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An example of resonance instability [(*)]

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Abstract. — We construct a semiclassical Schrödinger operator such that the imaginary part of its resonances closest to the real axis changes by a term of size h when a real compactly supported potential of size $o(h)$ is added.

Résumé. — On construit un opérateur de Schrödinger semiclassique dont la partie imaginaire des résonances les plus proches de l'axe réel est modifiée par un terme d'ordre h lorsqu'un potentiel réel à support compact de taille $o(h)$ lui est ajouté.

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1. Introduction

In this note, we consider a semiclassical Schrödinger operator P on $L^2(\mathbb{R}^n)$, $n \geqslant 1$,

$$P = -h^2 \Delta + V(x), \tag{1.1}$$

where $V \in C_0^\infty(\mathbb{R}^n;\mathbb{R})$ is a real-valued smooth compactly supported potential, and we study the stability of its resonances under a subprincipal perturbation of the form

$$h^{1+\delta} W(x), \tag{1.2}$$

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with $\delta > 0$ and $W \in C_0^\infty(\mathbb{R}^n;\mathbb{R})$. In order to realize the geometrical settings we need, it is sometimes easier to work with such operators outside compact smooth obstacles with Dirichlet boundary condition. Since P and $P+h^{1+\delta}W$ are compactly supported perturbations of $-h^2\Delta$, their resonances near the real axis are well-defined through the analytic distortion method or using the meromorphic extension of its truncated resolvent. We send the reader to the books of Sjöstrand [15] or Dyatlov and Zworski [7] for a general presentation of resonance theory, and we denote $\operatorname{Res}(Q)$ the set of resonances of the operator Q. In the semiclasssical limit (i.e. $h \to 0$), it is true that in many situations the distribution of the resonances near $E_0 > 0$ is governed by the geometry of the trapped set $K(E_0)$, that is the set of bounded Hamiltonian trajectories at energy E_0 (see (2.1)). In this paper, we shall see that this is not always the case.

It is well known that the spectrum of a self-adjoint operator is stable. This is a direct consequence of the spectral theorem. More precisely, for any self-adjoint operator P and any bounded perturbation W, the spectrum of $P + W$ satisfies

$$\sigma(P+W) \subset \sigma(P) + B(0, \|W\|).$$

Thus, a perturbation of size $h^{1+\delta}$ of a self-adjoint operator can not lead to a perturbation of size h of its spectrum.

On the contrary, the stability of resonances is a subtle problem as both stability results and instability results have been obtained. On one hand, the resonances tend to be stable as other spectral objects like the eigenvalues. This is particularly clear when the resonances are defined by complex distortion, since the usual perturbation theory of the discrete spectrum can directly be applied to the distorted operator. Even if the resonances are defined as the poles of the meromorphic extension of some weighted resolvent, Agmon [1, 2] has proved their stability. In the semiclassical setting, the stability of resonances under geometric perturbations has been obtained in particular settings, see e.g. Wunsch and Zworski [17] in the hyperbolic case or our previous paper [5, Section 4.2.4] in the homoclinic case. On the other hand, the resonances can be unstable since they come from a non self-adjoint problem: some typical non self-adjoint effects may occur concerning the resonances even if P is self-adjoint. For instance, the distorted operator may have a Jordan block or the truncated resolvent may have a pole of algebraic order greater than 1 (see e.g. Sjöstrand [14, Section 4]).

Our instability result is the following.

Theorem 1.1 (Resonance instability). — *In dimension $n = 2$, one can construct an operator P and a potential W as above satisfying the following property for all $\delta > 0$ small enough. There exist a set $\mathscr{H} \subset]0, 1]$ with $0 \in \overline{\mathscr{H}}$ and constants $D_0, E_0, \alpha > 0$ such that, for all $C > 0$ and $-C \leqslant A < B \leqslant C$,*

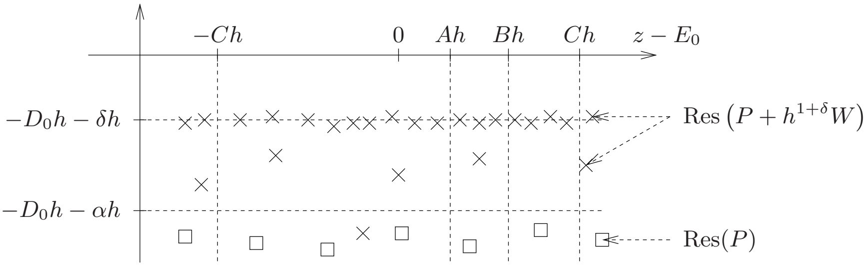


Figure 1.1. The spectral setting of Theorem 1.1.

(i) *On one hand, P has no resonance z with* $\operatorname{Re} z \in E_0+[-Ch,Ch]$ *and*

$$\operatorname{Im} z \geqslant -D_0h-\alpha h, \tag{1.3}$$

for $h\in\mathscr{H}$ small enough.

(ii) *On the other hand, the resonances z of $P+h^{1+\delta}W$ with* $\operatorname{Re} z\in E_0+[Ah,Bh]$ *closest to the real axis satisfy*

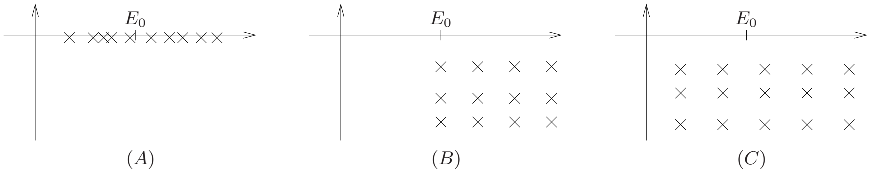
$$\operatorname{Im} z \sim -D_0h-\delta h, \tag{1.4}$$

for $h\in\mathscr{H}$ small enough.

The result is illustrated in Figure 1.1. In the statement of the previous result, we do not specify the subset of semiclassical parameters $\mathscr{H}$. In fact, depending on the geometric situation, the resonance instability may occur on the whole interval $\mathscr{H}=]0,1]$ or only near a sequence $\mathscr{H}$ like $\{j^{-1};\ j\in\mathbb{N}^*\}$. Operators corresponding to these different situations are given at the end of Section 2.

The constructions in the proof of Theorem 1.1 can be realized in any dimension $n\geqslant 2$, but our method of proof does not work in dimension $n=1$. Indeed, we need that the Hamiltonian vector field has a hyperbolic fixed point with many different trajectories in its stable manifold, which is only possible in dimension at least 2 (see Section 2). Yet we do not know if the resonance instability phenomenon described here occurs in dimension one.

Note that Theorem 1.1 (ii) provides at least one resonance z of $P+h^{1+\delta}W$ satisfying $\operatorname{Re} z\in E_0+[Ah,Bh]$ and $\operatorname{Im} z\sim -D_0h-\delta h$. But its proof shows that the number of such resonances is at least of order $|\ln h|$. In particular, let us define the *essential quantum trapping* of the operator Q in the interval



(A) (B) (C)

Figure 1.2. The resonances generated by (A) a well in the island, (B) a non-degenerate critical point and (C) a hyperbolic closed trajectory.

