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Research article

Bifurcation analysis in a discrete predator–prey model with herd behaviour and group defense

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Abstract: In this paper, we utilize the semi-discretization method to construct a discrete model from a continuous predator-prey model with herd behaviour and group defense. Specifically, some new results for the transcritical bifurcation, the period-doubling bifurcation, and the Neimark-Sacker bifurcation are derived by using the center manifold theorem and bifurcation theory. Novelty includes a smooth transition from individual behaviour (low number of prey) to herd behaviour (large number of prey). Our results not only formulate simpler forms for the existence conditions of these bifurcations, but also clearly present the conditions for the direction and stability of the bifurcated closed orbits. Numerical simulations are also given to illustrate the existence of the derived Neimark-Sacker bifurcation.

Keywords: discrete predator-prey system with herd behaviour and group defense; semi-discretization method; transcritical bifurcation; period-doubling bifurcation; Neimark-Sacker bifurcation

1. Introduction and preliminaries

Over the past several decades, the predator-prey interaction has become a hot point of studies in biomathematics [1-10]. Because differential equations can assume that generations overlap and that populations vary continously in time, the general model for predator-prey interaction may be written as

$$\begin{cases} \frac{dx}{dt} = f(x)x - g(x,y)y, \\ \frac{dy}{dt} = h(x,y)y - my, \end{cases} \tag{1.1}$$

where x and y are expressed as prey and predator population sizes (or densities), respectively, $f(x)$ denotes the growth rate of prey with the absence of predator, $g(x,y)$ represents the amount of prey consumed per predator per unit time, $h(x,y)$ is on behalf of per capita predator production, and m is the intrinsic death rate of predator. See also [1].

Due to the realistic meaning of $f(x)$, one can assume that the prey grows logistically with growth rate r and carrying capacity k in the absence of predator (i.e., $f(x) = r(1 - \frac{x}{k})$). Hence the system (1.1) can be written as

$$\begin{cases} \frac{dx}{dt} = rx(1 - \frac{x}{k}) - g(x, y)y, \\ \frac{dy}{dt} = eg(x, y)y - my, \end{cases} \tag{1.2}$$

where e is the conversion effciency.

As for the functional response $g(x, y)$, there are many different kinds of forms. Bian et al. proposed a system with the Beddington-DeAngelis funcional response [5]; De Assis et al. proposed a system with the square-root functional response [7] and so on. Notice the fact that in the natural ecosystem, many species may gather together and form herds to either search for food resources or to defend the predators, which means that all members of a group do not interact at one time. This behaviour is often called herd behaviour. In this paper, one talks about the following system [6,7]:

$$\begin{cases} \frac{dx}{dt} = rx(1 - \frac{x}{k}) - \frac{axy}{\sqrt{x+\tilde{h}}}, \\ \frac{dy}{dt} = \frac{eaxy}{\sqrt{x+\tilde{h}}} - my. \end{cases} \tag{1.3}$$

Here, the funcional response $\frac{ax}{\sqrt{x+\tilde{h}}}$ can be expressed as the function of the ratio of prey to predator, where $\tilde{h}$ is a threshold for the transition between herd grouping and solitary behaviour and a is the maximum value of prey consumed by each predator per unit time. In this system, all parameters are positive. The biological meanings for the parameters r, k, e, and m are the same as in (1.2).

For the sake of simplicity of mathematical analysis, let $\frac{x}{k} \to x, mt \to t, \frac{y}{ek} \to y, \frac{r}{m} \to \gamma, \frac{ae\sqrt{k}}{m} \to \beta, \frac{\tilde{h}}{k} \to h$, then one can derive an equivalent form of the system (1.3) as follows:

$$\begin{cases} \frac{dx}{dt} = x(\gamma(1 - x) - \frac{\beta y}{\sqrt{x+h}}), \\ \frac{dy}{dt} = y(\frac{\beta x}{\sqrt{x+h}} - 1). \end{cases} \tag{1.4}$$

This continuous system has been discussed in [6,7], but its discrete version has not been investgated as of yet. To be honest, it is very difficult to solve a complicate continuous equation or system without using computer. Therefore, one natuarally wishes to consider the corresponding discrete version of a continuous model. One tries to use various methods to derive the discrete model of the system (1.4) to make it easily studied [8–16]. In this paper, we adopt a semi-discretazation method, which does not need to consider the step size, to derive its discrete model. For this, suppose that $[t]$ denotes the greatest integer not exceeding t. Consider the average change rate of the system (1.4) at integer number points

$$\begin{cases} \frac{1}{x(t)}\frac{dx}{dt} = \gamma(1 - x([t])) - \frac{\beta y([t])}{\sqrt{x([t])+h}}, \\ \frac{1}{y(t)}\frac{dy}{dt} = \frac{\beta x([t])}{\sqrt{x([t])+h}} - 1. \end{cases} \tag{1.5}$$

It is easy to see that the system (1.5) has piecewise constant arguments, and that a solution $(x(t), y(t))$ of the system (1.5) for $t \in [0, +\infty)$ has the following characteristics:

1) on the interval $[0, +\infty)$, $x(t)$ and $y(t)$ are continuous;
2) when $t \in [0, +\infty)$ except possibly for the points $\{0, 1, 2, 3, \cdots\}$, $\frac{dx(t)}{dt}$ and $\frac{dy(t)}{dt}$ exist.

The following system can be obtained by integrating the system (1.5) over the interval [n,t] for any $t \in [n, n+1)$ and $n = 0, 1, 2, \cdots$

$$\begin{cases} x(t) = x_n e^{\gamma(1-x_n) - \frac{\beta y_n}{\sqrt{x_n+h}}}(t-n), \\ y(t) = y_n e^{\frac{\beta x_n}{\sqrt{x_n+h}}-1}(t-n), \end{cases} \tag{1.6}$$

where $x_n = x(n)$ and $y_n = y(n)$. Letting $t \to (n+1)^-$ in the system (1.6) produces

$$\begin{cases} x_{n+1} = x_n e^{\gamma(1-x_n) - \frac{\beta y_n}{\sqrt{x_n+h}}}, \\ y_{n+1} = y_n e^{\frac{\beta x_n}{\sqrt{x_n+h}}-1}, \end{cases} \tag{1.7}$$

where the parameters $h, \beta, \gamma > 0$, and their biological meanings are the same as in (1.4). The system (1.7) will be considered in the sequel.

The rest of the paper is organized as follows. In Section 2, we investigate the existence and stability of the fixed points of the system (1.7) in detail. In Section 3, we derive the sufficient conditions for transcritical bifurcation, period-doubling bifurcation, and Neimark-Sacker bifurcation of the system (1.7) to occur. In Section 4, numerical simulations are performed to illustrate the above theoretical results. In the end, some brief conclusions are stated in Section 5.

2. Existence and stability of fixed points

Considering the biological meaning of the system (1.7), we discuss the existence and stability of non-negative fixed points of the system (1.7) in this section. By solving the equations of fixed points of system (1.7)

$$x = xe^{\gamma(1-x) - \frac{\beta y}{\sqrt{x+h}}}, y = ye^{\frac{\beta x}{\sqrt{x+h}}-1},$$

it's easy to find that there are three nonnegative fixed points $E_0 = (0, 0)$, $E_1 = (1, 0)$, and $E_2 = (x_0, y_0)$ for $\beta > \sqrt{h+1}$, where

$$x_0 = \frac{1 + \sqrt{1 + 4h\beta^2}}{2\beta^2}, y_0 = \gamma x_0(1 - x_0).$$

The Jacobian matrix of the system (1.7) at a fixed point $E(x, y)$ is

$$J(E) = \begin{pmatrix} e^{\gamma(1-x) - \frac{\beta y}{\sqrt{h+x}}}\left(1 - \gamma x + \frac{\beta xy}{2(h+x)^{3/2}}\right) & -\frac{\beta x e^{\gamma(1-x) - \frac{\beta y}{\sqrt{h+x}}}}{\sqrt{h+x}} \\ y\, e^{\frac{\beta x}{\sqrt{h+x}}-1}\left(\frac{\beta}{\sqrt{h+x}} - \frac{\beta x}{2(h+x)^{3/2}}\right) & e^{\frac{\beta x}{\sqrt{h+x}}-1} \end{pmatrix},$$

whose charactertistic polynomial reads as

$$F(\lambda) = \lambda^2 - Tr(J(E))\lambda + Det(J(E)),$$

where

$$Tr(J(E)) = e^{\gamma(1-x) - \frac{\beta y}{\sqrt{h+x}}}\left(1 - \gamma x + \frac{\beta xy}{2(h+x)^{3/2}}\right) + e^{\frac{\beta x}{\sqrt{h+x}}-1},$$

$$Det(J(E)) = e^{\gamma(1-x) - \frac{\beta y}{\sqrt{h+x}} + \frac{\beta x}{\sqrt{h+x}} - 1}\left(1 - \gamma x + \frac{\beta xy}{2(h+x)^{3/2}} + \frac{\beta^2 xy}{h+x} - \frac{\beta^2 x^2 y}{2(h+x)^2}\right).$$

In order to analyze the properties of the fixed points of the system (1.7), we utilize the Appendix definition and Lemma [17–19].

By using Definition 5.1 and Lemma 5.2 in the Appendix, the following conclusions can be obtained.

Theorem 2.1. *The fixed point* $E_0 = (0, 0)$ *of the system* (1.7) *is a saddle.*

The proof for this theorem is simple and omitted here.

Theorem 2.2. *The type of the fixed point* $E_1 = (1, 0)$ *of the system* (1.7) *complies with the following results:*

Table 1. Properties of the positive fixed point E_1.

