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Research article

Generalized polynomial exponential sums and their fourth power mean

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Abstract: The study of the power mean of the generalized polynomial exponential sums plays a very important role in analytic number theory, and many classical number theory problems are closely related to it. In this article, we use the elementary methods and the properties of the exponential sums to study the calculating problem of one kind of fourth power mean of some special generalized polynomial exponential sums, and we give some exact calculating formulae for them.

Keywords: generalized polynomial exponential sums; fourth power mean; elementary method; exact calculating formula

1. Introduction

In this paper, we always assume that p denotes an odd prime, $f(x)$ is a polynomial with integral coefficients and χ is a Dirichlet character modulo p. The generalized polynomial exponential sums $S(f(x), \chi; p)$ are defined as

$$S\left(f(x), \chi; p\right) = \sum_{a=1}^{p-1} \chi(a) e\left(\frac{f(a)}{p}\right),$$

where $e(y) = e^{2\pi i y}$ and $i^2 = -1$.

Usually, if $f(x) = mx^k + nx^h$, we call $S(f(x), \chi; p)$ the two-term exponential sums. If $f(x) = mx^k + nx^h + tx^l$, then we call $S(f(x), \chi; p)$ the three-term exponential sums. The research on the properties of $S\left(f(x), \chi; p\right)$ is one of the important contents in analytic number theory, and these contents mainly involve two aspects. One is the upper bound estimate of $S(f(x), \chi; p)$. For example, Weil's classical works [1] and [2] obtained the best estimate:

$$\left|S\left(f(x), \chi; p\right)\right| \ll \sqrt{p}.$$

The generalized conclusion can also be found in [3].

Another aspect is about the calculation of the power mean of $S(f(x), \chi; p)$. For example, Zhang and Han [4] used the elementary and analytic methods to obtain the identity

$$\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{a^3+ma}{p}\right)\right|^6 = 5p^4 - 8p^3 - p^2,$$

where p denotes an odd prime with $3 \nmid (p-1)$.

Zhang and Meng [5] studied the sixth power mean of the two-term exponential sums and proved that for any odd prime p and integer t with $(t, p) = 1$, one has the identities

$$\sum_{n=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{na^3+ta}{p}\right)\right|^6 = \begin{cases} 5p^3 \cdot (p-1) & \text{if } p \equiv 5 \bmod 6; \\ p^2 \cdot \left(5p^2 - 23p - d^2\right) & \text{if } p \equiv 1 \bmod 6, \end{cases}$$

where $4p = d^2 + 27 \cdot b^2$, and d is uniquely determined by $d \equiv 1 \bmod 3$ and $b > 0$.

Chen and Wang [6] studied the fourth power mean of the two-term exponential sums, and proved the calculating formula

$$\sum_{n=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{na^4+a}{p}\right)\right|^4 = \begin{cases} 2p^2(p-2) & \text{if } p \equiv 7 \bmod 12; \\ 2p^3 & \text{if } p \equiv 11 \bmod 12; \\ 2p(p^2-10p-2\alpha^2) & \text{if } p \equiv 1 \bmod 24; \\ 2p(p^2-4p-2\alpha^2) & \text{if } p \equiv 5 \bmod 24; \\ 2p(p^2-6p-2\alpha^2) & \text{if } p \equiv 13 \bmod 24; \\ 2p(p^2-8p-2\alpha^2) & \text{if } p \equiv 17 \bmod 24, \end{cases}$$

where $\alpha = \alpha(p)$ is a constant depending only on p.

In fact, if $p \equiv 1 \bmod 4$, then we have the following identity (see Theorems 4–11 in [7]):

$$p = \alpha^2 + \beta^2 = \left(\sum_{a=1}^{\frac{p-1}{2}}\left(\frac{a+\overline{a}}{p}\right)\right)^2 + \left(\sum_{a=1}^{\frac{p-1}{2}}\left(\frac{a+r\overline{a}}{p}\right)\right)^2, \tag{1.1}$$

where $\left(\frac{*}{p}\right)$ is the Legendre's symbol and r is any quadratic non-residue modulo p.

