Electronic Research Archive

http://www.aimspress.com/journal/era

ERA, 31(8): 4773–4787.
DOI: 10.3934/era.2023244
Received: 29 March 2023
Revised: 03 June 2023
Accepted: 18 June 2023
Published: 07 July 2023

Research article

Some new results for B_1-matrices

Yan Li and Yaqiang Wang*

School of Mathematics and Information Science, Baoji University of Arts and Sciences, Baoji, Shaanxi 721013, China

* **Correspondence:** Email: yaqiangwang1004@163.com.

Abstract: The class of B_1-matrices is a subclass of P-matrices and introduced as a generalization of B-matrices. In this paper, we present several properties for B_1-matrices. Then, the infinity norm upper bound for the inverse of B_1-matrices is obtained. Furthermore, the error bound for the linear complementarity problem of B_1-matrices is presented. Finally, some numerical examples are given to illustrate our results.

Keywords: B_1-matrices; SDD_1-matrices; P-matrices; infinity norm; linear complementarity problem

1. Introduction

The linear complementarity problem is to find a vector $x \in R^n$ satisfying

$$x \geq 0, Ax + q \geq 0, x^T(Ax + q) = 0, \tag{1.1}$$

where A is an $n \times n$ real matrix, and $q \in R^n$. Usually, it is denoted by $LCP(A, q)$.

$LCP(A, q)$ arises in many applications such as finding a Nash equilibrium point of a bimatrix game, the network equilibrium problem, the contact problem etc. For details, see [1]. Moreover, it has been shown that the $LCP(A, q)$ has a unique solution for any vector $q \in R^n$ if and only if A is a P-matrix, where an $n \times n$ real matrix A is called a P-matrix if all its principal minors are positive. Therefore, the class of P-matrices plays an important role in $LCP(A, q)$; see [2–4].

The cases when the matrix A for the $LCP(A, q)$ belongs to P-matrices or some subclasses of P-matrices have been widely studied, such as B-matrices [5, 6], SB-matrices [7], DB-matrices [8] and so on [9–13]. The class of B_1-matrices is a subclass of P-matrices that contains B-matrices, which was proposed by Peña [14]. However, the error bound for the linear complementarity problem of B_1-matrices has not been reported yet.

At the end of this section, the structure of the article is given. Some properties of B_1-matrices are given in Section 2. In Section 3, the infinity norm upper bound for the inverse of B_1-matrices is obtained. In Section 4, the error bound for the linear complementarity problem corresponding to B_1-matrices is proposed.

2. Some properties for B_1-matrices

In this section, some properties for B_1-matrices are proposed. To begin with, some notations, definitions and lemmas are listed as follows.

Let n be an integer number, $N = \{1, 2, \ldots, n\}$ and $C^{n\times n}$ be the set of all complex matrices of order n.
$r_i(A) = \sum_{j\neq i} |a_{ij}|$,
$N_1(A) = \{i \in N : |a_{ii}| \leq r_i(A)\}$,
$N_2(A) = \{i \in N : |a_{ii}| > r_i(A)\}$,
$p_i(A) = \sum_{j\in N_1(A)\backslash\{i\}} |a_{ij}| + \sum_{j\in N_2(A)\backslash\{i\}} \frac{r_j(A)}{|a_{jj}|}|a_{ij}|$,
$r^+_{i_A} = max\{0, a_{ij}|i \neq j\}$,

$$B^+{=}(b^+_{ij})_{1\leq i,j\leq n}{=}\begin{bmatrix} a_{11} - r^+_{1_A} & a_{12} - r^+_{1_A} & \cdots & a_{1n} - r^+_{1_A} \\ a_{21} - r^+_{2_A} & a_{22} - r^+_{2_A} & \ddots & a_{2n} - r^+_{2_A} \\ \vdots & & \ddots & \vdots \\ a_{n1} - r^+_{n_A} & a_{n2} - r^+_{n_A} & \cdots & a_{nn} - r^+_{n_A} \end{bmatrix},$$

$p_i(B^+) = \sum_{j\in N_1(A)\backslash\{i\}} |a_{ij} - r^+_{i_A}| + \sum_{j\in N_2(A)\backslash\{i\}} \frac{r_j(B^+)}{|a_{jj}-r^+_{j_A}|}|a_{ij} - r^+_{i_A}|$.

Definition 2.1. *[2] A matrix $A = (a_{ij}) \in C^{n\times n}$ is a P-matrix if all its principal minors are positive.*

Definition 2.2. *[14] A matrix $A = (a_{ij}) \in C^{n\times n}$ is an SDD_1 by rows if for each $i \in N_1(A)$,*

$$|a_{ii}| > p_i(A). \tag{2.1}$$

Definition 2.3. *[14] A matrix $A = (a_{ij}) \in C^{n\times n}$ is a B_1-matrix if for all $i \in N$,*

$$a_{ii} - r^+_{i_A} > p_i(B^+). \tag{2.2}$$

Lemma 2.1. *[14] If a matrix $A = (a_{ij}) \in C^{n\times n}$ is an SDD_1 by rows, then it is also a nonsingular H-matrix.*

Lemma 2.2. *[15] If a matrix $A = (a_{ij}) \in C^{n\times n}$ is an H-matrix with positive diagonal entries, then it is also a P-matrix.*

In the following, some properties of B_1-matrices are derived.

First of all, utilizing Definitions 2.2 and 2.3, Theorems 2.1–2.4 can be easily obtained.