Conditions		Eigenvalues	Properties
	$0 < \beta < \sqrt{h+1}$	$\lvert\lambda_1\rvert < 1, \lvert\lambda_2\rvert < 1$	*sink*
$0 < \gamma < 2$	$\beta = \sqrt{h+1}$	$\lvert\lambda_1\rvert < 1, \lvert\lambda_2\rvert = 1$	*non − hyperbolic*
	$\beta > \sqrt{h+1}$	$\lvert\lambda_1\rvert < 1, \lvert\lambda_2\rvert > 1$	*saddle*
$\gamma = 2$		$\lvert\lambda_1\rvert = 1$	*non − hyperbolic*
	$0 < \beta < \sqrt{h+1}$	$\lvert\lambda_1\rvert > 1, \lvert\lambda_2\rvert < 1$	*saddle*
$\gamma > 2$	$\beta = \sqrt{h+1}$	$\lvert\lambda_1\rvert > 1, \lvert\lambda_2\rvert = 1$	*non − hyperbolic*
	$\beta > \sqrt{h+1}$	$\lvert\lambda_1\rvert > 1, \lvert\lambda_2\rvert > 1$	*source*

Proof. The Jacobian matrix $J(E_1)$ of the system (1.7) at the fixed point E_1 reads

$$J(E_1) = \begin{pmatrix} 1-\gamma & -\frac{\beta}{\sqrt{h+1}} \\ 0 & \mathrm{e}^{\frac{\beta}{\sqrt{h+1}}-1} \end{pmatrix}.$$

Obviously, $\lambda_1 = 1 - \gamma$ and $\lambda_2 = e^{\frac{\beta}{\sqrt{h+1}}-1}$.

When $0 < \gamma < 2$, $|\lambda_1| < 1$. If $0 < \beta < \sqrt{h+1}$, then $|\lambda_2| < 1$, so $E_1$is a sink; if $\beta = \sqrt{h+1}$, then $|\lambda_2| = 1$, therefore E_1 is non-hyperbolic; if $\beta > \sqrt{h+1}$, meaning $|\lambda_2| > 1$, then E_1 is a saddle.

When $\gamma = 2$, which reads $|\lambda_1| = 1$, E_1 is non-hyperbolic.

When $\gamma > 2$, $|\lambda_1| > 1$. If $0 < \beta < \sqrt{h+1}$, then $|\lambda_2| < 1$, so E_1 is a saddle; if $\beta = \sqrt{h+1}$, then $|\lambda_2| = 1$, therefore E_1 is non-hyperbolic; if $\beta > \sqrt{h+1}$, implying $|\lambda_2| > 1$, then E_1 is a source. The proof is complete.

We can easily derive the following result.

Lemma 2.3. *Consider the function* $f(x) = 4x^2 - 4x + 7 + (2x - 7)\sqrt{4x^2 + 20x + 1}$ *with* $x \in (1, \infty)$. *Then* $f(x)$ *is strictly increasing for* $x \in (1, \infty)$*, Furthermore,* $f(x)$ *has a unique positive root* X_0 *in (2, 2.5).*

Proof. Evidently, $f'(x) = 4(2x - 1) + \frac{16x^2+32x-68}{\sqrt{4x^2+20x+1}}$ and $f''(x) = 8 + \frac{64x^3+520x^2+912x712}{(\sqrt{4x^2+20x+1})^3} > 0$, so, for $x > 1$, $f'(x) > f'(1) = 0$. Hence, $f(x)$ is strictly increasing for $x \in (1, \infty)$. Again, $f(2) = 15 - 3\sqrt{57} < 0$ and $f(2.5) = 22 - 2\sqrt{76} > 0$. Therefore, $f(x)$ has a unique positive root X_0 in (2, 2.5).

Now consider the stability of the fixed point E_2.

Theorem 2.4. *For $\beta > \sqrt{h+1}$, $E_2 = (x_0, y_0) = (\frac{1+\sqrt{1+4h\beta^2}}{2\beta^2}, \gamma\frac{1+\sqrt{1+4h\beta^2}}{2\beta^2}(1 - \frac{1+\sqrt{1+4h\beta^2}}{2\beta^2}))$ is a positive fixed point of the system* (1.7).

Let X_0 be the unique positive root of the function $f(x) = 4x^2 - 4x + 7 + (2x-7)\sqrt{4x^2+20x+1}$ in (2, 2.5). Put $\beta_0 = \sqrt{4h+2}$ and $h_0 = \frac{4\beta^4 - 4\beta^2 + 7 + (2\beta^2-7)\sqrt{4\beta^4+20\beta^2+1}}{72\beta^2}$. Denote $\gamma_0 = \frac{8\beta^2(1+\sqrt{1+4h\beta^2})}{3(1+4h\beta^2)+(7-2\beta^2)\sqrt{1+4h\beta^2}+4(1-\beta^2)}$, where $h > h_0$. Then the following consequences hold about the fixed point E_2 illustrated in the Table 2.

Table 2. Properties of the fixed point E_2.

Conditions					Eigenvalues	Properties
$0 < h \le \frac{X_0-2}{4}$	$\sqrt{h+1} < \beta \le \sqrt{X_0}$		$0 < \gamma < \gamma_0$	$\beta < \beta_0$	$\lvert\lambda_1\rvert < 1, \lvert\lambda_2\rvert < 1$	*sink*
				$\beta = \beta_0$	$\lvert\lambda_1\rvert = 1, \lvert\lambda_2\rvert = 1$	*non − hyperbolic*
				$\beta > \beta_0$	$\lvert\lambda_1\rvert > 1, \lvert\lambda_2\rvert > 1$	*source*
			$\gamma = \gamma_0$		$\lambda_1 = -1, \lambda_2 \neq -1$	*non − hyperbolic*
			$\gamma > \gamma_0$		$\lvert\lambda_1\rvert > 1, \lvert\lambda_2\rvert < 1$	*saddle*
	$\beta > \sqrt{X_0}$	$h \le h_0$			$\lvert\lambda_1\rvert > 1, \lvert\lambda_2\rvert > 1$	*source*
		$h > h_0$	$0 < \gamma < \gamma_0$		$\lvert\lambda_1\rvert > 1, \lvert\lambda_2\rvert > 1$	*source*
			$\gamma = \gamma_0$		$\lambda_1 = -1, \lambda_2 \neq -1$	*non − hyperbolic*
			$\gamma > \gamma_0$		$\lvert\lambda_1\rvert > 1, \lvert\lambda_2\rvert < 1$	*saddle*
$h > \frac{X_0-2}{4}$	$\sqrt{h+1} < \beta \le \sqrt{X_0}$		$0 < \gamma < \gamma_0$		$\lvert\lambda_1\rvert < 1, \lvert\lambda_2\rvert < 1$	*sink*
			$\gamma = \gamma_0$		$\lambda_1 = -1, \lambda_2 \neq -1$	*non − hyperbolic*
			$\gamma > \gamma_0$		$\lvert\lambda_1\rvert > 1, \lvert\lambda_2\rvert < 1$	*saddle*
	$\beta > \sqrt{X_0}$	$h \le h_0$		$\beta < \beta_0$	$\lvert\lambda_1\rvert < 1, \lvert\lambda_2\rvert < 1$	*sink*
				$\beta = \beta_0$	$\lvert\lambda_1\rvert = 1, \lvert\lambda_2\rvert = 1$	*non − hyperbolic*
				$\beta > \beta_0$	$\lvert\lambda_1\rvert > 1, \lvert\lambda_2\rvert < 1$	*source*
		$h > h_0$	$0 < \gamma < \gamma_0$	$\beta < \beta_0$	$\lvert\lambda_1\rvert < 1, \lvert\lambda_2\rvert < 1$	*sink*
				$\beta = \beta_0$	$\lvert\lambda_1\rvert = 1, \lvert\lambda_2\rvert = 1$	*non − hyperbolic*
				$\beta > \beta_0$	$\lvert\lambda_1\rvert > 1, \lvert\lambda_2\rvert < 1$	*source*
			$\gamma = \gamma_0$		$\lambda_1 = -1, \lambda_2 \neq -1$	*non − hyperbolic*
			$\gamma > \gamma_0$		$\lvert\lambda_1\rvert > 1, \lvert\lambda_2\rvert < 1$	*saddle*

Proof. The Jacobian matrix $J(E_2)$ of the system (1.7) at the fixed point E_2 is

$$J(E_2) = \begin{pmatrix} 1 - \gamma x_0 + \frac{\gamma(1-x_0)}{2\beta^2 x_0} & -1 \\ \gamma(1-x_0)(1 - \frac{1}{2\beta^2 x_0}) & 1 \end{pmatrix},$$

whose characteristic polynomial can be written as

$$F(\lambda) = \lambda^2 - p\lambda + q, \tag{2.1}$$

where

$$p = 2 - \gamma x_0 + \frac{\gamma(1-x_0)}{2\beta^2 x_0}, \; q = 1 + \gamma(1 - 2x_0).$$

Note that $x_0 = \frac{1+\sqrt{1+4h\beta^2}}{2\beta^2}$, $y_0 = \gamma x_0(1 - x_0)$, and $E_2 = (x_0, y_0)$ is a positive fixed point, so $0 < x_0 < 1$. It's easy to calculate that

$$F(1) = \gamma(1 - x_0)\frac{\sqrt{1 + 4h\beta^2}}{1 + \sqrt{1 + 4h\beta^2}} > 0,$$

$$\begin{aligned}F(-1) &= 4 + \gamma[1 - 3x_0 + \frac{1 - x_0}{2\beta^2 x_0}]\\ &= 4 - \frac{\gamma(6\beta^2 x_0^2 - (2\beta^2 - 1)x_0 - 1)}{2\beta^2 x_0}\\ &= 4 - \frac{\gamma(3(\sqrt{1 + 4h\beta^2})^2 + (7 - 2\beta^2)\sqrt{1 + 4h\beta^2} + 4(1 - \beta^2))}{2\beta^2(1 + \sqrt{1 + 4h\beta^2})}.\end{aligned}$$

If $6\beta^2 x^{*2} - (2\beta^2 - 1)x^* - 1 = 0$ and $x^* > 0$, then $x^* = \frac{2\beta^2-1+\sqrt{4\beta^4+20\beta^2+1}}{12\beta^2}$. Simultaneously, it is easy to prove $x^* < \frac{1}{2}$.

