Numerical Solution for Generalized Fractional Huxley Equation by Using Two Dimensional Haar Wavelet Method

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□ ABSTRACT □

In this paper, we apply the two dimensional Haar Wavelet method to compute the numerical solutions of the generalized fractional Huxley equation. We present a technique to treat the nonlinear term in the equation based on the block pulse functions. The main feature of two methods is converting the generalized fractional Huxley equation to a system of nonlinear algebraic equations, which can be solved by using any computer software like Matlab. The results of comparison the numerical solution with the exact solution show that the proposed method is effective, simple, having low computation costs and the accuracy of the solution is quite high even in the case of a little number of collocation points.

Keywords: Fractional calculus, The generalized fractional Huxley equation, Haar wavelet, Caputo fractional derivative.

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حل عددي لمعادلة هيكسلي الكسرية المعمّمة باستخدام طريقة مويجة هار ثنائية البعد

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□ ملخّص □

نطبق في هذه المقالة طريقة مويجة هار ثنائية البعد لحساب حلول عددية لمعادلة هيكسلي الكسرية المعمّمة. نقدم تقنية لمعالجة الحد اللاخطي في المعادلة اعتماداً على دوال block pulse. الميّزة الأساسية للطريقتين هي تحويل معادلة هيكسلي الكسريّة المعمّمة لجملة من المعادلات الجبرية غير الخطية، والتي من الممكن حلّها باستخدام أي برنامج حاسوبي مثل Matlab. تظهر نتائج مقارنة الحل العددي مع الحل الدّقيق أن الطّريقة المقّدمة فعّالة وبسيطة ولها تكاليف حسابية منخفضة و دقة الحلّ عالية حتى في حالة عدد قليل من نقاط التّجميع.

الكلمات المفتاحيّة: حساب التّفاضل والتّكامل الكسري، معادلة هيكسلي الكسريّة المعمّمة، مويجة هار، مشتق كابوتو الكسري.

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Introduction:

The fractional calculus is dating back to 17th century, when it was mentioned, for the first time, by Leibnitz in a letter to L'Hopital in 1695. Subsequently, many mathematicians such as Liouville, Caputo, Riemann made principal contributions to the theory of fractional calculus. Fractional calculus is a field of applied mathematics that deals with derivatives and integrals of non-integer orders [1]. In contemporary years, fractional calculus has become one of the interest issues for many researchers in applied science and engineering because of the fact that realistic modeling of a physical phenomenon can be efficiently executed by way of using fractional calculus. Absolutely fractional derivatives provide an excellent tool for the description of hereditary properties of different materials and operations [2]. Therefore, various forms of fractional differential equations (FDEs) have been introduced, but exact solutions of most of the (FDEs) cannot be found easily, thus approximate and numerical methods must be used. Adomian decomposition method (ADM) was applied for solving the Fokker–Planck equation with space and time fractional derivatives by Odibat et al [3]. A solution of the fractional KdV–Burgers– Kuramoto equation by He's variational iteration method (VIM) and Adomian's decomposition method was obtained by Safari et al [4]. A numerical algorithm based on the Operational Tau method (OTM) was applied for solving the space and time fractional Fokker–Planck equation by Vanani et al [5]. A generalized block pulse operational matrix was used for solving fractional differential equations by Yuanlu et al [6]. Recently, different wavelet methods comprising the Haar wavelet method [7,8], Hermite wavelet method [9], Legendre wavelet method [10] and Chebyshev wavelet method [11].

In this paper, we investigated the generalized fractional Huxley equation

$$D_t^{\alpha} u - \frac{\partial^2}{\partial x^2} u = \beta u\left(1 - u^{\delta}\right)\left(u^{\delta} - \gamma\right); \qquad (1)$$

$$0 < \alpha \le 1;\ \beta \ne 0;\ \gamma \epsilon (0,1);\ \ 0 \le x \le 1;\ \ t \ge 0$$

where β, γ are parameters and D_t^{α} is the Caputo fractional derivative of order α. This equation describes nerve pulse propagation in nerve fibers and wall motion in liquid crystals. Hemida et al [12] used the Homotopy analysis method (HAM) to obtain an approximation solution of the fractional order generalized Huxley equation. El-Danaf et al [13] considered the possibility of extension to (ADM) and (VIM) for solving fractional Huxley equation. When $\alpha = 1$, the exact solution of the generalized Huxley equation

$$\frac{\partial}{\partial t} u - \frac{\partial^2}{\partial x^2} u = \beta u\left(1 - u^{\delta}\right)\left(u^{\delta} - \gamma\right), \qquad (2)$$

is given by Wang et al [14]

$$u(x,t) = \left\{\frac{\gamma}{2} + \frac{\gamma}{2}\tanh\left(\frac{\delta\gamma}{2}\sqrt{\frac{\beta}{1+\delta}}\left(x - \frac{1+\delta-\gamma}{1+\delta}\sqrt{\beta(1+\delta)}t\right)\right)\right\}^{\frac{1}{\delta}}. \qquad (3)$$

Many studies have used several methods to solve Eq.(2): Hashim et al [15] utilized (ADM), Hashemi et al [16] used [1]HPM, Batiha et al [17] applied variational iteration method (VIM), Gupta et al [18] used Haar wavelet method and Inan [19] presented a numerical solution based on an exponential finite difference method.

