$$\widetilde{\Psi}_N(z;k;\epsilon) = \begin{pmatrix} A & D \\ C & B \end{pmatrix} \ , \tag{E.19}$$

with elements

$$A_{ab} = \frac{k_a \cdot k_b}{z_a - z_b}, \quad B_{ab} = \frac{\epsilon_a \cdot \epsilon_b}{z_a - z_b}, \quad C_{ab} = \frac{\epsilon_a \cdot k_b}{z_a - z_b}, a \neq b, \quad 1 \le a, b \le N \tag{E.20}$$

$$A_{aa} = B_{aa} = 0, \quad C_{aa} = -\Sigma_a, \quad \Sigma_a = \sum_{\substack{c=1 \\ c \neq a}}^{N} \frac{\epsilon_a \cdot k_c}{z_a - z_c}, \quad 1 \le a \le N \tag{E.21}$$

$$D_{ab} = -C_{ba}, 1 \le a, b \le N \ . \tag{E.22}$$

4-point

$$A_4^{\text{YM, partial}} = \oint_{\mathcal{O}} \frac{dz_3}{h_1^{4,\text{ on-shell}}} \frac{1}{z_3\,(1-z_3)}\,\Psi_4^{\text{YM}} \tag{E.23}$$

$$\begin{aligned}
&\Psi_4^{\text{YM}}(z;k;\epsilon) \\
&\quad = (1-z_3)\,(\epsilon_1\cdot\epsilon_2)\,(k_2\cdot\epsilon_4)\,(k_4\cdot\epsilon_3) + (1-z_3)\,(\epsilon_1\cdot\epsilon_3)\,(k_3\cdot\epsilon_4)\,(k_4\cdot\epsilon_2) \\
&\qquad - (1-z_3)\,(\epsilon_1\cdot\epsilon_4)\,(k_4\cdot\epsilon_2)\,(k_4\cdot\epsilon_3) - (1-z_3)\,(\epsilon_2\cdot\epsilon_4)\,(k_2\cdot\epsilon_1)\,(k_4\cdot\epsilon_3) \\
&\qquad - (1-z_3)\,(\epsilon_3\cdot\epsilon_4)\,(k_3\cdot\epsilon_1)\,(k_4\cdot\epsilon_2) - z_3\,(\epsilon_1\cdot\epsilon_2)\,(k_2\cdot\epsilon_3)\,(k_2\cdot\epsilon_4) \\
&\qquad + z_3\,(\epsilon_1\cdot\epsilon_3)\,(k_2\cdot\epsilon_4)\,(k_3\cdot\epsilon_2) + z_3\,(\epsilon_1\cdot\epsilon_4)\,(k_2\cdot\epsilon_3)\,(k_4\cdot\epsilon_2) \\
&\qquad - z_3\,(\epsilon_2\cdot\epsilon_3)\,(k_2\cdot\epsilon_4)\,(k_3\cdot\epsilon_1) + z_3\,(\epsilon_2\cdot\epsilon_4)\,(k_2\cdot\epsilon_1)\,(k_2\cdot\epsilon_3) + z_3\,(\epsilon_2\cdot\epsilon_4)\,(k_2\cdot\epsilon_3)\,(k_3\cdot\epsilon_1) \\
&\qquad - z_3\,(k_2\cdot k_3)\,(\epsilon_1\cdot\epsilon_3)\,(\epsilon_2\cdot\epsilon_4) + \frac{z_3\,(k_2\cdot k_3)\,(\epsilon_1\cdot\epsilon_4)\,(\epsilon_2\cdot\epsilon_3)}{1-z_3} - (\epsilon_1\cdot\epsilon_2)\,(k_2\cdot\epsilon_3)\,(k_3\cdot\epsilon_4) \\
&\qquad + (k_2\cdot k_3)\,(\epsilon_3\cdot\epsilon_4)\,(\epsilon_1\cdot\epsilon_2) + (\epsilon_1\cdot\epsilon_3)\,(k_3\cdot\epsilon_2)\,(k_3\cdot\epsilon_4) - (\epsilon_1\cdot\epsilon_4)\,(k_3\cdot\epsilon_2)\,(k_4\cdot\epsilon_3) \\
&\qquad + (\epsilon_2\cdot\epsilon_3)\,(k_2\cdot\epsilon_1)\,(k_3\cdot\epsilon_4) - (\epsilon_3\cdot\epsilon_4)\,(k_2\cdot\epsilon_1)\,(k_3\cdot\epsilon_2) - (\epsilon_3\cdot\epsilon_4)\,(k_3\cdot\epsilon_1)\,(k_3\cdot\epsilon_2)
\end{aligned} \tag{E.24}$$

5-point

$$A_5^{\text{YM, partial}} = \oint_{\mathcal{O}} \frac{dz_3\,dz_4}{h_1^{5,\text{ on-shell}}\,h_2^{5,\text{ on-shell}}} \frac{z_3}{z_4} \frac{(1-z_4)}{(1-z_3)\,(z_3-z_4)}\,\Psi_5^{\text{YM}} \tag{E.25}$$

$$\begin{aligned}
&\Psi_5^{\text{YM}}(z;k;\epsilon) = -\frac{(k_2\cdot k_3)\,(k_4\cdot\epsilon_3)\,(\epsilon_1\cdot\epsilon_5)\,(\epsilon_2\cdot\epsilon_4)\,z_4}{2\,(1-z_4)} \\
&- \frac{1}{2}\,(k_2\cdot\epsilon_3)\,(k_3\cdot\epsilon_4)\,(k_4\cdot\epsilon_5)\,(\epsilon_1\cdot\epsilon_2) + \frac{1}{2}\,(k_2\cdot k_3)\,(k_4\cdot\epsilon_5)\,(\epsilon_3\cdot\epsilon_4)\,(\epsilon_1\cdot\epsilon_2) \\
&+ \frac{1}{2}\,(k_2\cdot\epsilon_3)\,(k_3\cdot k_4)\,(\epsilon_4\cdot\epsilon_5)\,(\epsilon_1\cdot\epsilon_2) - \frac{1}{2}\,(k_2\cdot k_3)\,(k_4\cdot\epsilon_3)\,(\epsilon_4\cdot\epsilon_5)\,(\epsilon_1\cdot\epsilon_2) \\
&- \frac{1}{2}\,(k_2\cdot\epsilon_5)\,(k_4\cdot\epsilon_3)\,(k_5\cdot\epsilon_4)\,(1-z_3)\,(\epsilon_1\cdot\epsilon_2) + \frac{1}{2}\,(k_2\cdot\epsilon_3)\,(k_2\cdot\epsilon_5)\,(k_5\cdot\epsilon_4)\,(z_3-z_4)\,(\epsilon_1\cdot\epsilon_2) \\
&+ \frac{(k_2\cdot\epsilon_3)\,(k_3\cdot\epsilon_5)\,(k_5\cdot\epsilon_4)\,(z_3-z_4)\,(\epsilon_1\cdot\epsilon_2)}{2z_3} - \frac{(k_2\cdot k_3)\,(k_5\cdot\epsilon_4)\,(\epsilon_3\cdot\epsilon_5)\,(z_3-z_4)\,(\epsilon_1\cdot\epsilon_2)}{2z_3} \\
&- \frac{(k_2\cdot\epsilon_5)\,(k_5\cdot\epsilon_3)\,(k_5\cdot\epsilon_4)\,(1-z_3)\,(z_3-z_4)\,(\epsilon_1\cdot\epsilon_2)}{2z_3} + \frac{(k_2\cdot\epsilon_4)\,(k_4\cdot\epsilon_5)\,(k_5\cdot\epsilon_3)\,(1-z_3)\,(z_3-z_4)\,(\epsilon_1\cdot\epsilon_2)}{2z_3\,(1-z_4)} \\
&- \frac{(k_2\cdot k_4)\,(k_5\cdot\epsilon_3)\,(\epsilon_4\cdot\epsilon_5)\,(1-z_3)\,(z_3-z_4)\,(\epsilon_1\cdot\epsilon_2)}{2z_3\,(1-z_4)} - \frac{1}{2}\,(k_2\cdot\epsilon_3)\,(k_2\cdot\epsilon_5)\,(k_3\cdot\epsilon_4)\,z_4\,(\epsilon_1\cdot\epsilon_2) \\
&+ \frac{(k_2\cdot\epsilon_4)\,(k_2\cdot\epsilon_5)\,(k_5\cdot\epsilon_3)\,(1-z_3)\,(z_3-z_4)\,z_4\,(\epsilon_1\cdot\epsilon_2)}{2z_3\,(1-z_4)} - \frac{(k_2\cdot\epsilon_3)\,(k_3\cdot\epsilon_4)\,(k_3\cdot\epsilon_5)\,z_4\,(\epsilon_1\cdot\epsilon_2)}{2z_3} \\
&+ \frac{(k_2\cdot k_3)\,(k_3\cdot\epsilon_4)\,(\epsilon_3\cdot\epsilon_5)\,z_4\,(\epsilon_1\cdot\epsilon_2)}{2z_3} + \frac{(k_2\cdot\epsilon_5)\,(k_3\cdot\epsilon_4)\,(k_5\cdot\epsilon_3)\,(1-z_3)\,z_4\,(\epsilon_1\cdot\epsilon_2)}{2z_3}
\end{aligned}$$

