Effect of Fermi Surface Curvature on Low-Energy Properties of Fermions with Singular Interactions

A. V. Chubukov[1] and D. V. Khveshchenko[2]

[1]*Department of Physics, University of Wisconsin, Madison, Wisconsin 53706, USA*
[2]*Department of Physics and Astronomy, University of North Carolina, Chapel Hill, North Carolina 27599, USA*

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We discuss the effect of Fermi surface curvature on long-distance or time asymptotic behaviors of two-dimensional fermions interacting via a gapless mode described by an effective gauge-field-like propagator. By comparing the predictions based on the idea of multidimensional bosonization with those of the strong-coupling Eliashberg approach, we demonstrate that an agreement between the two requires a further extension of the former technique.

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In recent years, the behavior of fermions coupled via singular (long-range and/or retarded) interactions has been at the forefront of theoretical research in condensed matter physics. Such singular interactions are often associated with the ground state instabilities and concomitant non-Fermi-liquid behaviors which might occur even if microscopic Hamiltonians involve only short-range couplings.

In a close proximity to the corresponding quantum phase transition, an effective singular coupling is mediated by (nearly gapless) collective excitations of the emergent order parameter of either charge or spin nature. Important examples include such extensively studied problems as antiferromagnetic [1,2] and charge [3] ordering transitions in hole-doped cuprates, quantum-critical ferromagnetic [4] and antiferromagnetic [5] instabilities in heavy fermion materials, the compressible quantum Hall effect (QHE) [6], and Pomeranchuk transitions in low-dimensional electron gases. The Pomeranchuk transition has been originally discussed in relation to the transport anisotropies reported in QHE systems at large half-integer filling factors [7]. In a more general setting, the idea of a spontaneous Pomeranchuk-like distortion of the Fermi surface (FS) associated with the transition to a rotationally anisotropic "nematic" state in a generic fermion system was put forward in Refs. [8–10].

Despite their different physical nature, the systems studied in Refs. [1,3–6,8] conform to the model of a finite density gas of two-dimensional fermions coupled via a collective bosonic mode, whose own dynamics is described by the (transverse) gauge-field-like propagator

$$\chi(i\omega, q) = -\frac{\chi_0}{\gamma|\omega|/q + q^2}. \tag{1}$$

A singular nature of the effective interaction (1) manifests itself in singular corrections to the fermion propagator $G^0(\omega, \vec{k}) = 1/(i\omega - \xi_k)$, where a generic fermion dispersion $\xi_k = v_F\tilde{k}_\perp + \beta\tilde{k}_\parallel^2$, $\tilde{\mathbf{k}} = \mathbf{k} - \mathbf{k}_F$, accounts for a nonzero FS curvature of order $\beta \sim v_F/k_F$.

To first order, the fermion self-energy defined as $\Sigma(\omega, \vec{k}) = G^{-1}(\omega, \vec{k}) - [G^0(\omega, \vec{k})]^{-1}$ takes the form [1]

$$\Sigma_1(i\omega, \vec{k}) = \int \frac{d\Omega d\vec{q}}{(2\pi)^3}\chi(i\Omega, \vec{q})G^0(i\omega + i\Omega, \vec{k} + \vec{q}) \sim i\omega_0^{1/3}\omega^{2/3}, \tag{2}$$

where $\omega_0 \sim \chi_0^3/(v_F^3\gamma)$. At energies $\omega < \omega_0$, the one-loop self-energy (2) exceeds the linear in energy term in the bare propagator. Therefore, it can no longer be treated as a perturbation, and the higher-order contributions must be considered as well.

If one chooses to completely neglect the FS curvature altogether, a naive perturbation series expansion for the self-energy appears to produce increasingly more and more divergent terms, the nth-order term behaving as $\Sigma_n \propto \omega^{1-n/3}$ (Refs. [11,12]). However, a finite FS curvature provides a regularization of such spurious divergences [9,11–13]. Namely, by treating the bosonic mode governed by the propagator (1) as a slow subsystem and invoking the generalized Migdal theorem, one finds that the vertex corrections are controlled by a (inverse) FS curvature and appear to be small in powers of $((\ln a)/a)^2$ for large values of the parameter $a = \beta k_F/v_F$.

