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http://www.aimspress.com/journal/era

ERA, 31(7): 4034–4056.
DOI: 10.3934/era.2023205
Received: 26 March 2023
Revised: 07 May 2023
Accepted: 14 May 2023
Published: 24 May 2023

Research article

Barycentric rational interpolation method for solving time-dependent fractional convection-diffusion equation

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Abstract: The time-dependent fractional convection-diffusion (TFCD) equation is solved by the barycentric rational interpolation method (BRIM). Since the fractional derivative is the nonlocal operator, we develop a spectral method to solve the TFCD equation to get the coefficient matrix as a full matrix. First, the fractional derivative of the TFCD equation is changed to a nonsingular integral from the singular kernel to a density function. Second, efficient quadrature of the new Gauss formula are constructed to simply compute it. Third, matrix equation of discrete the TFCD equation is obtained by the unknown function replaced by a barycentric rational interpolation basis function. Then, the convergence rate of BRIM is proved. Finally, a numerical example is given to illustrate our result.

Keywords: barycentric rational interpolation; collocation method; time-dependent fractional convection-diffusion equation; fractional derivative

1. Introduction

In this paper, we consider the time-dependent fractional convection-diffusion (TFCD) equation

$$\begin{cases} C_s^\alpha \phi(\mathbf{t}, s) - \triangle\phi(\mathbf{t}, s) + \nabla\phi(\mathbf{t}, s) = f(\mathbf{t}, s) & (\mathbf{t}, s) \in \Omega \times [0, T], \\ \phi(\mathbf{t}, 0) = \varphi_0(\mathbf{t}), \dfrac{\partial\phi(\mathbf{t}, 0)}{\partial s} = \varphi_1(\mathbf{t}), & \mathbf{t} \in \Omega, \\ \phi(\mathbf{t}, s)|_\Gamma = g(\mathbf{t}, s), & s \in [0, T], \end{cases} \tag{1.1}$$

where $1 < \alpha < 2$ and Ω are bounded domains in $\mathbb{R}^n$ with $n = 1, 2$ and $\Omega = [a, b]$ or $\Omega = [a, b] \times [c, d]$, Γ is the boundary of Ω. $f(\mathbf{t}, s), \varphi_0(\mathbf{t}), \varphi_1(\mathbf{t}), g(\mathbf{t}, s)$ are given functions and

$$\triangle\phi(\mathbf{t}, s) = \frac{\partial^2\phi(\mathbf{t}, s)}{\partial t_1^2} + \cdots + \frac{\partial^2\phi(\mathbf{t}, s)}{\partial t_n^2}, \nabla\phi(\mathbf{t}, s) = \frac{\partial\phi(\mathbf{t}, s)}{\partial t_1} + \cdots + \frac{\partial\phi(\mathbf{t}, s)}{\partial t_n} \tag{1.2}$$

The fractional derivative $C_s^\alpha = \frac{\partial^\alpha \phi(\mathbf{t},s)}{\partial \tau^\alpha}$ denotes the Caputo fractional derivative.

The Caputo fractional derivative of time is defined as

$$C_s^\alpha \phi(\mathbf{t}, s) = \begin{cases} \dfrac{1}{\Gamma(\xi-\alpha)} \displaystyle\int_0^s \dfrac{\partial^\xi \phi(\mathbf{t},\tau)}{\partial \tau^\xi} \dfrac{d\tau}{(s-\tau)^{\alpha+1-\xi}}, & m-1<\xi<m, \\ \dfrac{\partial^\xi \phi(\mathbf{t},\tau)}{\partial \tau^\xi}, & \xi = m, \end{cases} \tag{1.3}$$

and $\Gamma(\alpha)$ is the Γ function. The time fractional convection-diffusion equation has been widely applied in the modeling of the anomalous diffusive processes and in the description of viscoelastic damping materials.

In [1], a class of time fractional reaction diffusion equations with variable coefficients and the nonhomogeneous Neumann problem was solved by a compact finite difference method. It was proven that the method was unconditionally stable for the general case of variable coefficients, and the optimal error estimate for the numerical solution under the discrete L_2 norm was also given. In [2], by using Legendre spectral squares to discretize spatial variables, a high order numerical scheme for solving nonlinear time fractional reaction diffusion equations was proposed. Then, a priori estimate, existence, and uniqueness of the numerical solution were given, and the unconditional stability and convergence was proven. In [3] a fast and accurate numerical method for fractional reaction diffusion equations in unbounded domains using Fourier spectral method was constructed. In [4], an immersed finite element (IFE) method for solving time fractional diffusion equations with discontinuous coefficients was proposed. The singularity of the Caputo fractional derivative is approximated by the non-uniform L_1 scheme. In [5], a numerical method for diffusion problems with fractional derivatives in a bilateral Riemannian Liouville space was proposed. Under appropriate constraints, the monotonicity, positive retention, and linear stability of the method were proven. In [6], a locally discontinuous Galerkin and finite difference method for solving multiple variable order time fractional diffusion equations with variable order fractional derivatives was proposed, which proven that the scheme was unconditionally stable. In [7], a finite difference method for solving time fractional wave equations (TFWE) was proposed. For $\alpha \in (1, 2)$, the proposed difference scheme was of a second order accuracy in space and time, and the stability of the H-1-norm of the method was given. In [8], an hp discontinuous Galerkin method for solving nonlinear fractional differential equations with Caputo type fractional derivatives was proposed. This method converts fractional differential equations into either nonlinear Volterra or Fredholm integral equations, and then uses the hp discontinuous Galerkin method to solve the equivalent integral equations. Time-fractional diffusion equation [9] and nonlinear Caputo fractional differential equation [10] were studied by the finite difference scheme and optimal adaptive grid method.

The above methods such as the finite difference method, the Legendre spectral method, the Fourier spectral method, the finite element, and the discontinuous Galerkin method had been used to solve fractional partial equation with the time direction and space direction solved separatively in different directions. Different from the above methods, we construct the barycentric rational interpolation method (BRIM) to solve the time-dependent fractional convection-diffusion (TFCD) equation with time direction and space direction at the same time. For the barycentric interpolation method (BIM), there are BRIM and the barycentric Lagrange interpolation method (BLIM) which can be used to avoid the Runge phenomenon. In the recent years, linear rational interpolation (LRI) was proposed by

Floater [14–16] and the error of linear rational interpolation [11–13] was also proven. BIM has been developed by Wang et al. [17] and the algorithm of BIM has been used to linear/non-linear problems [18, 19]. In recent research, the Volterra integro-differential equation (VIDE) [20], heat equation (HE) [21], biharmonic equation (BE) [22, 23], telegraph equation (TE) [24], generalized Poisson equations [26], semi-infinite domain problems [27], fractional reaction-diffusion equation [28], and KPP equation [29] have been studied by linear BRIM and their convergence rate are also proven.

In this paper, BRIM has been used to solve the TFCD equation. As the fractional derivative is the nonlocal operator, spectral methods are developed to solve the TFCD equation and the coefficient matrix is a full matrix. The fractional derivative of the TFCD equation is changed to nonsingular integral by integral with order of density function plus one. The new Gauss formula is constructed to compute it simply and the matrix equation of discrete the TFCD equation is obtained by the unknown function replaced by the barycentric rational interpolation basis function. Then, the convergence rate of BRIM is proven.

2. Calculation of time fractional derivative

By the definition of (1.3), there are certain kinds of singularities in (1.1). Solving the TDFC equation is needed to efficiently calculate the Caputo fractional derivative. There are some methods to overcome the difficulty of singularity, we adopt the fractional integration as follow:

$$\begin{aligned} & C_s^{\alpha}\phi(\mathbf{t}, s) \\ & = \frac{1}{\Gamma(\xi-\alpha)} \int_0^s \frac{\partial^{\xi}\phi(\mathbf{t},\tau)}{\partial\tau^{\xi}} \frac{d\tau}{(s-\tau)^{\alpha+1-\xi}} \\ & = \frac{1}{(\xi-\alpha)\Gamma(\xi-\alpha)} \left[\frac{\partial^{\xi}\phi(\mathbf{t},0)}{\partial s^{\xi}} s^{\xi-\alpha} + \int_0^s \frac{\partial^{\xi+1}\phi(\mathbf{t},\tau)}{\partial\tau^{\xi+1}} \frac{d\tau}{(s-\tau)^{\alpha-\xi}} \right] \\ & = \Gamma_{\alpha}^{\xi} \left[\frac{\partial^{\xi}\phi(\mathbf{t},0)}{\partial s^{\xi}} s^{\xi-\alpha} + \int_0^s \frac{\partial^{\xi+1}\phi(\mathbf{t},\tau)}{\partial\tau^{\xi+1}} \frac{d\tau}{(s-\tau)^{\alpha-\xi}} \right], \end{aligned} \tag{2.1}$$

where $\Gamma_{\alpha}^{\xi} = \frac{1}{(\xi-\alpha)\Gamma(\xi-\alpha)}$.