Notice that $0 < h < \beta^2 - 1$. Moreover, $x_0 > (=, <)x^* \Leftrightarrow h > (=, <)h_0$. Additionally, $\beta^2 - 1 - h_0 = \frac{68\beta^4-68\beta^2-7-(2\beta^2-7)\sqrt{4\beta^4+20\beta^2+1}}{72\beta^2}$.

Set $x = \beta^2 > 1$. Denote

$$f(x) = 4x^2 - 4x + 7 + (2x - 7)\sqrt{4x^2 + 20x + 1}$$

and

$$g(x) = 68x^2 - 68x - 7 - (2x - 7)\sqrt{4x^2 + 20x + 1}.$$

Lemma 2.3 tells us that $f(x)$ is strictly increasing for $x > 1$ and has a unique positive root X_0 in (2, 2.5). From this one can see

$$h_0 < (=, >)0 \Leftrightarrow f(\beta^2) < (=, >)0 \Leftrightarrow \beta < (=, >)\sqrt{X_0}.$$

Obiviously, $g(1) = 18 > 0$, and $g'(x) = \frac{68(2x-1)\sqrt{4x^2+20x+1}-16x^2-32x+68}{\sqrt{4x^2+20x+1}} > \frac{68(2x-1)(2x+1)-16x^2-32x+68}{\sqrt{4x^2+20x+1}} = \frac{256x^2-32x}{\sqrt{4x^2+20x+1}} > 0$. So, $g(x) > g(1) > 0$ for $x > 1$. This implies that $h_0 < \beta^2 - 1$ always holds.

It is easy to see $x_0 > (=, <)x^* \Leftrightarrow 6\beta^2 x_0^2 - (2\beta^2 - 1)x_0 - 1 > (=, <)0$. From $F(-1) = 4 - \frac{\gamma(6\beta^2 x_0^2+2\beta^2 x_0+x_0-1)}{2\beta^2 x_0} = 0$, one has

$$\gamma = \frac{8\beta^2 x_0}{6\beta^2 x_0^2 - (2\beta^2 - 1)x_0 - 1} =: \gamma_0 = \frac{8\beta^2(1 + \sqrt{1 + 4h\beta^2})}{3(1 + 4h\beta^2) + (7 - 2\beta^2)\sqrt{1 + 4h\beta^2} + 4(1 - \beta^2)}.$$

Again, $\beta > (=, <)\beta_0 = \sqrt{4h + 2} \Leftrightarrow x_0 < (=, >)\frac{1+\sqrt{1+4h\beta_0^2}}{2\beta_0^2} = \frac{1}{2}$.

Now, one considers the following two cases:

1) Case I: $0 < h \leqslant \frac{X_0-2}{4}$. Then $\beta_0 = \sqrt{4h + 2} \leq \sqrt{X_0}$.
    (a) Subcase 1: $\sqrt{h + 1} < \beta \leq \sqrt{X_0}$. Then $h_0 \leq 0 < h$, implying $x^* < x_0$ and $\gamma_0 > 0$.

i. If $0 < \gamma < \gamma_0$, then $F(-1) > 0$.
- For $\beta < \beta_0$, $q < 1$, which reads $|\lambda_1| < 1$ and $|\lambda_2| < 1$ by Lemma 6.2(i.1). So, E_2 is a sink;
- For $\beta = \beta_0$, $q = 1$. Lemma 6.2(i.5) shows that $|\lambda_1| = |\lambda_2| = 1$, so E_2 is non-hyperbolic;
- For $\beta > \beta_0$, $q > 1$. Lemma 6.2(i.4) shows that $|\lambda_1| > 1$ and $|\lambda_2| > 1$, so E_2 is a source.

ii. If $\gamma = \gamma_0$, then $F(-1) = 0$. In other words, -1 is a root of the characteristic polynomial, namely E_2 is non-hyperbolic.

iii. If $\gamma > \gamma_0$, then $F(-1) < 0$. Using Lemma 6.2(i.3), we conclude that $|\lambda_1| < 1$ and $|\lambda_2| > 1$, so E_2 is a saddle.

(b) Subcase 2: $\beta > \sqrt{X_0}$. Then $h_0 > 0$.

i. If $0 < h \leq h_0$, then $0 < x_0 \leq x^*$, implying that $6\beta^2 x_0^2 - (2\beta^2 - 1)x_0 - 1 \leq 0$. So, $F(-1) \geq 4 > 0$. From $\beta > \sqrt{X_0} \geq \beta_0$, we see $q > 1$. Lemma 6.2(i.4) shows that $|\lambda_1| > 1$ and $|\lambda_2| > 1$, so E_2 is a source.

ii. If $h > h_0$, then $x^* < x_0 < 1$, implying that $\gamma_0 > 0$.

A. If $0 < \gamma < \gamma_0$, then $F(-1) > 0$. For $\beta > \sqrt{X_0} \geq \beta_0$, $q > 1$. Lemma 6.2(i.4) shows that $|\lambda_1| > 1$ and $|\lambda_2| > 1$, so E_2 is a source.

B. If $\gamma = \gamma_0$, then $F(-1) = 0$. In other words, -1 is one root of the characteristic polynomial, namely, E_2 is non-hyperbolic.

C. If $\gamma > \gamma_0$, then $F(-1) < 0$. Lemma 6.2(i.3) shows that $|\lambda_1| < 1$ and $|\lambda_2| > 1$, so E_2 is a saddle.

2) Case II: $h > \frac{X_0-2}{4}$. Then $\beta_0 = \sqrt{4h+2} > \sqrt{X_0}$.

(a) Subcase 1: $\sqrt{h+1} < \beta \leq \sqrt{X_0}$. Then $h_0 \leq 0 < h$, so, $x^* < x_0$ and hence $r_0 > 0$.

i. If $0 < \gamma < \gamma_0$, then $F(-1) > 0$. For $\sqrt{h+1} < \beta \leq \sqrt{X_0} < \beta_0$, $q < 1$, which reads $|\lambda_1| < 1$ and $|\lambda_2| < 1$ by Lemma 6.2(i.1). Therefore, E_2 is a sink.

ii. If $\gamma = \gamma_0$, then $F(-1) = 0$. Hence, E_2 is non-hyperbolic.

iii. If $\gamma > \gamma_0$, then $F(-1) < 0$. Lemma 6.2(i.3) shows that E_2 is a saddle.

(b) Subcase 2: $\beta > \sqrt{X_0}$. Then $h_0 > 0$.

i. If $\frac{X_0-2}{4} < h \leq h_0$, then $x_0 \leq x^*$, so, $F(-1) \geq 4 > 0$.
- For $\sqrt{X_0} < \beta < \beta_0$, $q < 1$, which reads $|\lambda_1| < 1$ and $|\lambda_2| < 1$ by Lemma 6.2(i.1), thus, E_2 is a sink;
- For $\beta = \beta_0$, $q = 1$. Lemma 6.2(i.5) shows that $|\lambda_1| = |\lambda_2| = 1$, so E_2 is non-hyperbolic;
- For $\beta > \beta_0$, $q > 1$. It follows from Lemma 6.2(i.4) that $|\lambda_1| > 1$ and $|\lambda_2| > 1$, hence E_2 is a source.

ii. If $h > h_0$, then $x^* < x_0$, so, $\gamma_0 > 0$.

A. If $0 < \gamma < \gamma_0$, then $F(-1) > 0$.
- For $\sqrt{X_0} < \beta < \beta_0$, $q < 1$. Lemma 6.2(i.1) tells us E_2 is a sink;
- For $\beta = \beta_0$, $q = 1$. Therefore, E_2 is non-hyperbolic;
- For $\beta > \beta_0$, $q > 1$. Lemma 6.2(i.4) shows that E_2 is a source.

B. If $\gamma = \gamma_0$, then $F(-1) = 0$, which shows E_2 is non-hyperbolic.

C. If $\gamma > \gamma_0$, then $F(-1) < 0$. Using Lemma 6.2(i.3), we conclude that $|\lambda_1| < 1$ and $|\lambda_2| > 1$, so E_2 is a saddle.

Summarizing the above analysis, the proof is complete.

3. Bifurcation analysis

In this section,we apply the center manifold theorem and bifurcation theory to investigate the local bifurcation problems of the system at the fixed points E_1 and E_2.

3.1. For fixed point $E_1 = (1, 0)$

It follows from Eq (1.4) that the fixed point E_1 always exists, regardless of what values the parameters β and γ take. One can see from Theorem 2.2 that the fixed point E_1 is a non-hyperbolic fixed point when $\beta = \sqrt{h+1}$ or $\gamma = 2$. As soon as the parameters β or γ goes through corresponding critical values, the dimensional numbers for the stable manifold and the unstable manifold of the fixed point E_1 vary. Therefore, a bifurcation probably occurs. Now, the considered parameter case is divided into the following three subcases:

Case I: $\beta = \sqrt{h+1}, \gamma \neq 2$;

Case II: $\beta \neq \sqrt{h+1}, \gamma = 2$;

Case III: $\beta = \sqrt{h+1}, \gamma = 2$.

First we consider Case I: $\beta = \sqrt{h+1}$, $\gamma \neq 2$, i.e., the parameters $(h, \beta, \gamma) \in \Omega_1 = \left\{(h, \beta, \gamma) \in R^3_+ \mid h > 0, \beta > 0, \gamma > 0, \gamma \neq 2\right\}$. Then, the following result is obtained.

Theorem 3.1. *Suppose the paramenters* $(h, \beta, \gamma) \in \Omega_1$. *Let* $\beta_1 = \sqrt{h+1}$. *If the parameter* β *varies in a small neighborhood of the critical value* β_1, *then the system* (1.7) *experiences a transcritical bifurcation at the fixed point* E_1 *when the parameter* β *goes through the critical value* β_1.

