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Research article

Boundedness of Hardy operators on grand variable weighted Herz spaces

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Abstract: In this paper, we will introduce the idea of grand variable weighted Herz spaces $\dot{K}^{\alpha(\cdot),\epsilon),\theta}_{q(\cdot)}(\tau)$ in which α is also a variable. Our main purpose in this paper is to prove the boundedness of Hardy operators on grand variable weighted Herz spaces.

Keywords: Hardy operators; grand Herz spaces; weighted Herz spaces; grand weighted Herz spaces
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1. Introduction

The notion of Herz spaces were introduced by C. Herz in [1]. Let $\alpha \in \mathbb{R}$, $1 \le p < \infty$ and $1 \le q < \infty$. The classical versions of non-homogeneous and homogeneous Herz spaces are defined by the norms

$$\|f\|_{K^{\alpha}_{p,q}(\mathbb{R}^n)} := \|f\|_{L^u(B(0,1))} + \left\{ \sum_{k\in\mathbb{N}} 2^{k\alpha q} \left(\int\limits_{2^k<|x|<2^{k+1}} |f(x)|^p dx \right)^{\frac{q}{p}} \right\}^{\frac{1}{q}}, \tag{1.1}$$

$$\|f\|_{\dot{K}^{\alpha}_{p,q}(\mathbb{R}^n)} := \left\{ \sum_{k\in\mathbb{Z}} 2^{k\alpha q} \left(\int\limits_{2^{k-1}<|x|<2^k} |f(x)|^p dx \right)^{\frac{q}{p}} \right\}^{\frac{1}{q}}, \tag{1.2}$$

respectively.

Variable exponent function spaces have been widely studied and have many important applications. Some examples of works in this area are [5, 6]. These variable exponent function spaces are important for studying problems in partial differential equations and applied mathematics. In particular, Herz spaces with variable exponent generalize classical Herz spaces, see [7]. The Herz-Morrey spaces $M\dot{K}^{\alpha,\lambda}_{q(\cdot),p}(\mathbb{R}^n)$ generalize the idea of Herz spaces with variable exponent $\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$. These function spaces were initially defined in [8]. Lu and Zhu [9] further studied the Morrey-Herz spaces $M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$ and established the boundedness of integral operators on these spaces.

Let g is a locally integrable function on $\mathbb{R}^n$. The n-dimensional Hardy operators can be defined as

$$\mathcal{H}g(z) := \frac{1}{|z|^n}\int_{|x|<|z|} g(x)dx, \quad \mathcal{H}^*g(z) := \int_{|x|\geq|z|} \frac{g(x)}{|x|^n}dx, \quad z \in \mathbb{R}^n \setminus \{0\}.$$

They were studied in many papers, see for instance [2–4].

Izuki and Noi introduced the concept of weighted variable Herz spaces $\dot{K}^{\alpha,r}_{s(\cdot)}(w)$ in their papers [10, 11]. The concept of grand Morrey spaces introduced in [12], has attracted significant attention from researchers. In [13], the idea of grand variable Herz spaces $\dot{K}^{\alpha(\cdot),p),\theta}_{q(\cdot)}(\mathbb{R}^n)$ was introduced, and boundedness of sublinear operators were obtained. Boundedness of other integral operators on grand variable Herz spaces can be seen in [14–18]. In [19], the definition of grand variable Herz-Morrey spaces introduced and obtained the boundedness of Riesz potential operator in these spaces. In [20], authors obtained the boundedness of variable Marcinkiewicz integral operator on grand variable Herz-Morrey spaces. Recently, in [21], the authors proved the boundedness of fractional integral operator on grand weighted Herz spaces $\dot{K}^{\alpha,p),\theta}_{q(\cdot)}(w)$ spaces. Grand weighted Herz-Morrey spaces are the generalization of grand weighted Herz spaces. In [22], Sultan et al. established the boundedness of fractional integral operator on grand weighted Herz-Morrey spaces.

Motivated by the study on grand weighted Herz spaces, our main purpose is to define grand variable weighted Herz spaces, which is the generalization of weighted Herz spaces with variable exponents. Our main purpose is to establish some boundedness results for the Hardy operators on grand variable weighted Herz spaces $\dot{K}^{\alpha(\cdot),\epsilon),\theta}_{q(\cdot)}(\tau)$.

Suppose that G is a measurable set in $\mathbb{R}^n$ with Lebesgue measure $|G| > 0$. The characteristic function of G is denoted by χ_G. It is important to note that in this paper, the symbol C represents a positive constant, which may vary in value at different occurrences.

2. Function spaces and weight classes

For this section we refer to [6, 24–26].

2.1. Definition of function spaces

We first recall some necessary definitions and notations.

Definition 2.1. Let G be a measurable set in $\mathbb{R}^n$ and $r(\cdot)$: $G \to [1, \infty)$ be a measurable function. We suppose that

$$1 \leq r_-(G) \leq r(g) \leq r_+(G) < \infty, \tag{2.1}$$

where $r_- := \operatorname{ess}\inf_{g\in G} r(g), \quad r_+ := \operatorname{ess}\sup_{g\in G} r(g)$.

(a) Variable Lebesgue space $L^{r(\cdot)}(G)$ can be defined as

$$L^{r(\cdot)}(G) = \left\{ f \ measurable : \int_G \left(\frac{|f(x)|}{\tau} \right)^{r(x)} dx < \infty, \ where \ \tau \ is \ a \ constant \right\}.$$

Norm in $L^{r(\cdot)}(G)$ can be defined as

$$||h||_{L^{r(\cdot)}(G)} = \inf \left\{ \gamma > 0 : \int_G \left(\frac{|f(x)|}{\tau} \right)^{r(x)} dx \leq 1 \right\}.$$

(b) The space $L^{r(\cdot)}_{\text{loc}}(G)$ can be defined as

$$L^{r(\cdot)}_{\text{loc}}(G) := \left\{ f : f \in L^{r(\cdot)}(K) \ for \ all \ compact \ subsets \ K \subset G \right\}.$$

The log-conditions may be stated as follows:

$$|r(x) - r(y)| \leq \frac{C(r)}{-\ln|x-y|}, \quad |x-y| \leq \frac{1}{2}, \quad x, y \in G, \tag{2.2}$$

where $C(r) > 0$.

