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Research Article

Connected domination number and traceable graphs

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Abstract

Let G be a simple connected graph with minimum degree δ, second minimum degree δ', and connected domination number $\gamma_c(G)$. It is shown that G has a spanning path whenever $\gamma_c(G) \geq n - \delta' - 1$. This result is best possible for $\delta' < 3$; that is, if $\gamma_c(G) \geq n - \delta' - 2$ and $\delta' < 3$, then G may or may not contain a spanning path. Also, this result settles completely a conjecture posed recently by Chellali and Favaron. In addition, for every choice of δ' and δ, an infinite family of non-traceable graphs satisfying $\delta' > \delta$ and $\gamma_c(G) \leq n - 2\delta'$ is provided, which shows that if another recent conjecture by Chellali and Favaron is true, then it is best possible in a sense. The obtained results, apart from addressing some stronger versions of conjectures generated by the computer program Graffiti.pc, improve some known results.

Keywords: connected domination; minimum degree; order; traceable graphs.

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1. Introduction

Let $G = (V, E)$ be a simple connected graph. The *degree* of a vertex $v \in V(G)$, denoted by $\deg_G(v)$, is the number of edges incident with v in G. The *minimum degree* of G, denoted by $\delta(G)$, is the smallest value of the degrees of vertices of G. The order n of G is the number of vertices in the vertex set $V(G)$ of G. The *size* m is the number of edges in the edge set $E(G)$ of G. If $d_1, d_2, d_3, \cdots, d_n$ are the degrees of all vertices in G with $d_1 \leq d_2 \leq d_3 \leq \cdots \leq d_n$, then $d_1, d_2, d_3, \cdots, d_n$ is called the *degree sequence* of G. The *second minimum degree*, denoted by δ', is the second value in the degree sequence, that is, $\delta' = d_2$. A vertex of degree 1 is called a *leaf*. The *leaf number* of G, denoted by $L(G)$, is the maximum number of leaf vertices contained in a spanning tree of G. A vertex in G which is not a leaf is called an *interior vertex* of G. A set $S \subseteq V(G)$ is *dominating* if every vertex in $V(G) \setminus S$ has a neighbour in S.

A subgraph $G[S]$ of G whose vertex set is S, is *induced* if its edge set consists of all edges in $E(G)$ that have both endpoints in S. The order of the smallest connected $G[S]$ of G, induced by a dominating set S, is the *connected domination number*, denoted by $\gamma_c(G)$. The complete bipartite $K_{1,3}$ graph is known as the *claw*. A graph G is claw-free if it has no induced subgraph isomorphic to the claw. The *circumference* of a graph G, denoted by $c(G)$, is the length of a longest cycle in G. A graph G is *Hamiltonian* if $c(G) = n$ and *traceable* if it has a spanning path.

The leaf number and the connected domination number introduced in a private communication[†] and in [37], respectively, are well-studied graph theoretical parameters, apart from their practical applications which are legion in network designs. The determination of these parameters is known to be NP-hard [13]. The aspect of bounds on the leaf number and connected domination number are well investigated in the literature [1,2,9,15–20,22,39]. For a survey and applications on the aforementioned parameters, the reader is referred to [4,5,17–19,35,39]. The leaf number and connected domination number of G are linked as $L(G) = n - \gamma_c(G)$.

We find it most convenient to work with the leaf number of a graph in settling the results of this paper. Recently, the bounds on the leaf number and connected domination number are found to be cornerstones in the establishment of sufficient conditions for a graph to be traceable or Hamiltonian [27,28,32]. The following theorem by Ding, Johnson, and Seymour [10] is important in proving the main theorem of this paper:

Theorem 1.1. [10] *Let G be a simple connected graph with size m and order n such that $m \geq n + \frac{1}{2}t(t-1)$ and $n \neq t + 2$. Then G has a spanning tree with at least t leaves.*

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[†]Private communication by L. Lovász and M. E. Saks

The pioneer for proving the existence of spanning paths or cycles in graphs is Dirac [11] who established the following result:

Theorem 1.2. [11] *Let G be a simple 2-connected graph with order $n \geq 3$ and minimum degree $\delta \geq 2$ such that $n \leq 2\delta$. Then G is Hamiltonian or $c(G) \geq 2\delta$.*