$I \subset \mathbb{R}$ by

$$\operatorname{ess-qt}_I(Q) = \lim_{n \to +\infty} \limsup_{\substack{h \to 0 \\ h \in \mathscr{H}}} \inf_{\substack{z_1, \dots, z_n \in \operatorname{Res}(Q) \\ \operatorname{Re} z_\bullet \in I}} \sup_{\substack{z \in \operatorname{Res}(Q) \setminus \{z_1, \dots, z_n\} \\ \operatorname{Re} z \in I}} \frac{h}{\vert \operatorname{Im} z \vert}. \qquad (1.5)$$

Roughly speaking, this means that $\mathbb{R} - ih \operatorname{ess-qt}_I(Q)^{-1}$ is the closest line to the real axis on which an infinity of resonances of Q accumulate below I as $h \in \mathscr{H}$ goes to 0. With this notation, $\operatorname{ess-qt}_{E_0 + [Ah, Bh]}(P)$ increases by at least $(\alpha - \delta)(D_0 + \alpha)^{-1}(D_0 + \delta)^{-1}$ when we add the perturbation $h^{1+\delta} W$ to the operator P. Thus, the resonance instability described here is not an anomaly due to an exceptional resonance or a Jordan block but a phenomenon mixing geometry and analysis.

We can also consider perturbations of size h of P. More precisely, for $0 < \kappa \ll 1$ fixed, one can show that the resonances z of $P + \kappa h W$ with $\operatorname{Re} z \in E_0 + [Ah, Bh]$ closest to the real axis satisfy $\operatorname{Im} z \sim -D_0 h$ for $h \in \mathscr{H}$ small enough. The proof of this point is similar to that of Theorem 1.1. On the contrary, for larger values of κ, some cancellations may appear and $P + \kappa h W$ may have a resonance free region of size $D_0 h + \alpha h$ below the real axis as for P.

The proof of Theorem 1.1 provides also resolvent estimates. Let P_θ denote the operator obtained from P after a complex distortion of angle $\theta = h \vert \ln h \vert$. It follows from a contradiction argument that its resolvent satisfies a polynomial estimate in $\Omega = E_0 + [-Ch, Ch] + i[-D_0 h - \alpha h, h]$. This means that, for some $M > 0$, we have

$$\big\Vert (P_\theta - z)^{-1} \big\Vert \lesssim h^{-M}, \qquad (1.6)$$

uniformly for $z \in \Omega$. By the usual perturbation theory, it implies that $P + W$ has no resonance in Ω for any distortable perturbation W of size $o(h^M)$. The stability of resonances under small enough perturbations has already been observed (see e.g. Agmon [1, 2]). Summing up, the resonances of P are stable for perturbations of size $o(h^M)$ and unstable for some perturbations of size $h^{1+\delta}$ (showing that $M \geqslant 1 + \delta$).

The instability phenomenon described here does not exist in the previously obtained asymptotics of resonances (see Figure 1.2). In the “well in the island” situation, the resonances are known to be exponentially close to the real axis (see Helffer and Sjöstrand [11] for globally analytic potentials and Lahmar-Benbernou, Martinez and the second author [8] for potentials analytic at infinity). Adding a subprincipal real potential $hW(x)$ does not change this properties. When the trapped set at energy E_0 consists of a non-degenerate critical point (say at $(x_0, 0) \in T^*\mathbb{R}^n$), Sjöstrand [14] has proved that the resonances form, modulo $o(h)$, a quarter of a rectangular lattice which is translated by $hW(x_0)$ when a subprincipal potential $hW(x)$ is added. Finally, the asymptotic of the resonances generated by a hyperbolic closed trajectory has been obtained by Gérard and Sjöstrand [10] (see also Ikawa [12] and Gérard [9] for obstacles). Modulo $o(h)$, they form half of a rectangular lattice which is translated by a real quantity after perturbation by a real potential $hW(x)$. Summing up, the imaginary part of the resonances is *very stable* in the three previous examples: it moves only by $o(h)$ when a perturbation by a real potential of size h is applied. In other words, if the *quantum trapping* (or maximum of the quantum lifetime) in $I = E_0 + [-Ch, Ch]$ of an operator Q is defined by

$$\operatorname{qt}_I(Q) = \limsup_{h \to 0} \sup_{\substack{z \in \operatorname{Res}(Q) \\ \operatorname{Re} z \in I}} \frac{h}{|\operatorname{Im} z|}, \tag{1.7}$$

with the conventions that $\operatorname{qt}_I(Q) = +\infty$ if the quantity diverges and $\operatorname{qt}_I(Q) = 0$ if Q has no resonance, we have $\operatorname{qt}_I(P) = \operatorname{qt}_I(P + hW)$ in these examples. The situation is completely opposite in Theorem 1.1 since a self-adjoint perturbation of size $o(h)$ induces a change of size 1 of the quantum trapping. By definition, we always have $\operatorname{qt}_I(Q) \in [0, +\infty]$ and $\operatorname{qt}_I(Q) \geqslant \text{ess-}\operatorname{qt}_I(Q)$. Moreover, if the resonance expansion of the quantum propagator holds, we have $\| \chi e^{-itQ/h} \varphi(Q) \chi \| \approx e^{t / \operatorname{qt}_I(Q)}$ for $t \gg 1$ and h in an appropriate sequence, justifying the name of quantum trapping. Other results in scattering theory provide resonance free regions, that is upper bounds on the quantum trapping, under geometric assumptions. In general, the bounds obtained do not depend on the subprincipal symbol, assumed to be self-adjoint in an appropriate class (see for instance Nonnenmacher and Zworski [13, Section 3.2]). In the present setting, Section 3.1 of [5] implies $\operatorname{qt}_I(P) \leqslant D_0^{-1}$, but Theorem 1.1 (i) shows that this inequality is not sharp.

From a broader point of view, Theorem 1.1 may seem natural since the distorted resolvent is generally large in the unphysical sheet and small perturbations may produce eigenvalues. More precisely, the norm of the distorted resolvent is known to be larger than h^{-1}, that is

$$\big\| (P_\theta - z)^{-1} \big\| \gg h^{-1},$$

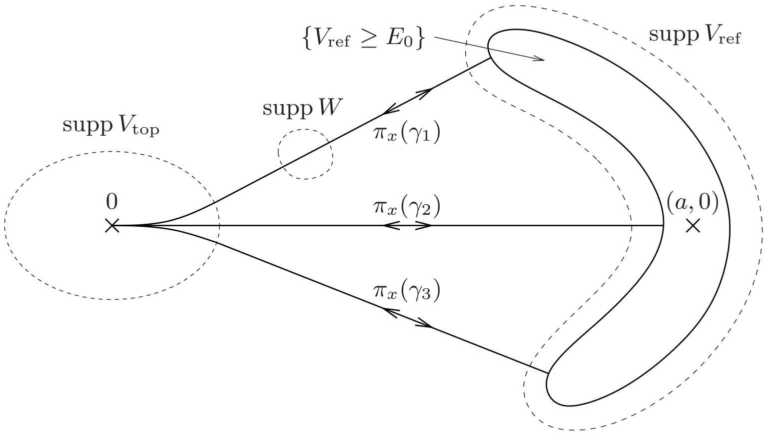


Figure 2.1. The potentials $V = V_{\text{top}} + V_{\text{ref}}$ and W.

with $\operatorname{Im} z < 0$, in many cases (see e.g. Burq and two of the authors [3] or Dyatlov and Waters [6]). By the pseudospectral theory (see e.g. Trefethen and Embree [16, Section I.4]), there exists a bounded operator W_θ of size $o(h)$ such that z is precisely an eigenvalue of $P_\theta + W_\theta$. Nevertheless, it is not clear that W_θ is the distortion of some operator W, that W is a potential and that W is self-adjoint. In fact, as explained in the previous paragraph, this is not always the case.

The present result is obtained for a Schrödinger operator whose trapped set at energy E_0 consists of a hyperbolic fixed point and homoclinic trajectories, following our recent paper [5]. The operator P and the potential W are constructed in Section 2. The instability phenomenon stated in Theorem 1.1 is proved in Section 3.

