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Research article

Semi-separation axioms associated with the Alexandroff compactification of the MW-topological plane

Sik Lee[1] **and Sang-Eon Han**[2,*]

[1] Department of Mathematics Education, Chonnam National University, Gwangju 61186, Republic of Korea

[2] Department of Mathematics Education, Institute of Pure and Applied Mathematics, Jeonbuk National University, Jeonju, Jeonbuk 54896, Republic of Korea

* **Correspondence:** Email: sehan@jbnu.ac.kr; Tel: +82632704449.

Abstract: The present paper aims to investigate some semi-separation axioms relating to the Alexandroff one point compactification (Alexandroff compactification, for short) of the digital plane with the Marcus-Wyse (MW-, for brevity) topology. The Alexandroff compactification of the MW-topological plane is called the infinite MW-topological sphere up to homeomorphism. We first prove that under the MW-topology on $\mathbb{Z}^2$ the connectedness of $X(\subset \mathbb{Z}^2)$ with $X^\sharp \geq 2$ implies the semi-openness of X. Besides, for the infinite MW-topological sphere, we introduce a new condition for the hereditary property of the compactness of it. In addition, we investigate some conditions preserving the semi-openness or semi-closedness of a subset of the MW-topological plane in the process of an Alexandroff compactification. Finally, we prove that the infinite MW-topological sphere is a semi-regular space; thus, it is a semi-T_3-space because it is a semi-T_1-space. Hence we finally conclude that an Alexandroff compactification of the MW-topological plane preserves the semi-T_3 separation axiom.

Keywords: Marcus-Wyse topology; semi-T_3-space; Alexandroff compactification; infinite MW-sphere; semi-regular, hereditary property

1. Introduction

Let (X, T) be a non-compact, locally compact, and Hausdorff topological space. Then it is well known that an Alexandroff one point compactification (Alexandroff compactification, for short) of (X, T), denoted by (X^*, T^*) and $* \in X^* \setminus X$, is also a Hausdorff topological space [1]. Namely, it is clear that an Alexandroff compactification preserves the Hausdorff topological structure. Meanwhile, unlike the Hausdorff compactification, let (Y, T) be neither compact nor Hausdorff but locally compact. Then we can also consider an Alexandroff (not Hausdorff) compactification (Y^*, T^*) of (Y, T) [2]. Then, it is well known that (Y^*, T^*) is a T_1-space (or topological space satisfying the T_1-separation axiom) if and only if (Y, T) is a T_1-space. However, up to now the study of the other cases remains open. For instance, assume that (Y, T) is not a compact space, but rather a locally compact space with a certain T_i-separation axiom, $i \neq 1$, or semi-T_j-separation axiom, $j \leq 3$. Then we may ask if an Alexandorff compactification of (Y, T) preserves the given T_i-separation axiom, $i \neq 1$, and the semi-T_j-separation axiom, $j \leq 3$, which remains open.

In the present paper, for convenience, we use the following notations. $\mathbb{N}$ and $\mathbb{Z}$ indicate the sets of natural numbers and integers, respectively. " :=" will often be used for indicating a new term. For the distinct integers $s, t \in \mathbb{Z}$, $[s, t]_{\mathbb{Z}} := \{x \in \mathbb{Z} \,|\, s \leq x \leq t\}$. $\aleph_0$ means the first infinite cardinality of a set, as usual and " $\subset$" (resp. $X^\sharp$) denotes a 'proper subset or equal' (resp. the cardinality of the given set X). Besides, for a subset A of X, A^c means the complement of A in X. In addition, given a topological space (X, T), for $A \subset X$ the closure of A in (X, T) is denoted by $Cl_T(A)$.

In order to study semi-separation axioms of topological spaces, we need many concepts such as semi-open [3], semi-regular [4], semi-closed [3, 4], s-regular [5–7] and so on. Since both separation axioms and semi-separation axioms take important roles in modern mathematics including pure and applied topology such as digital topology, computational topology, and so on, these axioms were studied in [4–6, 8–23]. In detail, in realation to the study of semi-topological properties of digital spaces, up to now there have been some works in the literature [16, 18, 19, 24–26]. Since the present paper examines what semi-separation is preserved by the Alexandroff compactification of the MW-topological plane, this trial is new and can facilitate some studies in the field of pure and applied topology and computer science.

The paper [27] introduced a topology on $\mathbb{Z}^2$, called the Marcus-Wyse topology, which can be used in applied topology as well as computer science. After that, many works investigated some properties of it and used them in applied sciences. Since we will often use "Marcus-Wyse" in the paper, hereafter we will use the term "MW-" instead of "Marcus-Wyse", if there is no danger of ambiguity. Since the MW-topological structure is one of the crucial platforms for studying the 2-dimensional lattice set $\mathbb{Z}^2$ from the viewpoints of pure and applied topology, the paper [19] proved that an MW-topological space is a semi-T_3-space. Besides, the recent paper [28] initially established an Alexandroff compactification of the MW-topological plane. Hereinafter, for convenience, let $((\mathbb{Z}^2)^*, \gamma^*)$ be an Alexandroff compactification of the MW-topological plane $(\mathbb{Z}^2, \gamma)$ and we call it the infinite MW-topological sphere up to homeomorphism [28]. Then we have the following queries.

(1) Under $(\mathbb{Z}^2, \gamma)$, are there some relationships between the connectedness of $X(\subset \mathbb{Z}^2)$ and the semi-openness of X?

(2) For $((\mathbb{Z}^2)^*, \gamma^*)$, is there a new condition for the hereditary property of the compactness of it?

(3) Are the semi-openness and semi-closedness in the MW-topological plane respectively preserved in the process of an Alexandroff compactification?

(4) Does the infinite MW-topological sphere satisfy the semi-T_3-separation axiom?

Equivalently, for the MW-topological plane, does an Alexandroff compactification of it preserve the semi-T_3-separation axiom?

To address the queries above, first of all, the present paper initially proves that the infinite MW-topological sphere is a semi-regular space so that we can finally confirm that an Alexandroff compactification of the MW-topological plane preserves the semi-T_3-separation axiom. Besides, the paper corrects some misprinted parts in the literature.

The rest of the paper is organized as follows. Section 2 provides some basic notions associated with the MW-topological space. Section 3 studies various properties of semi-closed and semi-open sets in the MW-topological plane. Section 4 investigates some properties associated with the semi-T_3-separation axiom of the MW-topological space. Furthermore, we introduce a condition for the hereditary property of the compactness of the infinite MW-topological sphere. Section 5 proves that the infinite MW-topological sphere is a semi-T_3-space. Thus we can confirm that an Alexandroff compactification of the MW-topological plane preserves the semi-T_3-separation axiom. Section 6 concludes the paper with a summary and further work.

2. Preliminaries

A topological space (X, T) is called an Alexandroff space [2,29] if for each $x \in X$, the intersection of all open sets of X containing x (denoted by $SN_T(x)$) is T-open [2]. To make the paper self-contained, let us now recall some properties of MW-topological spaces associated with the semi-separation axioms. The MW-topological plane was initially established in [27] and some studies of its various properties include [18–20, 26, 27, 30–32].

In detail, for a point $p := (x, y) \in \mathbb{Z}^2$, we will take the notation [33]

$$N_4(p) := \{(x \pm 1, y), p, (x, y \pm 1)\}, \tag{2.1}$$

called a 4-neighborhood of a given point $p := (x, y) \in \mathbb{Z}^2$.
In addition, we also use the notation $N_4(p)^* = N_4(p) \setminus \{p\}$ [33].

The MW-topology on $\mathbb{Z}^2$, denoted by $(\mathbb{Z}^2, \gamma)$, is generated by the set $\{M(p)\}$ in (2.2) below as a base [27], where for each point $p = (x, y) \in \mathbb{Z}^2$

$$M(p) := \begin{Bmatrix} N_4(p) \text{ if } x + y \text{ is even, and} \\ \{p\} : \text{ else.} \end{Bmatrix} \tag{2.2}$$

In relation to the further statement of a point in $\mathbb{Z}^2$, in the paper we call a point $p = (x_1, x_2)$ *doubly even* if $x_1 + x_2$ is an even number such that each x_i is even, $i \in \{1, 2\}$; and *even* if $x_1 + x_2$ is an even number. Then, in these cases we use the notation $p \in (\mathbb{Z}^2)_E$: the set of all doubly even or even points. Besides, we call a point $p = (x_1, x_2)$ *odd* if $x_1 + x_2$ is an odd number [34]. In this case we use the notation $p \in (\mathbb{Z}^2)_O$: the set of all odd points.