Proceeding along these lines, the authors of Refs. [9,11,12] developed a self-consistent, Eliashberg-type approach, according to which the all-order ansatz for the fermion self-energy demonstrates a distinctly non-Fermi-liquid behavior $\Sigma(\omega) \propto \omega^{2/3}$.

As a result, the equal-time fermion propagator

$$G(0, \vec{r}) = \int \frac{d\omega d^2\vec{k}}{(2\pi)^3}\frac{e^{i\vec{k}\vec{r}}}{i\omega + i\omega_0^{1/3}\omega^{2/3} - \xi_k} = \frac{v_F}{\pi}\int_0^\infty d\omega\int_{-\infty}^\infty d\xi_k\frac{J_0(k_Fr + r\xi_k/v_F)}{i\omega + i\omega_0^{1/3}\omega^{2/3} - \xi_k} \sim G^0(0, \vec{r})\left(\frac{r_0}{r}\right)^{1/2} \tag{3}$$

exhibits an algebraic decay which is faster than that of its free fermion counterpart, $G^0(0, r) \propto e^{ik_F r}/(k_F r^3)^{1/2}$. The asymptotic behavior (3) sets in at distances $r > r_0 \sim \gamma v_F^4/\chi_0^3$. Albeit being seemingly independent of the FS curvature, Eq. (3) was derived under the condition of convergence of the perturbative expansion for the self-energy $\Sigma(\omega)$, which can be guaranteed only provided that $a = \beta k_F/v_F \gg 1$ [12].

In the physically relevant situation ($a \sim 1$), the vertex corrections become of order one, and the perturbative expansion ceases to be controllable. Nevertheless, it has been conjectured in Ref. [11] that at the lowest energies the results of the Eliashberg theory remain qualitatively applicable and that the power-law behavior of $G(0, r)$ can be altered only in the unphysical limit of vanishing FS curvature, $a \to 0$ (see also Ref. [14]).

However, these findings have been recently challenged by the results obtained by the method of multidimensional bosonization in the context of the problem of 2D quantum nematic states. Devised as an alternative to the diagrammatic perturbation theory, the early version of multidimensional bosonization was based on a heuristic idea of dividing the FS onto small "patches" and introducing quasi-1D charge (spin) density operators [15]. While being sufficient for reproducing the correct behavior of the two-particle amplitudes at low energies and momenta, such a simplification has not been fully justified in the case of a single-particle amplitude or even the $2k_F$ behavior of a two-particle one [16].

Such an uncertainty notwithstanding, the bosonization technique of Ref. [16] was applied to the equal-time fermion propagator which was found to decay faster than any power law, $G(0, r) \propto \exp(-(r/r_0)^{1/3})$ [17]. The authors of Ref. [17] argued that this exponential behavior cannot be obtained within the Eliashberg theory, thus raising concerns that the latter might be intrinsically incomplete. On these grounds, the applicability of the Eliashberg theory was also questioned in other contexts, including the high-T_c superconductors [18].

In view of the continuing controversy, in this Letter we set out to revisit the status of the bosonization results pertinent to the gauge-fermion problem. Following the work of Ref. [11], we focus on the role of the FS curvature which was neglected in Ref. [17].

In the well-studied 1D case, any deviation from the linear fermion dispersion gives rise to cubic terms in the corresponding bosonized theory. While such terms do spoil the Gaussian form of the bosonic action, they appear to be subdominant, as far as the asymptotic long-range behavior of the one- and two-fermion amplitudes is concerned. When treated self-consistently, though, these cubic terms produce quadratic corrections to the spectrum of the 1D charge and spin density modes [19].

In D > 1 dimensions, the situation appears to be more involved, as the (potentially irrelevant) quadratic corrections to the fermion spectrum $\sim q_n^2 = (\vec{n}\,\vec{q})^2$, which depend solely on the component of the transferred fermion momentum $\vec{q}$ parallel to a unit vector $\vec{n}$ normal to the FS, are always complemented by those quadratic in terms of the tangential to the FS component, $q_t^2 = (\vec{n} \times \vec{q})^2$. The latter terms have no 1D analogs and represent a genuine effect of the FS curvature, as opposed to a merely nonlinear dispersion. Therefore, they cannot be *a priori* discarded on the same basis as those associated with the normal component of the transferred momentum.

