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Research article

Estimations and optimal censoring schemes for the unified progressive hybrid gamma-mixed Rayleigh distribution

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Abstract: Censoring is a common occurrence in reliability engineering tests. This article considers estimation of the model parameters and the reliability characteristics of the gamma-mixed Rayleigh distribution based on a novel unified progressive hybrid censoring scheme (UPrgHyCS), where experimenters are allowed more flexibility in designing the test and higher efficiency. The maximum likelihood estimates of the model parameters and reliability are provided using the stochastic expectation–maximization algorithm based on the UPrgHyCS. Further, the Bayesian inference associated with any parametric function of the model parameters is considered using the Markov chain Monte Carlo method with the Metropolis-Hastings (M-H) algorithm. Asymptotic confidence and credible intervals of the proposed quantities are also created. The maximum a posteriori estimates of the model parameters are acquired. Due to the importance of determining the optimal censoring scheme for reliability problems, different optimality criteria are proposed and derived to find it. This method can help to design experiments and get more information about unknown parameters for a given sample size. Finally, comprehensive simulation experiments are provided to investigate the performances of the considered estimates, and a real dataset is analyzed to elucidate the practical application and the optimality criterion work in real life scenarios. The Bayes estimates using the M-H technique show the best performance in terms of error values.

Keywords: Bayesian estimation; gamma-mixed Rayleigh distribution; MAPE; stochastic expectation-maximization algorithm; optimum sampling; unified progressive hybrid censoring

1. Introduction

The developments and needs in engineering, manufacturing and technology inspire more improved censoring schemes. The two fundamental censoring schemes are Type-I and Type-II schemes. Type-I censoring refers to the removal of units that have not failed after a prefixed time *T*. Type-II censoring is to discard the remaining units when the number of failed units reaches a prefixed number *m*. Furthermore, if Type-I and Type-II censoring schemes are combined together, it is a hybrid censoring scheme [1]. The main limitation of this distribution and its generalizations is that a small effective sample size may be acquired [2]. Therefore, Balakrishnan et al. [3] proposed Type-I and Type-II unified hybrid (UH) censoring schemes. Extensive studies have been conducted on Type-I, Type-II, hybrid and UH censoring schemes. For instance, Kundu and Howlader [4] focused on the inverse Weibull distribution and learned its Bayesian estimation and prediction problem under the Type-II censoring scheme. Panahi [5] studied the maximum likelihood and Bayesian statistical inference problems of a UH censored model for the Burr Type III distribution. Ghazal and Hasaballah [6] investigated the Bayesian prediction from an exponentiated Rayleigh distribution. Lone and Panahi [7] learned a constant-stress model using the Gompertz distribution on UH censored data. Unfortunately, none of these censoring schemes allow for the removal of the units during a life-testing experiment. The advent of progressive censoring schemes has greatly improved the occurrence of the situations above, which means that the random removal of survival units with pre-fixed numbers happens on the basis of Type-I or Type-II censoring at each failure. The progressive censoring can speed up the test process and makes the experiments more flexible and efficient in a certain sense [8–9]. So, Gorny and Cramer [10] introduced the unified progressive hybrid censoring scheme (UPrgHyCS). The life-testing experiment based on this censoring strategy can reduce both the total time spent on tests and the related costs due to unit failure. Furthermore, the effectiveness of statistical estimation is improved as a result of more failed observations.

The UPrgHyCS arises in reliability studies as follows [11]: Suppose that an independent and identically random sample of *n* units are placed for an experiment with two integers k and m; $(k < m)$, which are predetermined, and the censoring schemes ($R = (R_1, R_2, ..., R_m)$; $\sum_{i=1}^{m} R_i = n - m$) are also fixed in advance. Let T_1 and T_2 $(T_1 < T_2)$, be the two time points. This experiment will be stopped at $T^* = \max\{\min\{X_{k:m:n}, T_2\}, \min\{X_{m:m:n}, T_1\}\}$. In brief, there are four cases in the UPrgHyCS which are shown in Figure 1. Moreover, in life research and reliability engineering applications, it is of great significance to find the appropriate distribution. The gamma-mixed Rayleigh (GMR) distribution is a continuous distribution which plays a main role in a wide range of applications, involving life-testing, survival analysis and reliability. It is one of the life models suitable for products with non-monotonic loss efficiency. That is why this motivates us to consider the GMR distribution in this estimation problem setup. To our knowledge, no one has studied the GMR distribution in the estimation of the model parameters and reliability characteristics (RCH) under the UPrgHyCS, and particularly while determining the optimal censoring scheme. Determining the optimal censoring plan is one of the most important issues in estimation because it leads to efficient estimates for parameters. That is why this motivates us to consider different estimation methods for the model parameters and RCH of the GMR distribution, as well as to determine the optimal censoring scheme.

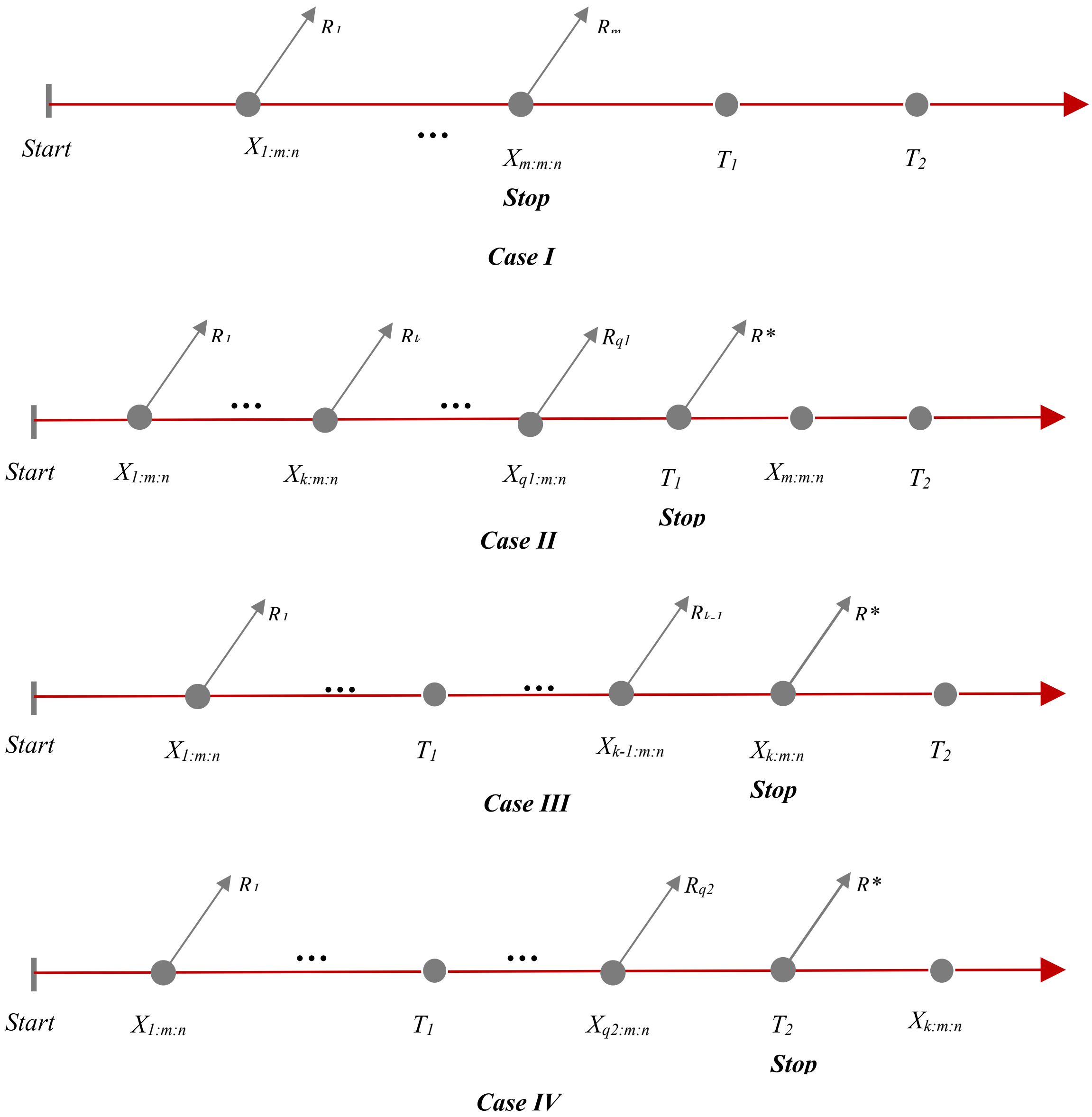

Figure 1. Different cases of the UPrgHyCS.