Proof. First, assume that $u_n = x_n - 1, v_n = y_n - 0$, which transforms the fixed point E_1 to the origin, and the system (1.7) to

$$\begin{cases} u_{n+1} = (u_n + 1)e^{-\gamma u_n - \frac{\beta v_n}{\sqrt{u_n+h+1}}} - 1, \\ v_{n+1} = v_n e^{\frac{\beta(u_n+1)}{\sqrt{u_n+h+1}} - 1}. \end{cases} \tag{3.1}$$

Second, giving a small perturbation β^* of the parameter β around β_1, i.e., $\beta^* = \beta - \beta_1$ with $0 < |\beta^*| \ll 1$, and letting $\beta^*_{n+1} = \beta^*_n = \beta^*$, the system (3.1) is perturbed into

$$\begin{cases} u_{n+1} = (u_n + 1)e^{-\gamma u_n - \frac{(\beta^*_n+\beta_1)v_n}{\sqrt{u_n+h+1}}} - 1, \\ v_{n+1} = v_n e^{\frac{(\beta^*_n+\beta_1)(u_n+1)}{\sqrt{u_n+h+1}} - 1}, \\ \beta^*_{n+1} = \beta^*_n. \end{cases} \tag{3.2}$$

By the Taylor expansion, the system (3.2) at $(u_n, v_n, \beta^*_n) = (0, 0, 0)$ can be written as

$$\begin{pmatrix} u_n \\ v_n \\ \beta^*_n \end{pmatrix} \to \begin{pmatrix} 1-\gamma & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_n \\ v_n \\ \beta^*_n \end{pmatrix} + \begin{pmatrix} g_1(u_n, v_n, \beta^*_n) + o(\rho_1^3) \\ g_2(u_n, v_n, \beta^*_n) + o(\rho_1^3) \\ 0 \end{pmatrix}, \tag{3.3}$$

where $\rho_1 = \sqrt{u_n^2 + v_n^2 + \beta^{*2}_n}$,

$$\begin{aligned}
g_1(u_n, v_n, \beta_n^*) = {} & u_n^2(\frac{\gamma^2}{2} - \gamma) + \frac{v_n^2}{2} + u_n v_n(\gamma - 1 + \frac{1}{2(h+1)}) - \frac{v_n \beta_n^*}{\sqrt{h+1}} - \frac{v_n^3}{6} \\
& + u_n^3(-\frac{\gamma^3}{6} + \frac{\gamma^2}{2}) + u_n^2 v_n(\frac{1}{2(h+1)} - \frac{3}{8(h+1)^2} + \gamma - \frac{\gamma}{2(h+1)} - \frac{\gamma^2}{2}) \\
& + u_n v_n^2(\frac{1}{2} - \frac{1}{2(h+1)} - \frac{\gamma}{2}) + \frac{v_n^2 \beta_n^*}{\sqrt{h+1}} \\
& + u_n v_n \beta_n^*(\frac{\gamma}{\sqrt{h+1}} - \frac{1}{\sqrt{h+1}} + \frac{1}{2(h+1)^{\frac{3}{2}}}), \\
g_2(u_n, v_n, \beta_n^*) = {} & u_n v_n(1 - \frac{1}{2(h+1)}) + \frac{v_n \beta_n^*}{\sqrt{h+1}} + \frac{u_n^2 v_n}{2}(1 - \frac{1}{h+1})^2 + \frac{v_n \beta_n^{*2}}{2(h+1)} \\
& + u_n v_n \beta_n^*(\frac{2}{\sqrt{h+1}} - \frac{1}{(h+1)^{\frac{3}{2}}}).
\end{aligned}$$

It is easy to derive the three eigenvalues of the matrix

$$A = \begin{pmatrix} 1-\gamma & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

to be $\lambda_1 = 1 - \gamma$ and $\lambda_2 = \lambda_3 = 1$ with corresponding eigenvectors

$$\xi_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \xi_2 = \begin{pmatrix} -\frac{1}{\gamma} \\ 1 \\ 0 \end{pmatrix}, \xi_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Notice $0 < \gamma \neq 2$ implies that $|\lambda_1| \neq 1$.

Set $T = (\xi_1, \xi_2, \xi_3)$, i.e.,

$$T = \begin{pmatrix} 1 & -\frac{1}{\gamma} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

then,

$$T^{-1} = \begin{pmatrix} 1 & \frac{1}{\gamma} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Taking the following transformation

$$(u_n, v_n, \beta_n^*)^T = T(X_n, Y_n, \delta_n)^T,$$

the system (3.3) is changed into the following form

$$\begin{pmatrix} X_n \\ Y_n \\ \delta_n \end{pmatrix} \to \begin{pmatrix} 1-\gamma & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X_n \\ Y_n \\ \delta_n \end{pmatrix} + \begin{pmatrix} g_3(X_n, Y_n, \delta_n) + o(\rho_2^3) \\ g_4(X_n, Y_n, \delta_n) + o(\rho_2^3) \\ 0 \end{pmatrix}, \tag{3.4}$$

where $\rho_2 = \sqrt{X_n^2 + Y_n^2 + \delta_n^2}$,

$$g_3(X_n, Y_n, \delta_n) = g_1(X_n - \frac{1}{\gamma}Y_n, Y_n, \delta_n) + \frac{1}{\gamma}g_2(X_n - \frac{1}{\gamma}Y_n, Y_n, \delta_n),$$
$$g_4(X_n, Y_n, \delta_n) = g_2(X_n - \frac{1}{\gamma}Y_n, Y_n, \delta_n).$$

Assume on the center manifold

$$X_n = h(Y_n, \delta_n) = a_{20}Y_n^2 + a_{11}Y_n\delta_n + a_{02}\delta_n^2 + o(\rho_3^2),$$

where $\rho_3 = \sqrt{Y_n^2 + \delta_n^2}$, then, from

$$\begin{aligned} X_{n+1} =&(1-\gamma)h(Y_n, \delta_n) + g_1(h(Y_n, \delta_n) - \frac{1}{\gamma}Y_n, Y_n, \delta_n) \\ &+\frac{1}{\gamma}g_2(h(Y_n, \delta_n) - \frac{1}{\gamma}Y_n, Y_n, \delta_n) + o(\rho_3^2), \\ h(Y_{n+1}, \delta_{n+1}) =&a_{20}Y_{n+1}^2 + a_{11}Y_{n+1}\delta_{n+1} + a_{02}\delta_{n+1}^2 + o(\rho_3^2) \\ =&a_{20}(Y_n + g_2(X_n - \frac{1}{\gamma}Y_n, Y_n, \delta_n)^2 \\ &+a_{11}(Y_n + g_2(X_n - \frac{1}{\gamma}Y_n, Y_n, \delta_n)\delta_n + a_{02}\delta_n^2 + o(\rho_3^2), \end{aligned}$$

and $X_{n+1} = h(Y_{n+1}, \delta_{n+1})$,we obtain the center manifold equation

$$\begin{aligned} (1-\gamma)h(Y_n, \delta_n)+&g_1(h(Y_n, \delta_n) - \frac{1}{\gamma}Y_n, Y_n, \delta_n) \\ &+\frac{1}{\gamma}g_2(h(Y_n, \delta_n) - \frac{1}{\gamma}Y_n, Y_n, \delta_n) + o(\rho_3^2) \\ &=a_{20}(Y_n + g_2(X_n - \frac{1}{\gamma}Y_n, Y_n, \delta_n)^2 \\ &+a_{11}(Y_n + g_2(X_n - \frac{1}{\gamma}Y_n, Y_n, \delta_n)\delta_n + a_{02}\delta_n^2 + o(\rho_3^2). \end{aligned}$$

By comparing the corresponding coefficients of terms with the same order in the above center manifold equation, it is easy to derive that

$$a_{20} = \frac{-2h-\gamma-1}{\gamma^3(2h+2)}, a_{11} = \frac{1-\gamma}{\gamma^2\sqrt{h+1}}, a_{02} = 0.$$

Therefore, the system (3.4) restricted to the center manifold is given by

$$\begin{aligned} Y_{n+1} = f_1(Y_n, \delta_n) :=&Y_n + g_2(h(Y_n, \delta_n) - \frac{1}{\gamma}Y_n, Y_n, \delta_n) + o(\rho_3^3) \\ =&Y_n + \frac{2h+1}{\gamma(2h+2)}Y_n^2 - \frac{Y_n\delta_n}{\sqrt{h+1}} + o(\rho_3^2) \end{aligned}$$

It is not difficult to calculate

$$f_1(Y_n,\delta_n)|_{(0,0)}=0,\frac{\partial f_1}{\partial Y_n}|_{(0,0)}=1,\frac{\partial f_1}{\partial \delta_n}|_{(0,0)}=0,$$
$$\frac{\partial^2 f_1}{\partial Y_n\partial\delta_n}|_{(0,0)}=-\frac{1}{\sqrt{h+1}}\neq 0,\frac{\partial^2 f_1}{\partial Y_n^2}|_{(0,0)}=\frac{2h+1}{\gamma(h+1)}\neq 0.$$

According to (21.1.43)–(21.1.46) in [24, p507], for a transcritical bifurication to occur, all conditions hold, hence, the system (1.7) undergoes a transcritical bifurcation at the fixed point E_1. The proof is over.

Next we consider Case II: $\beta\neq\sqrt{h+1}$, $\gamma=2$. By Theorem 2.2, one can see that $\lambda_1=-1$ and $|\lambda_2|\neq 1$when $\beta\neq\sqrt{h+1}$, $\gamma=2$. Thereout, the following result can be derived.