Theorem 2.1. *Let matrix $A = (a_{ij}) \in C^{n\times n}$ be a B_1-matrix. Then, B^+ is an SDD_1 matrix.*

Example 2.1. *Let matrix*

$$A = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 3 & -2 \\ 9 & 10 & 110 \end{bmatrix}.$$

We write $A = B^+ + C$, where

$$B^+ = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 2 & -3 \\ -1 & 0 & 100 \end{bmatrix},$$

and

$$C = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 10 & 10 & 10 \end{bmatrix}.$$

By calculation, we have that

$$N_1(B^+) = \{2\}, N_2(B^+) = \{1, 3\},$$

$$a_{11} - r^+_{1_A} = 2 > 1 = p_1(B^+) = |a_{12} - r^+_{1_A}| + \frac{r_3(B^+)}{|a_{33} - r^+_{3_A}|}|a_{13} - r^+_{1_A}|,$$

$$a_{22} - r^+_{2_A} = 2 > 0.0300 = p_2(B^+) = \frac{r_1(B^+)}{|a_{11} - r^+_{1_A}|}|a_{21} - r^+_{2_A}| + \frac{r_3(B^+)}{|a_{33} - r^+_{3_A}|}|a_{23} - r^+_{2_A}|,$$

and

$$a_{33} - r^+_{3_A} = 100 > 0.5000 = p_3(B^+) = |a_{32} - r^+_{3_A}| + \frac{r_1(B^+)}{|a_{11} - r^+_{1_A}|}|a_{31} - r^+_{3_A}|.$$

From Definition 2.3, it is easy to obtain that A is a B_1-matrix, and B^+ is an SDD_1 matrix since $|b^+_{22}| = 2 > 0.0300 = p_2(B^+) = \frac{r_1(B^+)}{|a_{11}-r^+_{1_A}|}|a_{21} - r^+_{2_A}| + \frac{r_3(B^+)}{|a_{33}-r^+_{3_A}|}|a_{23} - r^+_{2_A}|$ from Definition 2.2.

However, Theorem 2.1 is not true on the contrary, and the following Example 2.2 illustrates this fact.

Example 2.2. *Let us consider the matrix*

$$A = \begin{bmatrix} -3 & 1 & 1 \\ -4 & -2 & 1 \\ 3 & 1 & -2 \end{bmatrix}.$$

We write $A = B^+ + C$, where

$$B^+ = \begin{bmatrix} -4 & 0 & 0 \\ -5 & -3 & 0 \\ 0 & -2 & -5 \end{bmatrix},$$

and

$$C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 3 & 3 & 3 \end{bmatrix}.$$

By calculation, we have that

$$N_1(B^+) = \{2\}, N_2(B^+) = \{1, 3\}.$$

From Definitions 2.2 and 2.3, we obtain that B^+ is an SDD_1 matrix since $|b^+_{22}| = 3 > 0 = p_2(B^+) = \frac{r_1(B^+)}{|a_{11}-r^+_{1_A}|}|a_{21} - r^+_{2_A}| + \frac{r_3(B^+)}{|a_{33}-r^+_{3_A}|}|a_{23} - r^+_{2_A}|$, but A is not a B_1-matrix since $a_{11} - r^+_{1_A} = -4 < 0 = p_1(B^+) = |a_{12} - r^+_{1_A}| + \frac{r_3(B^+)}{|a_{33}-r^+_{3_A}|}|a_{13} - r^+_{1_A}|$.

Motivated by Example 2.2, one can easily obtain Theorem 2.2.

Theorem 2.2. *Let matrix $A = (a_{ij}) \in C^{n\times n}$ be a B_1-matrix if and only if B^+ is an SDD_1 matrix with positive diagonal entries.*

Note that B^+ is a Z-matrix with positive diagonal entries from the definition of B^+, and it is easy to obtain Theorem 2.3.

Theorem 2.3. *Let matrix $A = (a_{ij}) \in C^{n\times n}$ be a B_1-matrix if and only if B^+ is a B_1-matrix.*

Example 2.3. *Let us consider the matrix*

$$A = \begin{bmatrix} 75 & 74 & 23 \\ 10 & 15 & 12 \\ 9 & 10 & 110 \end{bmatrix}.$$

We write $A = B^+ + C$, where

$$B^+ = \begin{bmatrix} 1 & 0 & -51 \\ -2 & 3 & 0 \\ -1 & 0 & 100 \end{bmatrix},$$

and

$$C = \begin{bmatrix} 74 & 74 & 74 \\ 12 & 12 & 12 \\ 10 & 10 & 10 \end{bmatrix}.$$

By calculation, we have that

$$N_1(B^+) = \{1\}, N_2(B^+) = \{2, 3\},$$

$$a_{11} - r^+_{1_A} = 1 > 0.5100 = p_1(B^+) = \frac{r_2(B^+)}{|a_{22} - r^+_{2_A}|}|a_{12} - r^+_{1_A}| + \frac{r_3(B^+)}{|a_{33} - r^+_{3_A}|}|a_{13} - r^+_{1_A}|,$$

$$a_{22} - r^+_{2_A} = 3 > 2 = p_2(B^+) = |a_{21} - r^+_{2_A}| + \frac{r_3(B^+)}{|a_{33} - r^+_{3_A}|}|a_{23} - r^+_{2_A}|,$$

and

$$a_{33} - r^+_{3_A} = 100 > 1 = p_3(B^+) = |a_{31} - r^+_{3_A}| + \frac{r_2(B^+)}{|a_{22} - r^+_{2_A}|}|a_{32} - r^+_{3_A}|.$$

From Definition 2.3, we get that A is a B_1-matrix, and B^+ is also a B_1-matrix since $b_{11}^+ - r_{1_{B^+}}^+ = 1 > 0.5100 = p_1(V^+)$, $b_{22}^+ - r_{2_{B^+}}^+ = 3 > 2 = p_2(V^+)$ and $b_{33}^+ - r_{3_{B^+}}^+ = 100 > 1 = p_3(V^+)$, where $p_i(V^+) = p_i(B^+)$ since $r_{i_{B^+}}^+ = 0$. Consequently, Theorem 2.3 is shown to be valid by the example provided in Example 2.3.

Note that when A is a Z-matrix, we have that $r_{i_A}^+ = 0$ and $B^+ = A$, and therefore, it is easy to obtain Theorem 2.4.