$$+\frac{(k_2\cdot\epsilon_4)(k_2\cdot\epsilon_5)(k_4\cdot\epsilon_3)(1-z_3)z_4(\epsilon_1\cdot\epsilon_2)}{2(1-z_4)}+\frac{(k_2\cdot\epsilon_4)(k_3\cdot\epsilon_5)(k_4\cdot\epsilon_3)(1-z_3)z_4(\epsilon_1\cdot\epsilon_2)}{2z_3(1-z_4)}$$
$$-\frac{(k_2\cdot k_4)(k_3\cdot\epsilon_5)(\epsilon_3\cdot\epsilon_4)(1-z_3)z_4(\epsilon_1\cdot\epsilon_2)}{2z_3(1-z_4)}-\frac{(k_2\cdot\epsilon_4)(k_3\cdot k_4)(\epsilon_3\cdot\epsilon_5)(1-z_3)z_4(\epsilon_1\cdot\epsilon_2)}{2z_3(1-z_4)}$$
$$+\frac{(k_2\cdot k_4)(k_3\cdot\epsilon_4)(\epsilon_3\cdot\epsilon_5)(1-z_3)z_4(\epsilon_1\cdot\epsilon_2)}{2z_3(1-z_4)}+\frac{(k_2\cdot\epsilon_5)(k_3\cdot k_4)(\epsilon_3\cdot\epsilon_4)(1-z_3)z_4(\epsilon_1\cdot\epsilon_2)}{2(z_3-z_4)}$$
$$+\frac{(k_2\cdot\epsilon_3)(k_2\cdot\epsilon_4)(k_2\cdot\epsilon_5)z_4(z_4-z_3)(\epsilon_1\cdot\epsilon_2)}{2(1-z_4)}+\frac{(k_2\cdot\epsilon_3)(k_2\cdot\epsilon_4)(k_3\cdot\epsilon_5)z_4(z_4-z_3)(\epsilon_1\cdot\epsilon_2)}{2z_3(1-z_4)}$$
$$-\frac{(k_2\cdot k_3)(k_2\cdot\epsilon_4)(\epsilon_3\cdot\epsilon_5)z_4(z_4-z_3)(\epsilon_1\cdot\epsilon_2)}{2z_3(1-z_4)}+\frac{(k_2\cdot\epsilon_3)(k_2\cdot\epsilon_4)(k_4\cdot\epsilon_5)(z_4-z_3)(\epsilon_1\cdot\epsilon_2)}{2(1-z_4)}$$
$$-\frac{(k_2\cdot k_4)(k_2\cdot\epsilon_3)(\epsilon_4\cdot\epsilon_5)(z_4-z_3)(\epsilon_1\cdot\epsilon_2)}{2(1-z_4)}+\frac{(k_2\cdot\epsilon_4)(k_4\cdot\epsilon_3)(k_4\cdot\epsilon_5)(1-z_3)(\epsilon_1\cdot\epsilon_2)}{2(1-z_4)}$$
$$-\frac{(k_2\cdot k_4)(k_4\cdot\epsilon_3)(\epsilon_4\cdot\epsilon_5)(1-z_3)(\epsilon_1\cdot\epsilon_2)}{2(1-z_4)}+\frac{1}{2}(k_3\cdot\epsilon_2)(k_3\cdot\epsilon_4)(k_4\cdot\epsilon_5)(\epsilon_1\cdot\epsilon_3)$$
$$-\frac{1}{2}(k_3\cdot\epsilon_2)(k_4\cdot\epsilon_3)(k_4\cdot\epsilon_5)(\epsilon_1\cdot\epsilon_4)+\frac{1}{2}(k_3\cdot\epsilon_2)(k_4\cdot\epsilon_3)(k_5\cdot\epsilon_4)(\epsilon_1\cdot\epsilon_5)$$
$$+\frac{1}{2}(k_2\cdot\epsilon_1)(k_3\cdot\epsilon_4)(k_4\cdot\epsilon_5)(\epsilon_2\cdot\epsilon_3)-\frac{1}{2}(k_2\cdot\epsilon_1)(k_3\cdot\epsilon_2)(k_4\cdot\epsilon_5)(\epsilon_3\cdot\epsilon_4)$$
$$-\frac{1}{2}(k_3\cdot\epsilon_1)(k_3\cdot\epsilon_2)(k_4\cdot\epsilon_5)(\epsilon_3\cdot\epsilon_4)+\frac{1}{2}(k_2\cdot\epsilon_1)(k_3\cdot\epsilon_2)(k_4\cdot\epsilon_3)(\epsilon_4\cdot\epsilon_5)$$
$$+\frac{1}{2}(k_3\cdot\epsilon_1)(k_3\cdot\epsilon_2)(k_4\cdot\epsilon_3)(\epsilon_4\cdot\epsilon_5)+\frac{1}{2}(k_3\cdot\epsilon_2)(k_4\cdot\epsilon_1)(k_4\cdot\epsilon_3)(\epsilon_4\cdot\epsilon_5)$$
$$-\frac{1}{2}(k_3\cdot k_4)(k_3\cdot\epsilon_2)(\epsilon_1\cdot\epsilon_3)(\epsilon_4\cdot\epsilon_5)-\frac{1}{2}(k_2\cdot\epsilon_1)(k_3\cdot k_4)(\epsilon_2\cdot\epsilon_3)(\epsilon_4\cdot\epsilon_5)$$
$$+\frac{1}{2}(k_3\cdot\epsilon_4)(k_4\cdot\epsilon_5)(k_5\cdot\epsilon_2)(\epsilon_1\cdot\epsilon_3)(1-z_3)-\frac{1}{2}(k_4\cdot\epsilon_3)(k_4\cdot\epsilon_5)(k_5\cdot\epsilon_2)(\epsilon_1\cdot\epsilon_4)(1-z_3)$$
$$+\frac{1}{2}(k_4\cdot\epsilon_3)(k_5\cdot\epsilon_2)(k_5\cdot\epsilon_4)(\epsilon_1\cdot\epsilon_5)(1-z_3)+\frac{1}{2}(k_2\cdot\epsilon_1)(k_4\cdot\epsilon_3)(k_5\cdot\epsilon_4)(\epsilon_2\cdot\epsilon_5)(1-z_3)$$
$$-\frac{1}{2}(k_3\cdot\epsilon_1)(k_4\cdot\epsilon_5)(k_5\cdot\epsilon_2)(\epsilon_3\cdot\epsilon_4)(1-z_3)+\frac{1}{2}(k_3\cdot\epsilon_1)(k_4\cdot\epsilon_3)(k_5\cdot\epsilon_2)(\epsilon_4\cdot\epsilon_5)(1-z_3)$$
$$+\frac{1}{2}(k_4\cdot\epsilon_1)(k_4\cdot\epsilon_3)(k_5\cdot\epsilon_2)(\epsilon_4\cdot\epsilon_5)(1-z_3)-\frac{1}{2}(k_3\cdot k_4)(k_5\cdot\epsilon_2)(\epsilon_1\cdot\epsilon_3)(\epsilon_4\cdot\epsilon_5)(1-z_3)$$
$$-\frac{1}{2}(k_2\cdot\epsilon_5)(k_3\cdot\epsilon_2)(k_5\cdot\epsilon_4)(\epsilon_1\cdot\epsilon_3)(z_3-z_4)+\frac{1}{2}(k_2\cdot\epsilon_3)(k_4\cdot\epsilon_5)(k_5\cdot\epsilon_2)(\epsilon_1\cdot\epsilon_4)(z_3-z_4)$$
$$-\frac{1}{2}(k_2\cdot\epsilon_3)(k_5\cdot\epsilon_2)(k_5\cdot\epsilon_4)(\epsilon_1\cdot\epsilon_5)(z_3-z_4)+\frac{1}{2}(k_2\cdot\epsilon_5)(k_3\cdot\epsilon_1)(k_5\cdot\epsilon_4)(\epsilon_2\cdot\epsilon_3)(z_3-z_4)$$
$$-\frac{1}{2}(k_2\cdot\epsilon_1)(k_2\cdot\epsilon_3)(k_5\cdot\epsilon_4)(\epsilon_2\cdot\epsilon_5)(z_3-z_4)-\frac{1}{2}(k_2\cdot\epsilon_3)(k_3\cdot\epsilon_1)(k_5\cdot\epsilon_4)(\epsilon_2\cdot\epsilon_5)(z_3-z_4)$$
$$+\frac{1}{2}(k_2\cdot k_3)(k_5\cdot\epsilon_4)(\epsilon_1\cdot\epsilon_3)(\epsilon_2\cdot\epsilon_5)(z_3-z_4)-\frac{1}{2}(k_2\cdot\epsilon_3)(k_4\cdot\epsilon_1)(k_5\cdot\epsilon_2)(\epsilon_4\cdot\epsilon_5)(z_3-z_4)$$
$$+\frac{(k_2\cdot k_3)(k_4\cdot\epsilon_5)(\epsilon_1\cdot\epsilon_4)(\epsilon_2\cdot\epsilon_3)(z_3-z_4)}{2(1-z_3)}-\frac{(k_2\cdot k_3)(k_5\cdot\epsilon_4)(\epsilon_1\cdot\epsilon_5)(\epsilon_2\cdot\epsilon_3)(z_3-z_4)}{2(1-z_3)}$$
$$-\frac{(k_2\cdot k_3)(k_4\cdot\epsilon_1)(\epsilon_2\cdot\epsilon_3)(\epsilon_4\cdot\epsilon_5)(z_3-z_4)}{2(1-z_3)}-\frac{(k_3\cdot\epsilon_2)(k_3\cdot\epsilon_5)(k_5\cdot\epsilon_4)(\epsilon_1\cdot\epsilon_3)(z_3-z_4)}{2z_3}$$
$$-\frac{(k_3\cdot\epsilon_2)(k_4\cdot\epsilon_5)(k_5\cdot\epsilon_3)(\epsilon_1\cdot\epsilon_4)(z_3-z_4)}{2z_3}+\frac{(k_3\cdot\epsilon_2)(k_5\cdot\epsilon_3)(k_5\cdot\epsilon_4)(\epsilon_1\cdot\epsilon_5)(z_3-z_4)}{2z_3}$$
$$-\frac{(k_2\cdot\epsilon_1)(k_3\cdot\epsilon_5)(k_5\cdot\epsilon_4)(\epsilon_2\cdot\epsilon_3)(z_3-z_4)}{2z_3}+\frac{(k_2\cdot\epsilon_1)(k_3\cdot\epsilon_2)(k_5\cdot\epsilon_4)(\epsilon_3\cdot\epsilon_5)(z_3-z_4)}{2z_3}$$
$$+\frac{(k_3\cdot\epsilon_1)(k_3\cdot\epsilon_2)(k_5\cdot\epsilon_4)(\epsilon_3\cdot\epsilon_5)(z_3-z_4)}{2z_3}+\frac{(k_3\cdot\epsilon_2)(k_4\cdot\epsilon_1)(k_5\cdot\epsilon_3)(\epsilon_4\cdot\epsilon_5)(z_3-z_4)}{2z_3}$$
$$-\frac{(k_3\cdot\epsilon_5)(k_5\cdot\epsilon_2)(k_5\cdot\epsilon_4)(\epsilon_1\cdot\epsilon_3)(1-z_3)(z_3-z_4)}{2z_3}-\frac{(k_4\cdot\epsilon_5)(k_5\cdot\epsilon_2)(k_5\cdot\epsilon_3)(\epsilon_1\cdot\epsilon_4)(1-z_3)(z_3-z_4)}{2z_3}$$
$$+\frac{(k_5\cdot\epsilon_2)(k_5\cdot\epsilon_3)(k_5\cdot\epsilon_4)(\epsilon_1\cdot\epsilon_5)(1-z_3)(z_3-z_4)}{2z_3}+\frac{(k_2\cdot\epsilon_1)(k_5\cdot\epsilon_3)(k_5\cdot\epsilon_4)(\epsilon_2\cdot\epsilon_5)(1-z_3)(z_3-z_4)}{2z_3}$$
$$+\frac{(k_3\cdot\epsilon_1)(k_5\cdot\epsilon_2)(k_5\cdot\epsilon_4)(\epsilon_3\cdot\epsilon_5)(1-z_3)(z_3-z_4)}{2z_3}+\frac{(k_4\cdot\epsilon_1)(k_5\cdot\epsilon_2)(k_5\cdot\epsilon_3)(\epsilon_4\cdot\epsilon_5)(1-z_3)(z_3-z_4)}{2z_3}$$