In each subspace of $(\mathbb{Z}^2, \gamma)$ described in Figures 1 and 2, the symbol $\diamond$ (resp. $\bullet$) means a *doubly even or even point* (resp. an *odd point*). In view of (2.2), we can obviously obtain the following: under $(\mathbb{Z}^2, \gamma)$, the singleton with either a doubly even point or an even point is a closed set. In addition, the singleton with an odd point is an open set. Besides, for a subset $X \subset \mathbb{Z}^2$, the subspace induced by $(\mathbb{Z}^2, \gamma)$ is obtained, denoted by (X, γ_X), as usual, and it is also called an MW-topological space. It is clear that (X, γ_X) is an Alexandroff space.

In terms of this perspective, we clearly observe that the *smallest (open) neighborhood* of the point $p := (p_1, p_2)$ of $\mathbb{Z}^2$, denoted by $SN_\gamma(p) \subset \mathbb{Z}^2$, is the following [31]:

$$SN_\gamma(p) := \begin{Bmatrix} \{p\} \text{ if } p \text{ is an odd point} \\ N_4(p) \text{ if } p \text{ is a doubly even or even point.} \end{Bmatrix} \tag{2.3}$$

Hereafter, in (X, γ_X), for a point $p \in X$ we use the notation $SN_\gamma(p) := SN_\gamma(p) \cap X$ [31] for short. Using the smallest open set of (2.3), the notion of an MW-adjacency in $(\mathbb{Z}^2, \gamma)$ is defined, as follows: For distinct points $p, q \in (\mathbb{Z}^2, \gamma)$, we define that p is MW-adjacent to q [30] if

$$p \in SN_\gamma(p) \text{ or } q \in SN_\gamma(q)$$

Owing to the properties of (2.3), we obviously obtain the following: Given a point $p := (p_1, p_2)$ of $\mathbb{Z}^2$, the closure of the singleton $\{p\}$ in $(\mathbb{Z}^2, \gamma)$ is obtained and denoted by $Cl_\gamma(\{p\}) \subset \mathbb{Z}^2$, as follows [30]:

$$Cl_\gamma(\{p\}) := \begin{Bmatrix} \{p\} \text{ if } p \text{ is a doubly even or even point,} \\ N_4(p) \text{ if } p \text{ is an odd point.} \end{Bmatrix} \tag{2.4}$$

Definition 2.1. *[31] Let $X := (X, \gamma_X)$ be an MW-topological space.*

(1) Given distinct points x and y in X, we say that an MW-path from x to y in X is a sequence $(p_i)_{i\in[0,l]_\mathbb{Z}} \subset X, l \in \mathbb{N}$ such that $p_0 = x$, $p_l = y$ and each point p_i is MW-adjacent to p_{i+1} and $i \in [0, l-1]_\mathbb{Z}$. The number l is the length of this path. In particular, a singleton in $(\mathbb{Z}^2, \gamma)$ is assumed to be an MW-path with a length 0.

(2) Distinct points $x, y \in X$ are called MW-path connected (or MW-connected) if there is a finite MW-path $(p_0, p_1, ..., p_m)$ on X with $p_0 = x$ and $p_m = y$. For arbitrary points $x, y \in X$, if there is an MW-path $(p_i)_{i\in[0,m]_\mathbb{Z}} \subset X$ such that $p_0 = x$ and $p_m = y$; then, we say that X is MW-path connected (or MW-connected).

(3) We say that a simple MW-path in X is a finite MW-path $(p_i)_{i\in[0,m]_\mathbb{Z}}$ in X such that the points p_i and p_j are MW-adjacent if and only if $|\, i - j\,| = 1$.

Let us recall some concepts associated with a semi-open and semi-closed set. Namely, a subset A of a topological space (X, T) is said to be *semi-open* if there is an open set O in (X, T) such that $O \subset A \subset Cl(O)$. Besides, we say that a subset B of a topological space (X, T) is *semi-closed* if the complement of B in X (or B^c) is semi-open in (X, T). Then it turns out that a subset A of (X, T) is semi-open if and only if $A \subset Cl(Int(A))$ [3] and a subset B of (X, T) is semi-closed if and only if $Int(Cl(B)) \subset B$ [11]. Hence it is clear that both the whole set X and an empty set are both semi-open and semi-closed.

Based on the notions of a semi-open and a semi-closed set, we obviously obtain the following.

Remark 2.2. *[3, 26] (1) Given two semi-open sets, the union of them is semi-open.*

(2) Given two semi-closed sets, the intersection of them is semi-closed.

It is well known that any subspace of $(\mathbb{Z}^2, \gamma)$ is a semi-$T_{\frac{1}{2}}$ space which means that each singleton in $(\mathbb{Z}^2, \gamma)$ is either semi-closed or semi-open. Furthermore, since $(\mathbb{Z}^2, \gamma)$ is an Alexandroff space with the separation axiom T_0, the paper [24] proved that it is a semi-$T_{\frac{1}{2}}$ space. Owing to the topological structure of $(\mathbb{Z}^2, \gamma)$ (see the property of (2.3)), it is clear that an infinite subset of $(\mathbb{Z}^2, \gamma)$ is not compact in $(\mathbb{Z}^2, \gamma)$.

In relation to the study of semi-openess and semi-closedness of a set in $(\mathbb{Z}^2, \gamma)$, given a set X in $(\mathbb{Z}^2, \gamma)$, we will take the following notation [19]:

$$X_{op} := \{x \,|\, x \text{ is an odd point in } X\}.$$

Since the empty set is clearly both semi-open and semi-closed, the following two theorems will play important roles in discriminating semi-openness and semi-closedness of a subset of the MW-topological plane.

Lemma 2.3. *[26] In* $(\mathbb{Z}^2, \gamma)$*, a non-empty set* $X(\subset \mathbb{Z}^2)$ *is semi-open if and only if each* $x \in X$*,* $SN_\gamma(x) \cap X_{op} \neq \emptyset$*.*

Owing to the notion of semi-closedness, using Lemma 2.3, we obtain the following:

Lemma 2.4. *[26] In* $(\mathbb{Z}^2, \gamma)$*, a non-empty set* $Y(\subset \mathbb{Z}^2)$ *is semi-closed if and only if each* $x \in \mathbb{Z}^2 \setminus Y$*,* $SN_\gamma(x) \cap (\mathbb{Z}^2 \setminus Y)_{op} \neq \emptyset$*.*

By Lemmas 2.3 and 2.4, in $(\mathbb{Z}^2, \gamma)$, we obtain the following [26]:
(1) In the case of $p \in (\mathbb{Z}^2)_O$, the singleton $\{p\}$ is both semi-closed and semi-open.
(2) In the case of $q \in (\mathbb{Z}^2)_E$, the singleton $\{q\}$ is not semi-open but semi-closed. Namely, $\mathbb{Z}^2 \setminus \{q\}$ is semi-open, where $q \in (\mathbb{Z}^2)_E$.

In order to investigate the semi-T_3-separation axiom of the infinite MW-topological sphere and make the present paper self-contained, let us first recall the Alexandroff one point compactification of a locally compact, non-compact [29], and non-Hausdorff topological space. Let (X, T) be a non-compact and locally compact space. Take some object outside X, denote by the symbol $* \notin X$, adjoin it to X, and finally form the set $X \cup \{*\} := X^*$. Then topologize X^* by taking as open sets all the open subsets U of (X, T) together with all sets $V = X^* \setminus F = F^c$, where F is closed and compact in (X, T). This topological space is called an Alexandroff one point compactification (for brevity, Alexandroff extention or Alexandroff compactification, for brevity) of (X, T) and denoted by (X^*, T^*). According to (X^*, T^*), it is obvious that the singleton $\{*\}$ is not an open subset, but rather a closed subset of (X^*, T^*) [25].