Theorem 3.2. *Let $\gamma_1=2$. Suppose the paramenters $(h,\beta,\gamma)\in\Omega_2=\left\{(h,\beta,\gamma)\in R_+^3\mid h>0,\beta>0,\beta\neq\sqrt{h+1},\gamma>0\right\}$. If the parameter γ varies in a small neighborhood of the critical value γ_1, then the system* (1.7) *undergoes a period-doubling bifurcation at the fixed point E_1 when the parameter γ goes through the critical value γ_1.*

Proof. Shifting $E_1=(1,0)$ to the origin $O(0,0)$ and giving a small perturbation γ^* of the parameter γ at the critical value γ_1 with $0<|\gamma^*|\ll 1$, the system (3.1) is changed into the following form:

$$\begin{cases} u_{n+1}=(u_n+1)e^{-(\gamma^*+2)\,u_n-\frac{\beta v_n}{\sqrt{u_n+h+1}}}-1,\\ v_{n+1}=v_n e^{\frac{\beta(u_n+1)}{\sqrt{u_n+h+1}}-1}.\end{cases}\tag{3.5}$$

Set $\gamma_{n+1}^*=\gamma_n^*=\gamma^*$, then (3.5) can be seen as

$$\begin{cases} u_{n+1}=(u_n+1)e^{-(\gamma^*+2)\,u_n-\frac{\beta v_n}{\sqrt{u_n+h+1}}}-1,\\ v_{n+1}=v_n e^{\frac{\beta(u_n+1)}{\sqrt{u_n+h+1}}-1},\\ \gamma_{n+1}^*=\gamma_n^*.\end{cases}\tag{3.6}$$

By the Taylor expansion, the system (3.6) at $(u_n,v_n,\beta_n^*)=(0,0,0)$ can be expended into

$$\begin{pmatrix}u_n\\ v_n\\ \beta_n^*\end{pmatrix}\to\begin{pmatrix}-1 & -\frac{\beta}{\sqrt{h+1}} & 0\\ 0 & \mathrm{e}^{\frac{\beta}{\sqrt{h+1}}-1} & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}u_n\\ v_n\\ \beta_n^*\end{pmatrix}+\begin{pmatrix}g_5(u_n,v_n,\gamma_n^*)+o(\rho_4^3)\\ g_6(u_n,v_n,\gamma_n^*)+o(\rho_4^3)\\ 0\end{pmatrix},\tag{3.7}$$

where $\rho_4=\sqrt{u_n^2+v_n^2+\gamma_n^{*2}}$,

$$\begin{aligned} g_5(u_n,v_n,\gamma_n^*)&=v_n^2\frac{\beta^2}{2(h+1)}+u_nv_n(\frac{\beta}{\sqrt{h+1}}+\frac{\beta}{2(h+1)^{\frac{3}{2}}})-v_n\gamma_n^*\\ &+\frac{2u_n^3}{3}-v_n^3\frac{\beta^3}{6(h+1)^{\frac{3}{2}}}-u_nv_n(\frac{\beta}{2(h+1)^{\frac{3}{2}}}-\frac{3\beta}{8(h+1)^{\frac{5}{2}}})\\ &+u_n^2\gamma_n^*-u_nv_n^2(\frac{\beta^2}{2(h+1)^2}+\frac{\beta^2}{2(h+1)})+u_nv_n\gamma_n^*\frac{\beta}{\sqrt{h+1}},\\ g_6(u_n,v_n,\gamma_n^*)&=u_nv_n[\beta(\frac{1}{\sqrt{h+1}}-\frac{1}{2(h+1)^{\frac{3}{2}}})\,\mathrm{e}^{(\frac{\beta}{\sqrt{h+1}}-1)}]\end{aligned}$$

$$- u_n^2 v_n \mathrm{e}^{(\frac{\beta}{\sqrt{h+1}}-1)}[\beta(\frac{1}{2(h+1)^{\frac{3}{2}}} - \frac{3}{8(h+1)^{\frac{5}{2}}}) - \frac{\beta^2(\frac{1}{\sqrt{h+1}} - \frac{1}{2(h+1)^{\frac{3}{2}}})^2}{2}].$$

It is not difficult to derive the three eigenvalues of the matrix

$$A = \begin{pmatrix} -1 & -\frac{\beta}{\sqrt{h+1}} & 0 \\ 0 & \mathrm{e}^{\frac{\beta}{\sqrt{h+1}}-1} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

to be $\lambda_1 = -1$, $\lambda_2 = \mathrm{e}^{(\frac{\beta}{\sqrt{h+1}}-1)}$ and $\lambda_3 = 1$ with corresponding eigenvectors

$$\xi_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ \xi_2 = \begin{pmatrix} -\frac{\beta}{(\mathrm{e}^{\frac{\beta}{\sqrt{h+1}}-1}+1)\sqrt{h+1}} \\ 1 \\ 0 \end{pmatrix},\ \xi_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Notice $\beta \neq \sqrt{h+1}$ implies $|\lambda_2| \neq 1$.

Set $T = (\xi_1, \xi_2, \xi_3)$, i.e.,

$$T = \begin{pmatrix} 1 & -\frac{\beta}{(\mathrm{e}^{\frac{\beta}{\sqrt{h+1}}-1}+1)\sqrt{h+1}} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

then,

$$T^{-1} = \begin{pmatrix} 1 & \frac{\beta}{(\mathrm{e}^{\frac{\beta}{\sqrt{h+1}}-1}+1)\sqrt{h+1}} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Taking the following transformation

$$(u_n, v_n, \gamma_n^*)^T = T(X_n, Y_n, \delta_n)^T,$$

the system (3.7) is changed into the following form

$$\begin{pmatrix} X_n \\ Y_n \\ \delta_n \end{pmatrix} \to \begin{pmatrix} -1 & 0 & 0 \\ 0 & \mathrm{e}^{\frac{\beta}{\sqrt{h+1}}-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X_n \\ Y_n \\ \delta_n \end{pmatrix} + \begin{pmatrix} g_7(X_n, Y_n, \delta_n) + o(\rho_5^3) \\ g_8(X_n, Y_n, \delta_n) + o(\rho_5^3) \\ 0 \end{pmatrix}, \tag{3.8}$$

where $\rho_5 = \sqrt{X_n^2 + Y_n^2 + \delta_n^2}$,

$$\begin{aligned} g_7(X_n, Y_n, \delta_n) &= g_5(X_n - \frac{\beta}{(\mathrm{e}^{\frac{\beta}{\sqrt{h+1}}-1}+1)\sqrt{h+1}} Y_n, Y_n, \delta_n) \\ &\quad + \frac{\beta}{(\mathrm{e}^{\frac{\beta}{\sqrt{h+1}}-1}+1)\sqrt{h+1}} g_6(X_n - \frac{\beta}{(\mathrm{e}^{\frac{\beta}{\sqrt{h+1}}-1}+1)\sqrt{h+1}} Y_n, Y_n, \delta_n), \\ g_8(X_n, Y_n, \delta_n) &= g_6(X_n - \frac{\beta}{(\mathrm{e}^{\frac{\beta}{\sqrt{h+1}}-1}+1)\sqrt{h+1}} Y_n, Y_n, \delta_n). \end{aligned}$$

Suppose on this center manifold

$$Y_n = h(X_n, \delta_n) = b_{20}X_n^2 + b_{11}X_n\delta_n + b_{02}\delta_n^2 + o(\rho_6^2),$$

where $\rho_6 = \sqrt{X_n^2 + \delta_n^2}$, which must satisfy

$$Y_{n+1} = e^{\frac{\beta}{\sqrt{h+1}}-1}h(Y_n, \delta_n) + g_8(X_n, h(Y_n, \delta_n), \delta_n) + o(\rho_6^3).$$

Similar to Case I, one can establish the corresponding center manifold equation. Comparing the corresponding coefficients of terms with the same type in the equation produces

$$b_{20} = 0, b_{11} = 0, b_{02} = 0.$$

That is to say, $Y_n = h(X_n, \delta_n) = o(\rho_6^2)$. Therefore, the center manifold equation is given by

$$\begin{aligned} X_{n+1} =& f_2(X_n, \delta_n) := -X_n + g_7(X_n, h(Y_n, \delta_n) \\ =& -X_n + g_5(X_n - \frac{\beta}{(e^{\frac{\beta}{\sqrt{h+1}}-1}+1)\sqrt{h+1}}h(X_n, \delta_n), h(X_n, \delta_n), \delta_n) \\ &+ \frac{\beta}{(e^{\frac{\beta}{\sqrt{h+1}}-1}+1)\sqrt{h+1}}g_6(X_n - \frac{\beta}{(e^{\frac{\beta}{\sqrt{h+1}}-1}+1)\sqrt{h+1}}h(X_n, \delta_n), h(X_n, \delta_n), \delta_n) \\ &+ o(\rho_6^3) \\ =& -X_n - X_n\delta_n + X_n^2\delta_n + \frac{2}{3}X_n^3 + o(\rho_6^3). \end{aligned}$$

Thereout, one has

$$f_2^2(X_n, \delta_n) = f_2(f_2(X_n, \delta_n), \delta_n) = X_n + 2X_n\delta_n + X_n\delta_n^2 - \frac{4}{3}X_n^3 + o(\rho_6^3).$$

Therefore, the following results are derived:

$$f_2(0,0) = 0, \frac{\partial f_2}{\partial X_n}|_{(0,0)} = -1, \frac{\partial f_2^2}{\partial \delta_n}|_{(0,0)} = 0, \frac{\partial^2 f_2^2}{\partial X_n^2}|_{(0,0)} = 0,$$
$$\frac{\partial^2 f_2^2}{\partial X_n \partial \delta_n}|_{(0,0)} = 2 \neq 0, \frac{\partial^3 f_2^2}{\partial {X_n}^3}|_{(0,0)} = -8 \neq 0,$$

which, according to (21.2.17)–(21.2.22) in [24, p516], satisfy all conditions for a period-doubling bifurcation to occur. Therefore, the system (1.7) undergoes a period-doubling bifurcation at E_1. Again,

$$-\frac{\partial^3 f_2^2}{\partial X_n^3}|_{(0,0)} / \frac{\partial^2 f_2^2}{\partial X_n \partial \delta_n}|_{(0,0)} = 4(> 0).$$

Therefore, the period-two orbit bifurcated from E_1 lies on the right of $\gamma_1 = 2$.

