Convergence Analysis of Heterogeneous Decision-making Populations Under the Coordinating Best-response and Imitation Update Rules

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Abstract

This thesis emphasis is on coordination games. In a coordination game, selecting the same strategy or decision as the opponent is mutually beneficial for both parties. We studied the problem of equilibrium convergence in such games in both discrete and continuous (time) cases.

In the first Chapter, we provide a brief introduction to the field of game theory. We discuss different categories of agents based on their levels of rationality and decision-making strategies, along with a variety of games. Additionally, we address important issues and challenges within this field.

The second Chapter of this work is dedicated to a heterogeneous mixed population of imitators and best-responders. In this model, agents' update rules are assumed to be discrete functions of time. Imitators refer to agents who simply replicate the strategy of another agent with the highest payoff, while best-responders pick the strategies that maximise their individual outcomes. Suggesting the concept of 'sections'–a consecutive sequence of agents with similar strategies– helped us in establishing convergence to an equilibrium state. This convergence was demonstrated under any arbitrary asynchronous activation sequence within a linear network. The proof was then extended to networks with ring, starike, and sparse-tree structures. However, the question of equilibrium convergence for other network structures remains an open challenge.

In the third Chapter, we examined a large well-mixed population of imitators within a coordination context. Our analysis is grounded in the assumption that imitation here is driven by dissatisfaction. Equivalently, agents with lower payoffs are more dissatisfied and have more tendency to change and imitate higher earners within the population. The

analysis reveals the presence of three fixed points, of which two are stable and one is a saddle point. The stable manifold of the unstable fixed point is also calculated. Additionally, It is demonstrated that starting from any initial state, the population eventually converges towards one of these introduced fixed points.

Acknowledgements

Life is like a deck of cards; it's all about luck and how you play. Each card contains a symbol, a number, and a picture. You only play with the symbols and numbers, but pictures make your deck more beautiful. Gradually, you get used to the pictures, and your card without its picture looks so empty to you. You start learning that each picture conveys both the number and the symbol. You start counting on pictures instead of numbers in your game.

In the game of life, friends are the pictures. They make your life brighter and more delightful. You gradually learn that life is nothing but having friends you can rely on and being sure they will be there for you.

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Contents

List of Figures

Chapter 1

Introduction

Having a set of options, the situation where an individual has to choose a few from the given set creates a decision-making dilemma[11]. For example, a chef is planning a menu for a restaurant. They must choose a few dishes from many delicious options to feature.

Each decision-making problem consists of two main components, the decision-maker and the set of options. Back to our example, the chef is playing the role of the decision-maker, and the set of options includes all the dishes on the menu.

The limit on the number of options that can be chosen by the decision-maker leads them to this question: "Which subset of options should I choose?". The answer to this question depends extremely on the decision-maker's level of rationality.

Each individual puts effort into choosing the option that maximizes their outcome. The definition of outcome is context-dependent. In our example, the chef intends to increase the restaurant's income by choosing the features that influence the customers. The restaurant's income in this example is considered the outcome of the decision-making problem.

The more general case occurs when individuals' decisions are interdependent, meaning that an individual's decision is not only a function of their priorities over different subsets of all existing options but also influenced by others' decisions. Imagine a street with multiple restaurants. Each restaurant has some specialty dishes. To compete with other restaurants, choosing the same features as them seems tempting. However, it is not a way to stand out against the other restaurants. The chef must consider these factors along with their team's strengths and limitations in choosing the features that maximize the restaurant's income.

The impact of others' decisions on an individual's payoff (outcome) may drive the individual to change their decision once they have the opportunity; if a chef notices that a neighboring restaurant has chosen the same features, they may feel the need to differentiate themselves and select new, unique features.

Revising decisions may raise dissatisfaction among people who have already made their decision. We say the population (of individuals) is in satisfaction if each individual despite having the chance to change their decisions sticks to it. In other words, each individual believes their current decision maximizes their outcome regardless this is true or false. In our example, all chefs are already satisfied with their selected features.

One question is whether a population of decision-makers reaches such a satisfactory state. The answer is important from several aspects. For example, being aware of reaching a satisfactory state in economic systems enables us to adapt to changing circumstances and avoid major disruptions. In economics, this satisfactory state may lead to the stability of markets which can be reinforced by rules and incentives[5].

However, it may also be important how fast a population of decision-makers reaches such a satisfactory state if there is one. Moreover, if a population reaches such a satisfactory state, can it be guaranteed that by changing the decisions of a small proportion of the population, the population reaches that satisfactory state again? In other words, how stable this satisfactory state is.

The answer to all these questions depends on several aspects that one of them is how rational each individual is. For example, someone may just choose randomly among the options provided. In some cases, the decision-maker may look at others in the population and see whose outcome is more and what is their choice. Then they make the same decision hoping this choice increases their outcome. It means such an individual may believe the outcome only depends on the decision and no other factor. Some may discover others' decisions also affect their outcome. Therefore, they consider both their preferences and others' decisions in calculating the outcome. Someone may also find other affective factors and involve them in their final decision, such as the time of making a decision. Some people learn from made mistakes, and each round use this knowledge in their decision revisions.

The story and context behind the set of options each individual is given also play a vital role in the answer to questions we discussed before. For example, assume you are a student and your professor gives you the option to solve only three out of five questions in the textbook. In this context, if you choose the same set of questions that most of your friends have chosen, then you can discuss those problems together and solve them easier. In this case, choosing the same subset of options as your friends brings you a higher score on the assignment which is the payoff in this case.

There are some other cases where choosing a different option from that of other decision-makers results in higher payoffs. Consider a scenario where there are limited resources and individuals are allowed to choose one resource for their usage. If many people choose the same resource, each individual may receive a smaller proportion of that resource in the end. This implies choosing a resource that is chosen by fewer people is more profitable. Therefore, the story behind the options available is also important in understanding the dynamics of a population.

In this regard, scientists first came up with Classical Game Theory; mathematical models with the aim of studying the strategic interactions between rational agents[10]. In these models, each individual is referred to as an agent and the options that they can take as a set of strategies. Usually, it is assumed that the agents' strategies are singleton subsets of all the possible strategies.

In the network graph of a population, each agent corresponds to a node in a graph, and an edge between two nodes represents a neighborhood relationship between the corresponding agents. Specifically, two agents are considered neighbors if they participate in a game together. Two agents are said to be in a game if their decisions have an impact on each other.

The games between every two agents are defined in the format of an $n \times n, n \in \mathbf{N}$, matrix, known as the *payoff matrix*, that describes the outcomes associated with each possible combination of strategies that the players can choose. Entry a_{ij} is the outcome of playing the i'th strategy while your opponent plays the j'th strategy and totally there exist n different strategies:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

If the network of a population is complete, the population is said to be well-mixed. If the payoff matrices of games are different from each other the population is heterogeneous. In case all payoff matrices are the same, the population is homogeneous.

A game is coordination (resp. anti-coordination) if the agent earns more (resp. less) by choosing the same strategy as their opponent. It means the diagonal values of the matrix are greater (resp. less) than the non-diagonal values in the same matrix.

We refer to an agent as a best-responder if every time it updates its strategy, it maximizes its outcome. An agent is an imitator if it chooses the strategy of the neighbor with the highest outcome.

An example of coordination games is *Battle of the Sexes*[4]: Imagine a couple is planning to spend the night together, but has a choice between two events to attend: a prize fight and a ballet. Prize fight is the man's preference while the woman's preference is the ballet. However, both would prefer to spend the night together and go to the same event rather than different ones. The payoff matrix for such a game would look like:

$$\begin{array}{c} \\ \text{Prize Fight} \\ \text{Ballet} \end{array} \begin{array}{c} \begin{array}{cc} \text{Prize Fight} & \text{Ballet} \end{array} \\ \begin{bmatrix} 5,3 & 0,0 \\ 0,0 & 3,5 \end{bmatrix} \end{array}.$$

The values in each cell of the matrix represent the man's and woman's payoffs respectively when the man's and woman's decisions are the row and column corresponding to that cell, respectively.

An example of anti-coordination games is the *Snowdrift Game*[6]. In the snowdrift game, a situation is imagined where two drivers are stuck behind a snowdrift. Each driver's option is either shovelling the snow to free a path (Cooperate) or doing nothing and remaining in their car (Defect). The case when a player leaves their opponent to do shovelling all by themselves is the most rewarded situation, however, the opponent is still rewarded for their work. The situation involving both drivers in shovelling is also beneficial for both. The worst case is when non of the players participate in the shovelling and they are stuck behind the snowdrift forever!

$$\begin{array}{c} \\ \text{Cooperate} \\ \text{Defect} \end{array} \begin{array}{c} \begin{array}{cc} \text{Cooperate} & \text{Defect} \end{array} \\ \left[\begin{array}{cc} 2,2 & 1,3 \\ 3,1 & 0,0 \end{array}\right] \end{array}.$$

A game is called cooperative if the best outcome is achieved through mutual cooperation. The previously mentioned game, snowdrift, is also considered a cooperative game. Since the dominant strategy for both players is defecting, the final outcome will be the worst out of all the possible outcomes for the game. Cooperation is the best approach to avoid this outcome.