The probability density function and cumulative distribution function (cdf) of this distribution are shown in (1) and (2), respectively:

$$f(x;\alpha,\beta)=\frac{\alpha\beta^{\alpha}x}{(x^2+\beta^2)^{(\alpha/2)+1}};\quad x>0,\alpha,\beta>0, \tag{1}$$

and

$$F(x;\alpha,\beta)=1-\frac{\beta^{\alpha}}{(x^2+\beta^2)^{\alpha/2}};\quad x>0,\alpha,\beta>0, \tag{2}$$

Here, α and β are known as the shape and scale parameters respectively. The hazard function and reliability function of the GMR distribution are given, respectively, by $H(t)=\frac{x\alpha}{t^2+\beta^2};\ t>0,\alpha,\beta>0,$ and $R(t)=\frac{\beta^{\alpha}}{(t^2+\beta^2)^{\alpha/2}};\ t>0,\alpha,\beta>0$.

The hazards have the hump-shaped forms (see Figure 2), which is quite suitable for data-sets with a non monotonic failure rate. The hazard of the GMR distribution increases for $t<\beta$ and decreases for $t>\beta$ for any value of α. There are many researchers who have considered the issue of parameter estimation of the Rayleigh distribution and its generalizations [12–14].

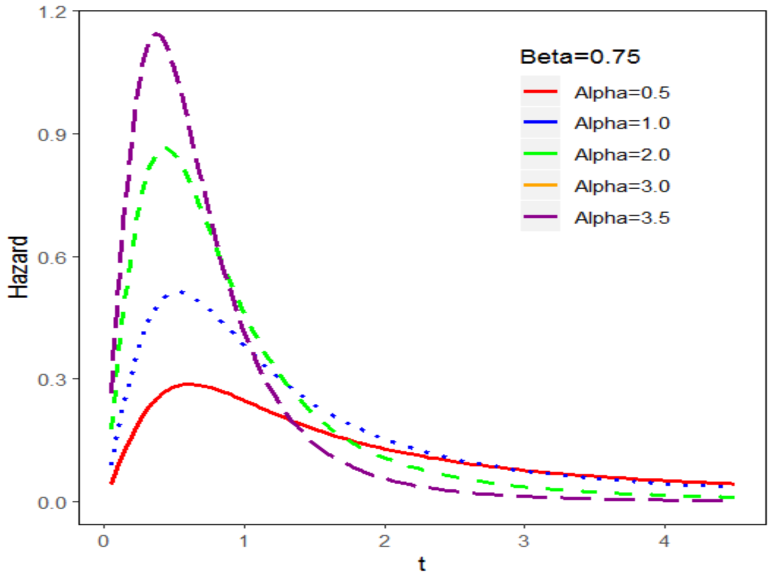

Figure 2. The hazard function of the GMR distribution.

Our objectives in this study to close this gap are as follows. First, the stochastic expectation–maximization algorithm (SEMA) is applied to attain the maximum likelihood estimates and the approximate confidence intervals (ACIs) for any function of the GMR distribution. This algorithm has good ergodic properties from which estimates can be attained. The main advantage of the SEMA is that it reduces the calculation time and complexity. The second objective is to discuss the Bayes estimates (BEs) of the unknown GMR parameters along with highest posterior density intervals (HPDI), by employing the Monte Carlo Markov chain sampling technique [15–21]. The optimal censoring scheme using different criteria has been studied. The maximum a posteriori estimates (MAPEs) are evaluated for the model parameters. Moreover, a comprehensive simulation study and real data analysis are presented to explain the methods of inference derived in this paper. The rest of the paper is organized as follows. In Section 2, we demonstrate use of the SEMA to obtain the MLEs of the model parameters and RCH and we provide the Fisher information matrix (see [22]) and ACIs under the UPrgHyCS. Bayesian analysis is explored in Section 3, where point estimates of the parameters along with the HPDI, are developed by using the MCMC sampling technique. The MAPEs for parameters are investigated in Section 4. Section 5 presents an extensive simulation study. In Section 6, different criteria for obtaining the optimal scheme are presented. One real dataset is analyzed for illustration in Section 7. Conclusions are given in Section 8.

2. Maximum likelihood estimation

Let $X_{1:m:n},\dots,X_{Q:m:n}$ be the UPrgHyCS for the GMR distribution. According to the UPrgHyCS, the likelihood function is represented as

$$L(\alpha,\beta\mid data)\propto\prod_{i=1}^{Q}\left(\left(\frac{\alpha\beta^{\alpha}x_{i:m:n}}{(x_{i:m:n}^{2}+\beta^{2})^{(\alpha/2)+1}}\right)\left(\frac{\beta^{\alpha}}{(x_{i:m:n}^{2}+\beta^{2})^{\alpha/2}}\right)^{R_i}\right)\times\left(\frac{\beta^{\alpha}}{(T^{2}+\beta^{2})^{\alpha/2}}\right)^{(n-Q-\sum_{i=1}^{Q}R_i)}, \tag{3}$$

where (Q,T) is equal to (m,x_m), (d_1,T_1), (k,x_k) and (d_2,T_2) for Cases I, II, III and IV respectively. Therefore, the log-likelihood function of (3) can be formulated as

$$\ln L(\alpha,\beta\mid data)=l(\alpha,\beta\mid data)=Q\ln\alpha+Q\alpha\ln\beta+\sum_{i=1}^{Q}\ln x_{i:m:n}-\sum_{i=1}^{Q}\left(\frac{\alpha}{2}(R_i+1)+1\right)\ln(x_{i:m:n}^{2}+\beta^{2})$$
$$-\frac{\alpha}{2}(n-Q-\sum_{i=1}^{Q}R_i)\ln(T^{2}+\beta^{2})+\alpha\ln\beta(\sum_{i=1}^{Q}R_i+(n-Q-\sum_{i=1}^{Q}R_i)). \tag{4}$$

Now, the MLEs can be acquired by differentiating the log-likelihood function with respect to the model parameters. These equations are shown as

$$\frac{\partial l(\alpha,\beta|data)}{\partial\alpha}=\frac{Q}{\alpha}+Q\ln\beta-\sum_{i=1}^{Q}((R_i+1)/2)\ln(x_{i:m:n}^{2}+\beta^{2})-((n-Q-\sum_{i=1}^{Q}R_i)/2)\ln(T^{2}+\beta^{2})$$
$$+\ln\beta(\sum_{i=1}^{Q}R_i+(n-Q-\sum_{i=1}^{Q}R_i))=0, \tag{5}$$

and

$$\frac{\partial l(\alpha,\beta|data)}{\partial\beta}=\frac{Q\alpha}{\beta}-\sum_{i=1}^{Q}\left(\frac{\alpha}{2}(R_i+1)+1\right)\frac{2\beta}{x_{i:m:n}^{2}+\beta^{2}}-\frac{\alpha}{2}(n-Q-\sum_{i=1}^{Q}R_i)\frac{2\beta}{T^{2}+\beta^{2}}+\frac{\alpha(\sum_{i=1}^{Q}R_i+(n-Q-\sum_{i=1}^{Q}R_i))}{\beta}=0. \tag{6}$$

For the sake of solving the nonlinear problem, we can use the Newton-Raphson algorithm. Nevertheless, it may not converge with small effective sample sizes. So, we apply the SEMA to acquire the MLEs, which is described below. Moreover, using the invariance property of the MLE, the MLE of the RCH can be written as

$$\hat{R}(t)=\frac{\hat{\beta}^{\hat{\alpha}}}{(t^2+\hat{\beta}^2)^{\hat{\alpha}/2}};\quad t>0 \quad . \tag{7}$$

Using the following theorem, we first show that the MLEs of α and β exist and are unique.

Theorem 2.1. The MLEs of the unknown parameters α and β for $(\alpha,\beta)\in(0,\infty)\times(0,\infty)$ exist and are unique.