Of course, one can also compute the following two quantities, which are the transversal condition and non-degenerate condition for judging the occurrence and stability of a period-doubling bifurcation, respectively (see [3,15–18]),

$$\alpha_1 = (\frac{\partial^2 f_2}{\partial X_n \partial \delta_n} + \frac{1}{2}\frac{\partial f_2}{\partial \delta_n}\frac{\partial^2 f_2}{\partial {X_n}^2})|_{(0,0)},$$

$$\alpha_2 = (\frac{1}{6}\frac{\partial^3 f_2}{\partial {X_n}^3} + (\frac{1}{2}\frac{\partial^2 f_2}{\partial {X_n}^2})^2)|_{(0,0)}.$$

It is easy to say $\alpha_1 = -1$ and $\alpha_2 = \frac{2}{3}$. Due to $\alpha_2 > 0$, the period-two orbit bifurcated from E_1 is stable. The proof is complete.

Finally, we consider Case III: $\beta = \sqrt{h+1}$, $\gamma = 2$. At this time, the two eigenvalues of the linearized matrix evaluated at this fixed point E_1 are $\lambda_1 = -1$ and $\lambda_2 = 1$. The bifurcation problem in this case is very complicated and will be considered future work.

3.2. *For fixed point* $E_2 = (\frac{1+\sqrt{1+4h\beta^2}}{2\beta^2}, \gamma\frac{1+\sqrt{1+4h\beta^2}}{2\beta^2}(1 - \frac{1+\sqrt{1+4h\beta^2}}{2\beta^2}))$

Consider the bifurcation of the system (1.7) at the fixed point E_2. The parameters are divided into the following three cases:

Case I: $\beta = \sqrt{4h+2}$, $\gamma \neq \frac{8(4h+2)}{4h+1}$;

Case II: $\beta \neq \sqrt{4h+2}$, $\gamma = \frac{8\beta^2(1+\sqrt{1+4h\beta^2})}{3(\sqrt{1+4h\beta^2})^2+(7-2\beta^2)\sqrt{1+4h\beta^2}+4(1-\beta^2)}$;

Case III: $\beta = \sqrt{4h+2}$, $\gamma = \frac{8(4h+2)}{4h+1}$.

According to our calculations, there is no bifurcation under Case II. Additionally, the bifurcation problem in case III is very complicated and will be considered future work. Therefore, we only consider Case I.

Suppose the paramenters

$$(h, \beta, \gamma) \in \Omega_3 = \{(h, \beta, \gamma) \in R^3_+ \mid h > 0, \beta > 0, \gamma > 0, \gamma \neq \frac{8(4h+2)}{4h+1}\}.$$

Then the following result may be obtained.

Theorem 3.3. *Suppose the paramenters* $(h, \beta, \delta) \in \Omega_3$ *and meet* $\gamma < \frac{8(4h+2)}{4h+1}$. *Let* $\beta_2 = \sqrt{4h+2}$. *Then the system* (1.7) *undergoes a Neimark-Sacker bifurcation at the fixed point* E_2 *when the parament* β *varies in a small neighborhood of the critical value* β_2. *Moreover, if* $L < (>)0$ *in* (3.13)*, then a (an) stable (unstable) invariant closed orbit is bifurcated out from the fixed point* E_2 *of system* (1.7) *when* $\beta > (<)\beta_2$.

Proof. First, give a small perturbation β^{**} of the parameter β around β_2 in the system (3.1), i.e., $\beta^{**} = \beta - \beta_2$ with $0 < |\beta^{**}| \ll 1$, and set $x_{01} = x_{01}(\beta^{**}) = \frac{1+\sqrt{1+4h(\beta^{**}+\beta_2)^2}}{2(\beta^{**}+\beta_2)^2}$ and $y_{01} = \gamma x_{01}(1 - x_{01})$. Under the perturbation, the system (3.1) reads

$$\begin{cases} u_{n+1} = (u_n + x_{01})e^{-\gamma(1-(u_n-x_{01}))-\frac{(\beta_{**}+\beta_2)[v_n+\gamma x_{01}(1-x_{01})]}{\sqrt{u_n+x_{01}+h}}} - x_{01}, \\ v_{n+1} = (v_n + y_{01})e^{(\frac{(\beta^{**}+\beta_2)(u_n+x_{01})}{\sqrt{u_n+x_{01}+h}}-1)} - y_{01}. \end{cases} \tag{3.9}$$

The characteristic equation of the linearized equation of the system (3.9) at the origin (0,0) is

$$F(\lambda) = \lambda^2 - p(\beta^{**})\lambda + q(\beta^{**}) = 0, \tag{3.10}$$

where

$$p(\beta^{**}) = 2 - \gamma x_{01} + \frac{\gamma(1 - x_{01})}{2(\beta_{**} + \beta_2)^2 x_{01}},$$

$$q(\beta^{**}) = 1 + \gamma(1 - 2x_{01}).$$

Notice $\beta_2 = \sqrt{4h+2}$. For $\gamma < \frac{8(4h+2)}{4h+1}$, $-2 < p(0) < 2$, $q(0) = 1$, so $p^2(0) - 4q(0) < 0$, and hence the two roots of $F(\lambda) = 0$ are

$$\lambda_{1,2}(\beta^{**}) = \omega \pm \mu i,$$

where $\omega = -\frac{1}{2}p(\beta^{**})$, $\mu = \frac{1}{2}\sqrt{4q(\beta^{**}) - p^2(\beta^{**})}$.

It is easy to obseve that $|\lambda_{1,2}(\beta^{**})| = \sqrt{q(\beta^{**})}$ and $(|\lambda_{1,2}(\beta^{**})|)\big|_{\beta^{**}=0} = \sqrt{q(0)} = 1$. Therefore, a Neimark-Sacker bifurcation probably occurs.

The occurrence of the Neimark-Sacker bifurcation requires the following two conditions to be satisfied:

1) $\left(\frac{d|\lambda_{1,2}(\beta^{**})|}{d\beta^{**}}\right)\Big|_{\beta^{**}=0} \neq 0$;
2) $\lambda_{1,2}^i(0) \neq 1, i = 1, 2, 3, 4.$

Notice

$$\left(\frac{d|\lambda_{1,2}(\beta^{**})|}{d\beta^{**}}\right)\Big|_{\beta^{**}=0} = \frac{\gamma\,(2h+1)}{(4\,h+1)\,\sqrt{4\,h+2}} \neq 0.$$

Obviously $\lambda_{1,2}^i(0) \neq 1$ for $i = 1, 2, 3, 4$, so the two conditions are satisfied.

Second, in order to derive the normal form of the system (3.9), one expands (3.9) in power series up to the third-order term around the origin to get

$$\begin{cases} u_{n+1} = & a_{10}u_n + a_{01}v_n + a_{20}u_n^2 + a_{11}u_nv_n + a_{02}v_n^2 \\ & +a_{30}u_n^3 + a_{21}u_n^2v_n + a_{12}u_nv_n^2 + a_{03}v_n^3 + o(\rho_7^3), \\ v_{n+1} = & b_{10}u_n + b_{01}v_n + b_{20}u_n^2 + b_{11}u_nv_n + b_{02}v_n^2 \\ & +b_{30}u_n^3 + b_{21}u_n^2v_n + b_{12}u_nv_n^2 + b_{03}v_n^3 + o(\rho_7^3), \end{cases} \tag{3.11}$$

where $\rho_7 = \sqrt{u_n^2 + v_n^2}$,

$$a_{10} = \frac{\gamma}{8h+4} - \frac{\gamma}{2} + 1, a_{01} = -1,$$

$$a_{20} = (\frac{\gamma}{2} - \frac{\gamma}{8h+4})^2 - \gamma + \frac{\gamma}{4h+2} - \frac{3\gamma}{32(h+\frac{1}{2})^2},$$

$$a_{11} = \gamma + -\frac{\gamma}{4h+2} + \frac{1}{2h+1} - 2, a_{02} = 1,$$

$$a_{30} = \left(\gamma - \frac{\gamma}{4\,h+2}\right)\left(\frac{\gamma}{2} - \frac{\gamma}{8\,h+4}\right) - \frac{3\,\gamma}{16\left(h+\frac{1}{2}\right)^2} + \frac{3\,\gamma\left(\frac{\gamma}{2} - \frac{\gamma}{8\,h+4}\right)}{32\left(h+\frac{1}{2}\right)^2}$$
$$+\frac{5\,\gamma}{64\left(h+\frac{1}{2}\right)^3} - \frac{\left(\gamma - \frac{\gamma}{4\,h+2}\right)\left(\left(\gamma - \frac{\gamma}{4\,h+2}\right)\left(\frac{\gamma}{6} - \frac{\gamma}{24\,h+12}\right) - \frac{3\,\gamma}{32\left(h+\frac{1}{2}\right)^2}\right)}{2},$$