Theorem 2.4. *If $A = (a_{ij}) \in C^{n\times n}$ is a Z-matrix with positive diagonal entries, then A is a B_1-matrix if and only if A is an SDD_1 matrix.*

Example 2.4. *Let us consider the Z-matrix with positive diagonal entries*

$$A = \begin{bmatrix} 1 & -3 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We write $A = B^+ + C$, where

$$B^+ = \begin{bmatrix} 1 & -3 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and

$$C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

That is, $A = B^+$. By calculation, we have that

$$N_1(B^+) = \{1\}, N_2(B^+) = \{2, 3\},$$

$$a_{11} - r_{1_A}^+ = 1 > 0 = p_1(B^+),$$

$$a_{22} - r_{2_A}^+ = 1 > 0 = p_2(B^+),$$

and

$$a_{33} - r_{3_A}^+ = 1 > 0 = p_3(B^+).$$

From Definition 2.3, we get that A is a B_1-matrix, and A is also an SDD_1 matrix since $|a_{11}| = 1 > 0 = p_1(A)$. Therefore, Example 2.4 illustrates that Theorem 2.4 is valid.

Next, some properties between B_1-matrix and nonnegative diagonal matrix, P-matrix are proposed.

Theorem 2.5. *If $A = (a_{ij}) \in C^{n\times n}$ is a B_1-matrix, and D is a nonnegative diagonal matrix of the same order, then $A + D$ is a B_1-matrix.*

Proof. Let $D = diag(d_1, d_2, \ldots, d_n)$ with $d_i \geq 0$, and $C = A + D$ with $C = (c_{ij}) \in C^{n\times n}$, where

$$c_{ij} = \begin{cases} a_{ii} + d_i, & i = j, \\ a_{ij}, & i \neq j. \end{cases}$$

Next, let us prove $c_{ii} - r^+_{i_C} > p_i(V^+)$ for all $i \in N$, where $V^+ = (v_{ij}) \in C^{n\times n}$ with $v_{ij} = c_{ij} - r^+_{i_C}$, and $r^+_{i_C} = max\{0, c_{ij} | i \neq j\} = r^+_{i_A}$.

Since A is a B_1-matrix, for all $i \in N$,

$$c_{ii} - r^+_{i_C} = a_{ii} + d_i - r^+_{i_A} \geq a_{ii} - r^+_{i_A} > p_i(B^+). \tag{2.3}$$

For $i \in N_1(V^+) \subseteq N_1(B^+)$,

$$\begin{aligned} p_i(V^+) &= \sum_{j\in N_1(V^+)\backslash\{i\}} |v_{ij}| + \sum_{j\notin N_1(V^+)\cup\{i\}} \frac{r_j(V^+)}{|v_{jj}|}|v_{ij}| \\ &= \sum_{j\in N_1(V^+)\backslash\{i\}} |a_{ij} - r^+_{i_A}| + \sum_{j\notin N_1(V^+)\cup\{i\}} \frac{r_j(B^+)}{|a_{jj} - r^+_{j_A} + d_j|}|a_{ij} - r^+_{i_A}| \\ &\leq \sum_{j\in N_1(B^+)\backslash\{i\}} |a_{ij} - r^+_{i_A}| + \sum_{j\notin N_1(B^+)\cup\{i\}} \frac{r_j(B^+)}{|a_{jj} - r^+_{j_A}|}|a_{ij} - r^+_{i_A}| \\ &= p_i(B^+). \end{aligned}$$

By (2.3), we deduce that $c_{ii} - r^+_{i_C} > p_i(B^+) \geq p_i(V^+)$, and this proof is completed.

Example 2.5. *Let the matrix*

$$A = \begin{bmatrix} 75 & 74 & 23 \\ 10 & 15 & 12 \\ 9 & 10 & 110 \end{bmatrix},$$

and

$$D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 15 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

By calculation, we have that

$$N_1(B^+) = \{1\}, N_2(B^+) = \{2, 3\},$$

$$a_{11} - r^+_{1_A} = 1 > 0.5100 = p_1(B^+),$$

$$a_{22} - r^+_{2_A} = 3 > 2 = p_2(B^+),$$

and

$$a_{33} - r^+_{3_A} = 100 > 1 = p_3(B^+).$$

From Definition 2.3, we get that A is a B_1-matrix. Further, Theorem 2.5 demonstrates that A + D satisfies the conditions required for a B_1-matrix.

Theorem 2.6. *If $A = (a_{ij}) \in C^{n\times n}$ is a B_1-matrix, then we write A as $A = B + C$, where B is a Z-matrix with positive diagonal entries, and C is a nonnegative matrix of rank 1. In particular, if A is a B_1-matrix and Z-matrix, then C is a zero matrix.*

Proof. Let us define $B = (b_{ij}) \in C^{n\times n}$ with $b_{ij} = a_{ij} - r^+_{i_A}$. Taking into account that $b_{ij} \le 0$ from definition of $r^+_{i_A}$, and $b_{ii} > 0$ since A is a B_1-matrix, then B is a Z-matrix with positive diagonal entries.

Let $C = (c_{ij}) \in C^{n\times n}$ with $c_{ij} = r^+_{i_A}$, and obviously C is a nonnegative matrix of rank 1. Hence, A can be decomposed as $A = B + C$.

Theorem 2.7. *If $A = (a_{ij}) \in C^{n\times n}$ is a B_1-matrix, then B^+ is a P-matrix.*

Proof. Since A is a B_1-matrix, it is easy to obtain that B^+ is an H-matrix by Theorem 2.1 and Lemma 2.1. $a_{ii} - r^+_{i_A} > p_i(B^+) \ge 0$, and it is equivalent to $b^+_{ii} > 0$ for all $i \in N$. We conclude that B^+ is a P-matrix from Lemma 2.2.