2. Construction of the operators

To construct a Schrödinger operator $P = -h^2\Delta + V(x)$ as in (1.1) with unstable resonances, we follow Example 4.23 and Example 4.24 (B) of [5]. We send back the reader to this paper for a slightly different presentation, some close geometric situations and general results about resonances generated by homoclinic trajectories. As usual, $p(x,\xi) = \xi^2 + V(x)$ denotes the symbol of P, its associated Hamiltonian vector field is

$$H_p = \partial_\xi p \cdot \partial_x - \partial_x p \cdot \partial_\xi = 2\xi \cdot \partial_x - \nabla V(x) \cdot \partial_\xi,$$

and the trapped set at energy E for P is

$$K(E) = \big\{(x,\xi) \in p^{-1}(E);\ t \mapsto \exp(tH_p)(x,\xi) \text{ is bounded}\big\}. \tag{2.1}$$

Recall that $K(E)$ is compact and stable by the Hamiltonian flow for $E > 0$.

In dimension $n = 2$, we consider the potential

$$V(x) = V_{\mathrm{top}}(x) + V_{\mathrm{ref}}(x), \tag{2.2}$$

as in Figure 2.1 and described below. On one hand, the potential V_{top} is of the form $V_{\mathrm{top}}(x) = V_1(x_1)V_2(x_2)$ where the functions $V_\bullet \in C_0^\infty(\mathbb{R})$ are single barriers (see Figure 2.2) with

$$V_1(x_1) = E_0 - \frac{\lambda_1^2}{4}x_1^2 + \mathcal{O}(x_1^3) \qquad \text{and} \qquad V_2(x_2) = 1 - \frac{\lambda_2^2}{4E_0}x_2^2 + \mathcal{O}(x_2^3),$$

near 0 and $0 < \lambda_1 < \lambda_2$. In particular, V_{top} is an anisotropic bump,

$$V_{\mathrm{top}}(x) = E_0 - \frac{\lambda_1^2}{4}x_1^2 - \frac{\lambda_2^2}{4}x_2^2 + \mathcal{O}(x^3),$$

near 0 and $(0,0)$ is a hyperbolic fixed point for H_p. The anisotropy of the fixed point is mandatory to have different incoming curves at $(0,0)$ with the same asymptotic direction. The stable/unstable manifold theorem ensures the existence of the incoming/outgoing Lagrangian manifolds $\Lambda_\pm$ characterized by

$$\Lambda_\pm = \big\{(x,\xi) \in T^*\mathbb{R}^2;\ \exp(tH_p)(x,\xi) \to (0,0) \text{ as } t \to \mp\infty\big\}.$$

They are stable by the Hamiltonian flow and included in $p^{-1}(E_0)$. Moreover, there exist two smooth functions $\varphi_\pm$, defined in a vicinity of 0, satisfying

$$\varphi_\pm(x) = \pm\sum_{j=1}^{2} \frac{\lambda_j}{4}x_j^2 + \mathcal{O}(x^3), \tag{2.3}$$

and such that $\Lambda_\pm = \{(x,\xi);\ \xi = \nabla\varphi_\pm(x)\}$ near $(0,0)$.

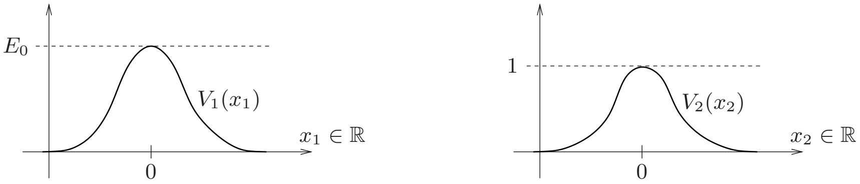


Figure 2.2. The potentials V_1 and V_2.

On the other hand, the reflecting potential V_{ref} is non-trapping and localized near $(a,0) \in \mathbb{R}^2$ with $a > 0$. If the support of V_{ref} is small enough and a is large enough, no Hamiltonian trajectory of energy E_0 can start from the support of V_{ref}, touch the support of V_{top} and then come back to the support of V_{ref}. Indeed, assume that such a trajectory γ exists. If γ gets away from γ_2 (see Figure 2.1), it will never come back to $\operatorname{supp} V_{\mathrm{ref}}$ since V_{top} is repulsive in the direction x_2. On the other hand, if γ stays close to γ_2, the existence of such a curve would contradict a property of hyperbolic fixed point given in Lemma B.1 of [5]. Thus, a trapped trajectory of energy E_0 is either $\{(0,0)\}$ or a Hamiltonian trajectory starting asymptotically from the origin, touching the support of V_{ref} and coming back to the origin; these latter trajectories are called homoclinic. In other words, $K(E_0)$ satisfies

$$K(E_0) = \Lambda_- \cap \Lambda_+,$$

and $\mathcal{H} = \Lambda_- \cap \Lambda_+ \setminus \{(0,0)\}$ denotes the set of homoclinic trajectories.

Giving to V_{ref} the form of a "croissant" barrier, we can make sure that $\mathcal{H}$ consists of a finite number of trajectories $\{\gamma_1, \dots, \gamma_K\}$ on which Λ_- and Λ_+ intersect transversally. In the sequel, we will need at least two homoclinic trajectories, that is $K \geqslant 2$. To achieve this geometric configuration, one can start replacing the potential barrier V_{ref} by an obstacle $\mathcal{O}$ such that the operator $\widetilde{P} = -h^2 \Delta_{\mathbb{R}^2 \setminus \mathcal{O}} + V_{\mathrm{top}}(x)$ on $\mathbb{R}^2 \setminus \mathcal{O}$ with Dirichlet boundary condition (see Figure 2.3) satisfies the expected geometric properties. Then, one can easily realize a situation where $K = 3$ whereas it seems complicated to have $K = 2$ (see [5, Example 4.14]). In Figure 2.3, the transversality of the intersection of Λ_- and Λ_+ is guaranteed if $\partial\mathcal{O}$ and the dash-dotted curves have a contact of order one on each point of $\partial\mathcal{O} \cap \pi_x(\mathcal{H})$. The presence of such obstacles does not affect the proof of Theorem 1.1 below since the trapped set contains no glancing trajectories, thus it does not modify the propagation of singularities. Therefore, the Schrödinger operator with obstacle $\widetilde{P}$ can be chosen as the operator P in Theorem 1.1. However, to get a Schrödinger operator without obstacle as in (1.1), one can approximate the obstacle $\mathcal{O}$ by a potential V_{ref} of the form $\chi(\operatorname{dist}(x, \mathcal{O})/\varepsilon)$ where $\chi \in C_0^\infty(\mathbb{R}^2)$ is a potential barrier as V_1 with $\chi(0) > E_0$ and $\varepsilon > 0$ is small enough. Indeed, first the reflection laws for Hamiltonian curves of energy E_0 on the obstacle $\mathcal{O}$ and on the potential V_{ref} are asymptotically the same as $\varepsilon \to 0$ outside the glancing region. Moreover, one can show that the Hamiltonian field with V_{ref} has a unique homoclinic trajectory, on which Λ_- and Λ_+ intersect transversally, in the neighborhood of each homoclinic trajectory existing in the case of the obstacle $\mathcal{O}$.

