ANNALES
DE LA FACULTÉ
DES SCIENCES
DE TOULOUSE
Mathématiques

TAKAYUKI KOIKE
Hermitian metrics on the anti-canonical bundle of the blow-up of the projective plane at nine points

Hermitian metrics on the anti-canonical bundle of the blow-up of the projective plane at nine points [(*)]

Takayuki Koike [(1)]

Abstract. — We investigate Hermitian metrics on the anti-canonical bundle of a rational surface obtained by blowing up the projective plane at nine points. For that purpose, we pose a modified variant of an argument made by Ueda on the complex analytic structure of a neighborhood of a subvariety by considering the deformation of the complex structure.

Résumé. — Nous étudions les métriques hermitiennes sur le faisceau anticanonique d'une surface rationnelle obtenue en éclatant le plan projectif en neuf points. Dans ce but nous utilisons une variante modifiée d'un argument de Ueda sur la structure analytique complexe d'un voisinage d'une sous-variété en considérant la déformation de la structure complexe.

1. Introduction

In this paper, we investigate the complex analytic structure of a small neighborhood of a subvariety of a complex manifold. As our motivation comes from a study of Hermitian metrics of the anti-canonical bundle of some concrete examples of complex projective manifolds, we explain and describe our main results in this context in this section.

First, let us explain our main interest on Hermitian metrics of line bundles, which is on semi-positivity criteria for nef line bundles. Let X be a complex projective manifold and L be a holomorphic line bundle on X. We say that L is *nef* if the intersection number $(L.C)$ is non-negative for any compact complex curve C of X, and that L is *semi-positive* if L admits a C^∞'ly smooth Hermitian metric h such that Chern curvature tensor $\sqrt{-1}\Theta_h$ is semi-positive. As is easily shown, L is nef if it is semi-positive. The first example, as far as the author knows, of (X, L) such that L is nef however is not semi-positive was constructed by Demailly, Peternell, and Schneider in [6, Example 1.7]. In [12], we also constructed another example of such (X, L) by choosing suitable nine points $Z = \{p_1, p_2, \ldots, p_9\}$ from the complex projective plane $\mathbb{P}^2$, letting X be the blow-up $\mathrm{Bl}_Z \mathbb{P}^2$ of $\mathbb{P}^2$ at Z, and by letting L be the anti-canonical line bundle K_X^{-1}. On the other hand, by the studies of Arnol'd, Ueda, and Brunella, it is known that the anti-canonical line bundle K_X^{-1} is semi-positive when $X = \mathrm{Bl}_Z \mathbb{P}^2$ is as above for almost every choice of the nine points Z in the sense of the Lebesgue measure [1, 2, 21] (see also [5, §1]). Let us note that, for any of the examples above, there exists a reduced divisor $Y \subset X$ which is included in the complete linear system $|L|$ such that the line bundle $L|_Y := i_Y^* L$ is topologically trivial, where $i_Y : Y \to X$ is the inclusion. Motivated by such circumstance, we are interested in the following (see also Conjecture 2.1):

PROBLEM 1.1. — *Let X be a complex manifold and $Y \subset X$ be a (reduced) hypersurface. Assume that Y is compact and $N_{Y/X} := i_Y^*[Y]$ is topologically trivial, where $[Y]$ is the holomorphic line bundle on X such that $\mathcal{O}_X([Y]) = \mathcal{O}_X(Y)$. When is $[Y]$ semi-positive?*

We will see previous results on this problem generally in Section 2.1. Here, let us focus on one of the most interesting cases, which is the case where $X = X_Z$ is the one obtained by blowing up $\mathbb{P}^2$ at nine points $Z = \{p_1, p_2, \ldots, p_9\} \subset \mathbb{P}^2$. As nothing is unclear on (singular) Hermitian metrics on $K_{X_Z}^{-1}$ if it is not nef (see [12, §7]), we assume that $K_{X_Z}^{-1}$ is nef. Then, according to [12, Proposition 7.10], there exists a reduced divisor $Y_Z \in |K_{X_Z}^{-1}|$ such that the restriction $K_{X_Z}^{-1}|_{Y_Z}$ admits a flat connection (i.e. all the transition functions are $\mathbb{C}^*$-constant for a suitable choice of local trivializations) and that Y_Z is the strict transform of either a smooth elliptic curve, a cycle of rational curves, a curve with a cusp, or three lines intersecting at a point of $\mathbb{P}^2$. Here we mean by *a cycle of rational curves* a one-dimensional compact reduced variety with only nodal (i.e. normal crossing) singularities whose normalization consists of finite numbers of $\mathbb{P}^1$'s and whose dual graph is a cycle graph (Note that a rational curve with a node is also a cycle of rational curves in our definition). We are mainly interested in the case where Y_Z is a

smooth elliptic curve or a cycle of rational curves, since the other cases have already been investigated in the proof of [12, Proposition 7.10 (ii)].

First, let us describe our main result for (X_Z, Y_Z) when Y_Z is a cycle of rational curves. In this case, there exists an isomorphism $\alpha \colon \operatorname{Pic}^0(Y_Z) \to \mathbb{C}^* := \mathbb{C} \setminus \{0\}$. Note that all the elements of $\operatorname{Pic}^0(Y_Z)$ admit flat connections (i.e. the natural map $H^1(Y, \mathbb{C}^*) \to \operatorname{Pic}^0(Y_Z)$ is surjective). Note also that an element $L \in \operatorname{Pic}^0(Y_Z)$ is unitary flat (i.e. L admits a Hermitian metric h with $\sqrt{-1}\Theta_h \equiv 0$) if and only if $|\alpha(L)| = 1$. In the present paper, we show the following:

THEOREM 1.2. — *Let (X_Z, Y_Z) be as above. Assume that the anti-canonical bundle $K_{X_Z}^{-1}$ is nef, and that Y_Z is a cycle of rational curves. Then the followings are equivalent:*

(i) *$K_{X_Z}^{-1}$ is semi-positive.*
(ii) *N_Z is unitary flat, where $N_Z := K_{X_Z}^{-1}|_{Y_Z}$ (i.e. $|\alpha(N_Z)| = 1$).*
(iii) *Y_Z admits a pseudoflat neighborhoods system (i.e. there exists a fundamental system $\{V_\varepsilon\}_\varepsilon$ of neighborhoods of Y_Z in X_Z such that the boundary ∂V_ε is Levi-flat for each ε).*
(iv) *The set $\{T \in c_1(K_{X_Z}^{-1}) \mid T : \text{closed semi-positive } (1,1)\text{-current}\}$ is not a singleton.*

Note that, when $K_{X_Z}^{-1}$ is nef and Y_Z has a singular point which is not a node (i.e. Y_Z is a curve with a cusp or three lines intersecting at a point), it follows from the argument in the proof of [12, Proposition 7.10] that the assertions (i), (iii), and (iv) in Theorem 1.2 are equivalent. Note also that, under the assumption in the theorem above, it is known that Y_Z admits a strongly pseudoconcave neighborhoods system when N_Z is *not* unitary flat (i.e. $|\alpha(N_Z)| \neq 1$, [22] for a rational curve with a node, [12, Theorem 1.6] in general). It is shown by Brunella that the assertions (i) and (iii) in Theorem 1.2 are equivalent to each other when Y_Z is non-singular and $X_Z \setminus Y_Z$ does not contain any compact complex curve [2, Theorem 1 (i)]. Therefore one can regard Theorem 1.2 as a singular analogue of Brunella's theorem.

Let us add some explanation on known results on the semi-positivity of $K_{X_Z}^{-1}$ when Y_Z is a cycle of rational curves. As is easily observed, $K_{X_Z}^{-1}$ is semi-ample if and only if $K_{X_Z}^{-1}|_{Y_Z}$ is torsion in $\operatorname{Pic}^0(Y_Z)$ (i.e. $\alpha(N_Z) = e^{2\pi\sqrt{-1}\theta}$ for some rational number θ). In this case, $K_{X_Z}^{-1}$ is semi-positive by a standard argument (see Section 2.1). In [12], we showed that the conditions (i), (iii) and (iv) in Theorem 1.2 hold if $\alpha(N_Z) = e^{2\pi\sqrt{-1}\theta}$ holds for some Diophantine irrational number θ [12, Theorem 1.4, Corollary 7.5]. Here we say that an irrational number θ is *Diophantine* if there exist positive numbers A and α such that $\min_{n\in\mathbb{Z}} |n - m\theta| \geqslant A \cdot m^{-\alpha}$. For example, any algebraic irrational

number is Diophantine according to Liouville's theorem. By combining [12, Theorem 1.6(ii)] and the argument in the proof of [10, Theorem 1.1] that $K_{X_Z}^{-1}$ is not semi-positive if $|\alpha(N_Z)| \neq 1$. Therefore, our contribution in the present paper is in the case where $\alpha(N_Z) = e^{2\pi\sqrt{-1}\theta}$ holds for some non-Diophantine irrational real number θ.

Next, let us describe our main result for (X_Z, Y_Z) when Y_Z is a smooth elliptic curve. In this case, any topologically trivial line bundle L on Y_Z admits a unitary flat structure (see [21, §1.1] for example). Denote by $\rho_L : \pi_1(Y_Z, *) \to U(1)$ the unitary representation of the fundamental group which corresponds to L: i.e. ρ_L is the one obtained by considering the holonomy of $\mathcal{F}_L$ along the zero section, where $U(1) := \{t \in \mathbb{C} \mid |t| = 1\}$ and $\mathcal{F}_L$ is the foliation of the total space of L which corresponds to the connection defined by a flat metric (see [13, §2.1] for the detail). Denote by $\text{rank}(L)$ the rank of the image of ρ_L as a finitely generated abelian group. Note that $\text{rank}(L) = 0$ if and only if L is torsion in $\text{Pic}^0(Y_Z)$ (i.e. there is a positive integer m such that $L^{\otimes m}$ is holomorphically trivial), that $\text{rank}(L) = 1$ if and only if any leaf of $\mathcal{F}_L$ is biholomorphic to $\mathbb{C}^*$ except for the zero section, and that $\text{rank}(L) = 2$ if and only if any leaf of $\mathcal{F}_L$ is biholomorphic to $\mathbb{C}$ except for the zero section. Our main result in the case where Y_Z is smooth is the following:

THEOREM 1.3. — *Let (X_Z, Y_Z) be as above. Assume that Y_Z is smooth and $\text{rank}(N_Z) < 2$, where $N_Z := K_{X_Z}^{-1}|_{Y_Z}$. Then $K_{X_Z}^{-1}$ is semi-positive, and Y_Z admits a pseudoflat neighborhoods system.*

Note that, as is classically known, $K_{X_Z}^{-1}$ is semi-ample if and only if $\text{rank}(N_Z) = 0$. As $K_{X_Z}^{-1}$ is semi-positive in this case, we are interested in the case where $\text{rank}(N_Z) > 0$ (i.e. N_Z is non-torsion). It follows from the studies of Arnol'd, Ueda, and Brunella [1, 2, 21] that $K_{X_Z}^{-1}$ is semi-positive if N_Z is Diophantine. Here we say that a topologically trivial line bundle L on Y_Z is *Diophantine* if there exist positive numbers A and α such that $d_{\text{Pic}^0(Y_Z)}(\mathbb{I}_Z, N_Z^{\otimes m}) \geqslant A \cdot m^{-\alpha}$, where $\mathbb{I}_Z$ is the holomorphically trivial line bundle on Y_Z and $d_{\text{Pic}^0(Y_Z)}$ is an invariant distance on $\text{Pic}^0(Y_Z)$ (see [21, §4.1]). Therefore, our contribution in the present paper when Y_Z is smooth is in the case where $\text{rank}(N_Z) = 1$ and N_Z is not Diophantine.

Let us briefly explain the idea how to construct a C^∞ Hermitian metric with semi-positive curvature under the condition (ii) of Theorem 1.2, or the assumption of Theorem 1.3. Based on an argument in the proof of [2, Theorem 1], or the argument which we described in [13, §5] (see also Section 2.1), the problem is reduced to show the existence of an open covering $\{V_j\}_j$ of a neighborhood of Y_Z and a system $\{\widehat{w}_j\}_j$ of holomorphic functions on V_j such that $\widehat{w}_j$ is a defining function of $Y_Z \cap V_j$ and that $|\widehat{w}_j/\widehat{w}_k| \equiv 1$ holds on

each $V_j \cap V_k$. Take a sufficiently fine covering $\{V_j\}$. Then, it follows from a simple argument that there exists a defining function w_j on V_j of $Y_Z \cap V_j$ for each j such that $|w_j/w_k| \equiv 1$ holds on $Y_Z \cap V_j \cap V_k$. Following the strategy of [1] or [21, §4], we will modify w_j by solving a functional equation in the form of

$$w_j = \widehat{w}_j + \sum_{m=2}^{\infty} F_{j,m} \cdot \widehat{w}_j^m$$

on each V_j (after shrinking V_j suitably), where each $F_{j,m}$ is a suitably constructed holomorphic function on V_j. As is described in [21, §4.2] or [12, §4.2.1], one can actually construct coefficient functions $\{F_{j,m}\}$ inductively on m in our setting (see also [18]). Thus the problem is reduced to the L^∞ estimate of each $A_m := \max_j \sup_{V_j} |F_{j,m}|$ so that the formal power series $\sum_{m=2}^{\infty} A_m X^m \in \mathbb{C}[[X]]$ has a positive radius of convergence (then one can actually solve the functional equation above by using the implicit function theorem, see Section 5.1 for details).

For such purpose, one need to estimate the L^∞ operator norm of the coboundary map $\delta \colon \check{C}^0(\{U_j\}, \mathcal{O}_{Y_Z}(N_Z^{-m})) \to \check{C}^1(\{U_j\}, \mathcal{O}_{Y_Z}(N_Z^{-m}))$ of the Čech cocycles, where $U_j := Y_Z \cap V_j$ and N_Z^{-m} is the m-th tensor power of the dual line bundle N_Z^{-1} of N_Z. According to Ueda's lemma [21, Lemma 4] or its singular analogue we will describe as Lemma 4.2, the operator norm of the coboundary map can be estimated from above by $K \cdot |1 - \alpha(N_Z^m)|^{-1}$ when Y_Z is a cycle of rational curves, and by $K \cdot (d_{\mathrm{Pic}^0(Y_Z)}(\mathbb{I}_Z, N_Z^{\otimes m}))^{-1}$ when Y_Z is smooth, where K is a constant which does not depend on m. This type of estimate is not enough for the case where N_Z can be "too-well" approximated by a sequences of torsion line bundles (for example when $\alpha(N_Z^m) = e^{2\pi\sqrt{-1}\theta}$ for some Liouville number $\theta \in \mathbb{R} \setminus \mathbb{Q}$). In order to overcome this difficulty, we alternatively consider a deformation family $\pi \colon \mathcal{X} \to S$ such that each fiber is isomorphic to X_Z for some nine points configuration Z. Then, by using the maximum principle suitably, we can improve the estimate in our situation (see also Section 2.4).

The organization of the paper is as follows. In Section 2, we will review the relationship between neighborhood theories and the semi-positivity of nef line bundles. Here we will also review Ueda theory, give an explanation on the original proof of Ueda's theorem, and explain our basic strategy to prove the main theorems. In Section 3, we describe a generalized configuration and state our main result as Theorem 3.1. In Section 4, we give two more concrete configurations as examples of the generalized configuration we describe in Section 3. In Section 5, we prove Theorem 3.1. In Section 6, we give some examples and prove Theorem 1.2 and 1.3.

2. Preliminaries

2.1. Semi-positivity of a nef line bundles and a neighborhood of the stable base locus

Let X be a complex manifold and L be a holomorphic line bundle on X. For a positive integer m, we denote by L^m the m-th tensor power $L^{\otimes m}$ and by L^{-m} the m-th tensor power $(L^*)^{\otimes m}$ of the dual bundle L^* of L. A line bundle L is said to be *semi-ample* if $\mathcal{O}_X(L^m)$ is globally generated for some positive integer m. If L is semi-ample, then L is semi-positive. Indeed, $h := \Phi_{|L^m|}^* h_{\mathrm{FS}}$ is a C^∞ Hermitian metric on L^m with $\sqrt{-1}\Theta_h \geqslant 0$, where m is a positive integer such that $\mathcal{O}_X(L^m)$ is globally generated, $\Phi_{|L^m|} \colon X \to \mathbb{P}(H^0(X, \mathcal{O}_X(L^m)))$ is the map defined by the complete linear system $|L^m|$, and h_{FS} is Fubini–Study metric on $\mathcal{O}_{\mathbb{P}(H^0(X,\mathcal{O}_X(L^m)))}(1)$. By considering the metric $h^{1/m}$, one can attach a C^∞ Hermitian metric on L with semi-positive curvature.

In what follows, we drop the condition that X is projective and assume that X is just a complex manifold. A holomorphic line bundle L on X is said to be *effective* if $H^0(X, \mathcal{O}_X(L)) \neq 0$. When L is effective, there exists a non-zero global section $f \in H^0(X, \mathcal{O}_X(L))$. Let D be the corresponding divisor: $D := \mathrm{div}(f)$. Note that $L \cong [D]$ and $\mathcal{O}_X(L) = \mathcal{O}_X(D)$ hold, where $[D]$ is the holomorphic line bundle on X which corresponds to the divisor D. In this case, the current T_D defined by $\langle T_D, \varphi \rangle := \int_D \varphi$ (for a compactly supported C^∞ form φ) is a closed semi-positive $(1,1)$-current such that $T_D \in c_1(L)$. Therefore we have that $\{T \in c_1(K_{X_z}^{-1}) \mid T : \text{closed semi-positive } (1,1)-\text{current}\}$ is not empty. Note that this set is not a singleton if L is semi-positive, since $\sqrt{-1}\Theta_h \in c_1(L)$ is also a closed semi-positive $(1,1)$-current for any C^∞ Hermitian metric h of L with semi-positive curvature.

We say that L is $\mathbb{C}^*$-*flat*, or L *admits a flat connection*, if L is an element of the image of the natural map $H^1(X, \mathbb{C}^*) \to H^1(X, \mathcal{O}_X^*)$, or equivalently, all the transition functions are taken as constant functions valued in $\mathbb{C}^*$ for a suitable choice of local trivializations. We say that L is *unitary flat* if L is an element of the image of the natural map $H^1(X, \mathrm{U}(1)) \to H^1(X, \mathcal{O}_X^*)$, or equivalently, all the transition functions are taken as constant functions valued in $\mathrm{U}(1)$ for a suitable choice of local trivializations. If L is unitary flat, then L is semi-positive. Actually one can easily construct a C^∞ Hermitian metric h on a unitary flat line bundle L such that $\sqrt{-1}\Theta_h \equiv 0$. Note that any topologically trivial holomorphic line bundle on a compact Kähler manifold is unitary flat (Kashiwara's theorem, see [21, §1.1]).

Let Y be a reduced divisor of X. In this paper, we denote by $N_{Y/X}$ the line bundle $[Y]|_Y := i_Y^*[Y]$ on Y, where $i_Y \colon Y \to X$ is the inclusion. As $N_{Y/X}$ coincides with the holomorphic normal bundle of Y when Y is non-singular, we call this bundle the normal bundle of Y in general. Even when $N_{Y/X}$ is unitary flat, it may possible that $L := [Y]$ is not semi-positive. Indeed, [6, Example 1.7] gives such an example.

Assume that there exists a neighborhood V of Y in X such that $L|_V$ is unitary flat, where $L := [Y]$. In this case, $L|_V$ admits a C^∞ Hermitian metric h_V with semi-positive curvature. On the other hand, L also admits a singular Hermitian metric h_{sing} such that $h_{\mathrm{sing}}|_{X \setminus Y}$ is a C^∞ Hermitian metric on $L|_{X \setminus Y}$, $h_{\mathrm{sing}} \to \infty$ holds when a point approaches to Y, and that Chern curvature current of h_{sing} is semi-positive. Indeed, the singular Hermitian metric h_{sing} defined by $|f_Y|^2_{h_{\mathrm{sing}}} \equiv 1$ enjoys these properties, where $f_Y \in H^0(X, \mathcal{O}_X(Y))$ is a canonical section. A C^∞ Hermitian metric h on L with semi-positive curvature can be constructed by using *the regularized maximum construction* for these two metrics h_V and h_{sing}, which is the same construction as we used for proving [9, Corollary 3.4] (see also [13, §5]). Here we briefly explain this construction. Fix a relatively compact open neighborhood V_0 of Y in V. For a sufficiently small positive constant C, one can easily see that the continuous Hermitian metric h on L defined by

$$h_x^{-1} := \begin{cases} \max\{C \cdot (h_V)_x^{-1}, \ (h_{\mathrm{sing}})_x^{-1}\} & \text{if } x \in V_0 \\ (h_{\mathrm{sing}})_x^{-1} & \text{if } x \in X \setminus \overline{V_0} \end{cases}$$

is well-defined, and that the curvature current $\sqrt{-1}\Theta_h$ is semi-positive. By replacing the function "max" in the construction above with "a regularized maximum function" (see [4, §5.E]), one can make h smooth.

Again, let X be a complex manifold and Y be a reduced divisor of X. For a singular Hermitian metric h on $L = [Y]$ with semi-positive curvature current, one can construct a plurisubharmonic function Φ_h on $X \setminus Y$ by $\Phi_h := -\log|f_Y|^2_h$, where $f_Y \in H^0(X, \mathcal{O}_X(Y))$ is a canonical section. As $\Phi_h(x) = O(-\log d(Y, x))$ as x approaches to Y if h is smooth, one can prove the non semi-positivity of L if one can deny the existence of such a plurisubharmonic function on $X \setminus Y$, which is the strategy of the proof of the main theorem in [10].

For investigating the flatness of $L|_V$ or the plurisubharmonic function on $V \setminus Y$ for a neighborhood V of Y, we apply Ueda theory [21, 22] and its analogues [11, 12, 13], which will be reviewed in the next subsection.

Motivated by the argument as above together with some results such as [2, Theorem 1 (i)] and Theorem 1.2, we pose the following:

CONJECTURE 2.1. — *Let X be a projective manifold and $Y \subset X$ be a reduced subvariety of codimension 1. Assume that the restriction $L|_Y(:=i_Y^* L)$ of the line bundle $L := [Y]$ to Y is unitary flat (or topologically trivial), where $i_Y : Y \to X$ is the inclusion. Then L is semi-positive if and only if there exists a neighborhood V of Y such that $L|_V$ is semi-positive.*

Note that, if the conjecture above is affirmative, then $L = [Y]$ is not semi-positive when Y is a smooth compact curve and the pair (Y, X) is of type (γ) (see Section 2.2 for the definition), which is actually the case for some examples as we showed in [14, 16].

2.2. Short review for Ueda theory and its analogues

Let X be a complex manifold and $Y \subset X$ be a holomorphically embedded compact complex subvariety with topologically trivial normal bundle.

In [21], Ueda investigated the complex analytic structure of a neighborhood of Y when X is surface and Y be a smooth (i.e. non-singular) complex curve (see also [18]). He defined the obstruction class $u_m(Y, X) \in H^1(Y, \mathcal{O}_X(N_{Y/X}^{-m}))$ $(m \geqslant 1)$, whose definition will be explained below in a generalized configuration. In broad strokes, he defined the obstruction classes by comparing $[Y]|_V$ and $\widetilde{N}$ on a small tubular neighborhood V of Y, where $\widetilde{N}$ is the flat extension of $N_{Y/X}$ (i.e. $\widetilde{N}$ is the unitary flat line bundle on V which corresponds to $N_{Y/X}$ via the natural isomorphism $H^1(V, U(1)) \to H^1(Y, U(1))$). The m-th obstruction class $u_m(Y, X)$, which is called *m-th Ueda class*, reflects the difference between $[Y]|_V$ and $\widetilde{N}$ in m-th order jet along Y. By using Ueda classes, he classified the pair (Y, X) into the following four cases: (α): The case where $u_m(Y, X) \neq 0$ for some $m \geqslant 0$, which means that $[Y]|_V$ and $\widetilde{N}$ are different to each other in a finite order jet along Y. In this case, the pair (Y, X) is said to be of *finite type*. Otherwise, the pair is said to be of *infinite type*. (β'): The case where there exists a proper holomorphic map $\pi : V \to \mathbb{D}$ onto the unit disk $\mathbb{D} := \{w \in \mathbb{C} \mid |w| < 1\}$ by shrinking V if necessary such that $\pi^*\{0\} = mY$ holds as divisors for some positive integer m. (β''): The case where $[Y]|_V$ is non-torsion and unitary flat by shrinking V if necessary. In these two cases, Y admits a pseudoflat fundamental neighborhoods system. (γ): The remaining case.