$$
\begin{aligned}
&-\frac{(k_3\cdot\epsilon_5)(k_4\cdot\epsilon_2)(k_5\cdot\epsilon_4)(\epsilon_1\cdot\epsilon_3)(1-z_3)(z_3-z_4)}{2z_3(1-z_4)}-\frac{(k_4\cdot\epsilon_2)(k_4\cdot\epsilon_5)(k_5\cdot\epsilon_3)(\epsilon_1\cdot\epsilon_4)(1-z_3)(z_3-z_4)}{2z_3(1-z_4)}\\
&+\frac{(k_4\cdot\epsilon_2)(k_5\cdot\epsilon_3)(k_5\cdot\epsilon_4)(\epsilon_1\cdot\epsilon_5)(1-z_3)(z_3-z_4)}{2z_3(1-z_4)}-\frac{(k_2\cdot\epsilon_1)(k_4\cdot\epsilon_5)(k_5\cdot\epsilon_3)(\epsilon_2\cdot\epsilon_4)(1-z_3)(z_3-z_4)}{2z_3(1-z_4)}\\
&+\frac{(k_3\cdot\epsilon_1)(k_4\cdot\epsilon_2)(k_5\cdot\epsilon_4)(\epsilon_3\cdot\epsilon_5)(1-z_3)(z_3-z_4)}{2z_3(1-z_4)}+\frac{(k_2\cdot\epsilon_1)(k_4\cdot\epsilon_2)(k_5\cdot\epsilon_3)(\epsilon_4\cdot\epsilon_5)(1-z_3)(z_3-z_4)}{2z_3(1-z_4)}\\
&+\frac{(k_4\cdot\epsilon_1)(k_4\cdot\epsilon_2)(k_5\cdot\epsilon_3)(\epsilon_4\cdot\epsilon_5)(1-z_3)(z_3-z_4)}{2z_3(1-z_4)}+\frac{1}{2}(k_2\cdot\epsilon_5)(k_3\cdot\epsilon_2)(k_3\cdot\epsilon_4)(\epsilon_1\cdot\epsilon_3)z_4\\
&-\frac{1}{2}(k_2\cdot\epsilon_5)(k_3\cdot\epsilon_2)(k_4\cdot\epsilon_3)(\epsilon_1\cdot\epsilon_4)z_4+\frac{1}{2}(k_2\cdot\epsilon_3)(k_3\cdot\epsilon_4)(k_5\cdot\epsilon_2)(\epsilon_1\cdot\epsilon_5)z_4\\
&-\frac{1}{2}(k_2\cdot\epsilon_5)(k_3\cdot\epsilon_1)(k_3\cdot\epsilon_4)(\epsilon_2\cdot\epsilon_3)z_4-\frac{1}{2}(k_2\cdot\epsilon_5)(k_3\cdot\epsilon_4)(k_4\cdot\epsilon_1)(\epsilon_2\cdot\epsilon_3)z_4\\
&+\frac{1}{2}(k_2\cdot\epsilon_5)(k_3\cdot k_4)(\epsilon_1\cdot\epsilon_4)(\epsilon_2\cdot\epsilon_3)z_4+\frac{1}{2}(k_2\cdot\epsilon_1)(k_2\cdot\epsilon_3)(k_3\cdot\epsilon_4)(\epsilon_2\cdot\epsilon_5)z_4\\
&+\frac{1}{2}(k_2\cdot\epsilon_3)(k_3\cdot\epsilon_1)(k_3\cdot\epsilon_4)(\epsilon_2\cdot\epsilon_5)z_4+\frac{1}{2}(k_2\cdot\epsilon_3)(k_3\cdot\epsilon_4)(k_4\cdot\epsilon_1)(\epsilon_2\cdot\epsilon_5)z_4\\
&-\frac{1}{2}(k_2\cdot k_3)(k_3\cdot\epsilon_4)(\epsilon_1\cdot\epsilon_3)(\epsilon_2\cdot\epsilon_5)z_4-\frac{1}{2}(k_2\cdot\epsilon_3)(k_3\cdot k_4)(\epsilon_1\cdot\epsilon_4)(\epsilon_2\cdot\epsilon_5)z_4\\
&+\frac{1}{2}(k_2\cdot k_3)(k_4\cdot\epsilon_3)(\epsilon_1\cdot\epsilon_4)(\epsilon_2\cdot\epsilon_5)z_4+\frac{1}{2}(k_2\cdot\epsilon_5)(k_3\cdot\epsilon_2)(k_4\cdot\epsilon_1)(\epsilon_3\cdot\epsilon_4)z_4\\
&-\frac{1}{2}(k_2\cdot k_3)(k_4\cdot\epsilon_1)(\epsilon_2\cdot\epsilon_5)(\epsilon_3\cdot\epsilon_4)z_4+\frac{(k_2\cdot k_3)(k_2\cdot\epsilon_4)(\epsilon_1\cdot\epsilon_5)(\epsilon_2\cdot\epsilon_3)(z_3-z_4)z_4}{2(1-z_3)(1-z_4)}\\
&+\frac{(k_2\cdot\epsilon_4)(k_3\cdot\epsilon_5)(k_5\cdot\epsilon_2)(\epsilon_1\cdot\epsilon_3)(1-z_3)(z_3-z_4)z_4}{2z_3(1-z_4)}-\frac{(k_2\cdot\epsilon_5)(k_4\cdot\epsilon_2)(k_5\cdot\epsilon_3)(\epsilon_1\cdot\epsilon_4)(1-z_3)(z_3-z_4)z_4}{2z_3(1-z_4)}\\
&-\frac{(k_2\cdot\epsilon_4)(k_5\cdot\epsilon_2)(k_5\cdot\epsilon_3)(\epsilon_1\cdot\epsilon_5)(1-z_3)(z_3-z_4)z_4}{2z_3(1-z_4)}+\frac{(k_2\cdot\epsilon_5)(k_4\cdot\epsilon_1)(k_5\cdot\epsilon_3)(\epsilon_2\cdot\epsilon_4)(1-z_3)(z_3-z_4)z_4}{2z_3(1-z_4)}\\
&-\frac{(k_2\cdot\epsilon_1)(k_2\cdot\epsilon_4)(k_5\cdot\epsilon_3)(\epsilon_2\cdot\epsilon_5)(1-z_3)(z_3-z_4)z_4}{2z_3(1-z_4)}-\frac{(k_2\cdot\epsilon_4)(k_4\cdot\epsilon_1)(k_5\cdot\epsilon_3)(\epsilon_2\cdot\epsilon_5)(1-z_3)(z_3-z_4)z_4}{2z_3(1-z_4)}\\
&+\frac{(k_2\cdot k_4)(k_5\cdot\epsilon_3)(\epsilon_1\cdot\epsilon_4)(\epsilon_2\cdot\epsilon_5)(1-z_3)(z_3-z_4)z_4}{2z_3(1-z_4)}-\frac{(k_2\cdot\epsilon_4)(k_3\cdot\epsilon_1)(k_5\cdot\epsilon_2)(\epsilon_3\cdot\epsilon_5)(1-z_3)(z_3-z_4)z_4}{2z_3(1-z_4)}\\
&+\frac{(k_2\cdot k_4)(k_3\cdot\epsilon_5)(\epsilon_1\cdot\epsilon_3)(\epsilon_2\cdot\epsilon_4)(1-z_3)(z_3-z_4)z_4}{2z_3(1-z_4)^2}-\frac{(k_2\cdot k_4)(k_5\cdot\epsilon_3)(\epsilon_1\cdot\epsilon_5)(\epsilon_2\cdot\epsilon_4)(1-z_3)(z_3-z_4)z_4}{2z_3(1-z_4)^2}\\
&-\frac{(k_2\cdot k_4)(k_3\cdot\epsilon_1)(\epsilon_2\cdot\epsilon_4)(\epsilon_3\cdot\epsilon_5)(1-z_3)(z_3-z_4)z_4}{2z_3(1-z_4)^2}+\frac{(k_2\cdot k_3)(k_3\cdot\epsilon_4)(\epsilon_1\cdot\epsilon_5)(\epsilon_2\cdot\epsilon_3)z_4}{2(1-z_3)}\\
&+\frac{(k_3\cdot\epsilon_2)(k_3\cdot\epsilon_4)(k_3\cdot\epsilon_5)(\epsilon_1\cdot\epsilon_3)z_4}{2z_3}-\frac{(k_3\cdot\epsilon_2)(k_3\cdot\epsilon_5)(k_4\cdot\epsilon_3)(\epsilon_1\cdot\epsilon_4)z_4}{2z_3}\\
&-\frac{(k_3\cdot\epsilon_2)(k_3\cdot\epsilon_4)(k_5\cdot\epsilon_3)(\epsilon_1\cdot\epsilon_5)z_4}{2z_3}+\frac{(k_2\cdot\epsilon_1)(k_3\cdot\epsilon_4)(k_3\cdot\epsilon_5)(\epsilon_2\cdot\epsilon_3)z_4}{2z_3}\\
&+\frac{(k_3\cdot\epsilon_2)(k_3\cdot\epsilon_5)(k_4\cdot\epsilon_1)(\epsilon_3\cdot\epsilon_4)z_4}{2z_3}-\frac{(k_2\cdot\epsilon_1)(k_3\cdot\epsilon_2)(k_3\cdot\epsilon_4)(\epsilon_3\cdot\epsilon_5)z_4}{2z_3}\\
&-\frac{(k_3\cdot\epsilon_1)(k_3\cdot\epsilon_2)(k_3\cdot\epsilon_4)(\epsilon_3\cdot\epsilon_5)z_4}{2z_3}-\frac{(k_3\cdot\epsilon_2)(k_3\cdot\epsilon_4)(k_4\cdot\epsilon_1)(\epsilon_3\cdot\epsilon_5)z_4}{2z_3}\\
&+\frac{(k_3\cdot k_4)(k_3\cdot\epsilon_2)(\epsilon_1\cdot\epsilon_4)(\epsilon_3\cdot\epsilon_5)z_4}{2z_3}+\frac{(k_3\cdot\epsilon_4)(k_3\cdot\epsilon_5)(k_5\cdot\epsilon_2)(\epsilon_1\cdot\epsilon_3)(1-z_3)z_4}{2z_3}\\
&-\frac{(k_3\cdot\epsilon_5)(k_4\cdot\epsilon_3)(k_5\cdot\epsilon_2)(\epsilon_1\cdot\epsilon_4)(1-z_3)z_4}{2z_3}-\frac{(k_3\cdot\epsilon_4)(k_5\cdot\epsilon_2)(k_5\cdot\epsilon_3)(\epsilon_1\cdot\epsilon_5)(1-z_3)z_4}{2z_3}\\
&-\frac{(k_2\cdot\epsilon_1)(k_3\cdot\epsilon_4)(k_5\cdot\epsilon_3)(\epsilon_2\cdot\epsilon_5)(1-z_3)z_4}{2z_3}+\frac{(k_3\cdot\epsilon_5)(k_4\cdot\epsilon_1)(k_5\cdot\epsilon_2)(\epsilon_3\cdot\epsilon_4)(1-z_3)z_4}{2z_3}\\
&-\frac{(k_3\cdot\epsilon_1)(k_3\cdot\epsilon_4)(k_5\cdot\epsilon_2)(\epsilon_3\cdot\epsilon_5)(1-z_3)z_4}{2z_3}-\frac{(k_3\cdot\epsilon_4)(k_4\cdot\epsilon_1)(k_5\cdot\epsilon_2)(\epsilon_3\cdot\epsilon_5)(1-z_3)z_4}{2z_3}\\
&+\frac{(k_3\cdot k_4)(k_5\cdot\epsilon_2)(\epsilon_1\cdot\epsilon_4)(\epsilon_3\cdot\epsilon_5)(1-z_3)z_4}{2z_3}-\frac{(k_2\cdot\epsilon_4)(k_3\cdot k_4)(\epsilon_1\cdot\epsilon_5)(\epsilon_2\cdot\epsilon_3)z_4}{2(1-z_4)}\\
&+\frac{(k_2\cdot k_4)(k_3\cdot\epsilon_4)(\epsilon_1\cdot\epsilon_5)(\epsilon_2\cdot\epsilon_3)z_4}{2(1-z_4)}+\frac{(k_2\cdot\epsilon_3)(k_3\cdot k_4)(\epsilon_1\cdot\epsilon_5)(\epsilon_2\cdot\epsilon_4)z_4}{2(1-z_4)}
\end{aligned}
$$

$$- \frac{(k_2\cdot k_4)(k_3\cdot\epsilon_2)(\epsilon_1\cdot\epsilon_5)(\epsilon_3\cdot\epsilon_4)\,z_4}{2(1-z_4)} + \frac{(k_2\cdot k_3)(k_4\cdot\epsilon_2)(\epsilon_1\cdot\epsilon_5)(\epsilon_3\cdot\epsilon_4)\,z_4}{2(1-z_4)}$$

$$+ \frac{(k_2\cdot\epsilon_5)(k_3\cdot\epsilon_4)(k_4\cdot\epsilon_2)(\epsilon_1\cdot\epsilon_3)(1-z_3)\,z_4}{2(1-z_4)} - \frac{(k_2\cdot\epsilon_5)(k_4\cdot\epsilon_2)(k_4\cdot\epsilon_3)(\epsilon_1\cdot\epsilon_4)(1-z_3)\,z_4}{2(1-z_4)}$$

$$- \frac{(k_2\cdot\epsilon_4)(k_4\cdot\epsilon_3)(k_5\cdot\epsilon_2)(\epsilon_1\cdot\epsilon_5)(1-z_3)\,z_4}{2(1-z_4)} + \frac{(k_2\cdot\epsilon_5)(k_3\cdot\epsilon_1)(k_4\cdot\epsilon_3)(\epsilon_2\cdot\epsilon_4)(1-z_3)\,z_4}{2(1-z_4)}$$

$$+ \frac{(k_2\cdot\epsilon_5)(k_4\cdot\epsilon_1)(k_4\cdot\epsilon_3)(\epsilon_2\cdot\epsilon_4)(1-z_3)\,z_4}{2(1-z_4)} - \frac{(k_2\cdot\epsilon_5)(k_3\cdot k_4)(\epsilon_1\cdot\epsilon_3)(\epsilon_2\cdot\epsilon_4)(1-z_3)\,z_4}{2(1-z_4)}$$

$$- \frac{(k_2\cdot\epsilon_1)(k_2\cdot\epsilon_4)(k_4\cdot\epsilon_3)(\epsilon_2\cdot\epsilon_5)(1-z_3)\,z_4}{2(1-z_4)} - \frac{(k_2\cdot\epsilon_4)(k_3\cdot\epsilon_1)(k_4\cdot\epsilon_3)(\epsilon_2\cdot\epsilon_5)(1-z_3)\,z_4}{2(1-z_4)}$$

$$- \frac{(k_2\cdot\epsilon_4)(k_4\cdot\epsilon_1)(k_4\cdot\epsilon_3)(\epsilon_2\cdot\epsilon_5)(1-z_3)\,z_4}{2(1-z_4)} + \frac{(k_2\cdot\epsilon_4)(k_3\cdot k_4)(\epsilon_1\cdot\epsilon_3)(\epsilon_2\cdot\epsilon_5)(1-z_3)\,z_4}{2(1-z_4)}$$

$$- \frac{(k_2\cdot k_4)(k_3\cdot\epsilon_4)(\epsilon_1\cdot\epsilon_3)(\epsilon_2\cdot\epsilon_5)(1-z_3)\,z_4}{2(1-z_4)} + \frac{(k_2\cdot k_4)(k_4\cdot\epsilon_3)(\epsilon_1\cdot\epsilon_4)(\epsilon_2\cdot\epsilon_5)(1-z_3)\,z_4}{2(1-z_4)}$$

$$- \frac{(k_2\cdot\epsilon_5)(k_3\cdot\epsilon_1)(k_4\cdot\epsilon_2)(\epsilon_3\cdot\epsilon_4)(1-z_3)\,z_4}{2(1-z_4)} + \frac{(k_2\cdot k_4)(k_3\cdot\epsilon_1)(\epsilon_2\cdot\epsilon_5)(\epsilon_3\cdot\epsilon_4)(1-z_3)\,z_4}{2(1-z_4)}$$

$$+ \frac{(k_3\cdot\epsilon_4)(k_3\cdot\epsilon_5)(k_4\cdot\epsilon_2)(\epsilon_1\cdot\epsilon_3)(1-z_3)\,z_4}{2z_3(1-z_4)} - \frac{(k_3\cdot\epsilon_5)(k_4\cdot\epsilon_2)(k_4\cdot\epsilon_3)(\epsilon_1\cdot\epsilon_4)(1-z_3)\,z_4}{2z_3(1-z_4)}$$

$$- \frac{(k_3\cdot\epsilon_4)(k_4\cdot\epsilon_2)(k_5\cdot\epsilon_3)(\epsilon_1\cdot\epsilon_5)(1-z_3)\,z_4}{2z_3(1-z_4)} - \frac{(k_2\cdot\epsilon_1)(k_3\cdot\epsilon_5)(k_4\cdot\epsilon_3)(\epsilon_2\cdot\epsilon_4)(1-z_3)\,z_4}{2z_3(1-z_4)}$$

$$+ \frac{(k_2\cdot\epsilon_1)(k_3\cdot\epsilon_5)(k_4\cdot\epsilon_2)(\epsilon_3\cdot\epsilon_4)(1-z_3)\,z_4}{2z_3(1-z_4)} + \frac{(k_3\cdot\epsilon_5)(k_4\cdot\epsilon_1)(k_4\cdot\epsilon_2)(\epsilon_3\cdot\epsilon_4)(1-z_3)\,z_4}{2z_3(1-z_4)}$$