As usual in the semiclassical regime, we will see in the next section that the asymptotic of the resonances is given in terms of geometric quantities related to the trapped set. Working from a quantization rule as in Example 4.24 of [5], it will appear that the resonance instability is governed here

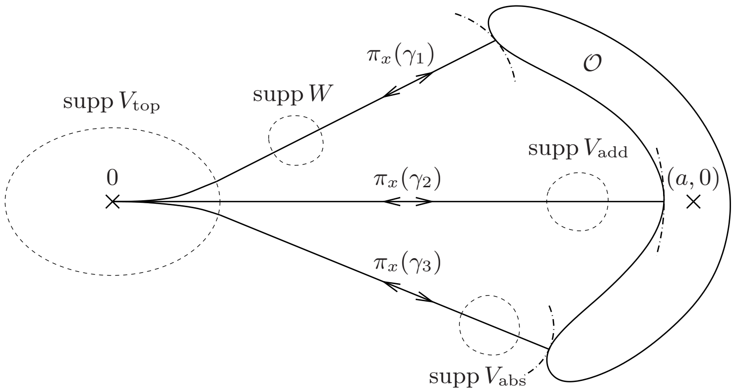


Figure 2.3. A realization with a potential V_{top} and an obstacle $\mathcal{O}$.

by the function

$$\mu(\sigma,h)=\Gamma\Big(\frac{\lambda_1+\lambda_2}{2\lambda_1}-i\frac{\sigma}{\lambda_1}\Big)e^{-\frac{\pi\sigma}{2\lambda_1}}\sum_{k=1}^{K}e^{iA_k/h}B_ke^{iT_k\sigma}, \tag{2.4}$$

for $\sigma\in\mathbb{C}$, where A_k,B_k,T_k are related to the curve $\gamma_k=(x_k,\xi_k)$ (see Lemma 3.2). We recall quickly how these quantities are defined and send the reader to [5, Section 4.1] for the proof of convergence of the various objects. First,

$$A_k=\int_{\gamma_k}\xi\cdot\mathrm{d}x,$$

is the action along γ_k. The function $x_k(t)$ has the following asymptotics

$$x_k(t)=g^k_\pm e^{\pm\lambda_1 t}+o\big(e^{\pm\lambda_1 t}\big),$$

as $t\to\mp\infty$ for some vector $g^k_\pm\in\mathbb{R}^2$. As a matter of fact, $g^k_\pm$ is collinear to the first vector of the canonical basis $(1,0)$ and do not vanish. Eventually, if $\gamma_k(t,y)=(x_k(t,y),\xi_k(t,y)):\mathbb{R}\times\mathbb{R}\longrightarrow T^*\mathbb{R}^2$ is a smooth parametrization of Λ_+ by Hamiltonian curves such that $\gamma_k(t,0)=\gamma_k(t)$, the limits of the Maslov determinants

$$\mathcal{M}^+_k=\lim_{s\to-\infty}\sqrt{\left|\det\frac{\partial x_k(t,y)}{\partial(t,y)}\Big|_{t=s,\ y=0}\right|}\,e^{-s\frac{\lambda_1+\lambda_2}{2}},$$

$$\mathcal{M}^-_k=\lim_{s\to+\infty}\sqrt{\left|\det\frac{\partial x_k(t,y)}{\partial(t,y)}\Big|_{t=s,\ y=0}\right|}\,e^{-s\frac{\lambda_2-\lambda_1}{2}},$$

exist and belong to $]0,+\infty[$. Let also ν_k denote the Maslov index of Λ_+ along γ_k. With these notations, we have

$$\begin{aligned} B_k &= \sqrt{\frac{\lambda_1}{2\pi}} \frac{\mathcal{M}_k^+}{\mathcal{M}_k^-} e^{-\frac{\pi}{2}(\nu_k+\frac{1}{2})i} \big|g_-^k\big| \big(i\lambda_1 |g_+^k||g_-^k|\big)^{-\frac{\lambda_1+\lambda_2}{2\lambda_1}}, \\ T_k &= \frac{\ln(\lambda_1 |g_+^k||g_-^k|)}{\lambda_1}. \end{aligned} \tag{2.5}$$

Note that $B_k \in \mathbb{C}\setminus\{0\}$ and $T_k \in \mathbb{R}$.

The idea is to find a geometric situation and a set $\mathscr{H} \subset]0,1]$ with $0 \in \overline{\mathscr{H}}$ such that

$$\mu(\sigma,h) = 0, \tag{2.6}$$

for all $\sigma \in \mathbb{C}$ and $h \in \mathscr{H}$. For simplicity, we take in the sequel $K = 3$ as in Figure 2.1 or 2.3 and assume that the trajectories γ_1 and γ_3 are symmetric. In particular, $A_1 = A_3$, $B_1 = B_3$ and $T_1 = T_3$. We consider two situations:

Case (I): $A_1 \neq A_2$ (say $A_2 > A_1$), $2B_1 = B_2 e^{i\nu}$, $\nu \in \mathbb{R}$, and $T_1 = T_2$. — Using (2.4) and the symmetry of γ_1 and γ_3, these relations imply that (2.6) holds true with

$$\mathscr{H} = \Big\{ \frac{A_2 - A_1}{(2j+1)\pi + \nu};\ j \in \mathbb{N} \Big\}.$$

The required relations can be realized since T_2 is only given by the potential V on the line $\mathbb{R}\times\{0\}$ if $\partial_{x_2} V(x_1,0) = 0$ for all $x_1 \in \mathbb{R}$, whereas B_2 is given by $\partial_{x_2}^2 V$ on $\mathbb{R}\times\{0\}$. If V_{ref} is replaced by an obstacle $\mathcal{O}$, one may need an additional potential V_{add} in order to satisfy these relations (see Figure 2.3).

Case (II): $A_1 = A_2$, $2B_1 = -B_2$ and $T_1 = T_2$. — In this setting, (2.6) holds true with $\mathscr{H} =]0,1]$. These relations can been obtained as before. More precisely, one can adjust V_{ref} on $\mathbb{R}\times\{0\}$ with $\partial_{x_2} V = 0$ on $\mathbb{R}\times\{0\}$ in order to have $A_1 = A_2$ and $T_1 = T_2$. Then, modifying $\partial_{x_2}^2 V_{\mathrm{ref}}$ on $\mathbb{R}\times\{0\}$, one can realize the condition $2B_1 = -B_2$. For that, we first perform a rotation around γ_2 in the plane (x_2,ξ_2), in such a way that Λ_+ makes a complete turn along γ_2. Thus, the Maslov indices satisfy $\nu_2 = \nu_1 \pm 2$. Finally, we can add another modification of $\partial_{x_2}^2 V_{\mathrm{ref}}$ so that $|2B_1| = |B_2|$ without producing caustics, therefore without changing the Maslov index ν_2.

Adding an absorbing potential $-ih|\ln h| V_{\mathrm{abs}}$, with $V_{\mathrm{abs}} \geqslant 0$, it is possible to artificially remove a homoclinic trajectory and thus to work with only $K = 2$ trajectories (see [5, Remark 2.1 (ii) and Example 4.14]). The resulting operator will be non self-adjoint (dissipative) but the conclusions of Theorem 1.1 will still hold.

For the perturbation W, we take any non-negative $C_0^\infty(\mathbb{R}^2;\mathbb{R})$ function supported away from the support of V and non-zero on the base space projection of only one homoclinic trajectory. In the sequel, we will assume that this trajectory is γ_1 as in Figures 2.1 and 2.3. We assume that $W = 0$ near the support of V only to simplify the discussion. The same way, $W \geqslant 0$ and W non-zero on $\pi_x(\gamma_1)$ can be weakened to $\int_{\mathbb{R}} W(x_1(t))\,\mathrm{d}t \neq 0$. Finally, $W = c_1 W_1 + c_2 W_2 + c_3 W_3$, with W_j non-zero only on γ_j, may be suitable for Theorem 1.1 generically with respect to $c_j \in \mathbb{R}$.