Classical game theory assumes that agents are rational and act in their own self-interest when choosing their strategies. In this framework, researchers define the game, and there is no inherent limitation on the agents' level of rationality except for the researcher's rationality! Each agent has only one chance to choose their strategy, and the game's outcome is determined by the agents' collective actions. The objective of classical game theory is to predict the outcomes of strategic interactions. Agents make strategic choices based on the expected payoffs of the available strategies to achieve this goal. Researchers use various methods, such as backward induction, to compute all possible outcomes and select the optimal strategy that maximizes the agents' payoff. Some of the applications of this branch can be seen in the robotic area, such as multi-agent systems, human-robot interaction, resource allocation, and security [7, 13, 1, 3, 8].

However, in the real world, the existence of a hyper-rational agent is rare. It is seen that cells and even humans are not hyper-rational in making their decisions. They have mutable strategies that can change over time. They are not necessarily rational or self-interested, but rather they have behavioural tendencies that determine their actions. The success of a strategy or a species in a given environment and understanding the evolution of behaviour over time matter in these studies [2],[12].

These findings yield scientists to come up with evolutionary game theory a new model that addresses these issues. In this new model, a population of agents with different strategies is assumed. A significant part of this new framework is the ratios of different strategies' populations. Each successful strategy reproduces more of itself and gradually makes that species population dominant. We say the whole population is in equilibrium if the ratio of each population's strategy does not change.

The scenario where agents have the ability to revise their decisions can be represented as follows: whenever an agent changes its strategy (e.g., from A to B), one member of the A population dies and a new member is produced in the B population.

In the evolutionary game theory, it is assumed that agents necessarily do not stick to a strategy. They often have the chance to revise their strategies. But how often? The answer to this question depends on the definition of time in the problem. The change in agents' strategies can be defined as a discrete or continuous function of time. In the discrete case usually, there is an activation sequence $\langle a_t \rangle_{t=0}^{\infty}$ where a_t denotes the set of agents active at time t. If for every $t > 0$, a_t is a singleton then the activation sequence is called asynchronous, otherwise, it is synchronous. Equilibrium in this case is a state that does not change under any activation sequence. In the continuous case, we are dealing with a system of differential equations that provide us with the rate of changes in each population size as a continuous function of time. Finding the fixed points of this system, the states where the rate of changes in all populations is zero provides us with the equilibrium states of the populations.

An equilibrium state is considered stable if small perturbations to the strategies' ratio result in the population returning to that equilibrium state. Under certain conditions and theorems, it is possible to experimentally investigate the stability of the fixed points of the systems.

In the case where the population size is large enough and the population is well-mixed, the discrete dynamic can be approximated by the continuous one and we can use tools in continuous cases for dealing with the discrete problem. This case is known as *mean dynamics*.

Our focus in this thesis is on coordination games. The second chapter is dedicated to the problem of equilibrium convergence in such games. We considered a heterogeneous mixed population of imitators and best-responders. The agents' update rules are assumed as discrete functions of time. We suggested the number of "sections" (a consecutive sequence of similar-strategies agents) in a linear graph as a potential function[1] for the given dynamics. A potential function is a defined scalar function over the strategy space of a game that assigns a numerical value to each possible combination of strategies chosen by the players[9]. The suggested potential function was the key idea for proving the convergence of the sparse-tree networks to an equilibrium point under any arbitrary asynchronous activation sequence. However, the equilibrium convergence for general graphs remains unsolved.

Chapter 3 focuses on analyzing a specific type of coordination game using mean dynamics. The analysis reveals the existence of three fixed points, of which two are stable and one is non-hyperbolic.

Chapter 2 has been accepted for publication in *The IEEE Control Systems Letters*. It is in the format of the manuscript submitted for publication. Chapter 3 is an ongoing work and requires further analysis.

[1] By a potential function we mean a positive upper bounded function that monotonically decreases over time.

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Chapter 2

Equilibrium Convergence in Discrete-time Coordination Games

2.1 Introduction

Evolutionary game theory has been successfully applied in different applications ranging from cancer and epidemiology to finance and rumour propagation [1, 12, 5, 2]. In the context of decision-making, individuals are modeled as game-playing agents who choose from a number of available strategies and accordingly earn payoffs against their matched opponents. The agents revise their decisions according to some update rules, the most common being *(myopic) best-response* and *imitation*. An agent following best-response, called a *best-responder*, chooses the strategy that maximizes its payoff against its neighbors given that they would not change their strategies. On the other hand, an agent following imitation, known as an *imitator*, simply imitates a neighbor with a higher payoff. The wide use of best response by human has been confirmed in experimental studies [13]. Similarly, imitation behavior emerges in several real-world scenarios, such as employees' "costumer sweethearting" [6], building cultural intelligence [15], and training language models [22].

Researchers have explored the existence and convergence towards an equilibrium point in both imitation and best-response dynamics [4, 9, 7, 10]. In the anti-coordination context where the highest-earning decision is the opposite of the opponent's, a population of best-responders converges to an equilibrium state [18]. The same holds for a population of best-responders in the coordination context, where the highest-earning strategy matches the opponent's [18, 17]. For populations of imitators, however, equilibration is guaranteed only in the coordination context [19, Theorem 1]. All of these studies used a potential function to prove equilibrium convergence. Clearly, a mixed population of imitators and best-responders may not equilibrate and undergo perpetual fluctuations [11]. The outcome is known for the anti-coordination case: equilibration can take place if and only if there exists an equilibrium [11]. What about a mixed population of imitators and best-responders in the coordination context? The existence of an activation sequence was established in [21] that would drive any such mixed population to an equilibrium state. It however remains open whether a potential function exists for such populations that guarantees the convergence of the population to an equilibrium state under any activation sequence, or if there is a counter example where an activation sequence can prevent a mixed population from equilibration.

We start tackling this problem for the simple *linear graph* and find that the number of the so-called "sections" (consecutive same-strategy playing agents) serves as a potential function. We then extend the results to a ring. Next, we proceed to a *starlike* graph, a central "branching node" connected to several linear graphs or "branches". We show that there always exists a branch where the number of sections in that branch will again be a potential function, establishing equilibration. Finally, we generalize the idea to *sparse trees*, i.e., trees where the distance between each two branching nodes is at least three.

2.2 Model

Consider an undirected network $\mathcal{G}$ over a finite set $\mathcal{N} = \{1, 2, \dots, n\}$ of agents who decide between strategies $\mathtt{A}$ and $\mathtt{B}$ over time $t = 0, 1, 2, \dots$. For each agent $i \in \mathcal{N}$, the network defines a set of *neighbors* $\mathcal{N}_i \subseteq \mathcal{N} \setminus \{i\}$ that are connected to agent i. At every time step, each agent $i \in \mathcal{N}$ plays a two-player *(row-column) coordination game* with each of its neighbors $j \in \mathcal{N}_i$

and earns a payoff according to their strategies and its payoff matrix

$$\boldsymbol{\pi}^i = \begin{bmatrix} R_i & S_i \\ T_i & P_i \end{bmatrix}, \ \min\{R_i, P_i\} > \max\{T_i, S_i\} \tag{2.1}$$

where R_i, S_i, T_i, and P_i are agent i's payoffs when agents i and j play strategy pairs $(\mathtt{A}, \mathtt{A})$, $(\mathtt{A}, \mathtt{B})$, $(\mathtt{B}, \mathtt{A})$, and $(\mathtt{B}, \mathtt{B})$. Then agent i's *utility* u_i is the accumulated payoff earned against all of its neighbors: $u_i(\boldsymbol{x}) = \sum_{j \in \mathcal{N}_i} \boldsymbol{\pi}^i_{x_i, x_j}$ where x_j is the strategy of agent j, the *state* $\boldsymbol{x} = [x_j]$ is the vector of all agents' strategies, and $\boldsymbol{X}_{pq}$ denotes the entry of matrix $\boldsymbol{X}$ at row p and column q. Agents update their strategies based on the type of *update rule* they follow, which is either *best response*, that is to choose the strategy that maximizes its utility, or *imitation*, that is to copy the strategy of its highest earning neighbor. The updates happen asynchronously over time, i.e., at each time step, a single agent becomes active to update its strategy at the next time step. More specifically, agent i active at time t updates its strategy at time $t+1$ to the following if it is an *imitator*:

$$x_i(t+1) = x_k(t), \qquad k = \arg\max_{j \in \mathcal{N}_i} u_j(\boldsymbol{x}(t)). \tag{2.2}$$

and to the following if it is a *best-responder*:

$$x_i(t+1) = \arg\max_{\mathtt{X} \in \{\mathtt{A}, \mathtt{B}\}} u_i(\boldsymbol{x}_{i=\mathtt{X}}(t)) \tag{2.3}$$

where $\boldsymbol{x}_{i=\mathtt{X}}$ is the vector $\boldsymbol{x}$ where the i^{th} entry is fixed to strategy $\mathtt{X}$. In the case where both strategies $\mathtt{A}$ and $\mathtt{B}$ maximize the utilities in (2.2) or (2.3), agent i does not switch strategies, i.e., $x_i(t+1) = x_i(t)$.