Ueda showed that Y admits a strongly pseudoconcave neighborhoods system if (Y, X) is of type (α) [21, Theorem 1]. In this case, he investigated

the detailed grouth properties of a plurisubharmonic function defined on $V \setminus Y$ [21, Theorem 2]. He also established a singular analogue of these theorems for the case where Y is a rational curve with a node and $N_{Y/X}$ is not unitary flat [22], which was slightly generalized by the author to, for example, the case where Y is a cycle of rational curves [12, Theorem 1.6]. By combining these and the argument we explained in the previous subsection, one obtains many examples of nef line bundles which are not semi-positive (see [10], [12, Corollary 1.3]). In the infinite type case, Ueda gave a sufficient condition for the pair (Y, X) to be of type (β') or (β'') [21, Theorem 3], whose proof also makes sense even when Y is a manifold of higher dimension. Note that [21, Theorem 3] can be regarded as a generalization of Arnol'd's theorem [1]. The following theorem is a generalized variant of Ueda's theorem:

THEOREM 2.2 ([21, Theorem 3], [12, Theorem 1.4], see also [13, Theorem 1.1]). — *Let X be a complex manifold and $Y \subset X$ be a compact reduced subvariety of codimension 1 such that the normal bundle is unitary flat. Assume one of the following two conditions: Y is non-singular, or Y is a cycle of rational curves. Then there exists a neighborhood V of Y such that the line bundle $[Y]|_V$ is unitary flat if the pair (Y, X) is of infinite type and the norml bundle $N_{Y/X} := [Y]|_Y$ is either torsion or Diophantine.*

Here we say that a topologically trivial line bundle L on a cycle of rational curve is Diophantine if $\alpha(N_{Y/X}) = e^{2\pi\sqrt{-1}\theta}$ for a Diophantine number $\theta \in \mathbb{R} \setminus \mathbb{Q}$ (see Section 1 for the definition when Y is non-singular). Especially, note that the pair (Y, X) is of type (β') if and only if it is of infinite type and $N_{Y/X}$ is a torsion element of $\mathrm{Pic}^0(Y)$. We will roughly review the strategy of the proof of Theorem 2.2 in the next subsection.

In [11, 13, 15], we investigated a higher codimensional analogue of Ueda theory. According to [12] and [13], we explain the definition of a generalized variant of Ueda classes. Let X be a complex manifold, and $Y \subset X$ be a compact reduced subvariety of codimension $r \geqslant 1$ such that $N_{Y/X}$ is unitary flat. Assume Y is a cycle of rational curves and $r = 1$ whenever Y is singular for simplicity. Take a finite open covering $\{U_j\}$ of Y and a neighborhood V_j of U_j in X. When Y is singular, one may assume the following condition by refining them if necessary: $U_j \cap U_k \neq \emptyset$ and $U_k \cap Y_{\mathrm{sing}} \neq \emptyset$ imply $U_j \cap Y_{\mathrm{sing}} = \emptyset$. Take a defining function $w_j \colon V_j \to \mathbb{C}^r$ of U_j for each j: i.e. w_j is a holomorphic function on V_j such that $\mathrm{div}(w_j^{(\lambda)})$'s transversally intersect along U_j, where $w_j^{(\lambda)} \colon V_j \to \mathbb{C}$ is the composition of w_j and λ-th projection map $\mathbb{C}^r \to \mathbb{C}$. By a simple argument, one may assume that $dw_j = T_{jk} dw_k$ holds on each $U_{jk} := U_j \cap U_k$ for some unitary matrix $T_{jk} \in \mathrm{U}(r)$ by changing

w_j's if necessary, where

$$dw_j := \begin{pmatrix} dw_j^{(1)} \\ dw_j^{(2)} \\ \vdots \\ dw_j^{(r)} \end{pmatrix}.$$

By shrinking V_j's if necessary again, we assume that, for each j with $U_j \cap Y_{\text{sing}} = \emptyset$, there exists a holomorphic surjection $\text{Pr}_{U_j} \colon V_j \to U_j$ such that $(w_j, z_j \circ \text{Pr}_{U_j})$ are coordinates of V_j, where z_j is a coordinate of U_j. In what follows, for any holomorphic function f on U_j, we denote by the same letter f the pull-buck $\text{Pr}_{U_j}^* f := f \circ \text{Pr}_{U_j}$. Take U_j and U_k such that $U_{jk} \neq \emptyset$. We may assume that $U_j \cap Y_{\text{sing}} = \emptyset$. In this case, one obtains the series expansion

$$T_{jk} \cdot \begin{pmatrix} w_k^{(1)} \\ w_k^{(2)} \\ \vdots \\ w_k^{(r)} \end{pmatrix} = \begin{pmatrix} w_j^{(1)} \\ w_j^{(2)} \\ \vdots \\ w_j^{(r)} \end{pmatrix} + \sum_{|a| \geqslant 2} \begin{pmatrix} f_{kj,a}^{(1)}(z_j) \\ f_{kj,a}^{(2)}(z_j) \\ \vdots \\ f_{kj,a}^{(r)}(z_j) \end{pmatrix} \cdot w_j^a,$$

where $a = (a_1, a_2, \ldots, a_r)$ is the multiple index running all the elements of $(\mathbb{Z}_{\geqslant 0})^r$ with $|a| := \sum_{\lambda=1}^r a_r \geqslant 2$, $f_{kj,a}^{(\lambda)}$'s are holomorphic functions on U_{jk} (we regard this also as a function defined by $(\text{Pr}_{U_j}|_{\text{Pr}_{U_j}^{-1}(U_{jk})})^* f_{kj,a}^{(\lambda)}$), and $w_j^a := \prod_{\lambda=1}^r (w_j^{(\lambda)})^{a_\lambda}$. For a positive integer m, we say that $\{(V_j, w_j)\}$ is of *type* m if $f_{kj,a} \equiv 0$ holds for any a with $|a| \leqslant m$ and any j, k such that $U_{jk} \neq \emptyset$ and $U_j \cap Y_{\text{sing}} = \emptyset$. If $\{(V_j, w_j)\}$ is of type m, it follows that $\{(U_{jk}, u_{jk})\}$ satisfies the 1-cocycle condition, where

$$u_{jk} := \sum_{\lambda=1}^r \sum_{|a|=m+1} f_{kj,a}^{(\lambda)} \frac{\partial}{\partial w_j^{(\lambda)}} \otimes dw_j^a,$$

and thus it defines an element of $H^1(Y, \mathcal{O}_Y(N_{Y/X} \otimes S^{m+1} N_{Y/X}^*))$. We denote by $u_m(Y, X)$ the class $[\{(U_{jk}, u_{jk})\}]$, which is the definition of the m-th Ueda class. Note that we define $u_{kj} := -u_{jk}$ for j with $U_j \cap Y_{\text{sing}} \neq \emptyset$ when Y is singular. Ueda class $u_m(Y, X)$ is well-defined up to the $\mathrm{U}(r)$-action of $H^1(Y, \mathcal{O}_Y(N_{Y/X} \otimes S^{m+1} N_{Y/X}^*))$: i.e. it does not depend on the choice of the system $\{(V_j, w_j)\}$ of type m as an element of $H^1(Y, \mathcal{O}_Y(N_{Y/X} \otimes S^{m+1} N_{Y/X}^*))/\mathrm{U}(r)$ [21, p. 588], [12, Proposition 3.6], [13, Lemma 3.6].

Remark 2.3. — The obstruction can be similarly defined even when the normal bundle is $\mathbb{C}^*$-flat if once we fix a system of type m. However, it is not the case on the well-definedness of $u_m(Y, X)$, see [3, Remark 2.2].

Finally, in the rest of this subsection, let us summarize the situation when $(Y, X) = (Y_Z, X_Z)$ is the pair as in Section 1: i.e. X is the blow-up of $\mathbb{P}^2$ at nine points $Z = \{p_1, p_2, \ldots, p_9\}$ such that $Z \subset C \setminus C_{\text{sing}}$, where C is either a smooth elliptic curve or a cycle of rational curves in $\mathbb{P}^2$, C_{sing} is the singular part of C, and Y is the strict transform of C. Assume that the anti-canonical bundle K_X^{-1} is nef. In this case, $N_{Y/X}$ is topologically trivial. By the arguments in [21, §1.1], $N_{Y/X}$ is unitary flat if Y is a smooth elliptic curve. As will be seen in Section 4.2, when Y is a cycle of rational curves, there exists an isomorphism $\alpha \colon \operatorname{Pic}^0(Y) \to \mathbb{C}^*$ such that an element $L \in \operatorname{Pic}^0(Y)$ is unitary flat if and only if $|\alpha(L)| = 1$ (see also [22, §1] for the case where Y is a rational curve with a node). If $N_{Y/X}$ is not unitary flat, Y admits a strongly pseudoconcave neighborhood and the line bundle $[Y]$ is not semi-positive. In this case, T_Y as in the previous subsection is only the element of the set $\{T \in c_1(K_{X_Z}^{-1}) \mid T \colon \text{closed semi-positive } (1,1)\text{-current}\}$ (It follows by combining the arguments in the proof of [10, Theorem 1.1] and [12, Theorem 1.6 (ii)]). In what follows, we will investigate the case where $N_{Y/X}$ is unitary flat. In this case, it follows from the argument as in [18] that (Y, X) is of infinite type even when $N_{Y/X}$ is torsion (it is clear that the pair is of infinite type if $N_{Y/X}$ is non-torsion, since $H^1(Y, \mathcal{O}_Y(N_{Y/X}^{-m})) = 0$ holds for any positive integer m in this case). Therefore, by Theorem 2.2, it follows that there exists a neighborhood V of Y such that $[Y]|_V$ is unitary flat if $N_{Y/X}$ is torsion or Diophantine. Thus, by using the regularized minimum construction as we explained in the previous subsection (see also [13, §5]), K_X^{-1} is semi-positive if $N_{Y/X}$ is torsion or Diophantine.

When Y is an elliptic curve, denote by $p = p(N_{Y/X})$ and $q = q(N_{Y/X})$ the real number such that the holonomy of the foliation defined by the flat metric along γ_1 and γ_2 is equal to $\exp(2\pi\sqrt{-1}p)$ and $\exp(2\pi\sqrt{-1}q)$, respectively, where γ_1 and γ_2 are generators of the fundamental group $\pi_1(Y, *)$ of Y. The normal bundle $N_{Y/X}$ is torsion if and only if both p and q are rational. The normal bundle $N_{Y/X}$ is Diophantine if either p or q is a Diophantine irrational number. According to Theorem 1.3, K_X^{-1} is semi-positive for (Y, X) as above if Y is smooth and either $p(N_{Y_Z/X_Z})$ or $q(N_{Y_Z/X_Z})$ is rational. Thus the remaining problem is as follows:

QUESTION 2.4. — *Is K_X^{-1} semi-positive when $(Y, X) = (Y_Z, X_Z)$ as above if Y_Z is smooth and neither $p(N_{Y_Z/X_Z})$ nor $q(N_{Y_Z/X_Z})$ is rational or Diophantine for any choice of the generator (γ_1, γ_2) of the fundamental group $\pi_1(Y, *)$ of Y?*

Note that, when $N_{Y/X}$ is neither torsion nor Diophantine, Ueda constructed an example of (Y, X) which is of type (γ) [21, §5.4]. For his example, we showed that $[Y]$ is not semi-positive [14] (see also [16]). However, as

this Ueda's example or some examples in [16] are essentially only the known examples of type (γ) and X is non-compact surface in these examples, we know nothing on the existence of such an example when X is compact.

2.3. Outline of the proof of Ueda type linearization theorems

In this subsection, we explain the outline of the proof of Ueda type linearization theorem such as Theorem 2.2 by using a toy model.

Let Y be a smooth elliptic curve $\mathbb{C}/\langle 1, \sqrt{-1}\rangle$: i.e. Y is the quotient $\mathbb{C}/\sim$, where "$\sim$" is the relation generated by $z \sim z+1 \sim z+\sqrt{-1}$. Denote by $[z]$ the image of z by the quotient map $\mathbb{C} \to Y$. Define an open covering $\{U_j\}_{j=1,2,3}$ by $U_j := \{[z] \in \mathbb{R}/\mathbb{Z} \mid 2j < \operatorname{Re} z < (2j+1)/6\}$. Let X be a non-singular surface which includes Y as a subvariety such that $N_{Y/X}^{-1} = [\{(U_{jk}, t_{jk})\}] \in \check{H}^1(\{U_j\}, \mathcal{O}_Y^*)$ holds, where

$$t_{jk} = \begin{cases} s & \text{if } (j,k) = (3,1) \\ s^{-1} & \text{if } (j,k) = (1,3) \\ 1 & \text{otherwise} \end{cases}$$

for some $s \in \mathrm{U}(1)$. Additionally, we assume for simplicity that there exists a neighborhood V of Y in X and a holomorphic map $\operatorname{Pr}_Y \colon V \to Y$ such that $\operatorname{Pr}_Y|_Y$ is the identity, and that there exists a defining function w_j on V_j of U_j for each j, where $V_j := \operatorname{Pr}_Y^{-1}(U_j)$. By a simple argument, it follows that we may assume $dw_j = t_{jk} dw_k$ holds on each U_{jk}. Fix a local coordinate z_j of Y on U_j, and regard (z_j, w_j) as coordinates of V_{jk} (Here we extend the domain of the function z_j to V_j by pulling buck by Pr_Y). Then, by letting $f_{kj,m} := \frac{\partial^m}{\partial w_j^m}|_{w_j=0}(t_{jk} \cdot w_k)$, one has that

$$t_{jk} \cdot w_k = w_j + f_{kj,2}(z_j) \cdot w_j^2 + f_{kj,3}(z_j) \cdot w_j^3 + \dots$$

holds on each V_{jk}. Our goal is to construct a new system $\{(V_j, \hat{w}_j)\}$ of local defining functions such that $\hat{w}_j = t_{jk}\hat{w}_k$ by modifying w_j's (and by shrinking V_j's if necessary).

The strategy of Ueda's proof of [21, Theorem 3] can be explained as follows: Define a suitable holomorphic functions $F_{j,m} \colon U_j \to \mathbb{C}$ for each $j = 1,2,3$ and for each $m \geqslant 2$ so that the solution of the functional equation

$$w_j = \hat{w}_j + \sum_{m=2}^{\infty} F_{j,m}(z_j) \cdot \hat{w}_j^m \tag{2.1}$$

satisfies $\hat{w}_j = t_{jk}\hat{w}_k$ on a neighborhood of U_{jk} (if exists). Here we are regarding $F_{j,m}$ as a function on V_j by pulling back by Pr_Y.

In order for $\widehat{w}_j$'s to satisfy $\widehat{w}_j = t_{jk}\widehat{w}_k$, both of the coefficients of $\widehat{w}_j^m$ in two expansions

$$t_{jk}w_k = w_j + \sum_{m=2}^{\infty} f_{kj,m} \cdot w_j^m$$

$$= \widehat{w}_j + \sum_{m=2}^{\infty} F_{j,m} \cdot \widehat{w}_j^m + \sum_{\ell=2}^{\infty} f_{kj,\ell} \cdot \left(\widehat{w}_j + \sum_{n=2}^{\infty} F_{j,n} \cdot \widehat{w}_j^n \right)^{\ell}$$

and

$$t_{jk}w_k = t_{jk}\widehat{w}_k + \sum_{m=2}^{\infty} t_{jk}F_{k,m} \cdot \widehat{w}_k^m = t_{jk}\widehat{w}_k + \sum_{m=2}^{\infty} t_{jk}^{1-m}F_{k,m} \cdot \widehat{w}_j^m$$

should coincide for each m. This condition can be reworded as $-F_{j,m} + t_{jk}^{1-m}F_{k,m} = h_{kj,m}$, where $h_{kj,m}$ is the coefficient of $\widehat{w}_j^m$ in the expansion of

$$\sum_{\ell=2}^{\infty} f_{kj,\ell} \cdot \left(\widehat{w}_j + \sum_{n=2}^{\infty} F_{j,n} \cdot \widehat{w}_j^n \right)^{\ell} \in \mathcal{O}_Y(U_j)[[\widehat{w}_j]].$$

Note that each $h_{kj,m}$ only depends on $f_{kj,\mu}$'s and $\{F_{j,\mu}\}_{\mu<m,j=1,2,3}$, and does not depend on $\{F_{j,\mu}\}_{\mu\geqslant m,j=1,2,3}$: Indeed, $h_{kj,2} = f_{kj,2}$ and $h_{kj,3} = f_{kj,3} + 2f_{kj,2} \cdot F_{j,2}$ hold for example. Conversely, it is observed by relatively simple inductive argument that a formal solution $\widehat{w}_j$ of the functional equation (2.1) satisfies $t_{jk}\widehat{w}_k = \widehat{w}_j$ in any order jet along U_{jk} if $F_{j,m}$'s satisfies $-F_{j,m} + t_{jk}^{1-m}F_{k,m} = h_{kj,m}$. Therefore what we should do is the following:

Step 1. — Solve the equation

$$\delta\{(U_j, F_{j,m})\}(:= \{(U_{jk}, -F_{j,m} + t_{jk}^{1-m}F_{k,m})\}$$
$$= \{(U_{jk}, h_{kj,m})\} \in \check{Z}^1(\{U_j\}, \mathcal{O}_Y(N_{Y/X}^{1-m}))$$

to obtain $F_{j,m}$'s inductively on m.

Step 2. — Estimate $B_m := \max_j \sup_{U_j} |F_{j,m}|$ and show that the formal power series $\sum_{m=2}^{\infty} B_m X^m \in \mathbb{C}[[X]]$ has a positive radius of convergence (if so, one actually shows the existence of the solution $\widehat{w}_j$ of the functional equation (2.1) by shrinking V_j by using the implicit function theorem, which completes the proof).

Note that the assumption that the pair is of infinite type is needed in Step 1 (since the class $[\{(U_{jk}, h_{kj,m})\}]$ coincides with m-th Ueda class $u_m(Y, X)$, see [21, p. 598]), and that the normal bundle is either torsion or Diophantine is needed in Step 2.

In the rest of this subsection, we will focus on Step 2. In what follows, we assume that each $f_{kj,m}$ is a constant function for simplicity. By a simple inductive observation, each $F_{j,m}$ is also constant in this case.

Remark 2.5. — When each $f_{kj,m}$ is a constant function, clearly there is a holomorphic foliation such that each leaves are defined by $\{w_j = \text{constant}\}$. In this case, the problem is reduced to the linearization problem of the holonomy function, which explains the strategy of original proof of Arnol′d's theorem [1] in our simple model.

For $\sigma \in \mathbb{C}^*$, denote by L_σ the unitary flat line bundle on Y defined by

$$L_\sigma := [\{(U_{12}, 1), (U_{21}, 1), (U_{23}, 1), (U_{32}, 1), (U_{31}, \sigma^{-1}), (U_{13}, \sigma)\}]$$
$$\in \check{H}^1(\{U_j\}, \mathbb{C}^*),$$

and by $\mathbb{C}(L_\sigma)$ the sheaf of locally constant sections of L_σ. In the case where each $f_{kj,m}$ is constant, $F_{j,m}$'s are inductively defined by solving

$$\delta\{(U_j, F_{j,m})\}(:= \{(U_{jk}, -F_{j,m} + t_{jk}^{1-m}F_{k,m})\}$$
$$= \{(U_{jk}, h_{kj,m})\} \in \check{Z}^1(\{U_j\}, \mathbb{C}(L_s^{m-1})).$$

The following lemmata are needed for the inductive estimate of $|F_{j,m}|$'s:

LEMMA 2.6. — *There exists a constant K which does not depend on σ such that, for any $\alpha = \{(U_{jk}, \alpha_{jk})\} \in \check{Z}^1(\{U_j\}, \mathbb{C}(L_\sigma))$ and $\beta = \{(U_j, \beta_j)\} \in \check{C}^0(\{U_j\}, \mathbb{C}(L_\sigma))$ with $\delta\beta = \alpha$, it holds that*

$$d(1, \sigma) \cdot \max_j |\beta_j| \leqslant K \cdot \max_{j,k} |\alpha_{j,k}|,$$

where d is the distance of $\mathrm{U}(1) = \mathbb{R}/\mathbb{Z}$ induced by the Euclidean distance of the universal covering $\mathbb{R}$.

Proof. — If $\sigma = 1$, nothing is non-trivial. Otherwise, by a simple calculation, one has that $\beta_1 = -\frac{A}{1-\sigma}$, $\beta_2 = -\alpha_{31} - \alpha_{23} - \frac{\sigma A}{1-\sigma}$, and $\beta_3 = -\alpha_{31} - \frac{\sigma A}{1-\sigma}$, where $A := \alpha_{12} + \alpha_{23} + \alpha_{31}$. The assertion follows from this and the equivalence of d and the distance of $\mathrm{U}(1)$ obtained by restricting the Euclidean distance of $\mathbb{C}$ ($\supset \mathrm{U}(1)$). □

LEMMA 2.7. — *For each $\sigma \in \mathrm{U}(1)$, there exists a constant $K_0(\sigma)$ which satisfies the following property: For any $\alpha = \{(U_{jk}, \alpha_{jk})\} \in \check{Z}^1(\{U_j\}, \mathbb{C}(L_\sigma))$ with $[\alpha] = 0 \in \check{H}^1(\{U_j\}, \mathbb{C}(L_\sigma))$, there exists an element $\beta = \{(U_j, \beta_j)\} \in \check{C}^0(\{U_j\}, \mathbb{C}(L_\sigma))$ such that $\delta\beta = \alpha$ and*

$$\max_j |\beta_j| \leqslant K_0(\sigma) \cdot \max_{j,k} |\alpha_{j,k}|$$

hold.

Proof. — When $\sigma \neq 1$, the assertion follows from Lemma 2.6 (Note that the solution β is unique in this case). When $\sigma = 1$, it follows by simple argument that $[\alpha] = 0$ holds if and only if $\alpha_{12} + \alpha_{23} + \alpha_{31} = 0$. As the solution β can be constructed by letting $\beta_1 := 0$, $\beta_2 := -\alpha_{31} - \alpha_{23}$, and $\beta_3 := -\alpha_{31}$, the assertion holds if one lets $K_0(1) := 2$. $\qquad \square$

Note that Lemma 2.6 can be regarded as an analogue of [21, Lemma 4], and Lemma 2.7 can be regarded as an analogue of [21, Lemma 3] (=[8, Lemma 2]).

Let $M := \max_j \sup_{V_j} |w_j|$, and assume that there exists a positive constant R such that $\{(z_j, w_j) \in V_j \mid z_j \in V_k, |w_j| \leqslant 1/R\} \Subset V_k$ holds for each j, k for simplicity (we mean by "$\Subset$" the relatively compact subset). Then, by an inductive argument as in [21, §4], one has that the formal power series $A(X) = X + \sum_{m=2}^\infty A_m X^m \in \mathbb{R}[[X]]$ defined by the equation

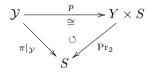$$\sum_{m=2}^\infty \frac{A_m}{K_0(s^{m-1})} \cdot X^m = \frac{MA(X)^2}{1 - A(X)}$$

satisfies $\max_j \sup_{U_j} |F_{j,m}| \leqslant A_m$ for any m, where $K_0(s^{m-1})$ is the one as in Lemma 2.7.

When s is a torsion element of U(1), $\{K_0(s^{m-1})\}_{m=2}^\infty$ is a finite set, and thus there exists a constant which is larger than any $K_0(s^{m-1})$. When $s = e^{2\pi\sqrt{-1}\theta}$ for a Diophantine irrational number, we have a suitable type of estimate of $K_0(s^{m-1})$'s by Lemma 2.6. In these cases, one can show that the formal power series $A(X)$ has a positive radius of convergence (by using the implicit function theorem suitably in the torsion case [21, §4.4], and by using the estimate as in [19] in the Diophantine case [21, §4.6]).

On the other hand, when $s = e^{2\pi\sqrt{-1}\theta}$ for a real number θ which is neither rational nor Diophantine, Lemma 2.6 is not enough to show the convergence of the formal power series $\sum_{m=2}^\infty \max_j |F_{j,m}| \cdot X^m \in \mathbb{C}[[X]]$ (Indeed, there is a counterexample by Ueda, see [21, §5.4]). We will explain our idea to improve the estimates under some special situations in the next subsection.

2.4. Our idea to improve the estimates

In order to improve the estimates of $|F_{j,m}|$'s in the previous subsection, we will consider a deformation of the complex structure of X. Let S be a neighborhood of U(1) in $\mathbb{C}^*$, $\mathcal{X}$ a complex manifold of dimension 3, and $\pi \colon \mathcal{X} \to S$ be a surjective holomorphic submersion. Assume that there exist a submanifold $\mathcal{Y} \subset \mathcal{X}$ and a biholomorphism $p \colon \mathcal{Y} \to Y \times S$ such that the

diagram

$$
\begin{array}{ccc}
\mathcal{Y} & \xrightarrow[\cong]{\ \ p\ \ } & Y \times S \\
& {\scriptstyle \pi|_{\mathcal{Y}}}\searrow \quad {\scriptstyle \circlearrowright} \quad \swarrow {\scriptstyle \mathrm{Pr}_2} & \\
& S &
\end{array}
$$

commutes, where $Y = \mathbb{C}/\langle 1, \sqrt{-1}\rangle$ and Pr_2 is the second projection. For each $s \in S$, denote by X_s the fiber $\pi^{-1}(s)$ and by Y_s the submanifold $\mathcal{Y} \cap X_s$. In what follows, we will identify $\mathcal{Y}$ with $Y \times S$ via p.