Several authors reported on cycle- and path-related properties based on different graph parameters such as large neighbourhood unions for non-adjacent vertices [3,36], connectivity and independence number [6,33], Harary and Wiener indices [21], large degree sums for non-adjacent vertices [34], degree, order and independence number [38]. For details on path- and cycle-related properties, see [12,14,35]. Li [23] proved the following theorem which we employ to settle results of the present paper:

Theorem 1.3. [23] *Let G be a simple 2-connected, claw-free graph with minimum degree δ and order n such that $n \leq 4\delta$. Then G is Hamiltonian or $G \in \mathcal{F}_1$, where $\mathcal{F}_1$ is a family of graphs defined as follows: If G is in $\mathcal{F}_1$, then G can be decomposed into three disjoint subgraphs, namely G_1, G_2 and G_3 such that for any $i \neq j$, $1 \leq i, j \leq 3$, $E_G(G_i, G_j) = \{u_i u_j, v_i v_j\}$ with $u_i, v_i \in V(G_i)$.*

Delaviña's computer program Graffiti.pc [7], which sorts through various graphs and looks for simple relations among parameters, posed conjectures Graffiti.pc 190 and Graffiti.pc 190a in 2006; where Conjecture Graffiti.pc 190a is a weaker version of Graffiti.pc 190. To the best of our knowledge, conjectures Graffiti.pc 190a and Graffiti.pc 190 [7], are the ones given here as Conjecture 1.1 and Conjecture 1.2, respectively, where $\delta' = \delta$. See also Conjecture 5 (which is Conjecture 1.2 here) in the paper [8]. Conjectures 1.1 and 1.2 have already been settled completely, see [27,31,32]. Recently, in a book chapter entitled "Connected domination" by Chellali and Favaron [5], we found that there are even stronger versions of both conjectures, Graffiti.pc 190a and Graffiti.pc 190 (see Conjectures 1.3 and 1.4 in this paper) in which δ is replaced in Conjectures 1.1 and 1.2 by δ', see Problem 7 under "Conjectures and open problems" in [5]. This motivated us to solve Conjecture 1.3 here completely as well as to provide an infinite family of graphs showing that if Conjecture 1.4 is true, then it is best possible in a sense.

Conjecture 1.1. [7] *Let G be a simple connected graph of order n, minimum degree δ, and connected domination number $\gamma_c(G)$, where $\gamma_c(G) \geq n - \delta - 1$. Then G is traceable.*

Conjecture 1.2. [7] *Let G be a simple connected graph of order n, minimum degree δ, and connected domination number $\gamma_c(G)$, where $\gamma_c(G) \geq n - 2\delta + 1$. Then G contains a spanning path.*

Conjecture 1.3. [5] *Let G be a simple connected graph of order n, second minimum degree δ', and connected domination number $\gamma_c(G)$, where $\gamma_c(G) \geq n - \delta' - 1$. Then G is traceable.*

Conjecture 1.4. [5] *Let G be a simple connected graph of order n, second minimum degree δ', and connected domination number $\gamma_c(G)$, where $\gamma_c(G) \geq n - 2\delta' + 1$. Then G has a spanning path.*

Motivated by the conjecture Graffiti.pc 190a, Mukwembi [30–32] established sufficient conditions for a graph to be either traceable or Hamiltonian based on the minimum degree and leaf number. In [31,32], the author settled completely Conjecture 1.1. Moreover, the results were shown to be best possible in a sense. More precisely, he proved the following result:

Theorem 1.4. [31,32] *Let G be a connected graph with minimum degree δ and connected domination number $\gamma_c(G)$, where $\gamma_c(G) \geq n - \delta - 1$. Then G is traceable.*

Later, Conjecture 1.2 together with a Hamiltonian analogue thereof was settled completely (see [26–28]). The results were also shown to be best possible in the sense that for any integer $p \geq 1$, the graphs $K_{\delta,\delta+p}$ and $K_{\delta,\delta+p+1}$ are non-Hamiltonian and non-traceable, respectively. Following the proofs in [26,29], for a given graph G, it is known that if the diameter $\mathrm{diam}(G) \neq 2$ and $\gamma_c(G) \geq n - 2\delta + 1$, then G is Hamiltonian or every longest cycle in G is a dominating cycle. The result is best possible in the sense that graphs in $\mathcal{F}_2$ [29] have $\mathrm{diam}(G) \neq 2$, $\gamma_c(G) = n - 2\delta + 1$ and every longest cycle is dominating. Extension of Conjectures 1.1 and 1.2 to graphs with forbidden subgraphs has also been made [24,29].

