PHYSICAL REALIZATION OF THE PARITY ANOMALY AND QUANTUM HALL EFFECT

D. KHVESHCHENKO
Landau Institute for Theoretical Physics, Kosyegina 2, 117 940 Moscow, USSR

and

P.B. WIEGMANN [1]
Institute for Theoretical Physics, University of California, Santa Barbara, CA 93106, USA

Received 2 May 1989

We present the lattice version of the anomalous Dirac operator in (2+1) dimensions. This version is associated with a group of magnetic translations rather than translations. This group naturally provided homologically nontrivial fiber bundles, the first Chern class of which is directly related with the number of zero modes and spectral asymmetry of the Dirac operator. The results are in agreement with the lattice fermions doubling phenomenon. The relation of the parity anomaly in (2+1) and the Hall effect is elucidated.

1. Introduction

It is known that the lattice formulation of fermionic gauge theory is highly problematic. In particular, there is the problem of the lattice formulation of the odd dimensional Dirac operator. In odd dimensions the Dirac operator has a spectrum asymmetry and possesses the parity anomaly [#1]. In the euclidean (2+1) space,

$$\operatorname{Im} \ln \det(\mathrm{i}\partial\!\!\!/ + a\!\!\!/ + m) = \frac{\operatorname{sgn} m}{16\pi}\,(a\,\mathrm{d}a - \tfrac{2}{3}a^3)\,. \qquad (1)$$

Several attempts undertaken to reproduce the Chern–Simons term in the right-hand side of (1) starting from some lattice theory, failed. For example, Semenoff [2] has shown that fermions of the honeycomb lattice become relativistic for the half-filling, and that there are two fermionic species with opposite signs of the mass, thus canceling the abnormal parity current. Namely, putting fermions on a lattice has the famous complication of doubling [3]. The doubling phenomenon, in particular, forbids the parity anomaly in the parity conserving theory: each species is accompanied by an opposite chirality partner, restoring the parity. This is a consequence of the translational group.

However, there is a simple way to have the parity anomaly from lattice fermions. The purpose of this paper is to show that this puzzle can be solved if one considers a projective representation of the group of translations or so-called "group of magnetic translations" [4] (it is also called the "finite Heisenberg group" [5]) [#2]

$$T_{t_1}\cdot T_{t_2} = T_{t_2}\cdot T_{t_1}\exp[\mathrm{i}\Phi(\boldsymbol{t}_1\times\boldsymbol{t}_2)]\,, \qquad (2)$$

where $\boldsymbol{t}_1$, $\boldsymbol{t}_2$ are some lattice vectors. The representation of this group forms a homologically nontrivial fiber bundle which is responsible for the parity anomaly. Actually we shall show that the Chern class of the fiber bundle is just the number of zero modes and spectral asymmetry of the Dirac operator. There is a simple physical realization of the quantum me-

[1] On leave from Landau Institute, Kosyegina 2, 117 940 Moscow, USSR.
[#1] For a review see ref. [1].

[#2] The interest in fiber bundles induced by the group (2) has been recently renewed in mathematics in the context of noncommutative geometry [6].

0370-2693/89/$ 03.50 © Elsevier Science Publishers B.V.
(North-Holland Physics Publishing Division)

chanics associated with this group. This is the Hall effect on a lattice, or so-called Hofstadter problem [7]. Several attempts were made to find analogues of the parity anomaly and the Hall effect (see ref. [8] for a review). However, only recently the relation has been understood in refs. [9–12]. The ideas of Haldane [9] and Wen, Wilczek and Zee [11] are close to ours presented here.

2. The model

Let us consider fermions, say, on a square lattice in the presence of an external abelian time-independent magnetic field A coupled with a slowly varying small gauge field a, which, generally speaking, can be non-abelian:

$$\mathscr{H}=\sum_{\langle ab\rangle}[c_a^\dagger\exp(\mathrm{i}A_{ab}+\mathrm{i}a_{ab})\,c_b+\mu(c_a^\dagger c_a-c_b^\dagger c_b)]\,, \tag{3}$$

where $\langle ab\rangle$ are nearest neighbors. For simplicity, we consider a homogeneous "magnetic" field with a flux $\phi=2\pi p/q$ per plaquette (p and q are incommensurate integers and q is even):

