Topics in contact Hamiltonian systems

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Chapter 3. Some examples of Hamiltonian systems

In this chapter, we give some physical examples of systems that can be described with contact Hamiltonian mechanics and contact Lagrangian mechanics. We consider three classes of systems on $(M = \mathbb{R}^{2n+1}, \eta = ds - p_i dq^i)$:

- Projectable systems: this class includes systems whose "Newtonian" differential equation is of the form

  $$\ddot{x} = f(x, \dot{x}),$$

  and the configuration dynamics is independent of the extra variable s.
- Non-Projectable systems: systems whose dynamical equations are explicitly dependent on s. Even if it seems physically counterintuitive, it turns out that these systems caught attention from a cosmological perspective [41, 40, 46].
- Liénard Systems: non-linear oscillations exhibit some peculiar behaviour, such as limit-cycles or chaos. It is possible to describe them as contact Hamiltonian systems [11]. For instance, Liénard equation can be lifted to a contact Hamiltonian system in $(M, \eta = ds - pdq)$.
- In the last section, we give the explicit form of the Hamiltonian systems on different contact manifolds: the 3-torus $\mathbb{T}^3$ and the 3-sphere $\mathbb{S}^3$.

For each of these classes, some benchmark systems are then introduced, these examples will be analysed with numerical and analytical techniques in the following chapters.

3.1. Projectable systems: Lagrangian description

Let us consider the second order ODE on $\mathbb{R}$:

$$\ddot{x} + f(x)\dot{x}^2 + g(t)\dot{x} + h(x) = 0, \tag{3.1}$$

Clearly, this can be rewritten as a first-order system involving only x and $\dot{x}$, without the need of additional degrees of freedom. In the following proposition, however, we show that introducing an external variable z equation (3.1) has a natural description in the contact manifold $(M, \eta = ds - \frac{\partial \mathcal{L}}{\partial \dot{x}} dx)$.

Proposition 3.1.1. *The differential equation*

$$\ddot{x} + f(x)\dot{x}^2 + g(t)\dot{x} + h(x) = 0, \tag{3.2}$$

can be obtained by the contact Lagrangian on $\mathbb{R} \times \mathbb{R}^3$

$$\mathcal{L}(t, x, \dot{x}, s) = \frac{1}{2}\dot{x}^2 - (2f(x)\dot{x} + g(t))s - U(x); \tag{3.3}$$

where $U(x)$ *solves the differential equation*

$$\frac{dU(x)}{dx} + 2f(x)U(x) = h(x). \tag{3.4}$$

Remark 13. *If we consider two solutions of the differential equation* (3.4)*, with different initial conditions in* x_0*, we recover the same equation of motion* (3.2) *but two different Lagrangians. This difference is inherited only into the dynamics of the variable* s*, while on* $(x, \dot{x})$ *it is irrelevant.*

Proof. Equation (3.2), immediately follows from Euler-Lagrange equations (Theorem 2.4.1) and a direct computation:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \dot{x}} &= \dot{x} - 2f(x)z, \\
\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} &= \ddot{x} - 2\frac{df(x)}{dx}\dot{x} - 2f(x)\mathcal{L} \\
&= \ddot{x} - 2\frac{df(x)}{dx}\dot{x}z - 2f(x)\left(\frac{1}{2}\dot{x}^2 - (2f(x)\dot{x} + g(t))z - U(x)\right), \\
\frac{\partial \mathcal{L}}{\partial x} &= -2\frac{df(x)}{dx}\dot{x}z - \frac{dU(x)}{dx}, \\
\frac{\partial \mathcal{L}}{\partial z}\frac{\partial \mathcal{L}}{\partial \dot{x}} &= -\left(\dot{x} - 2f(x)z\right)\left(2f(x) + g(t)\right).
\end{aligned}$$

□

In fact, the same construction works also for higher-dimensional version of (3.1). Let us consider:

$$\ddot{x}^\mu - f^\mu(x)\dot{x}^\nu\dot{x}^\nu + 2f^\nu(x)\dot{x}^\nu\dot{x}^\mu + g(t)\dot{x}^\mu + h^\mu(x) = 0. \tag{3.5}$$

where $x : \mathbb{R} \to \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R} \to \mathbb{R}$, and $h : \mathbb{R}^n \to \mathbb{R}^n$. It is easy to see that for $n = 1$ equation (3.5) reduces to equation (3.1). Also in this case the motion is projectable in the following sense.