Letting $\mathcal{U}_j := U_j \times S$, where U_j is the one in the previous subsection, we define a $\mathbb{C}^*$-flat line bundle $\mathcal{L}$ on $\mathcal{Y} = Y \times S$ by

$$
\mathcal{L} = [\{(\mathcal{U}_{12}, 1), (\mathcal{U}_{21}, 1), (\mathcal{U}_{23}, 1), (\mathcal{U}_{32}, 1), (\mathcal{U}_{31}, \mathrm{Pr}_2^{-1}), (\mathcal{U}_{13}, \mathrm{Pr}_2)\}]
$$
$$
\in \check{H}^1(\{\mathcal{U}_j\}, \mathcal{O}_{\mathcal{Y}}^*),
$$

where $\mathrm{Pr}_2 \colon Y \times S \to S(\subset \mathbb{C}^*)$ is the second projection. Note that $\mathcal{L}|_{Y_s} = L_s$ holds via natural isomorphism between $Y_s = Y \times \{s\}$ and Y. Here we assume that $N_{\mathcal{Y}/\mathcal{X}} = \mathcal{L}$. Then one can regard each pair (Y_s, X_s) as the one we observed in the previous subsection for each $s \in \mathrm{U}(1)$ under suitable additional assumptions.

Remark 2.8. — As will be seen in Section 6.1, the pair (Y_Z, X_Z) as in Section 1 is settled in a fiber of such a deformation $\pi \colon \mathcal{X} \to S$ when Y_Z is smooth. On the other hand, we constructed in [14, Example 4.3] a pathological example of such $\pi \colon \mathcal{X} \to S$, in which the line bundle $[Y_s]$ on X_s is semi-positive for almost every $s \in S$ in the sense of Lebesgue measure, whereas it is not semi-positive for uncountably many $s \in \mathrm{U}(1)$. In order to distinguish these two cases, Condition $(*)$ we will add below is important. Note that the pair (Y_s, X_s) is of infinite type for any torsion element s in $\mathrm{U}(1)$ in the former example [18], and that the pair (Y_s, X_s) is of finite type for any torsion element s in $\mathrm{U}(1)$ in the latter example [14, Example 4.3].

Assume, again for simplicity, that there exists a holomorphic retraction $\mathrm{Pr}_{\mathcal{Y}}$ from a neighborhood of $\mathcal{Y}$ onto $\mathcal{Y}$ such that $\mathrm{Pr}_{\mathcal{Y}}|_{\mathcal{Y}}$ is the identity, and that there exists a neighborhood $\mathcal{V}_j$ of each $\mathcal{U}_j$ in $\mathcal{X}$ and a defining function $w_j \colon \mathcal{V}_j \to \mathbb{C}$ of $\mathcal{U}_j$ such that $\{(V_{j,s}, w_j|_{V_{j,s}})\}$ is a system as in the previous subsection for each $s \in S$, where $V_{j,s} := \mathcal{V}_j \cap X_s$. Under such a situation, one can consider the "*simultaneous linearization problem*" for $\{(\mathcal{V}_j, w_j)\}$, which is the problem on constructing a new defining functions system $\{(\mathcal{V}_j, \widehat{w}_j)\}$ by shrinking $\mathcal{V}_j$'s if necessary such that $t_{jk} w_k = w_j$ holds on each $\mathcal{V}_{jk}$, where

$t_{jk}\colon \mathcal{V}_{jk} \to \mathbb{C}^*$ is the function defined by

$$
t_{jk}(x) := \begin{cases} \mathrm{Pr}_2 \circ \mathrm{Pr}_{\mathcal{Y}}(x) & \text{if } (j,k) = (3,1) \\ (\mathrm{Pr}_2 \circ \mathrm{Pr}_{\mathcal{Y}}(x))^{-1} & \text{if } (j,k) = (1,3) \\ 1 & \text{otherwise} \end{cases},
$$

where $\mathrm{Pr}_2\colon Y \times S \to S(\subset \mathbb{C}^*)$ is the second projection. Under some additional technical assumptions including the following Condition $(*)$ and $(**)$, this simultaneous linearization problem is reduced to the estimate of the L^∞ operator norm of the function

$$
\delta\colon \check{C}^0(\{\mathcal{U}_j\}, \mathcal{O}_{\mathcal{Y}}(\mathcal{L}^{m-1})) \to \check{C}^1(\{\mathcal{U}_j\}, \mathcal{O}_{\mathcal{Y}}(\mathcal{L}^{m-1}))
$$

for each $m \geqslant 2$ by the same argument as in the previous subsection and by an analogue of Lemma 2.7 (Proposition 2.9 below for a toy model case, Proposition 5.4 for the actual configuration).

Condition $(*)$. — The pair (Y_s, X_s) is of infinite type for any $s \in S$.

Condition $(**)$. — For each $m \geqslant 2$, $[\{(U_{jk} \times \{s\}, h_{kj,m}|_{U_{jk}\times\{s\}})\}] = 0 \in \check{H}^1(\{U_{j,s}\}, \mathcal{O}_Y(L_s^{m-1}))$ holds for any $s \in S$, where $h_{kj,m}$ is the function on $\mathcal{U}_{jk}$ defined in the same manner as in the previous subsection.

In fact, it turns out that Condition $(**)$ implies Condition $(*)$, since the class $[\{(U_{jk} \times \{s\}, h_{kj,m}|_{U_{jk}\times\{s\}})\}]$ coincides with m-th Ueda class for $s \in \mathrm{U}(1)$, as is mentioned in the previous subsection. On the other hand, even when Condition $(*)$ holds, it may possible that Condition $(**)$ does not hold, since the class $[\{(U_{jk} \times \{s\}, h_{kj,m}|_{U_{jk}\times\{s\}})\}]$ does not necessarily coincide with Ueda class (see also Remark 2.3). In the actual situation (i.e. in the case where each fiber X_s is realized as X_Z as in Section 1), Condition $(*)$ holds. Although Condition $(**)$ is much more useful for running an argument we will explain as Proposition 2.9 below for a toy model case (in order to explain the idea of the proof of Proposition 5.4 for actual configuration), we do not know whether or not Condition $(**)$ holds in the actual situation. Thus, we need to run the inductive argument by carefully shrinking S depending on m so that Condition $(**)$ holds at each step of the induction (Note that the cohomology group $\check{H}^1(\{U_{j,s}\}, \mathcal{O}_Y(L_s^{m-1}))$ itself vanishes if $s^{m-1} \neq 1$, which means that Condition $(*)$ implies Condition $(**)$ for $m = m_0$ if S is so small that $\{s \in S \mid s^{m_0-1} = 1\} \subset \mathrm{U}(1)$).

In the rest of this subsection, we will explain the idea how to establish an analogue of Lemma 2.7 for the family configuration and estimate the operator norm of $\delta\colon \check{C}^0(\{\mathcal{U}_j\}, \mathcal{O}_{\mathcal{Y}}(\mathcal{L}^{m-1})) \to \check{C}^1(\{\mathcal{U}_j\}, \mathcal{O}_{\mathcal{Y}}(\mathcal{L}^{m-1}))$. In what follows, we replace S and $\mathcal{O}_{\mathcal{Y}}(\mathcal{L}^{m-1})$ with $\mathrm{U}(1)$ and $\mathcal{A}_{Y\times\mathrm{U}(1)}(\mathcal{L}^m)$, respectively, and consider the coboundary map

$$
\delta\colon \check{C}^0(\{U_j \times \mathrm{U}(1)\}, \mathcal{A}_{Y\times\mathrm{U}(1)}(\mathcal{L}^m)) \to \check{C}^1(\{U_j \times \mathrm{U}(1)\}, \mathcal{A}_{Y\times\mathrm{U}(1)}(\mathcal{L}^m)).
$$

as a toy model, where $\mathcal{A}_{Y \times \mathrm{U}(1)}(\mathcal{L}^m)$ is the sheaf of the sections of $\mathcal{L}^m$ which are locally constant in Y direction and real analytic in $\mathrm{U}(1)$ direction.

For each positive integer m and for each $\nu = 0, 1, \ldots, m-1$, define a subset $W_{m,\nu}$ of $\mathrm{U}(1)$ by

$$W_{m,\nu} := \left\{ \exp(2\pi\sqrt{-1}\theta) \in \mathrm{U}(1) \;\middle|\; \theta \in \mathbb{R}, \; \left| \theta - \frac{\nu}{m} \right| \leqslant \frac{1}{2m} \right\}.$$

Note that $s = \zeta_m^\nu$ is the only element s of $W_{m,\nu}$ such that $s^m = 1$, where $\zeta_m := \exp(2\pi\sqrt{-1}/m)$. For an open neighborhood $\widetilde{W}_{m,\nu}$ defined by

$$\widetilde{W}_{m,\nu} := \left\{ \exp(2\pi\sqrt{-1}\theta) \in \mathrm{U}(1) \;\middle|\; \theta \in \mathbb{R}, \; \left| \theta - \frac{\nu}{m} \right| \leqslant \frac{3}{4m} \right\}$$

of $W_{m,\nu}$ in $\mathrm{U}(1)$, we first show the following:

PROPOSITION 2.9. — *Let* $\alpha = \{(U_{jk} \times \widetilde{W}_{m,\nu}, \alpha_{jk})\}$ *be an element of* $\check{Z}^1(\{U_j \times \widetilde{W}_{m,\nu}\}, \mathcal{A}_{Y \times \mathrm{U}(1)}(\mathcal{L}^m))$. *Denote by* $\alpha_s = \{(U_{jk} \times \{s\}, \alpha_{jk}|_{U_{jk} \times \{s\}})\} \in \check{Z}^1(\{U_{j,s}\}, \mathbb{C}(L_s^m))$ *the element obtained by restricting* α *to* Y_s *for each* $s \in \widetilde{W}_{m,\nu}$. *Then the followings are equivalent:*

(i) $[\alpha_{\zeta_m^\nu}] = 0 \in \check{H}^1(\{U_{j,\zeta_m^\nu}\}, \mathbb{C}(L_{\zeta_m^\nu}^m))$.
(ii) $[\alpha|_{Y \times \widetilde{W}_{m,\nu}}] = 0 \in \check{H}^1(\{U_j \times \widetilde{W}_{m,\nu}\}, \mathcal{A}_{Y \times \mathrm{U}(1)}(\mathcal{L}^m))$.

Proof. — As clearly (ii) implies (i), we show the converse. Assume (i) holds. Denote by $\widetilde{W}_{m,\nu}^*$ the set $\widetilde{W}_{m,\nu} \setminus \{\zeta_m^\nu\}$. It follows from an argument in the proof of Lemma 2.6 that the element $\beta = \{(U_j \times \widetilde{W}_{m,\nu}^*, \beta_j)\} \in \check{C}^0(\{U_j \times \widetilde{W}_{m,\nu}^*\}, \mathcal{A}_{Y \times \mathrm{U}(1)}(\mathcal{L}^m))$ defined by $\beta_1(s) = -\frac{A(s)}{1-s^m}$, $\beta_2(s) = -\alpha_{31}(s) - \alpha_{23}(s) - \frac{s^m A(s)}{1-s^m}$, and $\beta_3(s) = -\alpha_{31}(s) - \frac{s^m A(s)}{1-s^m}$ satisfies $\delta\beta = \alpha|_{Y \times \widetilde{W}_{m,\nu}^*}$, where $A(s) := \alpha_{12}(s) + \alpha_{23}(s) + \alpha_{31}(s)$. As it follows from the argument in the proof of Lemma 2.7 that $A(\zeta_m^\nu) = 0$, one can naturally extend β_j to define a primitive $\beta \in \check{C}^0(\{U_j \times \widetilde{W}_{m,\nu}\}, \mathcal{A}_{Y \times \mathrm{U}(1)}(\mathcal{L}^m))$. $\square$

Under Condition $(*)$, one has that the assertion (i) of Proposition 2.9 holds for any ν. If the function

$$W_{m,\nu} \ni s \mapsto \left| \frac{s^m A(s)}{1 - s^m} \right| \in \mathbb{C}$$

is convex, whose complex analytical counterpart is actually the case in the actual situation in some sense, one has that

$$\max_{s \in W_{m,\nu}} \left| \frac{s^m A(s)}{1 - s^m} \right| = \max_{s \in \partial W_{m,\nu}} \left| \frac{s^m A(s)}{1 - s^m} \right|,$$

where A is the one in the proof of Proposition 2.9. In this case, for a point $s_* \in \partial W_{m,\nu}$ which attains the maximum of the function above, it follows from Lemma 2.6 that
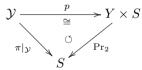
$$\max_{s \in W_{m,\nu}} \left| \frac{s^m A(s)}{1 - s^m} \right| = \left| \frac{s_*^m A(s_*)}{1 - s_*^m} \right| \leqslant \left| \frac{s_*^m A(s_*)}{\varepsilon d(1, s_*^m)} \right|$$

$$= \frac{|A(s_*)|}{\varepsilon \cdot md(\zeta_m^\nu, s_*)} = \frac{2}{\varepsilon} \cdot |A(s_*)| \leqslant \frac{6}{\varepsilon} \cdot \max_{j,k} \sup_{s \in W_{m,\nu}} |\alpha_{jk}(s)|,$$

where ε is a positive constant such that $\varepsilon d(1, \sigma) \leqslant |1 - \sigma|$. Note that ε depends on neither s nor m. As therefore the constant $6/\varepsilon$ depends on neither s nor m, one can regard this estimate as an improved variant of Lemma 2.6 under Condition $(*)$.

3. Main result for a suitable deformation configuration

According to the observation in Section 2.4, we will pose a generalized configuration, which can be regarded as a generalization of a configuration of the blow-up model of $\mathbb{P}^2$ at suitably chosen nine points as we described in Section 1 (see the next section), and state our main result in this section.

Let $S \subset \mathbb{C}^*$ be a neighborhood of U(1), $\mathcal{X}$ a complex manifold, $\pi \colon \mathcal{X} \to S$ a surjective holomorphic submersion, and $\mathcal{Y} \subset \mathcal{X}$ be a reduced subvariety of codimension $r > 0$ such that there exist a connected reduced compact complex variery Y and a biholomorphism $p \colon \mathcal{Y} \cong Y \times S$ such that the following diagram commutes.

Assume that Y is either a manifold (i.e. a non-singular variety) or an analytic space of dimension one with only nodal singularities. Also assume that $r = 1$ in the latter case (i.e. when Y is singular). Take a finite open covering $\{U_j\}$ of Y such that each U_j is Stein, connected and simply connected which satisfies the following properties: For each $p \in Y_{\text{sing}}$, there *uniquely* exists U_j such that $p \in U_j$, and it holds that $\bigcup_{U_j \cap Y_{\text{sing}} = \emptyset} U_j = Y \setminus Y_{\text{sing}}$. Note that especially it holds that $Y_{\text{sing}} \cap U_{jk} = \emptyset$ holds for any j and k if $j \neq k$. By considering a refinement of $\{U_j\}$ if necessary, we may assume that it also holds that $U_j \cap Y_{\text{sing}} = \emptyset$ if $U_{jk} \neq \emptyset$ and $U_k \cap Y_{\text{sing}} \neq \emptyset$.

Define an open covering $\{\mathcal{U}_j\}$ of $\mathcal{Y}$ by letting $\mathcal{U}_j := p^{-1}(U_j \times S)$. Take a neighborhood $\mathcal{V}_j$ of $\mathcal{U}_j$ in $\mathcal{X}$ such that $\mathcal{V}_j \cap \mathcal{Y} = \mathcal{U}_j$. We assume that

$\mathcal{V}_{jk} := \mathcal{V}_j \cap \mathcal{V}_k$ is empty if and only if $\mathcal{U}_{jk} := \mathcal{U}_j \cap \mathcal{U}_k$ is empty by shrinking $\mathcal{V}_j$'s if necessary. We use the following notation for each $s \in S$: $X_s := \pi^{-1}(s)$, $Y_s := X_s \cap \mathcal{Y} = p^{-1}(Y \times \{s\})$, $U_{j,s} := X_s \cap \mathcal{U}_j = p^{-1}(U_j \times \{s\})$, $V_{j,s} := X_s \cap \mathcal{V}_j$, $U_{jk,s} := U_{j,s} \cap U_{k,s}$, and $V_{jk,s} := V_{j,s} \cap V_{k,s}$. We denote by $\mathbb{I}_s$ the holomorphically trivial line bundle on Y_s.

In what follows, we give five assumptions. First one is the following:

Assumption 1 (Cohomology vanishing assumption). — For any topologically trivial holomorphic line bundle L on Y which is not holomorphically trivial, it holds that $H^1(Y, \mathcal{O}_Y(L)) = 0$.

The second one is:

Assumption 2 (Flatness assumption for the normal bundles). — There exists a holomorphic function

$$w_j = \begin{pmatrix} w_j^{(1)} \\ w_j^{(2)} \\ \vdots \\ w_j^{(r)} \end{pmatrix} : \mathcal{V}_j \to \mathbb{C}^r$$

such that the following six conditions hold:

(i) $\{w_j^{(\lambda)}\}_{\lambda=1}^r$ is a system of defining functions of $\mathcal{U}_j$ (i.e. $\{w_j^{(\lambda)}\}_{\lambda=1}^r$ generates the defining ideal sheaf $I_{\mathcal{U}_j} \subset \mathcal{O}_{\mathcal{V}_j}$ of $\mathcal{U}_j$).

(ii) $\max_\lambda \max_j \sup_{\mathcal{V}_j} |w_j^{(\lambda)}|$ is bounded.

(iii) For each j and k with $\mathcal{U}_{jk} \neq \emptyset$, there exists a holomorphic function $t_{jk}^{(\lambda)} : S \to \mathbb{C}^*$ for each $\lambda = 1, 2, \ldots, r$ such that $dw_j^{(\lambda)} = t_{jk}^{(\lambda)} dw_k^{(\lambda)}$ holds on $\mathcal{U}_{jk}$, where we denote by the same letter $t_{jk}^{(\lambda)}$ the composition of $p \colon \mathcal{Y} \to Y \times S$, the second projection, and $t_{jk}^{(\lambda)}$.

(iv) When U_k is singular, there exist coordinates (x_k, y_k, s) of $\mathcal{V}_k$ such that $w_k^{(1)} = x_k \cdot y_k$ and $\pi(x_k, y_k, s) = s$.

(v) When U_j is smooth (i.e. non-singular), $\mathcal{V}_j$ is embedded into $\mathbb{C}^r \times \mathcal{U}_j$ in the following manner: There exists a holomorphic retraction $\mathrm{Pr}_{\mathcal{U}_j} \colon \mathcal{V}_j \to \mathcal{U}_j$ (i.e. $\mathrm{Pr}_{\mathcal{U}_j} |_{\mathcal{U}_j}$ is the identity) such that $\pi \circ \mathrm{Pr}_{\mathcal{U}_j} = \pi$ and that $\mathcal{V}_j \ni x \mapsto (w_j(x), z_j \circ \mathrm{Pr}_{\mathcal{U}_j}(x), \pi(x)) \in \mathbb{C}^r \times U_j \times S$ defines coordinates of $\mathcal{V}_j$, where z_j is a coordinate of U_j.

(vi) When U_k is singular, $\mathcal{V}_k$ is embedded into $\mathbb{C}^2 \times S$ in the following manner: The images of the maps $\mathcal{V}_k \ni (x_k, y_k, s) \mapsto (x_k, 0, s) \in \mathcal{U}_k$ and $\mathcal{V}_k \ni (x_k, y_k, s) \mapsto (0, y_k, s) \in \mathcal{U}_k$ coincide with each of the irreducible components of $\mathcal{U}_k$.

In what follows, we often denote simply by w_j the function $w_j^{(1)}$ when $r = 1$. Denote by $\mathcal{N}^*$ the holomorphic vector bundle on $\mathcal{Y}$ of rank r defined by $\mathcal{N}^* := [\{(\mathcal{U}_{jk}, T_{jk})\}] \in H^1(\{\mathcal{U}_j\}, \mathrm{GL}_r(\mathcal{O}_{\mathcal{Y}}))$, where $T_{jk} := \mathrm{diag}(t_{jk}^{(1)}, t_{jk}^{(2)}, \dots, t_{jk}^{(r)})$ is the diagonal matrix, and, for each $s \in S$, by N_s^* the holomorphic vector bundle $\mathcal{N}^*|_{Y_s} = [\{(U_{jk,s}, T_{jk}(s))\}] \in H^1(\{U_{j,s}\}, \mathrm{GL}_r(\mathbb{C}))$ on Y_s of rank r. We regard them as conormal bundles of $\mathcal{Y}$ and Y_s, respectively. Note that $\mathcal{N}^* = \bigoplus_{\lambda=1}^r \mathcal{L}_\lambda$ and $N_s^* = \bigoplus_{\lambda=1}^r L_{\lambda,s}$ for each $s \in S$, where $\mathcal{L}_\lambda$ is the holomorphic line bundle on $\mathcal{Y}$ defined by $\mathcal{L}_\lambda := [\{(\mathcal{U}_{jk}, t_{jk}^{(\lambda)})\}] \in H^1(\{\mathcal{U}_j\}, \mathcal{O}_{\mathcal{Y}}^*)$, and $L_{\lambda,s}$ is the holomorphic line bundle on Y_s defined by $L_{\lambda,s} := \mathcal{L}_\lambda|_{Y_s} = [\{(U_{jk,s}, t_{jk}^{(\lambda)}(s))\}] \in H^1(\{U_{j,s}\}, \mathbb{C}^*)$ for each $\lambda = 1, 2, \dots, r$. Note that such a decomposability is actually the case for example when Y is an elliptic curve with unitary flat normal bundle, since any unitary flat vector bundle on an elliptic curve can be split as the direct sum of unitary flat line bundles.

For a multi-index $a = (a_1, a_2, \dots, a_r) \in \mathbb{Z}^r$, we denote by $|a|$ the sum $a_1 + a_2 + \cdots + a_r$, by $\mathcal{L}^a$ the line bundle $\mathcal{L}_1^{a_1} \otimes \mathcal{L}_2^{a_2} \otimes \cdots \otimes \mathcal{L}_r^{a_r}$, by L_s^a the line bundle $L_{1,s}^{a_1} \otimes L_{2,s}^{a_2} \otimes \cdots \otimes L_{r,s}^{a_r}$ for each $s \in S$, by t_{jk}^a the product $\prod_{\lambda=1}^r (t_{jk}^{(\lambda)})^{a_\lambda}$, and by dw_j^a the tensor $\bigotimes_{\lambda=1}^r (dw_j^{(\lambda)})^{\otimes a_\lambda}$.

Fix an integer $M_0 \geqslant 1$. Denote by $\widehat{S}_m$ the subset of S defined by

$$\widehat{S}_m := \left\{ s \in S \mid d(s, \mathrm{U}(1)) < \frac{1}{M_0 m} \right\}$$

for each $m \in \mathbb{Z}_{>0}$, where d is the distance of S attached by restricting the distance of $\mathbb{C}^*$ which is induced by the Euclidean distance of $\mathbb{C}$ via the covering map $\mathbb{C} \in \xi \mapsto \exp(2\pi\sqrt{-1}\xi) \in \mathbb{C}^*$: i.e. $d(\exp(2\pi\sqrt{-1}a), \exp(2\pi\sqrt{-1}b)) := |a - b|$ for $a, b \in \mathbb{C}$ when a is close to b.

Assumption 3 (Assumption on torsion points and the bound of the transition functions). — The following three conditions hold:

(i) The set $\widehat{S}_1$ is a relatively compact subset of S.
(ii) For each positive integer m and each $a = (a_1, a_2, \dots, a_r) \in \mathbb{Z}^r$ with $|a| = m$, $\{s \in \widehat{S}_m \mid L_s^a \cong \mathbb{I}_s\}$ is a subset of $\{s \in \mathrm{U}(1) \mid s^m = 1\}$.
(iii) There exists a positive constant $\Theta > 1$ such that

$$\max_{j,k} \sup_{s \in \widehat{S}_m} |t_{jk}^a(s)| \leqslant \Theta$$

holds for any positive integer m and any $a \in \mathbb{Z}^r$ with $|a| = m$.

Note that Assumption 3 (iii) implies $|t_{jk}^{(\lambda)}(s)| = 1$ for each $s \in \mathrm{U}(1)$ and each λ. For each positive integer m, denote by S_m the subset of $\widehat{S}_m$ defined by

$$S_m := \left\{ s \in S \ \middle| \ d(s, \mathrm{U}(1)) < \frac{1}{4M_0 m} \right\},$$

by $\zeta_{M_0 m}$ the point $\exp(2\pi\sqrt{-1}/(M_0 m))$ of $\mathrm{U}(1) \subset S$, by $W_{m,\nu}$ the closed subset of S defined by

$$W_{m,\nu} := \left\{ \exp(2\pi\sqrt{-1}\xi) \in \widehat{S}_m \ \middle| \ \begin{array}{l} \xi \in \mathbb{C}, \ \left| \mathrm{Re}\,\xi - \dfrac{\nu}{M_0 m} \right| \leqslant \dfrac{1}{2M_0 m}, \\ \qquad\qquad |\mathrm{Im}\,\xi| \leqslant \dfrac{1}{4M_0 m} \end{array} \right\}$$

for $\nu = 0, 1, 2, \ldots, M_0 m - 1$, and by $d_{a,\nu} \colon W_{m,\nu} \to \mathbb{R}_{\geqslant 0}$ the function defined by

$$d_{a,\nu}(s) := \begin{cases} d(s, \zeta_{M_0 m}^\nu) & \text{if } L_{\zeta_{M_0 m}^\nu}^a \text{ is holomorphically trivial} \\ \frac{1}{2M_0 m} & \text{otherwise} \end{cases}$$

for each m, ν, and a with $|a| = m$. Note that one simply has that $\bigcup_{\nu=0}^{M_0 m-1} W_{m,\nu} = \overline{S_m}$ and $d_{a,\nu}(s) \leqslant d(s, \{\sigma \in S \mid L_\sigma^a \cong \mathbb{I}_\sigma\})$ for each positive integer m, multi-index a with $|a| = m$, $\nu = 0, 1, 2, \ldots, M_0 m - 1$, and $s \in W_{m,\nu}$ (Here we use Assumption 3 (ii)).