In this paper, we settle completely Conjecture 1.3 and we give a direction for further study. Precisely, we show that Conjecture 1.3 is not only true but also is best possible in a sense. In [5], it is mentioned that the graphs $K_{\delta,\delta+p}$, $p \geq 2$ with $\delta' = \delta$ and $\gamma_c(G) \leq n - 2\delta$ would imply that Conjecture 1.4, if true, is best possible in a sense. In this paper, we construct an infinite family of non-traceable graphs satisfying $\delta' > \delta$ and $\gamma_c(G) \leq n - 2\delta'$ to show that if Conjecture 1.4 is true, then it is best possible in a sense for every δ'.

The following theorems are crucial in the establishment of the results in this paper:

Theorem 1.5. [27] *Let G be a simple connected graph with minimum degree δ and connected domination number $\gamma_c(G)$, where $\gamma_c(G) \geq n - 2\delta + 1$. Then G is traceable.*

Theorem 1.6. [30] *Let G be a simple connected graph of order n, minimum degree δ, and connected domination number $\gamma_c(G)$, where $\gamma_c(G) \geq n - 2\delta + 1$. Then G is 2-connected.*

We use the following notation, apart from those already defined: For vertices $u, v \in V(G)$, the *distance*, $d_G(u, v)$ is the length of a shortest path between u and v in G. The *open-neighbourhood*, $N_G(v)$, of a vertex v in G is given by

$$N_G(v) = \{u \in V(G) : \ d_G(u, v) = 1\}.$$

The *closed-neighbourhood*, $N_G[v]$, is given by $N_G[v] = \{v\} \cup N_G(v)$. Denote by $G - \{u\}$ the graph obtained from G by deleting a vertex u. Similarly, $G - e$ is the graph obtained from G by deleting the edge e. Let H be a subgraph of G. The set of neighbours of $v \in V(G)$ in H is denoted by $N_H(v)$. Denote by $V(G - H)$ the set of vertices in G which are not in H. Where there is no ambiguity, we drop the argument G.

2. Results

Let G be a simple connected graph with minimum degree δ, second minimum degree δ' and connected domination $\gamma_c(G)$ such that $\gamma_c(G) \geq n - \delta' - 1$. In this section, we prove that G is traceable and thereby settle Conjecture 1.3 completely. For $\delta = \delta'$ and $\delta \leq \delta' \leq 2(\delta - 1)$, the desired result follows from Theorems 1.4 and 1.5. So, we consider $\delta' \geq 2\delta - 1 > \delta$. Note that if $\delta' = \delta$ and $\gamma_c(G) \geq n - 2\delta' + 1$, then Conjecture 1.4 holds (see Theorem 1.5). Hence, in order to settle Conjectures 1.3 and 1.4, it suffices to consider $\delta' > \delta$. Assume that G satisfies the statements of the aforementioned conjectures with $\delta' > \delta$. Then the following facts and lemmas provide a cornerstone in settling the results of this paper as well as giving a guide to further studies on what we think might help to solve Conjecture 1.4:

Fact 2.1. *The graph G has only one vertex, say u, of degree δ, since $\delta' > \delta$. In addition, $\deg_G(x) \geq \delta'$, for all $x \in V(G - \{u\})$. Thus $\deg_{G-\{u\}}(x) \geq \delta' - 1$, for all $x \in V(G - \{u\})$. Moreover,*

(i) $\deg_G(x) \leq \delta' + 1$, *for all $x \in V(G)$ whenever $\gamma_c(G) \geq n - \delta' - 1$; otherwise, G has a spanning tree with at least $\delta' + 2$ leaves, which means that $\gamma_c(G) \leq n - \delta' - 2$, which is a contradiction.*

(ii) $\deg_G(x) \leq 2\delta' - 1$, *for all $x \in V(G)$ whenever $\gamma_c(G) \geq n - 2\delta' + 1$; otherwise, G has a spanning tree with at least $2\delta'$ leaves, which means that $\gamma_c(G) \leq n - 2\delta'$, which is a contradiction.*

The following lemma holds and its proof is similar to that of Lemma 2.2 which we provide shortly, and Lemma 2.2 is stronger than Lemma 2.1:

Lemma 2.1. *If $\gamma_c(G) \geq n - \delta' - 1$, $\delta' \geq 4$, and if G has a tree T' such that $L(T') = \delta' + 1$ then $V(G - T')$ cannot have two vertices of degree at least δ'. That is, $|V(G - T')| \leq 1$ if $V(G - T')$ has no vertex of degree δ. For $\delta' = 3$, $|V(G - T')| \leq 2$ if $V(G - T')$ has no vertex of degree δ.*

Lemma 2.2. *If $\gamma_c(G) \geq n - 2\delta' + 1$, $\delta' \geq 3$, and if G has a tree T' such that $L(T') = 2\delta' - 1$ then $V(G - T')$ cannot have more than two vertices of degree at least δ'. That is, $|V(G - T')| \leq 2$ if $V(G - T')$ has no vertex of degree δ. Moreover, if $V(G - T')$ has no vertex of degree δ and $|V(G - T')| = 2$, then the two vertices lying in $V(G - T')$ must be adjacent and do not share a neighbour in T'.*

Proof. Clearly, G has no tree with at least $2\delta'$ leaves. So, no interior vertex of T' has a neighbour in $V(G - T')$. Also, no leaf of T' has at least two neighbours in $V(G - T')$. In addition, no vertex of $V(G - T')$ has 3 neighbours in $V(G - T')$ by the same argument. Thus, each vertex in $V(G - T')$ of degree at least δ' has a neighbour in T', possibly a leaf vertex. These arguments imply that no vertex in $V(G - T')$ of degree at least δ' has two neighbours in $V(G - T')$. Thus, each vertex in $V(G - T')$ of degree at least δ' has at least $\delta' - 1$ neighbours in T', possibly leaf vertices. This in conjunction with the fact that no leaf of T' can have at least two neighbours outside T' and that $L(T') = 2\delta' - 1$ implies that there are at most two vertices outside T' of degree at least δ'; otherwise, $3(\delta' - 1) > 2\delta' - 1$, for all $\delta' \geq 3$, contradicting the fact that no leaf of T' has two neighbours out. Actually, the contradiction is in the sense that T' can only receive at most $2\delta' - 1$ edges from outside, whereas there are at least $3(\delta' - 1)$ edges from $G - T'$ to T', which in turn implies that there is a vertex in T' that has 2 neighbours out. Furthermore, if $V(G - T')$ has no vertex of degree δ and $|V(G - T')| = 2$, then the two vertices lying in $V(G-T')$ must be adjacent; otherwise, each would require δ' neighbours amongst the leaves of T', which is impossible. Consequently, such vertices cannot share a neighbour in T'; otherwise, we would have a leaf of T' which has two neighbours out. □

Lemma 2.3. *If $\gamma_c(G) \geq n-\delta'-1$ and $\delta' \geq 3$, then $n \leq \delta'+6$.*

Proof. Take $u \in V(G)$ with $\deg_G(u)=\delta$. By Fact 2.1, $\deg_G(x) \leq \delta'+1$ for all $x \in V(G)$ and every neighbour of u has degree at least δ'. Let $v \in N(u)$ be a vertex of maximum degree among neighbours of u. Let $K_{1,\deg(v)}$ be a star graph of v and its neighbours. By Fact 2.1, $V(G-K_{1,\deg(v)})$ has no vertex of degree δ, since u is the only vertex of degree δ. Hence, if $\deg(v)=\delta'+1$ then it follows by Lemma 2.1 that $n \leq \delta'+4$.

Assume that $\deg(v)=\delta'$. If x is a neighbour of v that has two neighbours in $V(G-K_{1,\deg(v)})$, then we obtain a tree with $\delta'+3$ vertices and $\delta'+1$ leaves. Hence by Lemma 2.1, $n \leq \delta'+5$. So, we assume that every neighbour of v has at most one neighbour in $V(G-K_{1,\deg(v)})$. We note first that if there is a vertex, say v', at distance 3 from v then v' has degree at least $\delta' \geq 3$, since it is in $V(G-K_{1,\deg(v)})$ and u is the only vertex of degree δ. Let R be a binary star whose interior vertices are those on a shortest $v-v'$ path $P_{v,v'}$ and its vertex set is $V(R)=N_G[v] \cup N_G[v']$. Evidently, $\deg_G(v)=\deg_G(v')=\delta'$; otherwise, $L(R) \geq 2\delta'-1 > \delta'+1$. So, R is a tree with $\delta'+5$ vertices and $\delta'+1$ leaves. For $\delta'=3$, one has $|V(G-R)| \leq 1$; otherwise, one constructs a tree with at least 5 leaves, which is a contradiction. This, in conjunction with Lemma 2.1, implies that $n \leq \delta'+6$ for all $\delta' \geq 3$.