$$\prod_{\mathrm{P}}\exp(\mathrm{i}A_{ab})=\exp(\mathrm{i}\phi)\,.$$

In the presence of a magnetic field, fermions exhibit the Hall current:

$$j_i(x)\equiv\langle[c^\dagger(x),c(x+\boldsymbol{e}_i)]\rangle=\sigma_{xy}\epsilon_{ij}a_j(x)\,,$$

with an integer Hall conductance σ_{xy}. [Here $\boldsymbol{e}_i$ are the lattice basis ($i=1,2$).] It means that the continuous Euler–Heisenberg action involves the Chern–Simons term:

$$\ln\det(\mathrm{i}\mathrm{d}_t-\mathscr{H})=\frac{\theta}{8\pi^2}\,\mathrm{tr}(a\mathrm{d}a-\tfrac{2}{3}a^3)\,,$$

$$\theta=2\pi\sigma_{xy}\,. \tag{4}$$

We shall show that for half-filling (the number of fermions is half the number of lattice sites), this simple model for *even q* provides q-species of Dirac fermions

$$\mathscr{L}=\sum_{f=1}^{q=\mathrm{even}}\bar\psi_f(\mathrm{i}\partial\!\!\!/+a\!\!\!/+m)\psi_f\,, \tag{5}$$

with the same sign of the mass

$$\mathrm{sgn}\,m=\sin\phi\,, \tag{6}$$

and thus q-anomalies:

$$\sigma_{xy}=\tfrac{1}{2}q\,. \tag{7}$$

The external flux can emerge spontaneously as a result of a spontaneous parity breaking [#3].

3. General properties of degeneracy points

The hamiltonian $\mathscr{H}$ on a bipartite lattice for even q is symmetric: for each state with energy $E^{(-)}<0$ there is a state with energy $E^{(+)}=-E^{(-)}>0$. This is a consequence of the hidden supersymmetry of the problem [10]. Let $\varDelta_{ab}=\exp(\mathrm{i}A_{ab})$ be an amplitude of hopping from a sublattice $\mathrm{A}\ni a$ to a sublattice $\mathrm{B}\ni b$. Then

$$Q=\begin{pmatrix}0&\varDelta_{ab}\\ \varDelta_{ba}&0\end{pmatrix},\quad \tilde Q=Q\Gamma,\quad \Gamma=\begin{pmatrix}1&0\\0&1\end{pmatrix},$$

are generators of the supersymmetry satisfying the algebra

$$\{Q,\tilde Q\}=\mathscr{H}^2-\mu^2\,,$$

and Γ is an operator which reverses the sign of the mass μ,

$$\mathscr{H}=m\Gamma+Q\,.$$

Let ϵ^2 and (χ_a,χ_b) be the spectrum and eigenstates of the positive operator $(\mathscr{H}^2-\mu^2)$, respectively, then $E^{(\alpha)}_{(\pm)}=\pm\sqrt{\mu^2+\epsilon_\alpha^2(k)}$ and

$$\psi_+=\left(\chi_a,\,-\frac{\chi_b}{|E|+|\mu|}\right)\sqrt{\frac{|E|+|\mu|}{2|E|}}\,,$$

$$\psi_-=\sqrt{\frac{|E|+|\mu|}{|E|-|\mu|}}\,\tilde Q\psi_+\,.$$

Let T be a translation $T\chi_a=\chi_b$ and let us suppose that the hamiltonian commutes with a subgroup T^q of the group of translations (q is even). Then the

[#3] This happened in the modern theory of antiferromagnetism [11–13] and probably in the lattice QCD with quarks. Actually it has been mentioned recently that lattice fermions coupled with a gauge field possess a magnetic flux in the ground state and thus dynamically break the parity in (2+1) dimensions and chirality in (3+1) dimensions [14].