[1] **He's homotopy perturbation method**

The importance of research and its objectives:

This research aims to provide numerical solutions to the generalized fractional Huxley equation using the Haar wavelet method. It is extremely important for researchers because these solutions play an important role for understanding and interpreting many physical and chemical phenomena.

Research methods and resources:

This research is classified within the competence of applied mathematics, especially in the field of numerical analysis and fractional partial differential equations, so the mathematical techniques used here mainly depend on the methods of numerical analysis and use it to treat the studied equation, where there isn't exact solution.

1. Preliminaries

We present in following some of the important concepts.

1.1 Fractional derivative and integration

Definition 1: [1,20]

The Riemann_Liouville fractional integral of order α is defined by:

$$J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau;\;\; \alpha > 0 \; and \; \alpha \in R^+, \tag{4}$$

where $\Gamma(\alpha)$ is the gamma function.

Some properties of the operator J^{α} are as follows:

$$J^{\alpha}\,(J^{\beta} f(t)) = J^{\beta}\big(J^{\alpha} f(t)\big) = J^{\alpha+\beta} f(t)\,,\;\; (\alpha > 0, \beta > 0) \tag{5}$$

$$J^{\alpha}\, t^{\gamma} = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma+\alpha)}\, t^{\alpha+\gamma}\,,\quad (\gamma > -1). \tag{6}$$

Definition 2: [1,20]

The Caputo fractional derivative of order α is defined as

$${}_0D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}}\, d\tau, \quad (n-1 < \alpha < n, n \epsilon N). \tag{7}$$

If $\alpha = n$, then ${}_0D_t^{\alpha} f(t)$ coincides with the ordinary derivative $d^n f/dx^n$.

Some properties of the operator D^{α} are given as follows

$$D^{\alpha}\,(\,D^{\beta} f(t)\,) = D^{\beta}\,\,(\,D^{\alpha} f(t)\,) = D^{\alpha+\beta} f(t), \tag{8}$$

$${}_0D_t^{\alpha}\, t^{\beta} = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)}\, t^{\beta-\alpha}\,, 0 < \alpha < \beta+1, \beta > -1, \tag{9}$$

$$D^{\alpha} J^{\beta} f(t) = D^{\alpha-\beta} f(t), \alpha \geq \beta, \tag{10}$$

$$J^{\alpha}\, D^{\alpha} f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0^+) \frac{t^k}{k!}\,, t \geq 0, n-1 < \alpha \leq n \; and \; n \epsilon N. \tag{11}$$

1.2 Haar Wavelet Method

Morlet (1982) [21] introduced the idea of wavelets as a family of functions created from dilation and translation of a single function called the mother wavelet, but from 1910 the Hungarian mathematician Alfred Haar [22] introduced Haar wavelet functions. Afterward, many other wavelet functions were introduced such as Shannon [23], Chebyshev [11], Hermite [9], Daubechies [24] and Legendre [10] wavelets and many others. Haar wavelets consist of piecewise constant functions on the real line that can take only three values i.e. 0, 1 and -1. However, Haar wavelets have the simplest orthogonal series with compact support and small computational costs.

For $t \epsilon [A, B]$, Haar wavelet functions are defined as follows [7]

$$h_0(t) = \begin{cases} 1\,, & t \in [A, B] \\ 0\;, & elsewhere \end{cases} \tag{12}$$

$$h_i(t)=\begin{cases}1\ , & \xi_1(i)\le t<\xi_2(i)\\ -1\ , & \xi_2(i)\le t<\xi_3(i)\\ 0\ , & elsewhere\end{cases} \tag{13}$$

where,

$$\xi_1(i)=A+\left(\frac{k-1}{2^j}\right)\ (B-A)=A+\left(\frac{k-1}{2^j}\right)m\Delta t,$$
$$\xi_2(i)=A+\left(\frac{k-(1/2)}{2^j}\right)\ (B-A)=A+\left(\frac{k-(1/2)}{2^j}\right)m\Delta t,$$
$$\xi_3(i)=A+\left(\frac{k}{2^j}\right)\ (B-A)=A+\left(\frac{k}{2^j}\right)m\Delta t,$$

$i=1,2,\dots,m-1, m=2^M$ and M is a positive integer which is called the maximum level of resolution. j and k represent the integer decomposition of the index i, i.e. $i=k+2^j-1,\ \ 0\le j<i$ and $1\le k<2^j+1$.

The integrals of the wavelets are calculated by Lepik [25] as follows

$$p_i(x)=\int_0^x h_i(x)dx\ ,q_i(x)=\int_0^x p_i(x)dx$$

where,

$$p_i(x)=\begin{cases}x-\xi_1 & ;x\in[\xi_1,\xi_2),\\ \xi_3-x & ;x\in[\xi_2,\xi_3),\\ 0 & elsewhere\end{cases} \tag{14}$$

$$q_i(x)=\begin{cases}0 & ;\ x\in[0,\xi_1),\\ 0.5\ (x-\xi_1)^2 & ;\ x\in[\xi_1,\xi_2),\\ \frac{1}{4m^2}-\frac{1}{2}(\xi_3-x)^2 & ;\ x\in[\xi_2,\xi_3),\\ \frac{1}{4m^2}\ ; & x\in[\xi_3,1],\end{cases} \tag{15}$$

To use the Haar wavelets for the numerical solutions, it must put them into a discrete form. There are different ways to do it; in this paper, the collocation method is applied.