Proof: We will show that for $(\alpha,\beta)\in(0,\infty)\times(0,\infty)$, the maximum of $l(\alpha,\beta\,|\,data)$ exists and is unique. Note the following

$$l_{\alpha\alpha}=\frac{\partial^2 l\left(\alpha,\beta|data\right)}{\partial\alpha^2}=-\frac{Q}{\alpha^2}, \tag{8}$$

$$l_{\beta\beta}=\frac{\partial^2 l\left(\alpha,\beta|data\right)}{\partial\beta^2}=-\frac{Q\alpha}{\beta^2}-\sum_{i=1}^{Q}(\frac{\alpha}{2}(R_i+1)+1)\frac{2(x_{i:m:n}^2-\beta^2)}{(x_{i:m:n}^2+\beta^2)^2}-\alpha(n-Q-\sum_{i=1}^{Q}R_i)\frac{T^2-\beta^2}{(T^2+\beta^2)^2}$$
$$-\frac{\alpha(\sum_{i=1}^{Q}R_i+(n-Q-\sum_{i=1}^{Q}R_i))}{\beta^2}. \tag{9}$$

Here, if $x_{i:m:n}>\beta$ and $T>\beta$, $l_{\alpha\alpha}<0$ and $l_{\beta\beta}<0$. Therefore, for a fixed $\alpha(\beta)$, $l(\alpha,\beta\,|\,data)$ is strictly a concave function of $\beta(\alpha)$. Moreover, for a fixed α,

$$\lim_{\beta\to 0} l(\alpha,\beta\,|\,data)=-\infty \quad\text{and}\quad \lim_{\beta\to\infty} l(\alpha,\beta\,|\,data)=-\infty,$$

and for a fixed β,

$$\lim_{\alpha\to 0} l(\alpha,\beta\,|\,data)=-\infty \quad\text{and}\quad \lim_{\alpha\to\infty} l(\alpha,\beta\,|\,data)=-\infty.$$

Therefore, for a fixed $\beta(\alpha)$, $l(\alpha,\beta\,|\,data)$ is unimodal with respect to $\alpha(\beta)$. Further,

$$\lim_{\alpha\to 0,\beta\to 0} l(\alpha,\beta\,|\,data)=-\infty,\quad \lim_{\alpha\to\infty,\beta\to 0} l(\alpha,\beta\,|\,data)=-\infty,\quad \lim_{\alpha\to 0,\beta\to\infty} l(\alpha,\beta\,|\,data)=-\infty \quad\text{and}$$

$$\lim_{\alpha\to\infty,\beta\to\infty} l(\alpha,\beta\,|\,data)=-\infty.$$

So, $l(\alpha,\beta\,|\,data)$ has a maximum value for some

$$(\alpha,\beta)\in\Im;\ \Im=\left\{(\alpha,\beta);(\alpha,\beta)\in(0,\infty)\times(0,\infty),l(\alpha,\beta\,|\,data)\geq l(\alpha_0,\beta_0\,|\,data)\right\}.$$

Moreover, $l(\alpha_k,\beta_k\,|\,data)>l(\alpha_k,\beta\,|\,data)>l(\alpha,\beta\,|\,data);\ (\alpha_k,\beta_k)\in(0,\infty)\times(0,\infty)$. Therefore, the MLEs of α and β exist and are unique.

2.1. SEMA

The expectation–maximization algorithm (EMA) is a considerable computational method based on incomplete data which is widely used to acquire the model parameters [23–25]. The complete sample $W=(W_1,W_2,...,W_n)$ includes the observed data $X=(X_{1:m:n},X_{2:m:n},...,X_{Q:m:n})$ and censored data $Z=(Z_1,...,Z_Q)$, where $Z_k=(Z_{k1},...,Z_{kR_k});\ k=\{1,2,...,Q\}$ is a $1\times R_k$ vector and $Z'=(Z_j^*)$ $j=1,...,R^*=n-Q-\sum_{i=1}^{Q}R$. The log-likelihood function based on $W=(X,Z)$ can be shown as

$$L_c(W;\alpha,\beta)\propto n\ln\alpha+n\alpha\ln\beta-\sum_{i=1}^{Q}(\frac{\alpha}{2}+1)\ln(x_{i:m:n}^2+\beta^2)-\sum_{i=1}^{Q}\sum_{j=1}^{R_i}(\frac{\alpha}{2}+1)\ln(z_{ij}^2+\beta^2)-\sum_{j=1}^{R^*}(\frac{\alpha}{2}+1)\ln(z_{Tj}^2+\beta^2), \tag{10}$$

where $R^*=n-Q-\sum_{i=1}^{Q}R_i$. Therefore, we have

$$\frac{\partial L_c(W;\alpha,\beta)}{\partial\alpha}=\frac{n}{\alpha}+n\ln\beta-\sum_{i=1}^{Q}((R_i+1)/2)\ln(x_{i:m:n}^2+\beta^2)-\sum_{i=1}^{Q}\sum_{j=1}^{R_i}\frac{\ln(z_{ij}^2+\beta^2)}{2}-\sum_{j=1}^{R^*}\frac{\ln(z_{Tj}^2+\beta^2)}{2}, \tag{11}$$

and

$$\frac{\partial L_c(W;\alpha,\beta)}{\partial\beta}=\frac{n\alpha}{\beta}-\sum_{i=1}^{Q}(\frac{\alpha}{2}(R_i+1)+1)\frac{2\beta}{x_{i:m:n}^2+\beta^2}\ -(\frac{\alpha}{2}+1)2\beta\left[\sum_{i=1}^{Q}\sum_{j=1}^{R_i}\frac{1}{z_{ij}^2+\beta^2}-\sum_{j=1}^{R^*}\frac{1}{z_{Tj}^2+\beta^2}\right]. \tag{12}$$

To solve the above equations under the conditions of the expectation step of the EMA, we should calculate the following conditional expectations (CEs): $E\left(\ln(Z_{ij}^2+\beta^2)\middle|Z_{ij}>x_i\right)=A(x_i;\alpha,\beta)$, $E\left(\ln(Z_{Tj}^2+\beta^2)\middle|Z_{Tj}>T\right)=B(T;\alpha,\beta)$. It is observed that the above CEs are complex. Thus, we use the SEMA to approximate them, as it does not require the CE of the missing data [26,27]. We generate the missing samples: $Z_{ij},\ i=1,...,Q,\ j=1,...,R_i$ and $Z'=Z_j^*,\ j=1,...,R^*$, whose conditional functions are given by, $\zeta(z_{ij}\,|z_{ij}>x_i)=\frac{F(z_{ij})-F(x_i)}{1-F(x_i)};\ z_{ij}>x_i$ and $\zeta(z_j^*\,|z_j^*>T)=\frac{F(z_j^*)-F(T)}{1-F(T)};\ z_j^*>T$. The estimates of the model parameters can be evaluated by applying an iterative method, which will be stopped after attaining certain accuracy.

2.2. The ACIs based on the MLE

In this subsection, we will create the $100(1-\gamma)\%,\ 0<\gamma<1$ ACIs for the model parameters based on the UPrgHyCS. The Fisher matrix is

$$I(\alpha,\beta)=E\begin{bmatrix}-l_{\alpha\alpha} & -l_{\alpha\beta}\\ -l_{\beta\alpha} & -l_{\beta\beta}\end{bmatrix},$$

where

$$l_{\alpha\alpha}=\frac{\partial^2 l\left(\alpha,\beta|data\right)}{\partial\alpha^2}=-\frac{Q}{\alpha^2}, \tag{13}$$

$$l_{\alpha\beta}=l_{\beta\alpha}=\frac{\partial^2 l\left(\alpha,\beta|data\right)}{\partial\beta\partial\alpha}=\frac{Q}{\beta}-\sum_{i=1}^{Q}(\frac{R_i+1}{2})\frac{2\beta}{x_{i:m:n}^2+\beta^2}-\frac{1}{2}(n-Q-\sum_{i=1}^{Q}R_i)\frac{2\beta}{T^2+\beta^2}+\frac{(\sum_{i=1}^{Q}R_i+(n-Q-\sum_{i=1}^{Q}R_i))}{\beta}, \tag{14}$$

and

$$l_{\beta\beta}=\frac{\partial^2 l\left(\alpha,\beta|data\right)}{\partial\beta^2}=-\frac{Q\alpha}{\beta^2}-\sum_{i=1}^{Q}(\frac{\alpha}{2}(R_i+1)+1)\frac{x_{i:m:n}^2-2\beta^2}{(x_{i:m:n}^2+\beta^2)^2}-\frac{\alpha}{2}(n-Q-\sum_{i=1}^{Q}R_i)\frac{T^2-2\beta^2}{(T^2+\beta^2)^2}$$
$$-\frac{\alpha(\sum_{i=1}^{Q}R_i+(n-Q-\sum_{i=1}^{Q}R_i))}{\beta^2}. \tag{15}$$

The MLE of $\theta=(\alpha,\beta)$ has the asymptotic distribution $\hat{\theta}\sim N(0,I^{-1}(\theta))$. $I^{-1}(\hat{\theta})$ is the inverse information matrix of $I(\hat{\theta})$, written as

$$\hat{I}^{-1}(\alpha,\beta)=\hat{I}^{-1}(\hat{\alpha},\hat{\beta})=\left\{\begin{bmatrix}-l_{\alpha\alpha} & -l_{\alpha\beta}\\ -l_{\beta\alpha} & -l_{\beta\beta}\end{bmatrix}_{\hat{\alpha},\hat{\beta}}\right\}^{-1}=\begin{bmatrix}var(\hat{\alpha}) & cov(\hat{\alpha},\hat{\beta})\\ cov(\hat{\alpha},\hat{\beta}) & var(\hat{\beta})\end{bmatrix}. \tag{16}$$