Assumption 4 (Ueda-type Lemma). — There exists a positive constant K such that the following holds: For any positive integer m, multi-index $a = (a_1, a_2, \ldots, a_r) \in \mathbb{Z}^r$ with $|a| = m$, $\nu = 0, 1, 2, \ldots, M_0 m - 1$, $s \in W_{m,\nu}$, a 1-cochain $\alpha = \{(U_{jk,s}, \alpha_{jk,s} dw_j^a)\} \in \check{Z}^1(\{U_{jk,s}\}, \mathcal{O}_{Y_s}(L_s^a))$, and a 0-cochain $\beta = \{(U_{j,s}, \beta_{j,s} dw_j^a)\} \in \check{C}^0(\{U_{j,s}\}, \mathcal{O}_{Y_s}(L_s^a))$ such that $\alpha = \delta\beta := \{(U_{jk,s}, (-\beta_{j,s} + t_{kj}^a(s) \cdot \beta_{k,s}) dw_j^a)\} \in \check{Z}^1(\{U_{jk,s}\}, \mathcal{O}_{Y_s}(L_s^a))$, it holds that

$$m \cdot d_{a,\nu}(s) \cdot \|\beta\|_{s,a} \leqslant K \cdot \|\alpha\|_{s,a},$$

where $\|\alpha\|_{s,a} := \max_{j,k} \sup_{U_{jk,s}} |\alpha_{jk,s}|$ and $\|\beta\|_{s,a} := \max_j \sup_{U_{j,s}} |\beta_{j,s}|$.

The final assumption is the following:

Assumption 5 (Assumption on the type of (Y_s, X_s) for $s \in \mathrm{U}(1)$). — For each $s \in \mathrm{U}(1)$, the pair (Y_s, X_s) is of infinite type in the sense of [12] when Y is singular, and of [13] when Y is smooth (i.e. Y_s admits a system of local defining functions of type m for any positive integer m, see also Section 2.2).

The following theorem is the main result:

THEOREM 3.1. — *Let s be an element of* $\mathrm{U}(1)(\subset S)$. *Then, under Assumptions 1, 2, 3, 4, and 5, the following holds by shrinking $V_{j,s}$'s if necessary: there exists a function*

$$
\widehat{w}_{j,s} = \begin{pmatrix} \widehat{w}_{j,s}^{(1)} \\ \widehat{w}_{j,s}^{(2)} \\ \vdots \\ \widehat{w}_{j,s}^{(r)} \end{pmatrix} : V_{j,s} \to \mathbb{C}^r
$$

on each $V_{j,s}$ such that $\{\widehat{w}_j^{(\lambda)}\}_{\lambda=1}^r$ generates the defining ideal sheaf $I_{U_{j,s}} \subset \mathcal{O}_{V_{j,s}}$ of $U_{j,s}$, and that $\widehat{w}_{j,s}^{(\lambda)} = t_{jk}^{(\lambda)}(s) \cdot \widehat{w}_{k,s}^{(\lambda)}$ holds on each $V_{jk,s}$ for $\lambda = 1, 2, \ldots, r$.

4. Two examples of configurations which satisfies five assumptions in Theorem 3.1

Before we give the proof of Theorem 3.1 in Section 5, we will give two examples of configurations which satisfy five assumptions in Section 3, so that one can apply Theorem 3.1 to the examples in Section 1.

4.1. A configuration in which Y_s's are smooth elliptic curves

Let Z be a manifold of dimension $r + 1$, L be a holomorphic line bundle on Z, $D_1^0, D_2^0, \ldots, D_r^0$ be prime divisors of Z such that $\mathcal{O}_Z(D_\lambda^0) \cong \mathcal{O}_Z(L)$ for any $\lambda = 1, 2, \ldots, r$. Assume that $Y^0 := D_1^0 \cap D_2^0 \cap \cdots \cap D_r^0$ is a smooth elliptic curve, D_λ^0's intersects to each other transversally along Y^0, and that the intersection number $(L.Y^0)$ is equal to 1. Denote by $p_0 \in Y^0$ the point such that $\mathcal{O}_{Y^0}(L|_{Y^0}) \cong \mathcal{O}_{Y^0}(p_0)$.

In what follows, we identify $\Sigma := \mathrm{Pic}^0(Y^0)$ with Y^0 via the map $Y^0 \ni q \mapsto \mathcal{O}_{Y^0}(q - p_0) \in \Sigma$. Let $\overline{\mathcal{X}}$ be the blow-up of $Z \times \Sigma$ along the subvariety

$$\{(z, s) \in Y^0 \times \Sigma \mid z \text{ coincides with } s \text{ via the identification above}\} \subset Z \times \Sigma,$$

and $\pi \colon \overline{\mathcal{X}} \to \Sigma$ be the morphism obtained by composing the blow-up morphism and the second projection $Z \times \Sigma \to \Sigma$. Note that π is a surjective holomorphic submersion, and that each fiber $X_s := \pi^{-1}(s)$ is the blow-up of Z at s for each $s \in \Sigma$. Denote by $\overline{\mathcal{D}}_\lambda \subset \overline{\mathcal{X}}$ the strict transform of $D_\lambda^0 \times \Sigma$ for each $\lambda = 1, 2, \ldots, r$, and by $\overline{\mathcal{Y}}$ the intersection $\bigcap_{\lambda=1}^r \overline{\mathcal{D}}_\lambda$. It is easily observed that $\{D_{\lambda,s} := X_s \cap \overline{\mathcal{D}}_\lambda\}_{\lambda=1}^r$ intersects transversally along $Y_s := X_s \cap \overline{\mathcal{Y}}$ for each $s \in \Sigma$. Note also that $N_{D_{\lambda,s}/X_s}|_{Y_s} \cong L_s^{-1}$ holds for any λ (and

thus it holds that $N^*_{Y_s/X_s} \cong L_s^{\oplus r}$) for each $s \in \Sigma \ (= Y^0$ by the identification we fixed in the above), where L_s is the line bundle on Y_s such that $\mathcal{O}_{Y_s}(L_s) \cong \mathcal{O}_{Y^0}(s - p_0)$ holds via the natural isomorphism between Y_s and Y^0 (i.e. via the composition of the blow-up morphism and the first projection $Z \times \Sigma \to Z$).

Take an element τ of the upper half plane $\mathbb{H} := \{t \in \mathbb{C} \,|\, \mathrm{Im}\, t > 0\}$ such that $Y := \mathbb{C}/\langle 1, \tau \rangle$ is isomorphic to Y^0. Let γ_1 and γ_2 be generators of $\pi_1(Y^0, p_0)$ which corresponds to the deck transformations $+1$ and $+\tau$ of the universal covering $\mathbb{C} \to Y^0$. Denote by $\mathcal{L}$ the line bundle on $Y^0 \times Y$ such that, for each $y \in Y$, the restriction $L_y := \mathcal{L}|_{Y^0 \times \{y\}}$ of it to $Y^0 \times \{y\}$ is the unitary flat line bundle which corresponds to the unitary representation $\pi_1(Y^0, *) \to$ U(1) defined by $\gamma_1 \mapsto \exp(2\pi\sqrt{-1} \cdot (-q(y)))$ and $\gamma_2 \mapsto \exp(2\pi\sqrt{-1}p(y))$ via the first projection $Y^0 \times \{y\} \to Y^0$, where $p(y)$ and $q(y)$ are the elements of $\mathbb{R}/\mathbb{Z}$ such that $y = [p(y) + q(y) \cdot \tau]$. Note that we can naturally regard $\mathcal{L}$ as a holomorphic line bundle, since each line bundle L_y is isomorphic (on $Y^0 \times \{y\}$ via the first projection as a holomorphic line bundle) to the $\mathbb{C}^*$-flat line bundle on Y^0 which corresponds to the $\mathbb{C}^*$-representation $\pi_1(Y^0, *) \to \mathbb{C}^*$ defined by $\gamma_1 \mapsto 1$ and $\gamma_2 \mapsto \exp(2\pi\sqrt{-1}(p(y) + q(y) \cdot \tau))$ (This $\mathbb{C}^*$-flat line bundle holomorphically depends on y. The isomorphism is easily obtained by considering the function $\sigma \colon z \mapsto \exp(2\pi\sqrt{-1} \cdot (q(s) - q_0)z)$ on the universal covering of Y). It follows from the fact that $\mathrm{Pic}^0(Y^0)$ has a property as the coarse moduli (see [17, §A.1] for example), there exists an isomorphism $i \colon Y \to \Sigma$ such that $(\mathrm{id}_{Y^0} \times i)^* N^{-1}_{\mathcal{D}_\lambda/\mathcal{X}}|_{\overline{y}} \cong \mathcal{L}$ via the isomorphism between $\mathcal{Y}$ and $Y^0 \times \Sigma$ which is defined by using the blow-up morphism (Note that it does not depend on λ, since $N^{-1}_{\mathcal{D}_\lambda/\mathcal{X}}|_{\overline{y}}$'s are isomorphic to each other for $\lambda = 1, 2, \ldots, r$).

In what follows, we identify Y with Y^0 (and thus also with Σ) via i. Note that the point $p_0 \in Y^0$ is identified with the point $[0] \in Y$. Note also that it follows from a simple argument that the following two conditions are equivalent to each other for an element of $y \in \Sigma = Y^0 = Y$ under our identifications:

(i) y is torsion as an element of Y (i.e. $[m(p(y) + q(y) \cdot \tau)] = [0] \in Y$ for some $m \in \mathbb{Z} \setminus \{0\}$),
(ii) y is torsion as an element of Σ (i.e. L_y^m is holomorphically trivial for some $m \in \mathbb{Z} \setminus \{0\}$).

In such case, we simply say that y is *torsion*.

Fix a rational number $q_0 \in \mathbb{Q}$ with $0 \leqslant q_0 < 1$. Set

$$\ell_0 := \{[p + q_0 \cdot \tau] \in Y \,|\, p \in \mathbb{R}\} \text{ and } \ell_\infty := \left\{ \left[p + \left(q_0 + \frac{1}{2} \right) \cdot \tau \right] \in Y \ \middle|\ p \in \mathbb{R} \right\},$$

and regard them as subsets of Σ via the identification. Define a base space S by $S := \Sigma \setminus \ell_\infty$. We regard S as a neighborhood $\{s \in \mathbb{C}^* \mid \exp(-\pi \operatorname{Im} \tau) < |s| < \exp(\pi \operatorname{Im} \tau)\}$ of $U(1)$ in $\mathbb{C}^*$ by using the embedding

$$S \ni [z] \mapsto \exp(2\pi\sqrt{-1}(z - q_0 \cdot \tau)) \in \mathbb{C}^*,$$

where we are regarding z as an element of $\{p + q\tau \in \mathbb{C} \mid p, q \in \mathbb{R}, \ |q - q_0| < 1/2\}$. In what follows, for each element $s \in S$, we let $p(s)$ and $q(s)$ be the real numbers such that $|q(s) - q_0| < 1/2$ and $s = [p(s) + q(s) \cdot \tau]$ hold as elements of Y ($p(s)$ is determined modulo $\mathbb{Z}$, whereas $q(s)$ is determined uniquely).

We define the total space $\mathcal{X}$ by $\mathcal{X} := \pi^{-1}(S) \subset \overline{\mathcal{X}}$, and the subvariety $\mathcal{Y}$ by $\mathcal{Y} := \overline{\mathcal{Y}} \cap \mathcal{X}$. Denote the restriction $\pi|_{\mathcal{X}}$ simply by the same letter π, and by $\mathcal{D}_\lambda$ the intersection $\overline{\mathcal{D}}_\lambda \cap \mathcal{X}$ for each $\lambda = 1, 2, \ldots, r$. Let $\{U_j\}, \{\mathcal{U}_j\}$, and $\{U_{j,s}\}$ be those as in Section 3. In the rest of this subsection, we will show that $\pi \colon \mathcal{X} \to S$ satisfies Assumptions 1, 2, 3, 4, and 5 under the following:

Assumption 5′. — The following holds for any torsion element $s \in \Sigma$: There exist a complex manifold B_s of dimension r, r divisors $\{E_\lambda\}_{\lambda=1}^r$ of B_s, a neighborhood V_s of Y_s in X_s, and a surjective proper holomorphic map $b_s \colon V_s \to B_s$ such that Y_s is a fiber of b_s, E_λ's intersects transversally to each other at the point $b_s(Y_s)$, and that $b_s^* E_\lambda = m_\lambda D_{\lambda,s}$ holds as divisors for some positive integer m_λ for each λ.

4.1.1. Assumption 1

As Y is an elliptic curve, the assertion of Assumption 1 clearly holds.

4.1.2. Assumption 2

As is obtained by a simple observation, the line bundle L_s is holomorphically isomorphic to the $\mathbb{C}^*$-flat line bundle which corresponds to the $\mathbb{C}^*$-representation $\pi_1(Y^0, *) \to \mathbb{C}^*$ defined by $\gamma_1 \mapsto \exp(2\pi\sqrt{-1}(-q_0))$ and $\gamma_2 \mapsto \exp(2\pi\sqrt{-1}(p(s)+(q(s)-q_0)\cdot\tau))$. Thus we have that, by taking a refinement of $\{U_j\}$ if necessary, there exists a holomorphic map $t_{jk} \colon S \to \mathbb{C}^*$ for each j and k such that each $t_{jk}(s)$ coincides with either 1, $\exp(\pm 2\pi\sqrt{-1}q_0)$, $\exp(\pm 2\pi\sqrt{-1}(p(s)+(q(s)-q_0)\cdot\tau))$, or $\exp(\pm 2\pi\sqrt{-1}q_0) \cdot \exp(\pm 2\pi\sqrt{-1}(p(s)+ (q(s) - q_0) \cdot \tau))$, and that $L_s = [\{(U_{jk,s}, t_{jk}(s))\}]$ holds for each $s \in S$.

Take a neighborhood $\mathcal{V}_j$ of each $\mathcal{U}_j$ by using [20, Corollary 1]. Then we have that each $\mathcal{V}_j$ is Stein, and there exists a defining functions system $\{w_j\}$ and a holomorphic retraction $\operatorname{Pr}_{\mathcal{U}_j}$ such that the assertions (ii) and (v) in

Takayuki Koike

Assumption 2 hold. By a standard argument, one can modify $\{w_j\}$ so that $w_j^{(\lambda)}$ is a defining function of $\mathcal{D}_\lambda \cap \mathcal{V}_j$ on $\mathcal{V}_j$, and that $dw_j^{(\lambda)} = t_{jk} dw_k^{(\lambda)}$ holds on each $\mathcal{U}_{jk}$, since $N_{\mathcal{D}_\lambda/\mathcal{X}}^{-1}|_\mathcal{Y} = [\{(\mathcal{U}_{jk}, t_{jk})\}]$ as elements of $H^1(\{\mathcal{U}_j\}, \mathcal{O}_\mathcal{Y}^*)$ for each λ (see [13, §2.2] for example). Thus we have that the assertions (i) and (iii) in Assumption 2 also hold by letting $t_{jk}^{(\lambda)} := t_{jk}$ for each λ.

4.1.3. Assumption 3

Fix a sufficiently large integer m_* such that $m_* > \max\{2, (\mathrm{Im}\,\tau)^{-1}\}$. By letting $M_0 := m_* \cdot \min\{m \in \mathbb{Z} \mid m > 0, mq_0 \in \mathbb{Z}\}$, we here show the assertions (i), (ii), and (iii) in Assumption 3.

First, the assertion (i) follows directly by our definition of M_0. Here let us note that $\mathrm{U}(1)$ is identified with ℓ_0 in the configuration in this subsection.

Next, for the assertion (ii), we will show that $\{s \in \widehat{S}_m \mid L_s^m \cong \mathbb{I}_s\} \subset \{s \in \mathrm{U}(1) \mid s^m = 1\}$ holds for each positive integer m (Note that $t_{jk}^a = t_{jk}^{|a|}$ and $L_s^a = L_s^{|a|}(= L_s^{\otimes|a|})$ hold for any $a \in \mathbb{Z}^r$). As L_s^m is holomorphically isomorphic to the unitary flat line bundle which corresponds to the unitary representation $\pi_1(Y_0, *) \to \mathrm{U}(1)$ defined by $\gamma_1 \mapsto \exp(2\pi\sqrt{-1}(mq(s)))$ and $\gamma_2 \mapsto \exp(2\pi\sqrt{-1}(mp(s)))$, one has that it is holomorphically trivial if and only if both $mp(s)$ and $mq(s)$ are integers, since a unitary flat line bundle on a compact complex manifold is trivial as a unitary flat line bundle if and only if it is holomorphically trivial (see also [13, Proposition 2.2]). Therefore one has that $M_0 mq(s)$ is an integer which is a multiple of m_* in this case. As it follows by definition that $M_0 q_0$ is also an integer which is a multiple of m_*, we have that either $q(s) = q_0$ or $|mM_0 q(s) - mM_0 q_0| \geqslant m_*$ holds if L_s^m is holomorphically trivial. The assertion (ii) follows from this, since

$$|mM_0 q(s) - mM_0 q_0| = \frac{mM_0}{\mathrm{Im}\,\tau} \cdot (|q(s)-q_0| \cdot \mathrm{Im}\,\tau) < mM_0 m_* \cdot d(s, \mathrm{U}(1)) < m_*$$

holds for any $s \in \widehat{S}_m$ (Note that $d(s, \mathrm{U}(1)) = |\mathrm{Im}(p(s) + (q(s)-q_0)\tau)|$ by definition).

Finally, we show the assertion (iii). Let m be a positive integer. Take an element $s \in \widehat{S}_m$. As $t_{jk}(s)$ is either 1, $\exp(\pm 2\pi\sqrt{-1}q_0)$, $\exp(\pm 2\pi\sqrt{-1}(p(s) + (q(s)-q_0)\cdot\tau))$, or $\exp(\pm 2\pi\sqrt{-1}q_0)\cdot\exp(\pm 2\pi\sqrt{-1}(p(s)+(q(s)-q_0)\cdot\tau))$, one has that

$$|t_{jk}(s)|^m \leqslant \max_\pm |\exp(\pm 2m\pi\sqrt{-1}(q(s)-q_0)\cdot\tau)| \leqslant \exp(2m\pi|q(s)-q_0|\cdot\mathrm{Im}\,\tau)).$$

As $|q(s)-q_0|\cdot\mathrm{Im}\,\tau \leqslant (M_0 m)^{-1}$ holds for any $s \in \widehat{S}_m$, one has the assertion by letting $\Theta := \exp(2\pi/M_0)$.

4.1.4. Assumption 4

Take a positive integer m, $\nu \in \{0, 1, 2, \ldots, M_0 m - 1\}$, and an element $s \in W_{m,\nu}$. When $s \in \mathrm{U}(1)$, it follows by [21, Lemma 4] that there exists a constant K which depends only on the choice of Y and $\{U_j\}$ such that the inequality

$$m \cdot d(s, \{\sigma \in S \mid L_\sigma^m \cong \mathbb{I}_\sigma\}) \cdot \max_j \sup_{U_{j,s}} |\beta_{j,s}| \leqslant K \cdot \max_{j,k} \sup_{U_{jk,s}} |\alpha_{jk,s}|$$

holds for any multi-index a with $|a| = m$, any 1-cochain $\alpha = \{(U_{jk,s}, \alpha_{jk,s} dw_j^a)\} \in \check{Z}^1(\{U_{jk,s}\}, \mathcal{O}_{Y_s}(L_s^m))$ and any 0-cochain $\beta = \{(U_{j,s}, \beta_{j,s} dw_j^a)\} \in \check{C}^0(\{U_{j,s}\}, \mathcal{O}_{Y_s}(L_s^m))$ such that $\alpha = \delta\beta$, since each $t_{jk}(s)$ is an element of $\mathrm{U}(1)$ by Assumption 3 (iii) (Note that such β is unique when $d(1, s^m) \neq 0$ by Assumption 3 (ii), since L_s^m is not holomorphically trivial in such case). As $d_{a,\nu}(s) \leqslant d(s, \{\sigma \in S \mid L_\sigma^m \cong \mathbb{I}_\sigma\})$, one has the assertion of Assumption 4 for such s.

In what follows, we investigate the case where s need not to be an element of $\mathrm{U}(1)$ by fixing such an element $s \in W_{m,\nu}$. Take a local trivialization e_j of L_s on each $U_{j,s}$ such that $e_j = t_{jk}(s) \cdot e_k$ holds on each $U_{jk,s}$.

As L_s is isomorphic to the line bundle which corresponds to the unitary representation $\pi_1(Y^0, *) \to \mathrm{U}(1)$ generated by $\gamma_1 \mapsto \exp(2\pi\sqrt{-1} \cdot (-q(s)))$ and $\gamma_2 \mapsto \exp(2\pi\sqrt{-1} p(s))$, it follows that there exists another local trivialization $\widehat{e}_j$ of L_s on each $U_{j,s}$ such that $\widehat{e}_j = \widehat{t}_{jk} \cdot \widehat{e}_k$ holds on each $U_{jk,s}$, where $\widehat{t}_{jk}$ is a constant function valued in $\mathrm{U}(1)$. Note that such $\widehat{e}_j$'s are constructed by considering the function $\sigma \colon z \mapsto \exp(2\pi\sqrt{-1} \cdot (q(s) - q_0)z)$ on the universal covering of Y. Therefore we may assume that the ratio $\sigma_j := \widehat{e}_j/e_j$ coincides with this function σ restricted to a suitable open subset of $\mathbb{C}$ via the covering map. As $|\sigma(z)| = \exp(-2\pi(q(s) - q_0)\mathrm{Im}\, z)$, one has the existence of a constant $M_1 > 1$ which only depends on M_0 and the manner how to choose domains of the universal covering of Y_s such that the restriction of σ to it coincides with σ_j (and thus depends on neither m nor $s \in \widehat{S}_m$) such that

$$M_1^{-\frac{1}{m}} \leqslant \inf_{U_{j,s}} |\sigma_j(z)| \leqslant \max_{U_{j,s}} |\sigma_j(z)| \leqslant M_1^{\frac{1}{m}}$$

holds for each j. Again by the inequality $d_{a,\nu}(s) \leqslant d(s, \{\sigma \in S \mid L_\sigma^m \cong \mathbb{I}_\sigma\})$, it is sufficient to show the following:

LEMMA 4.1. — *There exists a constant K which depends on neigher m nor $s \in \overline{S_m}$ such that, for any 1-cochain $\alpha = \{(U_{jk,s}, \alpha_{jk,s} e_j^m)\} \in \check{Z}^1(\{U_{jk,s}\}, \mathcal{O}_{Y_s}(L_s^m))$ and any 0-cochain $\beta = \{(U_{j,s}, \beta_{j,s} e_j^m)\} \in \check{C}^0(\{U_{j,s}\}, \mathcal{O}_{Y_s}(L_s^m))$ with $\alpha = \delta\beta := \{(U_{jk,s}, (-\beta_{j,s} + t_{kj}^m(s) \cdot \beta_{k,s}) e_j^m)\} \in$*

$\check{Z}^1(\{U_{jk,s}\}, \mathcal{O}_{Y_s}(L_s^m))$, *it holds that*

$$m \cdot \min_{a,b \in \frac{1}{m}\mathbb{Z}} |(p(s) + q(s)\tau) - (a + b\tau)| \cdot \|\beta\|_{s,m} \leqslant K \cdot \|\alpha\|_{s,m},$$

where $\|\alpha\|_{s,m} := \max_{j,k} \sup_{U_{jk,s}} |\alpha_{jk,s}|$ *and* $\|\beta\|_{s,m} := \max_j \sup_{U_{j,s}} |\beta_{j,s}|$.

Proof. — One may assume that L_s^m is not holomorphically trivial, since otherwise both $mp(s)$ and $mq(s)$ are integers by Assumption 3 (ii). Note that $H^0(\{U_{j,s}\}, \mathcal{O}_{Y_s}(L_s^m)) = H^1(\{U_{j,s}\}, \mathcal{O}_{Y_s}(L_s^m)) = 0$ in this case, which meas that β as in the assertion is uniquely exists for any α.

