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Research article

Two-grid methods of finite element approximation for parabolic integro-differential optimal control problems

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Abstract: In this paper, we present a two-grid scheme of fully discrete finite element approximation for optimal control problems governed by parabolic integro-differential equations. The state and co-state variables are approximated by a piecewise linear function and the control variable is discretized by a piecewise constant function. First, we derive the optimal a priori error estimates for all variables. Second, we prove the global superconvergence by using the recovery techniques. Third, we construct a two-grid algorithm and discuss its convergence. In the proposed two-grid scheme, the solution of the parabolic optimal control problem on a fine grid is reduced to the solution of the parabolic optimal control problem on a much coarser grid; additionally, the solution of a linear algebraic system on the fine grid and the resulting solution maintain an asymptotically optimal accuracy. Finally, we present a numerical example to verify the theoretical results.

Keywords: parabolic integro-differential equations; finite element methods; a priori error estimates; two-grid; superconvergence

1. Introduction

It is well known that optimal control problems play a very important role in the fields of science and engineering. In the operation of physical and economic processes, optimal control problems have a variety of applications. Therefore, highly effective numerical methods are key to the successful application of the optimal control problem in practice. The finite element method is an important method for solving optimal control problems and has been extensively studied in the literature. Many researchers have made various contributions on this topic. A systematic introduction to the finite element method for partial differential equations (PDEs) and optimal control problems can be found in [1, 2]. For example, a priori error estimates of finite element approximation were established for the optimal control problems governed by linear elliptic and parabolic state equations, see [3, 4]. Using adaptive finite element method to obtain posterior error estimation; see [5, 6]. Furthermore, some superconvergence results have been established by applying recovery techniques, see [7, 8].

The two-grid method based on two finite element spaces on one coarse and one fine grid was first proposed by Xu [9–11]. It is combined with other numerical methods to solve many partial differential equations, e.g., nonlinear elliptic problems [12], nonlinear parabolic equations [13], eigenvalue problems [14–16] and fractional differential equations [17].

Many real applications, such as heat conduction control of storage materials, population dynamics control and wave control problems governed by integro-differential equations, need to consider optimal control problems governed by elliptic integral equations and parabolic integro-differential equations. More and more experts and scholars began to pay attention to the numerical simulation of these optimal control problems. In [18], the authors analyzed the finite element method for optimal control problems governed by integral equations and integro-differential equations. In [19], the authors considered the error estimates of expanded mixed methods for optimal control problems governed by hyperbolic integro-differential equations. As far as we know, there is no research on a two-grid finite element method for parabolic integro-differential control problems in the existing literature.

In this paper, we design a two-grid scheme of fully discrete finite element approximation for optimal control problems governed by parabolic integro-differential equations. It is shown that when the coarse and fine mesh sizes satisfy $h = H^2$, the two-grid method achieves the same convergence property as the finite element method. We are interested in the following optimal control problems:

$$\min_{u\in K\subset U}\left\{\frac{1}{2}\int_0^T \|y-y_d\|^2+\|u\|^2dt\right\}, \tag{1.1}$$

$$y_t - div(A\nabla y) + \int_0^t div(B(t,s)\nabla y(s))ds = f + u,\ \forall\ x\in\Omega,\ t\in J, \tag{1.2}$$

$$y(x,t) = 0,\ \forall\ x\in\partial\Omega,\ t\in J, \tag{1.3}$$

$$y(x,0) = y_0(x),\ \forall\ x\in\Omega, \tag{1.4}$$

where Ω is a bounded domain in $\mathbb{R}^2$ and $J=(0,T]$. Let K be a closed convex set in $U = L^2(J;L^2(\Omega))$, $f\in L^2(J;L^2(\Omega))$, $y_d\in H^1(J;L^2(\Omega))$ and $y_0\in H^1(\Omega)$. K is a set defined by

$$K = \left\{u\in U : \int_\Omega u(x,t)dx \geq 0\right\}; \tag{1.5}$$

$A = A(x) = (a_{ij}(x))$ is a symmetric matrix function with $a_{ij}(x)\in W^{1,\infty}(\Omega)$, which satisfies the ellipticity condition

$$a_*|\xi|^2 \leq \sum_{i,j=1}^{2} a_{i,j}(x)\xi_i\xi_j \leq a^*|\xi|^2,\ \forall(\xi,x)\in\mathbb{R}^2\times\bar{\Omega},\ 0<a_*<a^*.$$

Moreover, $B(t,s) = B(x,t,s)$ is also a 2×2 matrix; assume that there exists a positive constant M such that

$$\|B(t,s)\|_{0,\infty} + \|B_t(t,s)\|_{0,\infty} \leq M.$$

In this paper, we adopt the standard notation $W^{m,p}(\Omega)$ for Sobolev spaces on Ω with a norm $\|\cdot\|_{m,p}$ given by $\|v\|_{m,p}^p = \sum_{|\alpha|\le m} \|D^\alpha v\|_{L^p(\Omega)}^p$, as well as a semi-norm $|\cdot|_{m,p}$ given by $|v|_{m,p}^p = \sum_{|\alpha|=m} \|D^\alpha v\|_{L^p(\Omega)}^p$. We set $W_0^{m,p}(\Omega) = \{v \in W^{m,p}(\Omega) : v|_{\partial\Omega} = 0\}$. For $p = 2$, we denote $H^m(\Omega) = W^{m,2}(\Omega)$, $H_0^m(\Omega) = W_0^{m,2}(\Omega)$, and $\|\cdot\|_m = \|\cdot\|_{m,2}$, $\|\cdot\| = \|\cdot\|_{0,2}$.

We denote by $L^s(J; W^{m,p}(\Omega))$ the Banach space of all L^s integrable functions from J into $W^{m,p}(\Omega)$ with the norm $\|v\|_{L^s(J;W^{m,p}(\Omega))} = \left(\int_0^T \|v\|_{W^{m,p}(\Omega)}^s dt\right)^{\frac{1}{s}}$ for $s \in [1, \infty)$ and the standard modification for $s = \infty$. For simplicity of presentation, we denote $\|v\|_{L^s(J;W^{m,p}(\Omega))}$ by $\|v\|_{L^s(W^{m,p})}$. Similarly, one can define the spaces $H^1(J; W^{m,p}(\Omega))$ and $C^k(J; W^{m,p}(\Omega))$. In addition C denotes a general positive constant independent of h and Δt, where h is the spatial mesh size and Δt is a time step.

The outline of this paper is as follows. In Section 2, we first construct a fully discrete finite element approximation scheme for the optimal control problems (1.1)–(1.4) and give its equivalent optimality conditions. In Section 3, we derive a priori error estimates for all variables, and then analyze the global superconvergence by using the recovery techniques. In Section 4, we present a two-grid scheme and discuss its convergence. In Section 5, we present a numerical example to verify the validity of the two-grid method.

2. Fully discrete finite element scheme

In this section, we shall construct a fully discrete finite element approximation scheme for the control problems (1.1)–(1.4). For sake of simplicity, we take the state space $Q = L^2(J; V)$ and $V = H_0^1(\Omega)$.

We recast (1.1)–(1.4) in the following weak form: find $(y, u) \in Q \times K$ such that

$$\min_{u\in K\subset U} \left\{\frac{1}{2}\int_0^T \|y - y_d\|^2 + \|u\|^2 dt\right\}, \tag{2.1}$$

$$(y_t, v) + (A\nabla y, \nabla v) = \int_0^t (B(t, s)\nabla y(s), \nabla v)ds + (f + u, v),\ \forall\ v \in V,\ t \in J, \tag{2.2}$$

$$y(x, 0) = y_0(x),\ \forall\ x \in \Omega, \tag{2.3}$$

where $(\cdot, \cdot)$ is the inner product of $L^2(\Omega)$.

Since the objective functional is convex, it follows from [2] that the optimal control problems (2.1)–(2.3) have a unique solution (y, u), and that (y, u) is the solution of (2.1)–(2.3) if and only if there is a co-state $p \in Q$ such that (y, p, u) satisfies the following optimality conditions:

$$(y_t, v) + (A\nabla y, \nabla v) = \int_0^t (B(t, s)\nabla y(s), \nabla v)ds + (f + u, v),\ \forall\ v \in V,\ t \in J, \tag{2.4}$$

$$y(x, 0) = y_0(x),\ \forall\ x \in \Omega, \tag{2.5}$$

$$-(p_t, q) + (A\nabla p, \nabla q) = \int_t^T (B^*(s, t)\nabla p(s), \nabla q)ds + (y - y_d, q),\ \forall\ q \in V,\ t \in J, \tag{2.6}$$

$$p(x, T) = 0,\ \forall\ x \in \Omega, \tag{2.7}$$

$$(u + p, \tilde{u} - u) \ge 0,\ \forall\ \tilde{u} \in K,\ t \in J. \tag{2.8}$$

As in [20], the inequality (Eq 2.8) can be expressed as

$$u = \max\{0, \bar{p}\} - p, \tag{2.9}$$

where $\bar{p} = \frac{\int_\Omega p dx}{\int_\Omega dx}$ denotes the integral average on Ω of the function p.

Let T_h denote a regular triangulation of the polygonal domain Ω, h_τ denote the diameter of τ and $h = \max_{\tau \in T_h} h_\tau$. Let $V_h \subset V$ be defined by the following finite element space:

$$V_h = \{v_h \in C^0(\bar{\Omega}) \cap V, v_h|_\tau \in P_1(\tau),\ \forall\ \tau \in T_h\}. \tag{2.10}$$

And the approximated space of control is given by

$$U_h := \{\tilde{u}_h \in U : \forall\ \tau \in T_h,\ \tilde{u}_h|_\tau = \text{constant}\}. \tag{2.11}$$

Set $K_h = U_h \cap K$.