Combining Eqs (2.1) and (1.1), we have

$$\Gamma_{\alpha}^{\xi} \left[\frac{\partial^{\xi}\phi(\mathbf{t},0)}{\partial s^{\xi}} s^{\xi-\alpha} + \int_0^s \frac{\partial^{\xi+1}\phi(\mathbf{t},\tau)}{\partial\tau^{\xi+1}} \frac{d\tau}{(s-\tau)^{\alpha-\xi}} \right] - \triangle\phi(\mathbf{t}, s) + \nabla\phi(\mathbf{t}, s) = f(\mathbf{t}, s) \tag{2.2}$$

The discrete formula of TFCD equation is obtained as

$$\phi(\mathbf{t}, s) = \sum_{j=1}^{m} R_j(\mathbf{t})\phi_j(s) \tag{2.3}$$

where

$$\phi(\mathbf{t}_i, s) = \phi_i(s), i = 1, 2, \cdots, m$$

and

$$R_j(\mathbf{t}) = \frac{\dfrac{\lambda_j}{\mathbf{t}-\mathbf{t}_j}}{\displaystyle\sum_{k=1}^{n} \frac{\lambda_k}{\mathbf{t}-\mathbf{t}_k}} \tag{2.4}$$

is the basis function, see [20]. Taking (2.3) into (2.2), we get

$$\begin{aligned}&\Gamma_{\alpha}^{\xi}\left[\frac{\partial^{\xi}\phi(\mathbf{t},0)}{\partial s^{\xi}}s^{\xi-\alpha}+\int_{0}^{s}\frac{\partial^{\xi+1}\phi(\mathbf{t},\tau)}{\partial\tau^{\xi+1}}\frac{d\tau}{(s-\tau)^{\alpha-\xi}}\right]-\left[\frac{\partial^{2}\phi(\mathbf{t},s)}{\partial\mathbf{t}^{2}}+\frac{\partial^{2}\phi(\mathbf{t},s)}{\partial s^{2}}\right]\\&+\left[\frac{\partial\phi(\mathbf{t},s)}{\partial\mathbf{t}}+\frac{\partial\phi(\mathbf{t},s)}{\partial s}\right]=f(\mathbf{t},s)\end{aligned}\tag{2.5}$$

Then we get

$$\begin{aligned}&\Gamma_{\alpha}^{\xi}\sum_{j=1}^{m}\left[R_{j}(\mathbf{t})\phi_{j}^{(\xi)}(0)s^{\xi-\alpha}+R_{j}(\mathbf{t})\int_{0}^{s}\frac{\phi^{(\xi+1)}(\tau)d\tau}{(s-\tau)^{\alpha-\xi}}\right]-\sum_{j=1}^{m}\left[R_{j}''(\mathbf{t})\phi_{j}(s)+R_{j}(\mathbf{t})\phi_{j}''(s)\right]\\&+\sum_{j=1}^{m}\left[R_{j}'(\mathbf{t})\phi_{j}(s)+R_{j}(\mathbf{t})\phi_{j}'(s)\right]=f(\mathbf{t},s),\end{aligned}\tag{2.6}$$

As for the discrete of t and s, we get

$$\phi_{j}(s)=\sum_{k=1}^{n}R_{k}(s)\phi_{ik}\tag{2.7}$$

where $\phi_i(s_j)=\phi(t_i,s_j)=\phi_{ij}, i=1,\cdots,m; j=1,\cdots,n$ and

$$R_{i}(s)=\frac{\dfrac{w_{i}}{s-s_{i}}}{\sum\limits_{k=1}^{m}\dfrac{w_{k}}{s-s_{k}}}\tag{2.8}$$

is the basis function.

Combining (2.6) and (2.7),

$$\begin{aligned}&\Gamma_{\alpha}^{\xi}\sum_{j=1}^{m}\sum_{k=1}^{n}\left[R_{j}(\mathbf{t})R_{k}^{(\xi)}(0)s^{\xi-\alpha}+R_{j}(\mathbf{t})\int_{0}^{s}\frac{R_{k}^{(\xi+1)}(\tau)d\tau}{(s-\tau)^{\alpha-\xi}}\right]\phi_{ik}-\sum_{j=1}^{m}\sum_{k=1}^{n}\left[R_{j}''(\mathbf{t})R_{k}(s)+R_{j}(\mathbf{t})R_{i}''(s)\right]\phi_{ik}\\&+\sum_{j=1}^{m}\sum_{k=1}^{n}\left[R_{j}'(\mathbf{t})R_{k}(s)+R_{j}(\mathbf{t})R_{k}'(s)\right]\phi_{ik}=f(\mathbf{t},s)\end{aligned}\tag{2.9}$$

where

$$R_{k}(\tau)=\frac{\dfrac{\lambda_{k}}{\tau-\tau_{k}}}{\sum\limits_{k=0}^{n}\dfrac{\lambda_{k}}{\tau-\tau_{k}}}$$

and

$$\begin{cases}R_{i}^{'}(\tau)=R_{i}(\tau)\left[-\dfrac{1}{\tau-\tau_{k}}+\dfrac{\sum\limits_{s=0}^{l}\dfrac{\lambda_{k}}{(\tau-\tau_{k})^{2}}}{\sum\limits_{s=0}^{l}\dfrac{\lambda_{k}}{\tau-\tau_{k}}}\right],\\ \quad\vdots\\ R_{i}^{(\xi+1)}(\tau)=[R_{i}^{(\xi)}(\tau)]^{'},\xi\in\mathbf{N}^{+}.\end{cases}$$

The term of (2.9) can be written as

$$\int_0^s \frac{R_j^{(\xi+1)}(\tau)d\tau}{(s-\tau)^{\alpha-\xi}} = Q_j^\alpha(s), \tag{2.10}$$

The integral (2.9) is calculated by

$$Q_j^\alpha(s) = \int_0^s \frac{R_j^{(\xi+1)}(\tau)d\tau}{(s-\tau)^{\alpha-\xi}} := \sum_{i=1}^{g} R_i^{(\xi+1)}(\tau_i^{\theta,\alpha})G_i^{\theta,\alpha}, \tag{2.11}$$

where $G_i^{\theta,\alpha}$ is Gauss weight and $\tau_i^{\theta,\alpha}$ is Gauss points with weights $(s-\tau)^{\xi-\alpha}$, see reference [26].

3. Matrix equation of TFCD equation

3.1. Matrix equation of (1+1) dimensional TFCD equation

For the (1+1) dimensional TFCD equation with $\Omega^1 = [a, b]$, (2.9) can be written as

$$\begin{aligned}
&\Gamma_\alpha^\xi \sum_{j_1=1}^{m_1}\sum_{k=1}^{n}\left[R_{j_1}(t_1)R_k^{(\xi)}(0)s^{\xi-\alpha} + R_{j_1}(t_1)\int_0^s \frac{R_k^{(\xi+1)}(\tau)d\tau}{(s-\tau)^{\alpha-\xi}}\right]\phi_{ik}\\
&-\sum_{j_1=1}^{m_1}\sum_{k=1}^{n}\left[R''_{j_1}(t_1)R_k(s) + R_{j_1}(t_1)R''_i(s)\right]\phi_{ik}\\
&+\sum_{j_1=1}^{m_1}\sum_{k=1}^{n}\left[R'_{j_1}(t_1)R_k(s) + R_{j_1}(t_1)R'_k(s)\right]\phi_{ik} = f(t_1, s)
\end{aligned} \tag{3.1}$$

Taking $a = t_{11} < t_{12} < \cdots < t_{1m_1} = b, 0 = s_1 < s_2 < \cdots < s_n = T$ with $h_t = (b-a)/m_1, h_s = T/n$ as either a uniform partition or uninform as a Chebychev point , $(t_{1i}, s_l), 1i = 1, 2, \cdots, m_1, l = 1, 2, \cdots, n$, we get

$$\begin{aligned}
&\Gamma_\alpha^\xi \sum_{j_1=1}^{m_1}\sum_{k=1}^{n}\left[R_{j_1}(t_{1i})R_k^{(\xi)}(0)s_l^{\xi-\alpha} + R_{j_1}(t_{1i})\int_0^{s_l} \frac{R_k^{(\xi+1)}(\tau)d\tau}{(s_l-\tau)^{\alpha-\xi}}\right]\phi_{ik}\\
&-\sum_{j_1=1}^{m_1}\sum_{k=1}^{n}\left[R''_{j_1}(t_{1i})R_k(s_l) + R_{j_1}(t_{1i})R''_i(s_l)\right]\phi_{ik}\\
&+\sum_{j_1=1}^{m_1}\sum_{k=1}^{n}\left[R'_{j_1}(t_{1i})R_k(s_l) + R_{j_1}(t_{1i})R'_k(s_l)\right]\phi_{ik} = f(t_{1i}, s_l)
\end{aligned} \tag{3.2}$$

By introducing the notation, $R_{j_1}(t_{1i}) = \delta_{j_1 i}, R_k(s_l) = \delta_{kl}, R'_{j_1}(t_{1i}) = R_{ij_1}^{(1,0)}, R'_k(s_l) = R_{ij}^{(0,1)}, R''_{j_1}(t_{1i}) = R_{ij_1}^{(2,0)}, R''_k(s_l) = R_{kl}^{(0,2)}$ where $R_{il}^{(0,2)}$ is the second order of the barycentric matrix.

$$\begin{aligned}
&\Gamma_\alpha^\xi \sum_{j_1=1}^{m_1}\sum_{k=1}^{n}\left[\delta_{ji}R_k^{(\xi)}(0)s_l^{\xi-\alpha} + \delta_{j_1 i}\int_0^{s_l} \frac{R_k^{(\xi+1)}(\tau)d\tau}{(s_l-\tau)^{\alpha-\xi}}\right]\phi_{ik} - \sum_{j_1=1}^{m_1}\sum_{k=1}^{n}\left[R_{ij}^{(2,0)}\delta_{kl} + \delta_{j_1 i}R_{kl}^{(0,2)}\right]\phi_{ik}\\
&+\sum_{j_1=1}^{m_1}\sum_{k=1}^{n}\left[R_{ij_1}^{(1,0)}\delta_{kl} + \delta_{j_1 i}R_{kl}^{(0,1)}\right]\phi_{ik} = f(t_{1i}, s_l)
\end{aligned} \tag{3.3}$$

by taking (2.11),

$$Q^{\alpha}_{j_1 l} = Q^{\alpha}_{j}(s_l) = \int_0^{s_l} \frac{R^{(\xi+1)}_{j_1}(\tau) d\tau}{(s_l - \tau)^{\alpha-\xi}} \tag{3.4}$$

then we get

$$\begin{aligned} &\Gamma^{\xi}_{\alpha} \sum_{j_1=1}^{m_1} \sum_{k=1}^{n} \left[\delta_{j_1 i} R^{(\xi)}_{k}(0) s_l^{\xi-\alpha} + \delta_{j_1 i} Q^{\alpha}_{kl} \right] \phi_{ik} \\ &- \sum_{j_1=1}^{m_1} \sum_{k=1}^{n} \left[R^{(2,0)}_{i j_1} \delta_{kl} + \delta_{j_1 i} R^{(0,2)}_{kl} - R^{(1,0)}_{i j_1} \delta_{kl} - \delta_{j_1 i} R^{(0,1)}_{kl} \right] \phi_{ik} = f(t_{1i}, s_l). \end{aligned} \tag{3.5}$$