Lemma 2.5. *[35] In* $((\mathbb{Z}^2)^*, \gamma^*)$*, the point* $*$ *does not have an open set* $U(\ni *)$ *that is homeomorphic to* $SN_\gamma(p)$*, where* $p \in \mathbb{Z}^2$*.*

Lemma 2.6. *In* $(\mathbb{Z}^2, \gamma)$*,* $A(\subset \mathbb{Z}^2)$ *is compact if and only if* A *is finite.*

Proof. Using the locally finite (LF-, for brevity) topological structure of $(\mathbb{Z}^2, \gamma)$ (see (2.3)), the proof is completed. More precisely, we first prove that with $(\mathbb{Z}^2, \gamma)$, in the case that $A(\subset \mathbb{Z}^2)$ is compact, A is finite. By the contrapositve logic, assume that A is not finite. Since $\mathbb{Z}^2$ is an infinite countable set, we may put $A := \{a_i \,|\, i \in M : \text{infinite countable set}\}$. Then it is clear that A is not compact because of the LF-topological structure of $(\mathbb{Z}^2, \gamma)$. To be specific, owing to the LF-topological structure of $(\mathbb{Z}^2, \gamma)$, it is obvious that A contains many countably infinite doubly even or even points, or many countably infinite odd points. Then, by (2.2), a certain open covering of A may not have a finite subcovering of it. For instance, let $A := \mathbb{Z} \times \{0\} \subset \mathbb{Z}^2$. Then take the open covering of A, $\{N_4(p) \,|\, p \in \{(2m, 0) \,|\, m \in \mathbb{Z}\}$ (see the set $N_4(p)$ in (2.2)). Then it is clear that this open covering of A does not have any finite subcovering of A.
Conversely, it is so obvious that a finite subset A is compact under $(\mathbb{Z}^2, \gamma)$. □

Owing to Lemmas 2.5 and 2.6, we have the following:

Corollary 2.7. *[25]* $((\mathbb{Z}^2)^*, \gamma^*)$ *is not an Alexandroff space.*

Before proving this assertion, while it was stated in [25], since this result is indeed an essential part of the present paper, to make the paper self-contained, let us prove it again more precisely.

Proof. Let us prove that for the point $* \in (\mathbb{Z}^2)^* \setminus \mathbb{Z}^2$, no smallest open set $SN_{\gamma^*}(*)(\ni *)$ exists in $((\mathbb{Z}^2)^*, \gamma^*)$, where $SN_{\gamma^*}(*)$ is the smallest open set containing the element $*$ in $((\mathbb{Z}^2)^*, \gamma^*)$. For the sake of a contradiction, suppose that there exists an $SN_{\gamma^*}(*)$. Then $(\mathbb{Z}^2)^* \setminus SN_{\gamma^*}(*)$ should be a closed and compact subset of $(\mathbb{Z}^2, \gamma)$. Let us take the notation $Y := (\mathbb{Z}^2)^* \setminus SN_{\gamma^*}(*) := \{y_1, y_2, \cdots, y_n\}$ (see Lemma 2.6). Owing to the MW-topological structure of $(\mathbb{Z}^2, \gamma)$, since the complement of Y in $\mathbb{Z}^2$ is an infinite set, there are doubly even or even points p in $(\mathbb{Z}^2, \gamma)$ such that $p \notin Y$. Then it is clear that the set $Y_1 := \{p\} \cup Y$ is also closed and compact (or finite) in $(\mathbb{Z}^2, \gamma)$. Hence $Y_2 := (\mathbb{Z}^2)^* \setminus Y_1$ is also an open set containing the point $*$ in $((\mathbb{Z}^2)^*, \gamma^*)$, and it is a proper subset of $SN_{\gamma^*}(*)$, i.e., $Y_2 \subsetneq SN_{\gamma^*}(*)$, which invokes a contradiction of the existence of $SN_{\gamma^*}(*)$. □

3. Some properties of MW-topological spaces

This section investigates various properties of the infinite MW-topological sphere with respect to the semi-topological structures. This work will play an important role in studying topological spaces related to the MW-topological structure. To do this work, let us now recall some notions associated with the s-T_3-separation axiom and the semi-T_3-separation axiom, and so on (see Definition 3.3 below). To be specific, the papers [5, 6] established the notion of s-regularity as follows: A topological space (X, T) is said to be s-regular if for each closed subset F of X and a point $x \in F^c$, there are $U, V \in SO(X, T)$ such that $F \subset U$, $x \in V$ and $U \cap V = \emptyset$ [6], where $SO(X, T) := \{U \mid U \text{ is semi-open in } (X, T)\}$. The paper [7] proved that this s-regularity has the finite product property. Unlike the s-regularity, let us now recall the notion of semi-regularity.

Definition 3.1. *[4] A topological space (X, T) is semi-regular if for each semi-closed set C and each $x \notin C$ there exist disjoint semi-open sets U and V in (X, T) such that $x \in U$ and $C \subset V$.*

Definition 3.2. *[5] A topological space (X, T) is said to be a semi-T_1-space if any distinct points $p, q \in X$ have their own semi-open sets $SO(p)$ and $SO(q)$ in (X, T) such that $q \notin SO(p)$ and $p \notin SO(q)$, where $SO(x)$ means a semi-open set containing the given point x.*

Besides, it turns out that a topological space (X, T) is a semi-T_1-space if and only if every singleton is semi-closed [5].

Based on the s-regularity and the semi-regularity above, it is clear that the semi-regularity implies the s-regularity. However, the converse does not hold. Let us now define the following:

Definition 3.3. *[19] (1) We say that a topological space (X, T) is an s-T_3-space if it is both a semi-T_1-space and an s-regular space.*

(2) We say thatl a topological space (X, T) is a semi-T_3-space if it is both a semi-T_1-space and a semi-regular space.

Based on Definition 3.3, it is clear that an s-T_3-space is a semi-T_3-space. However, the converse does not hold. A recent paper [19] studied some semi-properties of $(\mathbb{Z}^2, \gamma)$, as follows:

Lemma 3.4. *[19] Assume a simple MW-path X in $(\mathbb{Z}^2, \gamma)$.*
(1) X is semi-closed whenever $X^\sharp \leq 6$.
(2) X may not be semi-closed whenever $X^\sharp \geq 7$.

Motivated by Lemma 3.4(2), let us investigate some conditions for a simple MW-path to be semi-closed, as follows. Assume a simple MW-path $X = (p_0, p_1, \cdots, p_{l-1})$ in $(\mathbb{Z}^2, \gamma)$ with $X^\sharp \geq 7$ such that X does not have the subset $A := \{p_{t-3(mod\, l)}, p_{t-1(mod\, l)}, p_{t+1(mod\, l)}, p_{t+3(mod\, l)}\} \subset X$ satisfying that each element of A is an odd point and $A = N_4(c) \setminus \{c\}, c \in \mathbb{Z}^2 \setminus X$. Then the paper [19] proved that X is semi-closed. Thus we obtain the following:

Remark 3.5. *Under $(\mathbb{Z}^2, \gamma)$, consider the set $X(\subset \mathbb{Z}^2)$ such that $N_4(p) \setminus \{p\} \subset X$, $p \notin X$, and $p \in (\mathbb{Z}^2)_E$. Then X is not semi-closed.*

Unlike Remark 3.5, by Lemma 2.3, we have the following:

Remark 3.6. *[19] A simple MW-path need not be semi-open because the path $P := \{c_0\}$ as a singleton is not semi-open, where $c_0 \in (\mathbb{Z}^2)_E$.*

Motivated by Remark 3.6, for $p \in (\mathbb{Z}^2)_O$, while a singleton is semi-open, we need to study the following case.

Theorem 3.7. *In $(\mathbb{Z}^2, \gamma)$, any connected subset X with $X^\sharp \geq 2$ is semi-open.*

Proof. With the assumption, take $x \in X \subset \mathbb{Z}^2$. Then we can consider the following two cases.
(Case 1) Assume that $x \in X \cap (\mathbb{Z}^2)_O$. Since $x \in SN_\gamma(x) = \{x\}$, we obtain that $SN_\gamma(x) \cap X_{op} \neq \emptyset$.
(Case 2) Assume that $x \in X \cap (\mathbb{Z}^2)_E$. Since there is $x' \in SN_\gamma(x)$ such that $x' \in X \cap (\mathbb{Z}^2)_O$ because X is connected and $X^\sharp \geq 2$, we obtain that $SN_\gamma(x) \cap X_{op} \neq \emptyset$.
In view of these two cases, by Lemma 2.3, the proof is completed. □

Lemma 3.8. *[19] $(\mathbb{Z}^2, \gamma)$ is a semi-T_3-space.*

Since the notion of "s-regular" is more restrictive than that of "semi-rgular", the following is obtained.

Corollary 3.9. *$(\mathbb{Z}^2, \gamma)$ is an s-T_3-space.*

Let us now further recall some properties of $(\mathbb{Z}^2, \gamma)$ and $((\mathbb{Z}^2)^*, (\gamma)^*)$. Since $(\mathbb{Z}^2, \gamma)$ is neither compact nor Hausdorff but locally compact, an Alexandroff compactification of $(\mathbb{Z}^2, \gamma)$ was developed in [28]. Note that $(\mathbb{Z}^2, \gamma)$ is not a T_1-space. For convenience, we often call $((\mathbb{Z}^2)^*, (\gamma)^*)$ the infinite MW-topological sphere. Besides, under $((\mathbb{Z}^2)^*, \gamma^*)$, we need to remind that for an open set, say $O(*)$, containing the element $*$, the complement of $O(*)$ in $\mathbb{Z}^2$, i.e., $O(*)^c$, is a closed and compact subset of $(\mathbb{Z}^2, \gamma)$.