According to the bosonization recipe, a general formula for the 2D fermion propagator reads [16]

$$G(t, \vec{r}) = \oint_{\mathrm{FS}} \frac{d\vec{n}}{2\pi} \frac{d\omega}{2\pi} \frac{d\vec{q}}{(2\pi)^2} e^{i(\vec{q}\,\vec{r} - \omega t)} G^0_{\vec{n}}(\omega, \vec{q}) Z_{\vec{n}}(t, \vec{r}), \tag{4}$$

where the quasi-1D "patch" Green function

$$G^0_{\vec{n}}(\omega, \vec{q}) = \frac{1}{i\omega - v_F \vec{n}\,\vec{q}} = \int dt d\vec{r} \frac{e^{-i(\vec{q}\,\vec{r} - \omega t)}}{i v_F t - \vec{r}\,\vec{n}} \tag{5}$$

describes 1D fermion motion in the direction of the normal vector $\vec{n}$ defining a FS patch of linear size $\Lambda \ll p_F$.

The impact of the interaction on the fermion propagator is encoded in the "eikonal" (Debye-Waller-type) factor $Z_{\vec{n}}(t, \vec{r}) = \exp[-\Phi_{\vec{n}}(t, \vec{r})]$, where

$$\Phi_{\vec{n}}(t, \vec{r}) = \int \frac{d\omega d\vec{q}}{(2\pi)^3} \chi(\omega, \vec{q}) G^0_{\vec{n}}(\omega, \vec{q}) G^0_{\vec{n}}(-\omega, -\vec{q}) \times (1 - \cos(\omega t - \vec{r}\,\vec{q})), \tag{6}$$

which expression is common to any approximate scheme, where the system of interacting fermions is substituted by an effective single-particle environment composed of bosonic collective modes.

With the FS curvature neglected, Eq. (6) features a 1D effective interaction $\chi_{1D}(\omega, q_n) = \int (dq_t/2\pi)\chi(\omega, \vec{q}) \propto \omega^{-1/3}$ between the quasi-1D fermions belonging to a given patch. The applicability of the whole scheme hinges on the expectation that, for a sufficiently singular interaction function, such as that of Eq. (1), the scale Λ which sets the upper limit in the integration over the transverse momentum q_t will effectively drop out of Eq. (6).

However, a simple analysis shows that a typical value of the tangential component of the transferred momentum $q_t \sim \omega^{1/3}$ is by far greater than the normal one, $q_n \sim \omega$. Therefore, the validity of the assumption about the irrelevance of the FS curvature is anything but granted.

In order to assess the applicability of Eq. (6) in the case of a finite FS curvature, we compare it with the direct perturbative expansion for $G(0, r)$. To that end, we formally expand $Z_{\vec{n}}(0, r)$ in powers of $\Phi_{\vec{n}}(0, r)$ and make use of the identity

$$G^0(\omega,\vec{k})G^0(\omega+\Omega,\vec{k}+\vec{q})=\frac{G^0(\omega,\vec{k})-G^0(\omega+\Omega,\vec{k}+\vec{q})}{i\Omega-\xi_{k+q}+\xi_k}, \tag{7}$$

where $G^0(\omega,\vec{k})$ is the bare fermion Green function.

We explicitly verified that the first-order correction given by Eqs. (4) and (6) can be cast in the equivalent form

$$G(0,\vec{r}) = G_0(0,\vec{r}) + \int \frac{d\omega d\vec{k}}{(2\pi)^3} e^{i\vec{k}\vec{r}} G_0^2(\omega,\vec{k})\Sigma_1(\omega,\vec{k}), \tag{8}$$

where the lowest-order self-energy is given by Eq. (2). Thus, to first order, the bosonization and perturbation theory results agree with each other, and the FS curvature does not manifest itself.