3. Proof of the spectral instability

We consider the operators constructed in the previous section. In particular, we work in dimension $n = 2$, the trapped set of energy $E_0 > 0$ has $K = 3$ homoclinic trajectories and (2.6) holds true for $h \in \mathscr{H}$. Following Chapter 4 of [5], the resonances of P closest to the real axis are given by the 3×3 matrix $\mathcal{Q}$ whose entries are

$$\begin{aligned}\mathcal{Q}_{k,\ell}(z,h) = e^{iA_k/h}\Gamma\big(S(z,h)/\lambda_1\big)\sqrt{\frac{\lambda_1}{2\pi}}\frac{\mathcal{M}_k^+}{\mathcal{M}_k^-}\\ \times e^{-\frac{\pi}{2}(\nu_k+\frac{1}{2})i}\big|g_-^\ell\big|\big(i\lambda_1|g_+^k||g_-^\ell|\big)^{-S(z,h)/\lambda_1},\end{aligned} \tag{3.1}$$

with rescaled spectral parameter

$$S(z,h) = \frac{\lambda_1+\lambda_2}{2} - i\frac{z-E_0}{h}. \tag{3.2}$$

The same way, the entries of the corresponding matrix for $\widetilde{P} = P + h^{1+\delta}W$ are

$$\widetilde{\mathcal{Q}}_{k,\ell}(z,h) = \begin{cases} e^{-iwh^\delta}\mathcal{Q}_{k,\ell}(z,h) & \text{if } k = 1,\\ \mathcal{Q}_{k,\ell}(z,h) & \text{if } k \neq 1,\end{cases} \tag{3.3}$$

with the notation $w = \int_{\mathbb{R}} W(x_1(t))\,\mathrm{d}t \neq 0$.

Lemma 3.1. — *The matrices $\mathcal{Q}$ and $\widetilde{\mathcal{Q}}$ are of rank one with non-zero entries. Moreover, $\mathcal{Q}^2(z,h) = 0$ for all $z \in \mathbb{C}$ and $h \in \mathscr{H}$.*

Proof. — Since $\mathcal{M}_\bullet^\pm \neq 0$ and $g_\pm^\bullet \neq 0$, the entries of $\mathcal{Q}$ and $\widetilde{\mathcal{Q}}$ are always non-zero. From (3.1), the entries of $\mathcal{Q}$ can be written $\mathcal{Q}_{k,\ell} = \alpha_k\beta_\ell$ for some $\alpha_k, \beta_k \in \mathbb{C}\setminus\{0\}$. In particular, $\mathcal{Q} = \alpha(\beta,\cdot)$ with $\alpha,\beta \in \mathbb{C}^3\setminus\{0\}$ and $\mathcal{Q}$ is of rank one (the same thing for $\widetilde{\mathcal{Q}}$). Thus, 0 is an eigenvalue of $\mathcal{Q}$ of multiplicity at least 2 and the last eigenvalue is given by its trace, that is

$$\begin{aligned}\operatorname{tr}(\mathcal{Q}) &= \Gamma\big(S(z,h)/\lambda_1\big)\sum_{k=1}^{3} e^{iA_k/h}\sqrt{\frac{\lambda_1}{2\pi}}\frac{\mathcal{M}_k^+}{\mathcal{M}_k^-}\\ &\qquad\times e^{-\frac{\pi}{2}(\nu_k+\frac{1}{2})i}\big|g_-^k\big|\big(i\lambda_1|g_+^k||g_-^k|\big)^{-\frac{\lambda_1+\lambda_2}{2\lambda_1}+i\frac{z-E_0}{\lambda_1 h}}\\ &= \mu\Big(\frac{z-E_0}{h},h\Big).\end{aligned}$$

For $h \in \mathscr{H}$, all the eigenvalues of $\mathcal{Q}$ are zero from (2.6) and $\mathcal{Q}$ is nilpotent. Since $\mathcal{Q}^j = \alpha(\beta, \alpha)^{j-1}(\beta, \cdot)$, $\mathcal{Q}$ is nilpotent iff $(\beta, \alpha) = 0$ iff $\mathcal{Q}^2 = 0$. $\square$

Let $\mathcal{W}$ be the 3×3 diagonal matrix $\mathcal{W} = \operatorname{diag}(-iw, 0, 0)$. The eigenvalues of $\mathcal{W}\mathcal{Q}(z, h)$ are

$$-iw\mathcal{Q}_{1,1}(z,h), 0, 0. \tag{3.4}$$

In the present setting, the quantization rule for $\widetilde{P}$ takes the following form: we say that z is a pseudo-resonance of $\widetilde{P}$ when

$$1 \in \operatorname{sp}\big(h^{S(z,h)/\lambda_1 - 1/2 + \delta} \mathcal{W}\mathcal{Q}(z,h)\big). \tag{3.5}$$

From (3.4), this is equivalent to $h^{S(z,h)/\lambda_1 - 1/2+\delta} w \mathcal{Q}_{1,1}(z,h) = i$. The set of pseudo-resonances is denoted by $\operatorname{Res}_0(\widetilde{P})$. We show below that they provide the asymptotic of the resonances close to the real axis. Heuristically, at the level of powers of h, this equation amounts to $h^{S(z,h)/\lambda_1 - 1/2+\delta} = 1$ so that $\operatorname{Im} h = -\lambda_2 h/2 - \delta\lambda_1 h$. This explain why a perturbation of size $h^{1+\delta}$ changes the imaginary part of the resonances by a term of size h. More precisely, since (3.5) is similar to Definition 4.2 of [5], we can adapt Proposition 4.3 and Lemma 11.3 of [5] in our case and obtain the following asymptotic of the pseudo-resonances.

Lemma 3.2. — *Let* $0 < \delta < 1/2$, $C, \beta > 0$ *and* $\varepsilon(h)$ *be a function which goes to* 0 *as* $h \to 0$. *Then, uniformly for* $\tau \in [-C, C]$, *the pseudo-resonances* z *of* $\widetilde{P}$ *in*

$$E_0 + [-Ch, Ch] + i\Big[-\Big(\frac{\lambda_2}{2} + \delta\lambda_1\Big)h - C\frac{h}{|\ln h|}, h\Big], \tag{3.6}$$

with $\operatorname{Re} z \in E_0 + \tau h + h\varepsilon(h)[-1,1]$ *satisfy* $z = z_q(\tau) + o(h|\ln h|^{-1})$ *with*

$$z_q(\tau) = E_0 - \frac{A_1\lambda_1}{|\ln h|} + 2q\pi\lambda_1\frac{h}{|\ln h|} - ih\Big(\frac{\lambda_2}{2} + \delta\lambda_1\Big) + i\ln(\widetilde{\mu}(\tau))\lambda_1\frac{h}{|\ln h|}, \tag{3.7}$$

and

$$\widetilde{\mu}(\tau) = w\Gamma\Big(\frac{1}{2} - \delta - i\frac{\tau}{\lambda_1}\Big)\sqrt{\frac{\lambda_1}{2\pi}}\frac{\mathcal{M}_1^+}{\mathcal{M}_1^-}e^{-\frac{\pi}{2}(\nu_1 + \frac{3}{2})i}\big|g_-^1\big|\big(i\lambda_1|g_+^1||g_-^1|\big)^{-\frac{1}{2}+\delta+i\frac{\tau}{\lambda_1}},$$

for some $q \in \mathbb{Z}$. *On the other hand, for each* $\tau \in [-C, C]$ *and* $q \in \mathbb{Z}$ *such that* $z_q(\tau)$ *belongs to* (3.6) *with a real part lying in* $E_0 + \tau h + h\varepsilon(h)[-1,1]$, *there exists a pseudo-resonance* z *satisfying* $z = z_q(\tau) + o(h|\ln h|^{-1})$ *uniformly with respect to* q, τ. *Moreover, there exists* $M > 0$ *such that, for all* $z \in$ (3.6), *we have*

$$\operatorname{dist}\big(z, \operatorname{Res}_0(\widetilde{P})\big) > \beta\frac{h}{|\ln h|} \implies \big\|\big(1 - h^{S/\lambda_1 - 1/2+\delta}\mathcal{W}\mathcal{Q}\big)^{-1}\big\| \leqslant M. \tag{3.8}$$

In the lemma, we have used that the eigenvalues of $\mathcal{W}\mathcal{Q}$ are explicitly given by (3.4) and that two of them are zero. On the contrary, note that $\widetilde{\mu}(\tau)$ is a smooth function which does not vanish and that there are a lot of pseudo-resonances in (3.6). The assumption $0 < \delta < 1/2$ allows to avoid the poles of the Γ function. This result implies the following resolvent estimates at the classical level.