In view of the topological features of $((\mathbb{Z}^2)^*, \gamma^*)$, it is clear that $O \in \gamma$ implies that $O \in \gamma^*$. However, a closed set C in γ need not be a closed set in γ^* (see Lemma 3.10 below). Hence we need to investigate some semi-topological features of subsets of $((\mathbb{Z}^2)^*, \gamma^*)$, as follows:

Lemma 3.10. *Let F be a closed set in $(\mathbb{Z}^2, \gamma)$. Then F need not be a closed set in $((\mathbb{Z}^2)^*, \gamma^*)$.*

Proof. As an example, while the set $\mathbb{Z}^2$ is clearly a closed set in $(\mathbb{Z}^2, \gamma)$, it is not a closed set in $((\mathbb{Z}^2)^*, \gamma^*)$. □

Hereinafter, $Cl_{\gamma^*}(A)$ indicates the closure of the given set A in $((\mathbb{Z}^2)^*, \gamma^*)$. Motivated by Lemma 2.6, we have following:

Theorem 3.11. *(1) Assume that F is a closed set in $(\mathbb{Z}^2, \gamma)$ and that $F^\sharp \lneq \aleph_0$. Then $Cl_{\gamma^*}(F) = F$ under γ^*.*
(2) In the case that F is a closed set in $(\mathbb{Z}^2, \gamma)$ and that $F^\sharp = \aleph_0$, we have that $Cl_{\gamma^*}(F) = F \cup \{*\}$ under γ^*.

Proof. (1) With the hypothesis, we observe that F is also closed in $((\mathbb{Z}^2)^*, \gamma^*)$ because the set $(\mathbb{Z}^2)^* \setminus F$ is also open in $((\mathbb{Z}^2)^*, \gamma^*)$ owing to the finiteness of F (see Lemma 2.6).

(2) With the given hypothesis, because $F^\sharp = \aleph_0$, it is obvious that F is not closed in $((\mathbb{Z}^2)^*, \gamma^*)$ (see Lemma 2.6). For the sake of a contradiction, as an example consider the set $(\mathbb{Z}^2)_E$ which is a closed set in $(\mathbb{Z}^2, \gamma)$. Then it is obvious that the set $(\mathbb{Z}^2)_E$ is not closed in $((\mathbb{Z}^2)^*, \gamma^*)$ (see Lemma 2.6). However, the set $F \cup \{*\}$ is closed in $((\mathbb{Z}^2)^*, \gamma^*)$ because the set $(F \cup \{*\})^c$ in $(\mathbb{Z}^2)^*$ is equal to the set F^c in $\mathbb{Z}^2$ and $F^c \in \gamma \subset \gamma^*$, which completes the proof. □

4. Semi-topological properties of the infinite MW-topological sphere

In relation to the hereditary property in compactness, it is well known that a closed subset of a compact topological space is also compact. Unlike this property, let us now come back to the following query mentioned in Section 1. Under $(\mathbb{Z}^2, \gamma)$, is there another condition for the hereditary property of the compactness of $((\mathbb{Z}^2)^*, \gamma^*)$?

In relation to this work, we first need to recall that an infinite subset of $(\mathbb{Z}^2, \gamma)$ is not compact in $(\mathbb{Z}^2, \gamma)$ (see Lemma 2.6).

Theorem 4.1. *Let $A \subset \mathbb{Z}^2$. Then $(\mathbb{Z}^2)^* \setminus A$ is compact in $((\mathbb{Z}^2)^*, \gamma^*)$.*

Proof. With $((\mathbb{Z}^2)^*, \gamma^*)$, let $C := \{U_\alpha \,|\, U_\alpha \in \gamma^*, \alpha \in M\}$ be an arbitrary open covering of $(\mathbb{Z}^2)^* \setminus A$, $A \subset \mathbb{Z}^2$. Then, there is an open set containing the element $*$, say, $U_{\alpha_0}(\ni *) \in C$. Furthermore, we now obtain that $D := \mathbb{Z}^2 \setminus U_{\alpha_0}$ that is closed and compact in $(\mathbb{Z}^2, \gamma)$. By Lemma 2.6, we may put $D := \{d_i \,|\, i \in M' : \text{finite}\}$. Then, for each point $d_i \in D$, in C take a corresponding open set containing d_i, say, $U_{\alpha_i}(\ni d_i) \in C$, such that $U_{\alpha_i} \neq U_{\alpha_0}$. Then we have at least a finite subcovering $C_0 := \{U_{\alpha_0}, U_{\alpha_i} | \alpha_i \in M, i \in M'\}$ of $(\mathbb{Z}^2)^* \setminus A$, which completes the proof. □

In Theorem 4.1, note that the given set A need not be an open set in $(\mathbb{Z}^2, \gamma)$. To support Theorem 4.1, we have the following:

Example 4.1. *Both the sets $(\mathbb{Z}^2)^* \setminus (\mathbb{Z}^2)_O$ and $(\mathbb{Z}^2)^* \setminus (\mathbb{Z}^2)_E$ are compact in $((\mathbb{Z}^2)^*, \gamma^*)$.*

As a generalization of Theorem 4.1, the following is obtained by using the method used in the proof of Theorem 4.1.

Corollary 4.2. *Let (X^*, T^*) be an Alexandroff compactification of (X, T) and $A \subset X$, where (X, T) is neither compact nor Hausdorff but rather locally compact. Then $X^* \setminus A$ is compact in (X^*, T^*).*

By Theorem 4.1, we obtain the following:

Remark 4.3. *(Correction) (1) In Proposition 2.2 of [18], there are misprinted parts, i.e., the parts "but is not compact" in (1) of [18] and "neither closed nor compact" in (2) of [18]. Indeed, the former should be replaced by "compact" and the latter should be replaced with "not closed but compact" in (2).*

Namely, under $((\mathbb{Z}^2)^*, \gamma^*)$, we have the following:
(1-1) $(\mathbb{Z}^2)^* \setminus \{o\}$ *is closed and compact, where* $o \in (\mathbb{Z}^2)_O$.
(1-2) $(\mathbb{Z}^2)^* \setminus \{e\}$ *is not closed but is instead compact, where* $e \in (\mathbb{Z}^2)_E$.
(2) In Proposition 1(1) of [28], there is also a misprinted part, i.e., the part "not compact" should be replaced by "compact". Namely, under $((\mathbb{Z}^2)^*, \gamma^*)$*, the following holds.*
(2-1) $(\mathbb{Z}^2)^* \setminus A$ *is closed and compact, where A is a subset of* $(\mathbb{Z}^2)_O$.

Proof. By Theorem 4.1, the proofs of (1) and (2) are completed. □

By Theorem 4.1, we obtain the following:

Corollary 4.4. *(1) While not every subset of* $((\mathbb{Z}^2)^*, \gamma^*)$ *is always compact in* $((\mathbb{Z}^2)^*, \gamma^*)$*, any subset* $B(\ni *)$ *of* $(\mathbb{Z}^2)^*$ *is compact in* $((\mathbb{Z}^2)^*, \gamma^*)$.
(2) In the case that $p \in (\mathbb{Z}^2)_E \cup \{*\}$*, the singleton* $\{p\}$ *is not semi-open in* $((\mathbb{Z}^2)^*, \gamma^*)$.
(3) In $((\mathbb{Z}^2)^*, \gamma^*)$*,* $(\mathbb{Z}^2)_E$ *is not closed but is instead semi-closed.*

Proof. (1) Unlike Theorem 4.1, in the case that B is a subset of $(\mathbb{Z}^2)^*$ and it does not contain the element $*$, the set B need not be compact in $((\mathbb{Z}^2)^*, \gamma^*)$. For instance, take $B := (\mathbb{Z}^2)_O$. Then it is not compact in $((\mathbb{Z}^2)^*, \gamma^*)$.

However, by Theorem 4.1, a subset $B(\ni *)$ of $(\mathbb{Z}^2)^*$ is proved to be compact in $((\mathbb{Z}^2)^*, \gamma^*)$.
(2) The proof is straightforward from the notion of semi-openness.
(3) With the assumption, it is clear that $(\mathbb{Z}^2)_E$ is not closed in $((\mathbb{Z}^2)^*, \gamma^*)$ because it is not compact in $(\mathbb{Z}^2, \gamma)$. However, $(\mathbb{Z}^2)_E$ is semi-closed because $Cl_{\gamma^*}((\mathbb{Z}^2)_E) = (\mathbb{Z}^2)_E \cup \{*\}$; thus, $Int_{\gamma^*}(Cl_{\gamma^*}((\mathbb{Z}^2)_E) = \emptyset$, which implies the semi-closedness of $(\mathbb{Z}^2)_E$ under γ^*. □

By Lemma 2.3, we obtain the following:

Lemma 4.5. *(1) In* $(\mathbb{Z}^2, \gamma)$*, for any point* $p \in \mathbb{Z}^2$ *the set* $Cl_\gamma(SN_\gamma(p))$ *is semi-open [19].*
(2) $(\mathbb{Z}^2)_O$ *is a dense subset of* $((\mathbb{Z}^2)^*, \gamma^*)$.