On the other hand, Du and Han [8] discussed the fourth power mean of the three-term exponential sums and proved that

$$\sum_{m=0}^{p-1}\sum_{n=0}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^4+na^2+a}{p}\right)\right|^4 = 2p^4 - 11p^3 + 16p^2$$

and

$$\sum_{m=0}^{p-1}\sum_{n=0}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^6+na^2+a}{p}\right)\right|^4 = \begin{cases} 2p^4 - 11p^3 + 16p^2 & \text{if } p \equiv 3 \bmod 4; \\ 2p^4 - 15p^3 + 36p^2 & \text{if } p \equiv 1 \bmod 4. \end{cases}$$

Many other many results related to these contents can also be found in [9–18].

In this paper, we consider the following power mean of the generalized polynomial exponential sums:

$$H_{2k}(p) = \sum_{\chi \bmod p} \sum_{m=0}^{p-1} \left| \sum_{a=1}^{p-1} \chi(a) e\left(\frac{f\left(a^3\right) + ma^3 + a^2}{p} \right) \right|^{2k},$$

where k is a positive integer and $f(x)$ is a polynomial with integral coefficients.

We use the elementary methods and the properties of the exponential sums to give an exact calculating formula for $H_4(p)$ for all prime numbers. That is, we prove the following two calculating formulae:

Theorem 1. For any odd prime p with $3 \nmid (p-1)$, we have the identity

$$\sum_{\chi \bmod p} \sum_{m=0}^{p-1} \left| \sum_{a=1}^{p-1} \chi(a) e\left(\frac{f\left(a^3\right) + ma^3 + a^2}{p} \right) \right|^{4} = p(p-1)^2 (2p-3).$$

Theorem 2. For any odd prime p with $p \equiv 1 \bmod 3$, we have

$$\sum_{\chi \bmod p} \sum_{m=0}^{p-1} \left| \sum_{a=1}^{p-1} \chi(a) e\left(\frac{f\left(a^3\right) + ma^3 + a^2}{p} \right) \right|^{4}$$
$$= \begin{cases} p(p-1)\left(2p^2 - 5p + 15\right) & \text{if } p \equiv 7 \bmod 12; \\ p(p-1)\left(2p^2 - 5p + 15 - 12\sqrt{p}\right) & \text{if } p \equiv 1 \bmod 12 \text{ and } 3 \nmid \alpha; \\ p(p-1)\left(2p^2 - 5p + 15 + 4\sqrt{p}\right) & \text{if } p \equiv 1 \bmod 12 \text{ and } 3 \mid \alpha, \end{cases}$$

where the constant α is the same as defined in formula (1.1).

From these two theorems we may immediately deduce the following:

Corollary. For any odd prime p, we have the asymptotic formula

$$\sum_{\chi \bmod p} \sum_{m=0}^{p-1} \left| \sum_{a=1}^{p-1} \chi(a) e\left(\frac{f\left(a^3\right) + ma^3 + a^2}{p} \right) \right|^{4} = 2p^4 - 7p^3 + O\left(p^{\frac{5}{2}}\right).$$

Some note: For any integer $k \geq 3$, whether there exists an exact calculating formula for $H_{2k}(p)$ is an interesting open problem. Interested readers can continue this research.

2. Several lemmas

In this section, we give three simple lemmas. It is clear that the proofs of these lemmas need some basic knowledge of the elementary and analytic number theories, all of these can be found in [19] and [7], so we do not repeat them here. First, we have the following:

Lemma 1. For any odd prime p, we have the identity

$$\sum_{\substack{a=1 \\ a^3 \equiv 1 \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{d=1}^{p-1} e\left(\frac{-d^2\left(a^2-1\right)\left(b^2-1\right)}{p} \right)$$

$$= \begin{cases} (p-1)^2 & \text{if } 3 \nmid (p-1); \\ p^2+3 & \text{if } p \equiv 7 \bmod 12; \\ p^2+3-4\left(\frac{\omega-1}{p}\right)\sqrt{p} & \text{if } p \equiv 1 \bmod 12, \end{cases}$$

where $\omega \neq 1$ and $\omega^3 \equiv 1 \bmod p$.