Remark 1. The standard inequalities in a coordination game are $R_i > T_i$ and $P_i > S_i$ [20], implying that player i's payoff is maximized when playing the same strategy as that of its opponent. What condition (2.1) additionally imposes are the inequalities $P_i > T_i$ and

$R_i > S_i$, resulting in the so-called *opponent-coordination* payoff matrix [19]. Then agent i's payoff increases if its neighbor switches her strategy to that of agent i, which proves useful in constructing energy functions for imitation dynamics.

Example 1. ***[Programming languages]*** *Given the required effort to master a new programming language, programmers have to decide between two options each time they program an application:* (i) *the comfort of working in the already experienced language and* (ii) *the benefit of learning a new language. Some base their decisions on the prevalence of the language, because common languages are supported by a community of peers who can smoothen the learning experience via online forums. Others may focus on how successful other programmers were in terms of, e.g., their salaries or reputation of developed applications. The agents here are the community of App developers who interact via online networks. The programming languages* `Python` *and* `Java` *may be considered as the strategies and a programmer would earn more from his peers if they use the same language.*

Example 2. ***[Social media]*** `Telegram` *and* `WhatsApp` *are two social media applications. Individuals choosing one of them as their main communication stream may decide based on the (weighted) frequency or satisfaction of their friends on each platform, implying the best response and imitation update rules respectively. The individuals also have personal preferences over the apps because of their features, resulting in different payoff matrices.*

Define the agents' *activation sequence* as the sequence $\langle a_t \rangle_{t=0}^{\infty}$, where a_t is the active agent at time t. The activation sequence together with update rules (2.2) and (2.3) govern the state $\boldsymbol{x}(t)$ and define the *decision-making dynamics*, which we refer to as the *coordinating best-response and imitation dynamics*. A state $\boldsymbol{x}^* \in \{\mathtt{A}, \mathtt{B}\}^n$ is an *equilibrium* of the dynamics if under every activation sequence, $\boldsymbol{x}(0) = \boldsymbol{x}^*$ implies $\boldsymbol{x}(t) = \boldsymbol{x}^*$ for all $t \geq 0$. We are interested in determining whether the dynamics eventually equilibrate. We avoid trivial cases where the dynamics "get stuck" at a non-equilibrium state because one or more unsatisfied agents do not get the chance to become active. To this end, we assume the activation sequence is *persistent*, i.e., for every agent $i \in \mathcal{N}$ and every time $t > 0$, there exists some time $t' > t$ when agent i is active at time t' [16].

It follows from the coordination condition (2.1) and best-response update rule (2.3) that if agent i tends to play `A` at some state, so does it at any other state with more `A`-playing neighbors. In a more restrictive sense, it can be also shown that if an imitator tends to play `A` at some state, so does it at any other state where all of its `A`-playing neighbors still play `A`. This property is referred to as `A`-*coordinating* [21, Definition 2], based on which, the existence of an activation sequence that would drive the dynamics from a given initial condition to an equilibrium state was shown in [21, Lemma 1, Theorem 2]. However, it remains open whether the dynamics equilibrate under an arbitrary persistent activation sequence.

2.3 Equilibration results

The main result of this paper is about the equilibration of "sparse trees" as presented in the following theorem. The *distance* of two nodes in a graph is the number of edges in the shortest path connecting the two. Define a *branching agent* as an agent with more than two neighbors. We call a tree graph *sparse* if the distance between every pair of its branching agents is greater than two.

Theorem 1 (Sparse tree)**.** *A sparse-tree network equilibrates under the coordinating best-response and imitation dynamics with an arbitrary persistent activation sequence.*

Sparse trees are a generalization of starlikes, which in turn are a generalization of linear graphs, defined in what follows. We accordingly, first show the result for linear graphs (as well as rings), then starlikes, and finally sparse trees.

2.3.1 Linear graphs

Consider network $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ with edge set $\mathcal{E} = \{\{i, i+1\} \mid i = 1, \dots, n-1\}$, called a *linear graph*.

Definition 1 (Border agent)**.** *Given a linear graph, agent i is a* right-border *(resp.* left-border*) if it has a different strategy compared to agent $i+1$ (resp. $i-1$). A single agent with a strategy different from those of its two neighbors is both a right and left border agent. An agent is a* border *if it is right or left-border (or both).*

We consider the most "left" (resp. "right") agent, i.e., agent 1 (resp. n), as a left (resp. right) border agent. We can now define the notion of "section" as follows (Figure 2.1a).

Definition 2 (Section)**.** *A* section *in a linear graph at a given strategy state is a set of consecutive same-strategy playing agents $p, p+1, \dots, q$, where $q \geq p$ and agents p and q are borders. The size of the section is defined as $q - p + 1$.*

The special case of $p = q$ results in a size-one section consisting of a single agent. The number of sections appears to serve as a potential function according to the following lemma. The key idea of the proof is that the emergence of a new section requires the sequence $(\mathtt{A}, \mathtt{A}, \mathtt{A})$ (resp. $(\mathtt{B}, \mathtt{B}, \mathtt{B})$) to turn into $(\mathtt{A}, \mathtt{B}, \mathtt{A})$ (resp. $(\mathtt{B}, \mathtt{A}, \mathtt{B})$), which is impossible due to the coordinating nature of the population dynamics.

Lemma 2. *The number of sections in a linear graph does not increase under the coordinating best-response and imitation dynamics with an arbitrary activation sequence.*

Proof. A change in the population state takes place only if a border agent is active because other agents play the same strategy as their neighbors and hence do not switch strategies according to update rules (2.3) and (2.2). So the number of sections change at time t only if some border agent i becomes active at time $t-1$ and switches its strategy at time t to say strategy s. At least one neighbor of the border agent plays s at time $t-1$ as otherwise, the agent is not border. We have the following two cases, in neither of which the number of sections increases:

Case 1. Agent i has two neighbors, i.e., $i \notin \{1, n\}$. If both neighbors play s, then the border agent itself forms a section at time $t-1$, which disappears at time t. Since no other sections are generated, this results in a reduction in the number of sections. If only one neighbor plays s, then the number of sections does not change after the switch.

Case 2. Agent i has one neighbor, i.e., $i \in \{1, n\}$. Then the neighbor plays s at time $t-1$, implying that agent i itself again forms a section, which disappears at time t, resulting in a reduction.

□

As the number of sections are finite, in view of Lemma 2, there exists some time $T > 0$ when the number of sections becomes fixed and no longer changes. The sections may still expand or shrink though, preventing equilibration. However, one can show that once a section expands from a certain direction, say left (i.e., in the descending order of the agents' labels), then it may no longer shrink from left. Namely, if the left border of a section "moves" left after time T, it never "moves" right in the future. This idea is rigorously captured in the following lemma. For every time $t \geq T$, there is the same number of sections which we label as $1, 2, \ldots, S$ from left to right, that is in the ascending order of their left borders. Denote by $L_s(t)$ and $R_s(t)$ the left and right borders of section s at time $t \geq T$. Given a sequence of consecutive agents $p, p+1, \ldots, q$, where $q \geq p$, denote their strategies by $\boldsymbol{x}_{(p,p+1,\ldots,q)}$.

Lemma 3. *Consider the time T when the number of sections in the linear graph is fixed. Then for every section s and any time $t_1 \geq T$,*

$$L_s(t_1+1) = L_s(t_1) - 1 \Rightarrow \forall t \geq t_1 \;\; L_s(t+1) \leq L_s(t), \tag{2.4}$$

$$R_s(t_1+1) = R_s(t_1) + 1 \Rightarrow \forall t \geq t_1 \;\; R_s(t+1) \geq R_s(t).$$

Proof. We prove the first equation by contradiction; the proof of the second equation is similar. Assume the contrary and let $t_3 > t_1$ be the first time (2.4) is violated, i.e., $L_s(t_3+1) = L_s(t_3) + 1$. Let $t_2 \in [t_1, t_3 - 1]$ be the last time that the left border of s decreased, i.e., $L_s(t_2+1) = L_s(t_2) - 1$. Let agent i be the left border of section s at time t_2, i.e., $i = L_s(t_2)$.