1.3 Function Approximation

For arbitrary function $y(x,t)\epsilon L^2([0,1]\times[0,1])$, it can be expanded into Haar series by [26]

$$y(x,t)\approx\sum_{i=0}^{m-1}\sum_{j=0}^{m-1}C_{ij}\ \ h_i(x)h_j(t), \tag{16}$$

where,

$$C_{ij}=\int_0^1 h_i(x)h_j(x)dx, \tag{17}$$

are coefficients. Discrete $y(x,t)$ by choosing the same step of x and t we obtain,

$$Y(x,t)=H^T(x)\ C\ H(t), \tag{18}$$

here $Y(x,t)$ is the discrete form of $y(x,t)$ and

$$H=\begin{bmatrix}h_0\\ h_1\\ \vdots\\ h_{m-1}\end{bmatrix}=\begin{bmatrix}h_{0,0} & h_{0,1} & \cdots & h_{0,m-1}\\ h_{1,0} & h_{1,1} & \cdots & h_{1,m-1}\\ \vdots & \vdots & \vdots & \vdots\\ h_{m-1,0} & h_{m-1,1} & \cdots & h_{m-1,m-1}\end{bmatrix}, \tag{19}$$

where $h_0,h_1,\dots,h_{m-1}$ are the discrete forms of the Haar wavelet bases

$$C=\begin{bmatrix}c_{0,0} & c_{0,1} & \cdots & c_{0,m-1}\\ c_{1,0} & c_{1,1} & \cdots & c_{1,m-1}\\ \vdots & \vdots & \vdots & \vdots\\ c_{m-1,0} & c_{m-1,1} & \cdots & c_{m-1,m-1}\end{bmatrix}, \tag{20}$$

is the coefficient matrix of Y, and it can be got by formula :

$$C = (H^T)^{-1} . Y . H^{-1}. \tag{21}$$

1.4 Operational matrix of the general order integration

The integration of the $H_m(t) = [h_0(t), h_1(t), \dots, h_{m-1}(t)]^T$ can be approximated by [27]

$$\int_0^t H_m(\tau)dt \cong QH_m(t), \tag{22}$$

where Q is called the Haar wavelet operational matrix of integration which is a square matrix of m-dimension. The Haar wavelet operational matrix Q^α for the integration of the general order α is given by [8]

$$\begin{aligned} Q^\alpha H_m(t) = J^\alpha H_m(t) &= [J^\alpha h_0(t), J^\alpha h_1(t), \dots, J^\alpha h_{m-1}(t)]^T \\ &= [Qh_0(t), Qh_1(t), \dots, Qh_{m-1}(t)]^T, \end{aligned} \tag{23}$$

where,

$$Qh_0(t) = \begin{cases} \dfrac{t^\alpha}{\Gamma(1+\alpha)}, t\epsilon[A,B], \\ 0 \ , elsewhere \end{cases} \tag{24}$$

$$Qh_i(t) = \begin{cases} 0 \ , & A \le t < \xi_1(i), \\ \emptyset_1 \ , & \xi_1(i) \le t < \xi_2(i), \\ \emptyset_2 \ , & \xi_2(i) \le t < \xi_3(i), \\ \emptyset_3 \ , & \xi_3(i) \le t < B \quad , \end{cases} \tag{25}$$

where,

$$\emptyset_1 = \frac{\left(t - \xi_1(i)\right)^\alpha}{\Gamma(\alpha+1)} ,$$

$$\emptyset_2 = \frac{(t - \xi_1(i))^\alpha}{\Gamma(\alpha+1)} - 2\frac{\left(t - \xi_2(i)\right)^\alpha}{\Gamma(\alpha+1)} ,$$

$$\emptyset_3 = \frac{(t - \xi_1(i))^\alpha}{\Gamma(\alpha+1)} - 2\frac{\left(t - \xi_2(i)\right)^\alpha}{\Gamma(\alpha+1)} + \frac{\left(t - \xi_3(i)\right)^\alpha}{\Gamma(\alpha+1)} ,$$

for $i = 1,2,..,m$, $m = 2^J$ and J is a positive integer, called the maximum level of resolution. Here j and k represent the integer decomposition of the index i. i.e. $i = k + 2^j - 1$, $0 \le j < i$ and $1 \le k < 2^j + 1$.

Results and Discussion:

1. Application of Haar wavelet method

Consider the generalized fractional Huxley equation (1) subject to the initial condition

$$u(x,0) = \left[\frac{\gamma}{2} + \frac{\gamma}{2}\tanh\left(\frac{\delta\gamma}{2}\sqrt{\frac{\beta}{1+\delta}}x\right)\right]^{\frac{1}{\delta}}. \tag{26}$$

One will divide both space and time interval [0,1] into m equal subintervals; each of width $\Delta = \frac{1}{m}$.