Proof. It is clear that if $(3, p-1) = 1$, then $a^3 \equiv 1 \bmod p$ if and only if $a \equiv 1 \bmod p$. In this case, if a passes through a reduced residue system modulo p, then a^3 also passes through a reduced residue system modulo p. So, we have

$$\sum_{\substack{a=1 \\ a^3\equiv 1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{d=1}^{p-1} e\left(\frac{-d^2\left(a^2-1\right)\left(b^2-1\right)}{p}\right) = \sum_{b=1}^{p-1}\sum_{d=1}^{p-1} 1 = (p-1)^2. \tag{2.1}$$

If $3 \mid (p-1)$, then $a^3 \equiv 1 \bmod p$ if and only if $a \equiv 1,\ \omega,\ \omega^2 \bmod p$, where $\omega \neq 1$ and $\omega^3 \equiv 1 \bmod p$. In this case, from the properties of the reduced residue system modulo p and $\overline{\omega} \equiv \omega^2 \bmod p$, we have

$$\begin{aligned} &\sum_{\substack{a=1 \\ a^3\equiv 1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{d=1}^{p-1} e\left(\frac{-d^2\left(a^2-1\right)\left(b^2-1\right)}{p}\right) \\ = &\sum_{b=1}^{p-1}\sum_{d=1}^{p-1} 1 + \sum_{b=1}^{p-1}\sum_{d=1}^{p-1} e\left(\frac{-d^2\left(\omega^2-1\right)\left(b^2-1\right)}{p}\right) \\ &+\sum_{b=1}^{p-1}\sum_{d=1}^{p-1} e\left(\frac{-d^2\left(\omega-1\right)\left(b^2-1\right)}{p}\right) \\ = &(p-1)^2 + 2\cdot\left|\sum_{d=1}^{p-1} e\left(\frac{(\omega-1)d^2}{p}\right)\right|^2. \end{aligned} \tag{2.2}$$

Note that for any integer h with $(h, p) = 1$,

$$\sum_{a=0}^{p-1} e\left(\frac{ha^2}{p}\right) = 1 + \sum_{a=1}^{p-1}(1+\chi_2(a))\, e\left(\frac{ha}{p}\right) = \chi_2(h)\cdot\tau(\chi_2), \tag{2.3}$$

where $\chi_2 = \left(\frac{*}{p}\right)$ denotes the Legendre's symbol modulo p, $\tau(\chi)$ represents the classical Gauss sums and $\tau(\chi_2) = \sqrt{p}$ if $p \equiv 1 \bmod 4$, and $\tau(\chi_2) = i\cdot\sqrt{p}$, if $p \equiv 3 \bmod 4$.

So, we have the identity

$$\left|\sum_{d=1}^{p-1} e\left(\frac{(\omega-1)d^2}{p}\right)\right|^2 = \begin{cases} p+1 & \text{if } p \equiv 7 \bmod 12, \\ p+1-2\left(\frac{\omega-1}{p}\right)\cdot\sqrt{p} & \text{if } p \equiv 1 \bmod 12. \end{cases} \tag{2.4}$$

Combining (2.1), (2.2) and (2.4), we can get the calculating formula

$$\sum_{\substack{a=1 \\ a^3\equiv 1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{d=1}^{p-1} e\left(\frac{-d^2\left(a^2-1\right)\left(b^2-1\right)}{p}\right)$$

$$= \begin{cases} (p-1)^2 & \text{if } 3 \nmid (p-1); \\ p^2+3 & \text{if } p \equiv 7 \bmod 12; \\ p^2+3-4\left(\frac{\omega-1}{p}\right)\sqrt{p} & \text{if } p \equiv 1 \bmod 12. \end{cases}$$

This proves Lemma 1.

Lemma 2. For any odd prime p, we have the identities

$$\sum_{\substack{a=1 \\ a^3\equiv b^3\equiv 1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{d=1}^{p-1} e\left(\frac{-d^2\left(a^2-1\right)\left(b^2-1\right)}{p}\right)$$
$$= \begin{cases} p-1 & \text{if } 3 \nmid (p-1); \\ 5p-9 & \text{if } p \equiv 7 \bmod 12; \\ 5p-9+4\sqrt{p} & \text{if } p \equiv 1 \bmod 12. \end{cases}$$

Proof. It is clear that if $3 \nmid (p-1)$, then we have

$$\sum_{\substack{a=1 \\ a^3\equiv b^3\equiv 1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{d=1}^{p-1} e\left(\frac{-d^2\left(a^2-1\right)\left(b^2-1\right)}{p}\right) = p-1. \tag{2.5}$$