Systems of (3.5) can be written as

$$\begin{aligned} &\Gamma^{\xi}_{\alpha} \left[diag(s^{\xi-\alpha}) M^{(\xi 0)}_{1} \otimes I_n + I_{m_1} \otimes Q^{\alpha_2} \right] \begin{bmatrix} \phi_{11} \\ \vdots \\ \phi_{1n} \\ \phi_{m_1 1} \\ \vdots \\ \phi_{m_1 n} \end{bmatrix} \\ &- \left[M^{(2,0)} \otimes I_n + I_{m_1} \otimes M^{(0,2)} - M^{(1,0)} \otimes I_n - I_{m_1} \otimes M^{(0,1)} \right] \begin{bmatrix} \phi_{11} \\ \vdots \\ \phi_{1n} \\ \phi_{m_1 1} \\ \vdots \\ \phi_{m_1 n} \end{bmatrix} = \begin{bmatrix} f_{11} \\ \vdots \\ f_{1n} \\ f_{m_1 1} \\ \vdots \\ f_{m_1 n} \end{bmatrix}, \end{aligned} \tag{3.6}$$

where I_{m_1} and I_n are identity matrices, and $\otimes$ is Kronecker product.

Then, we get Eq (3.6) as

$$\left[\Gamma^{\xi}_{\alpha} \left(diag(s^{\xi-\alpha}) M^{(\xi 0)}_{1} \otimes I_n + I_{m_1} \otimes Q^{\alpha_2} \right) - \left(M^{(2,0)} \otimes I_n + I_{m_1} \otimes M^{(0,2)} - M^{(1,0)} \otimes I_n - I_{m_1} \otimes M^{(0,1)} \right) \right] \Phi = F \tag{3.7}$$

and

$$M\Phi = F, \tag{3.8}$$

with $M = \Gamma^{\xi}_{\alpha} \left(diag(s^{\xi-\alpha}) M^{(\xi 0)}_{1} \otimes I_n + I_{m_1} \otimes Q^{\alpha_2} \right) - \left(M^{(2,0)} \otimes I_n + I_{m_1} \otimes M^{(0,2)} - M^{(1,0)} \otimes I_n - I_{m_1} \otimes M^{(0,1)} \right)$ and $\Phi = [\phi_{11} \ldots \phi_{1n} \ldots \phi_{m_1 1} \ldots \phi_{m_1 n}]^T$, $F = [f_{11} \ldots f_{1n} \ldots f_{m_1 1} \ldots f_{m_1 n}]^T$.

3.2. Matrix equation of (2+1) dimensional TFCD equation

For the (2+1) dimensional TFCD equation with $\Omega^2 = [a,b]\times[c,d]$, then we have

$$\begin{aligned}
&\Gamma_\alpha^\xi \sum_{j_1=1}^{m_1}\sum_{j_2=1}^{m_2}\sum_{k=1}^{n}\left[R_{j_1}(t_1)R_{j_2}(t_2)R_k^{(\xi)}(0)s^{\xi-\alpha} + R_{j_1}(t_1)R_{j_2}(t_2)\int_0^s \frac{R_k^{(\xi+1)}(\tau)d\tau}{(s-\tau)^{\alpha-\xi}}\right]\phi_{ijk}\\
&-\sum_{j_1=1}^{m_1}\sum_{j_2=1}^{m_2}\sum_{k=1}^{n}\left[R''_{j_1}(t_1)R_{j_2}(t_2)R_k(s) + R_{j_1}(t_1)R''_{j_2}(t_2)R_i(s) + R_{j_1}(t_1)R_{j_2}(t_2)R''_i(s)\right]\phi_{ijk}\\
&+\sum_{j_1=1}^{m_1}\sum_{j_2=1}^{m_2}\sum_{k=1}^{n}\left[R'_{j_1}(t_1)R_{j_2}(t_2)R_k(s) + R_{j_1}(t_1)R'_{j_2}(t_2)R_i(s) + R_{j_1}(t_1)R_{j_2}(t_2)R'_i(s)\right]\phi_{ijk} = f(t_1,t_2,s)
\end{aligned} \tag{3.9}$$

By $a = t_{11} < t_{12} < \cdots < t_{1m_1} = b, c = t_{21} < t_{22} < \cdots < t_{2m_1} = d, 0 = s_1 < s_2 < \cdots < s_n = T$ with $h_{t_1} = (b-a)/m_1, h_{t_2} = (d-c)/m_2, h_s = T/n$ or uninform as a Chebychev point , $(t_{1i}, t_{2i}, s_l), 1i = 1,2,\cdots,m_1, 2i = 1,2,\cdots,m_2, l = 1,2,\cdots,n$, we get

$$\begin{aligned}
&\Gamma_\alpha^\xi \sum_{j_1=1}^{m_1}\sum_{j_2=1}^{m_2}\sum_{k=1}^{n}\left[R_{j_1}(t_{1i})R_{j_2}(t_{2j})R_k^{(\xi)}(0)s^{\xi-\alpha} + R_{j_1}(t_{1i})R_{j_2}(t_{2j})\int_0^s \frac{R_k^{(\xi+1)}(\tau)d\tau}{(s-\tau)^{\alpha-\xi}}\right]\phi_{ijk}\\
&-\sum_{j_1=1}^{m_1}\sum_{j_2=1}^{m_2}\sum_{k=1}^{n}\left[R''_{j_1}(t_{1i})R_{j_2}(t_{2j})R_k(s_l) + R_{j_1}(t_{1i})R''_{j_2}(t_{2j})R_i(s_l) + R_{j_1}(t_{1i})R_{j_2}(t_{2j})R''_i(s_l)\right]\phi_{ijk}\\
&+\sum_{j_1=1}^{m_1}\sum_{j_2=1}^{m_2}\sum_{k=1}^{n}\left[R'_{j_1}(t_{1i})R_{j_2}(t_{2j})R_k(s) + R_{j_1}(t_{1i})R'_{j_2}(t_{2j})R_i(s_l) + R_{j_1}(t_{1i})R_{j_2}(t_{2j})R'_i(s_l)\right]\phi_{ijk} = f(t_{1i},t_{2j},s_l)
\end{aligned} \tag{3.10}$$

By introducing the notation, $R_{j_1}(t_{1i}) = \delta_{j_1 i}, R_{j_2}(t_{1j}) = \delta_{j_2 j}, R_k(s_l) = \delta_{kl}, R'_{j_1}(t_{1i}) = R_{ij_1}^{(1,0,0)}, R'_{j_2}(t_{1j}) = R_{ij_1}^{(0,1,0)}, R'_k(s_l) = R_{ij}^{(0,0,1)}, R''_{j_1}(t_{1i}) = R_{ij_1}^{(2,0,0)}, R''_{j_2}(t_{1j}) = R_{ij_1}^{(0,2,0)}, R''_k(s_l) = R_{ij}^{(0,0,2)}$, we get

$$\begin{aligned}
&\Gamma_\alpha^\xi \sum_{j_1=1}^{m_1}\sum_{j_2=1}^{m_2}\sum_{k=1}^{n}\left[\delta_{j_1 i}\delta_{j_2 j}R_k^{(\xi)}(0)s^{\xi-\alpha} + \delta_{j_1 i}\delta_{j_2 j}\int_0^s \frac{R_k^{(\xi+1)}(\tau)d\tau}{(s-\tau)^{\alpha-\xi}}\right]\phi_{ijk}\\
&-\sum_{j_1=1}^{m_1}\sum_{j_2=1}^{m_2}\sum_{k=1}^{n}\left[R_{ij_1}^{(2,0,0)}\delta_{j_2 j}\delta_{kl} + \delta_{j_1 i}R_{ij_1}^{(0,2,0)}\delta_{kl} + \delta_{j_1 i}\delta_{j_2 j}R_{ij}^{(0,0,2)}\right]\phi_{ijk}\\
&+\sum_{j_1=1}^{m_1}\sum_{j_2=1}^{m_2}\sum_{k=1}^{n}\left[R_{ij_1}^{(1,0,0)}\delta_{j_2 j}\delta_{kl} + \delta_{j_1 i}R_{ij_1}^{(0,1,0)}\delta_{kl} + \delta_{j_1 i}\delta_{j_2 j}R_{ij}^{(0,0,1)}\right]\phi_{ijk} = f(t_{1i},t_{2j},s_l)
\end{aligned} \tag{3.11}$$