Then

$$L_s(t_2) = L_s(t_3 + 1) = i, \tag{2.5}$$

$$L_s(t) = i - 1 \quad \forall t \in [t_2 + 1, t_3], \tag{2.6}$$

Without loss of generality, assume that $x_i(t_2) = \mathtt{B}$. It is straightforward to show that if the agents of section s play a strategy, say B, at time T, then the agents of section s will play B at every future time step as well. Therefore, since agent i is the left border of section s at time t_2 and plays B at t_2, it follows that all the agents in section s play B at every time $t \geq T$. Thus, in view of (2.5) to (2.6),

$$\begin{aligned}
\boldsymbol{x}_{(i-2,i-1,i)}(t_2) &= (\mathtt{A}, \mathtt{A}, \mathtt{B}), \\
\boldsymbol{x}_{(i-2,i-1,i)}(t_2 + 1) &= (\mathtt{A}, \mathtt{B}, \mathtt{B}), \\
\boldsymbol{x}_{(i-2,i-1,i)}(t) &= (\mathtt{A}, \mathtt{B}, *) \quad \forall t \in [t_2 + 1, t_3 - 1], \\
\boldsymbol{x}_{(i-2,i-1,i)}(t_3) &= (\mathtt{A}, \mathtt{B}, \mathtt{B}), \\
\boldsymbol{x}_{(i-2,i-1,i)}(t_3 + 1) &= (\mathtt{A}, \mathtt{A}, \mathtt{B}).
\end{aligned}$$

The reason why $x_{i-2}(t_2) = \mathtt{A}$ is that otherwise a section would be removed at $t_2 + 1$, which is impossible as the number of sections is assumed to be fixed after time T. Similarly, $x_{i-2}(t_3) = \mathtt{A}$ as otherwise a new section would be generated at time $t_3 + 1$.

Now we show that the two switches of strategies of agent $i - 1$ at times $t_2 + 1$ and $t_3 + 1$ are in conflict. Note that at both times t_2 and t_3 agent i plays B but has at time t_2 at most and at time t_3 at least one other B-playing neighbor. So as the game is coordinating, i.e., in view of (2.1), $u_i(t_3) \geq u_i(t_2)$. We reach a contradiction in view of Lemma 6 and by letting $T = t_2$ and $T' = t_3$.

□

We are ready to prove the equilibration of linear graphs. Consider a section s at time T.

We say that *the left border of section s moves left at time $t \geq T+1$* if $L_s(t) = L_s(t-1) - 1$ and *moves right* if $L_s(t) = L_s(t-1) + 1$. Similarly, the movement of the right border is defined.

Proposition 1. *A linear graph equilibrates under the coordinating best-response and imitation dynamics with an arbitrary persistent activation sequence.*

Proof. Consider some section s at time T when the number of sections is fixed. If the left border of section s moves left at any future time, then it can only move left afterward according to Lemma 3. Since the linear graph is constrained from left by agent 1, the left border of section s will be fixed at some time. Similarly, the right border will be fixed if it moves right at some point. So if the left border moves left at some time and the right border moves right, then the borders of section s will be fixed for all future times.

Now if any of the borders, say right, becomes fixed but the left one only moves right after time T, then also the left border becomes fixed at some point as it is bounded from right by the right border (cannot pass it). On the other hand, if the right border only moves left after time T and the left only moves right, again the two will become fixed as they cannot pass each other. Therefore, the borders of section s will become fixed at some finite time. Since s was an arbitrary section, it holds that at some finite time, the borders of every section become fixed. This implies equilibration as the activation sequence is persistent.

□

2.3.2 Extension to rings

A network $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ with edge set $\mathcal{E} = \{\{1,2\}, \ldots, \{n, n+1\}, \{n+1, 1\}\}$ is a *ring.*

Proposition 2 (Rings)**.** *A ring network equilibrates under the coordinating best-response and imitation dynamics with an arbitrary persistent activation sequence.*

Proof. Following the same arguments used for the proof of the equilibration of a linear graph, it can be shown that the number of sections in a ring may not increase, and hence will become fixed at some time T, and that if the right border of a section moves right at some time, it may never move left afterwards, and vice versa. So the only possibility for the non-equilibration of a ring is that both borders of some section s only and infinitely often move right or only and infinitely often move left. Consider the second case, i.e., moving left. Then for every agent i in the ring, there exists a time when it belongs to the section s and a time when it does not belong to the section. Hence, it will undergo the switches from $\boldsymbol{x}_{(i-1,i,i+1)} = (\mathtt{A}, \mathtt{A}, \mathtt{B})$ to $(\mathtt{A}, \mathtt{B}, \mathtt{B})$ and from $(\mathtt{B}, \mathtt{B}, \mathtt{A})$ to $(\mathtt{B}, \mathtt{A}, \mathtt{A})$. So the agent decides differently at two states with the same number of A and B-playing neighbors. Thus, in view of (2.3), agent i, and hence, the whole ring are imitators. In view of the convergence result in [19, Theorem 1] for arbitrary networks of all coordinating imitators, the proof is complete.

□

2.3.3 Starlikes

We now proceed to a more general network: The *starlike* [14], that is a tree with at most one branching agent. Define a *branch* as a linear graph that begins from a neighbour of the branching agent and ends with a leaf but does not contain the branching agent itself (Figure 2.1b-a). Definition 3 and Lemma 4 are for general decision-making dynamics but are framed here according to Section 2.2. Definition 3 is based on the notion of *eventually periodic sequences* [3] and Lemma 4 follows standard induction arguments.

Definition 3 (Eventually periodic)**.** *The coordinating imitation and best-response dynamics are* eventually periodic *under the activation sequence* $\langle a_t \rangle_{t=0}^{\infty}$ *if both the activation sequence and the resulting state* $\boldsymbol{x}$ *become periodic after some finite time* t_0*, i.e.,*

$$\exists T, t_0 \in \mathbb{N}\ \forall t \in \mathbb{N}\ [a_{t_0+t+T} = a_{t_0+t}, \boldsymbol{x}_{t_0+t+T} = \boldsymbol{x}_{t_0+t}],$$

where T is the periodicity after time t_0*, and time interval* $[t_0, \infty)$ *is the* periodic interval*. The activation sequence* $\langle a_t \rangle_{t=0}^{\infty}$ *is called an* eventually periodic activation sequence*.*

Lemma 4. *If the coordinating imitation and best-response dynamics do not equilibrate under some persistent activation sequence, then there also exists a persistent eventually periodic activation sequence under which the population does not equilibrate.*

Lemma 5. *A starlike network equilibrates under the coordinating best-response and imitation dynamics with an arbitrary persistent activation sequence, if the branching agent does not switch strategies infinitely many times.*

Proof. It is straightforward to show that Lemmas 2 and 3 and hence proposition 1 hold if any of the end nodes in a linear graph fix their strategies. Consequently, every branch together with the branching agent forms a linear graph in the starlike that will equilibrate, leading to the equilibration of the whole starlike. □

The equilibration of starlike networks is established in Proposition 3. The idea of the proof is to focus on the times when the branching agent has the maximum number of same strategy, say B-playing, neighbors. The moment one of these neighbors, referred to as the "special agent" switches, the number of sections in the branch containing this special agent, referred to as the "special branch", will decrease, and this decrement will never be compensated in the future. So the number of sections in the the spacial branch is an energy-like function (see (2.7)). If before any of the neighbors switch, the branching agent itself switches, then the branching agent must be an imitator and the neighbor with the maximum utility will serve as the special agent. Given a linear graph P, denote the number of sections in P by $n(P)$, and more specifically by $n(P,t)$ to denote the number at time t.

Proposition 3 (Starlike)**.** *A starlike network equilibrates under the coordinating best-response and imitation dynamics with an arbitrary persistent activation sequence.*

Proof. We prove by contradiction. By assuming the contrary, it follows from Lemma 4, the existence of a persistent eventually periodic activation sequence denoted by $\langle b_t \rangle_{t=0}^{\infty}$ with periodic interval $[t_0, \infty)$. The branching agent, say i, switches strategies under $\langle b_t \rangle_{t=0}^{\infty}$ infinitely often; otherwise, the network equilibrates due to Lemma 5. Let $t_{\mathtt{B}} \geq t_0$ be the first time agent i plays $\texttt{B}$ and has the maximum number of $\texttt{B}$-playing neighbours during $[t_0, \infty)$. Denote by t_1 the first moment after time $t_{\texttt{B}}$ when either agent i or one of its neighbors switches strategies, resulting in the following two cases:

Case 1: An agent i's neighbour, say agent j, changes its strategy at time t_1. Consider the branch P (referred to as the "special" branch) including agent j (referred to as the "special" agent). Denote the times when agent j switches strategies after t_1 by $t_2, t_3 \dots$. Out of these time steps, let $\langle t_k^{\texttt{B}} \rangle_{k=1}^{\infty} \subset \langle t_k \rangle_{k=1}^{\infty}$ be those time steps such that agent i had its maximum number of $\texttt{B}$-playing neighbors at each time $t_k^{\texttt{B}} - 1$. Clearly, $t_1^{\texttt{B}} = t_1$. We show that

$$\forall k \geq 1 \quad n(P, t_{k+1}^{\texttt{B}}) - n(P, t_k^{\texttt{B}}) \leq -1. \tag{2.7}$$

At every time $t_k^{\texttt{B}}$ agent j switches from $\texttt{B}$ to $\texttt{A}$; otherwise, agent i will have more $\texttt{B}$-neighbors at time $t_k^{\texttt{B}}$ compared to $t_{\texttt{B}}$. Thus, as the dynamics are coordinating, agent j has at least one $\texttt{A}$-playing neighbour at $t_k^{\texttt{B}}$, who is not agent i. Because the network is starlike, agent j has at most two neighbors, so it has exactly one other neighbor, say agent k, who plays $\texttt{A}$ at $t_k^{\texttt{B}}$. So $n(P)$ reduces by one at time $t_k^{\texttt{B}}$. In view of Lemma 7 (in Appendix), $n(P)$ does not increase if any agent other than j switches strategies. So $n(P)$ may increase in the future, only at times when agent j switches strategies, i.e., $t_{k+1}, t_{k+2}, \dots$.