Example 2.6. *Let the matrix*

$$A = \begin{bmatrix} 1 & 0.1 & 0.1 \\ 3 & 4 & 2 \\ 0.1 & 0.1 & 1 \end{bmatrix}.$$

We write $A = B^+ + C$, where

$$B^+ = \begin{bmatrix} 0.9 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0.9 \end{bmatrix},$$

and

$$C = \begin{bmatrix} 0.1 & 0.1 & 0.1 \\ 3 & 3 & 3 \\ 0.1 & 0.1 & 0.1 \end{bmatrix}.$$

By calculation, we have that

$$N_1(B^+) = \{2\}, N_2(B^+) = \{1, 3\},$$

$$a_{11} - r^+_{1_A} = 0.9000 > 0 = p_1(B^+) = |a_{12} - r^+_{1_A}| + \frac{r_3(B^+)}{|a_{33} - r^+_{3_A}|}|a_{13} - r^+_{1_A}|,$$

$$a_{22} - r^+_{2_A} = 1 > 0 = p_2(B^+) = \frac{r_1(B^+)}{|a_{11} - r^+_{1_A}|}|a_{21} - r^+_{2_A}| + \frac{r_3(B^+)}{|a_{33} - r^+_{3_A}|}|a_{23} - r^+_{2_A}|,$$

and

$$a_{33} - r^+_{3_A} = 0.9000 > 0 = p_3(B^+) = |a_{32} - r^+_{3_A}| + \frac{r_1(B^+)}{|a_{11} - r^+_{1_A}|}|a_{31} - r^+_{3_A}|.$$

From Definition 2.3, we get that A is a B_1-matrix, and by Definition 2.1, one can easily verify that B^+ is a P-matrix. Therefore, Example 2.6 illustrates that Theorem 2.7 is valid.

Theorem 2.8. *If $A = (a_{ij}) \in C^{n\times n}$ is a B_1-matrix, and C is a nonnegative matrix of the form*

$$C = \begin{bmatrix} c_1 & c_1 & \cdots & c_1 \\ c_2 & c_2 & \cdots & c_2 \\ \vdots & \vdots & \cdots & \vdots \\ c_n & c_n & \cdots & c_n \end{bmatrix},$$

where $C = (c_{ij}) \in C^{n\times n}$ with $c_{ij} = r^+_{i_A}$, then $A + C$ is a B_1-matrix.

Proof. Note that for each $i \in N$, $r^+_{i_{A+C}} = r^+_{i_A} + c_i$, and moreover, $(A + C)^+ = B^+$. Since A is a B_1-matrix, from Theorem 2.3, we have that B^+ is a B_1-matrix, and then $(A + C)^+$ is also a B_1-matrix. Therefore, $A + C$ is a B_1-matrix.

3. Infinity norm upper bound for the inverse of B_1-matrix

In this section, an infinity norm upper bound for the inverse of B_1-matrices is obtained. Before that, some lemmas and theorems are listed.

Lemma 3.1. *[10] If $P = (p_1, \ldots, p_n)^T e$, where $e = (1, \ldots, 1)$ and $p_1, \ldots, p_n \geq 0$, then*

$$\|(I + P)^{-1}\|_\infty \leq n - 1, \tag{3.1}$$

where I is the $n \times n$ identity matrix.

Theorem 3.1. *[16] Let matrix $A = (a_{ij}) \in C^{n\times n}$ be an SDD_1 matrix, and then*

$$\|A^{-1}\|_\infty \leq \frac{\max\{1, \max_{i\in N_2(A)} \frac{p_i(A)}{|a_{ii}|} + \varepsilon\}}{\min\{\min_{i\in N_1(A)} H_i, \min_{i\in N_2(A)} Q_i\}}, \tag{3.2}$$

where
$H_i = |a_{ii}| - \sum\limits_{j\in N_1(A)\backslash\{i\}} |a_{ij}| - \sum\limits_{j\in N_2(A)\backslash\{i\}} (\frac{p_j(A)}{|a_{jj}|} + \varepsilon)|a_{ij}|, \qquad i \in N_1(A),$
$Q_i = \varepsilon(|a_{ii}| - \sum\limits_{j\in N_2(A)\backslash\{i\}} |a_{ij}|) + \sum\limits_{j\in N_2(A)\backslash\{i\}} \frac{r_j(A)-p_j(A)}{|a_{jj}|}|a_{ij}|, \qquad i \in N_2(A),$
and ε satisfies $0 < \varepsilon < \min_{i\in N} \frac{|a_{ii}|-p_i(A)}{\sum_{j\in N_2(A)\backslash\{i\}} |a_{ij}|}$.

Theorem 3.2. *Let $A = (a_{ij}) \in C^{n\times n}$ be a B_1-matrix, and then*

$$\|A^{-1}\|_\infty \leq (n-1)\frac{\max\{1, \max_{i\in N_2(B^+)}(\frac{p_i(B^+)}{|a_{ii}-r^+_{i_A}|} + \varepsilon)\}}{\min\{\min_{i\in N_1(B^+)} T_i, \min_{i\in N_2(B^+)} M_i\}}, \tag{3.3}$$

where $B^+ = (b^+_{ij})_{1\leq i,j\leq n}$ with $b^+_{ij} = a_{ij} - r^+_{i_A}$,
$T_i = |a_{ii} - r^+_{i_A}| - \sum\limits_{j\in N_1(B^+)\backslash\{i\}} |a_{ij} - r^+_{i_A}| - \sum\limits_{j\in N_2(B^+)\backslash\{i\}} (\frac{p_j(B^+)}{|a_{jj}-r^+_{j_A}|} + \varepsilon)|a_{ij} - r^+_{i_A}|, \qquad i \in N_1(B^+),$
$M_i = \varepsilon(|a_{ii} - r^+_{i_A}| - \sum\limits_{j\in N_2(B^+)\backslash\{i\}} |a_{ij} - r^+_{i_A}|) + \sum\limits_{j\in N_2(B^+)\backslash\{i\}} \frac{r_j(B^+)-p_j(B^+)}{|a_{jj}-r^+_{j_A}|}|a_{ij} - r^+_{i_A}|, \qquad i \in N_2(B^+),$
and ε satisfies $0 < \varepsilon < \min_{i\in N} \frac{|a_{ii}-r^+_{i_A}|-p_i(B^+)}{\sum_{j\in N_2(A)\backslash\{i\}} |a_{ij}-r^+_{i_A}|}$.