Before the fully discrete finite element scheme is given, we introduce some projection operators. First, we define the Ritz-Volterra projection [21] $R_h :\ V \to V_h$, which satisfies the following: for any $y, p \in V$

$$(A(\nabla(y - R_h y), \nabla v_h) - \int_0^t (B(t,s)\nabla(y - R_h y), \nabla v_h) ds = 0,\ \ \forall\ v_h \in V_h, \tag{2.12}$$

$$\left\| \frac{\partial^i (y - R_h y)}{\partial t^i} \right\| + h \left\| \nabla \frac{\partial^i (y - R_h y)}{\partial t^i} \right\| \leq Ch^2 \sum_{m=0}^{i} \left\| \frac{\partial^m y}{\partial t^m} \right\|_2,\ i = 0, 1. \tag{2.13}$$

$$(A(\nabla(p - R_h p), \nabla v_h) - \int_t^T (B^*(s,t)\nabla(p - R_h p), \nabla v_h) ds = 0,\ \ \forall\ v_h \in V_h, \tag{2.14}$$

$$\left\| \frac{\partial^i (p - R_h p)}{\partial t^i} \right\| + h \left\| \nabla \frac{\partial^i (p - R_h p)}{\partial t^i} \right\| \leq Ch^2 \sum_{m=0}^{i} \left\| \frac{\partial^m p}{\partial t^m} \right\|_2,\ i = 0, 1. \tag{2.15}$$

Next, we define the standard L^2-orthogonal projection [22] $Q_h :\ L^2(\Omega) \to U_h$, which satisfies the following: for any $\phi \in L^2(\Omega)$

$$(\phi - Q_h\phi, w_h) = 0,\ \ \forall\ w_h \in U_h, \tag{2.16}$$

$$\|\phi - Q_h\phi\|_{-s,2} \leq Ch^{1+s}\|u\|_1,\ \ s = 0, 1,\ \ \forall\ \phi \in H^1(\Omega), \tag{2.17}$$

At last, we define the element average operator [7] $\pi_h : L^2(\Omega) \to U_h$ by

$$\pi_h\psi|_\tau = \frac{\int_\tau \psi dx}{\int_\tau dx},\ \ \forall\ \psi \in L^2(\Omega),\ \ \tau \in T_h. \tag{2.18}$$

We have the approximation property

$$\|\psi - \pi_h\psi\|_{-s,r} \leq Ch^{1+s}\|\psi\|_{1,r},\ \ s = 0, 1,\ \ \forall\ \psi \in W^{1,r}(\Omega). \tag{2.19}$$

We now consider the fully discrete finite element approximation for the control problem. Let $\Delta t > 0$, $N = T/\Delta t \in \mathbb{Z}$ and $t_n = n\Delta t$, $n \in \mathbb{Z}$. Also, let

$$\psi^n = \psi^n(x) = \psi(x, t_n), \quad dt\psi^n = \frac{\psi^n - \psi^{n-1}}{\Delta t}, \quad \delta\psi^n = \psi^n - \psi^{n-1}.$$

Like in [23], we define for $1 \le s \le \infty$ and $s = \infty$, the discrete time dependent norms

$$|||\psi|||_{L^s(J;W^{m,p}(\Omega))} := \left(\sum_{n=1-l}^{N-l} \Delta t \|\psi^n\|_{m,p}^s \right)^{\frac{1}{s}}, \ |||\psi|||_{L^\infty(J;W^{m,p}(\Omega))} := \max_{1-l\le n\le N-l} \|\psi^n\|_{m,p},$$

where $l = 0$ for the control variable u and the state variable y, and $l = 1$ for the co-state variable p.

Then the fully discrete approximation scheme is to find $(y_h^n, u_h^n) \in V_h \times K_h$, $n = 1, 2, \cdots, N$, such that

$$\min_{u_h^n \in K_h} \left\{ \frac{1}{2} \sum_{n=1}^{N} \Delta t (\|y_h^n - y_d^n\|^2 + \|u_h^n\|^2) \right\}, \tag{2.20}$$

$$(dty_h^n, v_h) + (A\nabla y_h^n, \nabla v_h) = \left(\sum_{i=1}^{n} \Delta t B(t_n, t_{i-1}) \nabla y_h^i, \nabla v_h \right) + (f^n + u_h^n, v_h), \ \forall\, v_h \in V_h, \tag{2.21}$$

$$y_h^0 = R_h y^0. \tag{2.22}$$

Again, we can see that the above optimal control problem has a unique solution (y_h^n, u_h^n), and that $(y_h^n, u_h^n) \in V_h \times K_h$ is the solution of (2.20)–(2.22) if and only if there is a co-state $p_h^{n-1} \in V_h$ such that $(y_h^n, p_h^{n-1}, u_h^n)$ satisfies the following optimality conditions:

$$(dty_h^n, v_h) + (A\nabla y_h^n, \nabla v_h) = \left(\sum_{i=1}^{n} \Delta t B(t_n, t_{i-1}) \nabla y_h^i, \nabla v_h \right) + (f^n + u_h^n, v_h), \ \forall\, v_h \in V_h, \tag{2.23}$$

$$y_h^0 = R_h y^0, \tag{2.24}$$

$$\begin{aligned} &- (dtp_h^n, q_h) + (A\nabla p_h^{n-1}, \nabla q_h) \\ &\qquad = \left(\sum_{i=n}^{N} \Delta t B^*(t_i, t_{n-1}) \nabla p_h^{i-1}, \nabla q_h \right) + (y_h^n - y_d^n, q_h), \ \forall\, q_h \in V_h, \end{aligned} \tag{2.25}$$

$$p_h^N = 0, \tag{2.26}$$

$$(u_h^n + p_h^{n-1}, \tilde{u}_h - u_h^n) \ge 0, \ \forall\, \tilde{u}_h \in K_h. \tag{2.27}$$

Similarly, employing the projection (2.9), the optimal condition (2.27) can be rewritten as follows:

$$u_h^n = \max\{0, \overline{p_h^{n-1}}\} - \pi_h p_h^{n-1}, \tag{2.28}$$

where $\overline{p_h^{n-1}} = \frac{\int_\Omega p_h^{n-1}}{\int_\Omega 1}$.

In the rest of the paper, we shall use some intermediate variables. For any control function $\tilde{u} \in K$ satisfies the following:

$$
\begin{aligned}
&(d_t y_h^n(\tilde{u}), v_h) + (A\nabla y_h^n(\tilde{u}), \nabla v_h) \\
&= \left(\sum_{i=1}^{n} \Delta t B(t_n, t_{i-1}) \nabla y_h^i(\tilde{u}), \nabla v_h \right) + (f^n + \tilde{u}^n, v_h), \ \forall\, v_h \in V_h, \qquad (2.29)\\
&y_h^0(\tilde{u}) = R_h y^0, \qquad (2.30)\\
&-(d_t p_h^n(\tilde{u}), q_h) + (A\nabla p_h^{n-1}(\tilde{u}), \nabla q_h) \\
&= \left(\sum_{i=n}^{N} \Delta t B^*(t_i, t_{n-1}) \nabla p_h^{i-1}(\tilde{u}), \nabla q_h \right) + (y_h^n(\tilde{u}) - y_d^n, q_h), \ \forall\, q_h \in V_h, \qquad (2.31)\\
&p_h^N(\tilde{u}) = 0. \qquad (2.32)
\end{aligned}
$$

3. A priori error estimates and superconvergence

In this section, we will discuss a priori error estimates and superconvergence of the fully discrete case for the state variable, the co-state variable and the control variable. In order to do it, we need the following lemmas.

Lemma 3.1. *Let $(y_h^n(u), p_h^{n-1}(u))$ be the solution of (2.29)–(2.32) with $\tilde{u} = u$ and (y, p) be the solution of (2.4)–(2.8). Assume that the exact solution (y, p) has enough regularities for our purpose. Then, for Δt small enough and $1 \le n \le N$, we have*

$$
|||y - y_h(u)|||_{L^\infty(L^2)} + |||p - p_h(u)|||_{L^\infty(L^2)} \le C(\Delta t + h^2), \qquad (3.1)
$$

$$
|||\nabla(y - y_h(u))|||_{L^\infty(L^2)} + |||\nabla(p - p_h(u))|||_{L^\infty(L^2)} \le C(\Delta t + h). \qquad (3.2)
$$

Proof. For convenience, let

$$
\begin{aligned}
y^n - y_h^n(u) &= y^n - R_h y^n + R_h y^n - y_h^n(u) =: \eta_y^n + \xi_y^n,\\
p^n - p_h^n(u) &= p^n - R_h p^n + R_h p^n - p_h^n(u) =: \eta_p^n + \xi_p^n.
\end{aligned}
$$

Taking $t = t_n$ in (2.4), subtracting (2.29) from (2.4) and then using (2.12), we have

$$
\begin{aligned}
&(d_t \xi_y^n, v_h) + (A\nabla \xi_y^n, \nabla v_h) \\
=&(d_t y^n - y_t^n, v_h) - (d_t \eta_y^n, v_h) \\
&+ \left[\int_0^{t_n} (B(t_n, s)\nabla R_h y(s), \nabla v_h) ds - \left(\sum_{i=1}^{n} \Delta t B(t_n, t_{i-1}) \nabla y_h^i(u), \nabla v_h \right) \right]. \qquad (3.3)
\end{aligned}
$$

Choosing $v_h = d_t \xi_y^n$ in (3.3), we get

$$
\begin{aligned}
&(d_t \xi_y^n, d_t \xi_y^n) + (A\nabla \xi_y^n, d_t \nabla \xi_y^n) \\
=&(d_t y^n - y_t^n, d_t \xi_y^n) - (d_t \eta_y^n, d_t \xi_y^n) \\
&+ \left[\int_0^{t_n} (B(t_n, s)\nabla R_h y(s), d_t \nabla \xi_y^n) ds - \left(\sum_{i=1}^{n} \Delta t B(t_n, t_{i-1}) \nabla y_h^i(u), d_t \nabla \xi_y^n \right) \right]. \qquad (3.4)
\end{aligned}
$$