Now, consider the case when every vertex is at a distance of not more than 2 from v. Then each vertex in $V(G-K_{1,\deg(v)})$ has a neighbour in $K_{1,\deg(v)}$, possibly a leaf vertex. So, if there is a vertex in $V(G-K_{1,\deg(v)})$ that has two neighbours in $V(G-K_{1,\deg(v)})$, we join it to those two neighbours and one of its neighbours in $K_{1,\deg(v)}$ to get a tree with $\delta'+4$ vertices and $\delta'+1$ leaves, and hence the desired result follows from Lemma 2.1. Thus, it suffices to assume that every vertex in $V(G-K_{1,\deg(v)})$ has at most one neighbour in $V(G-K_{1,\deg(v)})$ and so it has at least $\delta'-1$ neighbours in $K_{1,\deg(v)}$. This, in conjunction with the fact that no leaf of $K_{1,\deg(v)}$ has two neighbours in $V(G-K_{1,\deg(v)})$, implies that $|V(G-K_{1,\deg(v)})| \leq 2$; otherwise $3(\delta'-1) > \delta'$, for all $\delta' \geq 2$, contradicting the fact that no leaf in $K_{1,\deg(v)}$ has two neighbours in $V(G-K_{1,\deg(v)})$. Note in simple terms that the contradiction lies in the fact that $K_{1,\deg(v)}$ receives at most δ' edges from $V(G-K_{1,\deg(v)})$ whereas $V(G-K_{1,\deg(v)})$ is sending at least $3(\delta'-1)$ edges to $K_{1,\deg(v)}$.

Therefore, in all cases Lemma 2.3 holds. □

Lemma 2.4. *If $\gamma_c(G) \geq n-\delta'-1$, $\deg_G(u)=\delta$, and $\delta' \geq 4$, then $G-\{u\}$ is 2-connected. In addition, for $\delta'=3$ and $n \leq 6$, the graph $G-\{u\}$ is 2-connected.*

Proof. Clearly, $G-\{u\}$ has minimum degree at least $\delta'-1$ and leaf number at most $\delta'+1$. So, for all $\delta' \geq 4$ we have $L(G-\{u\}) \leq 2(\delta'-1)-1$ and the result follows from Theorem 1.6. For $\delta'=3$, the required result is obtained by using similar arguments as in the proof of the main result in [25]. □

We are now in a position to prove the main theorem of this paper which improves and strengthens Theorem 1.4 and Theorem 1.5, apart from settling Conjecture 1.3, for $\delta' \geq 2\delta-1$.

Theorem 2.1. *Let G be a connected graph with second minimum degree δ', order n, and connected domination number $\gamma_c(G)$ such that $\gamma_c(G) \geq n-\delta'-1$. Then G is traceable.*

Proof. As mentioned before, for $\delta=\delta'$ and $\delta \leq \delta' \leq 2(\delta-1)$, the result follows from Theorem 1.4 and Theorem 1.5. So, here we prove the theorem for $\delta' > \delta$. It follows that $\delta' \geq 2$.

Consider first $\delta' \geq 4$. We start with G being a claw-free graph. Then $G-\{u\}$ is also claw-free. By Lemma 2.4, $G-\{u\}$ is 2-connected and by Fact 2.1 it has minimum degree at least $\delta'-1$. By Lemma 2.3, $n \leq \delta'+6$. So $G-\{u\}$ has order at most $\delta'+5$. Thus $4(\delta'-1) \geq \delta'+5$, for all $\delta' \geq 3$. Hence by Theorem 1.3, $G-\{u\}$ is Hamiltonian; since by the properties of G, $G \notin \mathcal{F}_1$. Therefore, G is traceable. Note here that this case also holds for $\delta'=3$ if $G-\{u\}$ is 2-connected.