Additionally, the decay condition: There exists a number $r_\infty \in (1, \infty)$, such that

$$|r(x) - r_\infty| \leq \frac{C}{\ln(e + |x|)}, \tag{2.3}$$

and also decay condition

$$|r(x) - r_0| \leq \frac{C}{\ln|x|}, |x| \leq \frac{1}{2}, \tag{2.4}$$

holds for some $r_0 \in (1, \infty)$.

We use these notations in this article:

(i) The set $\mathcal{P}(G)$ consists of all measuable functions $r(\cdot)$ satisfying $r_- > 1$ and $r_+ < \infty$.
(ii) $\mathcal{P}^{\log} = \mathcal{P}^{\log}(G)$ consists of all functions $r \in \mathcal{P}(G)$ satisfying (2.1) and (2.2).
(iii) $\mathcal{P}_\infty(G)$ and $\mathcal{P}_{0,\infty}(G)$ are the subsets of $\mathcal{P}(G)$ and values of these subsets lies in $[1, \infty)$ which satisfy the condition (2.3) and both conditions (2.3) and (2.4) respectively.
(iv) $\chi_k = \chi_{R_k}$, $R_k = D_k \setminus D_{k-1}$ and $D_k = D(0, 2^k) = \{x \in \mathbb{R}^n : |x| < 2^k\}$ for all $k \in \mathbb{Z}$.

We define the Hardy-Littlewood maximal operator M as

$$Mf(z) := \sup_{x>0} x^{-n} \int_{D(z,x)} |f(z)| dz,$$

where $f \in L^1_{\text{loc}}(G)$.

Definition 2.2. The weighted $L^{r(\cdot)}$ space is defined as the set of all measurable functions f on $\mathbb{R}^n$ such that $f\tau^{\frac{1}{r(\cdot)}} \in L^{r(\cdot)}(\mathbb{R}^n)$, where $r(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and τ is a weight. The norm of the Banach space $L^{r(\cdot)}(\tau)$ is denoted by

$$||f||_{L^{r(\cdot)}(\tau)} := ||f\tau^{\frac{1}{r(\cdot)}}||_{L^{r(\cdot)}},$$

where $r'(\cdot)$ is the conjugate exponent of $r(\cdot)$.

Definition 2.3. If $u \in [1, \infty)$, $\alpha \in \mathbb{R}$, $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, then the homogeneous Herz spaces $\dot{K}^{\alpha,u}_{q(\cdot)}(\mathbb{R}^n)$ and non-homogeneous $K^{\alpha,u}_{q(\cdot)}(\mathbb{R}^n)$ Herz spaces are defined respectively as

$$\dot{K}^{\alpha,u}_{q(\cdot)}(\mathbb{R}^n) = \left\{ g \in L^{q(\cdot)}_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) : \|g\|_{\dot{K}^{\alpha,u}_{q(\cdot)}(\mathbb{R}^n)} < \infty \right\}, \tag{2.5}$$

where

$$\|g\|_{\dot{K}^{\alpha,u}_{q(\cdot)}(\mathbb{R}^n)} = \left(\sum_{k=-\infty}^{k=\infty} \|2^{k\alpha} g \chi_k\|^u_{L^{q(\cdot)}} \right)^{\frac{1}{u}}.$$

$$K^{\alpha,u}_{q(\cdot)}(\mathbb{R}^n) = \left\{ g \in L^{q(\cdot)}_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) : \|g\|_{\dot{K}^{\alpha,u}_{q(\cdot)}(\mathbb{R}^n)} < \infty \right\}, \tag{2.6}$$

where

$$\|g\|_{K^{\alpha,u}_{q(\cdot)}(\mathbb{R}^n)} = \left(\sum_{k=-\infty}^{k=\infty} \|2^{k\alpha} g \chi_k\|^u_{L^{q(\cdot)}} \right)^{\frac{1}{u}} + \|g\|_{L^{q(\cdot)}(D(0,1))}.$$

Next, we define grand weighted variable $M\dot{K}^{\alpha,\epsilon),\theta}_{\lambda,q(\cdot)}(\tau)$ spaces.

Definition 2.4. If $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $0 \leq \lambda < \infty$, $0 < u < \infty$, $\theta > 0$, then the homogeneous grand weighted variable Herz-Morrey spaces denoted by $M\dot{K}^{\alpha,\epsilon),\theta}_{\lambda,q(\cdot)}(\tau)$ consist of locally integrable functions $f \in L^{q(\cdot)}_{\text{loc}}(\mathbb{R}^n \setminus 0, \tau)$ satisfying:

$$M\dot{K}^{\alpha,\epsilon),\theta}_{\lambda,q(\cdot)}(\tau) := \left\{ L^{q(\cdot)}_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{M\dot{K}^{\alpha,\epsilon),\theta}_{\lambda,q(\cdot)}(\tau)} < \infty \right\}, \tag{2.7}$$

where

$$\|f\|_{M\dot{K}^{\alpha,\epsilon),\theta}_{\lambda,q(\cdot)}(\tau)} = \sup_{\delta>0} \sup_{k_o \in \mathbb{Z}} 2^{-k_0\lambda} \left(\delta^\theta \sum_{k=-\infty}^{k_0} 2^{k\alpha u(1+\delta)} \|f\chi_k\|^{u(1+\delta)}_{L^{q(\cdot)}(\tau)} \right)^{\frac{1}{u(1+\delta)}}.$$

We can define non-homogeneous grand weighted Herz-Morrey spaces in a similar manner.