$$- \frac{(k_2\cdot\epsilon_1)(k_3\cdot\epsilon_4)(k_4\cdot\epsilon_2)(\epsilon_3\cdot\epsilon_5)(1-z_3)\,z_4}{2z_3(1-z_4)} - \frac{(k_3\cdot\epsilon_1)(k_3\cdot\epsilon_4)(k_4\cdot\epsilon_2)(\epsilon_3\cdot\epsilon_5)(1-z_3)\,z_4}{2z_3(1-z_4)}$$

$$- \frac{(k_3\cdot\epsilon_4)(k_4\cdot\epsilon_1)(k_4\cdot\epsilon_2)(\epsilon_3\cdot\epsilon_5)(1-z_3)\,z_4}{2z_3(1-z_4)} + \frac{(k_3\cdot k_4)(k_4\cdot\epsilon_2)(\epsilon_1\cdot\epsilon_4)(\epsilon_3\cdot\epsilon_5)(1-z_3)\,z_4}{2z_3(1-z_4)}$$

$$+ \frac{(k_2\cdot\epsilon_1)(k_3\cdot k_4)(\epsilon_2\cdot\epsilon_4)(\epsilon_3\cdot\epsilon_5)(1-z_3)\,z_4}{2z_3(1-z_4)} - \frac{(k_3\cdot k_4)(k_3\cdot\epsilon_2)(\epsilon_1\cdot\epsilon_5)(\epsilon_3\cdot\epsilon_4)\,z_4}{2(z_3-z_4)}$$

$$- \frac{(k_3\cdot k_4)(k_5\cdot\epsilon_2)(\epsilon_1\cdot\epsilon_5)(\epsilon_3\cdot\epsilon_4)(1-z_3)\,z_4}{2(z_3-z_4)} - \frac{(k_2\cdot\epsilon_1)(k_3\cdot k_4)(\epsilon_2\cdot\epsilon_5)(\epsilon_3\cdot\epsilon_4)(1-z_3)\,z_4}{2(z_3-z_4)}$$

$$- \frac{(k_3\cdot k_4)(k_4\cdot\epsilon_2)(\epsilon_1\cdot\epsilon_5)(\epsilon_3\cdot\epsilon_4)(1-z_3)\,z_4}{2(1-z_4)(z_3-z_4)} - \frac{(k_2\cdot k_4)(k_4\cdot\epsilon_3)(\epsilon_1\cdot\epsilon_5)(\epsilon_2\cdot\epsilon_4)(1-z_3)\,z_4}{2(1-z_4)^2}$$

$$- \frac{(k_2\cdot\epsilon_4)(k_2\cdot\epsilon_5)(k_3\cdot\epsilon_2)(\epsilon_1\cdot\epsilon_3)\,z_4(z_4-z_3)}{2(1-z_4)} - \frac{(k_2\cdot\epsilon_3)(k_2\cdot\epsilon_5)(k_4\cdot\epsilon_2)(\epsilon_1\cdot\epsilon_4)\,z_4(z_4-z_3)}{2(1-z_4)}$$

$$- \frac{(k_2\cdot\epsilon_3)(k_2\cdot\epsilon_4)(k_5\cdot\epsilon_2)(\epsilon_1\cdot\epsilon_5)\,z_4(z_4-z_3)}{2(1-z_4)} + \frac{(k_2\cdot\epsilon_4)(k_2\cdot\epsilon_5)(k_3\cdot\epsilon_1)(\epsilon_2\cdot\epsilon_3)\,z_4(z_4-z_3)}{2(1-z_4)}$$

$$- \frac{(k_2\cdot\epsilon_1)(k_2\cdot\epsilon_3)(k_2\cdot\epsilon_4)(\epsilon_2\cdot\epsilon_5)\,z_4(z_4-z_3)}{2(1-z_4)} - \frac{(k_2\cdot\epsilon_3)(k_2\cdot\epsilon_4)(k_3\cdot\epsilon_1)(\epsilon_2\cdot\epsilon_5)\,z_4(z_4-z_3)}{2(1-z_4)}$$

$$- \frac{(k_2\cdot\epsilon_3)(k_2\cdot\epsilon_4)(k_4\cdot\epsilon_1)(\epsilon_2\cdot\epsilon_5)\,z_4(z_4-z_3)}{2(1-z_4)} + \frac{(k_2\cdot k_3)(k_2\cdot\epsilon_4)(\epsilon_1\cdot\epsilon_3)(\epsilon_2\cdot\epsilon_5)\,z_4(z_4-z_3)}{2(1-z_4)}$$

$$+ \frac{(k_2\cdot k_4)(k_2\cdot\epsilon_3)(\epsilon_1\cdot\epsilon_4)(\epsilon_2\cdot\epsilon_5)\,z_4(z_4-z_3)}{2(1-z_4)} - \frac{(k_2\cdot\epsilon_4)(k_3\cdot\epsilon_2)(k_3\cdot\epsilon_5)(\epsilon_1\cdot\epsilon_3)\,z_4(z_4-z_3)}{2z_3(1-z_4)}$$

$$- \frac{(k_2\cdot\epsilon_3)(k_3\cdot\epsilon_5)(k_4\cdot\epsilon_2)(\epsilon_1\cdot\epsilon_4)\,z_4(z_4-z_3)}{2z_3(1-z_4)} + \frac{(k_2\cdot\epsilon_4)(k_3\cdot\epsilon_2)(k_5\cdot\epsilon_3)(\epsilon_1\cdot\epsilon_5)\,z_4(z_4-z_3)}{2z_3(1-z_4)}$$

$$- \frac{(k_2\cdot\epsilon_1)(k_2\cdot\epsilon_4)(k_3\cdot\epsilon_5)(\epsilon_2\cdot\epsilon_3)\,z_4(z_4-z_3)}{2z_3(1-z_4)} - \frac{(k_2\cdot\epsilon_4)(k_3\cdot\epsilon_5)(k_4\cdot\epsilon_1)(\epsilon_2\cdot\epsilon_3)\,z_4(z_4-z_3)}{2z_3(1-z_4)}$$

$$+ \frac{(k_2\cdot k_4)(k_3\cdot\epsilon_5)(\epsilon_1\cdot\epsilon_4)(\epsilon_2\cdot\epsilon_3)\,z_4(z_4-z_3)}{2z_3(1-z_4)} + \frac{(k_2\cdot\epsilon_3)(k_3\cdot\epsilon_5)(k_4\cdot\epsilon_1)(\epsilon_2\cdot\epsilon_4)\,z_4(z_4-z_3)}{2z_3(1-z_4)}$$

$$+ \frac{(k_2\cdot\epsilon_1)(k_2\cdot\epsilon_4)(k_3\cdot\epsilon_2)(\epsilon_3\cdot\epsilon_5)\,z_4(z_4-z_3)}{2z_3(1-z_4)} + \frac{(k_2\cdot\epsilon_4)(k_3\cdot\epsilon_1)(k_3\cdot\epsilon_2)(\epsilon_3\cdot\epsilon_5)\,z_4(z_4-z_3)}{2z_3(1-z_4)}$$

$$+ \frac{(k_2\cdot\epsilon_4)(k_3\cdot\epsilon_2)(k_4\cdot\epsilon_1)(\epsilon_3\cdot\epsilon_5)\,z_4(z_4-z_3)}{2z_3(1-z_4)} - \frac{(k_2\cdot k_4)(k_3\cdot\epsilon_2)(\epsilon_1\cdot\epsilon_4)(\epsilon_3\cdot\epsilon_5)\,z_4(z_4-z_3)}{2z_3(1-z_4)}$$

$$
\begin{aligned}
&+\frac{(k_2\cdot k_3)\,(k_4\cdot \epsilon_2)\,(\epsilon_1\cdot \epsilon_4)\,(\epsilon_3\cdot \epsilon_5)\,z_4\,(z_4-z_3)}{2z_3\,(1-z_4)}-\frac{(k_2\cdot k_3)\,(k_4\cdot \epsilon_1)\,(\epsilon_2\cdot \epsilon_4)\,(\epsilon_3\cdot \epsilon_5)\,z_4\,(z_4-z_3)}{2z_3\,(1-z_4)}\\
&-\frac{(k_2\cdot k_4)\,(k_2\cdot \epsilon_3)\,(\epsilon_1\cdot \epsilon_5)\,(\epsilon_2\cdot \epsilon_4)\,z_4\,(z_4-z_3)}{2\,(1-z_4)^{\,2}}-\frac{(k_2\cdot \epsilon_4)\,(k_3\cdot \epsilon_2)\,(k_4\cdot \epsilon_5)\,(\epsilon_1\cdot \epsilon_3)\,(z_4-z_3)}{2\,(1-z_4)}\\
&-\frac{(k_2\cdot \epsilon_3)\,(k_4\cdot \epsilon_2)\,(k_4\cdot \epsilon_5)\,(\epsilon_1\cdot \epsilon_4)\,(z_4-z_3)}{2\,(1-z_4)}+\frac{(k_2\cdot \epsilon_3)\,(k_4\cdot \epsilon_2)\,(k_5\cdot \epsilon_4)\,(\epsilon_1\cdot \epsilon_5)\,(z_4-z_3)}{2\,(1-z_4)}\\
&+\frac{(k_2\cdot \epsilon_4)\,(k_3\cdot \epsilon_1)\,(k_4\cdot \epsilon_5)\,(\epsilon_2\cdot \epsilon_3)\,(z_4-z_3)}{2\,(1-z_4)}-\frac{(k_2\cdot \epsilon_1)\,(k_2\cdot \epsilon_3)\,(k_4\cdot \epsilon_5)\,(\epsilon_2\cdot \epsilon_4)\,(z_4-z_3)}{2\,(1-z_4)}\\
&-\frac{(k_2\cdot \epsilon_3)\,(k_3\cdot \epsilon_1)\,(k_4\cdot \epsilon_5)\,(\epsilon_2\cdot \epsilon_4)\,(z_4-z_3)}{2\,(1-z_4)}+\frac{(k_2\cdot k_3)\,(k_4\cdot \epsilon_5)\,(\epsilon_1\cdot \epsilon_3)\,(\epsilon_2\cdot \epsilon_4)\,(z_4-z_3)}{2\,(1-z_4)}\\
&+\frac{(k_2\cdot \epsilon_1)\,(k_2\cdot \epsilon_3)\,(k_4\cdot \epsilon_2)\,(\epsilon_4\cdot \epsilon_5)\,(z_4-z_3)}{2\,(1-z_4)}+\frac{(k_2\cdot \epsilon_3)\,(k_3\cdot \epsilon_1)\,(k_4\cdot \epsilon_2)\,(\epsilon_4\cdot \epsilon_5)\,(z_4-z_3)}{2\,(1-z_4)}\\
&+\frac{(k_2\cdot \epsilon_3)\,(k_4\cdot \epsilon_1)\,(k_4\cdot \epsilon_2)\,(\epsilon_4\cdot \epsilon_5)\,(z_4-z_3)}{2\,(1-z_4)}+\frac{(k_2\cdot k_4)\,(k_3\cdot \epsilon_2)\,(\epsilon_1\cdot \epsilon_3)\,(\epsilon_4\cdot \epsilon_5)\,(z_4-z_3)}{2\,(1-z_4)}\\
&-\frac{(k_2\cdot k_3)\,(k_4\cdot \epsilon_2)\,(\epsilon_1\cdot \epsilon_3)\,(\epsilon_4\cdot \epsilon_5)\,(z_4-z_3)}{2\,(1-z_4)}-\frac{(k_2\cdot k_4)\,(k_3\cdot \epsilon_1)\,(\epsilon_2\cdot \epsilon_3)\,(\epsilon_4\cdot \epsilon_5)\,(z_4-z_3)}{2\,(1-z_4)}\\
&+\frac{(k_3\cdot \epsilon_4)\,(k_4\cdot \epsilon_2)\,(k_4\cdot \epsilon_5)\,(\epsilon_1\cdot \epsilon_3)\,(1-z_3)}{2\,(1-z_4)}-\frac{(k_4\cdot \epsilon_2)\,(k_4\cdot \epsilon_3)\,(k_4\cdot \epsilon_5)\,(\epsilon_1\cdot \epsilon_4)\,(1-z_3)}{2\,(1-z_4)}\\
&+\frac{(k_4\cdot \epsilon_2)\,(k_4\cdot \epsilon_3)\,(k_5\cdot \epsilon_4)\,(\epsilon_1\cdot \epsilon_5)\,(1-z_3)}{2\,(1-z_4)}-\frac{(k_2\cdot \epsilon_1)\,(k_4\cdot \epsilon_3)\,(k_4\cdot \epsilon_5)\,(\epsilon_2\cdot \epsilon_4)\,(1-z_3)}{2\,(1-z_4)}\\
&-\frac{(k_3\cdot \epsilon_1)\,(k_4\cdot \epsilon_2)\,(k_4\cdot \epsilon_5)\,(\epsilon_3\cdot \epsilon_4)\,(1-z_3)}{2\,(1-z_4)}+\frac{(k_2\cdot \epsilon_1)\,(k_4\cdot \epsilon_2)\,(k_4\cdot \epsilon_3)\,(\epsilon_4\cdot \epsilon_5)\,(1-z_3)}{2\,(1-z_4)}\\
&+\frac{(k_3\cdot \epsilon_1)\,(k_4\cdot \epsilon_2)\,(k_4\cdot \epsilon_3)\,(\epsilon_4\cdot \epsilon_5)\,(1-z_3)}{2\,(1-z_4)}+\frac{(k_4\cdot \epsilon_1)\,(k_4\cdot \epsilon_2)\,(k_4\cdot \epsilon_3)\,(\epsilon_4\cdot \epsilon_5)\,(1-z_3)}{2\,(1-z_4)}\\
&-\frac{(k_3\cdot k_4)\,(k_4\cdot \epsilon_2)\,(\epsilon_1\cdot \epsilon_3)\,(\epsilon_4\cdot \epsilon_5)\,(1-z_3)}{2\,(1-z_4)}+\frac{(k_2\cdot \epsilon_3)\,(k_2\cdot \epsilon_5)\,(k_4\cdot \epsilon_1)\,(\epsilon_2\cdot \epsilon_4)\,z_4\,(z_4-z_3)}{2\,(1-z_4)}
\end{aligned}
\tag{E.26}
$$

APPENDIX F

ACTION OF CONFORMAL YANGIAN $Y[\mathbf{SO}(2,n)]$ GENERATORS ON THE SCATTERING EQUATIONS FORMALISM : ADDITIONAL EXAMPLES

The following results have been checked computationally using the MATHEMATICA packages we have created.