Proof. Since A is a B_1-matrix, let $A = B^+ + C$, with $B^+ = (b_{ij}^+) \in C^{n\times n}$ and $b_{ij}^+ = a_{ij} - r_{i_A}^+$, $C = (c_{ij}) \in C^{n\times n}$ with $c_{ij} = r_{i_A}^+$. From Theorem 2.1 and Lemma 2.1, B^+ is an H-matrix, and it is also a nonsingular M-matrix. Then, B^+ has nonnegative inverse. According to $A = B^+ + C = B^+(I + (B^+)^{-1}C)$, it holds that $\|A^{-1}\|_\infty \le \|(I + (B^+)^{-1}C)^{-1}\|_\infty \|(B^+)^{-1}\|_\infty$. Observe that the matrix C is nonnegative, and $(B^+)^{-1} \ge 0$. Then, $(B^+)^{-1}C$ can be written as $(p_1, p_2, \dots, p_n)^T e$, where $p_i \ge 0$ and $e = (1, 1, \dots, 1)$, for $i = 1, 2, \dots, n$, and by Lemma 3.1,

$$\|(I + (B^+)^{-1}C)^{-1}\|_\infty \le n - 1. \tag{3.4}$$

However, B^+ is an SDD_1 matrix, by Theorem 3.1,

$$\|(B^+)^{-1}\|_\infty \le \frac{\max\{1, \max_{i\in N_2(B^+)} \frac{p_i(B^+)}{|a_{ii}-r_{i_A}^+|} + \varepsilon\}}{\min\{\min_{i\in N_1(B^+)} T_i, \min_{i\in N_2(B^+)} M_i\}}, \tag{3.5}$$

where
$T_i = |a_{ii} - r_{i_A}^+| - \sum\limits_{j\in N_1(B^+)\backslash\{i\}} |a_{ij} - r_{i_A}^+| - \sum\limits_{j\in N_2(B^+)\backslash\{i\}} (\frac{p_j(B^+)}{|a_{jj}-r_{j_A}^+|} + \varepsilon)|a_{ij} - r_{i_A}^+|, \qquad i \in N_1(B^+),$
$M_i = \varepsilon(|a_{ii} - r_{i_A}^+| - \sum\limits_{j\in N_2(B^+)\backslash\{i\}} |a_{ij} - r_{i_A}^+|) + \sum\limits_{j\in N_2(B^+)\backslash\{i\}} \frac{r_j(B^+)-p_j(B^+)}{|a_{jj}-r_{j_A}^+|}|a_{ij} - r_{i_A}^+|, \qquad i \in N_2(B^+),$
and ε satisfies $0 < \varepsilon < \min_{i\in N} \frac{|a_{ii}-r_{i_A}^+|-p_i(B^+)}{\sum\limits_{j\in N_2(A)\backslash\{i\}} |a_{ij}-r_{i_A}^+|}$.

By (3.4) and (3.5), we get (3.3).

4. Error bound for the linear complementarity problem of B_1-matrices

In this section, before an error bound for the linear complementarity problem corresponding to B_1-matrices is proposed, some lemmas are listed.

Lemma 4.1. *[11] Let $\gamma > 0$ and $\eta \ge 0$, and then for any $x \in [0, 1]$,*

$$\frac{1}{1 - x + x\gamma} \le \frac{1}{\min\{\gamma, 1\}}, \qquad \frac{\eta x}{1 - x + x\gamma} \le \frac{\eta}{\gamma}.$$

Lemma 4.2. *[12] Let $A = (a_{ij}) \in C^{n\times n}$ be an SDD_1 matrix with positive diagonal entries, and then $A_D = I - D + DA$ is also an SDD_1 matrix, where $D = diag(d_i)$ with $0 \le d_i \le 1$.*

Theorem 4.1. *Let $A = (a_{ij}) \in C^{n\times n} (n \ge 2)$ be an SDD_1 matrix with positive diagonal entries, and $A_D = I - D + DA$ is also an SDD_1 matrix, where $D = diag(d_i)$ with $0 \le d_i \le 1$. Then,*

$$\|A_D^{-1}\|_\infty \le \frac{\max\{1, \max_{i\in N_2(A_D)} \frac{p_i(A)}{|a_{ii}|} + \varepsilon\}}{\min\{\min_{i\in N_1(A_D)} L_i, \min_{i\in N_2(A_D)} G_i\}}, \tag{4.1}$$

where
$L_i = |d_i a_{ii}| - \sum\limits_{j\in N_1(A_D)\backslash\{i\}} |d_i a_{ij}| - \sum\limits_{j\in N_2(A_D)\backslash\{i\}} (\frac{p_j(A)}{|a_{jj}|} + \varepsilon)|d_i a_{ij}|, \qquad i \in N_1(A_D),$
$G_i = \varepsilon(|d_i a_{ii}| - \sum\limits_{j\in N_2(A_D)\backslash\{i\}} |d_i a_{ij}|) + \sum\limits_{j\in N_2(A_D)\backslash\{i\}} \frac{d_j r_j(A)}{|1-d_j+d_j a_{jj}|}|d_i a_{ij}| - \sum\limits_{j\in N_2(A_D)\backslash\{i\}} \frac{p_j(A)}{|a_{jj}|}|d_i a_{ij}|, \qquad i \in N_2(A_D),$
and ε satisfies $0 < \varepsilon < \min_{i\in N} \frac{|1-d_i+d_i a_{ii}|-p_i(A_D)}{\sum\limits_{j\in N_2(A_D)\backslash\{i\}} |d_i a_{ij}|}$.

Proof. Let $A_D = I - D + DA$ be an SDD_1 matrix, where

$$A_D = \begin{cases} 1 - d_i + d_i a_{ii}, & i = j, \\ d_i a_{ij}, & i \neq j. \end{cases}$$

By Theorem 3.1,

$$\|A_D^{-1}\|_\infty \leq \frac{\max\{1, \max_{i\in N_2(A_D)} \frac{p_j(A_D)}{|1-d_j+d_j a_{jj}|} + \varepsilon\}}{\min\{\min_{i\in N_1(A_D)} H_i, \min_{i\in N_2(A_D)} Q_i\}}, \tag{4.2}$$

where
$H_i = |1 - d_i + d_i a_{ii}| - \sum\limits_{j\in N_1(A_D)\setminus\{i\}} |d_i a_{ij}| - \sum\limits_{j\in N_2(A_D)\setminus\{i\}} (\frac{p_j(A_D)}{|1-d_j+d_j a_{jj}|} + \varepsilon)|d_i a_{ij}|, \qquad i \in N_1(A_D),$
$Q_i = \varepsilon(|1 - d_i + d_i a_{ii}| - \sum\limits_{j\in N_2(A_D)\setminus\{i\}} |d_i a_{ij}|) + \sum\limits_{j\in N_2(A_D)\setminus\{i\}} \frac{r_j(A_D)-p_j(A_D)}{|1-d_j+d_j a_{jj}|}|d_i a_{ij}|, \qquad i \in N_2(A_D),$
and ε satisfies $0 < \varepsilon < \min_{i\in N} \frac{|1-d_i+d_i a_{ii}|-p_i(A_D)}{\sum\limits_{j\in N_2(A_D)\setminus\{i\}} |d_i a_{ij}|}$.