Notice that

$$(dt\nabla\xi_y^n, A\nabla\xi_y^n) \geq \frac{1}{2\Delta t}(\|A^{\frac{1}{2}}\nabla\xi_y^n\|^2 - \|A^{\frac{1}{2}}\nabla\xi_y^{n-1}\|^2). \tag{3.5}$$

Multiplying Δt and summing over n from 1 to l $(1 \leq l \leq N)$ on both sides of (3.4), and by using (3.5) and $\xi_y^0 = 0$, we find that

$$\begin{aligned}
&\frac{1}{2}\|A^{\frac{1}{2}}\nabla\xi_y^l\|^2 + \sum_{n=1}^{l}\|dt\xi_y^n\|^2\Delta t \\
\leq &\sum_{n=1}^{l}(dty^n - y_t^n, dt\xi_y^n)\Delta t - \sum_{n=1}^{l}(dt\eta_y^n, dt\xi_y^n)\Delta t \\
&+ \sum_{n=1}^{l}\left[\int_0^{t_n}(B(t_n, s)\nabla R_h y(s), dt\nabla\xi_y^n)ds - \left(\sum_{i=1}^{n}\Delta t B(t_n, t_{i-1})\nabla y_h^i(u), dt\nabla\xi_y^n\right)\right]\Delta t \\
=: &\sum_{i=1}^{3}A_i.
\end{aligned} \tag{3.6}$$

Now, we estimate the right-hand terms of (3.6). For A_1, from the results given in [24], we have

$$\begin{aligned}
A_1 \leq &C\sum_{n=1}^{l}\left(\int_{t_{n-1}}^{t_n}\|y_{tt}\|dt\right)^2\Delta t + \|\sum_{n=1}^{l}\|dt\xi_y^n\|^2\Delta t \\
\leq &C(\Delta t)^2\int_0^{t_l}\|y_{tt}\|^2dt + \frac{1}{4}\sum_{n=1}^{l}\|dt\xi_y^n\|^2\Delta t \\
\leq &C(\Delta t)^2\|y_{tt}\|^2_{L^2(L^2)} + \frac{1}{4}\sum_{n=1}^{l}\|dt\xi_y^n\|^2\Delta t.
\end{aligned} \tag{3.7}$$

For A_2, using (2.13), the Hölder inequality and the Cauchy inequality, we have

$$\begin{aligned}
A_2 \leq &C\sum_{n=1}^{l}\|\frac{\eta_y^n - \eta_y^{n-1}}{\Delta t}\|^2\Delta t + \frac{1}{4}\sum_{n=1}^{l}\|dt\xi_y^n\|^2\Delta t \\
\leq &C\sum_{n=1}^{l}\frac{1}{\Delta t}\|\int_{t_{n-1}}^{t_n}(\eta_y)_t dt\|^2 + \frac{1}{4}\sum_{n=1}^{l}\|dt\xi_y^n\|^2\Delta t \\
\leq &C\sum_{n=1}^{l}\frac{1}{\Delta t}\Big((\int_{t_{n-1}}^{t_n}\|(\eta_y)_t\|^2dt)^{\frac{1}{2}}(\int_{t_{n-1}}^{t_n}1^2dt)^{\frac{1}{2}}\Big)^2 + \frac{1}{4}\sum_{n=1}^{l}\|dt\xi_y^n\|^2\Delta t \\
\leq &Ch^4\int_0^{t_l}\|y_t\|_2^2dt + \frac{1}{4}\sum_{n=1}^{l}\|dt\xi_y^n\|^2\Delta t \\
\leq &Ch^4\|y_t\|^2_{L^2(H^2)} + \frac{1}{4}\sum_{n=1}^{l}\|dt\xi_y^n\|^2\Delta t.
\end{aligned} \tag{3.8}$$

At last, for A_3, it follows from the Cauchy inequality, Cauchy mean value theorem and assumptions on A and B that

$$
\begin{aligned}
A_3 =& \sum_{n=1}^{l} \Big[\int_0^{t_n} (B(t_n, s)\nabla R_h y(s), dt\nabla \xi_y^n) ds - \left(\sum_{i=1}^{n} \Delta t B(t_n, t_i)\nabla y_h^i(u), dt\nabla \xi_y^n \right) \\
&+ \left(\sum_{i=1}^{n} \Delta t B(t_n, t_i)\nabla y_h^i(u), dt\nabla \xi_y^n \right) - \left(\sum_{i=1}^{n} \Delta t B(t_n, t_{i-1})\nabla y_h^i(u), dt\nabla \xi_y^n \right) \Big]\Delta t \\
\le& C(\Delta t)^2 (\|\nabla R_h y_t\|^2_{L^2(L^2)} + \|\nabla R_h y\|^2_{L^2(L^2)}) + C\sum_{n=1}^{l} \|\nabla \xi_y^n\|^2 \Delta t \\
&+ C\sum_{n=1}^{l} \Delta t \sum_{i=1}^{n} \|\nabla \xi_y^i\|^2 \Delta t + \frac{a_*}{4}\|\nabla \xi_y^l\|^2,
\end{aligned} \tag{3.9}
$$

where

$$
\begin{aligned}
&\sum_{n=1}^{l} \Big[\int_0^{t_n} (B(t_n, s)\nabla R_h y(s), dt\nabla \xi_y^n) ds - \left(\sum_{i=1}^{n} \Delta t B(t_n, t_i)\nabla y_h^i(u), dt\nabla \xi_y^n \right) \Big]\Delta t \\
=& \left(\int_0^{t_l} B(t_l, s)\nabla R_h y(s) ds - \sum_{i=1}^{l} B(t_l, t_i)\nabla R_h y^i \Delta t, \nabla \xi_y^l \right) + \sum_{i=1}^{l} \left(\Delta t B(t_l, t_i)\nabla \xi_y^i, \nabla \xi_y^l \right) \\
&+ \sum_{n=1}^{l-1} \left(\int_0^{t_n} (B(t_n, s) - B(t_{n+1}, s))\nabla R_h y ds, \nabla \xi_y^n \right) \\
&- \sum_{n=1}^{l-1} \left(\int_{t_n}^{t_{n+1}} B(t_{n+1}, s)(\nabla R_h y - \nabla R_h y^{n+1}) ds, \nabla \xi_y^n \right) \\
&- \sum_{n=1}^{l-1} \left(\sum_{i=1}^{n} \Delta t (B(t_n, t_i) - B(t_{n+1}, t_i))\nabla R_h y^i, \nabla \xi_y^n \right) - \sum_{n=1}^{l-1} \left(\Delta t B(t_{n+1}, t_{n+1})\nabla \xi_y^{n+1}, \nabla \xi_y^n \right) \\
&+ \sum_{n=1}^{l-1} \left(\sum_{i=1}^{n} \Delta t (B(t_n, t_i) - B(t_{n+1}, t_i))\nabla \xi_y^i, \nabla \xi_y^n \right) \\
=& \left(\int_0^{t_l} B(t_l, s)\nabla R_h y(s) ds - \sum_{i=1}^{l} B(t_l, t_i)\nabla R_h y^i \Delta t, \nabla \xi_y^l \right) + \left(\sum_{i=1}^{l} \Delta t B(t_l, t_i)\nabla \xi_y^i, \nabla \xi_y^l) \right) \\
&+ \sum_{n=1}^{l-1} \left(\int_0^{t_n} B_t(t_{n+1}^*, s)\Delta t \nabla R_h y ds, \nabla \xi_y^n \right) - \sum_{n=1}^{l-1} \left(\int_{t_n}^{t_{n+1}} \Delta t B(t_{n+1}, s)\nabla R_h y_t^{n+1} ds, \nabla \xi_y^n \right) \\
&- \sum_{n=1}^{l-1} \left(\sum_{i=1}^{n} (\Delta t)^2 B_t(t_{n+1}^*, t_i)\nabla R_h y^i ds, \nabla \xi_y^n \right) - \sum_{n=1}^{l-1} \left(\Delta t B(t_{n+1}, t_{n+1})\nabla \xi_y^{n+1}, \nabla \xi_y^n \right) \\
&+ \sum_{n=1}^{l-1} \left(\sum_{i=1}^{n} (\Delta t)^2 B_t(t_{n+1}^*, t_i)\nabla \xi_y^i, \nabla \xi_y^n \right)
\end{aligned}
$$