F.1 Level-Zero Generators on Scattering Polynomials

F.1.1 Action of Level-Zero Translation Generators

$$\begin{aligned} \mathbb{P}^{\mu}\left[h_m^N\right] &= \left(\sum_{i=1}^{N} k_i{}^{\mu}\right)\left[h_m^N\right] \qquad \mathbb{P}^{\mu}\left[\frac{1}{h_m^N}\right] = \left(\sum_{i=1}^{N} k_i{}^{\mu}\right)\left[\frac{1}{h_m^N}\right] \\ \mathbb{P}^{\mu}\left[H_N\right] &= \left(\sum_{i=1}^{N} k_i{}^{\mu}\right)\left[H_N\right] \qquad \mathbb{P}^{\mu}\left[\frac{1}{H_N}\right] = \left(\sum_{i=1}^{N} k_i{}^{\mu}\right)\left[\frac{1}{H_N}\right] \end{aligned} \tag{F.1}$$

$$\mathbb{P}^{\mu}\left[\delta^n\left(\textstyle\sum_{j=1}^N k_j\right)\, \frac{1}{H_N}\right] = \left(\sum_{i=1}^{N} k_i{}^{\mu}\right)\left[\delta^n\left(\textstyle\sum_{j=1}^N k_j\right)\, \frac{1}{H_N}\right] = 0 \tag{F.2}$$

F.1.2 Action of Level-Zero Lorentz Transformation Generators

$$\begin{aligned} \mathbb{L}^{\mu\nu}\left[h_m^N\right] &= 0 \qquad \mathbb{L}^{\mu\nu}\left[\frac{1}{h_m^N}\right] = 0 \\ \mathbb{L}^{\mu\nu}\left[H_N\right] &= 0 \qquad \mathbb{L}^{\mu\nu}\left[\frac{1}{H_N}\right] = 0 \end{aligned} \tag{F.3}$$

$$\mathbb{L}^{\mu\nu}\left[\delta^n\left(\textstyle\sum_{j=1}^N k_j\right)\, \frac{1}{H_N}\right] = 0 \tag{F.4}$$

F.1.3 Action of Level-Zero Dilatation Generator

$$\begin{aligned} \mathbb{D}\left[h_m^N\right] &= (N\,d+2)\left[h_m^N\right] \qquad \mathbb{D}\left[\frac{1}{h_m^N}\right] = (N\,d-2)\left[\frac{1}{h_m^N}\right] \\ \mathbb{D}\left[H_N\right] &= \left[N\,d+2\,(N-3)\right]\left[H_N\right] \qquad \mathbb{D}\left[\frac{1}{H_N}\right] = \left[N\,d-2\,(N-3)\right]\left[\frac{1}{H_N}\right] \end{aligned} \tag{F.5}$$

$$\mathbb{D}\left[\delta^n\left(\textstyle\sum_{j=1}^N k_j\right)\, \frac{1}{H_N}\right] = \left[N\,(d-2)-(n-6)\right]\left[\delta^n\left(\textstyle\sum_{j=1}^N k_j\right)\, \frac{1}{H_N}\right] = 0\,, \tag{F.6}$$

for $n = 6$ and $d = 2$.

F.1.4 Action of Level-Zero Special Conformal Transformation Generators

On a Single On-Shell Scattering Polynomial

$$\begin{aligned}\mathbb{K}^\mu\, h_m^{N,\text{ on-shell}} &= \sum_{S\subset A}\left[\mathbb{K}^\mu\,(k_S)^2\right]\left[\frac{\partial h_m^{N,\text{ on-shell}}}{\partial(k_S)^2}\right]\\ &= \sum_{S\subset A}\left[4\,d\,(|S|-1)\;k_S^\mu\right]\left[\frac{\partial h_m^{N,\text{ on-shell}}}{\partial(k_S)^2}\right]\end{aligned} \tag{F.7}$$

where $k_S \equiv \sum_{i\in S} k_i$, $k_S^\mu = \sum_{i\in S} k_i^\mu$ and $(k_S)^2 = \left(\sum_{i\in S} k_i\right)^2 = \sum_{\substack{i,j\in S\\ i\neq j}} k_i\cdot k_j$, on-shell.

On a Single Off-Shell Scattering Polynomial

$$\begin{aligned}\mathbb{K}^\mu\, h_m^{N,\text{ off-shell}} &= \sum_{S\subset A}\left[\mathbb{K}^\mu\,(k_S)^2\right]\left[\frac{\partial h_m^{N,\text{ off-shell}}}{\partial(k_S)^2}\right]\\ &= \sum_{S\subset A}\left[2\,(2\,d\,|S|-n+2)\,k_S^\mu\right]\left[\frac{\partial h_m^{N,\text{ off-shell}}}{\partial(k_S)^2}\right]\end{aligned} \tag{F.8}$$

where $k_S \equiv \sum_{i\in S} k_i$, $k_S^\mu = \sum_{i\in S} k_i^\mu$ and $(k_S)^2 = \left(\sum_{i\in S} k_i\right)^2$, off-shell.

Now note that from (6.3) we have:

$$\begin{aligned}\frac{\partial h_m^{N,\text{ off-shell}}}{\partial(k_S)^2} &= \sum_{J=2}^{N-2}\delta_{S,[1,J]}\,(z_J-z_{J+1})\,\Pi^{m-1}_{[1,J]^o} - \sum_{J=3}^{N-1}\delta_{S,[2,J]}\,(z_J-z_{J+1})\,\Pi^{m-1}_{[2,J]^o}\\ &\quad + \sum_{3\le I<J<N}\delta_{S,[I,J]}\,(z_I-z_{I-1})\,(z_J-z_{J+1})\,\Pi^{m-2}_{[I,J]'^o}\end{aligned} \tag{F.9}$$

and thus,

$$\begin{aligned}\mathbb{K}^\mu\, h_m^{N,\text{ off-shell}} &= \sum_{S\subset A}\left[2\,(2\,d\,|S|-n+2)\,k_S^\mu\right]\left[\frac{\partial h_m^{N,\text{ off-shell}}}{\partial(k_S)^2}\right]\\ &= \sum_{J=2}^{N-2} 2\,(2\,d\,J-n+2)\,k^\mu_{[1,J]}\,(z_J-z_{J+1})\,\Pi^{m-1}_{[1,J]^o}\\ &\quad - \sum_{J=3}^{N-1} 2\,(2\,d\,(J-1)-n+2)\,k^\mu_{[2,J]}\,(z_J-z_{J+1})\,\Pi^{m-1}_{[2,J]^o}\\ &\quad + \sum_{3\le I<J<N} 2\,(2\,d\,(J-I+1)-n+2)\,k^\mu_{[I,J]}\,(z_I-z_{I-1})\,(z_J-z_{J+1})\,\Pi^{m-2}_{[I,J]'^o}\end{aligned} \tag{F.10}$$

Similarly, for the Möbius covariant form of the off-shell polynomials (E.2), we have:

$$\frac{\partial \tilde{h}_m}{\partial (k_S)^2} = \sum_{\substack{1\le I<J<N \\ (I,J)\neq(1,N-1)}} \delta_{S,[I,J]}\,(z_I - z_{I-1})\,(z_J - z_{J+1})\,\Pi^{m-2}_{[I,J]^o} \tag{F.11}$$

and thus,

$$\begin{aligned}\mathbb{K}^\mu\,\tilde{h}_m &= \sum_{S\subset A}\left[2\,(2\,d\,|S| - n + 2)\,k_S^\mu\right]\left[\frac{\partial \tilde{h}_m}{\partial (k_S)^2}\right] \\ &= \sum_{\substack{1\le I<J<N \\ (I,J)\neq(1,N-1)}} 2\,(2\,d\,(J-I+1) - n + 2)\,k^\mu_{[I,J]}\,(z_I - z_{I-1})\,(z_J - z_{J+1})\,\Pi^{m-2}_{[I,J]^o}\end{aligned} \tag{F.12}$$

On the Inverse of an Off-Shell Scattering Polynomial

We propose the following rewriting of the action of $\mathbb{K}^\mu$ on the inverse of an off-shell scattering polynomial, which makes the vanishing of some of the terms for $d=2$ and $n=6$ manifest:

$$\begin{aligned}&\mathbb{K}^\mu\,\left[h_m^{N,\text{ off-shell}}\right]^{-1} \\ &\quad= -2\,\left[h_m^{N,\text{ off-shell}}\right]^{-2}\sum_{F\subset A}\left[2\,(d-2)\,|F| - (n-6)\right]\,k_F^\mu\left[\frac{\partial h_m^{N,\text{ off-shell}}}{\partial (k_F)^2}\right] \\ &\quad\quad -8\,\left[h_m^{N,\text{ off-shell}}\right]^{-2}\sum_{F\subset A}\left(|F|-2\right)\,k_F^\mu\left[\frac{\partial h_m^{N,\text{ off-shell}}}{\partial (k_F)^2}\right] \\ &\quad\quad -4\,\left[h_m^{N,\text{ off-shell}}\right]^{-3}\sum_{\substack{F,F'\subset A \\ F\neq F',\,F\cap F'\neq\emptyset}}\Big[k^2_{(F\setminus F')}\,k^\mu_{(F'\setminus F)} + k^2_{(F'\setminus F)}\,k^\mu_{(F\setminus F')} \\ &\quad\quad\quad - k^2_{(F\cap F')}\,k^\mu_{(F\cup F')} + k^2_{(F\cup F')}\,k^\mu_{(F\cap F')}\Big]\left[\frac{\partial h_m^{N,\text{ off-shell}}}{\partial (k_F)^2}\right]\left[\frac{\partial h_m^{N,\text{ off-shell}}}{\partial (k_{F'})^2}\right]\end{aligned} \tag{F.13}$$

Example: $\mathbb{K}^\mu$ on $\left[h_1^{4,\text{ off-shell}}\right]^{-1}$

$$\begin{aligned}\mathbb{K}^\mu\,\left[h_1^{4,\text{ off-shell}}\right]^{-1} &= -\frac{2\,(4\,d-n-2)\,\left[(1-z_3)\,(k_1^\mu + k_2^\mu) - z_3\,(k_2^\mu + k_3^\mu)\right]}{\left(h_1^{4,\text{ off-shell}}\right)^2} \\ &\quad - \frac{8(z_3-1)z_3\left(\left({k_3}^2 - {k_2}^2\right)k_1^\mu + \left({k_4}^2 - {k_2}^2\right)k_2^\mu + \left({k_1}^2 - {k_2}^2\right)k_3^\mu\right)}{\left(h_1^{4,\text{ off-shell}}\right)^3}\end{aligned} \tag{F.14}$$