Proof. (1) By Theorem 2.2, the proof is completed.
(2) Since $\mathbb{Z}^2 \subset Cl_{\gamma^*}((\mathbb{Z}^2)_O)$ and for any open set containing the element $*$, e.g., $O(*)(\ni *)$, in $((\mathbb{Z}^2)^*, \gamma^*)$ we obtain that $O(*) \cap (\mathbb{Z}^2)_O \neq \emptyset$; thus, $* \in Cl_{\gamma^*}((\mathbb{Z}^2)_O)$. Finally, we obtain that $Cl_{\gamma^*}((\mathbb{Z}^2)_O) = (\mathbb{Z}^2)^*$. □

Owing to Lemma 4.5(2), the following is obtained.

Corollary 4.6. *The infinite MW-topological sphere is separable.*

Let us now investigate some semi-topological properties of subsets of $(\mathbb{Z}^2)^*$.

Theorem 4.7. *Under* $((\mathbb{Z}^2)^*, \gamma^*)$*, the following is obtained.*
(1) $(\mathbb{Z}^2)^* \setminus A$ *need not be semi-open, where* $A(\neq \emptyset)$ *is a subset of* $(\mathbb{Z}^2)_O$.
(2) $(\mathbb{Z}^2)^* \setminus B$ *is not open but is instead semi-open, where B is a (infinitely) denumerable subset of* $(\mathbb{Z}^2)_E$.
(3) $(\mathbb{Z}^2)^* \setminus C$ *is not semi-closed but is instead semi-open, where* $C(\neq \emptyset) \subset (\mathbb{Z}^2)_E$.

Proof. (1) For a point $p \in (\mathbb{Z}^2)_E$, take $A := N_4(p)^* = N_4(p) \setminus \{p\} \subset \mathbb{Z}^2$. For instance, consider $p := (0, 0) \in (\mathbb{Z}^2)_E$ (see Figure 1(a)). Indeed, the set $N_4(p)^*$ is a subset of $(\mathbb{Z}^2)_O$. Then $(\mathbb{Z}^2)^* \setminus A$ is not semi-open in $((\mathbb{Z}^2)^*, \gamma^*)$ because

$$Cl_{\gamma^*}(Int_{\gamma^*}((\mathbb{Z}^2)^* \setminus A)) = Cl_{\gamma^*}((\mathbb{Z}^2)^* \setminus \bigcup_{q \in N_4(p)^*} N_4(q)) = (\mathbb{Z}^2)^* \setminus N_4(p),$$

so that $(\mathbb{Z}^2)^* \setminus A \not\subseteq Cl_{\gamma^*}(Int_{\gamma^*}((\mathbb{Z}^2)^* \setminus A))$; this is beacuse, while $p \in (\mathbb{Z}^2)^* \setminus A$, we observe that $p \notin Cl_{\gamma^*}(Int_{\gamma^*}((\mathbb{Z}^2)^* \setminus A))$.
Meanwhile, consider the set $A := \{(0, \pm 1), (-1, 0)\} \subset (\mathbb{Z}^2)_O$. Then $(\mathbb{Z}^2)^* \setminus A$ is semi-open.
(2) Note that $B = ((\mathbb{Z}^2)^* \setminus B)^c$ is not a closed set in γ^* because $* \in (\mathbb{Z}^2)^* \setminus B$ and further, by Lemma 2.6, B is not compact in $(\mathbb{Z}^2, \gamma)$. Thus $(\mathbb{Z}^2)^* \setminus B$ is not an open set in γ^*.
Next, we now prove that $(\mathbb{Z}^2)^* \setminus B$ is semi-open in $((\mathbb{Z}^2)^*, \gamma^*)$. More precisely, since

$$\begin{cases} Int_{\gamma^*}((\mathbb{Z}^2)^* \setminus B) = \mathbb{Z}^2 \setminus B \text{ and,} \\ Cl_{\gamma^*}(\mathbb{Z}^2 \setminus B) = (\mathbb{Z}^2)^* \text{ (see Lemma 4.5(2)),} \end{cases} \tag{4.1}$$

We obtain

$$Cl_{\gamma^*}(Int_{\gamma^*}((\mathbb{Z}^2)^* \setminus B)) = Cl_{\gamma^*}(\mathbb{Z}^2 \setminus B) = (\mathbb{Z}^2)^*,$$

which implies the semi-openness of $(\mathbb{Z}^2)^* \setminus B$. To be specific, to verify the assertion of (4.1), note that $Int_{\gamma^*}((\mathbb{Z}^2)^* \setminus B)$ should not contain the element $*$. For the sake of a contradiction, suppose $* \in Int_{\gamma^*}((\mathbb{Z}^2)^* \setminus B)$. Then there is certainly an open set $O(*) \in \gamma^* \setminus \gamma$ such that $O(*) \subset (\mathbb{Z}^2)^* \setminus B$, where $O(*)$ is an open set containing the element $*$. However, because $((O(*)^c)^\sharp = \aleph_0$, we obtain a contradiction to the compactness of $(O(*))^c$ in γ (see Lemma 2.6). In addition, we observe the property $(\mathbb{Z})^2 \setminus B \in \gamma \subset \gamma^*$.
(3) Let us now prove the non-semi-closedness of $(\mathbb{Z}^2)^* \setminus C$, where $C (\neq \emptyset) \subset (\mathbb{Z}^2)_E$. By Lemmas 3.10 and 4.5(2), since $(\mathbb{Z}^2)_O \subset (\mathbb{Z}^2)^* \setminus C$, we obtain

$$\begin{cases} (\mathbb{Z}^2)^* = Cl_{\gamma^*}((\mathbb{Z}^2)_O) \subset Cl_{\gamma^*}((\mathbb{Z}^2)^* \setminus C) \text{ so that} \\ Int_{\gamma^*}(Cl_{\gamma^*}((\mathbb{Z}^2)^* \setminus C)) \not\subseteq (\mathbb{Z}^2)^* \setminus C, \end{cases}$$

which implies the non-semi-closedness of $(\mathbb{Z}^2)^* \setminus C$.
Next, let us prove the semi-openness of $(\mathbb{Z}^2)^* \setminus C$. Since

$$\begin{cases} (\mathbb{Z}^2)_O \subset (\mathbb{Z}^2)^* \setminus C \text{ so that } (\mathbb{Z}^2)_O \subset Int_{\gamma^*}((\mathbb{Z}^2)^* \setminus C) \text{ and further,} \\ Cl_{\gamma^*}(Int_{\gamma^*}((\mathbb{Z}^2)^* \setminus C)) = (\mathbb{Z}^2)^* \text{(see Lemma 4.5(2)),} \end{cases}$$

which implies the semi-openness of $(\mathbb{Z}^2)^* \setminus C$. □

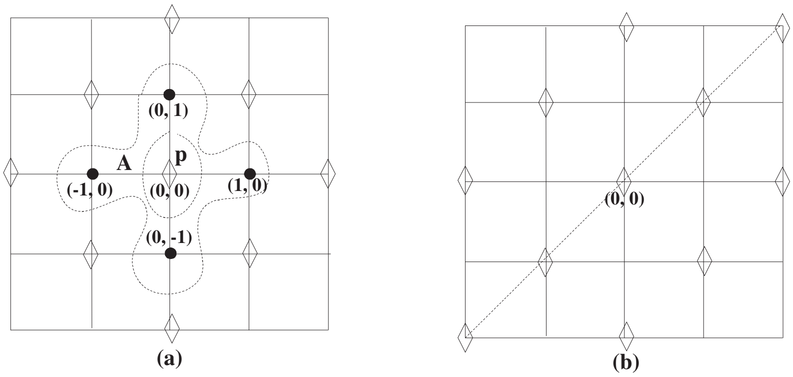

Figure 1. (a) Configuration of $A := N_4(p)^* = N_4(p) \setminus \{p\}, p = (0, 0)$ relating to the proof of Theorem 4.7(1). (b) $B(\subset \{p = (x, x) | x \in \mathbb{Z}\} \cup C, C \subset (\mathbb{Z}^2)_E)$ is a part of an (infinitely) denumerable subset of $(\mathbb{Z}^2)_E$ relating to the proof of Theorem 4.7(2).