Assume G has a claw as an induced subgraph. Let $\{x,x_1,x_2,x_3\}$ be the vertex set of such a claw with x as a center vertex. Then $\{x_1,x_2,x_3\}$ is an independent set. By Fact 2.1, at least one element in $\{x_1,x_2,x_3\}$, say x_1 has degree at least δ', since only one vertex of G is of degree δ. So, x_1 has $\delta'-1$ neighbours outside the claw. By attaching these to x_1 in a claw, we obtain a tree with $\delta'+3$ vertices and $\delta'+1$ leaves. Following Lemma 2.1, $n \leq \delta'+4$ if $u \in \{x,x_1,x_2,x_3\}$ and $n \leq \delta'+5$ if $u \notin \{x,x_1,x_2,x_3\}$.

Consider first $n \le \delta'+4$. Then $G-\{u\}$ has order at most $\delta'+3$ and minimum degree at least $\delta'-1$. Hence $2(\delta'-1) \ge \delta'+3$, for all $\delta' \ge 5$. So by Theorem 1.2, $G-u$ is Hamiltonian for $\delta' \ge 5$ and G is traceable. Consider $\delta' = 4$. Then $n \le 8$. So, $G-\{u\}$ has order at most 7. Recall here that $G-\{u\}$ has minimum degree at least $\delta'-1 = 3$. By Theorem 1.2 and Lemma 2.4, the circumference of $G-\{u\}$ is at least 6. Therefore, since $G-\{u\}$ has order at most 7 and minimum degree at least 3, one concludes that $G-\{u\}$ is Hamiltonian. Thus, G is traceable.

Consider $n \le \delta'+5$. Then by similar arguments as above $G-\{u\}$ is Hamiltonian for all $\delta' \ge 6$. Note here that it is enough to consider $n = \delta'+5$, for otherwise we are done by previous arguments. Let $\delta' = 5$. Then $n = 10$. So $G-\{u\}$ has order 9 and minimum degree at least 4. Again by Theorem 1.2, the circumference of $G-\{u\}$ is at least 8. Thus $G-\{u\}$ is Hamiltonian or every longest cycle in it is dominating. Assume in $G-\{u\}$ that every longest cycle is dominating. Take $C_8 = v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8$, and let v be the only vertex of $G-\{u\}$ which is not on C_8. Since $\deg_{G-\{u\}}(v) \ge 4$ and v cannot be adjacent to consecutive vertices on C_8, we assume that $vv_1, vv_3, vv_5, vv_7 \in E(G-\{u\})$. Then, $\{v_2, v_4, v_6, v_8\}$ is an independent set and none of its elements is adjacent to v; otherwise, we obtain a spanning cycle of $G-\{u\}$. This together with the fact that each vertex in $G-\{u\}$ has at least 4 neighbours in $G-\{u\}$, implies that every vertex in $\{v_2, v_4, v_6, v_8\}$ is adjacent to v_1. Now, take a star $K_{1,4}$ of v and its neighbours, and join each vertex in $\{v_2, v_4, v_6, v_8\}$ to v_1, we obtain a tree with 7 leaves, a contradiction. So, $G-\{u\}$ must be Hamiltonian. Similar arguments can be used for $\delta' = 4$, but in this case we change the approach.

Let $\delta' = 4$ and $n = 9$. Then, G has no vertex of degree $\delta'+1 = 5$; otherwise by Lemma 2.1 we have $n \le 8$, a contradiction. Now $\delta = 1, 2,$ or 3. Thus, by Fact 2.1, the only available degree sequences are $1,4,4,4,4,4,4,4,4$; $2,4,4,4,4,4,4,4,4$, and $3,4,4,4,4,4,4,4,4$. The first and third degree sequences are not graphical. By using the Havel-Hakimi theorem in conjunction with Theorem 1.1 as in [32], it is concluded that G with degree sequence $2,4,4,4,4,4,4,4,4$ is traceable. Similar arguments also establish the proof for $\delta' = 3$, see the discussion that follows.

Consider $\delta' = 3$. By Lemma 2.3, $n \le 9$. Recall here that $\delta = 1$ or 2. For $n \le 6$ the order of $G-\{u\}$ is at most 5 and by Lemma 2.4, $G-\{u\}$ is 2-connected. Also, it has minimum degree at least 2 and circumference at least 4 by Theorem 1.2. Thus, by the similar arguments as used in the last part for $\delta' = 5$, $G-u$ must be Hamiltonian and hence G is traceable. For $n = 7, 8,$ or 9, we apply Theorem 1.1 and the Havel-Hakimi theorem as in the case $\delta' = 4$ above.