Lemma 3.3. — *For all $0 < \delta < 1/2$, $\nu = \lambda_1/4 + \lambda_2/2$ and $C, \beta > 0$, the following properties are satisfied for $h \in \mathcal{H}$ small enough.*

(i) *For all $z \in E_0 + [-Ch, Ch] + i[-\nu h, h]$, we have*

$$\big\| \big(1 - h^{S/\lambda_1 - 1/2} \mathcal{Q}\big)^{-1} \big\| \lesssim \max \big(1, h^{\frac{\lambda_2}{2\lambda_1} + \frac{\operatorname{Im} z}{\lambda_1 h}}\big). \tag{3.9}$$

(ii) *For all $z \in$ (3.6) with $\operatorname{dist}(z, \operatorname{Res}_0(\widetilde{P})) > \beta h |\ln h|^{-1}$, we have*

$$\big\| \big(1 - h^{S/\lambda_1 - 1/2} \widetilde{\mathcal{Q}}\big)^{-1} \big\| \lesssim h^{-\delta}. \tag{3.10}$$

The particular value of ν in Lemma 3.3 has no particular meaning. We only need $\nu > D_0 = \lambda_2/2$ for Theorem 1.1 and $\nu < \lambda_1/2 + \lambda_2/2$ to avoid the poles of Γ.

Proof. — Since $\mathcal{Q}^2 = 0$ by Lemma 3.1, we get

$$\big(1 - h^{S/\lambda_1 - 1/2} \mathcal{Q}\big)^{-1} = 1 + h^{S/\lambda_1 - 1/2} \mathcal{Q}. \tag{3.11}$$

Using that $|h^{S/\lambda_1 - 1/2}| = h^{\frac{\lambda_2}{2\lambda_1} + \frac{\operatorname{Im} z}{\lambda_1 h}}$ and that $\mathcal{Q}(z, h)$ is uniformly bounded for $z \in E_0 + [-Ch, Ch] + i[-\nu h, h]$, this identity yields (3.9).

On the other hand, (3.3), $e^{-iwh^{\delta}} = 1 - iwh^{\delta} + \mathcal{O}(h^{2\delta})$ and $\mathcal{Q}^2 = 0$ give

$$\begin{aligned}
&1 - h^{S/\lambda_1 - 1/2} \widetilde{\mathcal{Q}} \\
&\quad = 1 - h^{S/\lambda_1 - 1/2} \mathcal{Q} - h^{S/\lambda_1 - 1/2 + \delta} \mathcal{W}\mathcal{Q} + \mathcal{O}(h^{\frac{\lambda_2}{2\lambda_1} + \frac{\operatorname{Im} z}{\lambda_1 h} + 2\delta}) \mathcal{Q} \\
&\quad = \Big(1 - h^{S/\lambda_1 - 1/2 + \delta} \mathcal{W}\mathcal{Q} + \mathcal{O}(h^{\frac{\lambda_2}{2\lambda_1} + \frac{\operatorname{Im} z}{\lambda_1 h} + 2\delta}) \mathcal{Q}\Big) \big(1 - h^{S/\lambda_1 - 1/2} \mathcal{Q}\big) \\
&\quad = \Big(1 + \mathcal{O}(h^{\frac{\lambda_2}{2\lambda_1} + \frac{\operatorname{Im} z}{\lambda_1 h} + 2\delta}) Q \big(1 - h^{S/\lambda_1 - 1/2 + \delta} \mathcal{W}\mathcal{Q}\big)^{-1}\Big) \\
&\qquad\qquad \times \big(1 - h^{S/\lambda_1 - 1/2 + \delta} \mathcal{W}\mathcal{Q}\big) \big(1 - h^{S/\lambda_1 - 1/2} \mathcal{Q}\big).
\end{aligned} \tag{3.12}$$

We have $|h^{S/\lambda_1 - 1/2}| = h^{\frac{\lambda_2}{2\lambda_1} + \frac{\operatorname{Im} z}{\lambda_1 h}} \leqslant h^{-\delta}$ for $z \in$ (3.6). Combining these estimates with (3.8), (3.9) and (3.11), (3.12) implies

$$\begin{aligned}
&\big(1 - h^{S/\lambda_1 - 1/2} \widetilde{\mathcal{Q}}\big)^{-1} \\
&\quad = \big(1 - h^{S/\lambda_1 - 1/2} \mathcal{Q}\big)^{-1} \big(1 - h^{S/\lambda_1 - 1/2 + \delta} \mathcal{W}\mathcal{Q}\big)^{-1} \big(1 + \mathcal{O}(h^{\delta})\big)^{-1} \\
&\quad = \big(1 + h^{S/\lambda_1 - 1/2} \mathcal{Q}\big) \big(1 - h^{S/\lambda_1 - 1/2 + \delta} \mathcal{W}\mathcal{Q}\big)^{-1} + \mathcal{O}(1),
\end{aligned} \tag{3.13}$$

if $\operatorname{dist}(z, \operatorname{Res}_0(\widetilde{P})) > \beta h |\ln h|^{-1}$. Then (3.10) follows. □

The next result provides a resonance free region for P and the asymptotic of the resonances closest to the real axis for $\widetilde{P} = P + h^{1+\delta} W$. Combined with Lemma 3.2, it implies directly Theorem 1.1 with $D_0 = \lambda_2/2$ if we choose $\lambda_1 = 1$.

Lemma 3.4. — *There exists $\alpha > 0$ such that, for all $\delta > 0$ small enough and $C > 0$, the following properties hold for $h \in \mathscr{H}$ small enough.*

(i) *P has no resonance in*

$$E_0 + [-Ch, Ch] + i\Big[-\Big(\frac{\lambda_2}{2} + \alpha\Big)h, h\Big].$$

(ii) *In the domain* (3.6), *we have*

$$\operatorname{dist}\big(\operatorname{Res}(\widetilde{P}), \operatorname{Res}_0(\widetilde{P})\big) = o\Big(\frac{h}{|\ln h|}\Big).$$

As in Definition 4.4 of [5], the notation $\operatorname{dist}(A, B) \leqslant \varepsilon$ in C means that

$$\begin{aligned} &\forall\, a \in A \cap C, \quad \exists\, b \in B, \qquad |a - b| \leqslant \varepsilon, \\ \text{and} \quad &\forall\, b \in B \cap C, \quad \exists\, a \in A, \qquad |a - b| \leqslant \varepsilon. \end{aligned}$$

The proof of Lemma 3.4 gives a polynomial estimate of the resolvents in the corresponding domains (at distance larger than $h|\ln h|^{-1}$ from the pseudo-resonances of $\widetilde{P}$).

Proof. — The first point of the lemma has already been obtained in Lemma 12.1 of [5]. In order to show the second point, we follow the strategy of Chapters 11 and 12 of [5] and summarized in the introduction of [5]. However, we cannot directly use [5] since we have to make clear the role of the perturbation $h^{1+\delta} W$ and since the resolvent of the quantization matrices are not uniformly bounded (see Lemma 3.3). Then, we first prove that $\widetilde{P}$ has no resonance and we show a polynomial estimate of its resolvent away from the pseudo-resonances.