$$
\begin{aligned}
a_{21} &= 2\gamma - \left(\gamma - \frac{\gamma}{4h+2}\right)\left(\frac{\gamma}{6} - \frac{\gamma}{24h+12}\right) + \frac{1}{h+\frac{1}{2}} - \frac{3}{8\left(h+\frac{1}{2}\right)^2} - \frac{\gamma}{2h+1} \\
&\quad - \frac{\frac{\gamma}{2} - \frac{\gamma}{8h+4}}{2h+1} - \frac{\left(\gamma - \frac{\gamma}{4h+2}\right)\left(\frac{2\gamma}{3} - \frac{\gamma}{6h+3} + \frac{1}{2h+1}\right)}{2} + \frac{3\gamma}{16\left(h+\frac{1}{2}\right)^2}, \\
a_{12} &= \frac{\gamma}{4h+2} - \frac{1}{h+\frac{1}{2}} - \gamma + 2, a_{03} = -\frac{2}{3}, \\
b_{10} &= \frac{\gamma\left(1 - \frac{1}{4h+2}\right)}{2}, b_{01} = 1, b_{20} = \frac{\gamma\,[2(1-\frac{1}{4h+2})^2 - \frac{1}{h+\frac{1}{2}} - \frac{3}{8(h+\frac{1}{2})^2}]}{4}, \\
b_{02} &= 0, b_{11} = 2 - \frac{1}{8h+4}, \\
b_{03} &= 0, b_{21} = 2\left(1 - \frac{1}{4h+2}\right)^2 - \frac{1}{\left(h+\frac{1}{2}\right)} + \frac{3}{8\left(h+\frac{1}{2}\right)^2}, b_{12} = 0,
\end{aligned}
$$

$$
\begin{aligned}
b_{30} &= \frac{\gamma}{4}\left(2 - \frac{1}{2h+1}\right)\left(\frac{2\left(1-\frac{1}{4h+2}\right)^2}{3} - \frac{1}{2h+1} + \frac{3}{16\left(h+\frac{1}{2}\right)^2},\right) \\
&\quad - \frac{\gamma}{4}\left(\frac{2}{\sqrt{h+\frac{1}{2}}} - \frac{1}{2\left(h+\frac{1}{2}\right)^{3/2}}\right)\left(\frac{1}{2\sqrt{\left(h+\frac{1}{2}\right)}} - \frac{3}{16\left(h+\frac{1}{2}\right)^{3/2}}\right) \\
&\quad + \frac{\gamma}{4}\left(\frac{3}{4\left(h+\frac{1}{2}\right)^2} - \frac{5}{16\left(h+\frac{1}{2}\right)^3}\right).
\end{aligned}
$$

Take matrix

$$
T = \begin{pmatrix} 0 & a_{01} \\ \mu & 1-\omega \end{pmatrix}, \text{ then } T^{-1} = \begin{pmatrix} \frac{\omega-1}{\mu a_{01}} & \frac{1}{\mu} \\ \frac{1}{a_{01}} & 0 \end{pmatrix}.
$$

Make a change of variables

$$
(u, v)^T = T(X, Y)^T,
$$

then the system (3.11) is changed to the following form:

$$
\begin{pmatrix} X \\ Y \end{pmatrix} \to \begin{pmatrix} \omega & -\mu \\ \mu & \omega \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} F(X, Y) + o(\rho_8^4) \\ G(X, Y) + o(\rho_8^4) \end{pmatrix}, \tag{3.12}
$$

where $\rho_8 = \sqrt{X^2 + Y^2}$,

$$
\begin{aligned}
F(X, Y) &= c_{20}u^2 + c_{11}uv + c_{02}v^2 + c_{30}u^3 + c_{21}u^2v + c_{12}uv^2 + c_{03}v^3, \\
G(X, Y) &= d_{20}u^2 + d_{11}uv + d_{02}v^2 + d_{30}u^3 + d_{21}u^2v + d_{12}uv^2 + d_{03}v^3, \\
u &= a_{01}Y, v = \mu X + (1-\omega)Y,
\end{aligned}
$$

$$c_{20}=\frac{a_{20}(\omega-1)}{\mu a_{01}}+\frac{b_{20}}{\mu},c_{11}=\frac{a_{11}(\omega-1)}{\mu a_{01}}+\frac{b_{11}}{\mu},c_{02}=\frac{a_{02}(\omega-1)}{\mu a_{01}}+\frac{b_{02}}{\mu},$$
$$c_{30}=\frac{a_{30}(\omega-1)}{\mu a_{01}}+\frac{b_{30}}{\mu},c_{21}=\frac{a_{21}(\omega-1)}{\mu a_{01}}+\frac{b_{21}}{\mu},c_{12}=\frac{a_{12}(\omega-1)}{\mu a_{01}}+\frac{b_{12}}{\mu},$$
$$c_{03}=\frac{a_{03}(\omega-1)}{\mu a_{01}}+\frac{b_{03}}{\mu},d_{20}=\frac{a_{20}}{a_{01}},d_{11}=\frac{a_{11}}{a_{01}},d_{02}=\frac{a_{02}}{a_{01}},d_{30}=\frac{a_{30}}{a_{01}},$$
$$d_{21}=\frac{a_{21}}{a_{01}},d_{12}=\frac{a_{12}}{a_{01}},d_{03}=\frac{a_{03}}{a_{01}}$$

Furthermore,

$$\begin{aligned}
&F_{XX}|_{(0,0)}=2c_{02}\mu^3,F_{XY}|_{(0,0)}=c_{11}a_{01}\mu+2c_{02}\mu(1-\omega),\\
&F_{YY}|_{(0,0)}=2c_{02}a_{01}^2+2c_{11}a_{01}(1-\omega),F_{XXX}|_{(0,0)}=6c_{03}\mu^3,\\
&F_{XXY}|_{(0,0)}=2c_{21}a_{01}\mu^2+6c_{03}\mu^2(1-\omega),\\
&F_{XYY}|_{(0,0)}=2c_{21}a_{01}^2\mu+4c_{12}a_{01}\mu(1-\omega)+6c_{03}\mu(1-\omega)^2,\\
&F_{YYY}|_{(0,0)}=4(1-\omega)^3+6c_{30}a_{01}^3+4c_{21}a_{01}^2(1-\omega)+6c_{12}a_{01}(1-\omega)^2,\\
&G_{XX}|_{(0,0)}=2d_{02}\mu^3,G_{XY}|_{(0,0)}=d_{11}a_{01}\mu+2d_{02}\mu(1-\omega),\\
&G_{YY}|_{(0,0)}=2d_{02}a_{01}^2+2d_{11}a_{01}(1-\omega),G_{XXX}|_{(0,0)}=6c_{03}\mu^3,\\
&G_{XXY}|_{(0,0)}=2d_{21}a_{01}\mu^2+6d_{03}\mu^2(1-\omega),\\
&G_{XYY}|_{(0,0)}=2d_{21}a_{01}^2\mu+4d_{12}a_{01}\mu(1-\omega)+6d_{03}\mu(1-\omega)^2,\\
&G_{YYY}|_{(0,0)}=4(1-\omega)^3+6d_{30}a_{01}^3+4d_{21}a_{01}^2(1-\omega)+6d_{12}a_{01}(1-\omega)^2.
\end{aligned}$$

To determine the stability and direction of the bifurcation curve (closed orbit) for the system (1.7), the discriminating quantity L should be calculated and not to be zero, where

$$L=-Re\Big(\frac{(1-2\lambda_1)\lambda_2^2}{1-\lambda_1}\zeta_{20}\zeta_{11}\Big)-\frac{1}{2}|\zeta_{11}|^2-|\zeta_{02}|^2+Re(\lambda_2\zeta_{21}),\tag{3.13}$$

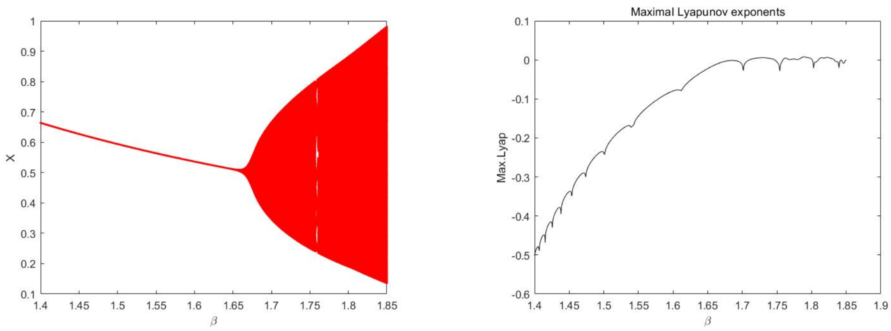
$$\begin{aligned}
\zeta_{20}&=\frac{1}{8}[F_{XX}-F_{YY}+2G_{XY}+i(G_{XX}-G_{YY}-2F_{XY})]|_{(0,0)},\\
\zeta_{11}&=\frac{1}{4}[F_{XX}+F_{YY}+i(G_{XX}+G_{YY})]|_{(0,0)},\\
\zeta_{02}&=\frac{1}{8}[F_{XX}-F_{YY}-2G_{XY}+i(G_{XX}-G_{YY}+2F_{XY})]|_{(0,0)},\\
\zeta_{21}&=\frac{1}{16}[F_{XXX}+F_{XYY}+G_{XXY}+G_{YYY}\\
&\quad+i(G_{XXX}+G_{XYY}-F_{XXY}-F_{YYY})]|_{(0,0)}.
\end{aligned}$$

Based on [24–26], we see that if $L<(>)0$, then an attracting (a repelling) invariant closed curve bifurcates from the fixed point for $\beta>(<)\beta_2$.

The proof of this theorem is complete.

4. Numerical simulation

In this section, we utilize Matlab to perform numerical simulations to validate the above theoretical analysis through utilizing bifurcation diagrams, phase portraits, maximum Lyapunov expoents, and fractal dimensions of the system (1.7) at the fixed point E_2.



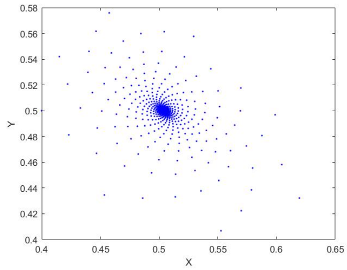
(a) $\beta \in (1.4, 1.85)$ (b) $\beta \in (1.4, 1.85)$

Figure 1. Bifurcation of the system (1.7) in (β, x)-plane and maximal Lyapunov exponents.