$$
\begin{aligned}
&\leq C(\Delta t)^2(\|\nabla R_h y_t\|^2_{L^2(L^2)} + \|\nabla R_h y\|^2_{L^2(L^2)}) + C\sum_{n=1}^{l}\|\nabla \xi_y^n\|^2\Delta t\\
&\quad + C\sum_{n=1}^{l}\Delta t\sum_{i=1}^{n}\|\nabla \xi_y^i\|^2\Delta t + \frac{a_*}{8}\|\nabla \xi_y^l\|^2
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{n=1}^{l}\Big[\left(\sum_{i=1}^{n}\Delta t B(t_n,t_i)\nabla y_h^i(u), dt\nabla\xi_y^n\right) - \left(\sum_{i=1}^{n}\Delta t B(t_n,t_{i-1})\nabla y_h^i(u), dt\nabla\xi_y^n\right)\Big]\Delta t\\
=&\left(\sum_{i=1}^{l}\Delta t(B(t_l,t_i)-B(t_l,t_{i-1}))\nabla R_h y^i, \nabla\xi_y^l\right) - \left(\sum_{i=1}^{l}\Delta t(B(t_l,t_i)-B(t_l,t_{i-1}))\nabla\xi_y^i, \nabla\xi_y^l\right)\\
&+\sum_{n=1}^{l-1}\left(\sum_{i=1}^{n}\Delta t(B(t_n,t_i)-B(t_n,t_{i-1}))\nabla R_h y^i, \nabla\xi_y^n\right)\\
&-\sum_{n=1}^{l-1}\left(\sum_{i=1}^{n}\Delta t(B(t_n,t_i)-B(t_n,t_{i-1}))\nabla\xi_y^i, \nabla\xi_y^n\right)\\
&-\sum_{n=1}^{l-1}\left(\sum_{i=1}^{n+1}\Delta t(B(t_{n+1},t_i)-B(t_{n+1},t_{i-1}))\nabla R_h y^i, \nabla\xi_y^n\right)\\
&+\sum_{n=1}^{l-1}\left(\sum_{i=1}^{n+1}\Delta t(B(t_{n+1},t_i)-B(t_{n+1},t_{i-1}))\nabla\xi_y^i, \nabla\xi_y^n\right)\\
=&\left(\sum_{i=1}^{l}(\Delta t)^2 B_t(t_l,t_i^*)\nabla R_h y^i, \nabla\xi_y^l\right) - \left(\sum_{i=1}^{l}(\Delta t)^2 B_t(t_l,t_i^*)\nabla\xi_y^i, \nabla\xi_y^l\right)\\
&\quad+\sum_{n=1}^{l-1}\left(\sum_{i=1}^{n}(\Delta t)^2 B_t(t_n,t_i^*)\nabla R_h y^i, \nabla\xi_y^n\right) - \sum_{n=1}^{l-1}\left(\sum_{i=1}^{n}(\Delta t)^2 B_t(t_n,t_i^*)\nabla\xi_y^i, \nabla\xi_y^n\right)\\
&\quad-\sum_{n=1}^{l-1}\left(\sum_{i=1}^{n+1}(\Delta t)^2 B_t(t_{n+1},t_i^*)\nabla R_h y^i, \nabla\xi_y^n\right) + \sum_{n=1}^{l-1}\left(\sum_{i=1}^{n+1}(\Delta t)^2 B_t(t_{n+1},t_i^*)\nabla\xi_y^i, \nabla\xi_y^n\right)\\
&\quad\leq C(\Delta t)^2\|\nabla R_h y\|^2_{L^2(L^2)} + C\sum_{n=1}^{l}\|\nabla\xi_y^n\|^2\Delta t + C\sum_{n=1}^{l}\Delta t\sum_{i=1}^{n}\|\nabla\xi_y^i\|^2\Delta t + \frac{a_*}{8}\|\nabla\xi_y^l\|^2,
\end{aligned}
$$

where t_i^* is located between t_{i-1} and t_i, and we also used

$$
\begin{aligned}
&\left\|\int_0^{t_n} B(t_n,s)\nabla R_h y(s)ds - \sum_{i=1}^{n} B(t_n,t_i)\nabla R_h y^i\Delta t\right\|\\
\leq& C\Delta t(\|\nabla R_h y_t\|_{L^2(L^2)} + \|\nabla R_h y\|_{L^2(L^2)}).
\end{aligned}
$$

From (3.7)–(3.9), we have

$$
\frac{1}{2}\|A^{\frac{1}{2}}\nabla\xi_y^l\|^2 + \frac{1}{2}\sum_{n=1}^{l}\|dt\xi_y^n\|^2\Delta t
$$

$$
\begin{aligned}
&\leq Ch^4\|y_t\|^2_{L^2(H^2)} + C(\Delta t)^2(\|y_{tt}\|^2_{L^2(L^2)} + \|\nabla R_h y_t\|^2_{L^2(L^2)} + \|\nabla R_h y\|^2_{L^2(L^2)}) \\
&\quad + C\sum_{n=1}^{l}\|\nabla\xi_y^n\|^2\Delta t + C\sum_{n=1}^{l}\Delta t\sum_{i=1}^{n}\|\nabla\xi_y^i\|^2\Delta t + \frac{a_*}{4}\|\nabla\xi_y^l\|^2.
\end{aligned} \tag{3.10}
$$

Adding $\sum_{n=1}^{l}\|\nabla\xi_y^n\|^2\Delta t$ to both sides of (3.10), by use of the assumption on A and discrete Gronwall's inequality, we have

$$
|||\nabla(R_h y - y_h(u))|||_{L^\infty(L^2)} \leq C(\Delta t + h^2). \tag{3.11}
$$

Using (2.13), the Poincare inequality and the triangle inequality, we get

$$
|||y - y_h(u)|||_{L^\infty(L^2)} \leq C(\Delta t + h^2), \ |||\nabla(y - y_h(u))|||_{L^\infty(L^2)} \leq C(\Delta t + h). \tag{3.12}
$$

Taking $t = t_{n-1}$ in (2.6), subtracting (2.31) from (2.6) and then using (2.14), we have

$$
\begin{aligned}
&-(dt\xi_p^n, q_h) + (A\nabla\xi_p^{n-1}, \nabla q_h) \\
=&-(dtp^n - p_t^{n-1}, q_h) + (dt\eta_p^n, q_h) \\
&+ \int_{t_{n-1}}^{T}(B^*(s, t_{n-1})\nabla R_h p(s), \nabla q_h)ds - \left(\sum_{i=n}^{N}\Delta t B^*(t_i, t_{n-1})\nabla p_h^{i-1}(u), \nabla q_h\right) \\
&+ (\delta y_d^n - \delta y^n + y^n - y_h^n(u), q_h).
\end{aligned} \tag{3.13}
$$

Choosing $q_h = -dt\xi_p^n$ in (3.13), multiplying by Δt and summing over n from $l+1$ to N $(0 \leq l \leq N-1)$ on both sides of (3.13), since $\xi_p^N = 0$, we find that

$$
\begin{aligned}
&\frac{1}{2}\|A^{\frac{1}{2}}\nabla\xi_p^l\|^2 + \sum_{n=l+1}^{N}\|dt\xi_p^n\|^2\Delta t \\
\leq& \sum_{n=l+1}^{N}(dtp^n - p_t^{n-1}, dt\xi_p^n)\Delta t - \sum_{n=l+1}^{N}(dt\eta_p^n, dt\xi_p^n)\Delta t \\
&- \sum_{n=l+1}^{N}\left[\int_{t_{n-1}}^{T}(B^*(s, t_{n-1})\nabla R_h p(s), dt\nabla\xi_p^n)ds - \left(\sum_{i=n}^{N}\Delta t B^*(t_i, t_{n-1})\nabla p_h^{i-1}(u), dt\nabla\xi_p^n\right)\right]\Delta t \\
&- \sum_{n=l+1}^{N}(\delta y_d^n - \delta y^n + y^n - y_h^n(u), dt\xi_p^n)\Delta t \\
=:& \sum_{i=1}^{4}B_i.
\end{aligned} \tag{3.14}
$$

Notice that

$$
-(A\nabla\xi_p^{n-1}, dt\nabla\xi_p^n) \geq \frac{1}{2\Delta t}(\|A^{\frac{1}{2}}\nabla\xi_p^{n-1}\|^2 - \|A^{\frac{1}{2}}\nabla\xi_p^n\|^2). \tag{3.15}
$$

Now, we estimate the right-hand terms of (3.14). Similar to (3.7), we have

$$
B_1 \leq C(\Delta t)^2\|p_{tt}\|^2_{L^2(L^2)} + \frac{1}{4}\sum_{n=l+1}^{N}\|dt\xi_p^n\|^2\Delta t. \tag{3.16}
$$

For B_2, using (2.15) and the Cauchy inequality, we have

$$B_2 \leq Ch^4\|p_t\|^2_{L^2(H^2)} + \frac{1}{4}\sum_{n=l+1}^{N}\|dt\xi_p^n\|^2\Delta t. \tag{3.17}$$

For B_3, applying the same estimates as A_3, we conclude that

$$\begin{aligned}
B_3 =& -\sum_{n=l+1}^{N}\Bigg[\int_{t_{n-1}}^{T}(B^*(s,t_{n-1})\nabla R_h p(s), dt\nabla\xi_p^n)ds - \left(\sum_{i=n}^{N}\Delta t B^*(t_{i-1},t_{n-1})\nabla p_h^{i-1}(u), dt\nabla\xi_p^n\right)\\
&+\left(\sum_{i=n}^{N}\Delta t B^*(t_{i-1},t_{n-1})\nabla p_h^{i-1}(u), dt\nabla\xi_p^n\right) - \left(\sum_{i=n}^{N}\Delta t B^*(t_i,t_{n-1})\nabla p_h^{i-1}(u), dt\nabla\xi_p^n\right)\Bigg]\Delta t\\
\leq& C(\Delta t)^2(\|\nabla R_h p_t\|^2_{L^2(L^2)} + \|\nabla R_h p\|^2_{L^2(L^2)}) + C\sum_{n=l+1}^{N}\|\nabla\xi_p^n\|^2\Delta t\\
&+ C\sum_{n=l+1}^{N}\Delta t\sum_{i=n}^{N}\|\nabla\xi_p^i\|^2\Delta t + \frac{a_*}{4}\|\nabla\xi_p^l\|^2,
\end{aligned} \tag{3.18}$$

where

$$\begin{aligned}
\|\nabla R_h p_t\|_{L^2(L^2)} + \|\nabla R_h p\|_{L^2(L^2)} \leq& \|\nabla(p_t - R_h p_t)\|_{L^2(L^2)} + \|\nabla p_t\|_{L^2(L^2)}\\
&+ \|\nabla(p - R_h p)\|_{L^2(L^2)} + \|\nabla p\|_{L^2(L^2)}.
\end{aligned}$$

For B_4, using the Cauchy inequality and the smoothness of y and y_d, we have

$$\begin{aligned}
B_4 =& -\sum_{n=l+1}^{N}(\delta y_d^n - \delta y^n + y^n - y_h^n(u), dt\xi_p^n)\Delta t\\
\leq& C(\Delta t)^2(\|y_t\|^2_{L^2(L^2)} + \|(y_d)_t\|^2_{L^2(L^2)}) + C\|y^n - y_h^n(u)\|^2_{L^2(L^2)}\\
&+ \frac{1}{4}\sum_{n=l+1}^{N}\|dt\xi_p^n\|^2\Delta t.
\end{aligned} \tag{3.19}$$