Lemma 3.5. — *For $\delta > 0$ small enough, $C, \beta > 0$ and $h \in \mathscr{H}$ small enough, $\widetilde{P}$ has no resonance in the domain*

$$\begin{aligned} &E_0 + [-Ch, Ch] \\ &\quad + i\Big[-\Big(\frac{\lambda_2}{2} + \delta\lambda_1\Big)h - C\frac{h}{|\ln h|}, h\Big] \setminus \Big(\operatorname{Res}_0(\widetilde{P}) + B\Big(0, \beta\frac{h}{|\ln h|}\Big)\Big), \end{aligned} \tag{3.14}$$

and there exists $M > 0$ such that the distorted operator $\widetilde{P}_\theta$ of angle $\theta = h|\ln h|$ satisfies

$$\big\|(\widetilde{P}_\theta - z)^{-1}\big\| \lesssim h^{-M},$$

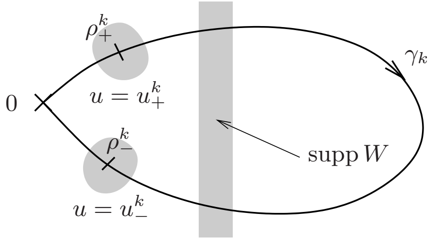


Figure 3.1. The geometric setting in the proof of Lemma 3.5.

uniformly for $h \in \mathscr{H}$ small enough and $z \in$ (3.14).

Proof of Lemma 3.5. — This result is just an adaptation of Proposition 11.4 of [5]. We only give the changes which have to be made in the present setting, sending back the reader to Section 11.2 of [5] for the technical details. From the general arguments of Chapter 8 of [5], it is enough to show that any $u = u(h) \in L^2(\mathbb{R}^2)$ and $z = z(h) \in$ (3.14) with

$$\begin{cases} (\widetilde{P}_\theta - z)u = \mathcal{O}(h^\infty), \\ \|u\|_{L^2(\mathbb{R}^2)} = 1, \end{cases} \tag{3.15}$$

vanishes microlocally near each point of $K(E_0)$. For $k = 1, 2, 3$, let $u_\pm^k$ be microlocal restrictions of u near $\rho_\pm^k$, where ρ_-^k (resp. ρ_+^k) is a point on γ_k just "before" (resp. "after") $(0,0)$ (see Figure 3.1). By microlocal restriction of $v \in L^2(\mathbb{R}^2)$ near $\rho \in T^*\mathbb{R}^2$, we mean a function $\operatorname{Op}(\psi)v$ where Op is the usual semiclassical Weyl–Hörmander quantization, $\psi \in C_0^\infty(T^*\mathbb{R}^2)$ and $\psi = 1$ near ρ. As in [5, (11.23)], they are Lagrangian distributions

$$u_-^k \in \mathcal{I}(\Lambda_+^{1,k}, h^{-N}) \qquad \text{and} \qquad u_+^k \in \mathcal{I}(\Lambda_+^0, h^{-N}),$$

associated to the Lagrangian manifold Λ_+ just after $(0,0)$ (denoted Λ_+^0) and after a turn along γ_k (denoted $\Lambda_+^{1,k}$) for some $N \in \mathbb{R}$. This property is proved in [5] computing u around the trapped set and using that the propagation through a hyperbolic fixed point transforms any function into a Lagrangian distribution (see [4]).

After an appropriate renormalization (see [5, (11.25)]), the symbols $a_-^k(x,h) \in S(h^{-N})$ of u_-^k satisfy the relation

$$a_-^k(x,h) = h^{S(z,h)/\lambda_1 - 1/2} \sum_{\ell=1}^{3} \mathcal{P}_{k,\ell}(x,h) a_-^\ell(x_-^\ell, h) + S(h^{-N+\zeta-\delta}), \tag{3.16}$$

near $x_-^k = \pi_x(\rho_-^k)$. In this expression, the symbols $\mathcal{P}_{k,\ell} \in S(1)$ (resp. the constant $\zeta > 0$) are independent of u (resp. δ, u) and $\mathcal{P}_{k,\ell}(x_-^k, h) = \widetilde{\mathcal{Q}}_{k,\ell}(z,h)$.

Compared with [5, (11.27)], $\mathcal{Q}$ is replaced by $\widetilde{\mathcal{Q}}$ in $\mathcal{P}_{k,\ell}(x_-^k, h)$. Indeed, no change has to be made for the propagation through the fixed point $(0,0)$ since W is supported away from V_{top} (see [5, (11.29)]), but the usual transport equation near γ_k

$$2\nabla\varphi_+ \cdot \nabla a_-^k + (\Delta\varphi_+ - i\sigma)a_-^k = \mathcal{O}(h^{-N+1}),$$

with $\sigma = (z - E_0)/h$ is replaced by

$$2\nabla\varphi_+ \cdot \nabla a_-^k + (\Delta\varphi_+ - i\sigma + ih^\delta W)a_-^k = \mathcal{O}(h^{-N+1}),$$

giving on the curve γ_k

$$\partial_t a_-^k(x_k(t)) + (\Delta\varphi_+ - i\sigma + ih^\delta W)a_-^k(x_k(t)) = \mathcal{O}(h^{-N+1}),$$

and leading to the additional factor $e^{-ih^\delta \int W(x_k(t))\,\mathrm{d}t}$ in the quantization matrix $\widetilde{\mathcal{Q}}$ (see [5, (11.31)]). On the other hand, the remainder term $\mathcal{O}(h^{-N+\zeta-\delta})$ in (3.16) comes from the fact that $|h^{S/\lambda_1-1/2}| \lesssim h^{-\delta}$ uniformly for $z \in$ (3.14) (see [5, Chapter 12.2] for a similar argument).

Applying (3.16) with $x = x_-^k$, we get

$$\big(1 - h^{S(z,h)/\lambda_1-1/2}\widetilde{\mathcal{Q}}(z,h)\big)a_-(x_-,h) = \mathcal{O}(h^{-N+\zeta-\delta}),$$

where $a_-(x_-,h)$ is a shortcut for the 3-vector with coefficients $a_-^k(x_-^k,h)$. From (3.10), it yields

$$|a_-(x_-,h)| \lesssim h^{-N+\zeta-2\delta},$$

uniformly for $z \in$ (3.14). Using again (3.16), we deduce $a_-^k \in S(h^{-N+\zeta-3\delta}) \subset S(h^{-N+\zeta/2})$ for $\delta > 0$ small enough. Thus, starting from $a_-^k \in S(h^{-N})$, we have proved $a_-^k \in S(h^{-N+\zeta/2})$. By a bootstrap argument (see [5, end of Chapter 9]), we obtain $u = \mathcal{O}(h^\infty)$ microlocally near $K(E_0)$ and the lemma follows. ☐

To finish the proof of Lemma 3.4, it remains to show that $\widetilde{P}$ has a resonance near each pseudo-resonance. That is

Lemma 3.6. — *For $\delta > 0$ small enough, $C, \beta > 0$ and $h \in \mathscr{H}$ small enough, the operator $\widetilde{P}$ has at least one resonance in $B(z, \beta h|\ln h|^{-1})$ for any pseudo-resonance $z \in$ (3.6).*

Proof of Lemma 3.6. — This result is equivalent to Proposition 11.6 of [5] in the present setting, and we only explain how to adapt its proof. If Lemma 3.6 did not hold, there would exist a sequence $z = z(h) \in$ (3.6) of pseudo-resonances where $h \in \mathscr{H}$ goes to 0 such that

$$\widetilde{P} \text{ has no resonance in } \mathcal{D} = B\Big(z, \beta\frac{h}{|\ln h|}\Big). \tag{3.17}$$

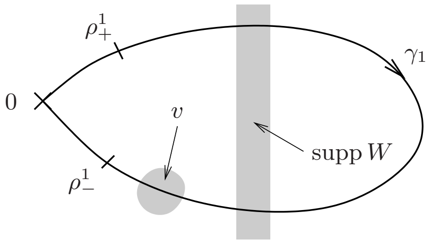


Figure 3.2. The geometric setting in the proof of Lemma 3.6.