Then, Eq (3.6) can be written as

$$\begin{aligned}
&\Gamma_\alpha^\xi\left(diag(s^{\xi-\alpha})M_1^{(\xi 0)}\otimes I_{m_1}\otimes I_{m_2} + I_{m_1}\otimes I_{m_2}\otimes Q^{\alpha_2}\right)\Phi\\
&-\left(M^{(2,0,0)}\otimes I_{m_2}\otimes I_n + I_{m_1}\otimes M^{(0,2,0)}\otimes I_n + I_{m_1}\otimes I_{m_2}\otimes M^{(0,0,2)}\right)\Phi\\
&+\left(M^{(1,0,0)}\otimes I_{m_2}\otimes I_n + I_{m_1}\otimes M^{(0,1,0)}\otimes I_n + I_{m_1}\otimes I_{m_2}\otimes M^{(0,0,1)}\right)\Phi = F
\end{aligned} \tag{3.12}$$

and

$$M\Phi = F, \tag{3.13}$$

with $M = \Gamma_\alpha^\xi\left(diag(s^{\xi-\alpha})M_1^{(\xi 0)}\otimes I_{m_1}\otimes I_{m_2} + I_{m_1}\otimes I_{m_2}\otimes Q^{\alpha_2}\right) - \left(M^{(2,0,0)}\otimes I_{m_2}\otimes I_n + I_{m_1}\otimes M^{(0,2,0)}\otimes I_n + I_{m_1}\otimes I_{m_2}\otimes M^{(0,0,2)}\right) +$

$\left(M^{(1,0,0)} \otimes I_{m_2} \otimes I_n + I_{m_1} \otimes M^{(0,1,0)} \otimes I_n + I_{m_1} \otimes I_{m_2} \otimes M^{(0,0,1)}\right)$ and $\Phi = [\phi_{111}\phi_{112}\cdots\phi_{11n}, \phi_{121}\phi_{122}\cdots\phi_{12n}, \ldots, \phi_{m_1m_21}\phi_{m_1m_22}\cdots\phi_{m_1m_2n}]^T$, $F = [f_{111}f_{112}\cdots f_{11n}, f_{121}f_{122}\cdots f_{12n}, \ldots, f_{m_1m_21}f_{m_1m_22}\cdots f_{m_1m_2n}]^T$.

The boundary condition can be solved by the substitution method, the additional method or the elimination method, see [17]. In the following, we adopt the substitution method and the additional method to add the boundary condition.

4. Convergence rate of TFCD equation

In this part, the error estimate of the TFCD equation is given with $r_n(s) = \sum_{i=1}^{n} r_i(s)\phi_i$ to replace $\phi(s)$, where $r_i(s)$ is defined as (2.8) and $\phi_i = \phi(s_i)$. We also define

$$e(s) := \phi(s) - r_n(s) = (s - s_i)\cdots(s - s_{i+d})\phi\left[s_i, s_{i+1}, \ldots, s_{i+d}, s\right], \tag{4.1}$$

see reference [20].

Then, we have

Lemma 1. *For $e(s)$ be defined by (4.1) and $\phi(s) \in C^{d+2}[a,b], d = 1, 2, \cdots$, there*

$$\left|e^{(k)}(s)\right| \le Ch^{d-k+1}, k = 0, 1, \cdots. \tag{4.2}$$

For the TFCD equation, the rational interpolation function of $\phi(t, s)$ is defined as $r_{mn}(t, s)$

$$r_{mn}(t, s) = \frac{\sum_{i=1}^{m+d_s}\sum_{j=1}^{n+d_t} \frac{w_{i,j}}{(s - s_i)\left(t - t_j\right)}\phi_{i,j}}{\sum_{i=1}^{m+d_s}\sum_{j=1}^{n+d_t} \frac{w_{i,j}}{(s - s_i)\left(t - t_j\right)}} \tag{4.3}$$

where

$$w_{i,j} = (-1)^{i-d_s+j-d_t} \sum_{k_1\in J_i} \prod_{h_1=k_1,h_1\neq j}^{k_1+d_s} \frac{1}{\left|s_i - s_{h_1}\right|} \sum_{k_2\in J_i} \prod_{h_2=k_2,h_2\neq j}^{k_2+d_t} \frac{1}{\left|t_j - t_{h_2}\right|}. \tag{4.4}$$

We define $e(t, s)$ be the error of $\phi(t, s)$ as

$$\begin{aligned} &e(t, s) := \phi(t, s) - r_{mn}(t, s) \\ &= (s - s_i)\cdots(s - s_{i+d_s})\,\phi\left[s_i, s_{i+1}, \ldots, s_{i+d_1}, s; t\right] \\ &+ \left(t - t_j\right)\cdots\left(t - t_{j+d_t}\right)\phi\left[s; t_j, t_{j+1}, \ldots, t_{j+d_2}, t\right] \\ &- (s - s_i)\cdots(s - s_{i+d_s})\left(t - t_j\right)\cdots\left(t - t_{j+d_t}\right)\phi\left[s_i, s_{i+1}, \ldots, s_{i+d_1}, s; t_j, t_{j+1}, \ldots, t_{j+d_2}, t\right]. \end{aligned} \tag{4.5}$$

With a similar analysis of Lemma 1, we have

Theorem 1. *For $e(t, s)$ defined as (4.5) and $\phi(t, s) \in C^{d_s+2}[a, b] \times C^{d_t+2}[0, T]$, then we have*

$$\left|e^{(k_1,k_2)}(s, t)\right| \le C(h_s^{d_s-k_1+1} + h_t^{d_t-k_2+1}), k_1, k_2 = 0, 1, \cdots. \tag{4.6}$$

Let $\phi(s_m, t_n)$ be the approximate function of $\phi(t, s)$ and L to be bounded operator, there holds

$$L\phi(t_m, s_n) = f(t_m, s_n) \tag{4.7}$$

and

$$\lim_{m,n\to\infty} L\phi(t_m, s_n) = \phi(t, s). \tag{4.8}$$

Then, we get

Theorem 2. *For $\phi(t_m, s_n) : L\phi(t_m, s_n) = \phi(t, s)$ and L defined as (4.7), there*

$$|\phi(t, s) - \phi(t_m, s_n)| \le C(h^{d_s-1} + \tau^{d_t-1}).$$

Proof. By the definition of (4.7), we have

$$\begin{aligned}
&L\phi(t, s) - L\phi(t_m, s_n)\\
&= C_s^\alpha \phi(t, s) - \triangle\phi(t, s) + \nabla\phi(t, s) - f(t, s)\\
&- \left[C_s^\alpha \phi(t_m, s_n) - \triangle\phi(t_m, s_n) + \nabla\phi(t_m, s_n) - f(t_m, s_n)\right]\\
&= C_s^\alpha \phi(t, s) - C_s^\alpha \phi(t_m, s_n) - \left[\triangle\phi(t, s) - \triangle\phi(s_m, t_n)\right]\\
&+ \left[\nabla\phi(t, s) - \nabla\phi(t_m, s_n))\right]\\
&-[f(t, s) - f(t_m, s_n)]\\
&:= E_1(t, s) + E_2(t, s) + E_3(t, s) + E_4(t, s)
\end{aligned} \tag{4.9}$$

here

$$E_1(t, s) = C_s^\alpha \phi(t, s) - C_s^\alpha \phi(t_m, s_n),$$

$$E_2(t, s) = \triangle\phi(t, s) - \triangle\phi(s_m, t_n),$$

$$E_3(t, s) = \nabla\phi(t, s) - \nabla\phi(t_m, s_n)),$$

$$E_4(t, s) = f(t, s) - f(t_m, s_n).$$

As for $E_1(t, s)$, we get

$$\begin{aligned}
&E_1(t, s) = C_s^\alpha \phi(t, s) - C_s^\alpha \phi(t_m, s_n)\\
&= \Gamma_\alpha^\xi \left[\frac{\partial^\xi \phi(0, s)}{\partial t^\xi} s^{\xi-\alpha} + \int_0^t \frac{\partial^{\xi+1}\phi(\tau, s)}{\partial \tau^{\xi+1}} \frac{d\tau}{(t-\tau)^{\alpha-\xi}}\right]\\
&-\Gamma_\alpha^\xi \left[\frac{\partial^\xi \phi(0, s_n)}{\partial t^\xi} s_n^{\xi-\alpha} + \int_0^{t_m} \frac{\partial^{\xi+1}\phi(\tau, s_n)}{\partial \tau^{\xi+1}} \frac{d\tau}{(t_m-\tau)^{\alpha-\xi}}\right]\\
&= \Gamma_\alpha^\xi \left[\frac{\partial^\xi \phi(0, s)}{\partial t^\xi} s^{\xi-\alpha} - \frac{\partial^\xi \phi(0, s_n)}{\partial t^\xi} s_n^{\xi-\alpha}\right]\\
&+\Gamma_\alpha^\xi \left[\int_0^t \frac{\partial^{\xi+1}\phi(\tau, s)}{\partial \tau^{\xi+1}} \frac{d\tau}{(t-\tau)^{\alpha-\xi}} - \int_0^{t_m} \frac{\partial^{\xi+1}\phi(\tau, s_n)}{\partial \tau^{\xi+1}} \frac{d\tau}{(t_m-\tau)^{\alpha-\xi}}\right]
\end{aligned} \tag{4.10}$$

and

$$
\begin{aligned}
&|E_1(t,s)| \\
&\le \left|\Gamma_\alpha^\xi\left[\frac{\partial^\xi\phi(0,s)}{\partial t^\xi}s^{\xi-\alpha}-\frac{\partial^\xi\phi(0,s_n)}{\partial t^\xi}s_n^{\xi-\alpha}\right]\right| \\
&+\left|\Gamma_\alpha^\xi\left[\int_0^t\frac{\partial^{\xi+1}\phi(\tau,s)}{\partial\tau^{\xi+1}}\frac{d\tau}{(t-\tau)^{\alpha-\xi}}-\int_0^{t_m}\frac{\partial^{\xi+1}\phi(\tau,s_n)}{\partial\tau^{\xi+1}}\frac{d\tau}{(t_m-\tau)^{\alpha-\xi}}\right]\right| \\
&\le |\Gamma_\alpha^\xi|\left|\frac{\partial^\xi\phi}{\partial t^\xi}(0,s)-\frac{\partial^\xi\phi}{\partial t^\xi}(0,s_n)\right|+|\Gamma_\alpha^\xi|\left|\frac{\partial^{\xi+1}\phi}{\partial t^{\xi+1}}(t,s)-\frac{\partial^{\xi+1}\phi}{\partial t^{\xi+1}}(t_m,s_n)\right| \\
&:= E_{11}(t,s)+E_{12}(t,s)
\end{aligned}
\tag{4.11}
$$

where

$$
\begin{aligned}
E_{11}(t,s) &= |\Gamma_\alpha^\xi|\left|\frac{\partial^\xi\phi}{\partial t^\xi}(0,s)-\frac{\partial^\xi\phi}{\partial t^\xi}(0,s_n)\right| \\
E_{12}(t,s) &= |\Gamma_\alpha^\xi|\left|\frac{\partial^{\xi+1}\phi}{\partial t^{\xi+1}}(t,s)-\frac{\partial^{\xi+1}\phi}{\partial t^{\xi+1}}(t_m,s_n)\right|
\end{aligned}
\tag{4.12}
$$