Definition 2.5. If $s(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $0 < \epsilon < \infty$, $\theta > 0$, then the homogeneous grand variable weighted Herz spaces denoted by $\dot{K}^{\alpha(\cdot),\epsilon),\theta}_{s(\cdot)}(\tau)$ consist of locally integrable functions $f \in L^{s(\cdot)}_{\text{loc}}(\mathbb{R}^n/0, \tau)$ satisfying:

$$\dot{K}^{\alpha(\cdot),\epsilon),\theta}_{s(\cdot)}(\tau) := \left\{ L^{s(\cdot)}_{\text{loc}}(\mathbb{R}^n/\{0\}) : \|f\|_{\dot{K}^{\alpha(\cdot),\epsilon),\theta}_{s(\cdot)}(\tau)} < \infty \right\}, \tag{2.8}$$

where

$$\|f\|_{\dot{K}^{\alpha(\cdot),\epsilon),\theta}_{s(\cdot)}(\tau)} = \sup_{\Delta>0} \left(\Delta^\theta \sum_{\ell=-\infty}^{\infty} 2^{\ell\alpha(\cdot)\epsilon(1+\Delta)} \|f\chi_\ell\|^{\epsilon(1+\Delta)}_{L^{s(\cdot)}(\tau)} \right)^{\frac{1}{\epsilon(1+\Delta)}}.$$

We can define non-homogeneous grand variable weighted Herz spaces in a similar manner.

2.2. The generalized Muckenhoupt class

Next, we will define the weight Muckenhoupt A_r.

Definition 2.6. For $1 < r < \infty$, $r' = \frac{r}{r-1}$ and any weight τ, we define $\tau \in A_r$ if there exists a constant $C > 0$ such that for every cube G

$$\left(\frac{1}{|Q|}\int_G \tau(z)\mathrm{d}z\right)\left(\frac{1}{|Q|}\int_G \tau(z)^{1-r'}\mathrm{d}z\right)^{r-1} \leq C < \infty.$$

We say that $\tau \in A_1$ if there exists a constant $C > 0$ such that $M\tau(z) \leq C\tau(z)$ for every $z \in \mathbb{R}^n$. We define $A_\infty = \bigcup_{1\leq r\leq\infty} A_r$.

Definition 2.7. Let $r(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. A weight τ is called an $A_{r(\cdot)}$ weight if

$$\sup_{D:\text{ball}} \frac{1}{|D|}\|\tau^{\frac{1}{r(\cdot)}}\chi_D\|_{L^{r(\cdot)}}\|\tau^{\frac{-1}{r(\cdot)}}\chi_D\|_{L^{r'(\cdot)}} < \infty. \tag{2.9}$$

Definition 2.8. If $r(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, then a weight τ is said to be an $A'_{r(\cdot)}$ weight if

$$\sup_{D:\text{ball}} |D|^{-P_D}\|\tau\chi_D\|_{L^1}\|\tau^{-1}\chi_D\|_{L^{r'(\cdot)/r(\cdot)}} < \infty. \tag{2.10}$$

The set $A'_{r(\cdot)}$ is defined to be the collection of all $A'_{r(\cdot)}$ weights.

Definition 2.9. Let $r_1(\cdot), r_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $1/r_1(\cdot) - 1/r_2(\cdot) = \alpha/n$. Then $\tau \in A(r_1(\cdot), r_2(\cdot))$ if

$$\|\tau\chi_D\|_{L^{r_2(\cdot)}}\|\tau^{-1}\chi_D\|_{L^{r_1'(\cdot)}} \leq |D|^{1-\frac{\alpha}{n}}.$$

2.3. Important lemmas

We require the following preliminary results to prove our main theorems.

Lemma 2.10. *[23, Generalized Hölder's inequality] If $f \in L^{p(\cdot)}(G)$, $g \in L^{q(\cdot)}(G)$ and $1/r(\cdot) = 1/p(\cdot) + 1/q(\cdot)$, then*

$$\|fg\|_{L^{r(\cdot)}(G)} \leq C\|f\|_{L^{p(\cdot)}(G)}\|g\|_{L^{q(\cdot)}(G)}.$$

Lemma 2.11. *Let decay conditions at origin and infinity be fulfilled. Then*

$$\frac{1}{t_0}2^{\frac{kn}{p(0)}} \leq \|\chi_{D(0,2^{k+1})\setminus D(0,2^k)}\|_{L^{p(\cdot)}(\tau)} \leq t_0 2^{\frac{kn}{p(0)}},\ \textit{for } 0 < k \leq 1 \tag{2.11}$$

and

$$\frac{1}{t_\infty}2^{\frac{kn}{p_\infty}} \leq \|\chi_{D(0,2^{k+1})\setminus D(0,2^k)}\|_{L^{p(\cdot)}(\tau)} \leq t_\infty 2^{\frac{kn}{p_\infty}},\ \textit{for } k \geq 1, \tag{2.12}$$

respectively, where $t_0 \geq 1$ and $t_\infty \geq 1$ independent of k.

Proof. We will prove (2.12) and other inequality can be estimated similarly. As we can see from [23] that

$$\tau(D(0,2^{k+1}) \setminus D(0,2^k)) \leq 2^{knp_\infty}.$$

Now right hand side inequality of (2.12) is given as

$$\int_{\mathbb{R}^n} \frac{\chi_{(D(0,2^{k+1})\backslash D(0,2^k))}\tau(x)dx}{[t_0 2^{\frac{kn}{p_\infty}}]^{p(x)}} \leq 1. \tag{2.13}$$

Using the decay condition we see that

$$\int_{\mathbb{R}^n} \frac{\chi_{(D(0,2^{k+1})\backslash D(0,2^k))}\tau(x)dx}{[t_0 2^{\frac{kn}{p_\infty}}]^{p(x)}} \leq \frac{1}{t_0^{p_-}} \int_{\mathbb{R}^n} \frac{\chi_{(D(0,2^{k+1})\backslash D(0,2^k))}\tau(x)dx}{[2^{\frac{kn}{p_\infty}}]^{p(x)}} \leq 2^{knp_\infty - kn}, \tag{2.14}$$

which determines the choice of $t_0^{p_-} = 2^{knp_\infty - kn}$ we will get our desired result. □

3. Boundedness of Hardy operators on grand variable weighted Herz spaces

In this section, we will prove the main results of this paper.