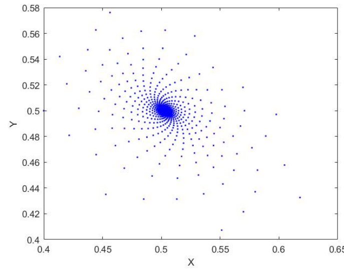
Consider the fixed point E_2. Vary β in the range $(1.4, 1.85)$, and fix $\gamma = 2, h = 0.2$ with the initial value $(x_0, y_0) = (0.4, 0.5)$. Figure 1(a) shows that the existence of a Neimark-Sacker bifurcation at the fixed point $E_2 = (0.5, 0.5)$ when $\beta = \beta_2 = \sqrt{2.8} \approx 1.6733$. Figure 1(b) describes the spectrum of maximum Lyapunov exponents, which are positive for the parameter $\beta \in (1.4, 1.85)$, which leads to chaos in system (1.7). For this, the interested readers may refer to [28] to create an electronic emulator to get immediate results.

The phase portraits associated with Figure 1(a) are drawn in Figure 2. When β increases, a circular curve enclosing the fixed point E_2 appears.

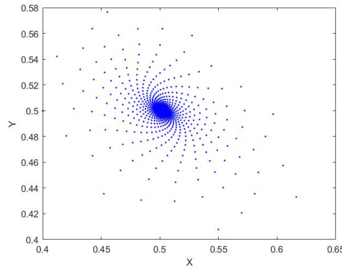
By choosing a different initial value $(x_0, y_0) = (0.52,\ 0.48)$ and three same values of β, the correspending phase portraits are plotted in Figure 3. Figure 2 implies that the closed curve is stable outside, while Figure 3 indicates that the closed curve is stable inside. That is to say, a stable invariant closed curve around the fixed point E_2 occurs. This agrees with the conclusion in Theorem 3.3.



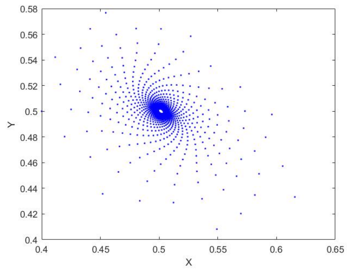
(a) $\beta = 1.665$



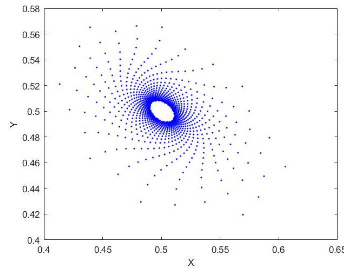
(b) $\beta = 1.667$



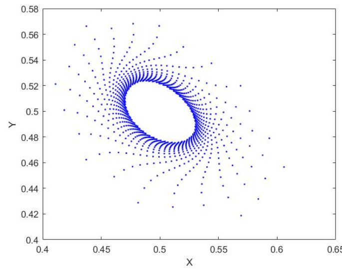
(c) $\beta = 1.669$



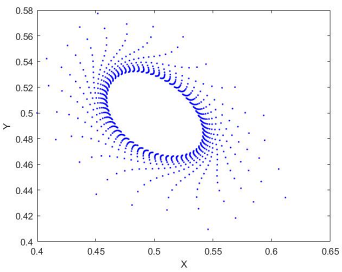
(d) $\beta = 1.67$



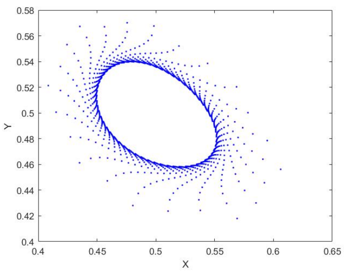
(e) $\beta = 1.672$



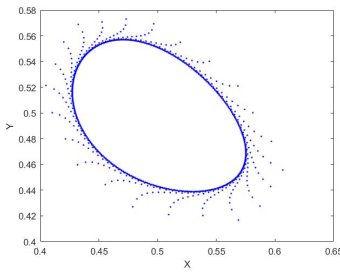
(f) $\beta = 1.674$



(g) $\beta = 1.675$

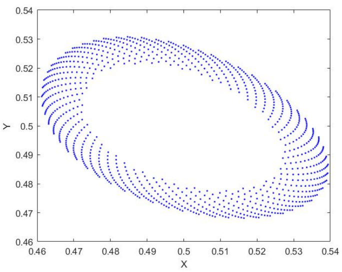


(h) $\beta = 1.676$

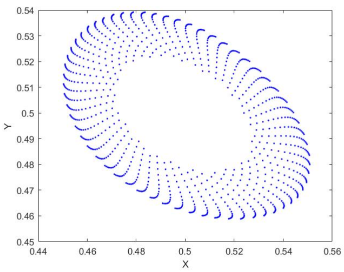


(i) $\beta = 1.679$

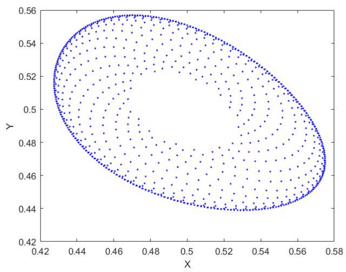
Figure 2. Phase portraits for the system (1.7) with $\gamma = 2, h = 0.2$ and different β with the initial value $(x_0, y_0) = (0.4, 0.5)$ outside the closed orbit.



(a) $\beta = 1.675$



(b) $\beta = 1.676$



(c) $\beta = 1.679$

Figure 3. Phase portraits for the system (1.7) with $\gamma = 2, h = 0.2$ and different β with the initial value $(x_0, y_0) = (0.52, 0.48)$ inside the closed orbit.

5. Conclusions

In this paper, we consider a predator–prey model with the prey individual behaviour and herd behaviour. By using the semi-discretization method, the continuous system (1.4) is transformed to the discrete system (1.7). Under the given parametric conditions, we demonstrate the existence and stability of three nonnegative fixed points $E_0 = (0, 0)$, $E_1 = (1, 0)$ and $E_2 = (\frac{1+\sqrt{1+4h\beta^2}}{2\beta^2}, \gamma\frac{1+\sqrt{1+4h\beta^2}}{2\beta^2}(1 - \frac{1+\sqrt{1+4h\beta^2}}{2\beta^2}))$. By using the center manifold theory, we determine the existence conditions of transcritical bifurcation and period-doubling bifurcation in the fixed point E_1 and the Neimark-Sacker bifurcation at the fixed point E_2 of system (1.7). we also derive that E_2 is asymptotically stable when $\beta > \beta_2 = \sqrt{4h + 2}$ and unstable when $\beta < \beta_2$. Additionally, the system (1.7) undergoes a Neimark-Sacker bifurcation when the parameter β goes through the critical value β_2. The occurrence for this phenomenon of Neimark-Sacker bifurcation indicates the coexistence of prey and predator when the parameter $\beta = \beta_2$.

Our findings indicate that the proposed discrete model shows a behaviour similar to the one found in the corresponding continuous model [27]. In particular, it gives rise to stable populations limit cycles. Ecologically, this means that the suggested response function may be adequate if we want to model the prey herd behaviour that takes place only for a sizable population, namely when the population level settles in a certain threshold (critical value).

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests. All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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Appendix

We here give a definition and a key Lemma.

Definition 5.1. *Let $E(x, y)$ be a fixed piont of the system* (1.7) *with multipliers λ_1 and λ_2.*

(i) *If $|\lambda_1| < 1$ and $|\lambda_2| < 1$, $E(x, y)$ is called sink, so a sink is locally asymptotically stable.*

(ii) *If $|\lambda_1| > 1$ and $|\lambda_2| > 1$, $E(x, y)$ is called source, so a source is locally asymptotically unstable.*

(iii) *If $|\lambda_1| < 1$ and $|\lambda_2| > 1$ (or $|\lambda_1| > 1$ and $|\lambda_2| < 1$), $E(x, y)$ is called saddle.*

(iv) *If either $|\lambda_1| = 1$ or $|\lambda_2| = 1$, $E(x, y)$ is called to be non-hyperbolic.*

Lemma 5.2. *Let $F(\lambda) = \lambda^2 + B\lambda + C$, where B and C are two real constants. Suppose λ_1 and λ_2 are two roots of $F(\lambda) = 0$. Then the following statements hold.*

(*i*) *If* $F(1) > 0$, *then*

(*i*.1) $|\lambda_1| < 1$ *and* $|\lambda_2| < 1$ *if and only if* $F(-1) > 0$ *and* $C < 1$*;*

(*i*.2) $\lambda_1 = -1$ *and* $\lambda_2 \neq -1$ *if and only if* $F(-1) = 0$ *and* $B \neq 2$*;*

(*i*.3) $|\lambda_1| < 1$ *and* $|\lambda_2| > 1$ *if and only if* $F(-1) < 0$*;*

(*i*.4) $|\lambda_1| > 1$ *and* $|\lambda_2| > 1$ *if and only if* $F(-1) > 0$ *and* $C > 1$*;*

(*i*.5) λ_1 *and* λ_2 *are a pair of conjugate complex roots and,* $|\lambda_1| = |\lambda_2| = 1$ *if and only if* $-2 < B < 2$ *and* $C = 1$*;*

(*i*.6) $\lambda_1 = \lambda_2 = -1$ *if and only if* $F(-1) = 0$ *and* $B = 2$.

(*ii*) *If* $F(1) = 0$, *namely,* 1 *is one root of* $F(\lambda) = 0$, *then another root* λ *satisfies* $|\lambda| = (<, >)1$ *if and only if* $|C| = (<, >)1$.

(*iii*) *If* $F(1) < 0$, *then* $F(\lambda) = 0$ *has one root lying in* $(1, \infty)$. *Moreover,*

(*iii*.1) *the other root* λ *satisfies* $\lambda < (=) - 1$ *if and only if* $F(-1) < (=)0$*;*

(*iii*.2) *the other root* $-1 < \lambda < 1$ *if and only if* $F(-1) > 0$.

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