Example: $\mathbb{K}^\mu$ on $\left[h_1^{5,\text{ off-shell}}\right]^{-1}$ and $\left[h_2^{5,\text{ off-shell}}\right]^{-1}$

$$\begin{aligned}\mathbb{K}^\mu \left[h_1^{5,\text{ off-shell}}\right]^{-1} &= -2\,(4\,d-n-2)\,\frac{(1-z_3)(k_1+k_2)^\mu-(z_3-z_4)(k_2+k_3)^\mu}{\left(h_1^{5,\text{ off-shell}}\right)^2}\\
&-2\,(6\,d-n-6)\,\frac{(z_3-z_4)(k_1+k_2+k_3)^\mu-z_4(k_2+k_3+k_4)^\mu}{\left(h_1^{5,\text{ off-shell}}\right)^2}\\
&+8\,z_4\,\frac{k_1^\mu+k_2^\mu+k_3^\mu+k_4^\mu}{\left(h_1^{5,\text{ off-shell}}\right)^2}-8\,\frac{\left(h_2^{5,\text{ off-shell}}\right)\,k_1^\mu}{\left(h_1^{5,\text{ off-shell}}\right)^3}\\
&+8\,\frac{\left(z_3(z_3-1)k_2^2-(z_3-z_4)\left((z_3-1)k_3^2-z_4k_4^2\right)\right)k_1^\mu}{\left(h_1^{5,\text{ off-shell}}\right)^3}\\
&+8\,\frac{\left(z_3(z_3-1)k_2^2-z_4(z_4-1)k_5^2\right)k_2^\mu}{\left(h_1^{5,\text{ off-shell}}\right)^3}+8\,\frac{z_4\left((z_3-1)k_2^2-(z_4-1)k_1^2\right)k_4^\mu}{\left(h_1^{5,\text{ off-shell}}\right)^3}\\
&+8\,\frac{\left(-z_3(z_3-1)k_1^2+z_3(z_3-1)k_2^2+z_4(z_3-z_4)k_5^2\right)k_3^\mu}{\left(h_1^{5,\text{ off-shell}}\right)^3}\end{aligned}\tag{F.15}$$

$$\begin{aligned}\mathbb{K}^\mu \left[h_2^{5,\text{ off-shell}}\right]^{-1} &= -2\,(4\,d-n-2)\,\frac{z_4(1-z_3)\left((k_1+k_2)^\mu-(k_3+k_4)^\mu\right)}{\left(h_2^{5,\text{ off-shell}}\right)^2}\\
&-2\,(6\,d-n-6)\,\frac{(z_3-z_4)(k_1+k_2+k_3)^\mu-z_3z_4(k_2+k_3+k_4)^\mu}{\left(h_2^{5,\text{ off-shell}}\right)^2}\\
&-8\,\frac{z_3k_1^\mu-z_3(z_4-1)k_2^\mu+(z_3-z_4)k_3^\mu}{\left(h_2^{5,\text{ off-shell}}\right)^2}+8\,z_3\,z_4\,\frac{\left(h_1^{5,\text{ off-shell}}\right)(k_1^\mu+k_2^\mu+k_3^\mu+k_4^\mu)}{\left(h_2^{5,\text{ off-shell}}\right)^3}\\
&+8\,\frac{z_4\left(z_3z_4(z_3-1)k_2^2+(z_3-z_4)\left((z_3-1)k_3^2+k_4^2\right)\right)k_1^\mu}{\left(h_2^{5,\text{ off-shell}}\right)^3}\\
&+8\,\frac{z_4\left((z_3-1)\left(z_3z_4k_2^2+(z_3-z_4)\left(k_3^2-k_4^2\right)\right)-z_3^2(z_4-1)k_5^2\right)k_2^\mu}{\left(h_2^{5,\text{ off-shell}}\right)^3}\\
&+8\,\frac{z_4\left(z_3z_4(z_3-1)\left(k_2^2-k_1^2\right)+(z_3-z_4)\left((z_3-1)k_3^2+k_5^2\right)\right)k_3^\mu}{\left(h_2^{5,\text{ off-shell}}\right)^3}\\
&+8\,\frac{z_4\left((z_3-1)\left(z_3z_4k_2^2+(z_3-z_4)k_3^2\right)-z_3^2(z_4-1)k_1^2\right)k_4^\mu}{\left(h_2^{5,\text{ off-shell}}\right)^3}\end{aligned}\tag{F.16}$$

For the Möbius covariant form of the off-shell polynomials, using (F.11), we can also write:

$$\begin{aligned}
&\mathbb{K}^{\mu}\,\left[\tilde{h}_m\right]^{-1} \\
&=-2\left[\tilde{h}_m\right]^{-2}\sum_{\substack{1\leq I<J<N\\(I,J)\neq(1,N-1)}}\left[2\,(d-2)\,(J-I+1)-(n-6)\right]\,k^{\mu}_{[I,J]}\,\left(z_I-z_{I-1}\right)\left(z_J-z_{J+1}\right)\Pi^{m-2}_{[I,J]^o} \\
&\quad-8\left[\tilde{h}_m\right]^{-2}\sum_{\substack{1\leq I<J<N\\(I,J)\neq(1,N-1)}}\left(J-I-1\right)\,k^{\mu}_{[I,J]}\,\left(z_I-z_{I-1}\right)\left(z_J-z_{J+1}\right)\Pi^{m-2}_{[I,J]^o} \\
&\quad-4\left[\tilde{h}_m\right]^{-3}\sum_{\substack{1\leq I<J<N\\(I,J)\neq(1,N-1)}}\;\sum_{\substack{1\leq I'<J'<N\\(I',J')\neq(1,N-1)}}\left(1-\delta_{[I,J]\cap[I',J'],\emptyset}\right)\left(1-\delta_{[I,J],[I',J']}\right) \\
&\quad\times\Bigg[k^2_{([I,J]\backslash[I',J'])}\,k^{\mu}_{([I',J']\backslash[I,J])}+k^2_{([I',J']\backslash[I,J])}\,k^{\mu}_{([I,J]\backslash[I',J'])} \\
&\qquad-k^2_{([I,J]\cap[I',J'])}\,k^{\mu}_{([I,J]\cup[I',J'])}+k^2_{([I,J]\cup[I',J'])}\,k^{\mu}_{([I,J]\cap[I',J'])}\Bigg] \\
&\quad\times\left[\left(z_I-z_{I-1}\right)\left(z_J-z_{J+1}\right)\left(z_{I'}-z_{I'-1}\right)\left(z_{J'}-z_{J'+1}\right)\Pi^{m-2}_{[I,J]^o}\,\Pi^{m-2}_{[I',J']^o}\right]
\end{aligned} \tag{F.17}$$

On the Product of Off-Shell Scattering Polynomials

We further propose the following rewriting of the action of $\mathbb{K}^\mu$ on the product of all $N-3$ off-shell scattering polynomials, $H_N^{\text{off-shell}} \equiv \prod_{m=1}^{N-3} h_m^{N,\,\text{off-shell}}$, appearing in the denominator of the N-point CHY integrand for any theory:

$$\begin{aligned}
\mathbb{K}^\mu\, H_N^{\text{off-shell}} &= \mathbb{K}^\mu \left[\prod_{m=1}^{N-3} h_m^{N,\,\text{off-shell}}\right] \\
&= \sum_{\rho\in\mathfrak{C}_{N-3}} \left[\prod_{j=1}^{N-4} h_{\rho(j)}^{N,\,\text{off-shell}}\right]\left[\mathbb{K}^\mu\, h_{\rho(N-3)}^{N,\,\text{off-shell}}\right] \\
&\quad + \sum_{\rho\in\mathfrak{S}_{N-3}} \left[\prod_{j=1}^{N-5} h_{\rho(j)}^{N,\,\text{off-shell}}\right]\left[\sum_{i=1}^{N} 2\,k_i^\nu \left(\partial_{i\nu}\, h_{\rho(N-4)}^{N,\,\text{off-shell}}\right)\left(\partial_i^\mu\, h_{\rho(N-3)}^{N,\,\text{off-shell}}\right)\right. \\
&\qquad\qquad \left. - k_i^\mu \left(\partial_{i\nu}\, h_{\rho(N-4)}^{N,\,\text{off-shell}}\right)\left(\partial_i^\nu\, h_{\rho(N-3)}^{N,\,\text{off-shell}}\right)\right] \\
&= \sum_{\rho\in\mathfrak{C}_{N-3}} \left[\prod_{j=1}^{N-4} h_{\rho(j)}^{N,\,\text{off-shell}}\right]\left[\mathbb{K}^\mu\, h_{\rho(N-3)}^{N,\,\text{off-shell}}\right] \\
&\quad + 2 \sum_{\rho\in\mathfrak{S}_{N-3}} \left\{\left[\prod_{j=1}^{N-5} h_{\rho(j)}^{N,\,\text{off-shell}}\right]\left[\sum_{\substack{F,F'\subset A\\ F\cap F'\neq\emptyset}} \left[\left(k_{F'}^2 + k_{(F\setminus F')}^2\right) k_F^\mu + \left(k_F^2 + k_{(F'\setminus F)}^2\right) k_{F'}^\mu\right.\right.\right. \\
&\quad \left.\left.\left. + \left(k_{(F\cap F')}^2 - k_{(F\setminus F')}^2 - k_{(F'\setminus F)}^2\right) k_{(F\cup F')}^\mu - k_{(F\cup F')}^2\, k_{(F\cap F')}^\mu\right]\left[\frac{\partial h_{\rho(N-4)}^{N,\,\text{off-shell}}}{\partial (k_F)^2}\right]\left[\frac{\partial h_{\rho(N-3)}^{N,\,\text{off-shell}}}{\partial (k_{F'})^2}\right]\right]\right\}
\end{aligned} \tag{F.18}$$

where $\mathfrak{S}_{N-3}$ is the symmetric group of order $N-3$, i.e. the group of all possible permutations of a given set of $N-3$ elements (here the $N-3$ off-shell scattering polynomials), and $\mathfrak{C}_{N-3}$ is the cyclic group of order $N-3$, i.e. the group of only the cyclic permutations of a given set of $N-3$ elements.

On the Inverse of the Product of Off-Shell Scattering Polynomials

Following similar methods to the ones we discussed in Chapter 4, we propose the following rewriting of the action of the level-zero generator $\mathbb{K}^\mu$ on the inverse of the product of all $N-3$ off-shell scattering polynomials, which makes the vanishing of some of the terms for $d=2$ and $n=6$

manifest:

$$
\begin{aligned}
&\mathbb{K}^{\mu}\,\left[H_N^{\text{off-shell}}\right]^{-1} = \\
&= -2\,\left[H_N^{\text{off-shell}}\right]^{-1} \sum_{\rho\in\mathfrak{C}_{N-3}} \left\{ \left[h_{\rho(N-3)}^{N,\text{ off-shell}}\right]^{-1} \sum_{F\subset A} \left[2\,|F|\,(d-2)-(n-6)\right]\,k_F^{\mu}\,\left[\frac{\partial h_{\rho(N-3)}^{N,\text{ off-shell}}}{\partial k_F^2}\right]\right\} \\
&\quad -4\,\left[H_N^{\text{off-shell}}\right]^{-1} \sum_{\rho\in\mathfrak{C}_{N-3}} \left\{ \left[h_{\rho(N-3)}^{N,\text{ off-shell}}\right]^{-1} \sum_{F\subset A} 2\,(|F|-2)\,k_F^{\mu}\,\left[\frac{\partial h_{\rho(N-3)}^{N,\text{ off-shell}}}{\partial k_F^2}\right]\right. \\
&\quad\quad + \left[h_{\rho(N-3)}^{N,\text{ off-shell}}\right]^{-2} \sum_{\substack{F,F'\subset A\\ F\neq F',\,F\cap F'\neq\emptyset}} \left[k^2_{(F\setminus F')}\,k^{\mu}_{(F'\setminus F)} + k^2_{(F'\setminus F)}\,k^{\mu}_{(F\setminus F')}\right. \\
&\quad\quad \left.\left. -k^2_{(F\cap F')}\,k^{\mu}_{(F\cup F')} + k^2_{(F\cup F')}\,k^{\mu}_{(F\cap F')}\right]\left[\frac{\partial h_{\rho(N-3)}^{N,\text{ off-shell}}}{\partial k_F^2}\right]\left[\frac{\partial h_{\rho(N-3)}^{N,\text{ off-shell}}}{\partial k_{F'}^2}\right]\right\} \\
&\quad\quad (\text{ The following terms contribute only when } N\geq 5\ ) \\
&\quad +2\,\left[H_N^{\text{off-shell}}\right]^{-1} \\
&\quad\quad \times \sum_{\rho\in\mathfrak{S}_{N-3}} \left\{ \left[h_{\rho(N-4)}^{N,\text{ off-shell}}\right]^{-1} \sum_{F\subset A} k_F^{\mu}\left[\frac{\partial h_{\rho(N-4)}^{N,\text{ off-shell}}}{\partial k_F^2}\right] + \left[h_{\rho(N-3)}^{N,\text{ off-shell}}\right]^{-1} \sum_{F\subset A} k_F^{\mu}\left[\frac{\partial h_{\rho(N-3)}^{N,\text{ off-shell}}}{\partial k_F^2}\right]\right. \\
&\quad\quad - \left[h_{\rho(N-4)}^{N,\text{ off-shell}}\right]^{-1}\left[h_{\rho(N-3)}^{N,\text{ off-shell}}\right]^{-1} \sum_{\substack{F,F'\subset A\\ F\neq F',\,F\cap F'\neq\emptyset}} \left[k^2_{(F\setminus F')}\,k^{\mu}_{(F'\setminus F)} + k^2_{(F'\setminus F)}\,k^{\mu}_{(F\setminus F')}\right. \\
&\quad\quad \left.\left. -k^2_{(F\cap F')}\,k^{\mu}_{(F\cup F')} + k^2_{(F\cup F')}\,k^{\mu}_{(F\cap F')}\right]\left[\frac{\partial h_{\rho(N-4)}^{N,\text{ off-shell}}}{\partial k_F^2}\right]\left[\frac{\partial h_{\rho(N-3)}^{N,\text{ off-shell}}}{\partial k_{F'}^2}\right]\right\}
\end{aligned}
\tag{F.19}
$$

where $\mathfrak{S}_{N-3}$ is the symmetric group of order $N-3$, i.e. the group of all possible permutations of a given set of $N-3$ elements (here the $N-3$ off-shell scattering polynomials), and $\mathfrak{C}_{N-3}$ is the cyclic group of order $N-3$, i.e. the group of only the cyclic permutations of a given set of $N-3$ elements.