Then, the two-sided $100(1-\gamma)\%,\ 0<\gamma<1$, ACIs for α and β are as follows

$$\left[\hat{\alpha}-Z_{\gamma/2}\sqrt{var(\hat{\alpha})},\hat{\alpha}+Z_{\gamma/2}\sqrt{var(\hat{\alpha})}\right], \tag{17}$$

and

$$\left[\hat{\beta}-Z_{\gamma/2}\sqrt{var(\hat{\beta})},\hat{\beta}+Z_{\gamma/2}\sqrt{var(\hat{\beta})}\right]. \tag{18}$$

Moreover, the ACI for the RCH is

$$\left[R - Z_{\gamma/2}\sqrt{var(\hat{R})}, R + Z_{\gamma/2}\sqrt{\mathrm{var}(\hat{R})}\right]. \tag{19}$$

Also, using the delta method, $\mathrm{var}(\hat{R})$ can be approximated by:

$$var(\hat{R}) = \left[\vartheta_R^T I^{-1}(\hat{\alpha},\hat{\beta})\vartheta_R\right]; \quad \vartheta_R^T = \left(\frac{\partial R(t)}{\partial \alpha}, \frac{\partial R(t)}{\partial \beta}\right).$$

3. Bayesian estimation

The Bayesian estimation method is based on the prior distribution (PD) provided by the parameters. The PD plays a decisive role in the Bayesian approach and must be appropriately determined. A common approach to gain a PD is to use the conjugate priors. The gamma distribution is versatile for adjusting different shapes of the density function. Therefore, we have chosen the independent gamma PDs for α and β with the hyper-parameters $(a_1,b_1),\ a_1,b_1 > 0$ and $(a_2,b_2),\ a_2,b_2 > 0$ respectively. The joint prior of α and β can be written as

$$\pi(\alpha,\beta) \propto \alpha^{a_1-1}e^{-b_1\alpha}\beta^{a_2-1}e^{-b_2\beta}, \qquad \alpha > 0,\ \beta > 0. \tag{20}$$

Here, the squared error loss function (SELF) is considered. According to the UPrgHyCS, and by using the SELF, the posterior distribution of α and β can be given by

$$\pi\left(\alpha,\beta\middle|data\right) \propto \alpha^{Q+a_1-1}\beta^{Q\alpha+a_2-1}e^{-\alpha b_1-\beta b_2}\prod_{i=1}^{Q}\left(\frac{\beta^{\alpha R_i}x_{i:m:n}}{(x_{i:m:n}^2+\beta^2)^{(\alpha/2)(1+R_i)+1}}\right)\times\left(\frac{\beta^{\alpha}}{(T^2+\beta^2)^{\alpha/2}}\right)^{(n-Q-\sum_{i=1}^{Q}R_i)} \tag{21}$$

Thus, the BE of any function with the model parameters say $g(\alpha,\beta)$, under the SELF is represented as

$$\hat{g}_{Bayes_{SE}} \propto \frac{\int_0^\infty\int_0^\infty g(\alpha,\beta)\pi(\alpha,\beta|data)d\alpha d\beta}{\int_0^\infty\int_0^\infty \pi(\alpha,\beta|data)d\alpha d\beta}. \tag{22}$$

The Metropolis-Hastings (M-H) algorithm [28,29] is proposed to evaluate the BEs of parameters; additionally, the corresponding intervals due to the multiple integrals in (22) are not computed analytically and there are difficulties in the numerical computations of these integrals. It is worth mentioning that the M-H algorithm has the ability to simulate complicated posteriors and is free from the reliance on conjugate priors. This algorithm is an efficient method to calculate the BEs, as it can be easily conducted and help to reduce the operation complexity of high dimensional distribution. To apply the M-H algorithm, the posterior density functions of α and β can be re-expressed as

$$\pi_1(\alpha|\beta, data) = \alpha^{Q+a_1-1}e^{-\alpha b_1}\prod_{i=1}^{Q}\left(\frac{\beta^{\alpha R_i}x_{i:m:n}}{(x_{i:m:n}^2+\beta^2)^{(\alpha/2)(1+R_i)+1}}\right)\times\left(\frac{\beta^{\alpha}}{(T^2+\beta^2)^{\alpha/2}}\right)^{(n-Q-\sum_{i=1}^{Q}R_i)}, \tag{23}$$

and

$$\pi_2(\beta|\alpha, data) = \beta^{Q\alpha+a_2-1}e^{-\beta b_2}\prod_{i=1}^{Q}\left(\frac{\beta^{\alpha R_i}x_{i:m:n}}{(x_{i:m:n}^2+\beta^2)^{(\alpha/2)(1+R_i)+1}}\right)\times\left(\frac{\beta^{\alpha}}{(T^2+\beta^2)^{\alpha/2}}\right)^{(n-Q-\sum_{i=1}^{Q}R_i)}. \tag{24}$$

The following steps can be employed to calculate consistent estimates of the model parameters and RCH.

Step 1: Set $z=1$ and start with $(\hat{\alpha}_{ML}, \hat{\beta}_{ML})$ and M = *MCMC* iteration with *NB* = burn in.

Step 2: Generate $\alpha^{(z)}$ and $\beta^{(z)}$ with the proposal distributions $N(\alpha^{(z-1)}, Variance(\alpha))$ and $N(\beta^{(z-1)}, Variance(\beta))$ respectively.

Step 3: Calculate $\Im_\alpha = \min\left(1, \dfrac{\pi_1(\alpha^*|\beta^{(z-1)}, data)}{\pi_1(\alpha^{(z-1)}|\beta^{(z-1)}, data)}\right)$; then, $\alpha^{(i)} = \begin{cases}\alpha^* & \varsigma_1 \le \Im_\alpha \\ \alpha^{(z\text{-}1)} & \varsigma_1 > \Im_\alpha\end{cases}$ and $\Im_\beta = \min\left(1, \dfrac{\pi_2(\beta^*|\alpha^{(z)}, data)}{\pi_2(\beta^{(z-1)}|\alpha^{(z)}, data)}\right)$; then, $\beta^{(i)} = \begin{cases}\beta^* & \varsigma_2 \le \Im_\beta \\ \beta^{(z\text{-}1)} & \varsigma_2 > \Im_\beta\end{cases}$, where $\varsigma_1 \sim Uniform(0,1)$ and $\varsigma_2 \sim Uniform(0,1)$.

Step 4: Set $z = z+1$.

Step 5: Redo Steps 1–4 *M* times.

Step 6: Discard the burn-in period (*NB*); then, the BEs of α and β and the RCH under the SELF are $\hat{\alpha}_{Metropolis-Gibbs_{SE}} = \dfrac{1}{M-NB}\sum_{z=NB+1}^{M}\alpha^{(z)}$, $\hat{\beta}_{Metropolis-Gibbs_{SE}} = \dfrac{1}{M-NB}\sum_{z=NB+1}^{M}\beta^{(z)}$ and $\hat{R}_{Metropolis-Gibbs_{SE}} = \dfrac{1}{M-NB}\sum_{z=NB+1}^{M}R^{(z)}$, where $R^{(z)} = \dfrac{\beta^{(z)\alpha^{(z)}}}{(t^2+(\beta^{(z)})^2)^{\alpha^{(z)}/2}}$, $t>0$.