Theorem 3.1. *Let* $1 < \epsilon < \infty$, $\theta > 0$, $\alpha, q \in \mathcal{P}_{0,\infty}(\mathbb{R}^n)$. *Let* α *satisfies* $\frac{-n}{q_\infty} < \alpha_\infty < \frac{n}{q'_\infty}$ *and* $\frac{-n}{q(0)} < \alpha(0) < \frac{n}{q'(0)}$. *Then the Hardy operator* $\mathcal{H}$ *will be bounded on* $\dot{K}^{\alpha(\cdot),\epsilon),\theta}_{q(\cdot)}(\tau)$.

Proof. Let $f \in \dot{K}^{\alpha(\cdot),\epsilon),\theta}_{q(\cdot)}(\tau)$, $f_j := f\chi_j$ for any $j \in \mathbb{Z}$, then $f = \sum\limits_{j=-\infty}^{\infty} f_j$, we have

$$\begin{aligned} |\mathcal{H}(f)(z).\chi_\ell(z)| &\leq \frac{1}{|z|^n} \int_{D_\ell} |f(x)|dx.\chi_\ell(z) \\ &\leq C2^{-\ell n} \sum_{j=-\infty}^{\ell} \|f_j\|_{L^{q(\cdot)}(\tau)} \|\chi_j\|_{L^{q'(\cdot)}(\tau)}.\chi_\ell(z). \end{aligned}$$

For E_1, we use the facts that, for each $\ell \in \mathbb{Z}$, $z \in R_\ell$ with $j \leq \ell$. Then Hölder's inequality and size condition imply

$$\begin{aligned} &\left\|\left(\mathcal{H}f_j\right)\chi_\ell\right\|_{L^{q(\cdot)}(\tau)} \\ &\leq \sup_{\Delta>0}\left(\Delta^\theta \sum_{\ell=-\infty}^{\infty} 2^{\ell\alpha(\cdot)\epsilon(1+\Delta)} \left(\sum_{j=-\infty}^{\ell} \|f_j\|_{L^{q(\cdot)}(\tau)} \|\chi_j\|_{L^{q'(\cdot)}(\tau)} 2^{-\ell n} \|\chi_\ell\|_{L^{q(\cdot)}(\tau)}\right)^{\epsilon(1+\Delta)}\right)^{\frac{1}{\epsilon(1+\Delta)}}. \end{aligned}$$

Splitting by using Minkowski's inequality we have

$$\begin{aligned} &\left\|\left(\mathcal{H}f_j\right)\chi_\ell\right\|_{L^{q(\cdot)}(\tau)} \\ &\leq \sup_{\Delta>0}\left(\Delta^\theta \sum_{\ell=-\infty}^{\infty} 2^{\ell\alpha(\cdot)\epsilon(1+\Delta)} \left(\sum_{j=-\infty}^{\ell} \|f_j\|_{L^{q(\cdot)}(\tau)} \|\chi_j\|_{L^{q'(\cdot)}(\tau)} 2^{-\ell n} \|\chi_\ell\|_{L^{q(\cdot)}(\tau)}\right)^{\epsilon(1+\Delta)}\right)^{\frac{1}{\epsilon(1+\Delta)}} \\ &\leq \sup_{\Delta>0}\left(\Delta^\theta \sum_{\ell=-\infty}^{-1} 2^{\ell\alpha(\cdot)\epsilon(1+\Delta)} \left(\sum_{j=-\infty}^{\ell} \|f_j\|_{L^{q(\cdot)}(\tau)} \|\chi_j\|_{L^{q'(\cdot)}(\tau)} 2^{-\ell n} \|\chi_\ell\|_{L^{q(\cdot)}(\tau)}\right)^{\epsilon(1+\Delta)}\right)^{\frac{1}{\epsilon(1+\Delta)}} \end{aligned}$$

$$+\sup_{\Delta>0}\left(\Delta^{\theta}\sum_{\ell=0}^{\infty}2^{\ell\alpha(\cdot)\epsilon(1+\Delta)}\left(\sum_{j=-\infty}^{\ell}\|f_j\|_{L^{q(\cdot)}(\tau)}\|\chi_j\|_{L^{q'(\cdot)}(\tau)}2^{-\ell n}\|\chi_\ell\|_{L^{q(\cdot)}(\tau)}\right)^{\epsilon(1+\Delta)}\right)^{\frac{1}{\epsilon(1+\Delta)}}$$
$$:=E_{11}+E_{12}.$$

For E_{11}, by virtue of Lemma 2.11 we get

$$2^{-\ell n}\|\chi_\ell\|_{L^{q(\cdot)}(\tau)}\|\chi_j\|_{L^{q'(\cdot)}(\tau)}\le C2^{-\ell n}2^{\frac{\ell n}{q(0)}}2^{\frac{jn}{q'(0)}}\le C2^{\frac{(j-\ell)n}{q'(0)}}. \tag{3.1}$$

Applying to E_{11} we can get

$$\begin{aligned}E_{11}\le&\sup_{\Delta>0}\left(\Delta^{\theta}\sum_{\ell=-\infty}^{-1}2^{\ell\alpha(\cdot)\epsilon(1+\Delta)}\left(\sum_{j=-\infty}^{\ell}\|\chi_\ell T(f_j)\|_{L^{q(\cdot)}(\tau)}\right)^{\epsilon(1+\Delta)}\right)^{\frac{1}{\epsilon(1+\Delta)}}\\ \le&C\sup_{\Delta>0}\left[\Delta^{\theta}\sum_{\ell=-\infty}^{-1}2^{\ell\alpha(\cdot)\epsilon(1+\Delta)}\left(\sum_{j=-\infty}^{\ell}\|\chi_\ell\|_{L^{q(\cdot)}(\tau)}2^{-\ell n}\|f_j\|_{L^{q(\cdot)}(\tau)}\|\chi_j\|_{L^{q'(\cdot)}(\tau)}\right)^{\epsilon(1+\Delta)}\right]^{\frac{1}{\epsilon(1+\Delta)}}.\end{aligned}$$