Combining (3.16)–(3.19), we have

$$\begin{aligned}
&\frac{1}{2}\|A^{\frac{1}{2}}\nabla\xi_p^l\|^2 + \frac{1}{4}\sum_{n=l+1}^{N}\|dt\xi_p^n\|^2\Delta t\\
\leq& C(\Delta t)^2\Big(\|p_{tt}\|^2_{L^2(L^2)} + \|\nabla R_h p_t\|^2_{L^2(L^2)} + \|\nabla R_h p\|^2_{L^2(L^2)} + \|y_t\|^2_{L^2(L^2)} + \|(y_d)_t\|^2_{L^2(L^2)}\Big)\\
&+ Ch^4\|p_t\|^2_{L^2(H^2)} + C\|y^n - y_h^n(u)\|^2_{L^2(L^2)} + \frac{a_*}{4}\|\nabla\xi_p^l\|^2\\
&+ C\sum_{n=l+1}^{N}\|\nabla\xi_p^n\|^2\Delta t + C\sum_{n=l+1}^{N}\Delta t\sum_{i=n}^{N}\|\nabla\xi_p^i\|^2\Delta t.
\end{aligned} \tag{3.20}$$

By adding $\sum_{n=l+1}^{N}\|\nabla\xi_p^n\|^2\Delta t$ to both sides of (3.20) and applying the assumption on A, discrete Gronwall's inequality and (3.12), we conclude that

$$|||\nabla(R_h p - p_h(u))|||_{L^\infty(L^2)} \leq C(\Delta t + h^2). \tag{3.21}$$

Using (2.15) and the triangle inequality, we get

$$|||p - p_h(u)|||_{L^\infty(L^2)} \le C(\Delta t + h^2),\ |||\nabla(p - p_h(u))|||_{L^\infty(L^2)} \le C(\Delta t + h); \tag{3.22}$$

we have completed the proof of the Lemma 3.1.

Lemma 3.2. *Choose $\tilde{u}^n = Q_h u^n$ and $\tilde{u}^n = u^n$ in (2.29)–(2.32) respectively. Then, for Δt small enough and $1 \le n \le N$, we have*

$$|||\nabla(y_h(u) - y_h(Q_h u))|||_{L^\infty(L^2)} + |||\nabla(p_h(u) - p_h(Q_h u))|||_{L^\infty(L^2)} \le Ch^2. \tag{3.23}$$

Proof. For convenience, let

$$\lambda_y^n = y_h^n(u) - y_h^n(Q_h u),\ \lambda_p^n = p_h^n(u) - p_h^n(Q_h u).$$

Taking $\tilde{u}^n = u^n$ and $\tilde{u}^n = Q_h u^n$ in (2.29), we easily get

$$\left(d t\lambda_y^n, v_h\right) + \left(A\nabla\lambda_y^n, \nabla v_h\right) = \sum_{i=1}^{n} \Delta t \left(B(t_n, t_{i-1})\nabla\lambda_y^i, \nabla v_h\right) + (u^n - Q_h u^n, v_h). \tag{3.24}$$

By choosing $v_h = dt\lambda_y^n$ in (3.24), multiplying by Δt and summing over n from 1 to l ($1 \le l \le N$) on both sides of (3.24), we find that

$$\begin{aligned}
&\frac{1}{2}\|A^{\frac{1}{2}}\nabla\lambda_y^l\|^2 + \sum_{n=1}^{l}\|dt\lambda_y^n\|^2\Delta t\\
\le& \sum_{n=1}^{l}\left(\sum_{i=1}^{n}\Delta t(B(t_n, t_{i-1})\nabla\lambda_y^i, dt\nabla\lambda_y^n\right)\Delta t + \sum_{n=1}^{l}\left(u^n - Q_h u^n, \lambda_y^n - \lambda_y^{n-1}\right)\\
=&\left(\sum_{i=1}^{l}\Delta t B(t_l, t_{i-1})\nabla\lambda_y^i, \nabla\lambda_y^l\right)\\
&+\sum_{n=1}^{l-1}\left(\sum_{i=1}^{n}\Delta t B(t_n, t_{i-1})\nabla\lambda_y^i - \sum_{i=1}^{n+1}\Delta t B(t_{n+1}, t_{i-1})\nabla\lambda_y^i, \nabla\lambda_y^n\right)\\
&+(u^l - Q_h u^l, \lambda_y^l) - \sum_{n=1}^{l-1}\left(u^{n+1} - Q_h u^{n+1} - (u^n - Q_h u^n), \lambda_y^n\right)\\
=&\left(\sum_{i=1}^{l}\Delta t B(t_l, t_{i-1})\nabla\lambda_y^i, \nabla\lambda_y^l\right) + \sum_{n=1}^{l-1}\left(\sum_{i=1}^{n}(\Delta t)^2 B_t(t_{n+1}^*, t_{i-1})\nabla\lambda_y^i, \nabla\lambda_y^n\right)\\
&-\sum_{n=1}^{l-1}\left(\Delta t B(t_{n+1}, t_n)\nabla\lambda_y^{n+1}, \nabla\lambda_y^n\right)\\
&+C\|u^l - Q_h u^l\|_{-1}\|\nabla\lambda_y^l\| + \sum_{n=1}^{l-1}\|(u - Q_h u)_t(\theta^n)\|_{-1}\|\nabla\lambda_y^n\|\Delta t\\
\le& C\sum_{n=1}^{l}\|\nabla\lambda_y^n\|^2\Delta t + C\sum_{n=1}^{l}\Delta t\sum_{i=1}^{n}\|\nabla\lambda_y^i\|^2\Delta t + \frac{a_*}{4}\|\nabla\lambda_y^l\|^2
\end{aligned}$$

$$+ Ch^4(\|u^l\|_1^2 + \|u_t\|_{L^2(H^1)}^2), \tag{3.25}$$

where we use (2.17) and the assumption on B; additionally, θ^n is located between t_n and t_{n+1}.

Add $\sum_{n=1}^{l} \|\nabla\lambda_y^n\|^2\Delta t$ to both sides of (3.25); then for sufficiently small Δt, combining (3.25) and the discrete Gronwall inequality, we have

$$|||\nabla(y_h(u) - y_h(Q_h u))|||_{L^\infty(L^2)} \leq Ch^2. \tag{3.26}$$

Similar to (3.24), we have

$$-(dt\lambda_p^n, q_h) + (A\nabla\lambda_p^{n-1}, \nabla q_h) = \left(\sum_{i=n}^{N} \Delta t B^*(t_i, t_{n-1})\nabla\lambda_p^{i-1}, \nabla q_h\right) + (\lambda_y^n, q_h),\ \forall\ q_h \in V_h. \tag{3.27}$$

By choosing $q_h = -dt\lambda_p^n$ in (3.27), multiplying by Δt and summing over n from $l+1$ to N $(0 \leq l \leq N-1)$ on both sides of (3.27), combining (3.26) and Poincare inequality gives

$$\begin{aligned}
&\frac{1}{2}\|A^{\frac{1}{2}}\nabla\lambda_p^l\|^2 + \sum_{n=1}^{l}\|dt\lambda_p^n\|^2\Delta t\\
\leq& -\sum_{n=l+1}^{N}\left(\sum_{i=n}^{N}\Delta t B^*(t_i, t_{n-1})\nabla\lambda_p^{i-1}, dt\nabla\lambda_p^n\right)\Delta t - \sum_{n=l+1}^{N}\left(\lambda_y^n, dt\lambda_p^n\right)\Delta t\\
=&\left(\sum_{i=l+1}^{N}\Delta t B^*(t_i, t_l)\nabla\lambda_p^{i-1}, \nabla\lambda_p^l\right) - \sum_{n=l+1}^{N-1}\left(\sum_{i=n}^{N}\Delta t B^*(t_i, t_{n-1})\nabla\lambda_p^{i-1}, \nabla\lambda_p^n\right)\\
&+\sum_{n=l+1}^{N-1}\left(\sum_{i=n+1}^{N}\Delta t B^*(t_i, t_n)\nabla\lambda_p^{i-1}, \nabla\lambda_p^n\right) - \sum_{n=l+1}^{N}\left(\lambda_y^n, dt\lambda_p^n\right)\Delta t\\
=&\left(\sum_{i=l+1}^{N}\Delta t B^*(t_i, t_l)\nabla\lambda_p^{i-1}, \nabla\lambda_p^l\right) - \sum_{n=l+1}^{N-1}\left(\sum_{i=n}^{N}(\Delta t)^2 B_t^*(t_i, t_n^*)\nabla\lambda_p^{i-1}, \nabla\lambda_p^n\right)\\
&-\sum_{n=l+1}^{N-1}\left(\Delta t B^*(t_n, t_n)\nabla\lambda_p^{n-1}, \nabla\lambda_p^n\right) - \sum_{n=l+1}^{N}\left(\lambda_y^n, dt\lambda_p^n\right)\Delta t\\
\leq& Ch^4 + \frac{a_*}{4}\|\nabla\lambda_p^l\|^2 + C\sum_{n=l+1}^{N}\|\nabla\lambda_p^{n-1}\|^2\Delta t\\
&+C\sum_{n=l+1}^{N-1}\Delta t\sum_{i=n}^{N}\|\nabla\lambda_p^i\|^2\Delta t + \frac{1}{2}\sum_{n=l+1}^{N}\|dt\lambda_p^n\|^2\Delta t.
\end{aligned} \tag{3.28}$$

Add $\sum_{n=l+1}^{N}\|\nabla\lambda_p^{n-1}\|^2\Delta t$ to both sides of (3.28); then for sufficiently small Δt, applying the discrete Gronwall inequality and the assumptions on A and B, we have

$$|||\nabla(p_h(u) - p_h(Q_h u))|||_{L^\infty(L^2)} \leq Ch^2. \tag{3.29}$$

Using the stability analysis as in Lemma 3.2 yields Lemma 3.3.