Proposition 3.1.2. *The differential equation* (3.5) *can be recovered by a contact Lagrangian system given by the Lagrangian function defined on* $\mathbb{R} \times \mathbb{R}^{2n} \times \mathbb{R}$*:*

$$\mathcal{L}(t, x, \dot{x}, s) = \frac{1}{2}\dot{x}^\nu\dot{x}^\nu - \left(2f^\nu(x)\dot{x}^\nu + g(t)\right)s - U(x); \tag{3.6}$$

where $h^\nu(x) = \frac{\partial U(x)}{\partial x^\nu} + f^\nu U(x)$*, and* $\frac{\partial f^\nu}{\partial x^\mu} = \frac{\partial f^\mu}{\partial x^\nu}$*.*

Proof. To recover equation (3.5), we directly compute the Euler-Lagrange equation (2.15):

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} &= \dot{x}^\mu - 2f^\mu(x)s, \\
\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial x^\mu} &= \ddot{x}^\mu - 2\frac{\partial f^\mu(x)}{\partial x^\xi}\dot{x}^\xi s - 2f^\mu \mathcal{L} = \\
&= \ddot{x}^\mu - 2\frac{\partial f^\mu(x)}{\partial x^\xi}\dot{x}^\xi s - 2f^\mu\left(\frac{1}{2}\dot{x}^\nu\dot{x}^\nu - \left(2f^\nu(x)\dot{x}^\nu + g(t)\right)s - U(x)\right), \\
\frac{\partial \mathcal{L}}{\partial x^\mu} &= -\frac{\partial U(x)}{\partial x^\mu} - 2\frac{\partial f^\nu(x)}{\partial x^\mu}\dot{x}^\nu s, \\
\frac{\partial \mathcal{L}}{\partial s} &= -2f^\mu(x)\dot{x}^\mu - g(t), \\
\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}\frac{\partial \mathcal{L}}{\partial s} &= -\left(+2f^\xi(x)\dot{x}^\xi + g(t)\right)\left(\dot{x}^\mu - 2f^\mu(x)s\right) = \\
&= -\left(2f^\xi(x)\dot{x}^\xi\dot{x}^\mu + g(t)\dot{x}^\mu - 4f^\xi(x)f^\mu(x)\dot{x}^\xi s - 2f^\mu(x)sg(t)\right).
\end{aligned}$$

Combining the previous terms in order to recover the contact Euler Lagrange equation we find out that the only term that is dependent on the variable s is

$$2\dot{x}^\nu\left(\frac{\partial f^\nu(x)}{\partial x^\mu} - \frac{\partial f^\mu(x)}{\partial x^\nu}\right);$$

that is zero if the second condition in the hypothesis is fulfilled. □

3.2. Projectable systems: Hamiltonian description

We want to describe the differential equation (3.1) from a contact Hamiltonian perspective. We compute the Legendre transform of the Lagrangian (3.3) to obtain the Hamiltonian description on the lift of the contact manifold $(\mathbb{R}^3, \eta = dz - pdx)$.

Proposition 3.2.1. *The Hamiltonian function on* $(\mathbb{R}\times\mathbb{R}^3), \eta = dz - pdx$ *related to the differential equation* (3.1)*, is*

$$H = \frac{1}{2}(p + 2f(x)\ z)^2 + g(t)z + U(x). \tag{3.7}$$

Proof. The momentum $p = p(x, \dot{x}, z)$ is defined by:

$$p = \frac{\partial \mathcal{L}}{\partial \dot{x}} = \dot{x} - 2f(x)z;$$

therefore $\dot{x}(x,p,z) = p + 2f(x)z$, and $p\dot{q} = p^2 + 2f(x)pz$. The Legendre transform is