Let $b=\frac{n}{q'(0)}-\alpha(0)$,

$$E_{11}\le C\sup_{\Delta>0}\left[\Delta^{\theta}\sum_{\ell=-\infty}^{-1}\left(\sum_{j=-\infty}^{\ell}2^{\alpha(0)j}\|f_j\|_{L^{q(\cdot)}(\tau)}2^{b(j-\ell)}\right)^{\epsilon(1+\Delta)}\right]^{\frac{1}{\epsilon(1+\Delta)}}. \tag{3.2}$$

Applying the fact $2^{-\epsilon(1+\Delta)}<2^{-\epsilon}$, Fubini's theorem for series and Hölder's inequality we get,

$$\begin{aligned}E_{11}\le&C\sup_{\Delta>0}\left[\Delta^{\theta}\sum_{\ell=-\infty}^{-1}\left(\sum_{j=-\infty}^{\ell}2^{\alpha(0)\epsilon(1+\Delta)j}\|f_j\|^{\epsilon(1+\Delta)}_{L^{q(\cdot)}(\tau)}2^{b\epsilon(1+\Delta)(j-\ell)/2}\right.\right.\\ &\left.\left.\times\sum_{j=-\infty}^{\ell}2^{b(\epsilon(1+\Delta))'(j-\ell)/2}\right)^{\frac{\epsilon(1+\Delta)}{(\epsilon(1+\Delta))'}}\right]^{\frac{1}{\epsilon(1+\Delta)}}\\ =&C\sup_{\Delta>0}\left(\Delta^{\theta}\sum_{\ell=-\infty}^{-1}\sum_{j=-\infty}^{\ell}2^{\alpha(0)\epsilon(1+\Delta)j}\|f_j\|^{\epsilon(1+\Delta)}_{L^{q(\cdot)}(\tau)}2^{b\epsilon(1+\Delta)(j-\ell)/2}\right)^{\frac{1}{\epsilon(1+\Delta)}}\\ =&C\sup_{\Delta>0}\left(\Delta^{\theta}\sum_{j=-\infty}^{-1}2^{\alpha(0)\epsilon(1+\Delta)j}\|f_j\|^{\epsilon(1+\Delta)}_{L^{q(\cdot)}(\tau)}\sum_{\ell=j+2}^{-1}2^{b\epsilon(1+\Delta)(j-\ell)/2}\right)^{\frac{1}{\epsilon(1+\Delta)}}\\ \le&C\sup_{\Delta>0}\left(\Delta^{\theta}\sum_{j=-\infty}^{-1}2^{\alpha(0)\epsilon(1+\Delta)j}\|f_j\|^{\epsilon(1+\Delta)}_{L^{q(\cdot)}(\tau)}\sum_{\ell=j+2}^{-1}2^{b\epsilon(j-\ell)/2}\right)^{\frac{1}{\epsilon(1+\Delta)}}\\ \le&C\sup_{\Delta>0}\left(\Delta^{\theta}\sum_{j=-\infty}^{-1}2^{\alpha(0)\epsilon(1+\Delta)j}\|f_j\|^{\epsilon(1+\Delta)}_{L^{q(\cdot)}(\tau)}\right)^{\frac{1}{\epsilon(1+\Delta)}}\end{aligned}$$

$$
\begin{aligned}
&= C\sup_{\Delta>0}\left(\Delta^{\theta}\sum_{j=-\infty}^{\infty}2^{a(\cdot)\epsilon(1+\Delta)j}\|f_j\|^{\epsilon(1+\Delta)}_{L^{q(\cdot)}(\tau)}\right)^{\frac{1}{\epsilon(1+\Delta)}}\\
&\le C\|f\|_{\dot K^{\alpha(\cdot),\epsilon),\theta}_{q(\cdot)}(\tau)}.
\end{aligned}
$$

Now for E_{12} using Minkowski's inequality we have

$$
\begin{aligned}
E_{12} \le& \sup_{\Delta>0}\left(\Delta^{\theta}\sum_{\ell=0}^{\infty}2^{\ell\alpha(\cdot)\epsilon(1+\Delta)}\left(\sum_{j=-\infty}^{-1}\|f_j\|_{L^{q(\cdot)}(\tau)}\|\chi_j\|_{L^{q'(\cdot)}(\tau)}2^{-\ell n}\|\chi_\ell\|_{L^{q(\cdot)}(\tau)}\right)^{\epsilon(1+\Delta)}\right)^{\frac{1}{\epsilon(1+\Delta)}}\\
&+\sup_{\Delta>0}\left(\Delta^{\theta}\sum_{\ell=0}^{\infty}2^{\ell\alpha(\cdot)\epsilon(1+\Delta)}\left(\sum_{j=0}^{\ell}\|f_j\|_{L^{q(\cdot)}(\tau)}\|\chi_j\|_{L^{q'(\cdot)}(\tau)}2^{-\ell n}\|\chi_\ell\|_{L^{q(\cdot)}(\tau)}\right)^{\epsilon(1+\Delta)}\right)^{\frac{1}{\epsilon(1+\Delta)}}\\
:=& A_1 + A_2.
\end{aligned}
$$

The estimate for A_2 can be followed in a similar manner to E_{11} with replacing $q'(0)$ by q'_∞ and using $\frac{n}{q'_\infty} - \alpha_\infty > 0$. For A_1 by virtue of Lemma 2.11 we obtain

$$
2^{-\ell n}\|\chi_\ell\|_{L^{q(\cdot)}(\tau)}\|\chi_j\|_{L^{q'(\cdot)}(\tau)} \le C2^{-\ell n}2^{\frac{\ell n}{q_\infty}}2^{\frac{jn}{q'(0)}} \le C2^{\frac{-\ell n}{q'_\infty}}2^{\frac{jn}{q'(0)}}. \tag{3.3}
$$