Haar wavelet solution of $u(x,t)$ is found by assuming that $\frac{\partial^2 u(x,t)}{\partial x^2}$ can be expanded in terms of Haar wavelets as [26]

$$\frac{\partial^2 u}{\partial x^2} = \sum_{i=0}^{m-1}\sum_{j=0}^{m-1} C_{ij} \ \ h_i(x)h_j(t). \tag{27}$$

Integrating Eq.(27) with respect to variable x from 0 to x, one can obtain

$$\frac{\partial u}{\partial x}-\frac{\partial}{\partial x}u(0,t)=\sum_{i=0}^{m-1}\sum_{j=0}^{m-1}C_{ij}\ Q_H\ h_i(x)h_j(t)\,. \tag{28}$$

Again, integrating Eq.(28) with respect to x from 0 to x, one can obtain

$$u(x,t)=\sum_{i=0}^{m-1}\sum_{j=0}^{m-1}C_{ij}\ {Q_H}^2\ h_i(x)h_j(t)+u(0,t)+x\frac{\partial}{\partial x}u(0,t). \tag{29}$$

To find $\frac{\partial}{\partial x}u(0,t)$, we put $x=1$ in Eq.(29)

$$\frac{\partial}{\partial x}u(0,t)=u(1,t)-u(0,t)-\sum_{i=0}^{m-1}\sum_{j=0}^{m-1}C_{ij}\ \left[{Q_H}^2\ h_i(x)\right]_{x=1}h_j(t)\,. \tag{30}$$

Substituting Eq.(30) in Eq.(29), one can get

$$u(x,t)=\sum_{i=0}^{m-1}\sum_{j=0}^{m-1}C_{ij}\ {Q_H}^2\ h_i(x)h_j(t)+u(0,t)+x\left[u(1,t)-u(0,t)-\sum_{i=0}^{m-1}\sum_{j=0}^{m-1}C_{ij}\ \left[{Q_H}^2\ h_i(x)\right]_{x=1}h_j(t)\right]. \tag{31}$$

From Eq. (15), it is obtained that $\left[{Q_H}^2\ h_i(x)\right]_{x=1}=q_i(1)=\begin{cases}0.5 & ;i=0\\ \frac{1}{4m^2} & ;i>0\end{cases}.$

The nonlinear term presented in Eq.(1) can be approximated using Haar wavelet function as [26]

$$\begin{aligned}\beta u(1-u^\delta)(u^\delta-\gamma)&=-\beta\left(u^{2\delta+1}-(1+\gamma)u^{\delta+1}+\gamma u\right)\\&=\sum_{i=0}^{m-1}\sum_{j=0}^{m-1}d_{ij}\ \ h_i(x)h_j(t).\end{aligned} \tag{32}$$

Substituting Eq.(31) in Eq.(32), one can get

$$\begin{aligned}-\beta\Bigg(&\Bigg[\sum_{i=0}^{m-1}\sum_{j=0}^{m-1}C_{ij}\ {Q_H}^2\ h_i(x)h_j(t)+u(0,t)\\&+x\left[u(1,t)-u(0,t)-\sum_{i=0}^{m-1}\sum_{j=0}^{m-1}C_{ij}\ \left[{Q_H}^2\ h_i(x)\right]_{x=1}h_j(t)\right]\Bigg]^{2\delta+1}\\&-(1+\gamma)\Bigg[\sum_{i=0}^{m-1}\sum_{j=0}^{m-1}C_{ij}\ {Q_H}^2\ h_i(x)h_j(t)+u(0,t)\\&+x\left[u(1,t)-u(0,t)-\sum_{i=0}^{m-1}\sum_{j=0}^{m-1}C_{ij}\ \left[{Q_H}^2\ h_i(x)\right]_{x=1}h_j(t)\right]\Bigg]^{\delta+1}\end{aligned}$$

$$+\gamma\left[\sum_{i=0}^{m-1}\sum_{j=0}^{m-1} C_{ij}\ {Q_H}^2\ h_i(x)h_j(t)+u(0,t)\right. \tag{33}$$

$$\left.\left.+x\left[u(1,t)-u(0,t)-\sum_{i=0}^{m-1}\sum_{j=0}^{m-1} C_{ij}\ \left[{Q_H}^2\ h_i(x)\right]_{x=1} h_j(t)\right]\right]\right)$$

$$=\sum_{i=0}^{m-1}\sum_{j=0}^{m-1} d_{ij}\ \ h_i(x)h_j(t)\,.$$

Substituting Eq.(27) and Eq.(32) in Eq.(1), one can obtain

$$D_t^{\alpha}u-\sum_{i=0}^{m-1}\sum_{j=0}^{m-1} C_{ij}\ \ h_i(x)h_j(t)\ \ -\sum_{i=0}^{m-1}\sum_{j=0}^{m-1} d_{ij}\ \ h_i(x)h_j(t)\ \ =0. \tag{34}$$

Now, applying J^{α} Eq.(11)to both sides of Eq.(34), one can get

$$u(x,t)-u(x,0)-\sum_{i=0}^{m-1}\sum_{j=0}^{m-1} C_{ij}\ \ h_i(x){Q_H}^{\alpha}h_j(t)$$

$$-\sum_{i=0}^{m-1}\sum_{j=0}^{m-1} d_{ij}\ \ h_i(x){Q_H}^{\alpha}h_j(t)=0. \tag{35}$$