We expect that the terms which vanish manifestly when we set $d=2$ and $n=6$, i.e. the terms of the first line, after being multiplied by the rest of the z-dependent factors in the CHY integrand, should exactly reproduce the action of $\mathbb{K}^{\mu}$ on the corresponding tree-level massless scalar $\lambda\,\phi^3$ theory off-shell partial amplitude or graph, for arbitrary d and n, by evaluation of the multi-variable contour integral around the solutions of the scattering equations. It must also be that the remaining terms, those that do not vanish manifestly when we set $d=2$ and $n=6$, after being multiplied by the rest of the z-dependent factors in the CHY integrand, will vanish by integration.

Example: $\mathbb{K}^\mu$ on $\left[H_5^{\text{off-shell}}\right]^{-1}$

$$
\begin{aligned}
&\mathbb{K}^\mu \left[h_1^{5,\text{ off-shell}}\, h_2^{5,\text{ off-shell}}\right]^{-1} \\
&= -2\,(4\,d-n-2)\left[\frac{(1-z_3)(k_1+k_2)^\mu-(z_3-z_4)(k_2+k_3)^\mu}{\left(h_1^{5,\text{ off-shell}}\right)^2\left(h_2^{5,\text{ off-shell}}\right)}\right. \\
&\qquad\qquad\left.+\frac{z_4(1-z_3)\left((k_1+k_2)^\mu-(k_3+k_4)^\mu\right)}{\left(h_1^{5,\text{ off-shell}}\right)\left(h_2^{5,\text{ off-shell}}\right)^2}\right] \\
&\quad -2\,(6\,d-n-6)\left[\frac{(z_3-z_4)(k_1+k_2+k_3)^\mu-z_4(k_2+k_3+k_4)^\mu}{\left(h_1^{5,\text{ off-shell}}\right)^2\left(h_2^{5,\text{ off-shell}}\right)}\right. \\
&\qquad\qquad\left.+\frac{(z_3-z_4)(k_1+k_2+k_3)^\mu-z_3z_4(k_2+k_3+k_4)^\mu}{\left(h_1^{5,\text{ off-shell}}\right)\left(h_2^{5,\text{ off-shell}}\right)^2}\right] \\
&\quad +\frac{8}{\left(h_1^{5,\text{ off-shell}}\right)^3\left(h_2^{5,\text{ off-shell}}\right)}\Big[\left(z_3(z_3-1)k_2^2-(z_3-z_4)\left((z_3-1)k_3^2-z_4k_4^2\right)\right)k_1^\mu \\
&\qquad +\left(z_3(z_3-1)k_2^2-z_4(z_4-1)k_5^2\right)k_2^\mu+z_4\left((z_3-1)k_2^2-(z_4-1)k_1^2\right)k_4^\mu \\
&\qquad +\left(-z_3(z_3-1)k_1^2+z_3(z_3-1)k_2^2+z_4(z_3-z_4)k_5^2\right)k_3^\mu\Big] \\
&\quad +\frac{8}{\left(h_1^{5,\text{ off-shell}}\right)\left(h_2^{5,\text{ off-shell}}\right)^3}\Big[z_4\left(z_3z_4(z_3-1)k_2^2+(z_3-z_4)\left((z_3-1)k_3^2+k_4^2\right)\right)k_1^\mu \\
&\qquad +z_4\left((z_3-1)\left(z_3z_4k_2^2+(z_3-z_4)\left(k_3^2-k_4^2\right)\right)-z_3^2(z_4-1)k_5^2\right)k_2^\mu \\
&\qquad +z_4\left(z_3z_4(z_3-1)\left(k_2^2-k_1^2\right)+(z_3-z_4)\left((z_3-1)k_3^2+k_5^2\right)\right)k_3^\mu \\
&\qquad +z_4\left((z_3-1)\left(z_3z_4k_2^2+(z_3-z_4)k_3^2\right)-z_3^2(z_4-1)k_1^2\right)k_4^\mu\Big] \\
&\quad +\frac{8}{\left(h_1^{5,\text{ off-shell}}\right)^2\left(h_2^{5,\text{ off-shell}}\right)^2}\Big[z_4\left(z_3(z_3-1)k_2^2+(z_3-z_4)k_4^2\right)k_1^\mu \\
&\qquad +z_3z_4\left((z_3-1)k_2^2-(z_4-1)k_5^2\right)k_2^\mu+z_3z_4\left((z_3-1)k_2^2-(z_4-1)k_1^2\right)k_4^\mu \\
&\qquad +z_4\left(z_3(z_3-1)\left(k_2^2-k_1^2\right)+(z_3-z_4)k_5^2\right)k_3^\mu\Big] \\
&\quad -8\,\frac{k_1^\mu}{\left(h_1^{5,\text{ off-shell}}\right)^3}-8\,z_3\,z_4\,\frac{k_5^\mu}{\left(h_2^{5,\text{ off-shell}}\right)^3}
\end{aligned}
\tag{F.20}
$$

F.2 Level-One Generators on Off-Shell Scattering Polynomials

F.2.1 Action of Level-One Translation Generators

On a Single Off-Shell Scattering Polynomial

$$\begin{aligned} \widehat{\mathbb{P}}^\mu\, h_m^{N,\text{ off-shell}} =& -d\left[\sum_{i=1}^{N}(N+1-2\,i)\,k_i^\mu\right] h_m^{N,\text{ off-shell}} \\ &+2\sum_{[I,J]\in\mathbb{A}}\left(-k^\mu_{[1,\,I-1]}+k^\mu_{[J+1,\,N]}\right)k^2_{[I,J]}\left(\frac{\partial h_m^{N,\text{ off-shell}}}{\partial k^2_{[I,J]}}\right) \end{aligned} \tag{F.21}$$

Example: $\widehat{\mathbb{P}}^\mu$ on $h_1^{4,\text{ off-shell}}$

$$\begin{aligned} \widehat{\mathbb{P}}^\mu\, h_1^{4,\text{ off-shell}} =& -d\,(3k_1^\mu+k_2^\mu-k_3^\mu-3k_4^\mu)\,h_1^{4,\text{ off-shell}} \\ &+2\big[(1-z_3)(k_1+k_2)^2\,(k_3^\mu+k_4^\mu)-z_3\,(k_2+k_3)^2\,(-k_1^\mu+k_4^\mu)\big] \end{aligned} \tag{F.22}$$

Example: $\widehat{\mathbb{P}}^\mu$ on $h_1^{5,\text{ off-shell}}$

$$\begin{aligned} \widehat{\mathbb{P}}^\mu\, h_1^{5,\text{ off-shell}} =& -d\,(4k_1^\mu+2k_2^\mu-2k_4^\mu-4k_5^\mu)\,h_1^{5,\text{ off-shell}} \\ &+2\big[(1-z_3)(k_1+k_2)^2\,(k_3^\mu+k_4^\mu+k_5^\mu)+(z_4-z_3)(k_2+k_3)^2\,(-k_1^\mu+k_4^\mu+k_5^\mu) \\ &+(z_3-z_4)(k_1+k_2+k_3)^2\,(k_4^\mu+k_5^\mu)-z_4(k_2+k_3+k_4)^2\,(-k_1^\mu+k_5^\mu)\big] \end{aligned} \tag{F.23}$$

Example: $\widehat{\mathbb{P}}^\mu$ on $h_2^{5,\text{ off-shell}}$

$$\begin{aligned} \widehat{\mathbb{P}}^\mu\, h_2^{5,\text{ off-shell}} =& -d\,(4k_1^\mu+2k_2^\mu-2k_4^\mu-4k_5^\mu)\,h_2^{5,\text{ off-shell}} \\ &+2\big[(1-z_3)z_4(k_1+k_2)^2\,(k_3^\mu+k_4^\mu+k_5^\mu)+(z_3-z_4)(k_1+k_2+k_3)^2\,(k_4^\mu+k_5^\mu) \\ &+(z_3-1)z_4(k_3+k_4)^2\,(-k_1^\mu-k_2^\mu+k_5^\mu)-z_3z_4(k_2+k_3+k_4)^2\,(-k_1^\mu+k_5^\mu)\big] \end{aligned} \tag{F.24}$$

Example: $\widehat{\mathbb{P}}^\mu$ on $h_1^{6,\text{ off-shell}}$

$$\begin{aligned}
&\widehat{\mathbb{P}}^\mu\, h_1^{6,\text{ off-shell}} \\
&= -\,d\,(5k_1^\mu + 3k_2^\mu + k_3^\mu - k_4^\mu - 3k_5^\mu - 5k_6^\mu)\, h_1^{6,\text{ off-shell}} \\
&\quad + 2\big[(1-z_3)(k_1+k_2)^2\,(k_3^\mu + k_4^\mu + k_5^\mu + k_6^\mu) + (z_4 - z_3)(k_2+k_3)^2\,(-k_1^\mu + k_4^\mu + k_5^\mu + k_6^\mu) \\
&\quad + (z_3 - z_4)(k_1+k_2+k_3)^2\,(k_4^\mu + k_5^\mu + k_6^\mu) + (z_5 - z_4)(k_2+k_3+k_4)^2\,(-k_1^\mu + k_5^\mu + k_6^\mu) \\
&\quad + (z_4 - z_5)(k_1+k_2+k_3+k_4)^2\,(k_5^\mu + k_6^\mu) - z_5(k_2+k_3+k_4+k_5)^2\,(-k_1^\mu + k_6^\mu)\,\big]
\end{aligned} \tag{F.25}$$

On the Inverse of an Off-Shell Scattering Polynomial

$$\begin{aligned}
\widehat{\mathbb{P}}^\mu\left[h_m^{N,\text{ off-shell}}\right]^{-1} = &-d\left[h_m^{N,\text{ off-shell}}\right]^{-1}\left[\sum_{i=1}^{N}(N+1-2\,i)\,k_i^\mu\right] \\
&-2\left[h_m^{N,\text{ off-shell}}\right]^{-2}\sum_{F\subset A}\left(-k^\mu_{[1,\,\min(F)-1]} + k^\mu_{[\max(F)+1,\,N]}\right)k_F^2\left(\frac{\partial h_m^{N,\text{ off-shell}}}{\partial k_F^2}\right)
\end{aligned} \tag{F.26}$$

Example: $\widehat{\mathbb{P}}^\mu$ on $\left[h_1^{4,\text{ off-shell}}\right]^{-1}$

$$\begin{aligned}
\widehat{\mathbb{P}}^\mu\left[h_1^{4,\text{ off-shell}}\right]^{-1} = &-\frac{d\,(3k_1^\mu + k_2^\mu - k_3^\mu - 3k_4^\mu)}{\left(h_1^{4,\text{ off-shell}}\right)} \\
&-\frac{2\left((1-z_3)(k_1+k_2)^2\,(k_3^\mu + k_4^\mu) - z_3\,(k_2+k_3)^2\,(-k_1^\mu + k_4^\mu)\right)}{\left(h_1^{4,\text{ off-shell}}\right)^2}
\end{aligned} \tag{F.27}$$

Example: $\widehat{\mathbb{P}}^\mu$ on $\left[h_1^{5,\text{ off-shell}}\right]^{-1}$

$$\widehat{\mathbb{P}}^\mu\left[h_1^{5,\text{ off-shell}}\right]^{-1}=-\frac{d\left(4k_1^\mu+2k_2^\mu-2k_4^\mu-4k_5^\mu\right)}{\left(h_1^{5,\text{ off-shell}}\right)}$$
$$-\frac{2\left((1-z_3)(k_1+k_2)^2\left(k_3^\mu+k_4^\mu+k_5^\mu\right)+(z_4-z_3)(k_2+k_3)^2\left(-k_1^\mu+k_4^\mu+k_5^\mu\right)\right)}{\left(h_1^{5,\text{ off-shell}}\right)^2}$$
$$-\frac{2\left((z_3-z_4)\left(k_4^\mu+k_5^\mu\right)(k_1+k_2+k_3)^2-z_4\left(k_5^\mu-k_1^\mu\right)(k_2+k_3+k_4)^2\right)}{\left(h_1^{5,\text{ off-shell}}\right)^2} \tag{F.28}$$