Lemma 3.3. *Let (y_h^n, p_h^n) and $(y_h^n(Q_hu), p_h^n(Q_hu))$ be the discrete solutions of (2.29)–(2.32) with $\tilde{u}^n = u_h^n$ and $\tilde{u}^n = Q_hu^n$, respectively. Then, for Δt small enough and $1 \le n \le N$, we have*

$$|||\nabla(y_h(Q_hu) - y_h)|||_{L^\infty(L^2)} + |||\nabla(p_h(Q_hu) - p_h)|||_{L^\infty(L^2)} \le C|||Q_hu - u_h|||_{L^2(L^2)}. \tag{3.30}$$

Next, we derive the following inequality.

Lemma 3.4. *Choose $\tilde{u}^n = Q_hu^n$ and $\tilde{u}^n = u_h^n$ in (2.29)–(2.32) respectively. Then, we have*

$$\sum_{n=1}^{N} \left(Q_hu^n - u_h^n, p_h^{n-1}(Q_hu) - p_h^{n-1}\right)\Delta t \ge 0. \tag{3.31}$$

Proof. For $n = 0, 1, \ldots, N$, let

$$r_p^n = p_h^n(Q_hu) - p_h^n,\ r_y^n = y_h^n(Q_hu) - y_h^n.$$

From (2.29)–(2.32), we have

$$\begin{aligned}
&\left(dtr_y^n, v_h\right) + \left(A\nabla r_y^n, \nabla v_h\right) - \sum_{i=1}^{n} \Delta t\left(B(t_n, t_{i-1})\nabla r_y^i, \nabla v_h\right) \\
&= \left(Q_hu^n - u_h^n, v_h\right),\ \forall\, v_h\ \in V_h, &(3.32)\\
&-\left(dtr_p^n, q_h\right) + \left(A\nabla r_p^{n-1}, \nabla q_h\right) - \sum_{i=n}^{N} \Delta t\left(B^*(t_i, t_{n-1})\nabla r_p^{i-1}, \nabla q_h\right) = \left(r_y^n, q_h\right),\ \forall\, q_h\ \in V_h. &(3.33)
\end{aligned}$$

Notice that

$$\sum_{n=1}^{N}\left(\Delta t\sum_{i=1}^{n} B(t_n, t_{i-1})\nabla r_y^i, \nabla r_p^{n-1}\right) = \sum_{n=1}^{N}\left(\Delta t\sum_{i=n}^{N} B^*(t_i, t_{n-1})\nabla r_p^{i-1}, \nabla r_y^n\right)$$

and

$$\sum_{n=1}^{N}\left(dtr_y^n, r_p^{n-1}\right)\Delta t + \sum_{n=1}^{N}\left(dtr_p^n, r_y^n\right)\Delta t = 0.$$

By choosing $v_h = -r_p^{n-1}$ in (3.32), $q_h = r_y^n$ in (3.33), and then multiplying the two resulting equations by Δt and summing it over n from 1 to N, we have

$$\sum_{n=1}^{N}\left(Q_hu^n - u_h^n, p_h^{n-1}(Q_hu) - p_h^{n-1}\right)\Delta t = \sum_{n=1}^{N} \|r_y^n\|^2\Delta t, \tag{3.34}$$

which completes the proof of the lemma.

Lemma 3.5. *Let u be the solution of (2.4)–(2.8) and u_h^n be the solution of (2.23)–(2.27). Assume that all of the conditions in Lemmas 3.1–3.4 are valid. Then, for Δt small enough and $1 \le n \le N$, we have*

$$|||Q_hu - u_h|||_{L^2(L^2)} \le C(h^2 + \Delta t). \tag{3.35}$$

Proof. Take $\tilde{u} = u_h^n$ in (2.8) and $\tilde{u}_h = Q_h u^n$ in (2.27) to get the following two inequalities:

$$(u^n + p^n, u_h^n - u^n) \geq 0 \tag{3.36}$$

and

$$\left(u_h^n + p_h^{n-1}, Q_h u^n - u_h^n\right) \geq 0. \tag{3.37}$$

Note that $u_h^n - u^n = u_h^n - Q_h u^n + Q_h u^n - u^n$. Adding the two inequalities (3.36) and (3.37), we have

$$(u_h^n + p_h^{n-1} - u^n - p^n, Q_h u^n - u_h^n) + (u^n + p^n, Q_h u^n - u^n) \geq 0. \tag{3.38}$$

Thus, by (3.38), (2.16), (2.8) and Lemma 3.4, we find that

$$\begin{aligned}
&|||Q_h u - u_h|||_{L^2(L^2)}^2 \\
&= \sum_{n=1}^{N} (Q_h u^n - u_h^n, Q_h u^n - u_h^n)\,\Delta t \\
&\leq \sum_{n=1}^{N} (Q_h u^n - u^n, Q_h u^n - u_h^n)\,\Delta t + \sum_{n=1}^{N} \left(p_h^{n-1} - p^n, Q_h u^n - u_h^n\right)\Delta t \\
&\quad + \sum_{n=1}^{N} (u^n + p^n, Q_h u^n - u^n)\,\Delta t \\
&= \sum_{n=1}^{N} \left(p_h^{n-1} - p_h^{n-1}(Q_h u), Q_h u^n - u_h^n\right)\Delta t + \sum_{n=1}^{N} \left(p^{n-1} - p^n, Q_h u^n - u_h^n\right)\Delta t \\
&\quad + \sum_{n=1}^{N} \left(p_h^{n-1}(u) - p^{n-1}, Q_h u^n - u_h^n\right)\Delta t + \sum_{n=1}^{N} (u^n + p^n, Q_h u^n - u^n)\,\Delta t \\
&\quad + \sum_{n=1}^{N} \left(p_h^{n-1}(Q_h u) - p_h^{n-1}(u), Q_h u^n - u_h^n\right)\Delta t \\
&\leq \sum_{n=1}^{N} \left(p^{n-1} - p^n, Q_h u^n - u_h^n\right)\Delta t + \sum_{n=1}^{N} \left(p_h^{n-1}(u) - p^{n-1}, Q_h u^n - u_h^n\right)\Delta t \\
&\quad + \sum_{n=1}^{N} \left(p_h^{n-1}(Q_h u) - p_h^{n-1}(u), Q_h u^n - u_h^n\right)\Delta t \\
&=: \sum_{i=1}^{3} F_i.
\end{aligned} \tag{3.39}$$

It follows from the Cauchy inequality, Lemma 3.1, Lemma 3.2 and Poincare's inequality that

$$F_1 \leq C(\Delta t)^2 \|p_t\|_{L^2(L^2)}^2 + \frac{1}{4} \sum_{n=1}^{N} \|Q_h u^n - u_h^n\|^2 \Delta t, \tag{3.40}$$

$$F_2 \leq C\left(h^4 + (\Delta t)^2\right) + \frac{1}{4} \sum_{n=1}^{N} \|Q_h u^n - u_h^n\|^2 \Delta t, \tag{3.41}$$

$$F_3 \leq Ch^4 + \frac{1}{4}\sum_{n=1}^{N} \|Q_h u^n - u_h^n\|^2 \Delta t. \tag{3.42}$$

Substituting the estimates for F_1–F_3 into (3.39), we derive (3.35).

Using (3.11), (3.21), Lemmas 3.2–3.5 and the triangle inequality, we derive the following superconvergence for the state variable.

Lemma 3.6. *Let u be the solution of (2.4)–(2.8) and u_h^n be the solution of (2.23)–(2.27). Assume that all of the conditions in Lemmas 3.1–3.5 are valid. Then, for Δt small enough and $1 \leq n \leq N$, we have*

$$|||\nabla(R_h y - y_h)|||_{L^\infty(L^2)} + |||\nabla(R_h p - p_h)|||_{L^\infty(L^2)} \leq C(h^2 + \Delta t). \tag{3.43}$$

Now, the main result of this section is given in the following theorem.

Theorem 3.1. *Let (y, p, u) and $(y_h^n, p_h^{n-1}, u_h^n)$ be the solutions of (2.4)–(2.8) and (2.23)–(2.27), respectively. Assume that y, p and u have enough regularities for our purpose; then, for Δt small enough and $1 \leq n \leq N$, we have*

$$|||y - y_h|||_{L^\infty(L^2)} + |||p - p_h|||_{L^\infty(L^2)} \leq C(h^2 + \Delta t), \tag{3.44}$$

$$|||\nabla(y - y_h)|||_{L^\infty(L^2)} + |||\nabla(p - p_h)|||_{L^\infty(L^2)} \leq C(h + \Delta t), \tag{3.45}$$

$$|||u - u_h|||_{L^2(L^2)} \leq C(h + \Delta t). \tag{3.46}$$

Proof. The proof of the theorem can be completed by using Lemmas 3.1–3.5, (2.17) and the triangle inequality.

To provide the global superconvergence for the control and state, we use the recovery techniques on uniform meshes. Let us construct the recovery operators P_h and G_h. Let $P_h v$ be a continuous piecewise linear function (without the zero boundary constraint). The value of $P_h v$ on the nodes are defined by a least squares argument on element patches surrounding the nodes; the details can be found in [25, 26].

We construct the gradient recovery operator $G_h v = (P_h v_x, P_h v_y)$ for the gradients of y and p. In the piecewise linear case, it is noted to be the same as the Z-Z gradient recovery (see [25, 26]). We construct the discrete co-state with the admissible set

$$\hat{u}_h^n = \max\{0, \overline{p_h^{n-1}}\} - p_h^{n-1}. \tag{3.47}$$

Now, we can derive the global superconvergence result for the control variable and state variable.