As $\alpha_\infty - \frac{n}{q'_\infty} < 0$ we have

$$
\begin{aligned}
A_1 \le& \sup_{\Delta>0}\left(\Delta^{\theta}\sum_{\ell=0}^{\infty}2^{\ell\alpha(\cdot)\epsilon(1+\Delta)}\left(\sum_{j=-\infty}^{-1}\|f_j\|_{L^{q(\cdot)}(\tau)}\|\chi_j\|_{L^{q'(\cdot)}(\tau)}2^{-\ell n}\|\chi_\ell\|_{L^{q(\cdot)}(\tau)}\right)^{\epsilon(1+\Delta)}\right)^{\frac{1}{\epsilon(1+\Delta)}}\\
\le& C\sup_{\Delta>0}\left[\Delta^{\theta}\sum_{\ell=0}^{\infty}2^{\ell a_\infty\epsilon(1+\Delta)}\times\left(\sum_{j=-\infty}^{-1}2^{\frac{-\ell n}{q'_\infty}}2^{\frac{jn}{q'(0)}}\|f_j\|_{L^{q(\cdot)}(\tau)}\right)^{\epsilon(1+\Delta)}\right]^{\frac{1}{\epsilon(1+\Delta)}}\\
\le& C\sup_{\Delta>0}\left[\Delta^{\theta}\sum_{\ell=0}^{\infty}2^{\frac{\ell\alpha-\ell n}{q'_\infty}\epsilon(1+\Delta)}\times\left(\sum_{j=-\infty}^{-1}2^{\frac{jn}{q'(0)}}\|f_j\|_{L^{q(\cdot)}(\tau)}\right)^{\epsilon(1+\Delta)}\right]^{\frac{1}{\epsilon(1+\Delta)}}\\
\le& C\sup_{\Delta>0}\left(\Delta^{\theta}\left(\sum_{j=-\infty}^{-1}2^{\frac{jn}{q'(0)}}\|f_j\|_{L^{q(\cdot)}(\tau)}\right)^{\epsilon(1+\Delta)}\right)^{\frac{1}{\epsilon(1+\Delta)}}\\
\le& C\sup_{\Delta>0}\left(\Delta^{\theta}\left(\sum_{j=-\infty}^{-1}2^{\frac{jn}{q'(0)}-\alpha(0)j}\|f_j\|_{L^{q(\cdot)}(\tau)}2^{\alpha(0)j}\right)^{\epsilon(1+\Delta)}\right)^{\frac{1}{\epsilon(1+\Delta)}}.
\end{aligned}
$$

By using $\frac{n}{q'(0)} - \alpha(0) > 0$ and Hölder's inequality we have

$$
A_1 \le C\sup_{\Delta>0}\left(\Delta^{\theta}\left(\sum_{j=-\infty}^{-1}2^{\frac{jn}{q'(0)}-\alpha(0)j}\|f_j\|_{L^{q(\cdot)}(\tau)}2^{\alpha(0)j}\right)^{\epsilon(1+\Delta)}\right)^{\frac{1}{\epsilon(1+\Delta)}}
$$

$$\leq C\sup_{\Delta>0}\left[\Delta^{\theta}\sum_{j=-\infty}^{-1}2^{\alpha(0)j\epsilon(1+\Delta)}\|f_j\|^{\epsilon(1+\Delta)}_{L^{q(\cdot)}(\tau)}\left(\sum_{j=-\infty}^{-1}2^{(\frac{jn}{q'(0)}-\alpha(0)j)(\epsilon(1+\Delta))'}\right)^{\frac{\epsilon(1+\Delta)}{(\epsilon(1+\Delta))'}}\right]^{\frac{1}{\epsilon(1+\Delta)}}$$

$$\leq C\sup_{\Delta>0}\left(\Delta^{\theta}\left(\sum_{j=-\infty}^{\infty}2^{\alpha(\cdot)j\epsilon(1+\Delta)}\|f_j\|^{\epsilon(1+\Delta)}_{L^{q(\cdot)}(\tau)}\right)\right)^{\frac{1}{\epsilon(1+\Delta)}}$$

$$\leq C\|f\|_{\dot{K}^{\alpha(\cdot),\epsilon),\theta}_{q(\cdot)}(\tau)},$$

which completes our desired results.

□

Theorem 3.2. *Let $1<\epsilon<\infty$, $\theta>0$, $\alpha, q\in\mathcal{P}_{0,\infty}(\mathbb{R}^n)$. Let α satisfies $\frac{-n}{q_\infty}<\alpha_\infty<\frac{n}{q'_\infty}$ and $\frac{-n}{q(0)}<\alpha(0)<\frac{n}{q'(0)}$. Then the Hardy operator $\mathcal{H}^*$ will be bounded on $\dot{K}^{\alpha(\cdot),\epsilon),\theta}_{q(\cdot)}(\tau)$.*

Proof. Let $f\in\dot{K}^{\alpha(\cdot),\epsilon),\theta}_{q(\cdot)}(\tau))$, $f_j:=f\chi_j$ for any $j\in\mathbb{Z}$, then $f=\sum_{j=-\infty}^{\infty}f_j$, we have

$$|\mathcal{H}^*(f)(z).\chi_\ell(z)|\leq\int_{\mathbb{R}^n\backslash D_\ell}\frac{|f(x)|}{|z|^n}dx.\chi_\ell(z)$$
$$\leq C\sum_{j=\ell+1}^{\infty}2^{-jn}\|f_j\|_{L^{q(\cdot)}(\tau)}\|\chi_j\|_{L^{q'(\cdot)}(\tau)}.\chi_\ell(z).$$

$$\left\|\left(\mathcal{H}^*f_j\right)\chi_\ell\right\|_{L^{q(\cdot)}(\tau)}=\sup_{\Delta>0}\left(\Delta^{\theta}\sum_{\ell=-\infty}^{\infty}2^{\ell\alpha(\cdot)\epsilon(1+\Delta)}\left(\sum_{j=\ell+1}^{\infty}2^{-jn}\|f_j\|_{L^{q(\cdot)}(\tau)}\|\chi_j\|_{L^{q'(\cdot)}(\tau)}.\chi_\ell(z)\right)^{\epsilon(1+\Delta)}\right)^{\frac{1}{\epsilon(1+\Delta)}}$$
$$\leq\sup_{\Delta>0}\left(\Delta^{\theta}\sum_{\ell=-\infty}^{-1}2^{\ell\alpha(\cdot)\epsilon(1+\Delta)}\left(\sum_{j=\ell+1}^{\infty}2^{-jn}\|f_j\|_{L^{q(\cdot)}(\tau)}\|\chi_j\|_{L^{q'(\cdot)}(\tau)}.\chi_\ell(z)\right)^{\epsilon(1+\Delta)}\right)^{\frac{1}{\epsilon(1+\Delta)}}$$
$$+\sup_{\Delta>0}\left(\Delta^{\theta}\sum_{\ell=0}^{\infty}2^{\ell\alpha(\cdot)\epsilon(1+\Delta)}\left(\sum_{j=\ell+1}^{\infty}2^{-jn}\|f_j\|_{L^{q(\cdot)}(\tau)}\|\chi_j\|_{L^{q'(\cdot)}(\tau)}.\chi_\ell(z)\right)^{\epsilon(1+\Delta)}\right)^{\frac{1}{\epsilon(1+\Delta)}}$$
$$:=E_1+E_2.$$