Example: $\widehat{\mathbb{P}}^\mu$ on $\left[h_2^{5,\text{ off-shell}}\right]^{-1}$

$$\widehat{\mathbb{P}}^\mu\left[h_2^{5,\text{ off-shell}}\right]^{-1}=-\frac{d\left(4k_1^\mu+2k_2^\mu-2k_4^\mu-4k_5^\mu\right)}{\left(h_2^{5,\text{ off-shell}}\right)}$$
$$-\frac{2\left((1-z_3)z_4(k_1+k_2)^2\left(k_3^\mu+k_4^\mu+k_5^\mu\right)+(z_3-z_4)\left(k_4^\mu+k_5^\mu\right)(k_1+k_2+k_3)^2\right)}{\left(h_2^{5,\text{ off-shell}}\right)^2}$$
$$-\frac{2\left((z_3-1)z_4(k_3+k_4)^2\left(-k_1^\mu-k_2^\mu+k_5^\mu\right)-z_3z_4\left(k_5^\mu-k_1^\mu\right)(k_2+k_3+k_4)^2\right)}{\left(h_2^{5,\text{ off-shell}}\right)^2} \tag{F.29}$$

Example: $\widehat{\mathbb{P}}^\mu$ on $\left[h_1^{6,\text{ off-shell}}\right]^{-1}$

$$\widehat{\mathbb{P}}^\mu\left[h_1^{6,\text{ off-shell}}\right]^{-1}$$
$$=-\frac{d\left(5k_1^\mu+3k_2^\mu+k_3^\mu-k_4^\mu-3k_5^\mu-5k_6^\mu\right)}{\left(h_1^{6,\text{ off-shell}}\right)}$$
$$-\frac{2\left((1-z_3)(k_1+k_2)^2\left(k_3^\mu+k_4^\mu+k_5^\mu+k_6^\mu\right)+(z_4-z_3)(k_2+k_3)^2\left(-k_1^\mu+k_4^\mu+k_5^\mu+k_6^\mu\right)\right)}{\left(h_1^{6,\text{ off-shell}}\right)^2}$$
$$-\frac{2\left((z_3-z_4)\left(k_4^\mu+k_5^\mu+k_6^\mu\right)(k_1+k_2+k_3)^2+(z_5-z_4)\left(-k_1^\mu+k_5^\mu+k_6^\mu\right)(k_2+k_3+k_4)^2\right)}{\left(h_1^{6,\text{ off-shell}}\right)^2}$$
$$-\frac{2\left((z_4-z_5)\left(k_5^\mu+k_6^\mu\right)(k_1+k_2+k_3+k_4)^2-z_5\left(k_6^\mu-k_1^\mu\right)(k_2+k_3+k_4+k_5)^2\right)}{\left(h_1^{6,\text{ off-shell}}\right)^2} \tag{F.30}$$

$$
\begin{aligned}
\widehat{\mathbb{P}}^\mu \left[H_N^{\text{off-shell}}\right]^{-1} &= -d \left[\sum_{i=1}^{N} (N+1-2\,i)\, k_i^\mu\right] \left[H_N^{\text{off-shell}}\right]^{-1} \\
&- 2 \left[H_N^{\text{off-shell}}\right]^{-1} \sum_{m=1}^{N-3} \left[h_m^{N,\text{ off-shell}}\right]^{-1} \sum_{[I,J]\in\mathbb{A}} \left(-k^\mu_{[1,\,I-1]} + k^\mu_{[J+1,\,N]}\right) k^2_{[I,J]} \left(\frac{\partial h_m^{N,\text{ off-shell}}}{\partial k^2_{[I,J]}}\right)
\end{aligned} \tag{F.31}
$$

Example: $\widehat{\mathbb{P}}^\mu$ on $\left[H_5^{\text{off-shell}}\right]^{-1}$

$$
\begin{aligned}
&\widehat{\mathbb{P}}^\mu \left[h_1^{5,\text{ off-shell}}\, h_2^{5,\text{ off-shell}}\right]^{-1} \\
&= - \frac{d\,(4k_1^\mu + 2k_2^\mu - 2k_4^\mu - 4k_5^\mu)}{\left(h_1^{5,\text{ off-shell}}\right)\left(h_2^{5,\text{ off-shell}}\right)} \\
&\quad - \frac{2\left((1-z_3)(k_1+k_2)^2\,(k_3^\mu+k_4^\mu+k_5^\mu) + (z_4-z_3)(k_2+k_3)^2\,(-k_1^\mu+k_4^\mu+k_5^\mu)\right)}{\left(h_1^{5,\text{ off-shell}}\right)^2\left(h_2^{5,\text{ off-shell}}\right)} \\
&\quad - \frac{2\left((z_3-z_4)\,(k_4^\mu+k_5^\mu)\,(k_1+k_2+k_3)^2 - z_4\,(k_5^\mu-k_1^\mu)\,(k_2+k_3+k_4)^2\right)}{\left(h_1^{5,\text{ off-shell}}\right)^2\left(h_2^{5,\text{ off-shell}}\right)} \\
&\quad - \frac{2\left((1-z_3)z_4(k_1+k_2)^2\,(k_3^\mu+k_4^\mu+k_5^\mu) + (z_3-z_4)\,(k_4^\mu+k_5^\mu)\,(k_1+k_2+k_3)^2\right)}{\left(h_1^{5,\text{ off-shell}}\right)\left(h_2^{5,\text{ off-shell}}\right)^2} \\
&\quad - \frac{2\left((z_3-1)z_4(k_3+k_4)^2\,(-k_1^\mu-k_2^\mu+k_5^\mu) - z_3z_4\,(k_5^\mu-k_1^\mu)\,(k_2+k_3+k_4)^2\right)}{\left(h_1^{5,\text{ off-shell}}\right)\left(h_2^{5,\text{ off-shell}}\right)^2}
\end{aligned} \tag{F.32}
$$

By contour integration for the 5-point partial amplitude we get

$$
\begin{aligned}
&- \widehat{\mathbb{P}}^\mu \oint \frac{dz_3dz_4}{h_1h_2}\,\frac{z_3(1-z_4)}{(1-z_3)(z_3-z_4)z_4} \\
&= -\widehat{\mathbb{P}}^\mu \oint \frac{dz_3dz_4}{h_1h_2}\left(\frac{1}{z_4} + \frac{1}{1-z_3} + \frac{1-z_4}{z_3-z_4} + \frac{z_3(1-z_4)}{(1-z_3)z_4} + \frac{z_3}{z_3-z_4} - 2\right) \\
&= -2k_1^\mu\left(\frac{2}{s_{23}s_{234}} + \frac{1}{s_{34}s_{12}} + \frac{1}{s_{123}s_{23}} + \frac{2}{s_{234}s_{34}}\right) - 2k_2^\mu\left(\frac{1}{s_{34}s_{12}} + \frac{1}{s_{234}s_{34}}\right) \\
&\quad + 2k_3^\mu\left(\frac{1}{s_{12}s_{123}} + \frac{1}{s_{34}s_{12}}\right) + 2k_4^\mu\left(\frac{2}{s_{12}s_{234}} + \frac{1}{s_{23}s_{234}} + \frac{1}{s_{34}s_{12}} + \frac{2}{s_{123}s_{23}}\right) \\
&\quad + 4k_5^\mu\left(\frac{1}{s_{12}s_{123}} + \frac{1}{s_{23}s_{234}} + \frac{1}{s_{34}s_{12}} + \frac{1}{s_{123}s_{23}} + \frac{1}{s_{234}s_{34}}\right) \\
&= -\widehat{\mathbb{P}}^\mu\left(\frac{1}{s_{12}s_{123}} + \frac{1}{s_{23}s_{234}} + \frac{1}{s_{34}s_{12}} + \frac{1}{s_{123}s_{23}} + \frac{1}{s_{234}s_{34}}\right)
\end{aligned} \tag{F.33}
$$

in agreement with (4.26).

Example: $\widehat{\mathbb{P}}^\mu$ on $\left[H_6^{\text{off-shell}}\right]^{-1}$

$$
\begin{aligned}
&\widehat{\mathbb{P}}^\mu \left[h_1^{6,\text{ off-shell}}\, h_2^{6,\text{ off-shell}}\, h_3^{6,\text{ off-shell}}\right]^{-1} \\
&= - \frac{d\left(5k_1^\mu + 3k_2^\mu + k_3^\mu - k_4^\mu - 3k_5^\mu - 5k_6^\mu\right)}{\left(h_1^{6,\text{ off-shell}}\right)\left(h_2^{6,\text{ off-shell}}\right)\left(h_3^{6,\text{ off-shell}}\right)} \\
&\quad - \frac{2\left((1-z_3)s_{12}\left(k_3^\mu + k_4^\mu + k_5^\mu + k_6^\mu\right) + (z_4 - z_3)s_{23}\left(-k_1^\mu + k_4^\mu + k_5^\mu + k_6^\mu\right)\right)}{\left(h_1^{6,\text{ off-shell}}\right)^2\left(h_2^{6,\text{ off-shell}}\right)\left(h_3^{6,\text{ off-shell}}\right)} \\
&\quad - \frac{2\left((z_3 - z_4)s_{123}\left(k_4^\mu + k_5^\mu + k_6^\mu\right) + (z_5 - z_4)s_{234}\left(-k_1^\mu + k_5^\mu + k_6^\mu\right)\right)}{\left(h_1^{6,\text{ off-shell}}\right)^2\left(h_2^{6,\text{ off-shell}}\right)\left(h_3^{6,\text{ off-shell}}\right)} \\
&\quad - \frac{2\left((z_4 - z_5)s_{1234}\left(k_5^\mu + k_6^\mu\right) - z_5\, s_{2345}\left(k_6^\mu - k_1^\mu\right)\right)}{\left(h_1^{6,\text{ off-shell}}\right)^2\left(h_2^{6,\text{ off-shell}}\right)\left(h_3^{6,\text{ off-shell}}\right)} \\
&\quad - \frac{2\left((1-z_3)(z_4+z_5)s_{12}\left(k_3^\mu + k_4^\mu + k_5^\mu + k_6^\mu\right) - z_5(z_3 - z_4)s_{23}\left(-k_1^\mu + k_4^\mu + k_5^\mu + k_6^\mu\right)\right)}{\left(h_1^{6,\text{ off-shell}}\right)\left(h_2^{6,\text{ off-shell}}\right)^2\left(h_3^{6,\text{ off-shell}}\right)} \\
&\quad - \frac{2\left((z_5+1)(z_3 - z_4)s_{123}\left(k_4^\mu + k_5^\mu + k_6^\mu\right) + (z_3 - 1)(z_4 - z_5)s_{34}\left(-k_1^\mu - k_2^\mu + k_5^\mu + k_6^\mu\right)\right)}{\left(h_1^{6,\text{ off-shell}}\right)\left(h_2^{6,\text{ off-shell}}\right)^2\left(h_3^{6,\text{ off-shell}}\right)} \\
&\quad - \frac{2\left(-z_3(z_4 - z_5)s_{234}\left(-k_1^\mu + k_5^\mu + k_6^\mu\right) + (z_3+1)(z_4 - z_5)s_{1234}\left(k_5^\mu + k_6^\mu\right)\right)}{\left(h_1^{6,\text{ off-shell}}\right)\left(h_2^{6,\text{ off-shell}}\right)^2\left(h_3^{6,\text{ off-shell}}\right)} \\
&\quad - \frac{2\left(z_5(z_4 - z_3)s_{45}\left(-k_1^\mu - k_2^\mu - k_3^\mu + k_6^\mu\right) - z_5(z_3 + z_4)s_{2345}\left(k_6^\mu - k_1^\mu\right)\right)}{\left(h_1^{6,\text{ off-shell}}\right)\left(h_2^{6,\text{ off-shell}}\right)^2\left(h_3^{6,\text{ off-shell}}\right)} \\
&\quad - \frac{2\left((z_3 - 1)z_5\, s_{345}\left(-k_1^\mu - k_2^\mu + k_6^\mu\right)\right)}{\left(h_1^{6,\text{ off-shell}}\right)\left(h_2^{6,\text{ off-shell}}\right)^2\left(h_3^{6,\text{ off-shell}}\right)} \\
&\quad - \frac{2\left((1-z_3)z_4 z_5 s_{12}\left(k_3^\mu + k_4^\mu + k_5^\mu + k_6^\mu\right) + z_5(z_3 - z_4)s_{123}\left(k_4^\mu + k_5^\mu + k_6^\mu\right)\right)}{\left(h_1^{6,\text{ off-shell}}\right)\left(h_2^{6,\text{ off-shell}}\right)\left(h_3^{6,\text{ off-shell}}\right)^2} \\
&\quad - \frac{2\left(z_3(z_4 - z_5)s_{1234}\left(k_5^\mu + k_6^\mu\right) + z_5(z_4 - z_3)s_{45}\left(-k_1^\mu - k_2^\mu - k_3^\mu + k_6^\mu\right)\right)}{\left(h_1^{6,\text{ off-shell}}\right)\left(h_2^{6,\text{ off-shell}}\right)\left(h_3^{6,\text{ off-shell}}\right)^2} \\
&\quad - \frac{2\left((z_3 - 1)z_4 z_5 s_{345}\left(-k_1^\mu - k_2^\mu + k_6^\mu\right) - z_3 z_4 z_5 s_{2345}\left(k_6^\mu - k_1^\mu\right)\right)}{\left(h_1^{6,\text{ off-shell}}\right)\left(h_2^{6,\text{ off-shell}}\right)\left(h_3^{6,\text{ off-shell}}\right)^2}
\end{aligned}
\tag{F.34}
$$

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