Step 7: Order $\alpha^{(i)}$, $\beta^{(i)}$ and $R^{(i)}$, $i=1,..,M$; then, the HPDI of any function (ξ) has the following form

$$\left[\xi_{(M\gamma/2)}, \xi_{(M(1-\gamma/2))}\right]. \tag{25}$$

4. MAPE

We know that the performance of the MAPE is better than that of the MLE based on small samples for heavy tailed or skewed distributions. Therefore, the MAPEs for the unknown parameters of the GMR distribution under the *UPrgHyCS* are investigated [30]. The MAPE can be obtained as

$$Map = \arg\max_{(\alpha,\beta)}\left(\pi(\alpha,\beta|data)\right) = \arg\max_{(\alpha,\beta)}\left(\ln\pi(data|\alpha,\beta) + \ln\pi(\alpha,\beta)\right). \tag{26}$$

where $\pi(data|\alpha,\beta)$ is the joint distribution. Using (21) and (26), the following is the maximum a posteriori estimator:

$$\hat{\Theta}_{MAP} = \arg\max_{\Theta}\left(\pi(\Theta|data)\right) \propto \arg\max_{\Theta}\Bigg((Q+a_1-1)\ln\alpha + (Q\alpha+a_2-1)\ln\beta - \alpha b_1 - \beta b_2 + \sum_{i=1}^{Q}\alpha R_i \ln\beta$$
$$-\sum_{i=1}^{Q}((\alpha/2)(1+R_i)+1)\ln(x_{i:m:n}^2+\beta^2) + (n-Q-\sum_{i=1}^{Q}R_i)\left[\alpha\ln\beta - ((\alpha/2)(1+R_i)+1)\ln(T^2+\beta^2)\right]\Bigg) \tag{27}$$

$\hat{\alpha}_{MAP}$ and $\hat{\beta}_{MAP}$, can be obtained by deriving the normal equations for the partial differentiation functions (27) and setting it to zero, as follows

$$\frac{Q+a_1-1}{\alpha} + Q\ln\beta + b_1 + \sum_{i=1}^{Q}R_i\ln\beta - \sum_{i=1}^{Q}((R_i+1)/2)\ln(x_{i:m:n}^2+\beta^2)$$
$$-((n-Q-\sum_{i=1}^{Q}R_i)/2)\ln(T^2+\beta^2) + \ln\beta(\sum_{i=1}^{Q}R_i + (n-Q-\sum_{i=1}^{Q}R_i)) = 0, \tag{28}$$

and

$$\frac{Q\alpha+a_2-1}{\beta} + b_2 - \sum_{i=1}^{Q}(\frac{\alpha}{2}(R_i+1)+1)\frac{2\beta}{x_{i:m:n}^2+\beta^2} - \frac{\alpha}{2}(n-Q-\sum_{i=1}^{Q}R_i)\frac{2\beta}{T^2+\beta^2}$$
$$+\frac{\alpha(\sum_{i=1}^{Q}R_i + (n-Q-\sum_{i=1}^{Q}R_i))}{\beta} = 0. \tag{29}$$

To evaluate $\hat{\alpha}_{MAP}$ and $\hat{\beta}_{MAP}$, we can apply the useful package, namely, the "*nleqslv*" package in *R* software.

5. Simulation study

This section is devoted to the comparative study of the proposed estimates under different cases of the UPrgHyCS. The simulation was carried out using *R* software. We considered different values for *n*, *m, k* and $T_i;\ i=1,2$ to generate 10^4 UPrgHyCSs from the $GMR(\alpha=2.5,\beta=1.25)$. Two different $T_i,\ i=1,2,$ i.e., $T_1=1.2,\ T_2=2.4$ and $T_1=1.5,\ T_2=2.8,$ and various combinations of (n,m), i.e., $(30,15),(30,20),(50,25),(50,30),$ are proposed. Also, for a given (n,m), different censoring schemes have been adopted:

CSI : $R_z=0$ *and* $R_m=n-m;\ z=1,...,m-1.$

CSII : $R_1=n-m$ *and* $R_z=0;\ z=2,...,m.$

CSIII $\begin{cases} R_z=1,\ R_{n-m+1}=R_{n-m+2}=...=R_m=0;\ n\le 2m,\ z=1,...,n-m \\ R_z=1,\ R_m=n-2m+1;\ n>2m,\ z=1,...,n-m. \end{cases}$

There are two kinds of Bayes estimation techniques that were simulated, i.e., use of the informative prior (INP) and non-informative prior (NOINP), respectively, where all

hyperparameters in the NOINP were chosen to be 0.0001 instead of 0, which is more appropriate since the hyperparameters are greater than 0, and the hyperparameters in the INP are selected according to this manner: the means of the PDs are equal to the original parameters ($a_1 = 5, b_1 = 2, a_2 = 1.25, b_2 = 1$). Using the M-H algorithm, 10000 MCMC samples were generated and then the first 1000 values (NB = 1000) were discarded. The trace plots for the MCMC chain of α and β for n = 30, m = 15, k = 10, T_1 = 1.2 and T_2 = 2.4 are shown in Figure 3. Different estimates were calculated and are displayed in the Tables 1–8. The mean squared errors (MSEs) of the point estimators (SEMA and Bayesian estimation under an INP) are presented in Tables 1 and 2, and the average length and coverage probability (*CP*) for the interval estimators are shown in Tables 3 and 4. The MSEs of the estimators and the average confidence lengths (ACLs) with their *CPs* for the ACI/credible intervals of $R(x)$ are also presented in Table 5. Also, to compare the BEs under the INP and NOINP, the heat-map plots for the MSEs of the Bayesian estimators are presented in Figures 4 and 5. The heat-map provides better visualization of the validity of different estimation methods.
The following results have been observed:

I. For fixed n and m, the MSEs of all attained estimates decrease when the values of T_1, T_2 and k increase. Larger k, T_1 and T_2 provide more effective samples which indicates more information about the parameters.
II. For a fixed n, the MSEs of the obtained estimates decrease with an increase in the values of m, k and pre-fixed time points T_1 and T_2.
III. Due to the fact that BEs include more information about the parameters compared to other estimates, the MCMC estimates are better than the frequentist estimates.
IV. Simulation results show that the increase in T_i for other fixed values, decrease the biases and MSE values in all cases since it provides more testing-time to let more failure occur.
V. The BEs under the INP perform better than the NOINP.
VI. The MAPEs have a better performance than the MLEs in terms of average MSEs.
VII. The results from the interval estimation suggest that with the increase of k, T_1 and T_2, the average interval lengths reduce, along with the average CPs approaching the nominal confidence level, by keeping n and m fixed.
VIII. The average lengths of the confidence intervals decrease with larger sample sizes. Also, the HPDIs perform better than the ACIs regarding the average lengths.

In summary, the performances of both BEs are more favorable than those obtained under the frequentist approach and MPAEs. Moreover, based on the CPs, ACIs are better than credible interval estimates.

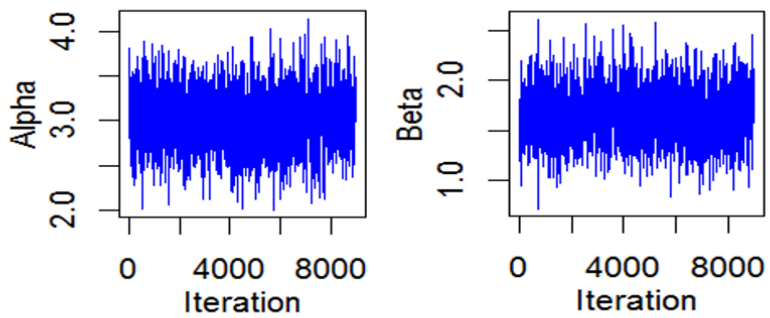

Figure 3. Trace plots for α and β (M = 10000, NB = 1000).

Table 1. The MSEs of α under different censoring schemes (C.S.) (T^*: (T_1,T_2) = (1.2,2.4) and T^{**}: (T_1,T_2) = (1.5,2.8)).

n	m	k	C.S.	T^*			T^{**}		
				MLE	*M-H*	*MAP*	*MLE*	*M-H*	*M-H*
30	15	5	I	0.24522	0.11398	0.13427	0.22543	0.10326	0.11875
30	15	5	II	0.11456	0.09432	0.10558	0.10109	0.07994	0.09654
30	15	5	III	0.18654	0.10986	0.13654	0.17003	0.10329	0.12305
30	15	10	I	0.22453	0.10942	0.12654	0.21743	0.09942	0.11198
30	15	10	II	0.10954	0.09122	0.10200	0.09327	0.07122	0.09254
30	15	10	III	0.18013	0.09985	0.12998	0.16532	0.09285	0.11548
30	20	10	I	0.22206	0.10435	0.12254	0.19858	0.09832	0.10548
30	20	10	II	0.10684	0.08828	0.09547	0.09004	0.06843	0.08543
30	20	10	III	0.17542	0.09784	0.12326	0.15643	0.09004	0.11034
30	20	15	I	0.19975	0.09335	0.11647	0.17674	0.08522	0.10365
30	20	15	II	0.10154	0.08207	0.09126	0.08756	0.06097	0.08312
30	20	15	III	0.15897	0.08426	0.10988	0.15111	0.08038	0.10587
50	30	15	I	0.19576	0.08894	0.10885	0.17214	0.07859	0.09942
50	30	15	II	0.10111	0.07857	0.08335	0.08342	0.05436	0.07690
50	30	15	III	0.15784	0.08066	0.10670	0.14587	0.07885	0.10216
50	30	20	I	0.16754	0.08095	0.09758	0.14732	0.04999	0.09112
50	30	20	II	0.08963	0.07214	0.08055	0.06353	0.03761	0.06546
50	30	20	III	0.13479	0.08037	0.09843	0.11955	0.06203	0.09492