We show that $n(P)$ decreases at each time $T = t_{2r}$ for an arbitrary $r \in \mathbb{N}$. Agent j switches from $\texttt{A}$ to $\texttt{B}$ at time t_{2r}. If neighbor k plays $\texttt{A}$ at time $T-1$, then agent i plays $\texttt{B}$ at the same time; otherwise, agent j does not tend to switch. So $\boldsymbol{x}_{(k,j,i)}(T-1) = (\texttt{A}, \texttt{A}, \texttt{B})$. Having the maximum number of $\texttt{B}$-playing neighbors at time t_1, agent i's utility at time $T-1$ is no more than at time $t_1 - 1$: $u_i(T-1) \leq u_i(t_1 - 1)$. But this is impossible according to Lemma 6. So neighbor k plays $\texttt{B}$ at time $T-1$. Then $n(P)$ reduces by the switch of agent j at time T.

On the other hand, $n(P)$ may increase by at most one at each time $t_{2r+1}, r \in \mathbb{N}$. Therefore, there is no finite time $T' > t_k^{\texttt{B}}$ when $n(P)$ equals its value at $t_k^{\texttt{B}}$, proving (2.7), a contradiction.

Case 2: Agent i switches from B *to* A *at time t_1.* There exists time $t_2 > t_1$ when agent i tends to switch back to B. However, the number of agent i's B-playing neighbors is maximized at time t_1, when it switched to A. Hence, because of the coordinating dynamics, agent i is an imitator.

Denote by $\langle T_r \rangle_{r=0}^{\infty}$ the time steps after t_0 that agent i changes its strategy, and let $\langle a_r \rangle_{r=0}^{\infty}$ be the corresponding neighbors imitated by agent i. Let a_j be an agent among $\langle a_r \rangle_{r=0}^{\infty}$ with the maximum utility, i.e., $a_j = \arg\max_r u_{a_r}$. So the maximum utility among the agent i's neighbors was earned by agent a_j at time T_j. Consider the branch P including agent a_j. We show that the number of sections in P decreases at least once after time T_j but never increases afterwards, which is in contradiction with T_j belonging to the periodic interval of the activation sequence.

First, we prove the following: *Statement 1. At any time T_r, $r \geq 0$, when agent i switches to* A*, agent a_j must also play* A*.* At time $T_j - 1$, agent a_j plays B and has at most one B-playing neighbor. At time $T_r - 1$, agent i plays B, so agent a_j has at least one B-playing neighbor. So if agent a_j plays B at time T_r, it earns no less than at time T_j because of the coordinating dynamics, i.e., $u_{a_j}(T_r) \geq u_{a_j}(T_j)$. Hence, according to the definition of a_j, agent a_j is a maximum earner at time T_r. Since agent i does switch at time T_r, it has to switch to the strategy of agent a_j according to (2.2). This is, however, impossible since both agents a_j and i play the same strategy B at time T_r. This proves Statement 1.

Next, we list and investigate the possible strategy states for the pair (i, a_j) starting from time T_j:

Case 2.1. $\boldsymbol{x}_{(i,a_j)}(t) = (\mathtt{B}, \mathtt{B})$. Then at the next time $T_r \geq t$ when agent i changes strategies, it switches to A. Hence, according to Statement 1, $\boldsymbol{x}_{(i,a_j)}(T_r - 1) = (\mathtt{B}, \mathtt{A})$. So according to Lemma 7, $n(P)$ reduces by at least 1 during $[t, T_r - 1]$ as agent i does not switch strategies in this interval. We reach Case 2.2 at time T_r as $\boldsymbol{x}_{(i,a_j)}(T_r) = (\mathtt{A}, \mathtt{A})$.

Case 2.2. $\boldsymbol{x}_{(i,a_j)}(t) = (\mathtt{A}, \mathtt{A})$. Then at the next time $T_s \geq t$ when agent i changes strategies, it switches to B. Now if agent a_j plays B at time $T_s - 1$, we have $\boldsymbol{x}_{(i,a_j)}(t) = (\mathtt{A}, \mathtt{B})$. So again according to Lemma 7, $n(P)$ reduces by at least 1 during $[t, T_s-1]$. We reach Case 1 at time T_s as $\boldsymbol{x}_{(i,a_j)}(T_s) = (\mathtt{B}, \mathtt{B})$. Now if agent a_j plays A at time $T_s - 1$, we have $\boldsymbol{x}_{(i,a_j)}(t) = (\mathtt{A}, \mathtt{A})$. So according to Lemma 7, $n(P)$ may not increase during $[t, T_s - 1]$. We reach Case 2.3 at time T_s as $\boldsymbol{x}_{(i,a_j)}(T_s) = (\mathtt{B}, \mathtt{A})$.

Case 2.3. $\boldsymbol{x}_{(i,a_j)}(t) = (\mathtt{B}, \mathtt{A})$. Then at the next time $T_p \geq t$ when agent i changes strategies, it switches to A. Hence, according to Statement 1, $\boldsymbol{x}_{(i,a_j)}(T_p - 1) = (\mathtt{B}, \mathtt{A})$ which is the same as the state at time t in this case. So according to Lemma 7, $n(P)$ does not increase during $[t, T_p - 1]$. We reach Case 2.2 at time T_p as $\boldsymbol{x}_{(i,a_j)}(T_p) = (\mathtt{A}, \mathtt{A})$.

At time T_j, the strategy state $\boldsymbol{x}_{(i,a_j)}$ matches Case 1, where $n(P)$ reduces. The proof is complete since it does not increase afterwards in any of the above cases. □

2.3.4 Sparse-trees

We are ready to prove Theorem 1. The idea is to show that the "special branches" of two branching nodes will intersect, resulting in the so-called "golden branch" (Figure 2.1b) which is guaranteed to equilibrate.

Proof of Theorem 1. Equilibration of starlike networks were shown in Proposition 3. So here we consider the case with at least two branching agents. We prove by contradiction and consider a persistent eventually periodic activation sequence denoted by $\langle b_t \rangle_{t=0}^{\infty}$ with periodic interval $[t_0, \infty)$. Similar to the proof of lemma 5, it can be shown that at least one branching agent changes its strategy during the periodic interval of the oscillation. We refer to the agents who change their strategy during $[t_0, \infty)$ a *settling* agent and otherwise *unsettling*. For each unsettling agent i, denote its special branch defined in the proof of Proposition 3 by P_i. Equilibration can be shown using Lemma 5 when there is no unsettling branching agent and similar to Proposition 3 when the special branches of no two branching agents overlap (no golden branch). So consider the case where there are two branching agents with the corresponding special neighbors i and j, and whose special branches intersect, denoted by P. In view of Lemma 7, $n(P)$ increases only at the time steps when either agent i or j switches. On the other hand, for both Case 1 and 2 in Proposition 3, it is guaranteed that there exists some infinite time series $\langle t_k^i \rangle_k^\infty$ (when agent i switches) such that $n(P, t_{k+1}^i) - n(P, t_k^i) \leq -1$ for all $k \geq 0$, and a some time series $\langle t_k^j \rangle_k^\infty$ (when agent j switches) such that $n(P, t_{k+1}^j) - n(P, t_k^j) \leq -1$ for all $k \geq 0$. This is a contradiction as then $n(P)$ is unbounded. □

2.4 Conclusion

We showed that every sparse tree network of coordinating heterogeneous imitators and best-responders equilibrates under any persistent activation sequence. This implies that neither the heterogeneity in the agents' perceptions of the coordination game (i.e., different payoff matrices), nor the order the agents become active can cause fluctuations in the mixed population, at least when their connections are as sparse as a sparse-tree. Whether dense trees or general graphs equilibrate under every activation sequence remains an open problem. For the proof, we introduced the number of sections in a linear graph as a potential function and generalized it to the starlike and then sparse tree networks. The potential functions may be tested in other decision making dynamics. For example, it is expected for the number of sections to increase and eventually become fixed in anti-coordination games under certain conditions [8].