spectrum is split into $\frac{1}{2}q$ positive and $\frac{1}{2}q$ negative subbands: $(\epsilon_\alpha(k), \chi_\alpha^n(k))$, where α is the number of the band α, $n=1, ..., \frac{1}{2}q$, a wave vector k belongs to the Brillouin zone B, and each level is q-fold degenerate. The theory becomes relativistic in the vicinity of the so-called degeneracy $\{k_f^*\}$, where positive and negative bands are crossing:

$$\epsilon_\pm(p) = \pm|p| , \tag{8}$$

where $p_i = R_{ij}(k-k_f^*)_j$ are some proper coordinates. The wave functions $\chi_\pm(p)$ of the crossing bands are holomorphic functions in the vicinity of $\{k_f^*\}$. Therefore, each degeneracy point is characterized by a topological invariant, referred to as "chirality", $\gamma_f = \pm 1$ [10]:

$$\begin{aligned}\gamma_f &= \frac{1}{4\pi \mathrm{i}} \sum_\sigma \oint \boldsymbol{\psi}_\sigma^*(p)\, \mathrm{d}\boldsymbol{\psi}_\sigma(p)\, \mathrm{sgn}\,\sigma \\ &= \frac{1}{4\pi \mathrm{i}} \oint \mathrm{tr}\, \Gamma \mathscr{H}^{-1}\, \mathrm{d}\mathscr{H} \\ &= \frac{1}{2\pi \mathrm{i}} \sum_\sigma \oint \boldsymbol{\chi}_\sigma^*(p)\, \mathrm{d}\chi_\sigma(p)\, \mathrm{sgn}\,\sigma \\ &= \frac{1}{2\pi \mathrm{i}} \oint \mathrm{tr}\, \mathscr{H}^{-2}\, \mathrm{d}\mathscr{H}^2 = \pm 1 .\end{aligned} \tag{9}$$

Here $\sigma = \pm$ refers to the lower or higher of the two central bands and the line integral is taken over a closed loop centered in the degeneracy point. If a_{ij} is a slowly varying gauge field, i.e. has momentum $k < 1/q$, then, in the vicinity of the degeneracy point the theory becomes q-flavor $(\mathrm{QCD})_3$:

$$\mathscr{H} = \bar{\psi}_f[(\mathrm{i}\partial_i + a_i)\gamma^i + m_f]\psi_f . \tag{10}$$

Here γ are Dirac matrices and $\psi_f(x) = \sum_{\sigma=\pm} S_{\sigma\sigma'}(p)\chi_{\sigma'}^n(p) c_{\sigma'}(p+k_f^*)$, where $c_\sigma(p)$ are original lattice fermions, $S_{\sigma\sigma'}(p) = (1+|\mu/2E|^{1/2}) \times [1+\mathrm{i}\gamma\cdot\boldsymbol{p}/(|E|+|\mu|)]$ and [#4]

$$m_f = |\mu|\gamma_f . \tag{11}$$

The total anomaly contribution is therefore

$$\frac{\theta}{2\pi} \equiv \sum_f \gamma_f = \sigma_{xy}^{(-)} - \sigma_{xy}^{(+)} , \tag{12}$$

where

$$\sigma_{xy}^{(\pm)} = \frac{1}{2\pi \mathrm{i}} \int_{\mathrm{B}} \mathrm{d}(\bar{\chi}_\pm^*(k)\, \mathrm{d}\chi_\pm(k)) = -\sigma_{xy}^{(\mp)} , \tag{13}$$

is the first Chern class of the fiber bundle based on the Brillouin zone B.

The number of the degeneracy points is always even [3], such that a point k_f^* has a partner $\bar{k}_f^* = k_f^* + Q$, where Q is a vector dual to the lattice vector. Therefore, $\theta/2\pi$ is an integer in agreement with the quantization of the Chern–Simons constant [8]. If the lattice theory conserves the parity P $(k_1 \to k_1, k_2 \to k_2)$, then chiralities of k_f^* and $\bar{k}_f^*$ are opposite: $\gamma_f = -\bar{\gamma}_f$ and the homotopy of the fiber bundle is trivial: $\theta = 0$. We shall show that in the presence of an external flux all fermion species have the same chirality