Theorem 3.2. *Let u and u_h^n be the solutions of (2.4)–(2.8) and (2.29)–(2.32), respectively. Assume that all of the conditions in Lemmas 3.1–3.5 are valid. Then we have*

$$|||u - \hat{u}_h|||_{L^2(L^2)} \leq C(h^2 + \Delta t). \tag{3.48}$$

Proof. Using (2.9), (3.47) and Theorem 3.1, we have

$$|||u - \hat{u}_h|||^2_{L^2(L^2)} = \sum_{n=1}^{N} \|u^n - \hat{u}_h^n\|^2 \Delta t$$

$$
\begin{aligned}
&\le C\sum_{n=1}^{N}\|\max\{0,\overline{p^n}\}-\max\{0,\overline{p_h^{n-1}}\}\|^2\Delta t + C\sum_{n=1}^{N}\|p^n-p_h^{n-1}\|^2\Delta t\\
&\le C\sum_{n=1}^{N}\|\overline{p^n}-\overline{p_h^{n-1}}\|^2\Delta t + C\sum_{n=1}^{N}\|p^n-p_h^{n-1}\|^2\Delta t\\
&\le C\sum_{n=1}^{N}\|p^n-p_h^{n-1}\|^2\Delta t\\
&\le C\sum_{n=1}^{N}\|p^n-p^{n-1}\|^2\Delta t + C\sum_{n=1}^{N}\|p^{n-1}-p_h^{n-1}\|^2\Delta t\\
&\le C(h^4+(\Delta t)^2).
\end{aligned}
\tag{3.49}
$$

Theorem 3.3. *Let (y,p) and (y_h^n, p_h^{n-1}) be the solutions of (2.4)–(2.8) and (2.29)–(2.32), respectively. Assume that all of the conditions in Lemmas 3.1–3.5 are valid. Then we have*

$$|||G_h y_h - \nabla y|||_{L^\infty(L^2)} + |||G_h p_h - \nabla p|||_{L^\infty(L^2)} \le C(h^2+\Delta t). \tag{3.50}$$

Proof. Notice that

$$|||G_h y_h - \nabla y|||_{L^\infty(L^2)} \le |||G_h y_h - G_h R_h y|||_{L^\infty(L^2)} + |||G_h R_h y - \nabla y|||_{L^\infty(L^2)}. \tag{3.51}$$

It follows from Lemma 3.6 that

$$|||G_h y_h - G_h R_h y|||_{L^\infty(L^2)} \le C|||\nabla(y_h - R_h y)|||_{L^\infty(L^2)} \le C(h^2+\Delta t). \tag{3.52}$$

It can be proved by the standard interpolation error estimate technique (see [1]) that

$$|||G_h R_h y - \nabla y|||_{L^\infty(L^2)} \le Ch^2. \tag{3.53}$$

Therefore, it follows from (3.52) and (3.53) that

$$|||G_h y_h - \nabla y|||_{L^\infty(L^2)} \le C(h^2+\Delta t). \tag{3.54}$$

Similarly, it can be proved that

$$|||G_h p_h - \nabla p|||_{L^\infty(L^2)} \le C(h^2+\Delta t). \tag{3.55}$$

Therefore, we complete the proof.

4. Two-grid scheme

In this section, we will present a two-grid scheme and analyze a priori error estimates. Now, we present our two-grid algorithm which has the following two steps:

Step 1. On the coarse grid T_H, find $(y_H^n, p_H^{n-1}, u_H^n) \in V_H^2 \times K_H$ that satisfies the following optimality conditions:

$$
\begin{aligned}
&(dty_H^n, v_H) + (A\nabla y_H^n, \nabla v_H)\\
&= \left(\sum_{i=1}^{n} \Delta t B(t_n, t_{i-1})\nabla y_H^i, \nabla v_H\right) + (f^n + u_H^n, v_H),\ \forall\ v_H\ \in V_H, && (4.1)\\
&y_H^0 = R_H y^0, && (4.2)\\
&-(dtp_H^n, q_H) + (A\nabla p_H^{n-1}, \nabla q_H)\\
&= \left(\sum_{i=n}^{N} \Delta t B^*(t_i, t_{n-1})\nabla p_H^{i-1}, \nabla q_H\right) + (y_H^n - y_d^n, q_H),\ \forall\ q_H\ \in V_H, && (4.3)\\
&p_H^N = 0, && (4.4)\\
&(u_H^n + p_H^{n-1}, u_H^* - u_H^n) \geq 0,\ \forall\ u_H^* \in K_H. && (4.5)
\end{aligned}
$$

Step 2. On the fine grid T_h, find $(\overline{\overline{y}}_h^n, \overline{\overline{p}}_h^{n-1}, \overline{\overline{u}}_h^n) \in V_h^2 \times K_h$ such that

$$
\begin{aligned}
&(dt\overline{\overline{y}}_h^n, v_h) + (A\nabla\overline{\overline{y}}_h^n, \nabla v_h)\\
&= \left(\sum_{i=1}^{n} \Delta t B(t_n, t_{i-1})\nabla\overline{\overline{y}}_h^i, \nabla v_h\right) + (f^n + \hat{u}_H^n, v_h),\ \forall\ v_h\ \in V_h, && (4.6)\\
&\overline{\overline{y}}_h^0 = R_h y^0, && (4.7)\\
&-(dt\overline{\overline{p}}_h^n, q_h) + (A\nabla\overline{\overline{p}}_h^{n-1}, \nabla q_h)\\
&= \left(\sum_{i=n}^{N} \Delta t B^*(t_i, t_{n-1})\overline{\overline{p}}_h^{i-1}, \nabla q_h\right) + (\overline{\overline{y}}_h^n - y_d^n, q_h),\ \forall\ q_h\ \in V_h, && (4.8)\\
&\overline{\overline{p}}_h^N = 0, && (4.9)\\
&(\overline{\overline{u}}_h^n + \overline{\overline{p}}_h^{n-1}, u_h^* - \overline{\overline{u}}_h^n) \geq 0,\ \forall\ u_h^* \in K_h. && (4.10)
\end{aligned}
$$

Combining Theorem 3.1 and the stability estimates, we easily get the following results.

Theorem 4.1. *Let (y, p, u) and $(\overline{\overline{y}}_h^n, \overline{\overline{p}}_h^n, \overline{\overline{u}}_h^n)$ be the solutions of (2.4)–(2.8) and (4.1)–(4.10), respectively. Assume that y, y_d, p, p_d and u have enough regularities for our purpose; then, for Δt small enough and $1 \leq n \leq N$, we have*

$$
\begin{aligned}
&|||\nabla(y - \overline{\overline{y}}_h)|||_{L^\infty(L^2)} + |||\nabla(p - \overline{\overline{p}}_h)|||_{L^\infty(L^2)} \leq C(h + H^2 + \Delta t), && (4.11)\\
&|||u - \overline{\overline{u}}_h|||_{L^2(L^2)} \leq C(h + H^2 + \Delta t). && (4.12)
\end{aligned}
$$

Proof. For convenience, let

$$
\begin{aligned}
y^n - \overline{\overline{y}}_h^n &= y^n - R_h y^n + R_h y^n - \overline{\overline{y}}_h^n =: \eta_y^n + e_y^n,\\
p^n - \overline{\overline{p}}_h^n &= p^n - R_h p^n + R_h p^n - \overline{\overline{p}}_h^n =: \eta_p^n + e_p^n.
\end{aligned}
$$

Taking $t = t_n$ in (2.4), subtracting (4.6) from (2.4) and then using (2.12), we have

$$\begin{aligned}
&\left(dte_y^n, v_h\right) + \left(A\nabla e_y^n, \nabla v_h\right) \\
=&\left(\int_0^{t_n} B(t_n, s)R_h\nabla y(s)ds - \sum_{i=1}^{n} \Delta t B(t_n, t_{i-1})\nabla\overline{\overline{y}}_h^i, \nabla v_h\right) + (dty^n - y_t^n, v_h) - \left(dt\eta_y^n, v_h\right) \\
&+ (u^n - \hat{u}_H^n, v_h), \ \forall\, v_h \in V_h.
\end{aligned} \tag{4.13}$$

Selecting $v_h = dte_y^n$ in (4.13), multiplying by Δt and summing over n from 1 to l $(1 \le l \le N)$ on both sides of (4.13), we find that

$$\begin{aligned}
&\frac{1}{2}\|A^{\frac{1}{2}}\nabla e_y^l\|^2 + \sum_{n=1}^{l}\|dte_y^n\|^2\Delta t \\
\le& -\sum_{n=1}^{l}\left(dt\eta_y^n, dte_y^n\right)\Delta t + \sum_{n=1}^{l}\left(dty^n - y_t^n, dte_y^n\right)\Delta t \\
&+ \sum_{n=1}^{l}\left(\int_0^{t_n} B(t_n, s)R_h\nabla y(s)ds - \sum_{i=1}^{n}\Delta t B(t_n, t_{i-1})\nabla\overline{\overline{y}}_h^i, dt\nabla e_y^n\right)\Delta t \\
&+ \sum_{n=1}^{l}(u^n - \hat{u}_H^n, dte_y^n)\Delta t \\
=:& \sum_{i=1}^{4} I_i.
\end{aligned} \tag{4.14}$$

Similar to Lemma 3.1, it is easy to show that

$$I_1 + I_2 \le Ch^4\|y_t\|_{L^2(H^2)}^2 + C(\Delta t)^2\|y_{tt}\|_{L^2(L^2)}^2 + \frac{1}{2}\sum_{n=1}^{l}\|dte_y^n\|^2\Delta t. \tag{4.15}$$

Similar to A_3, we find that

$$\begin{aligned}
I_3 \le& C(\Delta t)^2(\|\nabla R_h y_t\|_{L^2(L^2)}^2 + \|\nabla R_h y\|_{L^2(L^2)}^2) + C\sum_{n=1}^{l}\|\nabla e_y^n\|^2\Delta t \\
&+ C\sum_{n=1}^{l}\Delta t\sum_{i=1}^{n}\|\nabla e_y^i\|^2\Delta t + \frac{a_*}{4}\|\nabla e_y^l\|^2.
\end{aligned} \tag{4.16}$$

For I_4, using Theorem 3.2, we have

$$I_4 \le C(H^4 + (\Delta t)^2) + \frac{1}{4}\sum_{n=1}^{l}\|dte_y^n\|^2\Delta t. \tag{4.17}$$