However, an explicit comparison between the second-order term in the expansion of Eq. (6) and that of the perturbation theory for the self-energy demonstrates that the corresponding expressions can be reconciled only provided that one uses the fermion Green function

$$G_{\vec{n}}^0(\omega,\vec{q}) = \frac{1}{i\omega - v_F\vec{n}\,\vec{q} + \beta(\vec{n}\times\vec{q})^2} \tag{9}$$

instead of that with the linearized fermion dispersion. We also found that functional forms of the higher-order contributions to the self-energy $\Sigma(\omega)$ obtained by means of perturbation theory and eikonal approximation agree with each other, provided that G_0 is given by (9), although the corresponding prefactors do not necessarily match. This discrepancy should have been expected, though. Indeed, while being able to capture the main effect of small-angle scattering due to singular interactions, the eikonal approximation is not expected to be exact in 2D.

Taken at its face value, the above observation shows a relevance of the FS curvature. It also suggests a way to improve on the results obtained by virtue of the original bosonization technique. To explore the consequences of using Eq. (9) for $G_{\vec{n}}^0(\omega,\vec{q})$, we study a spatial dependence of Eq. (6) modified in accordance with the above prescription.

Substituting (9) into (6) and introducing $r_n = \vec{n}\,\vec{r}$, we readily obtain

$$\Phi_{\vec{n}}(t,\vec{r}) = \frac{\chi_0}{2\pi^3 v_F\gamma}\int_0^\infty \frac{dq_n}{\beta q_n}(1-\cos q_n r_n) S\left(\frac{v_F q_n}{\gamma^2\beta^3}\right), \tag{10}$$

where the integrand reads

$$S(z) = \int_0^\infty dx \int_0^\infty dy \frac{1}{\sqrt{zx^3+y}} \mathrm{Re}\left[\frac{1}{(1-iy)^2-x^2}\right]. \tag{11}$$

Evaluating the integrals in Eq. (11), we find that at $z \gg 1$ (e.g., in the limit of a vanishing FS curvature) $S(z) \approx (4\pi^2/27)z^{-1/3}$, thus yielding

$$\Phi_{\vec{n}}(0,\vec{r}) \approx \left(\frac{r_n}{r_0}\right)^{1/3}, \tag{12}$$

where $r_0 = [3\sqrt{3}v_F^{4/3}\gamma^{1/3}/(\Gamma(2/3)\chi_0)]^3$. This asymptotic behavior agrees with the result obtained in Ref. [17].

In the opposite limit $z \ll 1$, the function (11) attains a constant value $S(z) \approx (\pi^2/8)$ (Ref. [20]), thereby giving rise to the logarithmic behavior

$$\Phi_{\vec{n}}(0,r_n) \approx \frac{\chi_0}{16\pi\gamma\beta v_F}\log\left(\frac{r_n\beta^3\gamma^2}{v_F}\right). \tag{13}$$

Consequently, at the longest distances the equal-time fermion propagator shows a power-law decay $G(0,\vec{r}) \propto 1/r^{(3/2)+\eta}$ governed by the anomalous exponent $\eta = \chi_0/(16\pi\beta\gamma v_F)$. This behavior is in qualitative agreement with that predicted by the Eliashberg theory.

For a quantitative comparison, it might be instructive to evaluate the exponent η for the parameters of Eq. (1) computed (rather than postulated) under the assumption that the dynamics of the collective mode is governed by the fermion polarization itself, $\chi(\omega,q) \approx \chi_0/(q^2+\Pi(\omega,q))$ [12]. In this situation, which tends to be rather common in strongly correlated systems, one has $\gamma = m\chi_0/(2\pi v_F)$, with $m = k_F/v_F$. By making an additional assumption of a circular FS, i.e., $\beta = 1/(2m)$, one obtains $\eta = 1/4$, which is only a factor of 2 short of the Eliashberg result (3).

In light of the above, we conclude that the "minimal" way of accounting for a finite FS curvature through Eq. (9) allows one to arrive at a qualitative agreement with the results of Refs. [9,11,12], although any further progress towards a quantitative agreement is likely to require an even more drastic rectification of the original bosonization scheme.