$$\begin{aligned}
H(t,x,p,z) &= p\dot{x} - \mathcal{L}(t,x,\dot{x},z) \\
&= p^2 + 2f(x)pz - \frac{(p+2f(x)z)^2}{2} + \left(2f(x)\left(p + 2f(x)z\right) + g(t)\right)z + U(x) \\
&= \frac{1}{2}(p + 2zf(x))^2 + zg(t) + U(x),
\end{aligned}$$

for which we can compute the contact Hamilton equations:

$$\begin{cases} \dot{x} = p + 2zf(x) \\ \dot{p} = -2zf'(x)(2zf(x) + p) - U'(x) \\ \dot{z} = +\frac{p^2}{2} - z\left(2zf(x)^2 + g(t)\right) - U(x) \end{cases}$$

where $\mathcal{R}(\mathcal{H}) = 2f(x)(2zf(x) + p) + g(t)$. We can recover the equation of motion (3.1) by computing the time derivative $\ddot{x}$. □

It is interesting to see that even if the Hamiltonian (3.7) corresponds to a Lagrangian function linear in z and with the dynamics in $(x, \dot{x})$ independent of z, its dependence on z far from obvious. The Hamiltonian is quadratic in z and the dynamics in the subspace (x, p) is not independent of z.

In analogy with the previous section, we can carry another construction for ODEs in $\mathbb{R}^{2n} \times \mathbb{R}$. The Legendre transform of the Lagrangian (3.6) leads to the corresponding Hamiltonian system. The momentum associated to the velocity $\dot{x}^\mu$ is

$$p^\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = \dot{x}^\mu - 2f^\mu(x)s,$$

and to the Hamiltonian function is

$$\begin{aligned} \mathcal{H} &= \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}\dot{x}^\mu - \mathcal{L} = \\ &= \frac{\left(p^\mu + 2f^\mu(x)s\right)^2}{2} + g(t)s + U(x). \end{aligned}$$

The corresponding equations of motion are

$$\begin{cases} \dot{q}^\mu = p^\mu + 2f^\mu(x)s \\ \dot{p}^\mu = -\dfrac{\partial U(x)}{\partial x^\nu} - (p^\mu + 2f^\mu(x)s)\left(2\dfrac{\partial f^\mu}{\partial x^\nu}s + 2p^\mu f^\mu(x)\right) \\ \dot{s} = +\dfrac{p^\mu p^\mu}{2} - z\left(2zf(x)^2 + g(t)\right) - U(x) \end{cases}.$$

As observed in the previous example, also in this case, we find a contact Hamiltonian function that is quadratic in the action term but whose dynamics can be projected to the tangent bundle of the configuration space.

3.3. A dissipative dynamical system

A particular family of projectable systems are the dissipative dynamical systems. These are described by an equation similar to (3.5) where $f(x)$ (or $f^\nu(x)$ if we are in n-dimensions) is identically zero, and the potential can be time dependent. The dynamics is then described by the ODE:

$$\ddot{x}^\nu + g(t)\dot{x}^\nu + h^\nu(x,t) = 0; \tag{3.8}$$

which corresponds to Newton's equation for a system with time-dependent forcing and Rayleigh dissipation.

Proposition 3.3.1. *Equation* (3.8) *corresponds to the flow of the contact Hamiltonian*

$$\mathcal{H}(t,q,p_a,s) = \sum_{a=1}^{n} \frac{p_a^2}{2} + V(q,t) + f(t)\,s\,. \tag{3.9}$$

Proof. To prove the statement, observe that Hamilton's equations (2.19) in this case read

$$\begin{aligned} \dot{q}_a &= p_a \\ \dot{p}_a &= -\frac{\partial V(q,t)}{\partial q^a} - f(t)p_a \\ \dot{s} &= \sum_{a=1}^{n} \frac{p_a^2}{2} - V(q,t) - f(t)\,s\,. \end{aligned}$$

The system (3.10)–(3.11), gives exactly (3.8), while equation (3.12) decouples from the rest. □

Using contact geometry, we have immediately obtained a "Hamiltonisation" of all the dynamical systems of the form (3.8). This fact should not be underestimated: a Hamiltonian structure for all such systems allows us to benefit from the theory of Hamiltonian systems (extended to the contact case) and its powerful analytical and numerical tools. For instance, one can apply weak–KAM theorems and variational methods, as done e.g. in [47, 31, 48, 49].