By Theorem 4.7(1), we obtain the following:

Corollary 4.8. *Under $((\mathbb{Z}^2)^*, \gamma^*)$, let $A(\subset (\mathbb{Z}^2)^*)$ be a set containing the set $N_4(p)^*$, where $p \in (\mathbb{Z}^2)_E$ and $p \notin A$. Then $(\mathbb{Z}^2)^* \setminus A$ is not semi-open.*

Proof. While $p \notin Cl_{\gamma^*}(Int_{\gamma^*}((\mathbb{Z}^2)^* \setminus A))$, we observe that $p \in (\mathbb{Z}^2)^* \setminus A$. □

Lemma 4.9. *With $((\mathbb{Z}^2)^*, \gamma^*)$, for $q \in \mathbb{Z}^2$, we obtain*
$Cl_{\gamma^*}(SN_\gamma(q)) = Cl_\gamma(SN_\gamma(q))$.

Proof. Based on the notion of (2.3), $SN_\gamma(q) = SN_{\gamma^*}(q)$, $q \in \mathbb{Z}^2$; the proof is completed because

$$Cl_{\gamma^*}(SN_\gamma(q)) = \bigcup_{p \in SN_\gamma(q)} N_4(p) = Cl_\gamma(SN_\gamma(q)).$$

□

Example 4.2. *(1) The set $A := SN_\gamma(p) \cup \{*\}$ is not semi-open under γ^*, where $p \in \mathbb{Z}^2$. To be specific, in the case of $p \in (\mathbb{Z}^2)_E$, since $Int_{\gamma^*}(A) = SN_\gamma(p)$, we obtain*

$$Cl_{\gamma^*}(Int_{\gamma^*}(A)) = Cl_{\gamma^*}(SN_\gamma(p)) = \bigcup_{q \in SN_\gamma(p)} N_4(q) \not\supseteq A$$

because $Cl_{\gamma^*}(SN_\gamma(p))$ does not contain the element $*$.
In the case of $p \in (\mathbb{Z}^2)_O$, we obtain that $Cl_{\gamma^*}(Int_{\gamma^*}(A)) = N_4(p)$ so that $* \notin Cl_{\gamma^*}(Int_{\gamma^*}(A))$, which implies the non-semi-openess of A under γ^*.

(2) Let $B = (\mathbb{Z}^2)_E \cup \{p_1, p_2\}$, where $p_1 = (1, 0)$ and $p_2 = (5, 0)$. Then B is semi-closed in γ (see Lemma 2.4). Indeed, B is also semi-closed in γ^* because

$$Cl_{\gamma^*}(B) = B \cup \{*\} \cup (\bigcup_{q \in \{p_1, p_2\}} N_4(q)) \text{ and } Int_{\gamma^*}(Cl_{\gamma^*}(B)) = \{p_1, p_2\}.$$

As another example, since $SN_\gamma(p)$ is semi-closed in γ, it is also semi-closed in γ^* because $Int_{\gamma^*}(Cl_{\gamma^*}(SN_\gamma(p))) = SN_\gamma(p)$, Motivated by Lemma 4.9 and Example 4.2, the following is obtained.

Theorem 4.10. *(1) Let A be semi-open in* $(\mathbb{Z}^2, \gamma)$*. Then it is also semi-open in* $((\mathbb{Z}^2)^*, \gamma^*)$*.*
(2) Let B be semi-closed in $(\mathbb{Z}^2, \gamma)$*. Then it need not be semi-closed in* $((\mathbb{Z}^2)^*, \gamma^*)$*. However, in case B is finite and semi-closed in* $(\mathbb{Z}^2, \gamma)$*, it is semi-closed in* $((\mathbb{Z}^2)^*, \gamma^*)$*.*

Proof. (1) Based on the hypothesis, it suffices to prove that $A \subset Cl_{\gamma^*}(Int_{\gamma^*}(A))$. In the case of $A = \emptyset$, the assertion is trivial.

Let us now assume that $A(\neq \emptyset) \subset \mathbb{Z}^2$. Then, with the hypothesis, owing to $A \subset Cl_\gamma(Int_\gamma(A))$, for any $x \in A$, we obtain that

$$SN_\gamma(x) \cap (Int_\gamma(A))_{op} \neq \emptyset \Rightarrow SN_\gamma(x) \cap Int_\gamma(A) \neq \emptyset \tag{4.2}$$

under $(\mathbb{Z}^2, \gamma)$ because $(Int_\gamma(A))_{op} \subset Int_\gamma(A)$.

Meanwhile, under $((\mathbb{Z}^2)^*, \gamma^*)$, for $x \in A \subset \mathbb{Z}^2$ we have that $SN_{\gamma^*}(x) = SN_\gamma(x)$ and further, for $A \subset \mathbb{Z}^2$ we obtain that $Int_\gamma(A) = Int_{\gamma^*}(A)$ which implies that $A \subset Cl_{\gamma^*}(Int_{\gamma^*}(A))$ because for any $x \in A$, we obtain that (see (4.2))

$$SN_\gamma(x) \cap Int_\gamma(A) = SN_{\gamma^*}(x) \cap Int_{\gamma^*}(A) \neq \emptyset.$$

(2) For the sake of a contradiction, take $B := \mathbb{Z}^2$. Even though B is semi-closed in γ, it is not semi-closed in $((\mathbb{Z}^2)^*, \gamma^*)$. To be specific, we obtain that

$$Int_{\gamma^*}(Cl_{\gamma^*}(\mathbb{Z}^2)) = Int_{\gamma^*}((\mathbb{Z}^2)^*) = (\mathbb{Z}^2)^*,$$

which implies the non-semi-closedness of $\mathbb{Z}^2$ in γ^*.

However, in the case that $B(\subset \mathbb{Z}^2)$ is finite, we have that $Cl_\gamma(B) = Cl_{\gamma^*}(B)$ which is also finite. Thus we obtain

$$Int_{\gamma^*}(Cl_{\gamma^*}(B)) = Int_{\gamma^*}(Cl_\gamma(B)) = Int_\gamma(Cl_\gamma(B)) \subset B,$$

which completes the proof. □

5. The semi-regularity of the infinite *MW*-topological sphere

This section proves that the infinite *MW*-topological sphere has the semi-regularity, which addresses the last question raised in Section 1.

Lemma 5.1. *The infinite MW-topological sphere,* $((\mathbb{Z}^2)^*, \gamma^*)$*, is not a* T_1*-space.*

Proof. For the sake of a contradiction, take a point $p \in (\mathbb{Z}^2)_O$. Then the singleton $\{p\}$ is not closed in $((\mathbb{Z}^2)^*, \gamma^*)$. □

Unlike Lemma 5.1, let us investigate the semi-T_1-separation axiom of the infinite *MW*-topological sphere, as follows:

Theorem 5.2. *The infinite MW-topological sphere,* $((\mathbb{Z}^2)^*, \gamma^*)$*, is a semi-*$T_1$*-space.*

Proof. By Lemma 3.8, since $(\mathbb{Z}^2, \gamma)$ is a semi-T_1-space, to prove the semi-T_1-separation axiom of $((\mathbb{Z}^2)^*, \gamma^*)$, it suffices to consider the distinct points p and q such that either p or q is equal to the element '$*$'. Without loss of generality, we may assume that $p = * \in (\mathbb{Z}^2)^* \setminus \mathbb{Z}^2$ and $q \in \mathbb{Z}^2$. Then, owing to $\gamma \subset \gamma^*$, Lemma 4.9 and Theorem 4.10, take the set

$$V := Cl_{\gamma^*}(SN_{\gamma^*}(q)) = Cl_{\gamma^*}(SN_{\gamma}(q)) = Cl_{\gamma}(SN_{\gamma}(q)) \tag{5.1}$$

which is compact and closed in $(\mathbb{Z}^2, \gamma)$ because $SN_{\gamma}(q) \in \gamma \subset \gamma^*$. Furthermore, by Theorem 4.10, V is semi-open in $((\mathbb{Z}^2)^*, \gamma^*)$ because it is semi-open in $(\mathbb{Z}^2, \gamma)$. Then we clearly have that $q \in V$ and $p \notin V$. To be specific, in the case of $q \in (\mathbb{Z}^2)_E$, since $SN_{\gamma}(q) = N_4(q)$ and

$$Cl_{\gamma^*}(SN_{\gamma^*}(q)) = Cl_{\gamma^*}(SN_{\gamma}(q)) = \bigcup_{t \in SN_{\gamma}(q)} N_4(t),$$

we obtain that V is closed and semi-open (see Lemma 2.3) in $(\mathbb{Z}^2, \gamma)$ and further, in $((\mathbb{Z}^2)^*, \gamma^*)$.