Substituting Eq.(31) in Eq.(35), one can obtain

$$\sum_{i=0}^{m-1}\sum_{j=0}^{m-1} C_{ij}\ {Q_H}^2\ h_i(x)h_j(t)+u(0,t)+x[u(1,t)-u(0,t)$$

$$-\sum_{i=0}^{m-1}\sum_{j=0}^{m-1} C_{ij}\ \left[{Q_H}^2\ h_i(x)\right]_{x=1} h_j(t)]-u(x,0)$$

$$-\sum_{i=0}^{m-1}\sum_{j=0}^{m-1} C_{ij}\ \ h_i(x){Q_H}^{\alpha}h_j(t) \tag{36}$$

$$-\sum_{i=0}^{m-1}\sum_{j=0}^{m-1} d_{ij}\ \ h_i(x){Q_H}^{\alpha}h_j(t)=0.$$

Now, substituting the collocation points $x_l=\frac{l-0.5}{m}$ and $t_k=\frac{k-0.5}{m}$ for $l,k=0,1,\ldots,m-1$ in Eqs (33) and (36), one can obtain an algebraic system with c_{ij} and d_{ij} as unknowns. By solving these system using MATLAB command "*fsolve*" one can obtain the Haar wavelet coefficients c_{ij} and d_{ij}. Afterwards, the solution is calculated from Eq.(35).

2. Proposed technique

We introduce a technique to treat the nonlinear term in Eq. (32)

$$u(x,t)=\begin{bmatrix} u(x_0,t)\\ u(x_1,t)\\ \vdots\\ u(x_{m-1},t)\end{bmatrix}=\begin{bmatrix} u_0(t)\\ u_1(t)\\ \vdots\\ u_{m-1}(t)\end{bmatrix}, \tag{37}$$

where $x_l,\ l=0,1,\dots,m-1$ are the collocation points and $u_0(t),u_1(t),\dots,\ u_{m-1}(t)$ are functions which defined on $[0,1)$. It is well known that an integrable function $f(t)$ defined on the semi-open interval [0,1) can be expanded in an *m*-term Block Pulse Function (BPF) series as stated by [28],

$$f(t)\approx\sum_{i=1}^{m}c_i\,b_i(t), \tag{38}$$

where the coefficients c_i can be determined by

$$c_i=m\int_{\frac{i-1}{m}}^{\frac{i}{m}}f(t)dt\,; \tag{39}$$

c_i equal to average value of $f(t)$ over the interval $\frac{i-1}{m}<t\le\frac{i}{m}$ and b_i are the BPFs given as

$$b_i(t)=\begin{cases}1\,,\xi_1<t\le\xi_2,\\0\,,\quad elsewhere,\end{cases} \tag{40}$$

where $\xi_1=\frac{i-1}{m}$ and $\xi_2=\frac{i}{m}$, for $i=1,2,\dots,m$, and m is positive integer.
Eq.(38) can also be written as follows

$$f(t)=C_m^TB_m(t), \tag{41}$$

where $C_m^T=[\,c_1,c_2,\dots,c_m]$ is called the coefficient vector and $B_m(t)=[\,b_1(t),b_2(t),\dots,b_m(t)]^T$ is the block pulse function vector. The functions $b_i(t)$ are disjoint [29], that is

$$b_i(t)b_l(t)=\begin{cases}0; & i\ne l,\\b_i(t); & i=l,\end{cases} \tag{42}$$

Using Eq.(42), one can obtain

$$\begin{aligned}[f(t)]^\delta&=[c_1b_1(t)+c_2b_2(t)+\cdots+c_mb_m(t)]^\delta\\&=\left[{c_1}^\delta b_1(t)^\delta+{c_2}^\delta b_2(t)^\delta+\cdots+{c_m}^\delta b_m(t)^\delta\right]\\&=\left[{c_1}^\delta,\ {c_2}^\delta,\dots,{c_m}^\delta\right]B_m(t).\end{aligned} \tag{43}$$

Substituting Eq. (43) into Eq. (37), one can find $[u(x,t)]^\delta$ as follows

$$[u(x,t)]^\delta=\begin{bmatrix}{e_{1,1}}^\delta & {e_{1,2}}^\delta & \dots & {e_{1,m}}^\delta\\{e_{2,1}}^\delta & {e_{2,2}}^\delta & \dots & {e_{2,m}}^\delta\\\vdots & \vdots & \vdots & \vdots\\{e_{m,1}}^\delta & {e_{m,2}}^\delta & \dots & {e_{m,m}}^\delta\end{bmatrix}, \tag{44}$$

where $e_{i,j},\ i,j=1,2,\dots,m$ are the elements of the matrix which is calculated on the right side of Eq.(31).

3. Solution algorithm :

Inputs:
M is the maximum level of resolution ,
The interval $[\mathrm{a},\mathrm{b}]$ where $\mathrm{a}=0,\mathrm{b}=1$,
α is the order of fractional derivative and the parameters γ,β,δ ,
$u(x,0)$ is the initial solution ,
$u(x,t)$ is the exact solution when $\alpha=1$, to calculate the boundary solutions.
Using the approximation solution of equation from [12] to calculate the error.
Outputs:

$u(x_i, t_j)$ is the numerical solution in the collocation point (x_i, t_j) where $0 \le i \le m-1, 0 \le j \le m-1$

Computing the absolute error when $\alpha = 1$.

Steps of algorithm

Step 1: calculate the Haar matrix.

Step 2: calculate the operational matrix Q_H , then Q_H^2 and finally Q_H^α .

Step 3: formation the algebraic system from equations (33,36) or from (33,36,44).

Step 4: using the MATLAB command 'fsolve' to calculate the Haar coefficients .