Take a 1-cocycle $\alpha = \{(U_{jk,s}, \alpha_{jk,s}e_j^m)\} \in \check{Z}^1(\{U_{j,s}\}, \mathcal{O}_{Y_s}(L_s^m))$ such that $[\alpha] = 0 \in \check{H}^1(\{U_{j,s}\}, \mathcal{O}_{Y_s}(L_s^m))$. By letting $\hat{\alpha}_{jk,s} := \sigma_j^{-m} \cdot \alpha_{jk,s}$, one has that $\hat{\alpha}_{jk,s}\hat{e}_j^m = \alpha_{jk,s}e_j^m$. Take a primitive $\beta = \{(U_{j,s}, \beta_{j,s}e_j^m)\} = \{(U_{j,s}, \hat{\beta}_{j,s}\hat{e}_j^m)\}$, which uniquely exists as we mentioned above. Note that $\hat{\beta}_{j,s} \cdot \sigma_j^m = \beta_{j,s}$ for each j. It follows from [21, Lemma 4] that there exists a constant K_1 which depends only on Y and $\{U_j\}$ and on neither s, m, α, nor β such that

$$\max_j \sup_{U_{j,s}} |\hat{\beta}_{j,s}| \leqslant \frac{K_1}{d_\Sigma(\mathbb{I}_s, L_s^m)} \cdot \max_{j,k} \sup_{U_{jk,s}} |\hat{\alpha}_{jk,s}|,$$

where d_Σ is an invariant distance on Σ. As all the invariant distances on Σ are equivalent to each other, one can use our distance d on Y as d_Σ via the identification. As

$$\max_{j,k} \sup_{U_{jk,s}} |\hat{\alpha}_{jk,s}| = \max_{j,k} \sup_{U_{jk,s}} \left(|\alpha_{jk,s}| \cdot |\sigma_j|^{-m} \right) \leqslant M_1 \cdot \max_{j,k} \sup_{U_{jk,s}} |\alpha_{jk,s}|$$

and

$$\max_j \sup_{U_{j,s}} |\beta_{j,s}| = \max_j \sup_{U_{j,s}} \left(|\hat{\beta}_{j,s}| \cdot |\sigma_j|^m \right) \leqslant M_1 \cdot \max_j \sup_{U_{j,s}} |\hat{\beta}_{j,s}|$$

hold, one has the lemma by letting $K := K_1 \cdot M_1^2$. $\qquad\square$

4.1.5. Assumption 5

For any non-torsion element $s \in \Sigma$, one has that the pair (Y_s, X_s) is of infinite type, since it follows from Assumption 1 that $H^1(Y_s, \mathcal{O}_{Y_s}(N_s \otimes S^{m+1}N_s^*)) = 0$ for any $m \geqslant 1$. When $s \in U(1) = \ell_0$ is a torsion element, consider the foliation $\mathcal{F}_s$ on X_s defined by b_s as in Assumption 5′. Then, by a simple observation, it follows that the holonomy of $\mathcal{F}_s$ along a leaf Y_s is a diagonal matrix $\mathrm{diag}(t_1, t_2, \ldots, t_r)$ for m_λ-th roots of the unity t_λ's. This means that, for a suitable foliation chart $(z_j, \hat{w}_j^{(1)}, \hat{w}_j^{(2)}, \ldots, \hat{w}_j^{(r)})$ on a neighborhood of each $U_{j,s}$, the transition functions of $(\hat{w}_j^{(1)}, \hat{w}_j^{(2)}, \ldots, \hat{w}_j^{(r)})$ are unitary, which shows the assertion of Assumption 5.

4.2. A configuration in which Y_s's are cycles of rational curves

Let Z be a smooth complex surface and $Y^0 \subset Z$ be a holomorphically embedded cycle of N rational curves (i.e. Y^0 is a reduced subvariety of X with only nodal singularities whose normalization consists of N copies of $\mathbb{P}^1$ such taht the dual graph is a cycle graph). Denote by $i \colon C \to Y^0$ the normalization, and by $C_1, C_2, \ldots, C_N$ the irreducible components of C. We may assume that $i(C_\nu) \cap i(C_\mu) \neq \emptyset$ if $(\nu, \mu) = (1, 2), (2, 3), \ldots,$ $(N-1, N)$ or $(N, 1)$. Let p_0 be the intersection $i(C_N) \cap i(C_1)$, and p_ν be the intersection $i(C_\nu) \cap i(C_{\nu+1})$ for $\nu = 1, 2, \ldots, N-1$ when $N > 1$. In this case, we often identify each C_ν with the image $i(C_\nu)$, and regard each p_ν also as a point of C_ν or $C_{\nu-1}$ for $\nu = 0, 1, \ldots, N-1$ ($C_0 := C_N$). When $N = 1$, we denote by $\{p_0, p_1\}$ the preimage of the unique nodal point of Y^0 by i. We sometimes denotes the unique nodal point also by p_0 in this case.

Denote by L the normal bundle $N_{Y^0/Z} = [Y^0]|_{Y^0}$. Assume that the intersection number $(i^* L . C_\nu)$ is equal to 1 for $\nu = 1$ and to 0 for $\nu = 2, 3, \ldots, N$.

Set $\Sigma := C_1 \setminus \{p_0, p_1\}$. In this subsection, we identify Σ with $\mathbb{C}^*$ by using the non-homogeneous coordinate s of C_1 such that s maps p_0 to $0 \in \mathbb{P}^1$, p_1 to $\infty \in \mathbb{P}^1$, and the unique zero of a non section f_L of $H^0(Y^0, \mathcal{O}_{Y^0}(L))$ to $1 \in \mathbb{P}^1$. Note that it follows by a standard argument that $H^0(Y^0, \mathcal{O}_{Y^0}(L)) \cong \mathbb{C}, H^1(Y^0, \mathcal{O}_{Y^0}(L)) = 0$, and that, for an element $f \in H^0(Y^0, \mathcal{O}_{Y^0}(L))$, the following three conditions are equivalent to each other: $i^* f(p_0) = 0$, $i^* f(p_1) = 0$, and $f \equiv 0$.

We use a finite open covering $\{U_j\}$ of Y^0 such that $\bigcup_{U_j \cap Y_{\mathrm{sing}} = \emptyset} U_j = Y \setminus Y_{\mathrm{sing}}$ and that, for each $p_\nu \in Y_{\mathrm{sing}}^0$, it holds that $\#\{k \mid p_\nu \in U_k\} = 1$. We denote by $U_{(N1)}$ the unique element of $\{U_j\}$ which includes p_0, by $U_{(\nu, \nu+1)}$ the unique element of $\{U_j\}$ which includes p_ν for each $\nu = 1, 2, \ldots, N-1$, and by $U_{(N1)}^\pm$ the irreducible components of $U_{(N1)}$. We may assume that $U_{(N1)}^+ \cap \{s \in \Sigma \mid |s| < 1\} \neq \emptyset$. We also assume that $U_j \cap Y_{\mathrm{sing}}^0 = \emptyset$ holds if $U_k \cap Y_{\mathrm{sing}}^0 \neq \emptyset$ and $U_j \cap U_k \neq \emptyset$.

According to the proof of [12, Lemma 2.1] and [12, Remark 2.2], the map

$$\Sigma \ni s \mapsto \mathcal{O}_{Y^0}(L) \otimes \mathcal{O}_{Y^0}(-s) \in \mathrm{Pic}^0(Y^0)$$

is an isomorphism, by which we will identify Σ with $\mathrm{Pic}^0(Y^0)$ in this subsection. Let us see how are the transition functions of $\mathcal{O}_{Y^0}(L) \otimes \mathcal{O}_{Y^0}(-s_0)$ for $s_0 \in \Sigma$. Denote by $F_{s_0} \in H^0(C, \mathcal{O}_C(i^* L) \otimes \mathcal{O}_C(\{1\} - \{s_0\}))$ the section obtained by tensoring $i^* f_L$ and the element of $H^0(C, \mathcal{O}_C(\{1\} - \{s_0\}))$ which coincides with the rational map $s \mapsto \frac{s_0 - s}{1 - s}$ on C_1. Then it holds that $F_{s_0}(p_0) = s_0 \cdot F_{s_0}(p_1)$. Denote by $G \in H^0(U_{(N1)}, \mathcal{O}_{Y^0}(L) \otimes \mathcal{O}_{Y^0}(-i(s)))$ the

element defined by

$$G := \begin{cases} F_{s_0} & \text{on } U^+_{(N1)} \\ s_0 \cdot F_{s_0} & \text{on } U^-_{(N1)}. \end{cases}$$

By using G as a local frame on $U_{(N1)}$ and a section e_j such that $i^* e_j = F_{s_0}|_{U_j}$ as that on U_j for $j \neq (N1)$, one has that there exists a holomorphic function t_{jk} on Σ for each j and k such that the element of $\text{Pic}^0(Y^0)$ which corresponds to $s_0 \in \Sigma$ coincides with $[\{(U_{jk}, (t_{jk}(s_0))^{-1})\}]$. Note that $t_{jk}(s)$ is either $s^{\pm 1}$ or 1.

Let $\overline{\mathcal{X}}$ be the blow-up of $Z \times \Sigma$ along the subvariety

$$\{(z, s) \in Y^0 \times \Sigma \,|\, z \text{ coincides with } s \text{ via the identification above}\} \subset Z \times \Sigma,$$

and $\pi \colon \overline{\mathcal{X}} \to \Sigma$ be the morphism obtained by composing the blow-up morphism and the second projection $Z \times \Sigma \to \Sigma$. Note that π is a surjective holomorphic submersion, and that the fiber $X_s := \pi^{-1}(s)$ is the blow-up of Z at $s \in Y^0 \subset Z$ for each $s \in \Sigma$. Denote by $\overline{\mathcal{Y}}$ the strict transform of $Y^0 \times \Sigma$, by Y_s the intersection $X_s \cap \overline{\mathcal{Y}}$, and by N_s the line bundle $\mathcal{O}_{Y_s}(L) \otimes \mathcal{O}_{Y_s}(-s)$ on Y_s for each $s \in \Sigma$, where we are regarding L as a line bundle on Y_s by the natural identification of Y_s with Y^0. Note that $N_{Y_s/X_s} \cong N_s$.

In what follows, we identify Σ with $\mathbb{C}^*$ by using the coordinate s, and will use the distance d of Σ which is the restriction of that of $\mathbb{C}^*$ induced by the Euclidean distance of the universal covering $\mathbb{C} \ni \xi \mapsto \exp(2\pi\sqrt{-1}\xi) \in \Sigma$. Denote by S the subset defined by $S := \{s \in \Sigma \mid d(\mathrm{U}(1), s) < 2\}(= \{\exp(2\pi\sqrt{-1}\xi) \in \Sigma \mid |\,\mathrm{Im}\,\xi| < 2\})$. We define the total space $\mathcal{X}$ by $\mathcal{X} := \pi^{-1}(S) \subset \overline{\mathcal{X}}$, and the subvariety $\mathcal{Y}$ by $\mathcal{Y} := \overline{\mathcal{Y}} \cap \mathcal{X}$. Denote the restriction $\pi|_{\mathcal{X}}$ simply by the same letter π. Let $\{U_j\}, \{\mathcal{U}_j\}$, and $\{U_{j,s}\}$ be those as in Section 3. In the rest of this subsection, we will show that $\pi \colon \mathcal{X} \to S$ satisfies Assumptions 1, 2, 3, 4, and 5 under the following:

Assumption 5″. — Assume that an element $s \in \Sigma$ is torsion (i.e. $s^m = 1$ for some positive integer m). Then there exists a neighborhood V_s of Y_s in X_s and a surjective proper holomorphic map $b_s \colon V_s \to B_s$ onto a neighborhood B_s of the origin of $\mathbb{C}$ such that $b_s^* \{0\} = m \cdot Y_s$ holds as divisors for some positive constant m.

4.2.1. Assumption 1

One can easily show that the assertion of Assumption 1 holds by the calculation as in the proof of [12, Proposition 2.5].

4.2.2. Assumption 2

First we construct $\mathcal{V}_k$ for $k = (N1), (12), \ldots, (N, N-1)$. Denote by $\bar{\mathcal{U}}_k \subset \mathcal{X}$ the strict transform of $U_k \times \Sigma$. Let $\bar{\mathcal{U}}_k^{\pm}$ be each of the irreducible component of $\bar{\mathcal{U}}_k$. As each $\bar{\mathcal{U}}_k^+$ is stein, it follows from [20] that there exists a Stein neighborhood $\mathcal{W}_k$ of $\bar{\mathcal{U}}_k^+$ in $\mathcal{X}$. Take a defining function y_k of $\bar{\mathcal{U}}_k^+$ on $\mathcal{W}_k$, and x_k of $\bar{\mathcal{U}}_k^- \cap \mathcal{W}_k$ on $\mathcal{W}_k$. Set $w_k := x_k \cdot y_k$. Then, as $S \Subset \Sigma$, the assertion (ii), (iv), and (vi) of Assumption 2 holds for each $k = (N1), (12), \ldots, (N, N-1)$ by shrinking U_k's and $\mathcal{W}_k$'s and by letting $\mathcal{V}_k := \mathcal{W}_k \cap \mathcal{X}$. Note that $\{U_j\}$ is still an open covering of Y^0 after shrinking U_k's in such manner, since it holds that $\bigcup_{U_j \cap Y_{\text{sing}} = \emptyset} U_j = Y \setminus Y_{\text{sing}}$.

For j such that $U_j \cap Y_{\text{sing}} = \emptyset$, we define $\mathcal{V}_j$ by using [20, Corollary 1] in the same manner as in Section 4.1. Then, by the same argument as in, one has that the assertion (i), (ii), (iii), and (v) of Assumption 2 also holds by letting $t_{jk}^{(1)} := t_{jk}$ (Note that $N_{\mathcal{Y}/\mathcal{X}}^{-1} = [\{\mathcal{U}_{jk}, t_{jk}(s)\}]$).

4.2.3. Assumption 3

Here we show the assertion Assumption 3 holds by letting $M_0 := 1$. First let us note that the assertion (i) clearly holds.

It follows from a calculation by using a long exact sequence as in [12, Proposition 2.5] that, for $s \in S$, $H^0(Y_s, \mathcal{O}_{Y_s}(N_s)) = 0$ holds if and only if $s = 1$. Thus one has that the set $\{s \in S \mid N_s^m \cong \mathbb{I}_s\}$ coincides with $\{s \in U(1) \mid s^m = 1\}$, which proves the assertion (ii).

Denote by $p(s)$ and $q(s)$ the real numbers such that $\exp(2\pi\sqrt{-1}(p(s) + q(s)\sqrt{-1})) = s$ for each $s \in S$. Note that $p(s)$ is determined modulo $\mathbb{Z}$. Then, as $s \in \widehat{S}_m$ if and only if $|q(s)| < 1/m$, one has that $\max_{j,k} \sup_{s \in \widehat{S}_m} |t_{jk}(s)^m| \leqslant \exp(2\pi/m)^m = \exp(2\pi)$, since $t_{jk}(s)$ is either 1 or $s^{\pm 1}$. Thus the assertion (iii) holds by letting $\Theta := \exp(2\pi)$.

4.2.4. Assumption 4

Take a local frame e_j of $N_{\mathcal{Y}/\mathcal{X}}^{-1}$ on each $\mathcal{U}_j$ such that $e_j = t_{jk}e_k$ holds on each $\mathcal{U}_{jk}$ (or just let $e_j := dw_j|_{\mathcal{U}_j}$). It is sufficient to show the following:

LEMMA 4.2. — *There exists a constant K such that the following holds for any positive integer m and any element $s \in \widehat{S}_m$: for a 1-cochain*

$\alpha = \{(U_{jk,s}, \alpha_{jk,s} e_j^m)\} \in \check{Z}^1(\{U_{jk,s}\}, \mathcal{O}_{Y_s}(N_s^{-m}))$ *and a 0-cochain* $\beta = \{(U_{j,s}, \beta_{j,s} e_j^m)\} \in \check{C}^0(\{U_{j,s}\}, \mathcal{O}_{Y_s}(N_s^{-m}))$ *with* $\alpha = \delta\beta := \{(U_{jk,s}, (-\beta_{j,s} + t_{kj}^m(s) \cdot \beta_{k,s}) e_j^m)\}$, *it holds that*

$$m \cdot \min_{a \in \frac{1}{m}\mathbb{Z}} |(p(s) + q(s)\sqrt{-1}) - a| \cdot \|\beta\|_{s,m} \leqslant K \cdot \|\alpha\|_{s,m},$$

where $\|\alpha\|_{s,m} := \max_{j,k} \sup_{U_{jk,s}} |\alpha_{jk,s}|$ *and* $\|\beta\|_{s,m} := \max_j \sup_{U_{j,s}} |\beta_{j,s}|$.

Proof. — It is sufficient to show the lemma only when $s^m \neq 1$.

Let $J: \widetilde{C} \to Y^0$ be the morphism obtained by considering the normalization only at the singular point p_0. Note that $\widetilde{C}$ is a connected variety whose dual graph is tree, and that $J^{-1}(p_0)$ consists of two points, say $p_0^{\pm}$. Denote by $\widetilde{U}_j$ the preimage $J^{-1}(U_j)$ for each $j \neq (N1)$, and by $\widetilde{U}_{(N1\pm)}$ the neighborhood of $p_0^{\pm}$, respectively, such that $J^{-1}(U_{(N1)}) = \widetilde{U}_{(N1-)} \cup \widetilde{U}_{(N1+)}$ holds. We regard $\{U_j\}_{j \neq (N1)} \cup \{\widetilde{U}_{(N1+)}, \widetilde{U}_{(N1-)}\}$ as an open covering of $\widetilde{C}$.

Define a 0-cochain $\widehat{\beta} = \{(\widetilde{U}_j, \widehat{\beta}_j)\} \in \check{C}^0(\{\widetilde{U}_j\}, \mathcal{O}_{\widetilde{C}}(J^* N_s^{-m}))$ by $\widehat{\beta}_{(N1+)} := J^* \beta_{(N1)}$, $\widehat{\beta}_{(N1-)} := J^* \beta_{(N1)} \cdot s^m$, and by $\widehat{\beta}_j := J^* \beta_j$ for the other j's, where we are regarding N_s as a line bundle on Y^0 by using the natural identification of Y^0 and Y_s. As all the transition functions of $J^* N_s$ are trivial, one has that $\max_{j,k} \sup_{\widetilde{U}_{jk}} |\widehat{\alpha}_{jk}| = \|\alpha\|_{s,m}$ holds, where we are denoting $\delta\widehat{\beta}_j (= J^*\alpha)$ by $\widehat{\alpha} = \{(\widetilde{U}_{jk}, \widehat{\alpha}_{jk})\}$. Again by the fact that all the transition functions of $J^* N_s$ are trivial, it follows from [12, Lemma 4.1] that there exists an element $\widehat{\beta}' = \{(\widetilde{U}_j, \widehat{\beta}'_j)\} \in \check{C}^0(\{\widetilde{U}_j\}, \mathcal{O}_{\widetilde{C}}(J^* N_s^{-m}))$ such that $\delta\widehat{\beta}' = \widehat{\alpha}$ and $\max_j \sup_{\widetilde{U}_j} |\widehat{\beta}'_j| \leqslant K_0 \|\alpha\|_{s,m}$ holds for some constant K_0 which depends only on Y^0 and $\{U_j\}$ and on neither s nor m. Note that, as is clearly followed by the compactness of Y^0, one can take a constant $\ell \in \mathbb{C}$ such that $\widehat{\beta}_j = \widehat{\beta}'_j + \ell$. As it holds for $a_{\pm} := \widehat{\beta}'_{(N1\pm)}(p_0^{\pm})$ that $a_+ + \ell = s^{-m}(a_- + \ell)$, one has that $\ell = \frac{s^m a_+ - a_-}{1 - s^m}$. Therefore, by

$$\|\beta\|_s \leqslant \max\{1, |s|^m\} \cdot \left(\max_j \sup_{\widetilde{U}_j} |\widehat{\beta}'_j| + \frac{|s^m a_+ - a_-|}{|1 - s^m|} \right)$$

$$\leqslant e^{2\pi} \left(K_0 \|\alpha\|_{s,m} + \frac{2e^{2\pi} K_0 \|\alpha\|_{s,m}}{|1 - s^m|} \right),$$

one has the inequality $|1 - s^m| \cdot \|\beta\|_{s,m} \leqslant K_1 \cdot \|\alpha\|_{s,m}$ holds for a constant $K_1 := e^{2\pi} K_0 \cdot (1 + e^{2\pi} + 2e^{2\pi})$. The lemma follows form this, since d and the distance of S induced by restricting the Euclidean distance of $\mathbb{C}(\supset \mathbb{C}^* \supset S)$ are equivalent. $\square$

4.2.5. Assumption 5

Take $s \in \Sigma$. When s is non-torsion, it follows from Assumption 1 that the pair (Y_s, X_s) is of infinite type, since $H^1(Y_s, \mathcal{O}_{Y_s}(N_s^{-m})) = 0$ holds for any $m \geqslant 1$. When s is non-torsion, consider the fibration $b_s \colon V_s \to B_s$ as in Assumption $5''$. Then one can construct a system of local defining functions $\{(V_{j,s} \cap V_s, w_{j,s})\}$ of Y_s by considering m-th root of $b_s^* w|_{V_{j,s} \cap V_s}$, where w is the coordinate of B_s. As clearly the ratio $w_{j,s}/w_{k,s}$ is a constant function whose value is a m-th root of the unity for each j and k, one has that the assertion Assumption 5 holds.

5. Proof of Theorem 3.1

5.1. Outline of the proof

In this section, for proving Theorem 3.1, we will try to construct a new system of defining functions

$$
\widehat{w}_j = \begin{pmatrix} \widehat{w}_j^{(1)} \\ \widehat{w}_j^{(2)} \\ \vdots \\ \widehat{w}_j^{(r)} \end{pmatrix}
$$

by modifying w_j's as in Assumption 2. Modification is done in the following manner: construct a suitable holomorphic function $F_{j,a}^{(\lambda)}$ for each $\lambda = 1, 2, \ldots, r$ and for each multi-index $a \in (\mathbb{Z}_{\geqslant 0})^r$ with $|a| \geqslant 2$, and solve the functional equation

$$
w_j^{(\lambda)} = \widehat{w}_j^{(\lambda)} + \sum_{|a| \geqslant 2} F_{j,a}^{(\lambda)} \cdot \widehat{w}_j^a, \tag{5.1}
$$

where $\widehat{w}_j^a := \prod_{\lambda=1}^r (\widehat{w}_j^{(\lambda)})^{a_\nu}$.

We will construct the function $F_{j,a}^{(\lambda)}$ suitably as a function defined on $p^{-1}(Y \times S_{|a|-1})$ inductively on $|a|$, and extend it to a holomorphic function on a neighborhood of $p^{-1}(Y \times S_{|a|-1})$ in the manner we will explain in Remark 5.1 below.

Remark 5.1. — Here we explain our rule in this section how to extend a function $F = F_{j,a}^{(\lambda)}$ defined on $S_{|a|-1}$ to its neighborhood. For simplicity, we will explain on each $U_{j,s}$ ($s \in S_{|a|-1}$). Let F be a holomorphic

function defined on $U_{j,s}$. When U_j is non-singular, we use the pull-back $(\mathrm{Pr}_{\mathcal{U}_j}\,|_{V_{j,s}})^*F = F \circ \mathrm{Pr}_{\mathcal{U}_j}\,|_{V_{j,s}}$, which will be denoted by same letter F. For singular U_k, denote by p_k its singular point, and by $U_{k,s}^+$ and $U_{k,s}^-$ the irreducible components $\{(x_k, y_k) \in V_{j,s} \mid y_k = 0\}$ and $\{(x_k, y_k) \in V_{j,s} \mid x_k = 0\}$, respectively, of $U_{k,s}$. Letting $c := F(p_k)$, it is easily observed that there uniquely exist holomorphic functions $F^\pm$ on $U_{k,s}^\pm$ such that $F^\pm(p_k) = 0$ and

$$F - c = \begin{cases} F^+ & \text{on } U_{k,s}^+ \\ F^- & \text{on } U_{k,s}^- \end{cases}$$

hold. In this case, we use the function $\widetilde{F}$ on $V_{j,s}$ defined by $\widetilde{F}(x_k, y_k) := F^+(x_k) + F^-(y_k) + c$ as an extension of F. In what follows, we often denote $\widetilde{F}$ simply by the same letter F. Note that $\sup_{V_{j,s}} |F| \leqslant 3\sup_{U_{j,s}} |F|$ holds in both of the cases. □

The functions $\{F_{j,a}^{(\lambda)}\}_{|a|=m}$ will be constructed so that they enjoy the following (Property)$_m$ inductively when $\{F_{j,a}^{(\lambda)}\}_{|a|<m}$ have already been constructed in the manner that each of them enjoys (Property)$_{|a|}$.

(Property)$_m$. — For any choice of the remaining $\{F_{j,a}^{(\lambda)}\}_{|a|>m}$, the formal solution $\widehat{w}_j$ of the functional equation (5.1) satisfies $T_{jk}\widehat{w}_k = \widehat{w}_j + O(|\widehat{w}_j|^{m+1})$ as elements of $\mathcal{O}_{\mathcal{Y}}(p^{-1}(U_{jk} \times S_{m-1}))[[\widehat{w}_j^{(1)}, \widehat{w}_j^{(2)}, \dots, \widehat{w}_j^{(r)}]]$ for each j and k with $U_{jk} \neq \emptyset$ and $U_j \cap Y_{\mathrm{sing}} = \emptyset$.

We will describe how to construct $\{F_{j,a}^{(\lambda)}\}$ in Section 5.2. We here remark that we never shrink $\mathcal{V}_j$ and $\mathcal{U}_j$ anymore in Section 5.2. After finishing the construction of them, we will estimate each $|F_{j,a}^{(\lambda)}|$ suitably on each $U_{j,s}$ with $s \in \mathrm{U}(1)$ in Section 5.3 so that one can regard the right hand side of the functional equation (5.1) as a convergent series in a suitable sense.