In the case of $q \in (\mathbb{Z}^2)_O$, since $SN_{\gamma}(q) = \{q\}$ and $Cl_{\gamma^*}(SN_{\gamma}(q)) = N_4(q)$, by Theorem 4.10, we obtain that the set V of (5.1) is closed and semi-open in $(\mathbb{Z}^2, \gamma)$ and further, semi-open in $((\mathbb{Z}^2)^*, \gamma^*)$ (see Theorem 4.10).

Next, take another set

$$U := (\mathbb{Z}^2)^* \setminus Cl_{\gamma^*}(SN_{\gamma^*}(q)). \tag{5.2}$$

Then the set U is open in $((\mathbb{Z}^2)^*, \gamma^*)$ so that it is also semi-open in $((\mathbb{Z}^2)^*, \gamma^*)$.

Then, we obtain that $p \in U$ and $q \in V$ such that $U, V \in SO(((\mathbb{Z}^2)^*, \gamma^*))$, $q \notin U$, and $p \notin V$, the proof is completed. □

In the case that C is closed in $((\mathbb{Z}^2)^*, \gamma^*)$ and $(\mathbb{Z}^2)^* \setminus C$ contains the element $*$, C should be finite. However, we obtain the following:

Remark 5.3. *Assume that C is closed in* $((\mathbb{Z}^2)^*, \gamma^*)$ *and* $* \in C$.
Then C need not be finite.

Proof. For instance, take $C := (\mathbb{Z}^2)_E \cup \{*\}$. Although C is a closed set in $((\mathbb{Z}^2)^*, \gamma^*)$, i.e., $((\mathbb{Z}^2)^* \setminus C) \in \gamma \subset \gamma^*$, C is not finite.
Meanwhile, let $C := \{*\}$. Then, $C(\ni *)$ is clearly closed in γ^*, and C is finite. □

On top of Example 4.2(2) and Theorem 4.10(2), the following is obtained.

Lemma 5.4. *Under* $((\mathbb{Z}^2)^*, \gamma^*)$*, assume that*
(1) $D \subset (\mathbb{Z}^2)^*$ *with* $D^\sharp \lneq \aleph_0$ *and*
(2) there is

$$\text{no } N_4^*(p) \subset D \text{ where } p \notin D \text{ and } p \in (\mathbb{Z}^2)_E. \tag{5.3}$$

Then D is semi-closed.

Proof. For the set D with the hypothesis above, we need to point out that D need not be connected. Let us now consider the following two cases.
(Case 1) Assume the following case: $* \notin D$. Then, owing to the property of (5.3), we obtain the following:

$$\begin{cases} Int_{\gamma^*}(Cl_{\gamma^*}(D)), \\ = Int_{\gamma^*}[\bigcup_{p_i \in (\mathbb{Z}^2)_O \cap D} N_4(p_i) \cup \{d_j \,|\, d_j \in D \cap (\mathbb{Z}^2)_E\}], \\ = Int_{\gamma^*}(D) = Int_{\gamma}(D) \subset D. \end{cases} \tag{5.4}$$

(Case 2) Assume the following case: $* \in D$. Then, owing to the hypothesis, it is clear that $Cl_{\gamma^*}(D)(\ni *)$ is finite. Indeed, we need to confirm the finiteness of $Cl_{\gamma^*}(D)$. To be specific, we first observe that $* \in Cl_{\gamma^*}(D)$. And it is obvious that for $x(\neq *) \in \mathbb{Z}^2$, owing to the finiteness of D, we obtain

$$\{x \,|\, SN(x) \cap D \neq \emptyset\}^\sharp \text{ is finite,}$$

which implies the finiteness of $Cl_{\gamma^*}(D)$.

Let $O(*)$ be an open set containing the element $*$ in γ^*. While $O(*) \cap D \neq \emptyset$, since $O(*)^\sharp = \aleph_0$, there is not a certain open set $O(*)$ such that $O(*) \subset Cl_{\gamma^*}(D)$. Thus, using a method similar to the method used in (5.4), we obtain

$$Int_{\gamma^*}(Cl_{\gamma^*}(D)) = Int_{\gamma}(Cl_{\gamma^*}(D)) = Int_{\gamma}(Cl_{\gamma}(D)) \subset D. \tag{5.5}$$

The properties of (5.4) and (5.5) imply the semi-closedness of D. □

Corollary 5.5. *If $D(\subset \mathbb{Z}^2)$ is semi-closed in γ^*, then there is no $N_4^*(p) \subset D$ where $p \notin D$ and $p \in (\mathbb{Z}^2)_E$.*

Theorem 5.6. *The infinite MW-topological sphere is semi-regular.*

Proof. Let us take a semi-closed set C in $((\mathbb{Z}^2)^*, \gamma^*)$ and $x \notin C$. Then we need to take the following two cases (see Cases (1) and (2) below).
(Case 1) Assume the following case: $* \in C$ and $x \notin C$.
In relation to the choice of the element x, we can further consider the following two cases (see Cases (1-1) and (1-2) below):
(Case 1-1) Assume that $x \in (\mathbb{Z}^2)_O$ and $x \notin C$. Then, take the set $U := \{x\}$ which is both semi-closed (see Theorem 4.10(2)) and semi-open (see Theorem 4.10(1)) in $((\mathbb{Z}^2)^*, \gamma^*)$. Let us now take the set $V := (\mathbb{Z}^2)^* \setminus U$. Then this set V is semi-open in $((\mathbb{Z}^2)^*, \gamma^*)$, i.e., $Cl_{\gamma^*}(Int_{\gamma^*}(V)) = V$ and further, $C \subset V$. Thus we obtain that $U, V \in SO(((\mathbb{Z}^2)^*, \gamma^*))$ such that $U \cap V = \emptyset$.
(Case 1-2) Assume that $x \in (\mathbb{Z}^2)_E$ and $x \notin C$. Then, note that for any $A \subset (\mathbb{Z}^2)_E$, $(\mathbb{Z}^2)^* \setminus A$ is not semi-closed in $((\mathbb{Z}^2)^*, \gamma^*)$ because $Int_{\gamma^*}(Cl_{\gamma^*}((\mathbb{Z}^2)^* \setminus A)) = (\mathbb{Z}^2)^*$ (see Lemma 4.5(2)). Thus there is at least an MW-path $V := \{x, x'\}$ (see Figure 2(a)) such that $x' \in SN_\gamma(x)$ and $V \subset (\mathbb{Z}^2)^* \setminus C$. Then, by Theorem 3.7, V is semi-open in γ, thus it is also semi-open in γ^* (see Theorem 4.10(1)). Furthermore, V is also semi-closed in $((\mathbb{Z}^2)^*, \gamma^*)$ because $Int_{\gamma^*}(Cl_{\gamma^*}(V)) = \{x'\} \subset V$.
Next, take $U := (\mathbb{Z}^2)^* \setminus V$. Then U is semi-open in γ^* because

$$Cl_{\gamma^*}(Int_{\gamma^*}(U)) = (\mathbb{Z}^2)^* \setminus \{x'\} \supset U \text{(see Figure 2(b)).}$$

Then we obtain

$$C \subset U, x \in V, \text{ and } U \cap V = \emptyset.$$

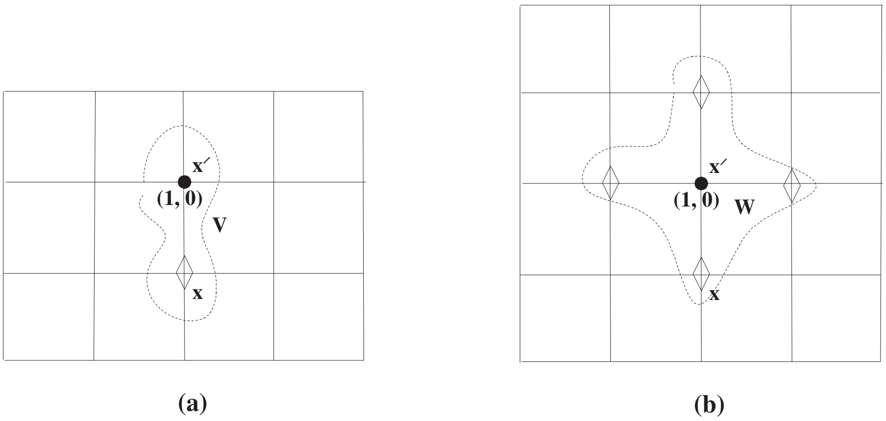

Figure 2. (a) Configuration of V in the proof of (Case 1-2) of Theorem 5.6. (b) $Int_{\gamma^*}(U) = (\mathbb{Z}^2)^* \setminus W$, where $U := (\mathbb{Z}^2)^* \setminus V$ and $V = \{x, x'\}$ as in (Case 1-2) of the proof of Theorem 5.6.