We now construct a "test function". Let $\widetilde{v}$ be a WKB solution near x_-^1 of

$$\begin{cases} (\widetilde{P}-\widetilde{z})\widetilde{v}=0 & \text{near } x_-^1, \\ \widetilde{v}(x)=e^{i\varphi_+^{1,1}(x)/h} & \text{on } |x|=|x_-^1| \text{ near } x_-^1, \end{cases}$$

for $\widetilde{z}\in\partial\mathcal{D}$, where $\varphi_+^{1,1}$ is a generating function of $\Lambda_+^{1,1}$. Note that $\widetilde{P}=P$ near x_-^1 and that $\widetilde{v}$ can be chosen holomorphic with respect to $\widetilde{z}$ near $\mathcal{D}$. After multiplication by a renormalization factor as in [5, (11.44)], this function is denoted $\widehat{v}$. Consider cut-off functions $\chi,\psi\in C_0^\infty(T^*\mathbb{R}^2)$ such that $\chi=1$ near ρ_-^1 and $\psi=1$ near the part of the curve $\operatorname{supp}(\nabla\chi)\cap\gamma_1$ before ρ_-^1. Then, we take as "test function"

$$v=\operatorname{Op}(\psi)\big[\widetilde{P},\operatorname{Op}(\chi)\big]\widehat{v},$$

whose microsupport is illustrated in Figure 3.2. We will now test the spectral projector associated to the presumed resonances close to z on v, and we show that it gives a non-zero contribution. The choice for v as a Lagrangian distribution associated to $\Lambda_+^{1,1}$ is natural since such functions already appear in the proof of the previous lemma.

Let $u\in L^2(\mathbb{R}^2)$ be the solution of

$$(\widetilde{P}_\theta-\widetilde{z})u=v, \tag{3.18}$$

for $\widetilde{z}\in\partial\mathcal{D}$. From Lemmas 3.2 and 3.5, u is well-defined and polynomially bounded. Let u_-^k be a microlocal restriction of u near ρ_-^k as before. Working as in Lemma 11.10 of [5], one can show that $u_-^k\in\mathcal{I}(\Lambda_+^{1,k},h^{-2\delta})$ with renormalized symbol a_-^k. Moreover, as in (3.16), we get

$$a_-^k(x,h)=h^{S(z,h)/\lambda_1-1/2}\sum_{\ell=1}^{3}\mathcal{P}_{k,\ell}(x,h)a_-^\ell(x_-^\ell,h) \\ +\widetilde{a}_k(x,h)+S(h^{\zeta-3\delta}), \tag{3.19}$$

near x_-^k, where $\widetilde{a}_k$ denotes the symbol of $\widetilde{v}$ near x_-^k. In particular, $\widetilde{a}_k(x,h) = 0$ for $k \neq 1$ and $\widetilde{a}_1(x_-^1,h) = 1$. This relation is obtained using the proofs of (3.16) and Lemma 11.8 of [5].

To obtain a contradiction with (3.17), we consider

$$\mathscr{I} = \frac{1}{2i\pi} \int_{\partial\mathcal{D}} u(\widetilde{z}) \,\mathrm{d}\widetilde{z}. \tag{3.20}$$

From the properties of u_-^k and $|\partial\mathcal{D}| = 2\pi\beta h |\ln h|^{-1}$, we have $\mathscr{I} \in \mathcal{I}(\Lambda_+^{1,k}, h^{1-2\delta}|\ln h|^{-1})$ microlocally near ρ_-^k, where its renormalized symbol $b_k(x,h)$ satisfies

$$b_k(x,h) = \frac{1}{2i\pi} \int_{\partial\mathcal{D}} a_-^k(x,h) \,\mathrm{d}\widetilde{z}. \tag{3.21}$$

Applying (3.19) with $x = x_-^k$ leads to

$$\big(1 - h^{S(z,h)/\lambda_1 - 1/2} \widetilde{\mathcal{Q}}(z,h)\big) a_-(x_-,h) = \widetilde{a}(x_-,h) + \mathcal{O}(h^{\zeta - 3\delta}),$$

where $c(x_-,h)$ is a generic shortcut for the 3-vector with coefficients $c^k(x_-^k,h)$. It implies

$$\begin{aligned} a_-(x_-,h) &= \big(1 - h^{S/\lambda_1 - 1/2}\widetilde{\mathcal{Q}}\big)^{-1} \widetilde{a}(x_-,h) + \mathcal{O}(h^{\zeta-4\delta}) \\ &= \big(1 + h^{S/\lambda_1 - 1/2}\mathcal{Q}\big)\big(1 - h^{S/\lambda_1 - 1/2 + \delta}\mathcal{W}\mathcal{Q}\big)^{-1} \widetilde{a}(x_-,h) \\ &\qquad\qquad + \mathcal{O}(1) + \mathcal{O}(h^{\zeta-4\delta}), \end{aligned}$$

from (3.10) and (3.13). We deduce

$$\begin{aligned} \mathcal{W}a_-(x_-,h) = {}& \mathcal{W}\big(1 - h^{S/\lambda_1 - 1/2 + \delta}\mathcal{W}\mathcal{Q}\big)^{-1} \widetilde{a}(x_-,h) - h^{-\delta}\widetilde{a}(x_-,h) \\ & + h^{-\delta}\big(1 - h^{S/\lambda_1 - 1/2 + \delta}\mathcal{W}\mathcal{Q}\big)^{-1} \widetilde{a}(x_-,h) + \mathcal{O}(1) + \mathcal{O}(h^{\zeta-4\delta}). \end{aligned}$$

Inserting this expression in (3.21) and using (3.8) yield

$$\begin{aligned} \mathcal{W}b(x_-,h) = {}& \frac{h^{-\delta}}{2i\pi} \int_{\partial\mathcal{D}} \big(1 - h^{S/\lambda_1 - 1/2 + \delta}\mathcal{W}\mathcal{Q}\big)^{-1} \widetilde{a}(x_-,h) \,\mathrm{d}\widetilde{z} \\ & + \mathcal{O}\Big(\frac{h}{|\ln h|}\Big) + \mathcal{O}\Big(\frac{h^{1+\zeta-4\delta}}{|\ln h|}\Big). \end{aligned}$$

Note that $\widetilde{a}(x_-,h) = {}^t(1,0,0)$ is an explicit eigenvector of $\mathcal{W}\mathcal{Q}$ associated to its non-zero eigenvalue $-iw\mathcal{Q}_{1,1}(z,h)$ (see (3.4)). Thus, computing the integral as in [5, (11.67)], we get

$$\mathcal{W}b(x_-,h) = i\lambda_1 \frac{h^{1-\delta}}{|\ln h|} \widetilde{a}(x_-,h) + o\Big(\frac{h^{1-\delta}}{|\ln h|}\Big) + \mathcal{O}\Big(\frac{h}{|\ln h|}\Big) + \mathcal{O}\Big(\frac{h^{1+\zeta-4\delta}}{|\ln h|}\Big).$$

Taking $\delta > 0$ small enough and using that $\mathcal{W}b \in S(h^{1-2\delta}|\ln h|^{-1})$, the previous asymptotic shows that $\mathcal{W}b \neq 0$ so that $\mathscr{I} \neq 0$. On the other hand, since $\widetilde{P}$ has no resonance in $\mathcal{D}$ (see (3.17)), the function u defined by (3.18) is holomorphic in $\mathcal{D}$ and (3.20) gives $\mathscr{I} = 0$. Eventually, we get a contradiction and the lemma follows. ☐

The second point of Lemma 3.4 is a direct consequence of Lemmas 3.5 and 3.6. ☐

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