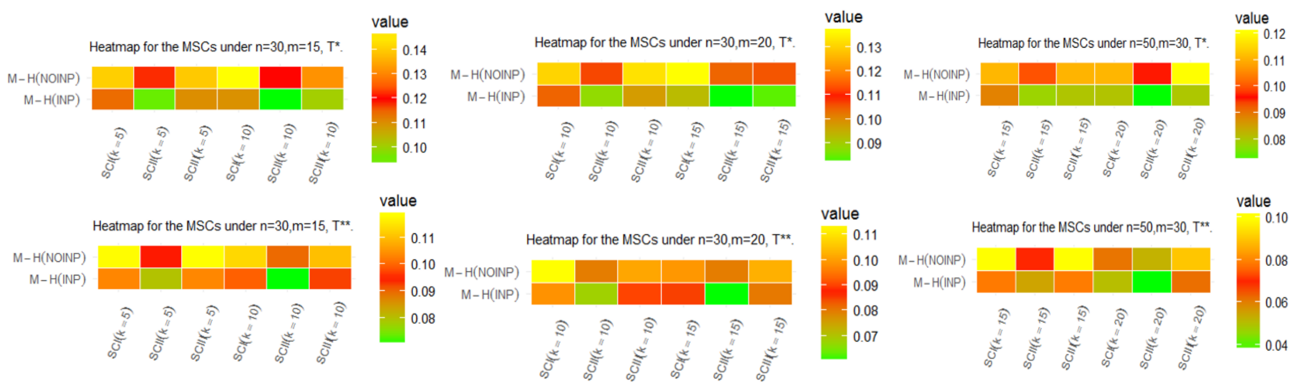

Figure 4. The heat-map results the MSEs of the BEs of α under the INP and NOINP.

Table 2. The MSEs of β under different censoring schemes (T^*: (T_1,T_2) = (1.2,2.4) and T^{**}: (T_1,T_2) = (1.5,2.8)).

n	*m*	*k*	C.S.	T^*			T^{**}		
				MLE	*M-H*	*MAP*	*MLE*	*M-H*	*MAP*
30	15	5	I	0.12932	0.10432	0.10783	0.11324	0.09470	0.09633
30	15	5	II	0.09843	0.08258	0.08959	0.09209	0.07742	0.07884
30	15	5	III	0.11442	0.09765	0.99842	0.11014	0.09365	0.09621
30	15	10	I	0.11705	0.09911	0.10076	0.10321	0.09416	0.09498
30	15	10	II	0.09121	0.07984	0.08169	0.08674	0.07121	0.07543
30	15	10	III	0.11032	0.08842	0.09327	0.10300	0.08205	0.08986
30	20	10	I	0.11409	0.09445	0.09782	0.09754	0.09008	0.09095
30	20	10	II	0.08432	0.07324	0.07893	0.08214	0.06885	0.06894
30	20	10	III	0.10999	0.08224	0.08674	0.09743	0.08054	0.08292
30	20	15	I	0.09953	0.09005	0.09366	0.09354	0.08685	0.08890
30	20	15	II	0.08123	0.07210	0.07632	0.07489	0.06236	0.06668
30	20	15	III	0.09674	0.08016	0.08417	0.09266	0.07985	0.08065
50	30	15	I	0.07854	0.07205	0.07341	0.07430	0.06957	0.07001
50	30	15	II	0.07094	0.06749	0.06867	0.06546	0.06028	0.06369
50	30	15	III	0.07645	0.07111	0.07389	0.07116	0.06934	0.06721
50	30	20	I	0.05894	0.05243	0.05757	0.05354	0.04778	0.04799
50	30	20	II	0.04776	0.04255	0.04509	0.04032	0.03858	0.03995
50	30	20	III	0.05214	0.04893	0.04928	0.05318	0.04643	0.04690

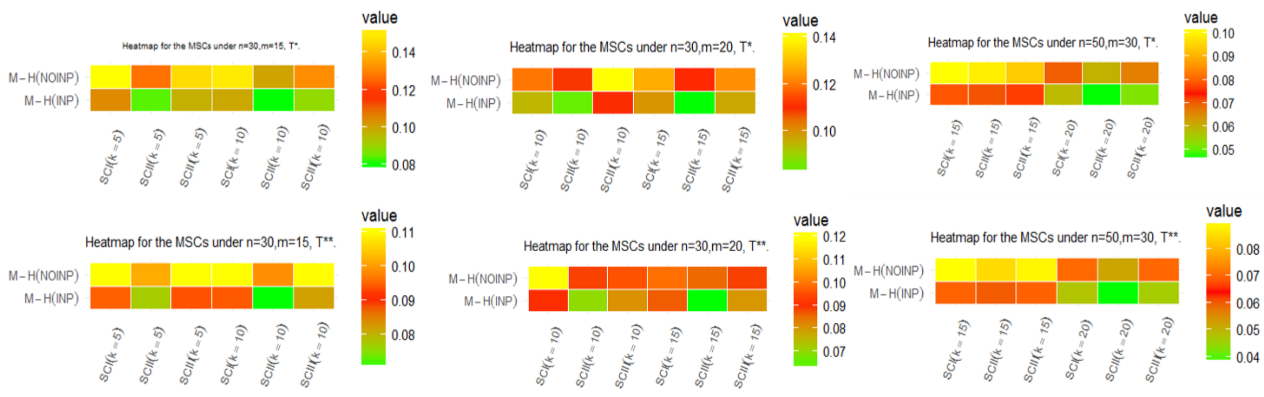

Figure 5: The heat-map results the MSEs of the BEs of β under the INP and NOINP.

Table 3. Average lengths and CPs of 95% intervals for α under different censoring schemes.

n	m	k	C.S.	T* ACIs	T* CP-ACI	T* HPD	T* CP-HPD	T** ACIs	T** CP-ACI	T** HPD	T** CP-HPD
30	15	5	I	0.96843	0.9532	0.67328	0.9719	0.94765	0.9588	0.66223	0.9632
30	15	5	II	0.95321	0.9536	0.65473	0.9612	0.94066	0.9532	0.65126	0.9611
30	15	5	III	0.96463	0.9531	0.67120	0.9634	0.94327	0.9565	0.67157	0.9637
30	15	10	I	0.96324	0.9495	0.66437	0.9632	0.94256	0.9435	0.65346	0.9721
30	15	10	II	0.09506	0.9511	0.65132	0.9715	0.93687	0.9588	0.63675	0.9598
30	15	10	III	0.95879	0.9478	0.66453	0.9643	0.93998	0.9485	0.64895	0.9600
30	20	10	I	0.95876	0.9573	0.65357	0.9632	0.93870	0.9600	0.65357	0.9665
30	20	10	II	0.94685	0.9524	0.64219	0.9487	0.93265	0.9517	0.64219	0.9498
30	20	10	III	0.95642	0.9537	0.65325	0.9634	0.93756	0.9565	0.65325	0.9701
30	20	15	I	0.95327	0.9533	0.64329	0.9465	0.93161	0.9428	0.63425	0.9675
30	20	15	II	0.93996	0.9539	0.63646	0.9599	0.92457	0.9515	0.62886	0.9589
30	20	15	III	0.95400	0.9478	0.64177	0.9621	0.92995	0.9600	0.63423	0.9623
50	30	15	I	0.94124	0.9600	0.63785	0.9654	0.92876	0.9423	0.61598	0.9609
50	30	15	II	0.93675	0.9564	0.63115	0.9622	0.91980	0.9488	0.61110	0.9567
50	30	15	III	0.93896	0.9576	0.63684	0.9653	0.92693	0.9436	0.61378	0.9411
50	30	20	I	0.92453	0.9488	0.60884	0.9642	0.89954	0.9600	0.58989	0.9633
50	30	20	II	0.91549	0.9525	0.59805	0.9578	0.89305	0.9490	0.56999	0.9589
50	30	20	III	0.92159	0.9596	0.60785	0.9633	0.89894	0.9449	0.58757	0.9610

Table 4. Average lengths and CPs of 95% intervals for β under different censoring schemes.

n	m	k	C.S.	T* ACIs	T* CP-ACI	T* HPD	T* CP-HPD	T** ACIs	T** CP-ACI	T** HPD	T** CP-HPD
30	15	5	I	0.62355	0.9616	0.58435	0.9700	0.61886	0.9588	0.57214	0.9675
30	15	5	II	0.59675	0.9563	0.57312	0.9615	0.59004	0.9587	0.56543	0.9421
30	15	5	III	0.61994	0.9475	0.58223	0.9655	0.61332	0.9600	0.56896	0.9622
30	15	10	I	0.61785	0.9477	0.58009	0.9421	0.60773	0.9456	0.58009	0.9700
30	15	10	II	0.59324	0.9537	0.56874	0.9446	0.57324	0.9565	0.56874	0.9597
30	15	10	III	0.61566	0.9600	0.57897	0.9632	0.60508	0.9600	0.57897	0.9413
30	20	10	I	0.61110	0.9600	0.57099	0.9621	0.58329	0.9566	0.56685	0.9426
30	20	10	II	0.58873	0.9522	0.56231	0.9453	0.56874	0.9469	0.55319	0.9411
30	20	10	III	0.60987	0.9589	0.56998	0.9600	0.58895	0.9429	0.56208	0.9589
30	20	15	I	0.60753	0.9548	056638	0.9459	0.58005	0.9539	0.56022	0.9586