Now we estimate $E_{11}(t,s)$ and $E_{12}(t,s)$ part by part, for the second part we have

$$
\begin{aligned}
&E_{12}(t,s) = |\Gamma_\alpha^\xi|\left|\frac{\partial^{\xi+1}\phi}{\partial t^{\xi+1}}(t,s)-\frac{\partial^{\xi+1}\phi}{\partial t^{\xi+1}}(t_m,s_n)\right| \\
&= |\Gamma_\alpha^\xi|\left|\frac{\partial^{\xi+1}\phi}{\partial t^{\xi+1}}(t,s)-\frac{\partial^{\xi+1}\phi}{\partial t^{\xi+1}}(t_m,s)+\frac{\partial^{\xi+1}\phi}{\partial t^{\xi+1}}(t_m,s)-\frac{\partial^{\xi+1}\phi}{\partial t^{\xi+1}}(t_m,s_n)\right| \\
&\le |\Gamma_\alpha^\xi|\left|\frac{\partial^{\xi+1}\phi}{\partial t^{\xi+1}}(t,s)-\frac{\partial^{\xi+1}\phi}{\partial t^{\xi+1}}(t_m,s)\right|+|\Gamma_\alpha^\xi|\left|\frac{\partial^{\xi+1}\phi}{\partial t^{\xi+1}}(t_m,s)-\frac{\partial^{\xi+1}\phi}{\partial t^{\xi+1}}(t_m,s_n)\right| \\
&= |\Gamma_\alpha^\xi|\left|\frac{\displaystyle\sum_{i=1}^{m-d_s}(-1)^i\frac{\partial^{\xi+1}\phi}{\partial t^{\xi+1}}[s_i,s_{i+1},\ldots,s_{i+d_1},s_n,t]}{\displaystyle\sum_{i=1}^{m-d_s}\lambda_i(s)}\right| \\
&+|\Gamma_\alpha^\xi|\left|\frac{\displaystyle\sum_{j=1}^{n-d_t}(-1)^j\frac{\partial^{\xi+1}\phi}{\partial t^{\xi+1}}[t_j,t_{j+1},\ldots,t_{j+d_2},s_n,t_m]}{\displaystyle\sum_{j=1}^{n-d_t}\lambda_j(t)}\right| \\
&= |\Gamma_\alpha^\xi|\left|\frac{\partial^{\xi+1}e}{\partial t^{\xi+1}}(t_m,s)\right|+|\Gamma_\alpha^\xi|\left|\frac{\partial^{\xi+1}e}{\partial t^{\xi+1}}(t_m,s_n)\right|.
\end{aligned}
$$

then we have

$$|E_{12}(t,s)| \le \left|\frac{\partial^{\xi+1} e}{\partial t^{\xi+1}}(t_m,s)\right| + \left|\frac{\partial^{\xi+1} e}{\partial t^{\xi+1}}(t_m,s_n)\right| \le C(h^{d_s-1}+\tau^{d_t-1}). \tag{4.13}$$

For $E_{11}(t,s)$, we get

$$|E_{11}(t,s)| \le C(h^{d_s+1-\xi}+\tau^{d_t-1}). \tag{4.14}$$

Similarly as $E_2(t,s)$, for $E_3(t,s)$ we have

$$|E_3(t,s)| \le C(h^{d_s}+\tau^{d_t}). \tag{4.15}$$

Combining (4.9), (4.14), and (4.15) together, the proof of theorem 2 is completed.

5. Numerical examples

In this part, two examples are presented to test the theorem.

Example 1. *Consider the time-dependent fractional convection-diffusion equation*

$$\begin{cases} \dfrac{\partial^\alpha \phi(t,s)}{\partial s^\alpha} - \dfrac{\partial^2 \phi(t,s)}{\partial t^2} + \dfrac{\partial \phi(t,s)}{\partial t} = f(t,s) & (t,s) \in [0,1]\times[0,1], \\ \phi(t,0)=0, \dfrac{\partial \phi(t,0)}{\partial t} = \sin \pi t, & t \in [0,1], \\ \phi(t,s)|_\Gamma = g(t,s), & s \in [0,1], \end{cases} \tag{5.1}$$

with the analysis solution is

$$\phi(t,s) = (s+s^3)\sin(\pi t)$$

with the initial condition

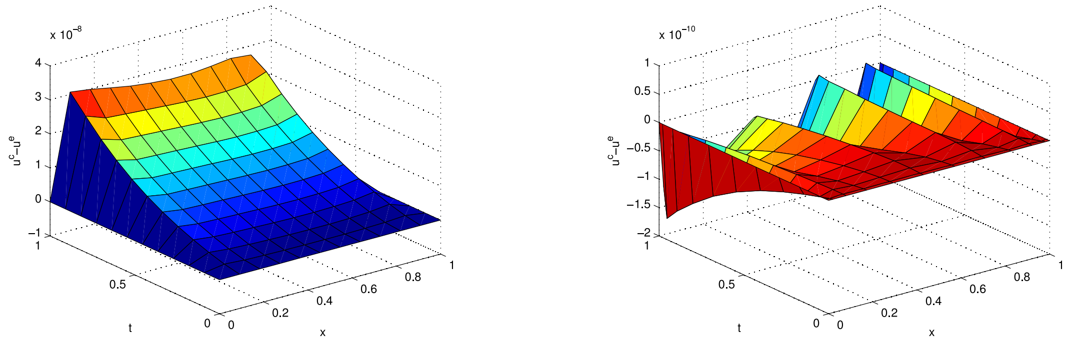
$$\phi(t,0)=0$$

and boundary condition

$$\phi(0,s)=\phi(1,s)=0$$

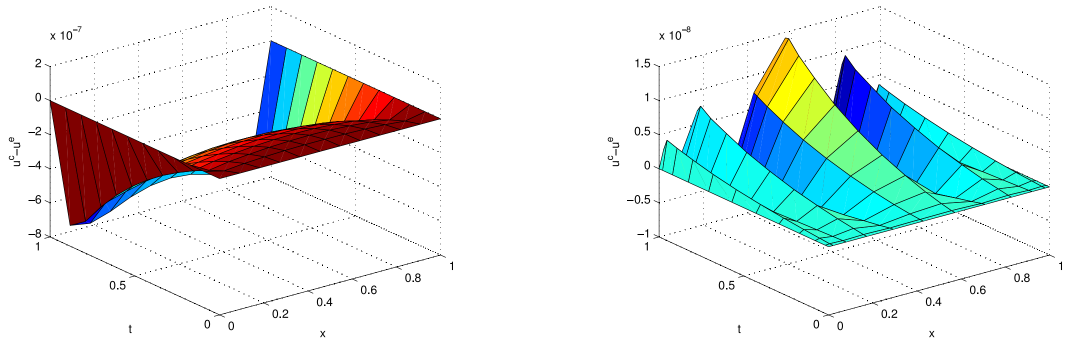
and

$$f(t,s) = \frac{6t^{3-\alpha}}{\Gamma(4-\alpha)}\sin(\pi t) + \pi^2(s+s^3)\sin(\pi t) + (s+s^3)\cos(\pi t)$$



(a) uniform (b) nonuniform

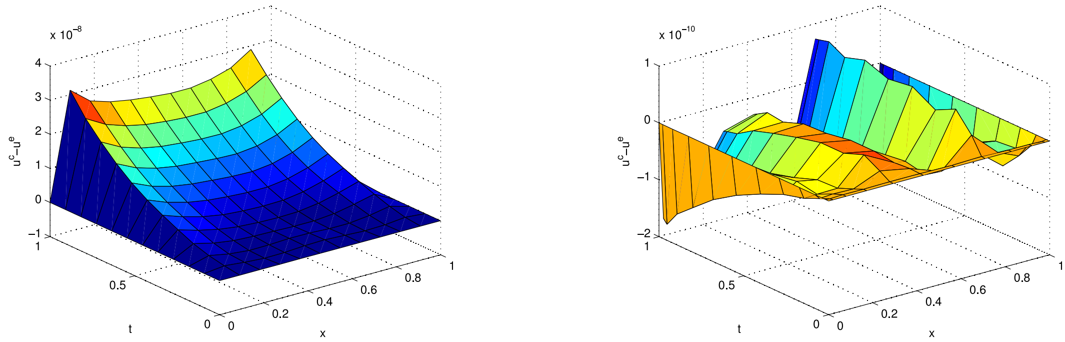
Figure 1. Errors of $m = n = 12$, $\Omega^1 = [0, 1], \alpha = 1.2$ in Example 1 (a) uniform; (b) nonuniform.