Step 5: find the numerical solution from equation (35).

Step 6: calculate the absolute error .

4. Numerical results and discussion

For comparison, we used the HAM solution after correcting the misprint in [12] to find $u(x,t)$ at some collocation points as follows

$$\begin{aligned} u(x,t) = \frac{-h}{\delta^2\,\Gamma(\alpha+1)} &((1+\tanh(\sigma\gamma x))^{\frac{1}{\delta}}\,(2^{\left(-\frac{1}{\delta}\right)}\gamma^{\left(\frac{1+2\delta}{\delta}\right)}\sigma^2 tanh^2(\sigma\gamma x) \\ &+2^{\left(-\frac{1}{\delta}\right)}\gamma^{\left(\frac{1+2\delta}{\delta}\right)}\sigma^2 + 2^{\left(-\frac{1}{\delta}\right)}\gamma^{\left(\frac{1+2\delta}{\delta}\right)}\sigma^2 tanh^2(\sigma\gamma x)\delta \\ &-2^{\left(\frac{-1+\delta}{\delta}\right)}\gamma^{\left(\frac{1+2\delta}{\delta}\right)}\sigma^2\ \tanh(\sigma\gamma x) \\ &+2^{\left(-\frac{1}{\delta}\right)}\gamma^{\left(\frac{1+\delta}{\delta}\right)}\beta\left(2^{\left(\frac{1}{\delta}\right)}\gamma^{\left(\frac{1}{\delta}\right)}(1+\tanh(\sigma\gamma x))^{\frac{1}{\delta}}\right)^{\delta}\ \delta^2 - 2^{\left(-\frac{1}{\delta}\right)}\gamma^{\left(\frac{1+\delta}{\delta}\right)}\beta\delta^2 \\ &-2^{\left(-\frac{1}{\delta}\right)}\gamma^{\left(\frac{1+2\delta}{\delta}\right)}\sigma^2\delta + 2^{\left(-\frac{1}{\delta}\right)}\gamma^{\left(\frac{1}{\delta}\right)}\beta\left(2^{\left(\frac{-1}{\delta}\right)}\gamma^{\left(\frac{1}{\delta}\right)}(1+\tanh(\sigma\gamma x))^{\frac{1}{\delta}}\right)^{\delta}\ \delta^2 \\ &-2^{\left(-\frac{1}{\delta}\right)}\gamma^{\left(\frac{1}{\delta}\right)}\beta\left(2^{\left(\frac{-1}{\delta}\right)}\gamma^{\left(\frac{1}{\delta}\right)}(1+\tanh(\sigma\gamma x))^{\frac{1}{\delta}}\right)^{2\delta}\ \delta^2)t^\alpha)\ + [\tfrac{\gamma}{2}+\tfrac{\gamma}{2}\tanh(\sigma\gamma x)]^{\frac{1}{\delta}}, \end{aligned}$$

Table 1: Absolut errors of Haar wavelet method, Haar wavelet with our technique and HAM [12] for $m = 16, \alpha = 1; \gamma = 0.001; \beta = 1, h = -0.3$.

x_i/t_i	$\delta = 1$			$\delta = 3$		
	$\lvert u_{Exact} - u_{Haar}\rvert$	$\lvert u_{Exact} - u_{our}\rvert$	$\lvert u_{Exact} - u_{HAM}\rvert$	$\lvert u_{Exact} - u_{Haar}\rvert$	$\lvert u_{Exact} - u_{our}\rvert$	$\lvert u_{Exact} - u_{HAM}\rvert$
0.0313	0.0187E-6	0.1084E-17	0.0054E-6	0.0004	0.3120E-11	0.0086E-4
0.0938	0.1041E-6	0.2059E-17	0.0164E-6	0.0022	0.1494E-11	0.0260E-4
0.1563	0.1962E-6	0.0867E-17	0.0273E-6	0.0041	0.3039E-11	0.0434E-4
0.2188	0.2844E-6	0.1084E-17	0.0383E-6	0.0059	0.1746E-11	0.0608E-4
0.2813	0.3546E-6	0.2602E-17	0.0492E-6	0.0073	0.1024E-11	0.0782E-4
0.3438	0.4069E-6	0.3794E-17	0.0602E-6	0.0083	0.0728E-11	0.0955E-4
0.4063	0.4418E-6	0.4336E-17	0.0711E-6	0.0089	0.0401E-11	0.1129E-4
0.4688	0.4598E-6	0.4770E-17	0.0820E-6	0.0093	0.0181E-11	0.1303E-4
0.5313	0.4617E-6	0.4878E-17	0.0930E-6	0.0093	0.0133E-11	0.1477E-4
0.5938	0.4483E-6	0.4878E-17	0.1039E-6	0.0090	0.0041E-11	0.1651E-4
0.6563	0.4197E-6	0.4445E-17	0.1149E-6	0.0085	0.0028E-11	0.1825E-4
0.7188	0.3763E-6	0.4119E-17	0.1258E-6	0.0076	0.0021E-11	0.1999E-4
0.7813	0.3183E-6	0.3577E-17	0.1368E-6	0.0064	0.0030E-11	0.2172E-4
0.8438	0.2455E-6	0.2710E-17	0.1477E-6	0.0050	0.0042E-11	0.2346E-4
0.9063	0.1582E-6	0.1626E-17	0.1587E-6	0.0032	0.0025E-11	0.2520E-4
0.9688	0.0564E-6	0.097E-17	0.1696E-6	0.0012	0.0133E-11	0.2694E-4