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n	m	k	C.S.	T* ACIs	T* CP-ACI	T* HPD	T* CP-HPD	T** ACIs	T** CP-ACI	T** HPD	T** CP-HPD
30	20	15	II	0.57999	0.9533	0.56008	0.9600	0.56206	0.9469	0.55812	0.9421
30	20	15	III	0.60442	0.9400	0.56539	0.9631	0.57649	0.9611	0.55998	0.9615
50	30	15	I	0.60044	0.9435	0.56154	0.9424	0.56995	0.9600	0.56154	0.9600
50	30	15	II	0.57325	0.9543	0.54997	0.9576	0.55369	0.9535	0.54997	0.9566
50	30	15	III	0.59745	0.9554	0.55686	0.9400	0.56560	0.9579	0.55686	0.9408
50	30	20	I	0.58432	0.9565	0.55648	0.9587	0.55210	0.9543	0.55648	0.9460
50	30	20	II	0.56894	0.9511	0.54026	0.9465	0.54767	0.9515	0.54026	0.9455
50	30	20	III	0.58207	0.9488	0.55622	0.9600	0.54989	0.9485	0.55622	0.9580

Table 5. The MSEs, average lengths and CPs of $R(x)$ under different censoring schemes.

n	m	k	C.S.	T* MLE	T* M-H	T* ACIs (CP)	T* HPD (CP)	T** MLE	T** M-H	T** ACIs (CP)	T** HPD (CP)
30	15	5	I	0.07985	0.06432	0.34873 (0.958)	0.21873 (0.965)	0.07143	0.06068	0.34643 (0.960)	0.21532 (0.972)
30	15	5	II	0.07324	0.05798	0.34531 (0.945)	0.21307 (0.970)	0.06897	0.05712	0.34144 (0.946)	0.20966 (0.936)
30	15	5	III	0.07744	0.06219	0.34855 (0.964)	0.21822 (0.932)	0.07007	0.06025	0.34623 (0.941)	0.21514 (0.934)
30	15	10	I	0.07173	0.05998	0.34507 (0.963)	0.21477 (0.974)	0.06895	0.05785	0.34205 (0.940)	0.21213 (0.931)
30	15	10	II	0.06537	0.05605	0.33867 (0.940)	0.21009 (0.964)	0.06650	0.05545	0.33532 (0.940)	0.20768 (0.960)
30	15	10	III	0.07009	0.05746	0.34216 (0.938)	0.21481 (0.972)	0.06822	0.05717	0.34100 (0.961)	0.21203 (0.935)
30	20	10	I	0.06996	0.05788	0.34094 (0.962)	0.20886 (0.935)	0.06485	0.05720	0.33658 (0.941)	0.20628 (0.972)
30	20	10	II	0.06312	0.05346	0.33256 (0.961)	0.20154 (0.964)	0.06131	0.05291	0.33026 (0.943)	0.20010 (0.962)
30	20	10	III	0.06704	0.05673	0.33965 (0.940)	0.20653 (0.971)	0.06467	0.05655	0.33537 (0.942)	0.20537 (0.973)
30	20	15	I	0.06494	0.05290	0.33658 (0.940)	0.19748 (0.933)	0.06212	0.05234	0.33302 (0.938)	0.19456 (0.975)
30	20	15	II	0.06010	0.05111	0.32760 (0.958)	0.19394 (0.940)	0.05899	0.05009	0.32445 (0.942)	0.19133 (0.940)
30	20	15	III	0.06486	0.05266	0.32999 (0.939)	0.19659 (0.972)	0.06178	0.05210	0.32876 (0.960)	0.19441 (0.973)

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n	m	k	C.S.	T* MLE	T* M-H	T* ACIs (CP)	T* HPD (CP)	T** MLE	T** M-H	T** ACIs (CP)	T** HPD (CP)
50	30	15	I	0.06231	0.04574	0.31887 (0.962)	0.18765 (0.964)	0.06180	0.04498	0.31366 (0.938)	0.18332 (0.974)
50	30	15	II	0.05769	0.04390	0.31539 (0.942)	0.18222 (0.940)	0.05713	0.04323	0.31199 (0.939)	0.17985 (0.935)
50	30	15	III	0.06318	0.04577	0.31889 (0.959)	0.18753 (0.938)	0.06156	0.04476	0.31358 (0.942)	0.18141 (0.960)
50	30	20	I	0.05873	0.03879	0.31266 (0.943)	0.18227 (0.960)	0.05800	0.03799	0.30807 (0.939)	0.17843 (0.961)
50	30	20	II	0.05266	0.03452	0.30687 (0.957)	0.17639 (0.959)	0.05189	0.03358	0.30256 (0.958)	0.17467 (0.939)
50	30	20	III	0.05831	0.03709	0.30890 (0.957)	0.18185 (0.960)	0.05790	0.03689	0.30775 (0.960)	0.17699 (0.939)

6. Optimum progressive censoring

Determination of the optimum progressive censoring scheme plan is a critical purpose for a reliability researcher. To determine the optimum progressive censoring scheme plan, some criteria for specified values of n, m, k, T_1, T_2 and $R_i, i=1,\ldots,m$ have been adopted as follows

- **Criterion 1:** $det(I^{-1}(\hat{q}))$; The objective is to minimize $det(I^{-1}(\hat{q}))$.
- **Criterion 2:** $trace(I^{-1}(\hat{\theta}))$; The objective is to minimize $trace(I^{-1}(\hat{\theta}))$.
- **Criterion 3:** $Variance(\log(\hat{V}_p))$; The objective is to minimize $Variance(\log(\hat{V}_p))$, $0<p<1$.
- **Criterion 4:** $\Im=\int_0^1 Variance(\log(\hat{\varsigma}_p))w(p)dp$; The objective is to minimize $\Im$, $w(p)\geq 0, \int_0^1 w(p)dp=1$.

Also, the logarithmic for ς_p of the GMR distribution can be represented as follows

$$\log(\varsigma_p)=\left[\left(\beta^{\alpha}(1-p)^{-1}\right)^{2/\alpha}-\beta^2\right]^{1/2}; \quad 0<p<1. \tag{30}$$

Using (30) and the delta method, the approximation of $Variance(\log(\hat{\varsigma}_p))$ can be shown as

$$Variance(\log(\hat{\varsigma}_p))=\left(\nabla\log(\hat{\varsigma}_p)\right)^T I^{-1}(\hat{\theta})\left(\nabla\log(\hat{\varsigma}_p)\right), \tag{31}$$

where, $\left(\nabla \log(\hat{\varsigma}_p)\right)^T = \left[\frac{\partial}{\partial \alpha} l\log(\hat{\varsigma}_p), \frac{\partial}{\partial \beta} l\log(\hat{\varsigma}_p)\right]_{(\alpha=\hat{\alpha}, \beta=\hat{\beta})}$ is the gradient of $\log(\hat{\varsigma}_p)$ with respect to α and β.

Note that the optimized censoring scheme corresponds to the lowest values of the proposed criteria.

7. Real data analysis

For the application of the UPrgHyCS in practice, we use real data analysis to validate the estimation methods acquired in this paper. The real life data contain the time between failures for 30 items of repairable mechanical equipment [31]. The fitted result of the dataset is compared by utilizing the Kolmogorov-Smirnov (K-S) distance, as well as the *p*-value. The K-S distance is 0.0862 with a *p*-value of 0.9788. We also presented the probability-probability (P-P) and empirical cdf plots for the fitted GMR distribution in Figures 6 and 7 respectively. It can be seen that the presentations follow from the numerical results. We also compared the GMR distribution with other well-known distributions by using the different model selection (MS) criteria such as the log-likelihood criterion, Akaike information criterion (AIC), Bayesian information criterion (BIC), K-S statistic, Cramer-von Mises (CVM) statistic and Anderson-Darling (AnDa) statistic. The results have been presented in Table 6. The computations of these criteria were carried out using *R* software by implementing the "*fitdistrplus*" package. From Table 6, it can be seen that the GMR is the best model, as compared to other fitted models in the literature, for fitting real-life data; this is because it has the lowest statistic values. Also, to show the existence and uniqueness of the MLEs, the contour plot of the joint $l(\alpha, \beta \mid data)$ using the complete dataset is plotted in Figure 8. It confirms these properties of the MLEs.