For $i \in N_1(A_D)$ and from $0 \leq d_i \leq 1$, we obtain that

$$|d_i a_{ii}| \leq |1 - d_i + d_i a_{ii}| \leq r_i(A_D) = d_i r_i(A),$$

which means that $i \in N_1(A)$, that is, $N_1(A_D) \subseteq N_1(A)$. Then,

$$\begin{aligned} p_i(A_D) &= \sum_{j\in N_1(A_D)\setminus\{i\}} |d_i a_{ij}| + \sum_{j\notin N_1(A_D)\bigcup\{i\}} \frac{d_j r_j(A)}{|1 - d_j + d_j a_{jj}|}|d_i a_{ij}| \\ &= d_i(\sum_{j\in N_1(A_D)\setminus\{i\}} |a_{ij}| + \sum_{j\notin N_1(A_D)\bigcup\{i\}} \frac{d_j r_j(A)}{|1 - d_j + d_j a_{jj}|}|a_{ij}|) \\ &\leq d_i(\sum_{j\in N_1(A)\setminus\{i\}} |a_{ij}| + \sum_{j\notin N_1(A)\bigcup\{i\}} \frac{r_j(A)}{|a_{jj}|}|a_{ij}|) \\ &= d_i p_i(A). \end{aligned}$$

By Lemma 4.1,

$$\frac{p_i(A_D)}{|1 - d_i + d_i a_{ii}|} \leq \frac{d_i p_i(A)}{1 - d_i + d_i a_{ii}} \leq \frac{p_i(A)}{|a_{ii}|}, \tag{4.3}$$

$$\begin{aligned} \frac{1}{H_i} &= \frac{1}{|1 - d_i + d_i a_{ii}| - \sum\limits_{j\in N_1(A_D)\setminus\{i\}} |d_i a_{ij}| - \sum\limits_{j\in N_2(A_D)\setminus\{i\}} (\frac{p_j(A_D)}{|1-d_j+d_j a_{jj}|} + \varepsilon)|d_i a_{ij}|} \\ &\leq \frac{1}{|d_i a_{ii}| - \sum\limits_{j\in N_1(A_D)\setminus\{i\}} |d_i a_{ij}| - \sum\limits_{j\in N_2(A_D)\setminus\{i\}} (\frac{p_j(A)}{|a_{jj}|} + \varepsilon)|d_i a_{ij}|} \\ &= \frac{1}{L_i} \leq \frac{1}{\min L_i}, \qquad i \in N_1(A_D), \end{aligned} \tag{4.4}$$

$$\begin{aligned}\frac{1}{Q_i} &= \frac{1}{\varepsilon(|1-d_i+d_ia_{ii}| - \sum\limits_{j\in N_2(A_D)\setminus\{i\}} |d_ia_{ij}|) + \sum\limits_{j\in N_2(A_D)\setminus\{i\}} \frac{r_j(A_D)-p_j(A_D)}{|1-d_j+d_ja_{jj}|}|d_ia_{ij}|} \\ &\le \frac{1}{\varepsilon[|d_ia_{ii}| - \sum\limits_{j\in N_2(A_D)\setminus\{i\}} |d_ia_{ij}|] + \sum\limits_{j\in N_2(A_D)\setminus\{i\}} \frac{d_jr_j(A)}{|1-d_j+d_ja_{jj}|}|d_ia_{ij}| - \sum\limits_{j\in N_2(A_D)\setminus\{i\}} \frac{p_j(A)}{|a_{jj}|}|d_ia_{ij}|} \\ &= \frac{1}{G_i} \le \frac{1}{\min G_i}, \qquad i\in N_2(A_D).\end{aligned} \tag{4.5}$$

Then, by (4.2)–(4.5), we get (4.1).

Example 4.1. *Let*

$$A = \begin{bmatrix} 16 & -8 & 4.1 & 8 \\ 0 & 8 & 3.1 & 1 \\ -8 & 8 & 20 & 8 \\ 1 & 1.2 & 5 & 8 \end{bmatrix},$$

and $D = diag(d_i)$ *with* $d_i = 0.9000$. *Then, we have*

$$A_D = I - D + DA = \begin{bmatrix} 14.5 & -7.2 & 3.69 & 7.2 \\ 0 & 7.3 & 2.79 & 0.9 \\ -7.2 & 7.2 & 18.1 & 7.2 \\ 0.9 & 1.08 & 4.5 & 7.3 \end{bmatrix}.$$

By calculation, $N_1(A_D) = \{1,3\}$, $N_2(A_D) = \{2,4\}$, $p_2(A) = 4$, $p_4(A) = 6.6150$ *and* $0 < \varepsilon < 0.0541$. *We choose* $\varepsilon = 0.0540$. *Then,* $L_1 = 0.3789$, $L_3 = 0.4689$, $G_2 = 0.3949$ *and* $G_4 = 0.3364$. *Hence,* $\|A_D^{-1}\|_\infty \le 2.9727$, *and the true value is* $\|A_D^{-1}\|_\infty = 0.3188$.

Next, an error bound for the linear complementarity problem corresponding to B_1-matrices is proposed.