$$\gamma_f = \mathrm{sgn}\,\phi . \tag{14}$$

4. Magnetic translations and Hall conductance

Wave functions of a particle in an external magnetic field form a representation of the group of magnetic translations (2). On the square lattice it is convenient to use the standard Landau gauge: let $\boldsymbol{a} = (n_1 n_2)$; $\boldsymbol{b} = \boldsymbol{a} + \boldsymbol{e}_i$, where $\boldsymbol{e}_1 = (1, 0)$, $\boldsymbol{e}_2 = (0, 1)$ are lattice vectors. One can put $A_{ab} = A_i(n_1, n_2) = (0, 2\pi(p/q)n_1)$. According to this gauge one can divide the lattice into "magnetic cells" $M_{n,n_2} = \{(n_1 = nq+m, n_2); m = 1, ..., q\}$. Then $\psi(n_1 n_2) = \exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{n}) \times \varphi_m(k)$, where m is a number of a site in the magnetic cell, and $\boldsymbol{k}$ takes values inside the Brillouin zone: $\mathrm{B} = \{-\pi/q \leqslant k_1 \leqslant \pi/q;\ -\pi \leqslant k_2 \leqslant \pi\}$.

Magneto Bloch wave functions $\varphi_m(\boldsymbol{k})$ satisfy the twisted conditions on the boundary of B:

$$\begin{aligned}&\varphi_m(k_1 k_2)\varphi_m\left(k_1 + \frac{2\pi}{q}, k_2\right) \\ &\quad = \varphi_m(k_1, k_2 + 2\pi)\exp(-\mathrm{i}mk_1 - \mathrm{i}qk_2) ,\end{aligned} \tag{15}$$

and the secular equations (Harper's equations):

[#4] The massive $(2+1)$ Dirac operator can be considered as a projection of the $(3+1)$ Weyl operator onto the $(2+1)$ space: $\psi(x, y, z) = \psi(x, y)\exp(\mathrm{i}z\mu)$. Then the sign of mass in the $(2+1)$ dimension is proportional to the chirality γ of the Weyl operator: $m = \gamma\mu$.

$$\exp(\mathrm{i}k_1)\,\varphi_{m+1}(k)+\exp(-\mathrm{i}k_2)\,\varphi_{m-1}(k)$$
$$+2\cos\left(2\pi\frac{p}{q}m+k_2\right)\varphi_m(k)=E(k)\varphi_m(k)\,,$$
$$\varphi_m(k)=\varphi_{m+q}(k)\,. \tag{16}$$

These equations have been studied extensively [7,15,16]. Below we enlist useful properties of the wave function $\varphi_m(k)$ in the Landau gauge.

(i) If $\varphi_m(k)$ is an eigenvalue with energy E, then $\tilde{\varphi}_m=\tilde{Q}_{mm'}\varphi_{m'}=(-1)^m\mathrm{i}^{q/2}\varphi_{m+q/2}\ (\mathrm{mod}\,q)$ is a state with $(-E)$.

(ii) $\det\mathscr{H}=4(\cos^2\frac{1}{2}qk_1+\cos^2\frac{1}{2}qk_2)$ [17]. Therefore the spectrum has isolated zeros at $k_f^*=(0,2\pi/q,f)$; $f=-\frac{1}{2}q,\ldots,\frac{1}{2}q$. These are the degeneracy points [10]. Their positions are gauge invariant.

(iii) The eigenstates of different degeneracy points are related by unitary transformation. In the Landau gauge

$$\varphi_m(k+k_f^*)=\varphi_{m+f}(k)\,. \tag{17}$$

(iv) $\varphi_m^{(-)}(k)$ in the band adjacent to the zero is a holomorphic function in the complex plane of $k_x+\mathrm{i}k_y$, except the points of degeneracy.

(v) Parity transformation P is compensated by the sign reversal of flux: $\phi\to-\phi$. The integer t in eq. (15) being the Chern class of a subband α, is the Hall conductance of the band:

$$t_\alpha=\frac{1}{2\pi\mathrm{i}}\oint_{\mathrm{B}}\varphi^*_{m,\alpha}(k)\,\mathrm{d}\varphi_{m,\alpha}(k)\,. \tag{18}$$