In this regard, our conclusions differ from those drawn in Ref. [21], where the first attempt to account for the effects of the FS curvature was made. The authors of Ref. [21] claimed that the FS curvature merely provides a cutoff for the infrared divergences, so that $Z(0,r_n)$ remains finite at $r_n \to \infty$, and the equal-time fermion propagator retains the free fermion behavior with $\eta = 0$. We believe that the logarithmic divergence (13) was overlooked in Ref. [21].

For completeness, we also consider the complementary limit of Eq. (6) at large t and $r = 0$. The corresponding asymptotic behavior of the Φ factor is then given by the formula

$$\Phi_{\vec{n}}(t,0) = \frac{\chi_0}{2\pi^2 v_F}\int_0^\Lambda \frac{d\omega}{\beta\omega}(1-\cos\omega t) \times \int_0^\infty \frac{du}{1+u^3}\mathrm{Im}\left[\frac{1}{\sqrt{u^2+\omega^{2/3}(\omega^{1/3}+i\beta u^2)^2}}\right]. \tag{14}$$

At $t < 1/\beta^3$, the curvature is unimportant, and a direct evaluation of Eq. (14) gives the following result:

$$\Phi_{\vec{n}}(t,0) = -\frac{\Gamma^2(\frac{3}{4})}{4\pi}\sqrt{\frac{\chi_0}{E_F}} + \frac{1}{6E_F t}, \tag{15}$$

where $E_F = \pi\gamma v_F^3/\chi_0$, which equals $mv_F^2/2$ if the value of γ is calculated self-consistently (see above).

Notably, the expression (15) is real, independent of the patch size Λ, and approaches its long-time asymptotic value as $1/t$. This is very different from the zero-curvature behavior of $\Phi_{\vec{n}}(0, \mathbf{r}) \propto r^{1/3}$. The difference stems from the fact that finite r and t provide two different regularizations of the double Green's function pole in Eq. (6). Also, contrary to the case of the equal-time behavior, the FS curvature affects only the subdominant term in (15) by replacing it with $\sim(1/t)\ln(t\beta^3\gamma^2)$ for $t > 1/(\gamma^2\beta^3)$.

The difference between $\Phi_{\vec{n}}(t, 0)$ and $\Phi_{\vec{n}}(0, \mathbf{r})$ has been previously observed in Refs. [11,17]. The authors of Ref. [11] neglected any terms with E_F in the denominator, thus arriving at the result $\Phi_{\vec{n}}(t, 0) = 0$. In contrast, the authors of Ref. [17] performed the same computation as we did but found that the time-dependent term in $\Phi_{\vec{n}}(t, 0)$ decays as $t^{-2/3}\ln t$, which we believe to be the result of a technical error.

A general failure of the heuristic $D > 1$-bosonization technique to properly account for the (apparently, important) effects of the FS curvature can be traced back to the substitution of the underlying W_∞ algebra of the phase space transformations [22] with the approximate $U(1)/SU(2)$ Kac-Moody commutation relations for the quasi-1D charge or spin density operators [16]. The importance of taking into account the true algebraic structure of the bosonized theory was elucidated in the "geometrical bosonization" approach, which strived to reformulate the dynamics of interacting fermions as a purely geometric theory of the fluctuating FS [23].

Notably, a full account of the exact W_∞ algebraic relations has already proven to be instrumental for calculating exact correlation functions of the 1D Calogero-Sutherland model (which also includes the case of noninteracting 1D fermions with parabolic dispersion) [24]. This algebraic structure is also present in a more recent reincarnation of the idea of geometric bosonization that has been independently developed in the theory of mesoscopic transport under the name of "ballistic σ model" [25].

In summary, we revisited the problem of two-dimensional fermions coupled to a gauge-field-like collective mode. Comparing the formula for the fermion propagator obtained by means of the multidimensional bosonization of Ref. [16] with that found in the framework of the Eliashberg approach [11,12], we observed that, in order for the two to agree, the bosonization prescription must be modified in order to incorporate the FS curvature. Contrary to the earlier claims, we find that including the FS curvature into Eq. (6) alters the predictions of the original bosonization approach, thereby resulting in a qualitative agreement with the Eliashberg theory.

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