(Case 2) Let us assume the following case: $* \notin C$ and $x \notin C$, i.e., $C \subset \mathbb{Z}^2$, and C is semi-closed in $((\mathbb{Z}^2)^*, \gamma^*)$.
Then we need to further consider the three cases as follows:
(Case 2-1) Assume that $x \in (\mathbb{Z}^2)_O$. Then we may state that $V := \{x\}$ and $U := (\mathbb{Z}^2)^* \setminus V$ so that both U and V are semi-open in $((\mathbb{Z}^2)^*, \gamma^*)$ and

$$C \subset U, x \in V, \text{ and } U \cap V = \emptyset.$$

(Case 2-2) Assume that $x \in (\mathbb{Z}^2)_E$. Note that for any $x \in (\mathbb{Z}^2)_E$, the set $\mathbb{Z}^2 \setminus \{x\}$ is not semi-closed in $((\mathbb{Z}^2)^*, \gamma^*)$ (see the proof of (Case 1-2) above) since

$$Int_{\gamma^*}(Cl_{\gamma^*}(\mathbb{Z}^2 \setminus \{x\})) = Int_{\gamma^*}((\mathbb{Z}^2)^*) = (\mathbb{Z}^2)^*.$$

Hence there is at least an MW-path $V := \{x, x'\}$ such that $x' \in SN_\gamma(x)$ and $V \subset \mathbb{Z}^2 \setminus C$. Then it is clear that V is semi-open in γ thus, it is also semi-open in γ^* (see Theorem 3.11(1)). Besides, V is also semi-closed in γ^*.
Then, take $U := (\mathbb{Z}^2)^* \setminus V$. Then U is semi-open in γ^* because V is semi-closed in γ^*. Besides, we obtain

$$C \subset U, x \in V, \text{ and } U \cap V = \emptyset.$$

(Case 2-3) Assume that $x = *$ and $C(\subset \mathbb{Z}^2)$. we now need to recall that the semi-closed set C in γ^* does not contain the set $N_4^*(p) \subset (\mathbb{Z}^2)_O$, where $p \in (\mathbb{Z}^2)_E$ and $p \notin C$ that is finite (see Lemma 5.4 and Corollary 5.5).
On top of this, as an infinite case of C, one of the important things is that the set C does not contain $(\mathbb{Z}^2)_O$ as a subset. For the sake of a contradiction, suppose that $(\mathbb{Z}^2)_O \subset C$. Then we obtain

$$Int_{\gamma^*}(Cl_{\gamma^*}(C)) \nsubseteq C,$$

because $Cl_{\gamma^*}(C) = (\mathbb{Z}^2)^*$, which implies the non-semi-closedness of C under γ^*. Besides, the given semi-closed set C is not a kind of the set $B := \mathbb{Z}^2 \setminus \{x_t \,|\, t \in M := [1, n]_{\mathbb{Z}}, x_t \in (\mathbb{Z}^2)_O\}$. If not, the set B is not semi-closed in γ^* at all. To be specific, since $Cl_{\gamma^*}(B) = (\mathbb{Z}^2)^* \setminus \{x_t \,|\, t \in M := [1, n]_{\mathbb{Z}}\}$ and

$$Int_{\gamma^*}(Cl_{\gamma^*}(B)) = (\mathbb{Z}^2)^* \setminus \bigcup_{t\in M} N_4(x_t),$$

which implies the non-semi-closedness of B in γ^* because $* \in Int_{\gamma^*}(Cl_{\gamma^*}(B))$ and $* \notin B$.
In view of this observation, it is clear that $\mathbb{Z}^2 \setminus C$ has a subset $D \subset (\mathbb{Z}^2)_O$ such that $D^\sharp = \aleph_0$ (see also Example 4.2(2) and Theorem 4.10(2)). Then we may take a set $V := D \cup \{*\}$ so that V is semi-open in γ^* because

$$V \subset Cl_{\gamma^*}(Int_{\gamma^*}(V)) = Cl_{\gamma^*}(D). \tag{5.6}$$

To be specific, since $Int_{\gamma^*}(V) = D$ and $* \in Cl_{\gamma*}(D)$, we obtain that $* \in Cl_{\gamma^*}(Int_{\gamma^*}(V))$, which guarantees the property of (5.6). Furthermore, V is also semi-closed in γ^*. Precisely, while $* \in Cl_{\gamma^*}(V)$, we obtain $* \notin Int_{\gamma^*}(Cl_{\gamma^*}(V)) = D$ so that $Int_{\gamma^*}(Cl_{\gamma^*}(V)) \subset V$.
Next, take $U := (\mathbb{Z}^2)^* \setminus V$. Then we have that $C \subset U$ and U is semi-open such that $U \cap V = \emptyset$. □

Corollary 5.7. *The infinite MW-topological sphere is a semi-T_3-space.*

By Corollary 5.7, the following is obtained.

Corollary 5.8. *The Alexandroff compactification preserves the semi-T_3-separation axiom of the MW-topological plane.*

Since the s-regularity is more restrictive than the semi-regularity, the following is obtained.

Corollary 5.9. *The infinite MW-topological sphere is s-regular.*

Since $(\mathbb{Z}^2, \gamma)$ satisfies the s-regularity and the semi-regularity, by Corollaries 5.7 and 5.8, the following is obtained.

Corollary 5.10. *The Alexandroff compactification preserves both the semi-T_3-separation axiom and the s-T_3-separation axiom of the MW-topological plane.*

Since neither the semi-T_3-separation axiom nor the s-T_3-separation axiom is related to the notion of a T_0-separation axiom, the following can be meaningful.

Theorem 5.11. *The Alexandroff compactification preserves the T_0-separation axiom of the MW-topological plane.*

Proof. Since the MW-topological space is a T_0-space, we need to prove that the infinite MW-topological sphere is also a T_0-space. To do this work, it suffices to prove that for distinct points $* \in (\mathbb{Z}^2)^* \setminus \mathbb{Z}^2$ and $p \in \mathbb{Z}^2$, there are two open sets U and V in $((\mathbb{Z}^2)^*, \gamma^*)$ such that $U \cap V = \emptyset$. To be specific, take the open set $SN_\gamma(p) \in \gamma \subset \gamma^*$ and the closed set $Cl_\gamma(SN_\gamma(p))$. Then put

$$U := (\mathbb{Z}^2)^* \setminus Cl_\gamma(SN_\gamma(p)).$$

Then it is clear that both U and V are open sets in $((\mathbb{Z}^2)^*, \gamma^*)$ and $U \cap V = \emptyset$, which implies that the infinite MW-topological sphere is a T_0-space. □

Unlike Theorem 5.11, since the MW-topological space is not a T_1-space, it is clear that the infinite MW-topological sphere is not a T_1-space. We say that a topological space (X, T) is a $T_{\frac{1}{2}}$-space (resp. $T_{\frac{1}{2}}$-space) [36] if for any $x \in X$ the singleton $\{x\}$ is either open (resp. semi-open) or closed (resp. semi-closed) in T. However, since the MW-topological space is a $T_{\frac{1}{2}}$-space, we obtain the following because the infinite MW-topological sphere is also a $T_{\frac{1}{2}}$-space owing to the closedness of the singleton $*$ in $((\mathbb{Z}^2)^*, \gamma^*)$.

Corollary 5.12. *The Alexandroff compactification preserves both the $T_{\frac{1}{2}}$- and semi-$T_{\frac{1}{2}}$-separation axiom of the MW-topological plane.*

6. Conclusions

Since the problem asking if the semi-T_3-separation axiom is preserved by the Alexandroff one point compactification, the present paper addressed the unsolved problem with a positive answer. In order to address this problem, we have initially investigated some relationships between the connectedness of a subset X and the semi-openness of X in the MW-topological space so that the connectedness of X with $X^\sharp \geq 2$ implies the semi-openness of X in $(\mathbb{Z}^2, \gamma)$. Next, we established a condition for the hereditary property in the compactness of the infinite MW-topological sphere. Using this approach, we finally proved that the semi-T_3-separation axiom is preserved by the Alexandroff one point compactification.

In the present paper, based on the semi-T_3-separation axiom of $(\mathbb{Z}^2, \gamma^2)$, we proved that $((\mathbb{Z}^2)^*, (\gamma^2)^*)$ also satisfies the semi-T_3-separation axiom. Unlike this case, as a general case, we still have the following query.

Open problem: Assume a topological space with the s-regularity or the semi-regularity. Does the Alexandroff compactification of it preserve each of the properties?

Use of AI tools declaration

The authors have not used Artificial Intelligence tools in the creation of this article.

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Conflicts of interest

The authors declare no conflict of interest.

Data availability

The authors confirm that the data supporting the findings of this study are available within this article.

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