In the rest of this subsection, we will explain how to solve the functional equation (5.1) after once the construction and the estimate of $F_{j,a}^{(\lambda)}$'s are finished. Take $s \in \mathrm{U}(1)$. We will construct a solution $\widehat{w}_j$ of the functional equation on each $V_{j,s}$ by shrinking $V_{j,s}$ to a smaller neighborhood of $U_{j,s}$ if $U_j \cap Y_{\mathrm{sing}} = \emptyset$, and $V_{k,s}$ to a smaller neighborhood of the nodal point if $U_j \cap Y_{\mathrm{sing}} \neq \emptyset$. Note that $\{V_{j,s}\}$ is an open covering of Y_s even after such shrinking, since it holds that $\bigcup_{U_j \cap Y_{\mathrm{sing}}=\emptyset} U_j = Y \setminus Y_{\mathrm{sing}}$.

First let us consider on $V_{j,s}$ for j such that $Y_{\mathrm{sing}} \cap U_j = \emptyset$. It follows by Assumption 2 (v) that one can embed $V_{j,s}$ into $U_{j,s} \times \mathbb{C}^r$ by regarding w_j as the coordinate of $\mathbb{C}^r$. For each $\lambda = 1, 2, \dots, r$, it will be turned out by the

estimate we will make in Section 5.3 that

$$G(z_j, w_j, \widehat{w}_j) := -w_j + \widehat{w}_j + \sum_{|a| \geqslant 2} \begin{pmatrix} F_{j,a}^{(1)}(z_j, s) \\ F_{j,a}^{(2)}(z_j, s) \\ \vdots \\ F_{j,a}^{(r)}(z_j, s) \end{pmatrix} \cdot \widehat{w}_j^a.$$

can be regarded as a $\mathbb{C}^r$-valued holomorphic function defined on a neighborhood of $U_{j,s} \times \{(0,0)\}$ in $U_{j,s} \times \mathbb{C}^r \times \mathbb{C}^r$. As the Jacobian matrix $(\partial G/\partial \widehat{w}_j)$ is invertible on each point of $U_{j,s} \times \{(0,0)\}$, one has by the implicit function theorem that there exists a holomorphic function $\widehat{w}_j(z_j, w_j)$ on a neighborhood of $U_{j,s}$ in $V_{j,s}$ such that $G(z_j, w_j, \widehat{w}_j(z_j, w_j))) \equiv 0$, which mean that $\widehat{w}_j$ is a solution of the functional equation (5.1).

Next, let us consider on $V_{k,s}$ for k such that $Y_{\mathrm{sing}} \cap U_k \neq \emptyset$. According to Assumption 2 (iv), one may regard $V_{k,s}$ as a subset of $\mathbb{C}^2$. Denoting $w_k^{(1)}$ simply by w_k, consider

$$G(x_k, y_k, \widehat{x}_k) := -x_k + \widehat{x}_k + \sum_{m=2}^{\infty} F_{k,m}^{(1)}(x_k, y_k, s) \cdot \widehat{x}_k^m y_k^{m-1}.$$

By the estimate we will make in Section 5.3, it will be turned out G defines a holomorphic function on a neighborhood of the point $(0,0,0)$ in $V_{k,s}$. As $\frac{\partial}{\partial \widehat{x}_k} G(0,0,0) = 1 \neq 0$, it follows from the implicit function theorem that there exists a holomorphic function $\widehat{x}_k = \widehat{x}_k(x_k, y_k)$ defined on a neighborhood of the nodal point in $V_{k,s}$ such that $G(x_k, y_k, \widehat{x}_k(x_k, y_k)) \equiv 0$ holds, which means that $\widehat{w}_k := \widehat{x}_k y_k$ is a solution of the functional equation (5.1).

5.2. Construction of $F_{j,a}^{(\lambda)}$'s

5.2.1. Outline of the construction

Take w_j's as in Assumption 2. Let

$$t_{jk} w_k^{(\lambda)} = w_j^{(\lambda)} + \sum_{|a| \geqslant 2} f_{kj,a}^{(\lambda)}(z_j, s) \cdot w_j^a$$

be the expansion of $t_{jk} w_k^{(\lambda)}$ on $\mathcal{V}_{jk}$ for each $\lambda = 1, 2, \ldots, r$, where $a = (a_1, a_2, \ldots, a_r)$ runs all the elements of $(\mathbb{Z}_{\geqslant 0})^r$ with $|a| \geqslant 2$. We always assume that U_j is non-singular whenever we consider such kind of expansion

Takayuki Koike

(it may possible that U_k is singular). We note that each coefficient functions are obtained by

$$f_{kj,a}^{(\lambda)}(z_j,s) := \frac{1}{|a|!} \left. \frac{\partial^{|a|}(t_{jk}w_k^{(\lambda)})}{\partial(w_j^{(1)})^{a_1}\partial(w_j^{(2)})^{a_2}\cdots\partial(w_j^{(r)})^{a_r}} \right|_{(w_j,z_j,s)=(0,0,\ldots,0,z_j,s)}.$$

We are also regarding this function as the one defined on $\mathcal{V}_{jk}$ according to the rule we mentioned in Remark 5.1 (i.e. by pulling back by $\mathrm{Pr}_{\mathcal{U}_j}$).

In order for all $F_{j,a}^{(\lambda)}$'s to satisfy (Property)$_m$, the following two formal expansions of $t_{jk}w_k^{(\lambda)}$ around a point $(0,0,\ldots,0,z_j,s)$ for each $z_j \in U_{jk}$ and each $s \in \mathrm{U}(1)$ should coincide:

$$t_{jk}^{(\lambda)}w_k^{(\lambda)} = w_j^{(\lambda)} + \sum_{|a|\geqslant 2} f_{kj,a}^{(\lambda)}(z_j,s)\cdot w_j^a$$

$$= \widehat{w}_j^{(\lambda)} + \sum_{|a|\geqslant 2} F_{j,a}^{(\lambda)}(z_j,s)\cdot \widehat{w}_j^a$$

$$+ \sum_{|a|\geqslant 2} f_{kj,a}^{(\lambda)}(z_j,s)\cdot \prod_{\mu=1}^{r}\left(\widehat{w}_j^{(\mu)} + \sum_{|b|\geqslant 2} F_{j,b}^{(\mu)}(z_j,s)\cdot \widehat{w}_j^b\right)^{a_\mu}$$

and

$$t_{jk}^{(\lambda)}w_k^{(\lambda)}$$

$$= t_{jk}^{(\lambda)}\widehat{w}_k^{(\lambda)} + \sum_{|a|\geqslant 2} t_{jk}^{(\lambda)} F_{k,a}^{(\lambda)}(z_k,s)\cdot \widehat{w}_k^a$$

$$= t_{jk}^{(\lambda)}\widehat{w}_k^{(\lambda)} + \sum_{|a|\geqslant 2} t_{jk}^{(\lambda)} F_{k,a}^{(\lambda)}(z_k(w_j,z_j,s),s)\cdot t_{kj}^a \widehat{w}_j^a$$

$$= t_{jk}^{(\lambda)}\widehat{w}_k^{(\lambda)} + \sum_{|a|\geqslant 2} t_{jk}^{(\lambda)} t_{kj}^a\left(F_{k,a}^{(\lambda)}(z_k(0,z_j,s),s) + \sum_{|b|\geqslant 2} F_{kj,a,b}^{(\lambda)}(z_j,s)\cdot w_j^b\right)\cdot \widehat{w}_j^a$$

$$= t_{jk}^{(\lambda)}\widehat{w}_k^{(\lambda)} + \sum_{|a|\geqslant 2} t_{jk}^{(\lambda)} t_{kj}^a F_{k,a}^{(\lambda)}(z_k(0,z_j,s),s)\cdot \widehat{w}_j^a$$

$$+ \sum_{|a|\geqslant 2} t_{jk}^{(\lambda)} t_{kj}^a \sum_{|b|\geqslant 2} F_{kj,a,b}^{(\lambda)}(z_j,s)\cdot \widehat{w}_j^a \cdot \prod_{\mu=1}^{r}\left(\widehat{w}_j^{(\mu)} + \sum_{|c|\geqslant 2} F_{j,c}^{(\mu)}(z_j,s)\cdot \widehat{w}_j^c\right)^{b_\mu}$$

when U_k is smooth, where $F_{kj,a,b}^{(\lambda)}(z_j,s)$'s are the function defined by the expansion

$$F_{k,a}^{(\lambda)}(z_k(w_j,z_j,s),s) = F_{k,a}^{(\lambda)}(z_k(0,z_j,s),s) + \sum_{|b|\geqslant 2} F_{kj,a,b}^{(\lambda)}(z_j,s)\cdot w_j^b.$$

When U_k is singular, replace the second expansion with

$$
\begin{aligned}
&t_{jk}^{(\lambda)} w_k^{(\lambda)}\\
&= t_{jk}^{(\lambda)} \widehat{w}_k^{(\lambda)} + \sum_{|a|\geqslant 2} t_{jk}^{(\lambda)} F_{k,a}^{(\lambda)}(x_k, y_k, s) \cdot \widehat{w}_k^a\\
&= t_{jk}^{(\lambda)} \widehat{w}_k^{(\lambda)} + \sum_{|a|\geqslant 2} t_{jk}^{(\lambda)} F_{k,a}^{(\lambda)}(x_k(w_j, z_j, s), y_k(w_j, z_j, s), s) \cdot t_{kj}^a \widehat{w}_j^a\\
&= t_{jk}^{(\lambda)} \widehat{w}_k^{(\lambda)}\\
&\quad + \sum_{|a|\geqslant 2} t_{jk}^{(\lambda)} t_{kj}^a \left(F_{k,a}^{(\lambda)}(x_k(0, z_j, s), y_k(0, z_j, s), s) + \sum_{|b|\geqslant 2} F_{kj,a,b}^{(\lambda)}(z_j, s) \cdot w_j^b \right) \cdot \widehat{w}_j^a\\
&= t_{jk}^{(\lambda)} \widehat{w}_k^{(\lambda)} + \sum_{|a|\geqslant 2} t_{jk}^{(\lambda)} t_{kj}^a F_{k,a}^{(\lambda)}(x_k(0, z_j, s), y_k(0, z_j, s), s) \cdot \widehat{w}_j^a\\
&\quad + \sum_{|a|\geqslant 2} t_{jk}^{(\lambda)} t_{kj}^a \sum_{|b|\geqslant 2} F_{kj,a,b}^{(\lambda)}(z_j, s) \cdot \widehat{w}_j^a \cdot \prod_{\mu=1}^{r} \left(\widehat{w}_j^{(\mu)} + \sum_{|c|\geqslant 2} F_{j,c}^{(\mu)}(z_j, s) \cdot \widehat{w}_j^c \right)^{b_\mu},
\end{aligned}
$$

where $F_{kj,a,b}^{(\lambda)}(z_j, s)$'s are the function defined by the expansion

$$
\begin{aligned}
&F_{k,a}^{(\lambda)}(x_k(w_j, z_j, s), y_k(w_j, z_j, s), s)\\
&\qquad = F_{k,a}^{(\lambda)}(x_k(0, z_j, s), y_k(0, z_j, s), s) + \sum_{|b|\geqslant 2} F_{kj,a,b}^{(\lambda)}(z_j, s) \cdot w_j^b
\end{aligned}
$$

in this case. In the following, we will construct $F_{j,a}^{(\lambda)}$'s according to the observation based on the comparison of these expansions.

Remark 5.2. — On each X_s, the idea of the construction of $F_{j,a}^{(\lambda)}$'s is the same one as in the proof of [21, Theorem 3]. Indeed, one can run just the same argument when $s \in \mathrm{U}(1)$ as the one described in [12, 13, 21]. In the construction of $F_{j,a}^{(\lambda)}$'s, the condition that $s \in \mathrm{U}(1)$ is important since otherwise the transition functions $t_{jk}(s)$ need not to be elements of $\mathrm{U}(1)$ and this cause a serious problem when we compare a cohomology class what we will denote by $[\{(U_{jk,s}, h_{kj,a}^{(\lambda)}(-, s))\}]$ in the notation below with Ueda classes. This problem is caused by the difficulty on the well-definedness of Ueda classes when $t_{jk}(s)$'s are not unitary (see also Remark 2.3). However, in our configuration, it follows by Assumption 1 and 3 that $\check{H}^1(\{U_{j,s}\}, S^m N_s^* \otimes N_s) = 0$ holds for any $s \in S_{m-1} \setminus \mathrm{U}(1)$, which helps us to overcome this kind of difficulty.

5.2.2. Construction on S_m^*

Denote by $h_{1,kj,a}^{(\lambda)}(z_j, s)$ the coefficient of $\widehat{w}_j^a$ in the expansion

$$\sum_{|a| \geqslant 2} f_{kj,a}^{(\lambda)}(z_j, s) \cdot \prod_{\mu=1}^r \left(\widehat{w}_j^{(\mu)} + \sum_{|b| \geqslant 2} F_{j,b}^{(\mu)}(z_j, s) \cdot \widehat{w}_j^b \right)^{a_\mu}.$$

Then one has that

$$\widehat{w}_j^{(\lambda)} + \sum_{|a| \geqslant 2} F_{j,a}^{(\lambda)}(z_j, s) \cdot \widehat{w}_j^a$$

$$+ \sum_{|a| \geqslant 2} f_{kj,a}^{(\lambda)}(z_j, s) \cdot \prod_{\mu=1}^r \left(\widehat{w}_j^{(\mu)} + \sum_{|b| \geqslant 2} F_{j,b}^{(\mu)}(z_j, s) \cdot \widehat{w}_j^b \right)^{a_\mu}$$

$$= \widehat{w}_j^{(\lambda)} + \sum_{|a| \geqslant 2} \left(F_{j,a}^{(\lambda)}(z_j, s) + h_{1,kj,a}^{(\lambda)}(z_j, s) \right) \cdot \widehat{w}_j^a.$$

Denote by $h_{2,kj,a'}^{(\lambda)}(z_j, s)$ the coefficient of $\widehat{w}_j^{a'}$ in the expansion

$$\sum_{|a| \geqslant 2} t_{jk}^{(\lambda)} t_{kj}^a \sum_{|b| \geqslant 1} F_{kj,a,b}^{(\lambda)}(z_j, s) \cdot \widehat{w}_j^a \cdot \prod_{\mu=1}^r \left(\widehat{w}_j^{(\mu)} + \sum_{|c| \geqslant 2} F_{j,c}^{(\mu)}(z_j, s) \cdot \widehat{w}_j^c \right)^{b_\mu}.$$

Note that

$$\widehat{w}_j^{(\lambda)} + \sum_{|a| \geqslant 2} t_{jk}^{(\lambda)} t_{kj}^a F_{k,a}^{(\lambda)}(z_k(0, z_j, s), s) \cdot \widehat{w}_j^a$$

$$+ \sum_{|a| \geqslant 2} t_{jk}^{(\lambda)} t_{kj}^a \sum_{|b| \geqslant 1} F_{kj,a,b}^{(\lambda)}(z_j, s) \cdot \widehat{w}_j^a \cdot \prod_{\mu=1}^r \left(\widehat{w}_j^{(\mu)} + \sum_{|c| \geqslant 2} F_{j,c}^{(\mu)}(z_j, s) \cdot \widehat{w}_j^c \right)^{b_\mu}$$

$$= \widehat{w}_j^{(\lambda)} + \sum_{|a| \geqslant 2} \left(t_{jk}^{(\lambda)} t_{kj}^a F_{k,a}^{(\lambda)}(z_k(0, z_j, s), s) + h_{2,kj,a}^{(\lambda)}(z_j, s) \right) \cdot \widehat{w}_j^a.$$

It is simply observed that each $h_{1,kj,a}^{(\lambda)}(z_j, s)$ and $h_{2,kj,a}^{(\lambda)}(z_j, s)$ depends only on $\{F_{j,b}^{(\lambda)}\}_{|b|<|a|}$ and some known functions, and thus especially it does not depend on $\{F_{j,b}^{(\lambda)}\}_{|b| \geqslant |a|}$. Therefore, according to the observation we made above, we define $\{F_{j,a}^{(\lambda)}\}_{|a|=m+1}$ by Lemma 5.3 and Lemma 5.4 below so that they are the solution of the equation

$$F_{j,a}^{(\lambda)}(z_j, s) + h_{1,kj,a}^{(\lambda)}(z_j, s) = t_{jk}^{(\lambda)} t_{kj}^a F_{k,a}^{(\lambda)}(z_k(0, z_j, s), s) + h_{2,kj,a}^{(\lambda)}(z_j, s),$$

or equivalently,

$$\delta\{(\mathcal{U}_j^{(m)}, F_{j,a}^{(\lambda)})\} = \{(\mathcal{U}_{jk}^{(m)}, h_{kj,a}^{(\lambda)}(z_j, s))\} \in \check{Z}^1(\{(\mathcal{U}_j^{(m)}\}, \mathcal{O}_{\mathcal{Y}_m}(\mathcal{L}^{a-e_\lambda})),$$

where $h_{kj,a}^{(\lambda)}(z_j, s) := h_{1,kj,a}^{(\lambda)}(z_j, s) - h_{2,kj,a}^{(\lambda)}(z_j, s)$, when $\{F_{j,b}\}_{|b|\leqslant m}$ is already decided in a manner such that each (Property)$_{m'}$ holds for $m' = 1, 2, \ldots, m$ (inductive assumption). See the proof of Lemma 5.3 for the fact that $\{(\mathcal{U}_{jk}^{(m)}, h_{kj,a}^{(\lambda)}(z_j, s))\} \in \check{Z}^1(\{(\mathcal{U}_j^{(m)}\}, \mathcal{O}_{\mathcal{Y}_m}(\mathcal{L}^{a-e_\lambda}))$. Here we are denoting by e_λ the multi-index $(0, 0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^r$ whose λ-th entry is one and the other entries are zero, by $\mathcal{Y}_m$ the set $p^{-1}(Y \times S_m)$, by $\mathcal{U}_j^{(m)}$ the set $p^{-1}(U_j \times S_m)$, and by $\mathcal{U}_{jk}^{(m)}$ the intersection $\mathcal{U}_j^{(m)} \cap \mathcal{U}_k^{(m)}$.

First we construct $F_{j,a}^{(\lambda)}$'s on $S_m^* := S_m \setminus \mathrm{U}(1)$.

LEMMA 5.3. — *Let m be a positive integer. Assume that $\{F_{j,b}\}_{|b|\leqslant m}$ is already decided in a manner such that each (Property)$_{m'}$ holds for $m' = 1, 2, \ldots, m$. Then there uniquely exist holomorphic functions $\{F_{j,a}^{(\lambda)}\}_{|a|=m+1}$ on $p^{-1}(Y \times S_m^*)$ such that*

$$F_{j,a}^{(\lambda)}(z_j, s) - t_{jk}^{(\lambda)} t_{kj}^a F_{k,a}^{(\lambda)}(z_k(0, z_j, s), s) = -h_{1,kj,a}^{(\lambda)}(z_j, s) + h_{2,kj,a}^{(\lambda)}(z_j, s)$$

holds on each $U_{jk,s}$ with $s \in S_m^*$, and that $\{F_{j,a}^{(\lambda)}\}_{|a|=m+1}$ satisfies (Property)$_{m+1}$.

Proof. — As it follows by the same argument as in the end of Section 5.1 that one can solve the functional equation

$$w_j^{(\lambda)} = u_j^{(\lambda)} + \sum_{2 \leqslant |a| \leqslant m} F_{j,a}^{(\lambda)} \cdot u_j^a$$

to define a new system $\{(\widetilde{\mathcal{V}}_j, u_j)\}$ of local defining functions of $p^{-1}(Y \times S_{m-1}^*)$ by using the implicit function theorem. As it holds that $T_{jk}u_k = u_j + O(|u_j|^{m+1})$ by (Property)$_m$, it follows from the calculation as we described above that

$$T_{jk}u_k = u_j - \sum_{|a|=m+1} h_{kj,a}^{(\lambda)}(z_j, s) \cdot u_j^a + O(|u_j|^{m+2}).$$

Denote by $h_{jk,a}^{(\lambda)}$ the function $-t_{jk}^{a-e_\lambda} \cdot h_{kj,a}^{(\lambda)}$ when $U_k \cap Y_{\mathrm{sing}} \neq \emptyset$. Then one easily obtains, by comparing the both hands sides of the expansion of $T_{jk}\widehat{w}_k - \widehat{w}_j = T_{j\ell}(T_{\ell k}\widehat{w}_k - \widehat{w}_\ell) + (T_{j\ell}\widehat{w}_\ell - \widehat{w}_j)$, that $\{(\mathcal{U}_{jk}^{(m)}, h_{kj,a}^{(\lambda)}(z_j, s))\} \in \check{Z}^1(\{(\mathcal{U}_j^{(m)}\}, \mathcal{O}_{\mathcal{Y}_m}(\mathcal{L}^{a-e_\lambda}))$.

It follows from Assumptions 1 and 3 that $H^1(Y_s, L_s^{a-e_\lambda}) = 0$ for any $s \in S_m^*$. Thus one has that $R^j(\mathrm{Pr}_2 \circ p|_{\mathcal{Y}_m^*})_* \mathcal{L}^{a-e_\lambda} = 0$ for each $j > 0$ for $\mathcal{Y}_m^* := p^{-1}(Y \times S_m^*)$, by which it follows that $H^1(\mathcal{Y}_m^*, \mathcal{L}^{a-e_\lambda}) =$

$H^1(S_m^*, (\mathrm{Pr}_2 \circ p)_* \mathcal{L}^{a-e_\lambda})$. As S_m^* is Stein and $(\mathrm{Pr}_2 \circ p)_* \mathcal{L}^{a-e_\lambda}$ is coherent by Grauert's theorem, one has that $H^1(\mathcal{Y}_m^*, \mathcal{L}^{a-e_\lambda}) = 0$, which implies the lemma. Note that the uniqueness of the solution follows by $H^0(Y_s, L_s^a) = 0$ for each $s \in S_m^*$ (Here we used Assumption 3 (ii)). $\qquad\square$

5.2.3. Construction on S_m and preliminary L^∞ estimates

In this subsection, we fix a positive integer m and $\nu \in \{0, 1, 2, \ldots, mM_0 - 1\}$, and use the following simple notation: $\xi_0 := \nu/(M_0 m)$, $s_0 := \exp(2\pi\sqrt{-1}\xi_0)$. Let W_0 be the set defined by

$$W_0 := \left\{ \xi \in \mathbb{C} \,\Big|\, |\mathrm{Re}\,\xi - \xi_0| \leqslant \frac{1}{2M_0 m}, \ |\mathrm{Im}\,\xi| < \frac{1}{4M_0 m} \right\}.$$

Note that it holds that $\exp(2\pi\sqrt{-1}W_0) = W_{m,\nu}$.

In the previous subsection, under the inductive assumption and Lemma 5.3, we have seen that there exists a holomorphic function $F_{j,a}^{(\lambda)}$: $p^{-1}(Y \times S_m^*) \to \mathbb{C}$ for each a with $|a| = m+1$ which enjoys (Property)$_{m+1}$, which is the solution of the functional equation

$$-F_{j,a}^{(\lambda)}(z_j, s) + t_{jk}^{(\lambda)} t_{kj}^a F_{k,a}^{(\lambda)}(z_k(0, z_j, s), s) = h_{kj,a}^{(\lambda)}(z_j, s).$$

Note that $h_{kj,a}^{(\lambda)}(z_j, s)$ is holomorphic on $\mathcal{Y}_{m-1}$. In what follows, we show that $F_{j,a}^{(\lambda)}$ holomorphically extends to $\mathcal{Y}_m$, or equivalently, that $F_{j,a}^{(\lambda)}|_{p^{-1}(Y \times W_{m,\nu}^*)}$ holomorphically extends to $p^{-1}(Y \times W_{m,\nu})$, where $W_{m,\nu}^* := W_{m,\nu} \setminus \{s_0\}$ (for each ν).