It is important to also compare the simplicity and generality of the formulation provided here against previous attempts in the literature. For instance, in [50] an algorithm for the symplectic Hamiltonisation of systems of the type (3.8) has been provided. However, the construction suggested there is based on a non–trivial reparametrisation that requires solving an additional differential equation in order to obtain the new time variable (which in many cases cannot be done exactly, cf. [50]). We stress that in our analysis we do not encounter any such complication.

3.3.1. General Rayleigh dissipation

In autonomous symplectic Lagrangian and Hamiltonian mechanics, it is well known that the dissipation functions cannot be included in the Lagrangian nor in the Hamiltonian by maintaining the time-independence [8]. In practice, the generalized d'Alambert principle allows us to consider a modification of the Euler-Lagrange equation [8]

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}^\mu} - \frac{\partial \mathcal{L}}{\partial \dot{q}^\mu} = Q_\mu,$$

where the right-hand side represents the forces that do not arise from a potential, and thus cannot be written as gradients of a potential function. These usually represent friction forces. The Rayleigh dissipation is one example. It is described by damping terms

$$Q_\mu = -\frac{\partial}{\partial \dot{q}^\mu}\left(\frac{k_i \dot{q}_i^{\mu\ 2}}{2}\right).$$

For a free particle subject to this kind of dissipation, we obtain Newton's equations

$$m\ddot{q}^{\mu} = -k_i \dot{q}_i^{\mu}.$$

The same equations can be recovered by the contact technique, as we have shown in the previous section 3.3. In the next section we present some mechanical systems with Rayleigh dissipation which will come back multiple times in this manuscript.

3.3.2. Linear Damped Particle

The first example is the linear-damped particle: a free particle of mass m under a linear dissipation dependent on the velocity, characterized by a parameter γ. The equation of motion then is:

$$m\ddot{x} + \gamma\dot{x} = 0,$$

the solution and the characterization is well known from every bachelor physics book. The contact Hamiltonian is provided by exploiting the results of section 3.3.1 or section 3.3, and it reads

$$\mathcal{H}_{LDO} = \frac{p^2}{2m} + \gamma s;$$

The corresponding equations of motion are:

$$\begin{cases} \dot{q} = \dfrac{p}{m} \\ \dot{p} = -\gamma p \\ \dot{s} = \dfrac{p^2}{2m} - \gamma s \end{cases}.$$

In this example we can apply the results of section 2.2 and observe that in this case the function in involution is p, and thus the ratio between the momenta and the Hamiltonian is a conserved quantity.

3.3.3. Damped Harmonic Oscillator

A classical example in physics is the harmonic oscillator, the prototypical complete integrable system [16, 51, 8, 17]. The inclusion of Rayleigh dissipation leads to the damped harmonic oscillator [52]. The "Newtonian" equation is:

$$m\ddot{x} + \gamma\dot{x} + kx = 0.$$

The corresponding contact Hamiltonian is

$$\mathcal{H}_{LDO} = \frac{p^2}{2m} + k\frac{q^2}{2} + \gamma s;$$

with Hamiltonian equations:

$$\begin{cases} \dot{q} = p/m \\ \dot{p} = -kq - \gamma p \\ \dot{s} = \dfrac{p^2}{2m} - k\dfrac{q^2}{2} - \gamma s \end{cases}.$$