If $3 \mid (p-1)$, then note that $\omega^2+\omega+1 \equiv 0 \bmod p$; from (2.3) and the properties of the reduced residue system modulo p, we have

$$\sum_{\substack{a=1 \\ a^3\equiv b^3\equiv 1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{d=1}^{p-1} e\left(\frac{-d^2\left(a^2-1\right)\left(b^2-1\right)}{p}\right)$$
$$= 3\sum_{d=1}^{p-1}1 + \sum_{d=1}^{p-1}\left(1+e\left(\frac{-d^2\left(\omega^2-1\right)^2}{p}\right)+e\left(\frac{-d^2\left(\omega^2-1\right)(\omega-1)}{p}\right)\right)$$
$$+\sum_{d=1}^{p-1}\left(1+e\left(\frac{-d^2(\omega-1)^2}{p}\right)+e\left(\frac{-d^2(\omega-1)\left(\omega^2-1\right)}{p}\right)\right)$$
$$= 5(p-1)+2\sum_{d=1}^{p-1}e\left(\frac{-d^2}{p}\right)+2\sum_{d=1}^{p-1}e\left(\frac{-d^2(\omega+1)}{p}\right)$$
$$= 5(p-1)-4+2\chi_2(-1)\tau(\chi_2)+2\tau(\chi_2)$$
$$= \begin{cases} 5p-9 & \text{if } p \equiv 7 \bmod 12; \\ 5p-9+4\sqrt{p} & \text{if } p \equiv 1 \bmod 12, \end{cases} \tag{2.6}$$

where we have used the identity $\left(\frac{-1}{p}\right) = -1$ if $p \equiv 7 \bmod 12$.

Now, Lemma 2 follows from identities (2.5) and (2.6).

Lemma 3. Let p be a prime with $p \equiv 1 \bmod 12$. Then, for integer ω with $(\omega-1, p) = 1$ and $\omega^3 \equiv 1 \bmod p$, we have

$$\left(\frac{1-\omega}{p}\right) = \begin{cases} 1 & \text{if } 3 \nmid \alpha; \\ -1 & \text{if } 3 \mid \alpha, \end{cases}$$

where α is the same as defined in (1.1).

Proof. Since $p \equiv 1 \bmod 12$, we may assume that $p = 12k + 1$. Let g be any primitive root modulo p. Take $\omega = g^{\frac{p-1}{3}} = g^{4k}$. It is clear that we have $g^{12k} \equiv 1 \bmod p$ or

$$\left(g^{4k} - 1\right)\left(g^{8k} + g^{4k} + 1\right) \equiv 0 \bmod p. \tag{2.7}$$

Since g is a primitive root modulo p, from (2.7), we have

$$g^{8k} + g^{4k} + 1 \equiv 0 \bmod p$$

or

$$\left(g^{4k} - 1\right)^2 \equiv -3g^{4k} \bmod p. \tag{2.8}$$

Since $\left(\frac{-3}{p}\right) = 1$, from [20] and [21], we have

$$(-3)^{\frac{p-1}{4}} \equiv \begin{cases} 1 \mod p & \text{if } 12 \mid (p-1) \text{ and } 3 \nmid \alpha; \\ -1 \mod p & \text{if } 12 \mid (p-1) \text{ and } 3 \mid \alpha. \end{cases} \tag{2.9}$$

If $3 \nmid \alpha$, then, from (2.9), we know that -3 is a fourth residue modulo p. That is, there exists an integer x such that $x^4 \equiv -3 \bmod p$. Applying (2.8), we have

$$\left(g^{4k} - x^2g^{2k} - 1\right)\left(g^{4k} + x^2g^{2k} - 1\right) \equiv 0 \bmod p$$

or it is equivalent to

$$g^{4k} - x^2g^{2k} - 1 \equiv 0 \bmod p \quad \text{or} \quad g^{4k} + x^2g^{2k} - 1 \equiv 0 \bmod p. \tag{2.10}$$

If $g^{4k} - x^2g^{2k} - 1 \equiv 0 \bmod p$, then we have

$$\left(\frac{\omega - 1}{p}\right) = \left(\frac{g^{4k} - 1}{p}\right) = \left(\frac{x^2g^{2k}}{p}\right) = 1. \tag{2.11}$$

If $g^{4k} + x^2g^{2k} - 1 \equiv 0 \bmod p$, then note that $\left(\frac{-1}{p}\right) = 1$; we also have

$$\left(\frac{\omega - 1}{p}\right) = \left(\frac{g^{4k} - 1}{p}\right) = \left(\frac{-x^2g^{2k}}{p}\right) = 1. \tag{2.12}$$

Similarly, if $3 \mid \alpha$, then, from (2.9), we can also deduce

$$\left(\frac{\omega - 1}{p}\right) = \left(\frac{g^{4k} - 1}{p}\right) = -1. \tag{2.13}$$

Now, Lemma 3 follows from (2.11), (2.12) and (2.13).