Combining (4.15)–(4.17), the discrete Gronwall inequality, the triangle inequality and (2.13), we get

$$|||\nabla(y - \overline{\overline{y}}_h)|||_{L^\infty(L^2)} \le C(h + H^2 + \Delta t). \tag{4.18}$$

By taking $t = t_{n-1}$ in (2.6), subtracting (4.8) from (2.6) and using (2.12), we have

$$
\begin{aligned}
&-(dte_p^n, q_h) + (A\nabla e_p^{n-1}, \nabla q_h) \\
=&\Big(\int_{t_{n-1}}^{T} B^*(s, t_{n-1})\nabla R_h p(s)ds - \sum_{i=n}^{N} B^*(t_i, t_{n-1})\overline{\overline{p}}_h^{\,i-1}\Delta t, \nabla q_h\Big) - (dtp^n - p_t^{n-1}, q_h) \\
&+ (dt\eta_p^n, q_h) + (\delta y_d^n - \delta y^n, q_h) + (y^n - \overline{\overline{y}}_h^{\,n}, q_h), \ \forall\, q_h \in V_h.
\end{aligned} \tag{4.19}
$$

By selecting $q_h = -dte_p^n$ in (4.19), multiplying by Δt and summing over n from $l+1$ to N $(0 \le l \le N-1)$ on both sides of (4.19), we find that using (2.15), (4.18) and the triangle inequality, similar to (3.14), gives

$$
|||\nabla(p - \overline{\overline{p}}_h)|||_{L^\infty(L^2)} \le C(h + H^2 + \Delta t). \tag{4.20}
$$

Note that

$$
\begin{aligned}
\overline{\overline{u}}_h^{\,n} &= \max\{0, \overline{\overline{\overline{p}}_h^{\,n-1}}\} - \pi_h \overline{\overline{p}}_h^{\,n-1}, \\
u^n &= \max\{0, \overline{p^n}\} - p^n.
\end{aligned}
$$

Using (2.19), (4.20) and the mean value theorem, we have

$$
\begin{aligned}
|||u - \overline{\overline{u}}_h|||_{L^2(L^2)}^2 =& \sum_{n=1}^{N} \|u^n - \overline{\overline{u}}_h^{\,n}\|^2 \Delta t \\
\le& C\sum_{n=1}^{N} \|\max\{0, \overline{p^n}\} - \max\{0, \overline{\overline{\overline{p}}_h^{\,n-1}}\}\|^2 \Delta t + C\sum_{n=1}^{N} \|p^n - \pi_h \overline{\overline{p}}_h^{\,n-1}\|^2 \Delta t \\
\le& C\sum_{n=1}^{N} \|\overline{p^n} - \overline{\overline{\overline{p}}_h^{\,n-1}}\|^2 \Delta t + C\sum_{n=1}^{N} \|p^n - p^{n-1}\|^2 \Delta t \\
&+ C\sum_{n=1}^{N} \|p^{n-1} - \pi_h p^{n-1}\|^2 \Delta t + C\sum_{n=1}^{N} \|\pi_h p^{n-1} - \pi_h \overline{\overline{p}}_h^{\,n-1}\|^2 \Delta t \\
\le& C\sum_{n=1}^{N} \|p^n - \overline{\overline{p}}_h^{\,n-1}\|^2 \Delta t + C\sum_{n=1}^{N} \|p^n - p^{n-1}\|^2 \Delta t \\
&+ C\sum_{n=1}^{N} \|p^{n-1} - \pi_h p^{n-1}\|^2 \Delta t + C\sum_{n=1}^{N} \|\pi_h p^{n-1} - \pi_h \overline{\overline{p}}_h^{\,n-1}\|^2 \Delta t \\
\le& C\sum_{n=1}^{N} \|p^n - p^{n-1}\|^2 \Delta t + C\sum_{n=1}^{N} \|p^{n-1} - \pi_h p^{n-1}\|^2 \Delta t \\
&+ C\sum_{n=1}^{N} \|p^{n-1} - \overline{\overline{p}}_h^{\,n-1}\|^2 \Delta t \\
\le& C(h^2 + H^4 + (\Delta t)^2),
\end{aligned} \tag{4.21}
$$

which completes the proof.

5. Numerical experiments

In this section, we present the following numerical experiment to verify the theoretical results. We consider the following two-dimensional parabolic integro-differential optimal control problems

$$\min_{u\in K}\left\{\frac{1}{2}\int_0^1(\|y-y_d\|^2+\|u\|^2)dt\right\}$$

subject to

$$\begin{aligned}(y_t,v)+(\nabla y,\nabla v)&=\int_0^t(\nabla y(s),\nabla v)ds+(f+u,v),\ \forall\ v\in V,\\ y(x,0)&=y_0(x),\ \forall\ x\in\Omega,\end{aligned}$$

where $\Omega=(0,1)^2$.

We applied a piecewise linear finite element method for the state variable y and co-state variable p. The stopping criterion of the finite element method was chosen to be the abstract error of control variable u between two adjacent iterates less than a prescribed tolerance, i.e.,

$$\|u_h^{l+1}-u_h^l\|\le\epsilon,$$

where $\epsilon=10^{-5}$ was used in our numerical tests. For the linear system of equations, we used the algebraic multigrid method with tolerance 10^{-9}.

The numerical experiments were conducted on a desktop computer with a 2.6 GHz 4-core Intel i7-6700HQ CPU and 8 GB 2133 MHz DDR4 memory. The MATLAB finite element package iFEM was used for the implementation [27].

Example: We chose the following source function f and the desired state y_d as

$$\begin{aligned}f(x,t)&=\left(2e^{2t}+4\pi^2e^{2t}+4\pi^2+\sin(\pi t)\right)\sin(\pi x)\sin(\pi y)-\frac{4}{\pi^2}\sin(\pi t),\\ y_d(x,t)&=\left(\pi\cos(\pi t)-8\pi^2\sin\pi t+8\pi^2(\frac{\cos(\pi t)}{\pi})-\frac{\cos\pi T}{\pi}+e^{2t}\right)\sin(\pi x)\sin(\pi y)\end{aligned}$$

such that the exact solutions for y, p, u are respectively,

$$\begin{aligned}y&=e^{2t}\sin(\pi x)\sin(\pi y),\\ p&=\sin(\pi t)\sin(\pi x)\sin(\pi y),\\ u&=\sin(\pi t)\Big(\frac{4}{\pi^2}-\sin(\pi x)\sin(\pi y)\Big).\end{aligned}$$

In order to see the convergence order with respect to time step size $\triangle t$ and mesh size h, we choose $\triangle t=h$ or $\triangle t=h_2$ with $h=\frac{1}{4},\frac{1}{16},\frac{1}{64}$. To see the convergence order of the two-grid method, we choose the coarse and fine mesh size pairs $(\frac{1}{2},\frac{1}{4}),(\frac{1}{4},\frac{1}{16}),(\frac{1}{8},\frac{1}{64})$. Let us use y^h, p^h and u^h as two-grid solutions in the following tables. In Tables 1 and 2, we let $\triangle t=h_2$ and present the errors of the finite element method and two-grid method for y and p in the L^2-norm. Next, in Tables 3 and 4, we set $\triangle t=h$ and show the errors of the two methods for y and p in the H^1-norm and u in the L^2-norm. We can see that the two-grid method maintains the same convergence order as the finite element method. Moreover, we also display the computing times of the finite element method and the two-grid method in these tables. By comparison, we find that the two-grid method is more effective for solving the optimal control problems (1.1)–(1.4).

Table 1. Errors of finite element method with $\triangle t = h^2$ at $t = 0.5$.

h	$\|\|y - y_h\|\|$	$\|\|p - p_h\|\|$	$CPU\ time\ (s)$
$\frac{1}{4}$	0.1095	0.0856	0.7031
$\frac{1}{16}$	0.0079	0.0045	8.8702
$\frac{1}{64}$	0.0005	0.0002	2253.6396

Table 2. Errors of two-grid method with $\triangle t = h^2$ at $t = 0.5$.

(h, H)	$\|\|y - y^h\|\|$	$\|\|p - p^h\|\|$	$CPU\ time\ (s)$
$(\frac{1}{4}, \frac{1}{2})$	0.1059	0.0853	0.4335
$(\frac{1}{16}, \frac{1}{4})$	0.0056	0.0043	5.0842
$(\frac{1}{64}, \frac{1}{8})$	0.0006	0.0002	1027.9740

Table 3. Errors of finite element method with $\triangle t = h$ at $t = 0.5$.

h	$\|\|y - y_h\|\|_1$	$\|\|p - p_h\|\|_1$	$\|\|u - u_h\|\|$	$CPU\ time\ (s)$
$\frac{1}{4}$	1.6604	1.1385	0.1358	0.4720
$\frac{1}{16}$	0.6187	0.2143	0.0367	0.6320
$\frac{1}{64}$	0.1687	0.0578	0.0090	24.0800

Table 4. Errors of two-grid method with $\triangle t = h$ at $t = 0.5$.

(h, H)	$\|\|y - y^h\|\|_1$	$\|\|p - p^h\|\|_1$	$\|\|u - u^h\|\|$	$CPU\ time\ (s)$
$(\frac{1}{4}, \frac{1}{2})$	1.6755	1.1375	0.0988	0.2880
$(\frac{1}{16}, \frac{1}{4})$	0.6288	0.2142	0.0346	0.3870
$(\frac{1}{64}, \frac{1}{8})$	0.1716	0.0579	0.0089	7.3120

6. Conclusions

In this paper, we presented a two-grid finite element scheme for linear parabolic integro-differential control problems (1.1)–(1.4). A priori error estimates for the two-grid method and finite element method have been derived. We have used recovery operators to prove the superconvergence results.

These results seem to be new in the literature. In our future work, we will investigate a posteriori error estimates. Furthermore, we shall consider a priori error estimates and a posteriori error estimates for optimal control problems governed by hyperbolic integro-differential equations.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare there is no conflict of interest.

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