For estimating E_2, now by using the fact that for each $\ell\in\mathbb{Z}$ and $j\geq\ell+1$ with $z\in R_\ell$ to get

$$2^{-jn}\|\chi_\ell\|_{L^{q(\cdot)}(\tau)}\|\chi_j\|_{L^{q'(\cdot)}(\tau)}\|\leq C2^{-jn}2^{\frac{\ell n}{q_\infty}}2^{\frac{jn}{q'_\infty}}\leq C2^{\frac{(\ell-j)n}{q_\infty}}.\tag{3.4}$$

$$E_2\leq\sup_{\Delta>0}\left(\Delta^{\theta}\sum_{\ell=0}^{\infty}2^{\ell\alpha(\cdot)\epsilon(1+\Delta)}\left(\sum_{j=\ell+1}^{\infty}2^{-jn}\|f_j\|_{L^{q(\cdot)}(\tau)}\|\chi_j\|_{L^{q'(\cdot)}(\tau)}.\chi_\ell(z)\right)^{\epsilon(1+\Delta)}\right)^{\frac{1}{\epsilon(1+\Delta)}}$$
$$\leq C\sup_{\Delta>0}\left(\Delta^{\theta}\sum_{\ell=0}^{\infty}\left(\sum_{j\geq\ell+1}^{\infty}2^{\alpha_\infty j}\|f_j\|_{L^{q(\cdot)}(\tau)}2^{d(\ell-j)}\right)^{\epsilon(1+\Delta)}\right)^{\frac{1}{\epsilon(1+\Delta)}},$$

where $d = \frac{n}{q_\infty} + \alpha_\infty > 0$. Then we use Hölder's theorem for series and $2^{-\epsilon(1+\Delta)} < 2^{-\epsilon}$ to obtain

$$\begin{aligned}
E_2 \leq& C \sup_{\Delta>0}\left[\Delta^\theta \sum_{\ell=0}^{\infty}\left(\sum_{j\geq\ell+1}^{\infty} 2^{\alpha_\infty\epsilon(1+\Delta)j}\|f_j\|_{L^{q(\cdot)}(\tau)}^{\epsilon(1+\Delta)}2^{d\epsilon(1+\Delta)(\ell-j)/2}\right)\right.\\
&\left.\times\left(\sum_{j\geq\ell+1}^{\infty}2^{d(\epsilon(1+\Delta))'(\ell-j)/2}\right)^{\frac{\epsilon(1+\Delta)}{(\epsilon(1+\Delta))'}}\right]^{\frac{1}{\epsilon(1+\Delta)}}\\
\leq& C\sup_{\Delta>0}\left[\Delta^\theta\sum_{\ell=0}^{\infty}\sum_{j\geq\ell+1}^{\infty}2^{\alpha_\infty\epsilon(1+\Delta)j}\|f_j\|_{L^{q(\cdot)}(\tau)}^{\epsilon(1+\Delta)}2^{d\epsilon(1+\Delta)(\ell-j)/2}\right]^{\frac{1}{\epsilon(1+\Delta)}}\\
\leq& C\sup_{\Delta>0}\left(\Delta^\theta\sum_{j=0}^{\infty}2^{\alpha_\infty\epsilon(1+\Delta)j}\|f_j\|_{L^{q(\cdot)}(\tau)}^{\epsilon(1+\Delta)}\sum_{\ell=0}^{l-2}2^{d\epsilon(1+\Delta)(\ell-j)/2}\right)^{\frac{1}{\epsilon(1+\Delta)}}\\
\leq& C\sup_{\Delta>0}\left(\Delta^\theta\sum_{j\in\mathbb{Z}}2^{\alpha_\infty\epsilon(1+\Delta)j}\|f_j\|_{L^{q(\cdot)}(\tau)}^{\epsilon(1+\Delta)}\sum_{k=-\infty}^{l-2}2^{d\epsilon(1+\Delta)(\ell-j)/2}\right)^{\frac{1}{\epsilon(1+\Delta)}}\\
=& C\sup_{\Delta>0}\left(\Delta^\theta\sum_{j\in\mathbb{Z}}2^{a(\cdot)\epsilon(1+\Delta)j}\|f_j\|_{L^{q(\cdot)}(\tau)}^{\epsilon(1+\Delta)}\right)^{\frac{1}{\epsilon(1+\Delta)}}\\
\leq& C\|f\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),\epsilon),\theta}(\tau)}.
\end{aligned}$$