First, let us note that, when $L_{s_0}^{a-e_\lambda}$ is not holomorphically trivial, one can construct the function $F_{j,a}^{(\lambda)}$ as the one defined on $p^{-1}(Y \times W_{m,\nu})$ by exactly the same argument as in the proof of Lemma 5.3. Note also that, in this case, it follows from Assumption 4 that

$$\max_j \sup_{U_{j,s}} |F_{j,a}^{(\lambda)}| \leqslant \frac{K}{m \cdot (2M_0 m)^{-1}} \max_{j,k} \sup_{U_{jk,s}} |h_{kj,a}^{(\lambda)}|$$

$$= 2M_0 K \cdot \max_{j,k} \sup_{U_{jk,s}} |h_{kj,a}^{(\lambda)}|. \quad (5.2)$$

When $L_{s_0}^{a-e_\lambda}$ is holomorphically trivial, we show the following:

PROPOSITION 5.4. — *Let a be a multi-index with $|a| = m+1$, and λ be an element of $\{1, 2, \ldots, r\}$. Denote by a' the multi-index $\alpha - e_\lambda$. Assume that $L_{s_0}^{a'}$ is holomorphically trivial. Let $\alpha = \{(U_{jk} \times W_{m,\nu}, \alpha_{jk} dw_j^{a'})\} \in \check{Z}^1(\{U_j \times$*

$W_{m,\nu}\}, \mathcal{O}_{\mathcal{Y}}(\mathcal{L}^{a'}))$ *be a* 1*-cocycle with* $M := \max_{j,k} \sup_{U_{jk} \times W_{m,\nu}} |\alpha_{jk}| < \infty$, *where we are identifying* $\mathcal{L}$ *with* $p^*\mathcal{L}$ *via* p. *For each* $\zeta \in W_0$, *denote by*

$$\alpha_\xi = \{(U_{jk} \times \{\exp(2\pi\sqrt{-1}\xi)\}), \alpha_{\zeta,jk} dw_j^{a'})\}$$
$$\in \check{Z}^1\Big(\{U_j \times \{\exp(2\pi\sqrt{-1}\xi)\}\}, \mathcal{O}_{Y_{\exp(2\pi\sqrt{-1}\xi)}}\big(L^{a'}_{\exp(2\pi\sqrt{-1}\xi)}\big)\Big)$$

the 1*-cocycle obtained by restricting* α *to* $Y \times \{\exp(2\pi\sqrt{-1}\xi)\}$. *Then the followings are equivalent:*

(i) $[\alpha_{\xi_0}] = 0 \in \check{H}^1(\{U_{j,s_0}\}, \mathcal{O}_{Y_{s_0}})$.
(ii) $[\alpha] = 0 \in \check{H}^1(\{U_j \times W_{m,\nu}\}, \mathcal{O}_{\mathcal{Y}}(\mathcal{L}^{a'}|_{Y \times W_{m,\nu}}))$.

Proof. — As clearly (ii) implies (i), we show the converse. Assume that the assertion (i) holds. In what follows, we use the estimate $\max_{j,k} \sup_{s \in W_{m,\nu}} |t_{jk}^{(\lambda)}(s)| \leqslant \Theta^{1/m} \leqslant \Theta$ follows from Assumption 3 (iii).

By the assertion (i), it follows from [21, Lemma 3] (=[8, Lemma 2], for the smooth case) or [12, Lemma 4.1] (for the singular case) that there exists a 0-cochain $\beta_{\xi_0} = \{(U_j \times \{s_0\}, \beta_{\xi_0,j} dw_j^{a'})\} \in \check{C}^0(\{U_j \times \{s_0\}\}, \mathcal{O}_{Y_{s_0}}(L^{a'}_{s_0}))$ with $\delta\beta_{\xi_0} = \alpha_{\xi_0}$ such that

$$\max_j \sup_{U_{j,s_0}} |\beta_{\xi_0,j}| \leqslant K_{\mathrm{KS}} M$$

holds, where K_{KS} is the constant which depends only on Y and $\{U_j\}$. Define a holomorphic function $\beta_j \colon U_j \times W_{m,\nu} \to \mathbb{C}$ by $\beta_j(z_j, s) := \beta_{\xi_0,j}(z_j)$, a 0-cochain β by

$$\beta := \{(U_j \times W_{m,\nu}, \beta_j dw_j^{a'})\} \in \check{C}^0(\{U_j \times W_{m,\nu}\}, \mathcal{O}_{\mathcal{Y}}(\mathcal{L}^{a'})),$$

and a 1-cocycle

$$\gamma = \{(U_{jk} \times W_{m,\nu}, \gamma_{jk})\} \in \check{Z}^1(\{U_j \times W_{m,\nu}\}, \mathcal{O}_{\mathcal{Y}}(\mathcal{L}^{a'}))$$

by $\gamma := \alpha - \delta\beta$. In what follows, we will construct a primitive $\widetilde{\beta}$ of γ (Then clearly it holds that $\delta(\beta + \widetilde{\beta}) = \alpha$, which proves the lemma). Note that

$$\max_{j,k} \sup_{U_{jk} \times W_{m,\nu}} |\gamma_{jk}| \leqslant M + (1 + \Theta) \cdot K_{\mathrm{KS}} M = (1 + K_{\mathrm{KS}} + \Theta K_{\mathrm{KS}})M.$$

Denote by W_0^* the set $W_0 \setminus \{\zeta_0\}$ and by $W_{m,\nu}^*$ the image $\exp(2\pi\sqrt{-1}W_0^*)$. It follows by the same argument as in the proof of Lemma 5.3, one has that there uniquely exists a holomorphic function $\widetilde{\beta}_j \colon U_j \times W_{m,\nu}^* \to \mathbb{C}$ such that the 0-cochain $\widetilde{\beta}$ defined by

$$\widetilde{\beta} := \{(U_j \times W_{m,\nu}^*, \widetilde{\beta}_j dw_j^{a'})\} \in \check{C}^0(\{U_j \times W_{m,\nu}^*\}, \mathcal{O}_{\mathcal{Y}}(\mathcal{L}^{a'}))$$

satisfies $\delta\widetilde{\beta} = \alpha|_{Y \times W_{m,\nu}^*}$. According to Assumption 4, one has the inequality

$$m \cdot |\xi - \xi_0| \cdot \|\widetilde{\beta}\|_{\exp(2\pi\sqrt{-1}\xi),a'} \leqslant K \cdot \|\gamma\|_{\exp(2\pi\sqrt{-1}\xi),a'}$$

for each $\xi \in W_0^*$. Therefore, for each element $z_j \in U_j$ and each $\xi \in W_0^*$, one has that

$$|\widetilde{\beta}(z_j, \exp(2\pi\sqrt{-1}\xi))| \leqslant \frac{K}{m|\xi - \xi_0|} \cdot \max_{j,k} \sup_{p \in U_{jk}} |\gamma_{jk}(p, \exp(2\pi\sqrt{-1}\xi))|$$

$$= \frac{K}{m} \cdot \max_{j,k} \sup_{p \in U_{jk}} \left| \frac{\gamma_{jk}(p, \exp(2\pi\sqrt{-1}\xi))}{\xi - \xi_0} \right|$$

holds. As $\gamma_{jk}|_{\xi = \xi_0} \equiv 0$ by construction, one has that the function

$$U_{jk} \times W_0^* \ni (p, \xi) \mapsto \frac{\gamma_{jk}(p, \exp(2\pi\sqrt{-1}\xi))}{\xi - \xi_0} \in \mathbb{C}$$

can also be regarded as a holomorphic function defined on $U_{jk} \times W_0$. Thus it follows by the maximum principle that there exists a point (p_*, ξ_*) of the boundary $\partial U_{jk} \times \partial W_0$ which attains the maximum

$$\sup_{(p,\xi) \in U_{jk} \times W_0} \left| \frac{\gamma_{jk}(p, \exp(2\pi\sqrt{-1}\xi))}{\xi - \xi_0} \right|.$$

Therefore, by $|\xi_* - \xi_0| \geqslant \frac{1}{4M_0 m}$, we have that

$$\sup_{(p,\xi) \in U_{jk} \times W_0} \left| \frac{\gamma_{jk}(p, \exp(2\pi\sqrt{-1}\xi))}{\xi - \xi_0} \right| = \frac{|\gamma_{jk}(p_*, \exp(2\pi\sqrt{-1}\xi_*))|}{|\xi_* - \xi_0|}$$

$$\leqslant \frac{(1 + K_{\mathrm{KS}} + \Theta K_{\mathrm{KS}})M}{(4M_0 m)^{-1}}$$

$$= 4M_0 m \cdot (1 + K_{\mathrm{KS}} + \Theta K_{\mathrm{KS}})M$$

holds. Thus we obtain the estimate

$$|\widetilde{\beta}(z_j, \exp(2\pi\sqrt{-1}\xi))| \leqslant \frac{K}{m} \cdot \max_{j,k} \sup_{p \in U_{jk}} \left| \frac{\gamma_{jk}(p, \exp(2\pi\sqrt{-1}\xi))}{\xi - \xi_0} \right|$$

$$\leqslant 4M_0 K \cdot (1 + K_{\mathrm{KS}} + \Theta K_{\mathrm{KS}})M$$

for each $z_j \in U_j$ and $\xi \in W_0^*$, by which one can apply Riemann's extension theorem to conclude that $\widetilde{\beta}$ can be holomorphically extended to $\xi = \xi_0$. $\quad\square$

As is seen in Section 5.3, the assumption "$M < \infty$" in Proposition 5.4 will be inductively assured for the case where α is the one defined by $h_{kj,a}^{(\lambda)}$. For such case, the assertion (i) of Proposition 5.4 actually holds by Assumption 5 and the argument as in [21, p. 598] (see also [12, Remark 3.5] and [12, §4.2.1] for singular case, [13, §3.2] and [13, Claim 4.3, 4.4] for higher codimensional case), by which we obtain that the solution $F_{j,a}^{(\lambda)}$'s of

$$-F_{j,a}^{(\lambda)}(z_j, s) + t_{jk}^{(\lambda)} t_{kj}^a F_{k,a}^{(\lambda)}(z_k(0, z_j, s), s) = h_{kj,a}^{(\lambda)}(z_j, s)$$

actually exist on $\mathcal{Y}_m$. Note also that it follows by the argument in the proof of Proposition 5.4 that the inequality

$$\max_j \sup_{U_{j,s}} |F_{j,a}^{(\lambda)}|$$

$$\leqslant ((1+\Theta)K_{\mathrm{KS}} + 4M_0 K \cdot (1 + K_{\mathrm{KS}} + \Theta K_{\mathrm{KS}})) \cdot \max_{j,k} \sup_{U_{jk,s}} |h_{kj,a}^{(\lambda)}| \quad (5.3)$$

holds for each $s \in W_{m,\nu}$.

5.3. Estimates of $|F_{j,a}^{(\lambda)}|$

Here we will construct positive constants $\{A_m\}_{m=2}^\infty$ such that

$$\max_j \max_{|a|=m+1} \sup_{p^{-1}(U_j \times S_m)} |F_{j,a}^{(\lambda)}| \leqslant A_{m+1}$$

holds and that the formal series

$$A(X^1, X^2, \ldots, X^r) := \sum_{|a|\geqslant 2} A_{|a|} X^a := \sum_{|a|\geqslant 2} A_{|a|} \prod_{\nu=1}^r (X^\nu)^{a_\nu}$$

is convergent.

First, we define a new open covering $\{U_j^*\}$ of Y whose index set coincides with that of $\{U_j\}$ as follows: Set $U_j^* := U_k$ if $U_k \cap Y_{\mathrm{sing}} \neq \emptyset$, and U_j^* is a relatively compact subset of U_j if $U_j \cap Y_{\mathrm{sing}} = \emptyset$ such that $U_{jk} \neq \emptyset$ if and only if $U_j^* \cap U_k^* \neq \emptyset$. By Assumption 3(i), one has that

$$\left\{ (z_j, w_j, s) \in \mathcal{V}_j \;\middle|\; \begin{array}{l} s \in S_1, \; z_j \in U_j \cap U_k^*, \\ |w_j^{(\lambda)}| < 1/R \text{ for each } \lambda \in \{1, 2, \ldots, r\} \end{array} \right\} \Subset \mathcal{V}_k$$

holds for sufficiently large constant R if both U_j and U_k are smooth.

When U_k is singular, we need to modify U_k and some of the elements of $\{U_j\}$ as follows if necessary (see also [12, Remark 4.3]). Take a sufficiently small positive constant ε such that $\mathcal{V}_{(k,3)} \Subset \mathcal{V}_k$, where $\mathcal{V}_{(k,\mu)} := \{(x_k, y_k, s) \in \mathcal{V}_k \mid \max\{|x_k|, |y_k|\} < \mu \cdot \varepsilon, \; s \in S_1\}$ for $\mu = 1, 2, 3$. Denote by $U_{(k,\mu)}$ the subset $\{(x_k, y_k) \in U_k \mid \max\{|x_k|, |y_k|\} < \mu \cdot \varepsilon\}$ for $\mu = 1, 2, 3$. For non-singular U_j such that $U_{(k,3)} \cap U_j \neq \emptyset$ for some singular U_k, we will divide U_j into $U_{(j,3)} := U_{(k,3)} \cap U_j$ and $U_{(j,2)} := U_j \setminus \overline{U_{(k,2)}}$, and use

$$\{U_j \mid U_j \text{ is smooth, } U_{(k,3)} \cap U_j = \emptyset \text{ for any singular } U_k\}$$
$$\cup \{U_{(j,2)} \mid U_{(k,3)} \cap U_j \neq \emptyset \text{ for some singular } U_k\}$$
$$\cup \{U_{(j,3)} \mid U_{(k,3)} \cap U_j \neq \emptyset \text{ for some singular } U_k\}$$
$$\cup \{U_{(k,1)} \mid U_k \text{ is singular}\}$$

as a new open covering of Y. We will use the restriction of the original coordinate (x_k, y_k, s) on a neighborhood $\mathcal{V}_{(k,1)}$ of $p^{-1}(U_{(k,1)} \times S)$ for each k such that U_k is singular, the restriction of the original coordinate (z_j, w_j, s) on a neighborhood of $\mathcal{V}_{(j,2)}$ of $p^{-1}(U_{(j,2)} \times S)$ for each $U_{(j,2)}$ ($\mathcal{V}_{(j,2)}$ is, for example, a subset of $\mathcal{V}_j \setminus \overline{\mathcal{V}_{(k,2)}}$). For each $U_{(j,3)}$, we use the restriction of (x_k, y_k, s) as the coordinate of a neighborhood. In this case, either $x_k \equiv 0$ or $y_k \equiv 0$ holds on $p^{-1}(U_{(j,3)} \times S)$. When, for example, $y_k \equiv 0$ holds on $p^{-1}(U_{(j,3)} \times S)$, we use the set $\mathcal{V}_{(j,3)} := \{(x_k, y_k, s) \in \mathcal{V}_k \mid x_k \in U_{(j,3)},\ |y_k| < \varepsilon\}$ as a neighborhood of $p^{-1}(U_{(j,3)} \times S)$, and use $w_{(j,3)} := y_k$ and $z_{(j,3)} := x_k$ as coordinates.

After modifying $\{U_j\}$ and $\{\mathcal{V}_j\}$ in such manner, we may assume that

$$\left\{ (z_j, w_j, s) \in \mathcal{V}_j \ \middle| \ \begin{array}{l} s \in S_1,\ z_j \in U_j \cap U_k, \\ |w_j^{(\lambda)}| \leqslant 1/R \text{ for each } \lambda \in \{1, 2, \ldots, r\} \end{array} \right\} \subset \mathcal{V}_k$$

holds for any smooth U_j even if U_k is singular (by letting $1/R < \varepsilon/2$).

Denote by M the constant $\Theta \cdot \max_\lambda \max_j \sup_{\mathcal{V}_j} |w_j^{(\lambda)}|$, which is finite by Assumption 2 (ii). By Cauchy's inequality, one has that

$$\max_{j,k} \sup_{(U_j \cap U_k^*) \times S} |f_{kj,a}^{(\lambda)}| \leqslant MR^{|a|}.$$

For any a. It follows by $h_{kj,a}^{(\lambda)} = f_{kj,a}^{(\lambda)}$ that $\max_{j,k} \sup_{(U_j \cap U_k^*) \times S_1} |h_{kj,a}^{(\lambda)}| \leqslant MR^2$. Therefore one has by the cocycle condition and Assumption 3 (iii) that, for each $(p, s) \in U_{jk} \times S_1$, it holds that

$$|h_{kj,a}^{(\lambda)}(p, s)| \leqslant |t_{jk}^{(\lambda)} t_{kj}^a h_{k\ell,a}^{(\lambda)}(p, s)| + |t_{j\ell}^{(\lambda)} t_{\ell j}^a h_{\ell j,a}^{(\lambda)}(p, s)|$$
$$= |h_{\ell k,a}^{(\lambda)}(p, s)| + |t_{j\ell}^{(\lambda)} t_{\ell j}^a h_{\ell j,a}^{(\lambda)}(p, s)| \leqslant (1 + \Theta)MR^2$$

even if $p \notin U_k^*$, where ℓ is the one such that $p \in U_\ell^*$ (Note that U_k is smooth if $p \notin U_k$ by construction of $\{U_j^*\}$). Thus we obtain the inequality

$$\max_{jk} \max_{|a|=2} \sup_{p^{-1}(U_{jk} \times S_1)} |h_{kj,a}^{(\lambda)}| \leqslant B_1 := (1 + \Theta)MR^2.$$

Consider the division $\overline{S_m} = \bigcup_{\nu=0}^{mM_0-1} W_{m,\nu}$ for $m = 1$. Take a multi-index a with $|a| = 2$, $\lambda \in \{1, 2, \ldots, \lambda\}$, and $\nu \in \{0, 1, \ldots, M_0 - 1\}$. When $L_{\exp(2\pi\sqrt{-1}\nu/M_0)}^{a-e_\lambda}$ is holomorphically trivial, it follows from the inequality (5.3) that

$$\max_j \sup_{p^{-1}(U_j \times W_{1,\nu})} |F_{j,a}^{(\lambda)}| \leqslant ((1 + \Theta)K_{\mathrm{KS}} + 4M_0 K \cdot (1 + K_{\mathrm{KS}} + \Theta K_{\mathrm{KS}})) B_1$$

holds (Note again that here we used Assumption 5 to assure the assertion (i) of Proposition 5.4. Here we used the argument as in [21, p. 598], [12, Remark 3.5], [12, §4.2.1], [13, §3.2], [13, Claim 4.3, 4.4] in order to see that the 1-cocycle defined by $h^{(\lambda)}_{kj,a}$ coincides with the Ueda class, see also the proof of Lemma 5.3). In what follows, we let $K > K_{\mathrm{KS}} > 1$. Then one has that

$$\max_{j} \max_{|a|=2} \sup_{p^{-1}(U_j \times W_{1,\nu})} |F_{j,a}| \leqslant 14 M_0 \Theta K^2 \cdot B_1$$

holds. When $L^{a-e_\lambda}_{\exp(2\pi\sqrt{-1}\nu/M_0)}$ is not holomorphically trivial, it follows from the inequality (5.2) that

$$\max_{j} \sup_{p^{-1}(U_j \times W_{1,\nu})} |F_{j,a}| \leqslant 2 M_0 K B_1$$

holds. Thus one has that $\max_j \max_{|a|=2} \sup_{p^{-1}(U_j \times S_1)} |F_{j,a}| \leqslant A_2$ holds if one lets $A_2 := 14 M_0 \Theta K^2 \cdot (1+\Theta) M R^2$.

Assume that we have decided $A_2, A_3, \ldots, A_m$ suitably. Then, as $h^{(\lambda)}_{1,kj,a}$'s are defined by

$$\sum_{|a|\geqslant 2} f^{(\lambda)}_{kj,a}(z_j,s) \cdot \prod_{\mu=1}^{r} \left(\widehat{w}^{(\mu)}_j + \sum_{|b|\geqslant 2} F^{(\mu)}_{j,b}(z_j,s) \cdot \widehat{w}^b_j \right)^{a_\mu} = \sum_{|a|\geqslant 2} h^{(\lambda)}_{1,kj,a}(z_j,s) \cdot \widehat{w}^a_j,$$

it follows that the value of $\max_{jk} \max_{|a|=m+1} \sup_{p^{-1}((U_j \cap U^*_k) \times S_m)} |h^{(\lambda)}_{1,kj,a}|$ is bounded from above by the coefficient of X^a in the expansion of

$$\sum_{|a|\geqslant 2} M R^{|a|} \cdot \prod_{\mu=1}^{r} (X^\mu + A(X))^{a_\mu}$$

$$= M \prod_{\nu=1}^{r} \frac{1}{1 - R(X^\nu + A(X))} - M - M R \sum_{\nu} (X^\nu + A(X)),$$

where $A(X) = A(X^1, X^2, \ldots, X^r)$. Note that the coefficient of X^a in the expansion of the above depends only on $A_2, A_3, \ldots, A_m$ if $|a| = m+1$. Similarly, as $h^{(\lambda)}_{2,kj,a}$'s are defined by

$$\sum_{|a|\geqslant 2} t^{(\lambda)}_{jk} t^a_{kj} \sum_{|b|\geqslant 1} F^{(\lambda)}_{kj,a,b}(z_j,s) \cdot \widehat{w}^a_j \cdot \prod_{\mu=1}^{r} \left(\widehat{w}^{(\mu)}_j + \sum_{|c|\geqslant 2} F^{(\mu)}_{j,c}(z_j,s) \cdot \widehat{w}^c_j \right)^{b_\mu}$$

$$= \sum_{|a|\geqslant 2} h^{(\lambda)}_{2,kj,a}(z_j,s) \cdot \widehat{w}^a_j,$$

the value of $\max_{jk} \max_{|a|=m+1} \sup_{p^{-1}((U_j \cap U_k^*) \times S_m)} |h_{2,kj,a}^{(\lambda)}|$ is bounded from above by the coefficient of X^a in the expansion of

$$\sum_{|a| \geqslant 2} \Theta \sum_{|b| \geqslant 1} 3A_{|a|} R^{|b|} \cdot X^a \cdot \prod_{\mu=1}^{r} (X^\mu + A(X))^{b_\mu}$$

$$= 3\Theta A(X) \cdot \sum_{|b| \geqslant 2} \prod_{\mu=1}^{r} (RX^\mu + RA(X))^{b_\mu}.$$

Here we used the estimate $|F_{kj,c,d}^{(\lambda)}| \leqslant 3A_{|c|} R^{|d|}$ which is obtained by Cauchy's inequality and the rule we described in Remark 5.1.

Therefore, one has that the value of

$$\max_{jk} \max_{|a|=m+1} \sup_{p^{-1}((U_j \cap U_k^*) \times S_m)} |h_{kj,a}^{(\lambda)}|$$

is bounded from above by the coefficient of X^a in the expansion of

$$M \cdot \left(\prod_{\nu=1}^{r} \frac{1}{1 - R(X^\nu + A(X))} - 1 - R \sum_\nu (X^\nu + A(X)) \right)$$

$$+ 3\Theta A(X) \cdot \left(\prod_{\nu=1}^{r} \frac{1}{1 - R(X^\nu + A(X))} - 1 \right).$$

Take a point $(p, s) \in U_{jk} \times S_m$. When $p \notin U_k$, as U_k is smooth by construction of $\{U_j^*\}$ in this case, it follows from the cocycle condition and Assumption 3 (iii) that

$$|h_{kj,a}^{(\lambda)}(p, s)| \leqslant |t_{jk}^{(\lambda)} t_{kj}^a h_{k\ell,a}^{(\lambda)}(p, s)| + |t_{j\ell}^{(\lambda)} t_{\ell j}^a h_{\ell j,a}^{(\lambda)}(p, s)|$$

$$= |h_{\ell k,a}^{(\lambda)}(p, s)| + |t_{j\ell}^{(\lambda)} t_{\ell j}^a h_{\ell j,a}^{(\lambda)}(p, s)|$$

$$\leqslant (1 + \Theta) \cdot \max_{jk} \max_{|a|=m+1} \sup_{p^{-1}((U_j \cap U_k^*) \times S_m)} |h_{kj,a}^{(\lambda)}|,$$

where ℓ is an index such that $p \in U_\ell^*$.

Thus one has that the value of $\max_{jk} \max_{|a|=m+1} \sup_{p^{-1}(U_{jk} \times S_m)} |h_{kj,a}^{(\lambda)}|$ is bounded by the coefficient of X^a in the expansion of

$$(1+\Theta)M \cdot \left(\prod_{\nu=1}^{r} \frac{1}{1 - R(X^\nu + A(X))} - 1 - R \sum_\nu (X^\nu + A(X)) \right)$$

$$+ 3(1+\Theta)\Theta A(X) \cdot \left(\prod_{\nu=1}^{r} \frac{1}{1 - R(X^\nu + A(X))} - 1 \right),$$

which will be denoted by B_m.

Take a multi-index a with $|a| = m + 1$, an element $\nu \in \{0, 1, \dots, mM_0 - 1\}$, and $\lambda \in \{1, 2, \dots, r\}$. When $L^{a-e_\lambda}_{\exp(2\pi\sqrt{-1}\nu/(mM_0))}$ is not holomorphically trivial, it follows by (5.3) that

$$\max_{j} \sup_{p^{-1}(U_j \times W_{m,\nu})} |F^{(\lambda)}_{j,a}(p, s)| \leqslant 14 M_0 \Theta K^2 \cdot B_m$$

(again we remark that here we used Assumption 5). When $L^{a-e_\lambda}_{\exp(2\pi\sqrt{-1}\nu/(mM_0))}$ is holomorphically trivial, it follows by (5.2) that

$$\max_{j} \sup_{p^{-1}(U_j \times W_{m,\nu})} |F^{(\lambda)}_{j,a}(p, s)| \leqslant 2 M_0 K \cdot B_m$$

holds.