(a) uniform (b) nonuniform

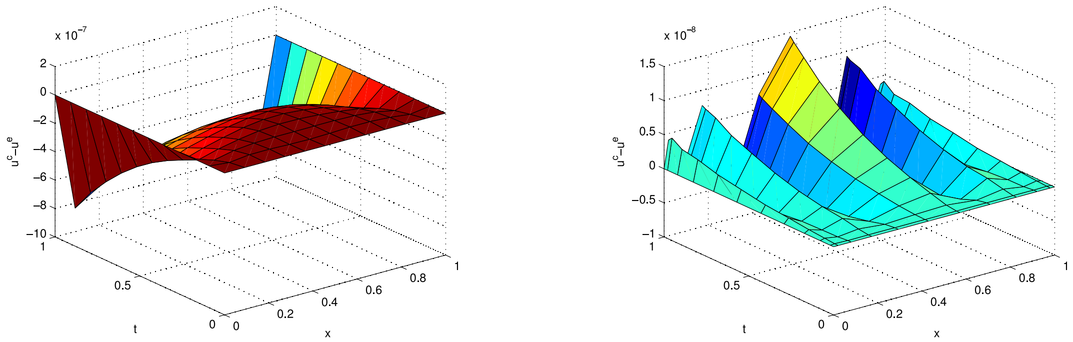
Figure 2. Errors of $m = n = 12, d_t = d_s = 8$, $\Omega^1 = [0, 1], \alpha = 1.2$ in Example 1 (a) uniform; (b) nonuniform.

In Figures 1 and 2, errors of $m = n = 12$, $\Omega^1 = [0, 1], \alpha = 1.2$ and $m = n = 12, d_t = d_s = 8$, $\Omega^1 = [0, 1], \alpha = 1.2$ in Example 1 with uniform and nonuniform partition for the TFCD equation by BRIM are presented, respectively. From the Figure, we know that the precision can reach to 10^{-8} for both the uniform and nonuniform partition.



(a) uniform (b) nonuniform

Figure 3. Errors of $m = n = 12$, $\Omega^1 = [0, 1], \alpha = 1.8$ in Example 1 (a) uniform; (b) nonuniform.



(a) uniform (b) nonuniform

Figure 4. Errors of $m = n = 12, d_t = d_s = 8, \Omega^1 = [0, 1], \alpha = 1.8$ in Example 1 (a) uniform; (b) nonuniform.

In Figures 3 and 4, errors of $m = n = 12, \Omega^1 = [0, 1], \alpha = 1.8$ and $m = n = 12, d_t = d_s = 8, \alpha = 1.8$, $\Omega^1 = [0, 1]$ in Example 1 with uniform and nonuniform partition for the TFCD equation by BRIM are presented, respectively. From the Figure, we know that the precision can reach to 10^{-8} for both uniform and nonuniform partition. For different value of α, BRIM can be used to solve the TFCD equation efficiently.

Table 1. Errors of TFCD equation for $\alpha_1 = 1.8, d_t = d_s = 5$.

	uniform	nonuniform	uniform	nonuniform
t	$(12, 12)d_t = d_s = 5$	$(12, 12)d_t = d_s = 5$	$(12, 12)$	$(12, 12)$
0.5	1.6077e-05	3.4641e-06	9.4012e-09	5.4710e-11
0.9	7.6161e-06	1.2065e-06	1.1950e-08	1.0220e-11
1	3.3826e-05	3.2595e-06	6.1614e-08	4.5688e-11
5	2.7710e-04	2.3571e-05	8.8436e-07	9.3036e-10
10	4.0780e-03	3.8953e-04	2.8067e-05	1.4820e-08
15	3.3288e-03	2.8728e-04	2.1309e-04	2.2781e-07

In Table 1, errors of the TFCD equation for $\alpha_1 = 1.8, d_t = d_s = 5$ with $t = 0.1, 0.9, 1, 5, 10, 15$ are presented under the uniform and nonuniform partition with BRIM and BLIM. As the time variable increases from 0.5 to 15, there is still high accuracy. For BRIM, we can choose the parameters d_t, d_s and m, n approximately to get high accuracy. Under the same partition of m, n, the accuracy of BLIM is higher than BRIM.

Table 2. Errors of different α under BRIM with $m = n = 10, d_t = 5, d_s = 5$.

	uniform	nonuniform
1.05	1.3605e-05	5.0592e-05
1.1	1.5511e-06	1.0653e-05
1.3	3.7907e-06	2.0445e-05
1.5	2.9437e-07	3.9908e-06
1.6	1.5585e-06	7.4171e-06
1.8	1.7836e-07	1.7089e-06
1.9	2.5754e-07	3.4347e-06
1.99	6.0471e-08	9.9797e-07

In Table 2, for BRIM, the errors of different $\alpha_1 = 1.05, 1.1, 1.3, 1.5, 1.6, 1.8, 1.9, 1.99$ under uniform with $m = n = 10, d_t = 5, d_s = 5$ are presented under the uniform and nonuniform partition. From the table, we know that for different α, BRIM has a high accuracy with decreased values for m and n.

In the following table, numerical results are presented to test our theorem.

From Tables 3 and 4, the error of BRIM under uniform for $\alpha = 1.8, d_s = 5$ with different d_t are given, and the convergence rate is $O(h^{d_t})$. From Table 4, with space variable uniform for $\alpha = 1.8, d_t = 5$, the convergence rate is $O(h^7)$, which we will investigate in future paper.

Table 3. Errors of BRIM under uniform for $\alpha = 1.8, d_s = 5$.

m, n	$d_t = 2$		$d_t = 3$		$d_t = 4$		$d_t = 5$	
8	1.0091e-03		1.0123e-03		1.0227e-03		1.0394e-03	
10	2.0466e-04	7.1497	2.0526e-04	7.1511	2.0654e-04	7.1692	2.0796e-04	7.2107
12	5.5556e-05	7.1521	5.7191e-05	7.0089	5.7426e-05	7.0204	5.7744e-05	7.0278
14	1.9062e-05	6.9393	2.1790e-05	6.2599	2.8246e-05	4.6029	3.3826e-05	3.4693

Table 4. Errors of BRIM under uniform for $\alpha = 1.8, d_t = 5$.

m, n	$d_s = 2$		$d_s = 3$		$d_s = 4$		$d_s = 5$	
8	1.4494e-02		4.3112e-03		2.0427e-03		1.0394e-03	
10	7.1283e-03	3.1802	1.4415e-03	4.9096	6.7844e-04	4.9395	2.0796e-04	7.2107
12	3.9852e-03	3.1894	6.0013e-04	4.8062	2.7092e-04	5.0349	5.7744e-05	7.0278
14	2.9746e-03	1.8973	1.4504e-03	-	6.3278e-04	-	3.3826e-05	3.4693

For Tables 5 and 6, the errors of Chebyshev partition for s and t are presented. For $\alpha = 1.8, d_t = 5$, the convergence rate is $O(h^{d_s})$ in Table 5, while in Table 6, the convergence rate is $O(h^{d_t})$, which agrees with our theorem.

Table 5. Errors of BRIM under Chebyshev partition with $\alpha = 1.8, d_t = 5$.

m, n	$d_s = 2$		$d_s = 3$		$d_s = 4$		$d_s = 5$	
8	1.9490e-02		4.4626e-03		7.1364e-04		1.0394e-03	
10	8.1224e-03	3.9224	5.4856e-04	9.3939	4.5776e-04	1.9899	2.0796e-04	7.2107
12	3.9100e-03	4.0098	2.0389e-04	5.4284	1.0292e-04	8.1858	5.7744e-05	7.0278
14	2.1533e-03	3.8697	6.4616e-05	7.4546	2.0776e-05	10.380	3.3826e-05	3.4693

Table 6. Errors of BRIM under Chebyshev partition $\alpha = 1.8, d_s = 5$.

m, n	$d_t = 2$		$d_t = 3$		$d_t = 4$		$d_t = 5$	
8	7.4953e-05		7.4985e-05		7.4823e-05		7.4663e-05	
10	4.4669e-05	2.3195	4.4515e-05	2.3369	4.4571e-05	2.3216	4.4558e-05	2.3133
12	1.3867e-05	6.4158	1.4149e-05	6.2868	1.4072e-05	6.3235	1.4030e-05	6.3383
14	4.0908e-06	7.9196	3.3018e-06	9.4397	3.4105e-06	9.1944	3.2595e-06	9.4687

In the following table, $\alpha = 1.2$ is chosen to present numerical results. From Tables 7 and 8, the error of BRIM under uniform for $d_t = 5$ with different d_s is given, and the convergence rate is $O(h^7)$. From Table 7, with space variable $s, d_s = 5$, the convergence rate is $O(h^{d_t})$, which agrees with our theorem.