For E_1 by using Minkowski's inequality

$$\begin{aligned}
E_1 \leq& \sup_{\Delta>0}\left(\Delta^\theta\sum_{\ell=-\infty}^{-1}2^{\ell\alpha(\cdot)\epsilon(1+\Delta)}\left(\sum_{j=\ell+1}^{\infty}2^{-jn}\|f_j\|_{L^{q(\cdot)}(\tau)}\|\chi_j\|_{L^{q'(\cdot)}(\tau)}.\chi_\ell(z)\right)^{\epsilon(1+\Delta)}\right)^{\frac{1}{\epsilon(1+\Delta)}}\\
\leq& \sup_{\Delta>0}\left(\Delta^\theta\sum_{\ell=-\infty}^{-1}2^{\ell\alpha(\cdot)\epsilon(1+\Delta)}\left(\sum_{j=\ell+1}^{-1}2^{-jn}\|f_j\|_{L^{q(\cdot)}(\tau)}\|\chi_j\|_{L^{q'(\cdot)}(\tau)}.\chi_\ell(z)\right)^{\epsilon(1+\Delta)}\right)^{\frac{1}{\epsilon(1+\Delta)}}\\
&+\sup_{\Delta>0}\left(\Delta^\theta\sum_{\ell=-\infty}^{-1}2^{\ell\alpha(\cdot)\epsilon(1+\Delta)}\left(\sum_{j=0}^{\infty}2^{-jn}\|f_j\|_{L^{q(\cdot)}(\tau)}\|\chi_j\|_{L^{q'(\cdot)}(\tau)}.\chi_\ell(z)\right)^{\epsilon(1+\Delta)}\right)^{\frac{1}{\epsilon(1+\Delta)}}\\
:=& D_1 + D_2.
\end{aligned}$$

The estimate for D_1 can be obtained by similar way to E_2 by replacing q_∞ with $q(0)$ and using the fact that $\frac{n}{q(0)} + \alpha(0) > 0$. For D_2 using Lemma 2.11 we have

$$2^{-jn}\|\chi_\ell\|_{L^{q(\cdot)}(\tau)}\|\chi_j\|_{L^{q'(\cdot)}(\tau)} \leq C2^{-jn}2^{\frac{kn}{q(0)}}2^{\frac{jn}{q'_\infty}} \leq C2^{\frac{\ell n}{q(0)}}2^{\frac{-jn}{q_\infty}} \tag{3.5}$$

$$D_2 \leq \sup_{\Delta>0}\left(\Delta^\theta\sum_{\ell=-\infty}^{-1}2^{\ell\alpha(0)\epsilon(1+\Delta)}\left(\sum_{j=0}^{\infty}\|\chi_\ell T(f_j)\|_{L^{q(\cdot)}(\tau)}\right)^{\epsilon(1+\Delta)}\right)^{\frac{1}{\epsilon(1+\Delta)}}$$

$$
\begin{aligned}
&\le C\sup_{\Delta>0}\left(\Delta^{\theta}\sum_{\ell=-\infty}^{-1}2^{\ell\alpha(0)\epsilon(1+\Delta)}\times\left(\sum_{j=0}^{\infty}2^{-jn}2^{\frac{\ell n}{q(0)}}2^{\frac{jn}{q'_\infty}}\|f_j\|_{L^{q(\cdot)}(\tau)}\right)^{\epsilon(1+\Delta)}\right)^{\frac{1}{\epsilon(1+\Delta)}}\\
&\le C\sup_{\Delta>0}\left(\Delta^{\theta}\sum_{\ell=-\infty}^{-1}2^{\ell\alpha(0)\epsilon(1+\Delta)}\times\left(\sum_{j=0}^{\infty}2^{\frac{\ell n}{q(0)}}2^{\frac{-jn}{q_\infty}}\|f_j\|_{L^{q(\cdot)}(\tau)}\right)^{\epsilon(1+\Delta)}\right)^{\frac{1}{\epsilon(1+\Delta)}}\\
&\le C\sup_{\Delta>0}\left(\Delta^{\theta}\sum_{\ell=-\infty}^{-1}2^{\ell(\alpha(0)+n)/q(0)\epsilon(1+\Delta)}\times\left(\sum_{j=0}^{\infty}2^{\frac{-jn}{q_\infty}}\|f\chi_j\|_{L^{q(\cdot)}(\tau)}\right)^{\epsilon(1+\Delta)}\right)^{\frac{1}{\epsilon(1+\Delta)}}\\
&\le C\sup_{\Delta>0}\left(\Delta^{\theta}\left(\sum_{j=0}^{\infty}2^{\frac{-jn}{q_\infty}}\|f\chi_j\|_{L^{q(\cdot)}(\tau)}\right)^{\epsilon(1+\Delta)}\right)^{\frac{1}{\epsilon(1+\Delta)}}\\
&\le C\sup_{\Delta>0}\left(\Delta^{\theta}\left(\sum_{j=0}^{\infty}2^{j\alpha_\infty}\|f\chi_j\|_{L^{q(\cdot)}(\tau)}2^{j(nq_\infty+\alpha_\infty)}\right)^{\epsilon(1+\Delta)}\right)^{\frac{1}{\epsilon(1+\Delta)}}.
\end{aligned}
$$

Now by applying Hölder's inequality and using the fact that $\frac{n}{q_\infty}+\alpha_\infty>0$ we have

$$
\begin{aligned}
D_2\le& C\sup_{\Delta>0}\left(\Delta^{\theta}\left(\sum_{j=0}^{\infty}2^{j\alpha_\infty\epsilon(1+\Delta)}\|f_j\|^{\epsilon(1+\Delta)}_{L^{q(\cdot)}(\tau)}\right)^{\epsilon(1+\Delta)}\right.\\
&\qquad\left.\times\left(\sum_{j=0}^{\infty}2^{j(nq_\infty+\alpha_\infty)\epsilon(1+\Delta)}\right)^{\frac{\epsilon(1+\Delta)}{(\epsilon(1+\Delta))'}}\right)^{\frac{1}{\epsilon(1+\Delta)}}\\
\le& C\sup_{\Delta>0}\left(\Delta^{\theta}\left(\sum_{l\in\mathbb{Z}}2^{\alpha_\infty j\epsilon(1+\Delta)}\|f_j\|^{\epsilon(1+\Delta)}_{L^{q(\cdot)}(\tau)}\right)\right)^{\frac{1}{\epsilon(1+\Delta)}}\\
\le& C\|f\|_{\dot{K}^{\alpha(\cdot),\epsilon),\theta}_{q(\cdot)}(\tau)},
\end{aligned}
$$

which completes our desired results.

□

4. Conclusions

In this paper, we defined the idea of grand variable weighted Herz spaces. We proved the boundedness of Hardy operators on grand variable weighted Herz spaces by using the properties of exponents. These results hold for weighted Herz spaces with variable exponent.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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