3. Proofs of the theorems

In this section we complete the proofs of our theorems. First, we prove Theorem 1. For any integer n, note the trigonometrical identities

$$\sum_{a=0}^{p-1} e\left(\frac{na}{p}\right) = \begin{cases} p & \text{if } p \mid n; \\ 0 & \text{if } p \nmid n, \end{cases}$$

as well as the orthogonality of the characters modulo p:

$$\sum_{\chi \bmod p} \chi(a) = \begin{cases} p-1 & \text{if } p \mid (a-1); \\ 0 & \text{otherwise.} \end{cases}$$

If $3 \nmid (p-1)$, then, for any polynomial $f(x)$ with integral coefficients, from (2.3), Lemma 1, Lemma 2 and the properties of the reduced residue system modulo p we have

$$\begin{aligned}
& \frac{1}{p(p-1)} \sum_{\chi \bmod p} \sum_{m=0}^{p-1} \left| \sum_{a=1}^{p-1} \chi(a) e\left(\frac{f\left(a^3\right) + ma^3 + a^2}{p}\right) \right|^4 \\
= & \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{\substack{d=1 \\ a^3+b^3 \equiv c^3+d^3 \bmod p \\ ab \equiv cd \bmod p}}^{p-1} e\left(\frac{f\left(a^3\right) + f\left(b^3\right) - f\left(c^3\right) - f\left(d^3\right) + a^2 + b^2 - c^2 - d^2}{p}\right) \\
= & \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{\substack{d=1 \\ a^3+b^3 \equiv c^3+1 \bmod p \\ ab \equiv c \bmod p}}^{p-1} e\left(\frac{f\left((ad)^3\right) + f\left((bd)^3\right) - f\left((cd)^3\right) - f\left(d^3\right)}{p}\right) \\
& \times e\left(\frac{d^2\left(a^2 + b^2 - c^2 - 1\right)}{p}\right) \\
= & \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{\substack{d=1 \\ \left(a^3-1\right)\left(b^3-1\right) \equiv 0 \bmod p}}^{p-1} e\left(\frac{f\left((ad)^3\right) + f\left((bd)^3\right) - f\left((abd)^3\right) - f\left(d^3\right)}{p}\right) \\
& \times e\left(\frac{-d^2\left(a^2 - 1\right)\left(b^2 - 1\right)}{p}\right) \\
= & 2 \sum_{\substack{a=1 \\ a^3 \equiv 1 \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{d=1}^{p-1} e\left(\frac{-d^2\left(a^2 - 1\right)\left(b^2 - 1\right)}{p}\right) \\
& - \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{\substack{d=1 \\ a^3 \equiv b^3 \equiv 1 \bmod p}}^{p-1} e\left(\frac{-d^2\left(a^2 - 1\right)\left(b^2 - 1\right)}{p}\right) \\
= & 2(p-1)^2 - (p-1) = (p-1)(2p-3).
\end{aligned}$$

This proves Theorem 1.

Now, we prove Theorem 2. If $p \equiv 7 \bmod 12$, then, from Lemma 1, Lemma 2 and the methods of proving Theorem 1, we have

$$\begin{aligned}
&\sum_{\chi \bmod p} \sum_{m=0}^{p-1} \left| \sum_{a=1}^{p-1} \chi(a) e\left(\frac{f\left(a^3\right) + ma^3 + a^2}{p} \right) \right|^4 \\
= \;& p(p-1) \cdot \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{\substack{c=1 \\ a^3+b^3 \equiv c^3+1 \bmod p \\ ab \equiv c \bmod p}}^{p-1} \sum_{d=1}^{p-1} e\left(\frac{f\left((ad)^3\right) + f\left((bd)^3\right) - f\left((cd)^3\right) - f\left(d^3\right)}{p} \right) \\
&\times e\left(\frac{d^2\left(a^2+b^2-c^2-1\right)}{p} \right) \\
= \;& 2p(p-1) \cdot \sum_{\substack{a=1 \\ a^3 \equiv 1 \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{d=1}^{p-1} e\left(\frac{-d^2\left(a^2-1\right)\left(b^2-1\right)}{p} \right) \\
&- p(p-1) \sum_{a=1}^{p-1} \sum_{\substack{b=1 \\ a^3 \equiv b^3 \equiv 1 \bmod p}}^{p-1} \sum_{d=1}^{p-1} e\left(\frac{-d^2\left(a^2-1\right)\left(b^2-1\right)}{p} \right) \\
= \;& 2p(p-1)(p^2+3) - p(p-1)(5p-9) = p(p-1)\left(2p^2-5p+15\right). \qquad (3.1)
\end{aligned}$$