Table 2: Our results compared to Haar wavelet method and HAM [12] for $m = 16, \delta = 1, \alpha = 0.25, 0.75, \ \gamma = 0.001; \beta = 1, h = -0.3.$

x_i/t_i	$\delta = 1$					
	$\alpha = 0.25$			$\alpha = 0.75$		
	u_{Haar}	u_{our}	u_{HAM}	u_{Haar}	u_{our}	u_{HAM}
0.0313	0.5000E-3	0.4999E-3	0.4999E-3	0.5000E-3	0.4999E-3	0.4999E-3
0.0938	0.5001E-3	0.4999E-3	0.4999E-3	0.5001E-3	0.4999E-3	0.5000E-3
0.1563	0.5001E-3	0.4999E-3	0.4999E-3	0.5001E-3	0.4999E-3	0.5000E-3
0.2188	0.5002E-3	0.4999E-3	0.4999E-3	0.5002E-3	0.4999E-3	0.5000E-3
0.2813	0.5003E-3	0.4999E-3	0.4999E-3	0.5003E-3	0.4999E-3	0.5000E-3
0.3438	0.5003E-3	0.4999E-3	0.4999E-3	0.5003E-3	0.4999E-3	0.5000E-3
0.4063	0.5003E-3	0.4999E-3	0.5000E-3	0.5003E-3	0.4999E-3	0.5000E-3
0.4688	0.5003E-3	0.4999E-3	0.5000E-3	0.5003E-3	0.4999E-3	0.5000E-3
0.5313	0.5003E-3	0.4999E-3	0.5000E-3	0.5003E-3	0.4999E-3	0.5000E-3
0.5938	0.5003E-3	0.4999E-3	0.5000E-3	0.5003E-3	0.4999E-3	0.5000E-3
0.6563	0.5003E-3	0.4999E-3	0.5000E-3	0.5003E-3	0.4999E-3	0.5000E-3
0.7188	0.5002E-3	0.4999E-3	0.5000E-3	0.5003E-3	0.4999E-3	0.5000E-3
0.7813	0.5002E-3	0.4999E-3	0.5000E-3	0.5002E-3	0.4999E-3	0.5000E-3
0.8438	0.5001E-3	0.4999E-3	0.5000E-3	0.5001E-3	0.4999E-3	0.5000E-3
0.9063	0.5000E-3	0.4999E-3	0.5000E-3	0.5000E-3	0.4999E-3	0.5000E-3
0.9688	0.4999E-3	0.4999E-3	0.5000E-3	0.4999E-3	0.4999E-3	0.5000E-3

Table 3: Our results compared to Haar wavelet method and HAM [12] for $m = 16, \delta = 3, \alpha = 0.25, 0.75, \ \gamma = 0.001; \beta = 1, h = -0.3.$

x_i/t_i	$\delta = 3$					
	$\alpha = 0.25$			$\alpha = 0.75$		
	u_{Haar}	u_{our}	u_{HAM}	u_{Haar}	u_{our}	u_{HAM}
0.0313	0.078376	0.079368	0.079365	0.078731	0.079369	0.079369
0.0938	0.076497	0.079366	0.079364	0.076737	0.079367	0.079369
0.1563	0.074894	0.079365	0.079364	0.075029	0.079366	0.079369
0.2188	0.073562	0.079363	0.079365	0.073561	0.079364	0.079370
0.2813	0.072496	0.079362	0.079366	0.072383	0.079363	0.079370
0.3438	0.071693	0.079361	0.079366	0.071486	0.079362	0.079371
0.4063	0.071151	0.079359	0.079367	0.070870	0.079361	0.079371
0.4688	0.070868	0.079358	0.079368	0.070535	0.079360	0.079372
0.5313	0.070844	0.079357	0.079369	0.070481	0.079359	0.079372
0.5938	0.071080	0.079357	0.079370	0.070709	0.079357	0.079373
0.6563	0.071580	0.079356	0.079371	0.071223	0.079356	0.079373
0.7188	0.072349	0.079355	0.079372	0.072024	0.079355	0.079374
0.7813	0.073393	0.079354	0.079373	0.073119	0.079354	0.079374
0.8438	0.074720	0.079353	0.079374	0.074512	0.079353	0.079375
0.9063	0.076342	0.079352	0.079375	0.076211	0.079352	0.079376
0.9688	0.078268	0.079350	0.079376	0.078223	0.079350	0.079376

We present in Table 1 the absolute errors of the Haar wavelet method, the Haar wavelet method with our technique for the nonlinear term and the HAM [12] at some collocation points for $m = 16, \quad \alpha = 1; \quad \gamma = 0.001; \quad \beta = 1,$
$h = -0.3$ and for two different values of δ. We see that our method gives more accurate results than the mentioned methods and the absolute error becomes more and more small when we increase m.
In Table 2 and the Table 3, we compare between the numerical solutions of Eq.(1) using the Haar wavelet method, the Haar wavelet method with our technique for the nonlinear

term and the HAM [12] at some collocation points for $m = 16$, $\gamma = 0.001; \beta = 1, h = -0.3$ and for different non-integer values of α and for two different values of δ.