Appendix

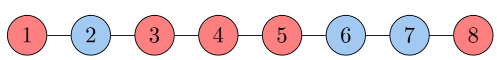
We skip the proofs of the following lemmas.

Lemma 6. *Consider a network governed by the coordinating best-response and imitation dynamics with an arbitrary activation sequence. Assume that the network includes neighboring agents $p-1$ and p, each of degree two, and denote the other neighbor of agent p by $p+1$. If there exists some time $T \geq 0$ when agent p tends to switch strategies and*

$\boldsymbol{x}_{(p-1,p,p+1)}(T) = (\mathtt{A}, \mathtt{A}, \mathtt{B})$, *then agent* p *does not tend to switch strategies at any time* T' *when* $\boldsymbol{x}_{(p-1,p,p+1)}(T') = (\mathtt{A}, \mathtt{B}, \mathtt{B})$ *and when agent* $p+1$ *earns non-less, i.e.,* $u_{p+1}(T') \geq u_{p+1}(T)$.

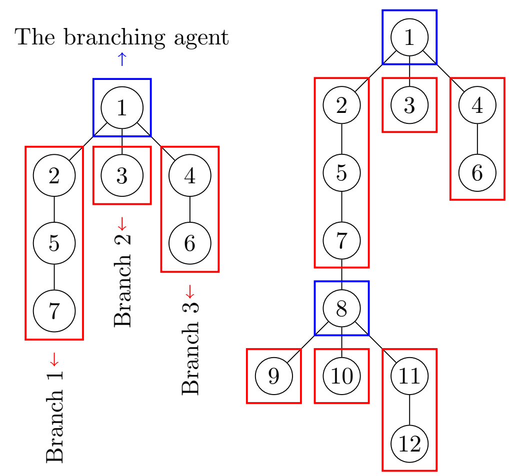
We say that a network *admits a linear graph* $(1, 2, \ldots, m)$ if there is a link between node i and $i+1$ for all $i = 1, \ldots, m-1$ and the degree of every node $2, \ldots, m-1$ is two. We refer to $(2, \ldots, m-1)$ as the *interior* of the linear graph.

Lemma 7. *Consider a network admitting the linear graph* $(1, \ldots, m)$ *governed by the coordinating best-response and imitation dynamics. Then the number of sections in the interior of the linear graph does not increase if each of the ending agents* 1 *and* 2 *either are a leaf or its strategy does not change under the activation sequence.*



(a) Line graphs

A linear graph with five sections. The sections in this linear graph are $\{1\}$, $\{2\}$, $\{3, 4, 5\}$, $\{6, 7\}$, and $\{8\}$. Blue and red are used for strategies A and B, respectively. Agents 1, 2, and 8 are each both left and right borders. Agents 3, and 6 are only left-borders, while agents 5 and 7 are right-borders. Agent 4 is a non-border agent.



(b) Starlike and sparse-tree networks

Top left graph: A starlike network. The agent on the top is the branching agent. The graph has three branches. **Top right graph: A sparse-tree network.** Each red section demonstrates a line in the population. Blue agents are branching agents of the population.

Figure 2.1: Chapter 2 figures.

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Chapter 3

Mean Field Dynamics for Diagonal Coordination Games

In the preceding chapter, we assumed a network with size N where each agent's neighbours could be any subset of the total population as long as the network's graph serves as a sparse tree. In this chapter, we revise these two assumptions by letting N be a large enough number and the network be complete–the population is well-mixed. These two assumptions transfer the underlying dynamics of the game into mean dynamics. The mean dynamics describes the rate of change in the proportions of the strategies in use which is determined by the payoff matrix and the agents' updating rules.

In this chapter, the focus is on imitation dynamics. According to imitation dynamics, upon receiving a revision opportunity, an agent randomly chooses an opponent and imitates their strategy with a probability which is a function of payoffs and the social states–the current share of each strategy among the players [4]. Under some conditions, the imitation dynamics can be approximated by *replicator dynamics* where the growth rate of the proportions of agents adopting a strategy depends on the excess payoff of that strategy. Replicator dynamics are among the most common dynamics in evolutionary game theory. The dynamics have also been extended to the multi-population game, where the population is heterogeneous in terms of payoff matrices [3]. In [2], the convergence of a two-population replicator dynamic under non-constant payoff matrices has been investigated. The mentioned works are based on the assumption that the agents of each sub-population are only interacting with themselves. In other words, the learning only occurs among sub-population members.

The utility could, however, be dependent on the whole social state. The stability of the fixed-points of replicator dynamics where the interactions among the agents depend on the sub-populations they belong to has been investigated in [7]. Some research also considers a multi-population society where the members of each sub-population only interact with members of the other sub-populations [5]. None of the above studies, however, considered a population game where the members of each sub-population interact anonymously–they interact with and are influenced by others regardless of which sub-population they belong to. In this chapter, we formulate this population game where the agents use imitation dynamics to update their decisions. In this case, the imitation dynamics do not reduce to the well-known replicator dynamics [1]. The imitation model is assumed to be driven by dissatisfaction. Particularly, an agent receiving a revision opportunity chooses an opponent randomly and mimics the opponent with a probability. In this type of imitation, the probability of imitation decreases as the revising agent's payoff increases. In other words, their dissatisfaction reduces and they are less willing to change their decision. We show that the extreme equilibria–where all agents are playing the same strategy–are locally stable. Under a special structure of the payoff matrix, It is shown that the interior equilibrium point–where some agents play strategy 1 and the rest play strategy 2–is not stable.

3.1 Model formulation

Consider a large well-mixed population of size $N \in \mathcal{N}$ agents choosing between strategies 1 and 2. Agents sharing a common payoff matrix are categorized into the same *type*. Having m different payoff matrices implies m sub-populations. The population proportion of type-i agents is denoted by n_i, which satisfies the equality $\sum_i^m n_i = 1$. The agents are playing a two-strategy game where the payoff matrix of an agent of type i, $i \in \{1, \cdots, m\}$, is

$$\pi_i = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} a_i & 0 \\ 0 & b_i \end{bmatrix} \end{array}, \qquad a_i, b_i > 0. \tag{3.1}$$

The proportion of type-i agents playing strategy 1 (resp. 2) is shown as x_i (resp. $n_i - x_i$). The proportion of agents playing strategy 1 at time t is represented by $X(t) = \Sigma_1^m x_i(t)$. By presenting the utility of a type-i agent who plays strategy $k \in \{1, 2\}$ as $u_i^k(X)$, we have

$$u_i^1(X) = a_i X,$$
$$u_i^2(X) = b_i(1 - X).$$

By having these, and under the assumption that agents interact with each other with the same probability (perfect mixing), the mean field dynamics of the sub-population i can be described as follows [4]:

$$\dot{x}_i(t) = (n_i - x_i(t))\, \rho_{21}^i - x_i(t)\rho_{12}^i, \tag{3.2}$$

where ρ_{12}^i (resp. ρ_{21}^i) represents the switching rate from strategy 1 to 2 in the type-i sub-population (resp. from strategy 2 to 1).

3.2 Imitation dynamics

Imitation is the behaviour of replicating the best-outcome observed decision among agents[6]. In a discrete-time model, each agent activates according to an activation sequence and updates its decision based on some imitation protocols. For example, the protocol in which the activated agent observes a neighbour randomly and then replicates it if its outcome is higher is different from when the activated agent checks the outcome of all its neighbours and then replicates the one with the highest outcome[4]. An imitation protocol is driven by dissatisfaction whenever a player switches strategies with a probability that is linearly decreasing in their current payoff, given the opportunity[4].

In the continuous version, the rate of change in each sub-population and consequently the state of the population at a time can be estimated. Under the mentioned mean-field approximation assumption and according to imitation driven by dissatisfaction, when agents update their strategies the switching rate will be formulated as:

$$\begin{aligned} \rho^i_{12} &= (1-X)(C-u^1_i(X)), \\ \rho^i_{21} &= X(C-u^2_i(X)), \end{aligned} \tag{3.3}$$

where C is a large enough constant to ensure that ρ_{12} and ρ_{21} remain non-negative, i.e., $C > \max\{a_i, b_i\}$, for all $i \in \{1, \cdots, m\}$.

The switching rate from type i to $j \neq i$ (ρ_{ij}), when i and $j \in \{1, 2\}$ is increasing in type j's population (X when $j = 1$ and $1 - X$ when $j = 2$). Since the chance of encountering and replicating a type-j agent in the overall population increases. It is also decreasing in type i's current payoff (u_i). Because agents tend to change their strategy less if they earn more. As a result, changing the strategy is more tempting when the agents are more dissatisfied with their current strategy.