If $p \equiv 1 \bmod 12$, then, from Lemma 1, Lemma 2 and Lemma 3, we have

$$\begin{aligned}
&\sum_{\chi \bmod p} \sum_{m=0}^{p-1} \left| \sum_{a=1}^{p-1} \chi(a) e\left(\frac{f\left(a^3\right) + ma^3 + a^2}{p} \right) \right|^4 \\
= \;& p(p-1) \cdot \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{\substack{c=1 \\ a^3+b^3 \equiv c^3+1 \bmod p \\ ab \equiv c \bmod p}}^{p-1} \sum_{d=1}^{p-1} e\left(\frac{f\left((ad)^3\right) + f\left((bd)^3\right) - f\left((cd)^3\right) - f\left(d^3\right)}{p} \right) \\
&\times e\left(\frac{d^2\left(a^2+b^2-c^2-1\right)}{p} \right) \\
= \;& 2p(p-1) \cdot \sum_{\substack{a=1 \\ a^3 \equiv 1 \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{d=1}^{p-1} e\left(\frac{-d^2\left(a^2-1\right)\left(b^2-1\right)}{p} \right) \\
&- p(p-1) \sum_{a=1}^{p-1} \sum_{\substack{b=1 \\ a^3 \equiv b^3 \equiv 1 \bmod p}}^{p-1} \sum_{d=1}^{p-1} e\left(\frac{-d^2\left(a^2-1\right)\left(b^2-1\right)}{p} \right) \\
= \;& 2p(p-1)\left(p^2+3-4\left(\frac{\omega-1}{p}\right)\sqrt{p}\right) - p(p-1)(5p-9) - 4p(p-1)\sqrt{p}
\end{aligned}$$

$$= \begin{cases} p(p-1)\left(2p^2-5p+15-12\sqrt{p}\right) & \text{if } 3 \nmid \alpha; \\ p(p-1)\left(2p^2-5p+15+4\sqrt{p}\right) & \text{if } 3 \mid \alpha. \end{cases} \tag{3.2}$$

Combining (3.1) and (3.2), we obtain Theorem 2.

This completes the proofs of all of our results.

4. Conclusions

The main result of this paper is an exact calculating formula for one kind of special fourth power mean of the polynomial exponential sums. That is, for any polynomial $f(x)$ with integral coefficients, we have the identity

$$\sum_{\chi \bmod p} \sum_{m=0}^{p-1} \left| \sum_{a=1}^{p-1} \chi(a) e\left(\frac{f\left(a^3\right) + ma^3 + a^2}{p} \right) \right|^4$$
$$= \begin{cases} p(p-1)\left(2p^2-5p+3\right) & \text{if } 3 \nmid (p-1); \\ p(p-1)\left(2p^2-5p+15\right) & \text{if } p \equiv 7 \bmod 12; \\ p(p-1)\left(2p^2-5p+15-12\sqrt{p}\right) & \text{if } p \equiv 1 \bmod 12 \text{ and } 3 \nmid \alpha; \\ p(p-1)\left(2p^2-5p+15+4\sqrt{p}\right) & \text{if } p \equiv 1 \bmod 12 \text{ and } 3 \mid \alpha. \end{cases}$$

where p is an odd prime and $\alpha = \sum_{a=1}^{\frac{p-1}{2}} \left(\frac{a+\overline{a}}{p} \right)$ is a constant depending only on p.

In this paper, we also propose an open problem. That is, for any integer $k \geq 3$, does there exist an exact calculating formula for the $2k$-th power mean

$$\sum_{\chi \bmod p} \sum_{m=0}^{p-1} \left| \sum_{a=1}^{p-1} \chi(a) e\left(\frac{f\left(a^3\right) + ma^3 + a^2}{p} \right) \right|^{2k} ?$$

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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