Theorem 4.2. *Let* $A = (a_{ij}) \in C^{n\times n} (n \ge 2)$ *be a* B_1*-matrix satisfying the hypotheses of Theorem 4.1. Then,*

$$\max_{d\in[0,1]^n} \|(I-D+DA)^{-1}\|_\infty \le \max_{d\in[0,1]^n} \frac{(n-1)\max\{1, \max_{i\in N_2(B_D^+)}(\frac{p_i(B^+)}{|b_{ii}^+|}+\varepsilon)\}}{\min\{\min_{i\in N_1(B_D^+)} F_i, \min_{i\in N_2(B_D^+)} Z_i\}},$$

where

$B^+ = (b_{ij}^+) \in C^{n\times n}$ *with* $b_{ij}^+ = a_{ij} - r_{i_A}^+$,

$F_i = |d_ib_{ii}^+| - \sum\limits_{j\in N_1(B_D^+)\setminus\{i\}} |d_ib_{ij}^+| - \sum\limits_{j\in N_2(B_D^+)\setminus\{i\}} (\frac{p_j(B^+)}{|b_{jj}^+|} + \varepsilon)|d_ib_{ij}^+|, \qquad i\in N_1(B_D^+),$

$Z_i = \varepsilon(|d_ib_{ii}^+| - \sum\limits_{j\in N_2(B_D^+)\setminus\{i\}} |d_ib_{ij}^+|) + \sum\limits_{j\in N_2(B_D^+)\setminus\{i\}} \frac{d_jr_j(B^+)}{|1-d_j+d_jb_{jj}^+|}|d_ib_{ij}^+| - \sum\limits_{j\in N_2(B_D^+)\setminus\{i\}} \frac{p_j(B^+)}{|b_{jj}^+|}|d_ib_{ij}^+|, \qquad i\in N_2(B_D^+),$

and ε *satisfies* $0 < \varepsilon < \min_{i\in N} \frac{|1-d_i+d_ib_{ii}^+|-p_i(B_D^+)}{\sum\limits_{j\in N_2(B_D^+)\setminus\{i\}} |d_ib_{ij}^+|}$.

Proof. Let $A = B^+ + C$, where $B^+ = (b_{ij}^+) \in C^{n\times n}$ with $b_{ij}^+ = a_{ij} - r_{i_A}^+$, $C = (c_{ij}) \in C^{n\times n}$ with $c_{ij} = r_{i_A}^+$. B^+ is an SDD_1 matrix with positive diagonal entries. Thus for each diagonal matrix $D = diag(d_i)$ with $0 \le d_i \le 1$,

$$A_D = I - D + DA = (I - D + DB^+) + DC = B_D^+ + C_D,$$

where $B_D^+ = I - D + DB^+$ and $C_D = DC$. Similar to the proof of Theorem 3.2,

$$\|A_D^{-1}\|_\infty \le \|[I + (B_D^+)^{-1}C_D]\|_\infty \|(B_D^+)^{-1}\|_\infty \le (n-1)\|(B_D^+)^{-1}\|_\infty.$$

Notice that B^+ is an SDD_1 matrix, and by Lemma 4.2, $B_D^+ = I - D + DB^+$ is also an SDD_1 matrix. Hence, by (4.1), it holds that

$$\|(B_D^+)^{-1}\|_\infty \le \frac{\max\{1, \max_{i\in N_2(B_D^+)}(\frac{p_i(B^+)}{|b_{ii}^+|} + \varepsilon)\}}{\min\{\min_{i\in N_1(B_D^+)} F_i, \min_{i\in N_2(B_D^+)} Z_i\}},$$

where

$$F_i = |d_i b_{ii}^+| - \sum_{j\in N_1(B_D^+)\backslash\{i\}} |d_i b_{ij}^+| - \sum_{j\in N_2(B_D^+)\backslash\{i\}} (\frac{p_j(B^+)}{|b_{jj}^+|} + \varepsilon)|d_i b_{ij}^+|, \qquad i \in N_1(B_D^+),$$

$$Z_i = \varepsilon(|d_i b_{ii}^+| - \sum_{j\in N_2(B_D^+)\backslash\{i\}} |d_i b_{ij}^+|) + \sum_{j\in N_2(B_D^+)\backslash\{i\}} \frac{d_j r_j(B^+)}{|1-d_j+d_j b_{jj}^+|}|d_i b_{ij}^+| - \sum_{j\in N_2(B_D^+)\backslash\{i\}} \frac{p_j(B^+)}{|b_{jj}^+|}|d_i b_{ij}^+|, \qquad i \in N_2(B_D^+),$$

and ε satisfies $0 < \varepsilon < \min_{i\in N} \frac{|1-d_i+d_i b_{ii}^+| - p_i(B_D^+)}{\sum_{j\in N_2(B_D^+)\backslash\{i\}} |d_i b_{ij}^+|}$.

Example 4.2. *Let the matrix*

$$A = \begin{bmatrix} 8 & -2 & -1 & -1 \\ 4 & 13 & 4 & 5 \\ -8 & -8 & 15 & -8 \\ -4 & -4 & -2 & 6 \end{bmatrix},$$

and

$$B^+ = \begin{bmatrix} 8 & -2 & -1 & -1 \\ -1 & 8 & -1 & 0 \\ -8 & -8 & 15 & -8 \\ -4 & -4 & -2 & 6 \end{bmatrix},$$

where we set $D = diag(d_i)$ with $d_i = 0.7000$. Then,

$$B_D^+ = I - D + DB^+ = \begin{bmatrix} 5.9 & -1.4 & -0.7 & -0.7 \\ -0.7 & 5.9 & -0.7 & 0 \\ -5.6 & -5.6 & 10.8 & -5.6 \\ -2.8 & -2.8 & -1.4 & 4.5 \end{bmatrix}.$$

By the definitions of B-matrix and B_1-matrix, it is easy to get that A is not a B-matrix but is a B_1-matrix. Therefore, the existing bounds (such as the bound (13) in Theorem 4 [10]) cannot be used to compute the error bound for the linear complementarity problem for matrix A. However, the error *bound for the linear complementarity problem for matrix A can be computed by Theorem 4.2.*

By simple calculation, $N_1(B_D^+) = \{3,4\}$, $N_2(B_D^+) = \{1,2\}$, $p_1(B^+) = 2.5000$, $p_2(B^+) = 1.5000$ *and* $0 < \varepsilon < 0.1084$. *Let* $\varepsilon = 0.1083$. *Then, from our bound in Theorem 4.2, the error bound for the linear complementarity problem for matrix A is given as* $\max_{d\in[0,1]^n} \|(I - D + DA)^{-1}\|_\infty \leq 5.7186$*, and the true value is* $\|(I - D + DA)^{-1}\|_\infty = 0.3359$.