Thus now we have that one can obtain constants $\{A_m\}_{m=2}^{\infty}$ such that

$$\max_{j} \max_{|a|=m+1} \sup_{p^{-1}(U_j \times S_m)} |F^{(\lambda)}_{j,a}(p, s)| \leqslant A_{m+1}$$

by letting $A(X) \in \mathbb{C}[[X^1, X^2, \dots, X^r]]$ be the formal power series defined by the equation

$$\frac{1}{14 M_0 \Theta K^2} A(X)$$

$$= (1 + \Theta) M \cdot \left(\prod_{\nu=1}^{r} \frac{1}{1 - R(X^\nu + A(X))} - 1 - R \sum_{\nu} (X^\nu + A(X)) \right)$$

$$+ 3(1 + \Theta) \Theta A(X) \cdot \left(\prod_{\nu=1}^{r} \frac{1}{1 - R(X^\nu + A(X))} - 1 \right).$$

Denoting by C the constant $14 M_0 \Theta K^2 (1+\Theta)$, we let $F(X^1, X^2, \dots, X^r, Y) \in \mathcal{O}_{\mathbb{C}^{r+1},0}$ be the one defined by

$$F(X, Y) := -Y + CM \cdot \left(\prod_{\nu=1}^{r} \frac{1}{1 - R(X^\nu + Y)} - 1 - R \sum_{\nu} (X^\nu + Y) \right)$$

$$+ 3C\Theta Y \cdot \left(\prod_{\nu=1}^{r} \frac{1}{1 - R(X^\nu + Y)} - 1 \right).$$

Then, as $\frac{\partial}{\partial Y} F(0, 0) = -1 \neq 0$, it follows from the implicit function theorem that $A(X)$ is convergent.

5.4. End of the proof

Let s be an element of U(1). Then, by the estimates in Remark 5.1 and Section 5.3, one can carry out the argument as in the end of Section 5.1 on

each $U_{j,s}$ to obtain the solution $\widehat{w}_j$ of the functional equation (5.1) defined on a neighborhood $\widehat{V}_{j,s}$ of $U_{j,s}$ in X_s. Note that clearly $\{(\widehat{V}_{j,s}, \widehat{w}_j)\}$ is also a local defining functions system of Y_s, since w_j and $\widehat{w}_j$ coincide in the first order jet. It follows by (Property)$_m$'s that $\widehat{w}_j = T_{jk}(s) \cdot \widehat{w}_k$ holds on each $\widehat{V}_{j,s} \cap \widehat{V}_{k,s}$, the theorem holds. □

6. Examples and Proof of Theorem 1.2 and 1.3

6.1. Some examples

First, we give an example of the configuration we described in Section 4.1.

Example 6.1. — Let (V, F) be a del Pezzo manifold of degree 1: i.e. V is a projective manifold of dimension n and F is an ample line bundle on V with $K_V^{-1} \cong F^{n-1}$ and the self-intersection number (F^n) is equal to 1. In this case, as we wrote in [13, §6.3], it follows from [7, 6.4] that dim $H^0(V, L) = n$. Take general elements $D_1^0, D_2^0, \dots, D_n^0 \in |F|$. By [7, 4.2] and $(D_1^0, D_2^0, \dots, D_n^0) = (F^n) = 1$, it holds that the intersection $\bigcap_{\lambda=1}^n D_\lambda^0$ is a point, which we denote by p_0. It is clear that D_λ^0's intersect each other transversally at p_0. From this fact and Bertini's theorem, we may assume that each D_λ^0 is non-singular.

Consider an sequence of the subvarieties $V_n := V, V_{n-1} := D_1^0, V_{n-2} := D_1^0 \cap D_2^0, \cdots, V_1 := D_1^0 \cap D_2^0 \cdots, \cap D_{n-1}^0$. Denote by F_λ the restriction $F|_{V_\lambda}$ for each $\lambda = 1, 2, \cdots, n-1$. Note that it follows from a simple inductive argument that (V_λ, F_λ) is also a del Pezzo manifold of degree 1 for each λ. Especially, for $\lambda = 1$, it holds that V_1 is an elliptic curve and $\deg F_1 = 1$. Let $\pi \colon \mathcal{X} \to S$ be the one obtained by the construction we described in Section 4.1 starting from $Z := V$, $L := F$, and $Y^0 := V_1$. By running the same argument as in [18, §5] inductively on λ, one has that this model actually enjoys Assumption 5′ (One has that the anti-canonical bundle of X_s is semi-ample, and that the morphism defined by the complete linear system $K_{X_s}^{-m}$ can be used as the fibration b_s for each torsion s, where m is a suitable positive integer).

Next, we give an example of the configuration we described in Section 4.2.

Example 6.2. — Let $Y \subset \mathbb{P}^2$ be a cycle of rational curves which is of degree three, and $p_1, p_2, \dots, p_8$ be points of $Y \setminus Y_{\text{sing}}$. Assume one of the following four conditions:

(1) Y is a rational curve with one node,

(2) Y consists of two irreduceble components C_1 and C_2 such that the degree of C_1 is one and of C_2 is two, $p_1, p_2 \in C_1$ and $p_3, p_4, \ldots,$ $p_8 \in C_2$,

(3) Y consists of two irreduceble components C_1 and C_2 such that the degree of C_1 is two and of C_2 is three, $p_1, p_2, \ldots, p_5 \in C_1$ and $p_6, p_7, p_8 \in C_2$, or

(4) Y consists of three irreducible components C_1, C_2 and C_3, $p_1, p_2 \in C_1$, $p_3, p_4, p_5 \in C_2$, and $p_6, p_7, p_8 \in C_3$.

Let $\pi \colon \mathcal{X} \to S$ be the one obtained by the construction we described in Section 4.1 by using the blow-up of $\mathbb{P}^2$ at $p_1, p_2, \ldots, p_8$ as Z, and the strict transform of Y as Y^0. By the same argument as in [18, §5], one has that this model actually enjoys Assumption $5''$.

6.2. Proof of Theorem 1.2

Let (X_Z, Y_Z) be as in Section 1 such that the anti-canonical bundle $K_{X_Z}^{-1}$ is nef, and that Y_Z is a cycle of rational curves.

When $K_{X_Z}^{-1}|_{Y_Z}$ is unitary flat (i.e. $|\alpha(K_{X_Z}^{-1}|_{Y_Z})| = 1$), by the argument as in [12, §7], $(X_Z, Y_Z) = (X_s, Y_s)$ holds for some deformation $\pi \colon \mathcal{X} \to S$ as in Example 6.2 and for some $s \in \mathrm{U}(1) \subset S$. Thus one has by Theorem 3.1 that there exists a local defining functions system $\{(V_j, w_j)\}$ of Y_Z such that each ratio w_j/w_k is a unitary constant. It follows by regarding w_j as a local frame of $[Y_Z]$ on each V_j that $[Y_Z]$ is flat on a neighborhood of Y_Z. Thus, by the argument as in the proof of [9, Corollary 3.4] or [13, §5] (see also Section 2.1), one has the assertion (i). Define a function $\Phi \colon \bigcup_j V_j \to \mathbb{R}$ by $\Phi := \log |w_j|$ on each V_j. By considering neighborhoods $\{\Phi < \varepsilon\}$ of Y_Z for sufficiently small positive numbers ε, one has that the assertion (iii). The assertion (iv) follows by considering Chern curvature forms of smooth Hermitian metrics on $[Y]$ with semi-positive curvature.

When $K_{X_Z}^{-1}|_{Y_Z}$ is not unitary flat, neither the assertion (i) nor (iv) holds by [12, Theorem 1.1 (ii)]. In this case, there exists a strongly plurisubharmonic function Ψ on $V \setminus Y_Z$ for some neighborhood V of Y_Z according to [12, Theorem 1.6 (i)], by which one has that the assertion (iii) does not hold. □

6.3. Proof of Theorem 1.3

Let (X_Z, Y_Z) be as in Section 1 such that Y_Z is smooth and $\mathrm{rank}(N_Z) = 1$, where $N_Z := K_{X_Z}^{-1}|_{Y_Z}$ ($Z = \{p_1, p_2, \ldots, p_9\}$). Denote by V the blow-up of $\mathbb{P}^2$

at the eight points $p_1, p_2, \ldots, p_7$, and p_8. Then, as V is a del Pezzo manifold of degree 1, one can obtain the deformation $\pi \colon \mathcal{X} \to S$ as in Example 6.1. Note that, it follows by the condition $\mathrm{rank}(N_Z) = 1$ that one may assume that $X_Z \cong X_s$ for some element s of $\mathrm{U}(1) \subset S$ by choosing τ and q_0 suitably in the construction in Section 4.1. Thus one has by Theorem 3.1 that there exists a local defining functions system $\{(V_j, w_j)\}$ of Y_Z such that each ratio w_j/w_k is a unitary constant.

It follows by regarding w_j as a local frame of $[Y_Z]$ on each V_j that $[Y_Z]$ is flat on a neighborhood of Y_Z. Thus, by the argument as in the proof of [9, Corollary 3.4] or [13, §5] (see also Section 2.1), one has the former assertion.

Define a function $\Phi \colon \bigcup_j V_j \to \mathbb{R}$ by $\Phi := \log |w_j|$ on each V_j. By considering neighborhoods $\{\Phi < \varepsilon\}$ of Y_Z for sufficiently small positive numbers ε, one has that the latter assertion. □

Appendix. A correction to the paper "Toward a higher codimensional Ueda theory"

We found that the main theorem [11, Theorem 1] was not correct. We have to add some assumptions in [11, Theorem 1]. The main application [11, Corollary 1] needs no correction.

A.1. Corrected form of [11, Theorem 1]

Corrected form of [11, Theorem 1] is the following:

THEOREM A.1. — *Let X be a complex manifold, S a smooth hypersurface of X, and C be a smooth compact hypersurface of S such that $N_{S/X}|_V$ is flat, where V is a sufficiently small neighborhood of C in S. Assume one of the following two conditions holds:*

(i) *$N_{C/S} \in \mathcal{E}_0(C)$, $N_{S/X}|_C \in \mathcal{E}_0(C)$,*
(ii) *$N_{C/S}$ and $N_{S/X}|_C$ are isomorphic to each other and they are elements of $\mathcal{E}_1(C)$.*

Further assume that $u_{n,m}(C, S, X; \{w_j\}) = 0$ holds for all $n \geqslant 1, m \geqslant 0$ and for all system $\{w_j\}$ of order (n, m), and that there exists a system of local defining functions of C in V of extension type infinity. Then there exists a neighborhood W of C in X such that $\mathcal{O}_X(S)|_W$ is flat. Moreover, there exists a smooth hypersurface Y of W which intersects S transversally along C.

In the above statement, we removed the case (iii) from [11, Theorem 1] and added the assumption on the existence of a system of local defining functions of C in V *of extension type infinity*, which is the notion we posed in [15]. As a result, we could also add the conclusion on the existence of the transversal Y to [11, Theorem 1]. For the proof of Theorem A.1, see [15, §3.4].

Let us explain some terms in Theorem A.1. We say the line bundle L on a manifold M is *flat* if the transition functions are chosen as constant functions valued in $U(1) := \{t \in \mathbb{C} \mid |t| = 1\}$ (i.e. $L \in H^1(M, U(1))$). We denote by $\mathcal{E}_0(C)$ the set of all flat line bundles F such that there exists a positive integer n with $F^n = \mathbb{I}$, where $\mathbb{I}$ is the holomorphically trivial line bundle. We denote by $\mathcal{E}_1(C)$ the set of all flat line bundles F which satisfies the condition $|\log d(\mathbb{I}, F^n)| = O(\log n)$ as $n \to \infty$, where d is an invariant distance of the Picard group ($\mathcal{E}_1(C)$ does not depend on the choice of d, see [21, §4.1]). Let (C, S, X) be as in Theorem A.1. In [11, §3.1], we defined the obstruction class $u_{n,m}(C, S, X) = u_{n,m}(C, S, X; \{w_j\}) \in H^1(C, N_{S/X}|_C^{-n} \otimes N_{C/S}^{-m})$ for each $n \geqslant 1, m \geqslant 0$ and for each *system* $\{w_j\}$ *of order* (n, m). We here explain the meaning of our new assumption "there exists a system of local defining functions of C in V of extension type infinity". Let V be a sufficiently small tubular neighborhood of C in S and W be a sufficiently small tubular neighborhood of C in X such that $W \cap S = V$. Take a sufficiently fine open covering $\{U_j\}$ of C, $\{V_j\}$ of V, and $\{W_j\}$ of W such that $V_j = W_j \cap S$, $U_j = V_j \cap C$, and $U_{jk} := U_j \cap U_k = \emptyset$ iff $W_{jk} := W_j \cap W_k = \emptyset$. Extend a coordinates system x_j of U_j to W_j. Let y_j be a defining function of U_j in V_j and w_j a defining function of V_j in W_j. As both $N_{S/X}$ and $N_{C/S}$ are flat in our settings, we may assume that $t_{jk}w_k = w_j + O(w_j^2)$ holds on W_{jk} and $s_{jk}y_k = y_j + O(y_j^2)$ holds on V_{jk} for some constants $t_{jk}, s_{jk} \in U(1)$. The assumption "there exists a system of local defining functions of C in V of extension type infinity" means that we can choose such $\{y_j\}$ with the following two additional properties:

(a) $s_{jk}y_k = y_j$ holds on V_{jk} for each j and k, and
(b) $\{y_j\}$ is *of extension type infinity* in the sense of [15, Definition 3.2]: i.e. the class $v_{n,m}(C, S, X; \{z_j\}) \in H^1(C, N_{S/X}|_C^{-n} \otimes N_{C/S}^{-m+1})$ is equal to zero for each $n \geqslant 1, m \geqslant 0$ and for *any* type (n, m) extension $\{z_j\}$ of $\{y_j\}$ (the class $v_{n,m}(C, S, X; \{z_j\})$ is the obstruction class we posed in [15]).

We here remark that, as we will see later, [11, Corollary 1] needs no correction. It is because the example of general 8 points blow-up of $\mathbb{P}^3$ automatically satisfies this condition.

A.2. Details of the mistakes

There are following three mistakes in [11]: one is on the well-definedness of the obstruction classes, another one is in the statement of [11, Lemma 1], and the other one is in a Taylor expansion in Lemma 6 and Lemma 7. In this section, we explain the details of these mistakes.

A.2.1. Mistake on the well-definedness of the obstruction classes

The first one is on the well-definedness of the obstruction classes. In [11, Proposition 3], we stated that the (n,m)-th Ueda class $u_{n,m}(C, S, X)$ of the triple (C, S, X) is independent of the choice of a system $\{w_j\}$ of order (n,m) up to non-zero constant multiples. However, we found a critical mistake in the proof. Thus, now we should denote the obstruction class by $u_{n,m}(C, S, X; \{w_j\})$. See also [15, §2.2.2, §3.2.2].

A.2.2. Mistake in the statement of [11, Lemma 1]

We also found a mistake in [11, Lemma 1], which is an open analogue of [8, Lemma 2]. The corrected form of [11, Lemma 1] should be stated as follows:

LEMMA A.2 (Corrected form of [11, Lemma 1]). — *Let C be a compact complex manifold embedded in a complex manifold S Fix a sufficiently small connected neighborhood V of C in S and a sufficiently fine open covering $\{V_j\}$ of V which consists of a finite number of open sets. Fix also a relatively compact open domain $V_0 \subset V$ which contains C. For each flat line bundle E on V, there exists a positive constant $K = K(E)$ such that, for each 1-cocycle $\alpha = \{(V_{jk}, \alpha_{jk})\}$ of E which can be realized as the coboundary of some 0-cochain, there exists a 0-cochain $\beta = \{(V_j \cap V_0, \beta_j)\}$ of E such that $\alpha|_{V_0}$ is equal to the coboundary $\delta(\beta)$ of β and the inequality*

$$\max_j \sup_{V_0 \cap V_j} |\beta_j| \leqslant K \cdot \max_{jk} \sup_{V_0 \cap V_{jk}} |\alpha_{jk}|$$

holds.

This mistake is critical for proving [11, Theorem 1] for the case (iii), which is why we had to remove this case.

A.2.3. Mistake in a Taylor expansion in Lemma 6 and Lemma 7

Here we explain the mistake under the configuration of Lemma 7. Lemma 7 is the lemma for defining the system of functions $\{G_j^{(n,m)}\}$ inductively: i.e. assuming that $\{G_j^{(\nu,\mu)}\}$ has already defined for each $(\nu,\mu) < (n,m)$, we are stating how to define $\{G_j^{(n,m)}\}$ in this lemma. For the definition of $\{G_j^{(n,m)}\}$, we regard $G_j^{(\nu,\mu)} z_j^\mu$ as the function defined on W_j which does not depend on the variable w_j and considered the expansion

$$
\begin{aligned}
G_j^{(\nu,\mu)}(x_j) \cdot z_j^\mu &= G_j^{(\nu,\mu)}(x_j(x_k, z_k, w_k)) \cdot z_j(x_k, z_k, w_k)^\mu \\
&= G_j^{(\nu,\mu)}(x_j(x_k, 0, 0)) \cdot s_{jk}^\mu z_k^\mu \\
&\quad + \sum_{q=1}^{\infty} G_{jk,0,q}^{(\nu,\mu)}(x_k) \cdot z_k^{\mu+q} + \sum_{p=1}^{\infty} \sum_{q=0}^{\infty} G_{jk,p,q}^{(\nu,\mu)}(x_k) \cdot z_k^{\mu+q} w_k^p
\end{aligned}
$$

on W_{jk}, in which we made a mistake. This expansion should be

$$
\begin{aligned}
G_j^{(\nu,\mu)}(x_j) \cdot z_j^\mu &= G_j^{(\nu,\mu)}(x_j(x_k, z_k, w_k)) \cdot z_j(x_k, z_k, w_k)^\mu \\
&= G_j^{(\nu,\mu)}(x_j(x_k, 0, 0)) \cdot s_{jk}^\mu z_k^\mu \\
&\quad + \sum_{q=1}^{\infty} G_{jk,0,q}^{(\nu,\mu)}(x_k) \cdot z_k^{\mu+q} + \sum_{p=1}^{\infty} \sum_{q=0}^{\infty} G_{jk,p,q}^{(\nu,\mu)}(x_k) \cdot z_k^{q} w_k^p .
\end{aligned}
$$

Even after this correction, the inductive definition of $\{G_j^{(n,m)}\}$ can be executed just as in Lemma 7. However, the norm estimate problem occurs in this case so that we can not show the convergence of the functional equation (8) in [11, §4]. To avoid this difficulty, we have to refine not only the system $\{w_j\}$, but also the extension $\{z_j\}$ of $\{y_j\}$ by using a suitable functional equation at the same time (with fixing only $\{x_j\}$ and $\{y_j\}$), see [15, §3.4] for the details.

We here remark that, by the same reason, we also have to correct [11, Proposition 4, Lemma 5].

A.3. Proof of [11, Corollary 1]

Here we prove the following:

COROLLARY A.3 (=[11, Corollary 1]). — *Let $C_0 \subset \mathbb{P}^3$ be a complete intersection of two quadric surfaces of $\mathbb{P}^3$ and let $p_1, p_2, \ldots, p_8 \in C_0$ be 8 points different from each other. Assume $\mathcal{O}_{\mathbb{P}^3}(-2)|_{C_0} \otimes \mathcal{O}_{C_0}(p_1 + p_2 + \cdots + p_8) \in \mathcal{E}_1(C_0)$. Then the anti-canonical bundle of the blow-up of $\mathbb{P}^3$ at $\{p_j\}_{j=1}^{8}$*

is not semi-ample, however admits a smooth Hermitian metric with semi-positive curvature.

Proof of Corollary A.3. — We use the notations in [11, §5.2]. We apply Theorem A.1 to the triple (C, S_0, X). We here remark that the existence of the transversal Y is clear in this example (consider $Y := S_\infty$).

All we have to do here is to check the added condition "there exists a system of local defining functions of C in V of extension type infinity". Let $\{s_{jk}\}$ and $\{y_j\}$ be as in Section 1 here. As $u_n(C, S_0) \in H^1(C, N^{-n}) = 0$ for each $n \geqslant 1$, we can conclude from [21, Theorem 3] that we may assume the condition (a) $s_{jk}y_k = y_j$ holds on V_{jk} for each j and k. We will check the condition (b) the class $v_{n,m}(C, S, X; \{z_j\}) \in H^1(C, N_{S/X}|_C^{-n} \otimes N_{C/S}^{-m+1})$ is equal to zero for each $n \geqslant 1, m \geqslant 0$ and for *any* type (n, m) extension $\{z_j\}$ of $\{y_j\}$. First we will check the case where $(n, m) = (1, 0)$. Note that the class $v_{1,0}(C, S, X; \{z_j\})$ does not depend on the choice of an extension $\{z_j\}$ of $\{y_j\}$ (nor a system $\{w_j\}$). It can be shown by just the same (and much more simple) argument as in [15, §3.2.2]. Thus, it is sufficient to show that $v_{1,0}(C, S, X; \{z_j\}) = 0$ for a suitably fixed extension $\{z_j\}$ of $\{y_j\}$. For this purpose, let us fix an extension z_j of y_j such that z_j is a defining function of $W_j \cap S_\infty$. Let

$$s_{jk}z_k = z_j + p_{jk}^{(1)}(x_j, z_j) \cdot w_j + O(w_j^2)$$

be the expansion in w_j and

$$p_{jk}^{(1)}(x_j, y_j) = q_{jk}^{(1,0)}(x_j) + O(y_j)$$

be the expansion of $p_{jk}^{(1)}|_{V_{jk}}$ in y_j for each j and k. As $s_{jk}z_k/z_j$ is holomorphic around $W_{jk} \cap S_\infty$, we obtain that $p_{jk}^{(1)}(x_j, z_j)$ can be divided by z_j. Therefore we obtain that $v_{1,0}(C, S, X; \{z_j\}) = [\{q_{jk}^{(1,0)}\}] \equiv [\{0\}] = 0$. Next we will check the case where $(n, m) > (1, 0)$. In this case, as N is non-torsion and $n + m - 1 > 0$, we obtain that $H^1(C, N_{S/X}|_C^{-n} \otimes N_{C/S}^{-m+1}) = H^1(C, N^{-n-m+1}) = 0$ holds, which proves the assertion. □

Bibliography

[1] V. I. ARNOL'D, "Bifurcations of invariant manifolds of differential equations and normal forms in neighborhoods of elliptic curves", *Funkts. Anal. Prilozh.* **10** (1976), no. 4, p. 1-12.

[2] M. BRUNELLA, "On Kähler surfaces with semipositive Ricci curvature", *Riv. Mat. Univ. Parma* **1** (2010), no. 2, p. 441-450.

[3] B. CLAUDON, F. LORAY, J. V. PEREIRA & F. TOUZET, "Compact leaves of codimension one holomorphic foliations on projective manifolds", *Ann. Sci. Éc. Norm. Supér.* **51** (2018), no. 6, p. 1457-1506.

[4] J.-P. Demailly, "Complex analytic and differential geometry", `https://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/agbook.pdf`.

[5] ———, "Structure Theorems for Compact Kähler Manifolds with Nef Anticanonical Bundles", in *Complex analysis and geometry. KSCV 10*, Springer Proceedings in Mathematics & Statistics, vol. 144, Springer, 2015, p. 119-133.

[6] J.-P. Demailly, T. Peternell & M. Schneider, "Compact complex manifolds with numerically effective tangent bundles", *J. Algebr. Geom.* **3** (1994), no. 2, p. 295-345.

[7] T. Fujita, *Classification theories of polarized varieties*, London Mathematical Society Lecture Note Series, vol. 155, London Mathematical Society, 1990.

[8] K. Kodaira & D. C. Spencer, "A theorem of completeness of characteristic systems of complete continuous systems", *Am. J. Math.* **81** (1959), p. 477-500.

[9] T. Koike, "On minimal singular metrics of certain class of line bundles whose section ring is not finitely generated", *Ann. Inst. Fourier* **65** (2015), no. 5, p. 1953-1967.

[10] ———, "On the minimality of canonically attached singular Hermitian metrics on certain nef line bundles", *Kyoto J. Math.* **55** (2015), no. 3, p. 607-616.

[11] ———, "Toward a higher codimensional Ueda theory", *Math. Z.* **281** (2015), no. 3-4, p. 967-991.

[12] ———, "Ueda theory for compact curves with nodes", *Indiana Univ. Math. J.* **66** (2017), no. 3, p. 845-876.

[13] ———, "Higher codimensional Ueda theory for a compact submanifold with unitary flat normal bundle", *Nagoya Math. J.* **238** (2018), p. 104-136.

[14] ———, "Plurisubharmonic functions on a neighborhood of a torus leaf of a certain class of foliations", *Forum Math.* **31** (2019), no. 6, p. 1457-1466.

[15] T. Koike & N. Ogawa, "Local criteria for non embeddability of Levi-flat manifolds", *J. Geom. Anal.* **28** (2018), no. 2, p. 1052-1077.

[16] ———, "On the neighborhood of a torus leaf and dynamics of holomorphic foliations", `https://arxiv.org/abs/1808.10219`, 2018.

[17] T. Koike & T. Uehara, "A gluing construction of K3 surfaces", `https://arxiv.org/abs/1903.01444`, 2019.

[18] A. Neeman, *Ueda theory: theorems and problems*, Memoirs of the American Mathematical Society, no. 415, American Mathematical Society, 1989, 123 pages.

[19] C. L. Siegel, "Iterations of analytic functions", *Ann. Math.* **43** (1942), p. 607-612.

[20] Y.-T. Siu, "Every Stein subvariety admits a Stein neighborhood", *Invent. Math.* **38** (1976), p. 89-100.

[21] T. Ueda, "On the neighborhood of a compact complex curve with topologically trivial normal bundle", *J. Math. Kyoto Univ.* **22** (1983), p. 583-607.

[22] ———, "Neighborhood of a rational curve with a node", *Publ. Res. Inst. Math. Sci.* **27** (1991), no. 4, p. 681-693.