If the Fermi level is located in the αth gap, the total Hall conductance is equal to

$$\sigma_{xy}(\alpha)=\sum_{\alpha'=1}^{\alpha}t_{\alpha'}\,.$$

Thouless et al. [15] have shown that $\sigma_{xy}(\alpha)$ is determined by the Diophantine equations:

$$p\sigma_{xy}(\alpha)-\alpha=s\cdot q\,, \tag{19}$$

where the integer $|\sigma_{xy}|\leqslant\frac{1}{2}q$ for the square lattice. This determines the Hall conductance uniquely, except the case of the half-filling $\alpha=\frac{1}{2}q$, where the spectrum has the points of degeneracy. At these points σ_{xy} has a discontinuity with a universal jump $\sigma_{xy}\ (\frac{1}{2}\pm0)=\pm\frac{1}{2}q$. For example, in the case $p=1$: $\sigma_{xy}(\alpha)=\alpha-\theta\times(\alpha-\frac{1}{2}q)$. This jump corresponds to eq. (13) and is recognized as a zero modes contribution.

5. Chirality of degeneracy points

Therefore at half-filling $\sigma_{xy}=\frac{1}{2}q\,\mathrm{sgn}\,\phi$ and, according to eq. (12) all points of degeneracy have the same chirality (14). This result can be easily obtained from the simple properties (iii) and (iv):

$$\gamma_f=\frac{1}{2\pi\mathrm{i}}\sum_m\oint_{k_f^*}\varphi_m^{*(-)}(p+k_f^*)\,\mathrm{d}\varphi_m^{(-)}(p+k_f)=\mathrm{sgn}\,\phi\,.$$

6. Concluding remarks

The result presented here shows that one can get the lattice version of relativistic anomalous operators, making the space discrete according to the group of magnetic translations. The "no-go" theorem [3] can be extended to this case and corresponds to the quantization of the Chern–Simons coefficient q [1,8]. The same results are reached for arbitrary lattice, arbitrary flux distribution with slight modifications. For example, the honeycomb lattice provides $2q$ species of relativistic fermions, whereas the triangle lattice provides $\frac{1}{2}q$ species (in this case $\frac{1}{2}q$ is supposed to be even).

The result can be generalized for arbitrary odd and even dimensions: the Berry's phase [i.e. $\mathrm{Im}\ln\det(\mathrm{i}d_t-\mathscr{H})$] of fermions in the presence of external flux reproduces the hierarchy of anomalies.

Acknowledgement

This research was supported in part by the National Science Foundation under Grant No. PHY82-17853, supplemented by funds from the National Aeronautics and Space Administration, at the University of California at Santa Barbara.

References

[1] R. Jackiw, in: Current algebra and anomalies (World Scientific, Singapore, 1985), and references therein.

[2] G.W. Semenoff, Phys. Rev. Lett. 53 (1983) 2449.
[3] H.B. Nielsen and M. Ninomiya, Nucl. Phys. B 185 (1981) 20; B 193 (1981) 173.
[4] J. Zak, Phys. Rev. A 134 (1964) 1602.
[5] D. Mamford, Tata Lectures on theta (Birkhäuser, Basel, 1983).
[6] A. Connes, Publ. Math. IHES 62 (1985).
[7] D.R. Hofstadter, Phys. Rev. B 14 (1976) 2239.
[8] I. Krive and K. Rozavsky, Usp. Fiz. Nauk 64 (1987) 49; A. Niemi and G.W. Semenoff, Phys. Rep. 135 (1986) 99.
[9] D.F.M. Haldane, Phys. Rev. Lett. 61 (1988) 2015.
[10] X.G. Wen and A. Zee, ITP preprint 88-115.
[11] X.G. Wen, F. Wilczek and A. Zee, ITP preprint 88-148.
[12] P.B. Wiegmann, in: Proc. Nobel Symp. 73 (1988); Physica C 153–155 (1988) 103.
[13] P.W. Anderson, in: Proc. Nobel Symp. 73 (1988).
[14] P.B. Wiegmann, to be published.
[15] M.Ya. Azbel, Zh. Eksp Teor. Fiz. 46 (1964) 929; M. Wilkinson, Proc. R. Soc. London A 391 (1984) 305.
[16] D.J. Thouless, in: The quantum Hall effect, eds. R.E. Prange and S.M. Girvin (Springer, Berlin, 1987) p. 101, and references therein; D.J. Thouless et al., Phys. Rev. Lett. 49 (1982) 405.
[17] A. Abanov, private communication.