The exact contact transformation

$$\begin{pmatrix} q \\ p \\ s \end{pmatrix} \xrightarrow{\phi} \begin{pmatrix} Q \\ P \\ S \end{pmatrix} = \begin{pmatrix} q\sqrt{\omega m} \\ \frac{p}{\sqrt{\omega m}} \\ s \end{pmatrix},$$

where $\omega = \sqrt{\frac{k}{m}}$, maps the contact Hamiltonian into

$$H = \omega(Q^2 + P^2) + \gamma S.$$

We can verify that it consist in an exact contact transformation by computing the pullback

$$\phi^*\eta' = \phi^*(dS - PdQ) = \eta = ds - pdq.$$

The new Hamiltonian generates the push-forward of the old Hamiltonian vector field. One can also consider the mapping:

$$\begin{pmatrix} Q \\ P \\ S \end{pmatrix} \to \begin{pmatrix} \tilde{Q} \\ \tilde{P} \\ \tilde{S} \end{pmatrix} = \begin{pmatrix} \frac{Q}{\sqrt{\omega}} \\ \frac{P}{\sqrt{\omega}} \\ \frac{S}{\omega} \end{pmatrix}$$

from which one obtain a new Hamiltonian:

$$\mathscr{H} = \frac{\tilde{P}^2}{2} + \frac{\tilde{Q}^2}{2} + \frac{\gamma}{\omega}\tilde{S}. \tag{3.13}$$

However, this is not an exact contactomorphism since the computation of the pull-back gives rise to a scale factor ω:

$$\phi^*\eta' = \phi^*(d\tilde{S} - \tilde{P}d\tilde{Q}) = \omega\eta = \omega\left(dS - PdQ\right).$$

In this case we preserve both the Reeb and the contact distributions, but the new Reeb vector field is the push-forward is scaled by the scale factor. This last Hamiltonian depends just on a parameter $\Gamma := \frac{\gamma}{\omega}$.

Remark 14. *The Newtonian equation of the damped harmonic oscillator is related to circuit theory. A RLC(Resistor - Inductor - Capacitor) circuit without a generator is described by*

$$\ddot{I} + 2\alpha\dot{I} + \omega_0^2 I = 0,$$

where the constants α and ω depend on the shape of the circuit and I is the current.

3.3.4. The Lane–Emden equation

The Lane–Emden equation

$$y''(x) + \frac{2}{x}y'(x) + y^n(x) = 0, \qquad y(0) = 1, \quad y'(0) = 0 \tag{3.14}$$

is a family of nonlinear singular equations widely used in physics to model isothermal gas spheres, such as e.g. stars [53]. In equation (3.14), y is a dimensionless variable related to the density of the star, and x is a dimensionless distance from the center, and the density is normalised so that the central density is 1. Finally, the integer n, called the barotropic index, depends on the nature of the gas.

Clearly, for $n \neq 0, 1$, the Lane–Emden equation is nonlinear. Besides, due to the $1/x$ term, it is singular in the initial condition. Therefore, the study of solutions of equation (3.14) is at the same time physically relevant and mathematically challenging. For this reason a large number of numerical schemes have been proposed in the literature, based on different approaches such as e.g. series expansions, spectral methods, perturbation techniques, neural network methods, and so on (see [54, 55] for comprehensive lists of the techniques used so far).

For us the relevance of (3.14) lies in the observation that such equation belongs to the class (3.9) and therefore it has a natural description in terms of contact geometry. The contact Hamiltonian for the Lane–Emden equation is

$$\mathscr{H}(x, y, p, s) = \frac{p^2}{2} + \frac{y^{n+1}}{n+1} + \frac{2}{x} s \,, \tag{3.15}$$

which is of the type (3.9).

3.3.5. Perturbed Kepler Problem

After planar reduction, the form of a perturbed Kepler problem is [56]

$$\ddot{x} + \frac{x}{|x|^3} = F(x, \dot{x}, t; \alpha),$$

where $x \in \mathbb{R}^2$. Here, the perturbation $F(x, \dot{x}, t; \alpha)$ is usually assumed to be either of the form $F = \alpha \partial V(x, t)/\partial x$, where $V(x, t)$ is a periodic function in t, or F may include different types of dissipation, the simplest one being a linear drag [57, 58, 59, 60]. In the case of a linear drag, the corresponding equation

$$\ddot{x} + \alpha \dot{x} + \frac{x}{|x|^3} = 0$$

is obviously of the form (3.9), with

$$\mathscr{H}(x, p, s) = \frac{|p|^2}{2} - \frac{1}{|x|} + \alpha\, s \,.$$

We refer to [59] for a detailed analysis of the dynamics in this case. Here, to show the usefulness of our integrators, we study the slightly more general case of a linear drag that also depends explicitly on time.