By plugging eq. (3.2) in eq. (3.3), the mean field dynamics for type-i agents can be rewritten as

$$\dot{x}_i(t) = (n_i - x_i)\sum_{i=1}^{m} x_i(C - b_i(1-X)) - x_i(t)\sum_{i=1}^{m}(n_i - x_i)(C - a_i x), \tag{3.4}$$

which is simplified to

$$\dot{x}_i(t) = (n_i - x_i)\, X\, (C - b_i\,(1-X)) - x_i(t)\,(1-X)\,(C - a_i X)\,. \tag{3.5}$$

The following Lemma describes the basic properties of the above dynamics.

Lemma 8. *The dynamics described by eq.* (3.5) *are deterministic evolutionary dynamics, i.e., the solution exists, is forward invariant w.r.t. The simplex* $\{(x_1, x_2, \cdots, x_m)| \quad 0 \leq x_i \leq n_i\}$ *is unique, and system* (3.5) *is Lipschitz continuous.*

Proof. The proof is similar to those of Theorems 4.4.1 and 4.4.2 [4, Chapter 4] and is omitted here. □

3.2.1 The equilibrium points

Let x^* denote the equilibrium state of the dynamics (3.5). By definition,

$$\dot{x}_i^*(t) = 0, \quad \forall i \in \{1, \ldots, m\}. \tag{3.6}$$

Solving eq. (3.6) for system (3.5) results in

$$x_i^*(t) = \frac{n_i X^* \left(b_i \left(1 - X^*\right) - C\right)}{-X^{*2} \left(a_i + b_i\right) + X^* \left(a_i + b_i\right) - C}, \quad \forall i \in \{1, \ldots, m\}. \tag{3.7}$$

It can be shown by basic calculations that x_i increases with the value of C, and as C increases, x_i converges to $n_i X$.

In view of $\Sigma_i x_i^* = X^*$ we obtain the following constraint:

$$X^* = \Sigma_i \frac{n_i X^* \left(b_i \left(1 - X^*\right) - C\right)}{-X^{*2} \left(a_i + b_i\right) + X^* \left(a_i + b_i\right) - C}. \tag{3.8}$$

It will be shown the scenarios where all agents adopt the same strategy are equilibrium points of the system, $X^* = 0$ or $X^* = 1$. Referring to these equilibrium points as *extreme (exterior)*, any other equilibrium point of the system is *non-extreme (interior)*. To investigate non-extreme equilibrium points, we will have the following assumption.

Assumption 9. *For each sub-population i, $i \in \{1, 2, \cdots, m\}$, the trace of the payoff matrix is a constant equal to R.*

Example 3. ***[Global Linguistic Connectivity]*** *Imagine a scenario where countries around the world decide to teach a second language, either language 1 or 2, at school to increase global connectivity. Each of these two languages comes with its own unique advantages for the adopting country. If two countries choose different languages, neither of them gains any benefit from the linguistic relationship. Countries with the same first and second official languages are categorized into the same type. Assume there exist m distinct types among all the countries. Countries within the same type share a common payoff matrix. The payoff matrix associated with a country of type $i \in \{1, \ldots, m\}$ is as follows:*

$$\pi_i = \begin{array}{c c} & \begin{array}{cc} \mathit{1} & \mathit{2} \end{array} \\ \begin{array}{c} \mathit{1} \\ \mathit{2} \end{array} & \begin{bmatrix} a_i & 0 \\ 0 & b_i \end{bmatrix} \end{array}. \tag{3.9}$$

Here, a_i and b_i represent the advantages of adopting each language for the country and are scaled, so that $a_i + b_i = 1$ for all $i \in 1, \ldots, m$.

Proposition 4. *Under Assumption 9, dynamics* (3.5) *admit three equilibrium states $x^{*,1}$, $x^{*,2}$, and $x^{*,3}$, located on the hyperplanes $X = 0$, $X = 1$, and $X = \frac{\Sigma_i n_i b_i}{R}$, respectively.*

Proof. Under Assumption 9, the denominator in equation (3.7) is identical for all types i in the given state of the population. It implies

$$\begin{aligned} X^* &= \Sigma_i \left(\frac{n_i X^* \left(b_i \left(1 - X^*\right) - C\right)}{-(X^*)^2 \left(a_i + b_i\right) + X^* \left(a_i + b_i\right) - C} \right) \\ &= \frac{\Sigma_i \left(n_i X^* \left(b_i \left(1 - X^*\right) - C\right)\right)}{-R(X^*)^2 + RX^* - C} \\ &= \frac{X^* \left((1 - X^*)\Sigma_i(n_i b_i) - C\right)}{-R(X^*)^2 + RX^* - C}. \end{aligned}$$

Simplifying the above equation results in

$$X^* \left(R(X^*)^2 - (R+\alpha)\, X^* + \alpha\right) = 0,$$

where $\alpha = \Sigma_i n_i b_i$. Equating the above relation to zero provides us with the candidate planes $X^{*,1} = 0$, $X^{*,2} = 1$, and $X^{*,3} = \frac{\alpha}{R}$, which include the equilibrium points of the system. Based on the formula obtained in eq. (3.7), each $X^{*,i}$ is a function of X^* and can be calculated. Solutions associated to planes $X^{*,1} = 0$ and $X^{*,2} = 1$ are extreme equilibria for the system and their feasibility is trivial. Furthermore, the constraints for each sub-population i such that $0 \le x_i^{*,1} \le n_i$ and $0 \le x_i^{*,2} \le n_i$ imply $X^{*,1} = 0$ and $X^{*,2} = 1$ result single points that serve as the equilibrium points of the system. However, the feasibility of the corresponding solution to the plane $X^{*,3} = \frac{\alpha}{R}$ needs to be validated. First, we show that $0 \le X^{*,3} \le 1$, and then we show that by a sufficiently large C each $x_i^{*,3}$ corresponding to $X^{*,3}$ will be valid, i.e., $0 \le x_i^{*,3} \le n_i$.

By defining $b = \max(b_i)$, and $a = R - b$, we have

$$\begin{aligned} X^{*,3} &= \frac{\alpha}{R} \\ &= \frac{\Sigma_k (n_k b_k)}{R} \\ &\le \frac{\Sigma_k (n_k b)}{R} \\ &= \frac{b}{b+a} \\ &\le 1. \end{aligned}$$

In what follows, it is shown that by choosing $C > \frac{R}{4}$, each $x_i^{*,3}$ remains non-negative and not greater than n_i. The non-negativeness of $x_i^{*,3}$ can be concluded by showing that the nominator and denominator of eq. (3.7) are non-positive, i.e,

$$-R\left(X^{*,3}\right)^2 + RX^{*,3} - C < -R\left(X^{*,3}\right)^2 + RX^{*,3} - \frac{R}{4} = -R\left(X^{*,3} - \frac{1}{2}\right)^2 \le 0, \qquad (3.10)$$

and

$$b_i\left(1 - X^{*,3}\right) - C < 0.$$

In addition, we need to show that for all $i \in \{1, \ldots, m\}$, $x_i^{*,3} \leq n_i$. This can be concluded by the following relations:

$$\begin{aligned}
x_i^{*,3}(t) = \frac{n_i X^{*,3}\left(b_i\left(1 - X^{*,3}\right) - C\right)}{-R\left(X^{*,3}\right)^2 + RX^{*,3} - C} \leq n_i \\
\iff \frac{X^{*,3}\left(b_i\left(1 - X^{*,3}\right) - C\right)}{-R\left(X^{*,3}\right)^2 + RX^{*,3} - C} \leq 1 \\
\overset{(3.10)}{\iff} X^{*,3}\left(b_i\left(1 - X^{*,3}\right) - C\right) \geq -R\left(X^{*,3}\right)^2 + RX^{*,3} - C \\
\iff \left(1 - X^{*,3}\right)\left(C - a_i X^{*,3}\right) \geq 0.
\end{aligned}$$

Therefore, there exists a valid fixed point corresponding to $X^{*,3} = \frac{\alpha}{R}$. □

3.2.2 Stability analysis of the equilibrium points

Lemma 10. *Consider the population game described by system* (3.5)*, and assume that Assumption 9 holds. Then, the extreme equilibria $x^{*,1}$ and $x^{*,2}$ are globally asymptotically stable in the regions $[0, \frac{\alpha}{R})$ and $(\frac{\alpha}{R}, 1]$, respectively, while the interior equilibrium point $x^{*,3}$ is unstable.*

Proof. In view of $\dot{X} = \Sigma_i \dot{x}_i$, we have

$$\dot{X} = X\left(1 - X\right)\left(X - \frac{\alpha}{R}\right). \tag{3.11}$$

For the equilibria $x^{*,1}$ and $x^{*,2}$ note that from eq. (3.11), the sign of $\dot{X}$ is negative in the right ϵ-neighborhood of equilibrium point $X^{*,1} = x^{*,1}$ and positive in the left ϵ-neighborhood $X^{*,2} = x^{*,2}$. These imply any trajectory in the existing neighborhood of the extreme fixed points will converge to them. Furthermore, since $[\lim_{X\to 0} x_i = 0]$ and $[\lim_{X\to 1} x_i = n_i]$, equilibrium points $x^{*,1}$ and $x^{*,2}$ are globally asymptotically stable.