Conclusions and Recommendations:

We discuss the numerical solutions for the generalized fractional Huxley equation using two dimensional Haar Wavelet method with the block pulse functions. The obtained results are compared to the results of other methods, these comparisons show that our technique have a high accuracy and less computational errors. It is clear from the tables that the numerical solutions obtained by proposed method are in good agreement with the exact solutions and better than numerical solutions obtained by some other methods in literature. Moreover, the error may be reduced if we increase the level of resolution, which requires more number of collocation points.

MATLAB CODING FOR HAAR WAVELET :

```
J=input('J='); M=pow2(J); M2=2*M; a=input('a='); b=input('b=');
dx=(b-a)/M2;
for L=1:M2
    X(L)=a+(L-0.5)*dx;
    H(1,L)=1;
     if (X(L)<(0.5*(2*a+b)))
        H(2,L)=1;
        elseif(X(L)>=(0.5*(2*a+b)))
         H(2,L)=-1;
        end;
end;
for j=1:J
    m=pow2(j);
    for K1=1:m
        K=K1-1;
        i=m+K1;
        ksi1=a+K/m;
        ksi2=a+(K+0.5)/m;
        ksi3=a+(K+1)/m;
       for L=1:M2
            if X(L)<ksi1
                H(i,L)=0;
            elseif X(L)<ksi2
                H(i,L)=1;
              elseif X(L)<ksi3
                H(i,L)=-1;
              elseif X(L)>=ksi3
               H(i,L)=0;
              else;
            end;
        end;
    end;
end;
```

MATLAB CODING FOR HAAR WAVELET OPERATIONAL MATRIX Q_H^α

```
function IF= operational( af )
global J
M=pow2(J);
M2=2*M;
a=0;
b=1;
dx=(b-a)/M2;
for L=1:M2
   X(L)=a+(L-0.5)*dx;
   IF(1,L)=(1/gamma(1+af))*X(L)^af;
      if (X(L)<(0.5*(2*a+b)))
     IF(2,L)=(1/gamma(1+af))*X(L)^af;
   elseif(X(L)>=(0.5*(2*a+b)))
     IF(2,L)=(1/gamma(af+1))*((X(L)-0)^af-2*(X(L)-0.5)^af);
   end;
end;
for j=1:J
   m=pow2(j);
   for K1=1:m
      K=K1-1;
      i=m+K1;
      ksi1=a+K/m;
      ksi2=a+(K+0.5)/m;
      ksi3=a+(K+1)/m;
           for L=1:M2
         if X(L)<ksi1
         IF(i,L)=0;
         elseif X(L)<ksi2
            IF(i,L)=(1/gamma(af+1))*((X(L)-ksi1)^af);
        elseif X(L)<ksi3
            IF(i,L)=(1/gamma(af+1))*((X(L)-ksi1)^af-2*(X(L)-ksi2)^af);
            elseif X(L)>=ksi3
            IF(i,L)=(1/gamma(af+1))*((X(L)-ksi1)^af-2*(X(L)-ksi2)^af+(X(L)-ksi3)^af);
           else;
         end;
      end;
   end;
end;
end
```

MATLAB CODING FOR THE ALGEBRAIC SYSTEM AND SOLVE IT

```
de=input(' enter delta=   ');
ga=0.001;
be=1;
syms x t
```

```
fe=@(x,t)((ga/2)*(1+tanh((de*ga/2)*(sqrt(be/(1+de)))*(x-(((1+de-ga)/(1+de))*(sqrt
(be*(1+de))))*t))))^(1/de);
for i=1:M2
   for j=1:M2
       UE(i,j)=fe(X(i),X(j));
   end
end
for i=1:M2
   ufx0(i)=fe(X(i),0);
   uf0t(i)=fe(0,X(i));
   uf1t(i)=fe(1,X(i));
end
 for i=1:M2
   U0T(i,:)=uf0t;
   UX0(:,i)=ufx0;
end
q(1)=0.5;
for j=0:J
   m=pow2(j);
   for K1=1:m
      i=m+K1;
      q(i)=1/(4*m*m);
        end
end
Q1= operational(1);
Q2= operational(2);
af=input('af=');
QF= operational(af);
C=sym('C_%d_%d',[2*M2 M2]);
func=@(C)[-((Q2'*C(1:M2,1:M2)*H+X'*(uf1t-uf0t-
q*C(1:M2,1:M2)*H)+U0T)^(2*de+1)-(1+ga)*(Q2'*C(1:M2,1:M2)*H+X'*(uf1t-uf0t-
q*C(1:M2,1:M2)*H)+U0T)^(de                +1)+ga*(Q2'*C(1:M2,1:M2)*H+X'*(uf1t-uf0t-
q*C(1:M2,1:M2)*H)+U0T))-
H'*C(M2+1:M2*2,1:M2)*H;(Q2'*C(1:M2,1:M2)*H+X'*(uf1t-uf0t-
q*C(1:M2,1:M2)*H)+U0T)-H'*C(1:M2,1:M2)*QF-H'*C(M2+1:M2*2,1:M2)*QF-UX0];
options = optimoptions('fsolve','Display','iter');
x0=zeros(2*M2,M2);
[x1,fval,exitflag,output] = fsolve(func,x0,options);
u=H'*x1(1:M2,1:M2)*QF+H'*x1(M2+1:M2*2,1:M2)*QF+UX0
```

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