Example 4.3. *Consider the matrix*

$$A = \begin{bmatrix} 0.5 & -0.24 & -0.22 \\ -0.05 & 0.2 & 0.01 \\ 0.01 & -0.06 & 0.2 \end{bmatrix},$$

and we write $A = B^+ + C$*, where*

$$B^+ = \begin{bmatrix} 0.5 & -0.24 & -0.22 \\ -0.06 & 0.19 & 0 \\ 0 & -0.07 & 0.19 \end{bmatrix}.$$

It is easy to verify that A is a B-matrix. Then, it is also a B_1*-matrix [14]. By the bound (13) in Theorem 4 [10], we have*

$$\max_{d\in[0,1]^n} \|(I - D + DA)^{-1}\|_\infty \leq 50.$$

By simple calculation, we have that

$$B_D^+ = I - D + DB^+ = \begin{bmatrix} 0.5005 & -0.2398 & -0.2198 \\ -0.0599 & 0.1908 & 0 \\ 0 & -0.0699 & 0.1908 \end{bmatrix},$$

and $p_1(B^+) = 0.1568$, $p_2(B^+) = 0.0552$, $p_3(B^+) = 0.0221$ *and* $0 < \varepsilon < 0.8154$. *Let* $\varepsilon = 0.8153$, *and then from our bound in Theorem 4.2, we get that* $\max_{d\in[0,1]^n} \|(I - D + DA)^{-1}\|_\infty \leq 24.2275 < 50$. *Therefore, Example 4.3 shows that the error bound of a* B_1*-matrix is sharper than the error bound of a B-matrix under some cases.*

5. Conclusions

In this paper, some properties for B_1-matrices and the infinity norm upper bound for the inverse of B_1-matrices are presented. Based on these results, the error bound for the linear complementarity problem of B_1-matrices is obtained. Moreover, numerical examples are also presented to illustrate the corresponding results.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was partly supported by the National Natural Science Foundations of China (31600299), Natural Science Basic Research Program of Shaanxi, China (2020JM-622).

Conflict of interest

The authors declare there is no conflict of interest.

References

1. B. C. Eaves, The Linear complementarity problem, *Manage. Sci.*, **17** (1971), 507–660. https://doi.org/10.1287/mnsc.17.9.612
2. H. B. Li, T. Z. Huang, H. Li, On some subclasses of P-matrices, *Numer. Linear Algebra Appl.*, **14** (2007), 391–405. https://doi.org/10.1002/nla.524
3. X. J. Chen, S. H. Xiang, Computation of error bounds for P-matrix linear complementarity problems, *Math. Program.*, **106** (2006), 513–525. https://doi.org/10.1007/s10107-005-0645-9
4. R. Mathias, J. S. Pang, Error bounds for the linear complementarity problem with a P-matrix, *Linear Algera Appl.*, **132** (1990), 123–136. https://doi.org/10.1016/0024-3795(90)90058-K
5. C. M. Araújo, J. R. Torregrosa, Some results on B-matrices and doubly B-matrices, *Linear Algebra Appl.*, **459** (2014), 101–120. https://doi.org/10.1016/j.laa.2014.06.048
6. F. Wang, D. S. Sun, New error bound for linear complementarity problems for B-matrices, *Linear Multilinear Algebra*, **66** (2018), 2156–2167. https://doi.org/10.1080/03081087.2017.1389847
7. P. F. Dai, C. J. Lu, Y. T. Li, New error bounds for the linear complementarity problem with an SB-matrix, *Numerical Algorithms*, **64** (2013), 741–757. https://doi.org/10.1007/s11075-012-9691-6
8. P. F. Dai, Error bounds for linear complementarity problems of DB-matrices, *Linear Algebra Appl.*, **434** (2011), 830–840. https://doi.org/10.1016/j.laa.2010.09.049
9. Z. Z. Bai, On the convergence of the multisplitting methods for the linear complementarity problem, *SIAM J. Matrix Anal. Appl.*, **21** (1999). https://doi.org/10.1137/S0895479897324032
10. M. Esnaola, J. Peña, $B-Nekrasov$ matrices and error bounds for linear complementarity problems, *Numerical Algorithms*, **72** (2016), 435–445. https://doi.org/10.1007/s11075-015-0054-y
11. L. Gao, An alternative error bound for linear complementarily problems involving B^S-matrices, *J. Inequal. Appl.*, **2018** (2018). https://doi.org/10.1186/s13660-018-1618-x
12. Y. X. Zhao, L. L. Liu, F. Wang, Error bounds for linear complementarity problems of SDD_1 matrices and $SDD_1 - B$ matrices, *AIMS Math.*, **7** (2022), 11862–11878.
13. C. Q. Li, Schur complement-based infinity norm bounds for the inverse of SDD matrices, *Bull. Malays. Math. Sci. Soc.*, **43** (2020), 3829–3845. https://doi.org/10.1007/s40840-020-00895-x
14. J. M. Peña, Diagonal dominance, Schur complements and some classes of H-matrices and P-matrices, *Adv. Comput. Math.*, **32** (2011), 357–373. https://doi.org/10.1007/s10444-010-9160-5

15. R. Bru, C. Corral, I. Gimenez, J. Mas, Classes of general H-matrices, *Linear Algebra Appl.*, **429** (2008), 2358–2366. https://doi.org/10.1016/j.laa.2007.10.030

16. X. Y. Chen, Y. T. Li, L. Liu, Y. Q. Wang, Infinity norm upper bounds for the inverse of SDD_1 matrices, *AIMS Math.*, **2022** (2022), 8847–8860.

AIMS Press