Table 7. Errors of BRIM under uniform partition for $\alpha = 1.2, d_t = 5$.

m, n	$d_s = 2$		$d_s = 3$		$d_s = 4$	
8	9.3201e-04		9.4352e-04		9.4689e-04	
10	1.9149e-04	7.0919	1.8804e-04	7.2283	1.8804e-04	7.2443
12	4.9055e-05	7.4696	5.2968e-05	6.9491	5.1073e-05	7.1490
14	2.2723e-05	4.9923	2.0827e-05	6.0553	2.1242e-05	5.6910

Table 8. Errors of BRIM under uniform partition for $\alpha = 1.2, d_s = 5$.

m, n	$d_t = 1$		$d_t = 2$		$d_t = 3$		$d_t = 4$	
8	1.3533e-02		3.9763e-03		1.8858e-03		9.5103e-04	
10	6.6743e-03	3.1676	1.3072e-03	4.9852	6.1744e-04	5.0035	1.8959e-04	7.2270
12	3.7253e-03	3.1983	5.3934e-04	4.8559	2.4381e-04	5.0965	5.1750e-05	7.1218
14	2.5987e-03	2.3364	3.0681e-04	3.6595	1.3060e-04	4.0495	2.0609e-05	5.9726

For Tables 9 and 10, the errors of BRIM under the Chebyshev partition for with $\alpha = 1.2$ are presented. For $d_t = 5$, the convergence rate is $O(h^7)$ in Table 9, while in Table 10, the convergence rate is $O(h^{d_t})$, which agrees with our theorem.

Table 9. Errors of BRIM under Chebyshev partition with $\alpha = 1.2, d_t = 5$.

m, n	$d_s = 2$		$d_s = 3$		$d_s = 4$		$d_s = 5$	
8	7.3421e-05		7.3288e-05		7.3555e-05		7.3699e-05	
10	4.5834e-05	2.1115	4.5522e-05	2.1341	4.6189e-05	2.0852	4.6041e-05	2.1083
12	1.4338e-05	6.3739	1.4995e-05	6.0906	1.4208e-05	6.4662	1.4082e-05	6.4975
14	2.8314e-06	10.523	3.3197e-06	9.7819	4.2225e-06	7.8714	4.4239e-06	7.5113

Table 10. Errors of BRIM under Chebyshev partition with $\alpha_1 = 1.2, d_s = 5$.

m, n	$d_t = 1$		$d_t = 2$		$d_t = 3$	
8	1.9844e-02		4.6715e-03		7.3397e-04	
10	8.1292e-03	3.9994	5.3572e-04	9.7050	4.7628e-04	1.9380
12	3.9786e-03	3.9191	1.8927e-04	5.7066	9.7191e-05	8.7172
14	2.4670e-03	3.1002	9.6933e-05	4.3409	3.4887e-05	6.6466

Example 2. *Consider the time-dependent fractional convection-diffusion equation*

$$\begin{cases} \frac{\partial^\alpha \phi(t_1,t_2,s)}{\partial s^\alpha} - \frac{\partial^2 \phi(t_1,t_2,,s)}{\partial t_1^2} - \frac{\partial^2 \phi(t_1,t_2,,s)}{\partial t_2^2} & \\ +\frac{\partial \phi(t_1,t_2,s)}{\partial t_1} + \frac{\partial \phi(t_1,t_2,s)}{\partial t_2} = f(t_1,t_2,s) & (t_1,t_2,s) \in \Omega^2 \times [0,1] \\ \phi(t_1,t_2,0) = 0, \frac{\partial \phi(t_1,t_2,0)}{\partial s} = 0, & t_1,t_2 \in \Omega^2 \\ \phi(t_1,t_2,s)|_\Gamma = 0, & s \in [0,1], \end{cases} \tag{5.2}$$

with the analysis solution is

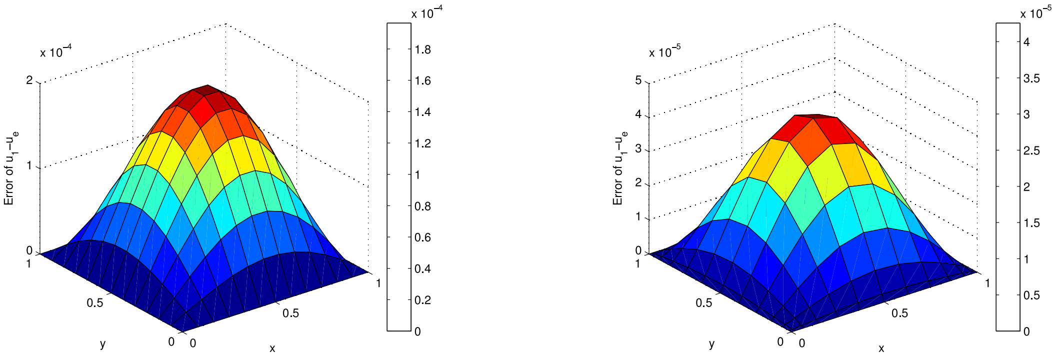
$$\phi(t_1,t_2,s) = s^{3+\alpha}\sin(\pi t_1)\sin(\pi t_2)$$

with the initial condition

$$\phi(t_1,t_2,0) = 0$$

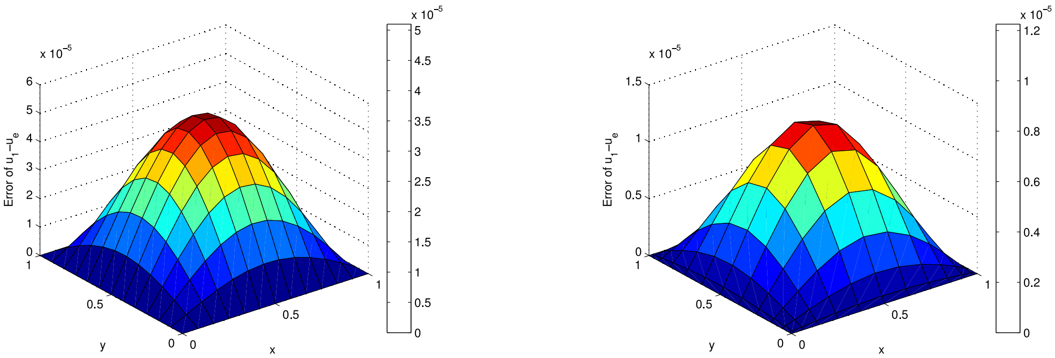
and

$$f(t_1, t_2, s) = \left(\frac{\Gamma(4+\alpha)s^3}{6} + 2\pi^2 s^{3+\alpha}\right)\sin(\pi t_1)\sin(\pi t_2) + \pi s^{3+\alpha}[\cos(\pi t_1)\sin(\pi t_2) + \sin(\pi t_1)\cos(\pi t_2)].$$



(a) uniform (b) nonuniform

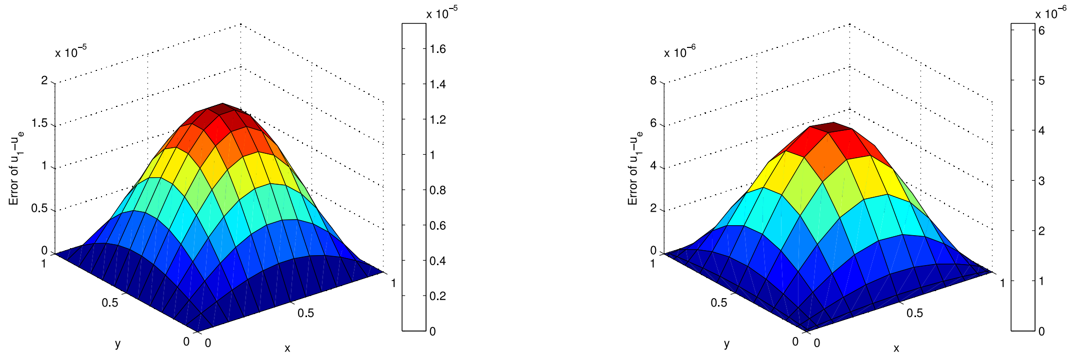
Figure 5. Errors of $m = n = l = 13$, $\Omega^2 = [0,1]\times[0,1]$, $\alpha = 1.2$ in Example 2 (a) uniform; (b) nonuniform.



(a) uniform (b) nonuniform

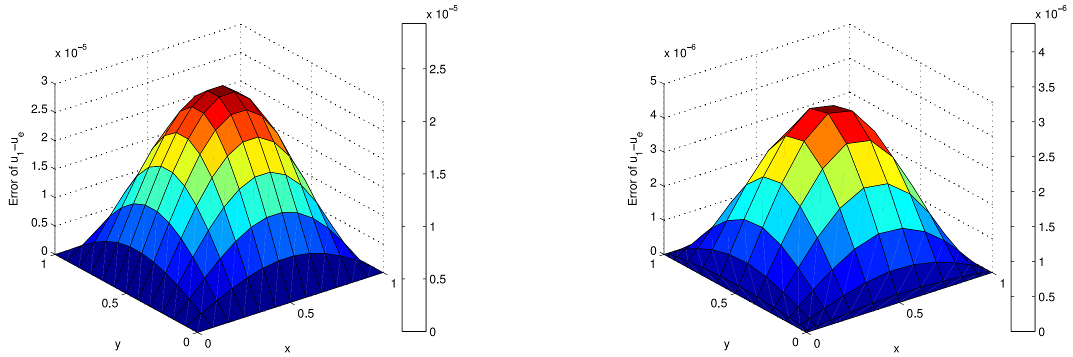
Figure 6. Errors of $m = n = l = 13, d_{t1} = d_{t2} = d_s = 6$, $\Omega^2 = [0,1]\times[0,1]$, $\alpha = 1.2$ in Example 2 (a) uniform; (b) nonuniform.

In Figures 5 and 6, errors of $m = n = 13$, $\Omega^2 = [0,1]\times[0,1]$, $\alpha = 1.2$ and $m = n = 13, d_t = d_s = 7$, $\Omega^2 = [0,1]\times[0,1]$, $\alpha = 1.2$ in Example 2(a) uniform and 2(b) nonuniform for the TFCD equation by the rational interpolation collocation methods are presented, respectively. From the Figure, we know that the precision can reach to 10^{-6} for both the uniform and nonuniform partition.