Consider $\delta' = 2$. Since $\delta' \ne \delta$, we have $\delta = 1$ in this case. Let $C_k = v_1, v_2, v_3, \cdots, v_k, v_1$ be a longest cycle in G. Further, let $P_l = v_i, u_1, u_2, \cdots, u_l$ be a tail of C_k; that is, a longest path such that all of its vertices are not on C_k, except only one which is v_i for some fixed i. Then, the set $V(G) - (V(C_k) \cup V(P_l))$ is empty; otherwise we have a tree with 4 leaves, which contradicts either $\delta' > \delta$ or the choice of P_l. Hence, $V(G) = V(C_k) \cup V(P_l)$. If $l = 0$, then G is Hamiltonian and therefore traceable. For $l \ge 1$, G has a spanning subgraph isomorphic to tadpole and so it is traceable.

In all cases, G is traceable as desired. □

To see that Theorem 2.1 is best possible in a sense, for fixed $i, j \in \mathbb{N}$ and $k \ge 6$, take a cycle $C_k = v_1, v_2, v_3, \cdots, v_k$ and add a vertex v as well as edges vv_j, vv_{j+2} with the condition $j, j+2 \notin \{i-1, i, i+1\}$ and then add a path $P_{u_1v_i} = u_1, u_2, u_3, \cdot, u_s, v_i$ where all the vertices of $P_{u_1v_i}$, except v_i, are not in $\{v\} \cup V(C_k)$. Then $\delta' = 2$, the leaf number is $4 = \delta'+2$, and all these graphs are non-traceable. In fact, these graphs together with a graph family reported in [29] and [32] imply that our result is best possible in a sense for $\delta' \le 2$. For $\delta' \ge 3$, no construction yielding the sharpness of the inequality has been found; this suggests that the stronger version of Conjecture 1.4 might be true. Note that Conjectures 1.3 and 1.4 are the same for $\delta' = 2$ and so Conjecture 1.4 is true for $\delta' \le 2$.

In order to show that if Conjecture 1.4 is true then it is best possible in a sense, we construct an infinite family of non-traceable graphs with leaf number at least $2\delta'$ satisfying $\delta' \ne \delta$. For finite integers p' and p with $1 \le p' < \delta'$, $p \ge 2$, $\delta \ge 1$, and $\delta' = \delta + p' \ge 2$, let $G = K_{\delta',\delta'+p} - p'e$ be the graph obtained from the complete bipartite graph $K_{\delta',\delta'+p}$ by deleting p' edges incident to a vertex x that belongs to the larger partite set. That is, if $v_1, v_2, \ldots, v_{\delta'}$ are the vertices in the smaller partite set, then

$$K_{\delta',\delta'+p} - p'e = K_{\delta',\delta'+p} - \{xv_i : 1 \le i \le p'\}.$$

Here, G is a non-traceable graph of minimum degree δ, second minimum degree $\delta' = \delta + p'$, and leaf number

$$\begin{aligned} L(G) &= \delta' + p - 1 + \delta' - 1 \\ &= 2\delta' + p - 2 \\ &\ge 2\delta' \qquad \text{since } p \ge 2. \end{aligned}$$

Note that for $p' = 1$, one has $G = K_{\delta+1,\delta+1+p} - e$, that is the complete bipartite graph $K_{\delta+1,\delta+1+p}$ minus an edge.

3. Conclusion

A conjecture by Chellali and Favaron [5] has been settled completely in this paper for all $\delta' \geq \delta$. The result (Theorem 2.1) has been shown to be best possible for $\delta' \leq 2$; there are non-traceable graphs with leaf number at least $\delta'+2$ (see the graphs constructed in the previous section as well as the ones constructed in [29] and [32]). The result also implies that Conjecture 1.4, which is a stronger version of the conjecture Graffiti.pc 190, is true for $\delta' \leq 2$. Furthermore, a family of graphs has been constructed to indicate that if Conjecture 1.4 is true, then the result generated from it, would be best possible in a sense for every choice of δ and δ'. However, for $\delta' \geq 3$, no graph has been found to show that the results are best possible in a sense. This gives an indication that Conjecture 1.4 might be true for all $\delta' \geq \delta$. It would be interesting if Conjecture 1.4 could be solved completely. It is hoped that the approach used in this paper together with the ideas presented in [27] and [28] will certainly help in settling Conjecture 1.4 completely. As part of further research, it would be exciting to find results on Hamiltonian graphs, analogous to the results of this paper and Conjecture 1.4.

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