We consider the following UPrgHyCSs, which are given as follows

- ***Scheme I:*** $n = 30, m = 22, k = 18, T_1 = 1.9, T_2 = 2.5, R = (8, 0^{*21})$,
- ***Scheme II:*** $n = 30, m = 22, k = 21, T_1 = 1.79, T_2 = 1.95, R = (8, 0^{*21})$,
- ***Scheme III:*** $n = 30, m = 22, k = 20, T_1 = 1.8, T_2 = 2.5, R = (8, 0^{*21})$.

Table 6: Different MS criteria for the fitted distributions.

Distribution	*Log-likelihood*	*AIC*	*BIC*	*K-S*	*CVM*	*AnDa*
GMR	-39.84937	83.69873	86.50113	0.086269	0.019857	0.146416
Inverse Weibull	-46.37561	96.75122	99.55361	0.133855	0.183039	1.229709
Inverse Gamma	-45.50735	95.01469	97.81709	0.157632	0.173521	1.066590
Log-Normal	-40.73513	85.47025	88.27265	0.098692	0.036932	0.257797
Inverted Exp-Rayleigh	-52.88474	109.7695	112.5719	0.259787	0.620545	3.236685

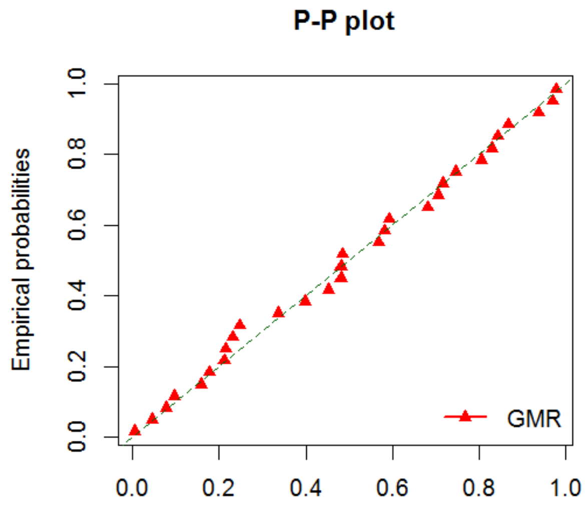

Figure 6. The P-P plot for the GMR.

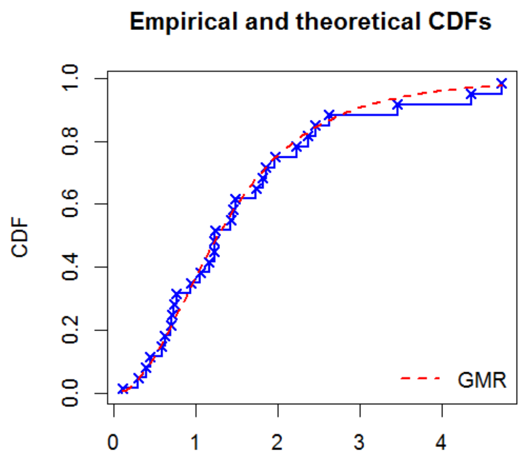

Figure 7. The empirical and the fitted GMR.

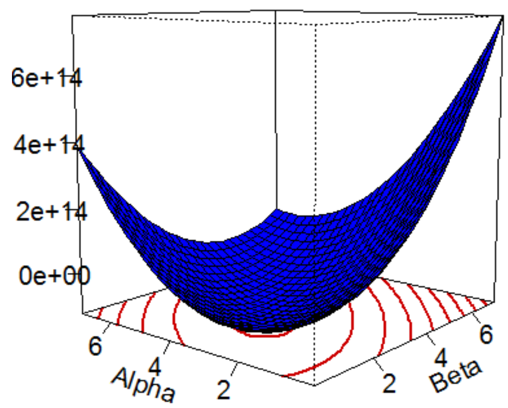

Figure 8. The 3D plot of negative $l(\alpha, \beta \mid data)$.

According to the censoring schemes described above, the MLEs based on the SEMA, Bayesian point estimation, maximum a posteriori estimation, ACIs and HPDIs were acquired. Borrowing the idea from [32], the hyperparameters under the NOINP were taken as $a_i = b_i = 0.0001,\ i = 1,2$. Table 7 displays the point and interval estimates of the parameters. It is clear that the BEs perform marginally better than the MLEs. Also, the Bayesian approach features the shortest credible interval while the length of the ACI is the longest.

To express the concept of optimal censoring, the different criteria are evaluated based on the following three generated samples:

$$((1): R = (0^{*21}, 8),\ (2): R = (8, 0^{*21}),\ (3): R = (0^{*11}, 8, 0^{*10})).$$

Based on two values of $p; (p = 0.4, 0.7)$, the results were obtained as reported in Table 8. It is clear that $R = (8, 0^{*21})$ is optimal scheme under criteria 1, 3 and 4, while $R = (0^{*11}, 8, 0^{*10})$ is optimal based on criterion 2. We notice that these outcomes support our results in the simulation section.

Table 7: Different estimates for α and β based on different censoring schemes.

Schemes		*MLEs*	*BEs*	*ACI*	*HPD*
CSI	α	4.54394	4.19734	(4.2455,5.7984)	(4.3874,5.3258)
CSI	β	2.43283	2.20064	(1.9784,3.1109)	(2.0406,3.0244)
CSII	α	3.85462	3.82115	(3.4318,4.1732)	(3.5433,4.1263)
CSII	β	1.99794	1.89873	(1.1178,2.7943)	(1.3642,2.5648)
CSIII	α	4.75328	4.26964	(4.1658,5.2154)	(4.3275,5.2295)
CSIII	β	2.31763	2.19835	(1.9226,2.9974)	(1.9998,2.6893)

Table 8: Optimum censoring schemes for the proposed samples.

Schemes	*Criterion* 1	*Criterion* 2	*p*	*Criterion* 3	*Criterion* 4
(1)	0.000672	0.189035	*p = 0.4*	0.0585438	0.0732473
			p = 0.7	0.0511043	0.0634898
(2)	**0.000264**	0.150968	*p = 0.4*	**0.0328412**	**0.0532815**
			p = 0.7	**0.0299632**	**0.0477685**
(3)	0.000348	**0.1495437**	*p = 0.4*	0.0489853	0.0599854
			p = 0.7	0.0402164	0.0564893

6. Conclusions

It is of great significance to find the appropriate lifetime distribution and optimal censoring scheme. This is also the basis of life-testing research and reliability research. In this article, the estimation and optimal censoring problems of the parameters and RCH of the GMR distribution are studied based on the UPrgHyCS. The SEMA is provided to calculate the MLEs of the model parameters and RCH, as well as the ACIs. We provided conditions under which the MLEs exist and are unique. Moreover, the Bayesian approach has been investigated with the gamma priors; since the Bayesian estimation cannot be acquired in closed form, the MCMC method was utilized to provide the

BEs and the HPDIs. Also, we have obtained the MAPEs for the unknown parameters. An elaborate simulation study was performed for the comparison of the considered estimators. Simulation results indicate that the Bayes estimator outperforms the MLE in terms of their accuracies as reported in Tables 1–5. Also, the classical and Bayes estimations based on the NOINP show similar performances. Different information measures are utilized to achieve the optimal censoring scheme. To elucidate the practical application of the proposed methods to real-life phenomenon, and to determine the optimal censoring plan, a real dataset was analyzed. According to the research, the GMR distribution is a suitable model for the proposed data, and the left censoring scheme is optimal in most cases. Future work will explore estimation for stress-strength reliability models and accelerated life-testing models under an UPrgHyCS.

Use of AI tools declaration

The authors declare that they have not used artificial intelligence tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interest.

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Supplementary

Nomenclature	Acronyms
Approximate Confidence Intervals	ACIs
Bayes Estimates	BEs
Censoring Scheme	CS
Coverage Probability	CP
Gamma-Mixed Rayleigh	GMR
Highest Posterior Density Interval	HPDI
Informative Prior	INP
Metropolis-Hastings	M-H
Mean Square Error	MSE
Maximum Likelihood	ML
Maximum a Posteriori Estimate	MAPE
Monte Carlo Markov Chain	MCMC
Non-Informative Prior	NOINP
Optimal Censoring Plan	OCP
Prior Distribution	PD
Progressive Censoring	Prg

Continue to next page

Nomenclature	Acronyms
Reliability Characteristic	RCH
Stochastic Expectation-Maximization Algorithm	SEMA
Squared Error Loss Function	SELF
Unified Hybrid	UH
Unified Progressive Hybrid Censoring Scheme	UPrgHyCS