The equilibrium $X = \alpha/R$ is an interior root for eq. (3.11) and hence the first two terms of eq. (3.11) are positive around the ϵ-neighborhood of X. The last term, however, is positive for $X > \alpha/R$ and negative for $X < \alpha/R$, implying that starting from a ϵ-neighborhood of X makes the system diverge from $X = \alpha/R$. More specifically, let $X = \alpha/R + \epsilon_1$ for an arbitrarily small $\epsilon_1 > 0$. Then, $\dot{X}$ will be and remain positive till X approaches 1. Similarly, $X = \alpha/R - \epsilon_1$ for an arbitrarily small $\epsilon_1 > 0$. Then, $\dot{X}$ will be and remain negative till X approaches 0. Therefore, no matter how close the initial condition is to α/R, a trajectory will escape from the neighborhood of plane $X = \alpha/R$.

□

Proposition 5. *The hyperplane $X = \alpha/R$ serves as a stable manifold for the interior fixed point placed on it, $x^{*,3}$.*

Proof. Equality $\dot{X} = 0$ demonstrates the invariance of plane $X = \frac{\alpha}{R}$ under dynamics (3.5). Therefore, any trajectory on it remains so. We additionally demonstrate the convergence of any trajectory x on this plane towards the interior fixed point. First, let us calculate vector $v = x^{*,3} - x$. The i'th element of this vector is:

$$v_i = x_i^{*,3} - x_i = \frac{n_i \frac{\alpha}{R}\left(b_i\left(1 - \frac{\alpha}{R}\right) - C\right)}{-\frac{\alpha^2}{R} + \alpha - C} - x_i, \quad \forall i \in \{1, \ldots, m\} \tag{3.12}$$

Rearranging the left-hand side of eq. (3.5), yields a new equality for $\dot{x}_i^{*,3}$:

$$\dot{x}_i = n_i \frac{\alpha}{R}\left(-b_i(1 - \frac{\alpha}{R}) + C\right) + x_i\left(-\frac{\alpha^2}{R} + \alpha - C\right) \tag{3.13}$$

It is evident that $\dot{x}$ is a positive multiply of v and therefore both share the same direction. This implies that trajectory x converges toward $x^{*,3}$. □

Example 4. ***[An example of a two-type population]*** *Consider a population containing two types 1 and 2 with proportions $n_1 = 0.6$ and $n_2 = 0.4$, respectively, having the following* *payoff matrices:*

$$\pi_1 = \begin{array}{c} \\ 1 \\ 2 \end{array}\!\!\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 20 & 0 \\ 0 & 880 \end{bmatrix} \end{array}, \quad \pi_2 = \begin{array}{c} \\ 1 \\ 2 \end{array}\!\!\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 850 & 0 \\ 0 & 50 \end{bmatrix} \end{array}. \tag{3.14}$$

It follows R for this population equals $20 + 880 = 850 + 50 = 900$, *and the dynamics of the given system are*

$$\dot{x}_1(t) = (0.6 - x_1)\, X\, (10000 - 880\,(1 - X)) - x_1(t)\,(1 - X)\,(10000 - 20X)\,,$$
$$\dot{x}_2(t) = (0.4 - x_2)\, X\, (10000 - 50\,(1 - X)) - x_2(t)\,(1 - X)\,(10000 - 850X)\,,$$

when $C = 10^4$. *The behaviour of the interior equilibrium point* $X^{*,3} \approx 0.60$ $\left(x_1^{*,3} \approx 0.36, x_2^{*,3} \approx 0.24\right)$ *is demonstrated in phase portrait 3.2(a). The green line* $x + y \approx 0.60$ *demonstrates the stable manifold of the system. The Jacobian matrix for X is:*

$$\begin{bmatrix} -3741.33 & 6044.33 \\ 3955.66 & -5830.00 \end{bmatrix}.$$

This matrix provides us with the approximated eigenvalues 214.32 *and* -9785.67, *along with their corresponding eigenvectors* $(0.83, 0.54)^T$ *and* $(-1, 1)^T$. *As can also be observed from the system's phase portrait, the system's unstable manifold aligns with the corresponding eigenvector of the positive eigenvalue. This manifold is presented with blue color in phase portrait 3.1.*

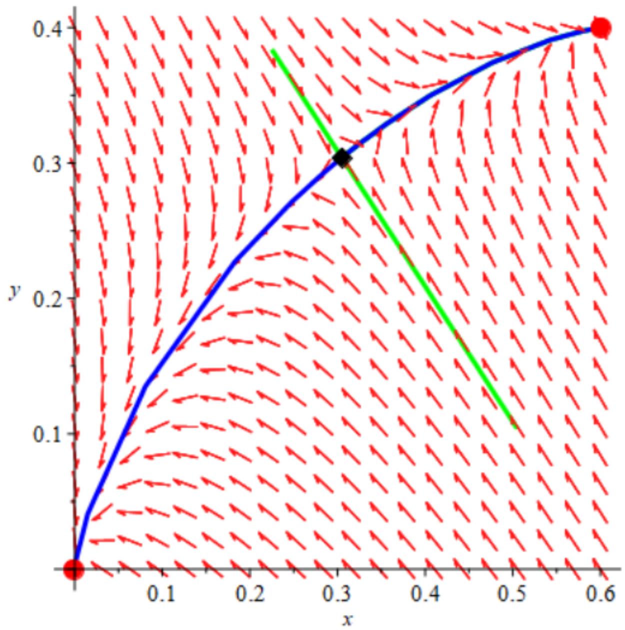

Figure 3.1: Stable and unstable manifolds of the system in example 4
a)Phase portrait of the system in example 4 with stable (green curve) and unstable manifold (blue curve) of the interior fixed point. Red dots represent the stable equilibrium points of the system, while the black diamond represents the interior unstable saddle fixed point.

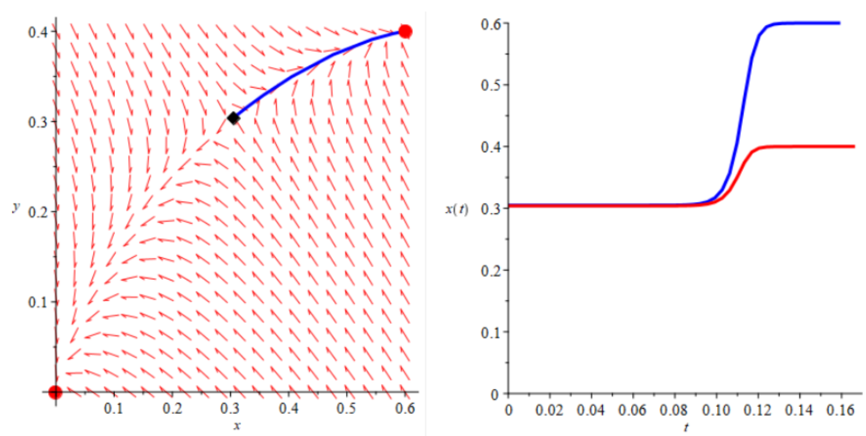

Figure 3.2: System's dynamics of the system in example 4
a)Phase portrait of the system with an initial value near the interior fixed point. In this case, the blue trajectory started its travel close to fixed point $x^{*,3}$ and still diverged from it and converged to $x^{*,2}$. **b)The trajectory solution for the given system.** All agents of each type progressively adopt strategy 2.

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Chapter 4

Concluding Remarks

- In Chapter 2, we analyzed coordination games in discrete time and demonstrated that every sparse tree network of coordinating imitators and best-responders reaches equilibrium under any activation sequence, using a semi-potential function. However, we did not examine the general case.

- The results of Chapter 2 demonstrate that even in the presence of heterogeneity, coordination within a social context is strong enough to drive a sparse-network population toward an equilibrium.

- Moving on to Chapter 3, we assumed a large, well-mixed network and used this to transfer the discrete dynamics from Chapter 2 to continuous mean field dynamics. When the payoff matrices are diagonal, we found that the system has three fixed points, two of which are trivial and one that is an interior fixed point. We proved that the interior fixed point is a saddle fixed point while the others are stable. We also calculate the stable manifold associated with the interior fixed point.

- Despite the existence of a saddle fixed point, we can demonstrate that under some perturbations, the population will converge to one of the three fixed points of the system, eventually.

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