(a) uniform (b) nonuniform

Figure 7. Errors of $m = n = l = 13, \Omega^2 = [0, 1] \times [0, 1], \alpha = 1.9$ in Example 2 (a) uniform; (b) nonuniform.



(a) uniform (b) nonuniform

Figure 8. Errors of $m = n = l = 13, d_t = d_s = 6, \Omega^2 = [0, 1] \times [0, 1], \alpha = 1.8$ in Example 2 (a) uniform; (b) nonuniform.

In Figures 7 and 8, the errors of $m = n = 13, \Omega^2 = [0, 1] \times [0, 1], \alpha = 1.9$ and $m = n = 13, d_t = d_s = 6, \alpha = 1.9, \Omega^2 = [0, 1] \times [0, 1]$ in Example 2(a) uniform and 2(b) nonuniform for the TFCD equation by rational interpolation collocation methods are presented, respectively. From the figure, we know that the precision can reach to 10^{-6} for both the uniform and nonuniform partition.

Table 11. Errors of TFCD equation with $d_{t1} = d_{t2} = d_s = 5, \alpha = 1.9$.

	method of substitution		method of additional	
	uniform	nonuniform	uniform	nonuniform
8	7.0419e-04	3.3178e-04	3.1465e-03	3.3304e-03
10	3.3310e-04	1.0079e-04	9.2704e-04	3.2072e-04
12	1.8129e-04	3.1367e-05	5.3770e-04	1.0461e-04
14	1.0696e-04	1.3069e-05	3.2444e-04	2.7111e-05

In Table 11, the errors of the TFCD equation with $d_{t1} = d_{t2} = d_s = 5, \alpha = 1.9$ for substitution methods and additional methods are presented, and there are nearly no differences for the two methods. Compared with two methods, the additional method is more simple than the substitution methods. In the following, we chose the substitution method to deal with the boundary condition.

Table 12. Errors of non-uniform with $\alpha = 1.2, d_s = 5$.

m, n, l	$d_{t1} = d_{t2} = 1$		$d_{t1} = d_{t2} = 2$		$d_{t1} = d_{t2} = 3$		$d_{t1} = d_{t2} = 4$	
8	2.7562e-02		1.2846e-02		2.8232e-03		2.1145e-04	
10	2.4880e-02	0.4586	4.2585e-03	4.9481	4.1631e-04	8.5782	4.1373e-04	-
12	1.3801e-02	3.2323	2.2876e-03	3.4084	9.6620e-05	8.0115	1.0619e-04	7.4594
14	1.0876e-02	1.5456	1.2425e-03	3.9594	4.6241e-05	4.7805	3.9039e-05	6.4913

Table 13. Errors of non-uniform with $\alpha = 1.2, d_{t1} = d_{t2} = 5$.

m, n, l	$d_s = 2$		$d_s = 3$		$d_s = 4$		$d_s = 5$	
8	1.3243e+00		7.8057e-02		1.5961e-02		6.2422e-04	
10	7.3310e-01	2.6500	3.5876e-02	3.4837	4.9632e-03	5.2349	3.0553e-04	3.2017
12	6.2810e-01	0.8479	2.2361e-02	2.5930	2.1901e-03	4.4870	1.1816e-04	5.2105
14	5.5624e-01	0.7881	1.5276e-02	2.4719	1.1022e-03	4.4542	6.8114e-05	3.5737

From Tables 12 and 13, the error of BRIM under non-uniform for $\alpha = 1.2, d_s = 5$ with different d_{t1}, d_{t2} are given, and the convergence rate is $O(h^{d_1})$. From Table 13, with space variable uniform for $\alpha = 1.2, d_{t1} = d_{t2} = 5$, the convergence rate is $O(h^{d_s})$, which we will investigate in future paper.

Table 14. Errors of uniform with $\alpha = 1.2, d_{t1} = d_{t2} = 5$.

m, n, l	$d_s = 2$		$d_s = 3$	$d_s = 4$		$d_s = 5$	
8	1.4288e+00		7.6992e-01	7.8669e-02		2.0025e-03	
10	3.3357e-01	6.5191	1.2495e+00	3.3837e-02	3.7810	1.0038e-03	3.0947
12	1.4418e-01	4.6005	2.8110e+00	1.6731e-02	3.8627	5.9571e-04	2.8621
14	1.0264e-01	2.2045	4.1671e+01	1.0120e-02	3.2616	5.0537e-04	1.0670

Table 15. Errors of uniform with $\alpha_1 = 1.2, d_s = 5$.

m,n,l	$d_{t1} = d_{t2} = 1$		$d_{t1} = d_{t2} = 2$		$d_{t1} = d_{t2} = 3$		$d_{t1} = d_{t2} = 4$	
8	1.2826e-02		5.7354e-03		1.5229e-03		1.2495e-03	
10	9.0437e-03	1.5660	2.9311e-03	3.0082	4.9942e-04	4.9966	5.6185e-04	3.5819
12	6.2085e-03	2.0631	1.6990e-03	2.9911	2.0744e-04	4.8189	2.9431e-04	3.5465
14	4.8193e-03	1.6431	1.0705e-03	2.9963	1.1045e-04	4.0887	1.9707e-04	2.6017

For Tables 14 and 15, the errors of the uniform partition for s and t are presented. For $\alpha = 1.2, d_s = 5$, the convergence rate is $O(h^{d_s})$ in Table 14, while in Table 15, the convergence rate is $O(h^{d_{t1}})$, which agrees with our theorem.

Table 16. Errors of uniform with $\alpha = 1.9, d_s = 5$.

m,n,l	$d_{t1} = d_{t2} = 2$		$d_{t1} = d_{t2} = 3$		$d_{t1} = d_{t2} = 4$		$d_{t1} = d_{t2} = 5$	
8	7.2024e-03		4.2245e-03		1.1282e-03		7.7258e-04	
10	4.6350e-03	1.9754	2.3361e-03	2.6550	4.1889e-04	4.4402	3.3536e-04	3.7399
12	3.2040e-03	2.0251	1.4242e-03	2.7142	1.8938e-04	4.3540	1.8214e-04	3.3481
14	2.3575e-03	1.9902	9.3114e-04	2.7567	1.0609e-04	3.7595	1.0722e-04	3.4378

Table 17. Errors of uniform with $\alpha = 1.9, d_{t1} = d_{t2} = 5$.

m,n,l	$d_s = 1$	$d_s = 2$		$d_s = 3$		$d_s = 4$	
8	7.1413e-01	1.9907e-01		6.9366e-02		1.2212e-03	
10	7.5039e-01	1.7041e-01	0.6966	4.4086e-02	2.0312	8.0096e-04	1.8900
12	7.7490e-01	1.4576e-01	0.8571	3.0184e-02	2.0778	5.3284e-04	2.2356
14	7.8155e-01	1.2601e-01	0.9444	2.1918e-02	2.0758	3.6584e-04	2.4392

In the following table, $\alpha = 1.9$ is chosen to present numerical results. From Tables 16 and 17, the error of BRIM under uniform for $d_s = 5$ with different d_{t1}, d_{t2} are given, and the convergence rate is $O(h^{d_{t1}})$. From Table 17, with space variable $d_{t1} = d_{t2} = 5$, the convergence rate is $O(h^{d_s-1})$, which agrees with our theorem.

Table 18. Errors of non-uniform with $\alpha_1 = 1.9, d_s = 5$.

m,n,l	$d_{t1} = d_{t2} = 2$		$d_{t1} = d_{t2} = 3$		$d_{t1} = d_{t2} = 4$		$d_{t1} = d_{t2} = 5$	
8	1.8544e-02		9.4605e-03		1.8420e-03		3.1671e-04	
10	1.4747e-02	1.0267	3.2891e-03	4.7346	3.5472e-04	7.3821	2.6826e-04	0.7440
12	8.6541e-03	2.9234	1.4864e-03	4.3563	1.0556e-04	6.6478	7.3391e-05	7.1092
14	5.9605e-03	2.4189	8.7234e-04	3.4574	3.7193e-05	6.7671	1.8804e-05	8.8340

Table 19. Errors of non-uniform with $\alpha = 1.9, d_{t1} = d_{t2} = 5$.

m, n, l	$d_s = 2$		$d_s = 3$		$d_s = 4$		$d_s = 5$	
8	5.8112e-01		1.1023e-01		2.9033e-02		6.2495e-04	
10	6.2713e-01	-	7.3871e-02	1.7937	1.2478e-02	3.7842	2.6071e-04	3.9179
12	6.4611e-01	-	5.1865e-02	1.9399	6.0291e-03	3.9897	1.0664e-04	4.9032
14	6.5178e-01	-	3.7744e-02	2.0616	3.1919e-03	4.1257	4.9371e-05	4.9957

For Tables 18 and 19, the errors of BRIM under thev Chebyshev partition for with $\alpha = 1.9$ are presented. For $d_s = 5$, the convergence rate is $O(h^{t1})$ in Table 18, while in Table 19, the convergence rate is $O(h^{d_s})$, which agrees with our theorem.

6. Concluding remarks

In this paper, BRIM is used to solve the TFCD equation. The singularity of fractional derivative is overcome by thre integral to the density function from the singular kernel. For arbitrary fractional derivative new Gauss formula is constructed to simply calculate it. For the Diriclet boundary condition, the TFCD equation is changed to discrete the TFCD equation and the matrix equation. In the future, the TFCD equation with the Nuemann condition can be solved by BRIM, and a high dimensional TFCD equation can be studied by our methods.

Acknowledgments

The work of Jin Li was supported by Natural Science Foundation of Shandong Province (Grant No. ZR2022MA003).

Conflicts of interest

The authors declare that they have no conflicts of interest.

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