The equation for the modified Kepler problem that we consider is

$$\ddot{x} + \alpha \sin(\Omega t)\, \dot{x} + \gamma \frac{x}{|x|^3} = 0, \quad \alpha, \Omega, \gamma \in \mathbb{R}. \tag{3.16}$$

Clearly equation (3.16) is of the type (3.9), the corresponding contact Hamiltonian being

$$\mathscr{H}(x, p, s, t) = \frac{|p|^2}{2} - \frac{\gamma}{|x|} + \alpha \sin(\Omega t)\, s \,. \tag{3.17}$$

3.3.6. The Spin–Orbit model

In this section, we consider the so–called spin–orbit model in the version presented in [50], trying to use the same notation as in the referenced paper as much as possible.

This model describes the motion of a small body, e.g. a satellite, that moves around a larger body on a Keplerian orbit and rotates around its shorter principal axis with zero obliquity (see also [61]). The corresponding Newton equation is of the form (3.8) and is given by a second–order time–dependent differential equation in the angle that describes the relative orientation of the longer principal axis with respect to a preassigned direction. The time variations in the moment of inertia of the satellite introduce an angular velocity–dependent term that accounts for the body's rotation in addition to the external torques (in Figure 3.1 it is depicted the situation). More precisely, the equation

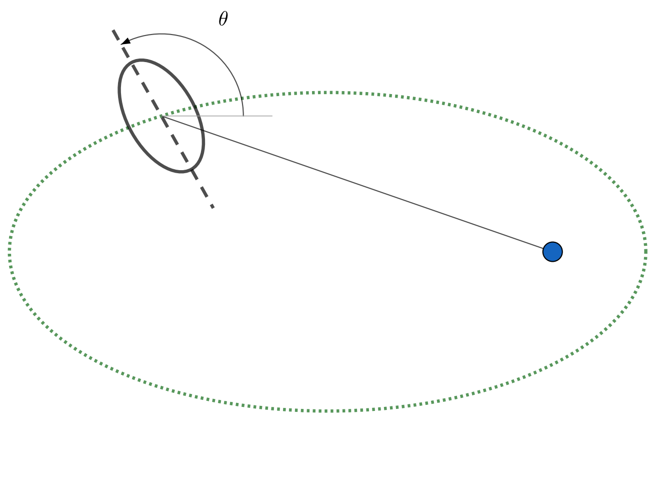

Figure 3.1: The figure depicts the spin orbit model: the green dotted line is the Kepler trajectory, the blue circle the big central body while the ellipsoid the extended body that orbits around the central one.

$$\frac{d\Gamma}{dt} = \frac{dC}{dt}\dot{\theta} + C\ddot{\theta} = N_z(\theta, t) \tag{3.18}$$

describes the rotation of the body around its principal axis, with moment of inertia

C. In the equation, Γ represents the angular momentum of the body, $\dot{\theta}$ the angular velocity and N_z the external torques.

For $C \neq 0$, equation (3.18) can be rewritten as

$$\ddot{\theta} + \frac{dC}{dt}\frac{\dot{\theta}}{C} - \frac{N_z(\theta,t)}{C} = 0\,,$$

which is clearly of the type (3.9), with contact Hamiltonian

$$\mathscr{H}(t,\theta,p,s) = \frac{p^2}{2} + \frac{N_z(\theta,t)}{C} + \frac{dC}{dt}\frac{1}{C}\,s\,. \tag{3.19}$$

In the examples that follow, as in [50], we will consider a moment of inertia that varies periodically around an average value $\widetilde{C}$ with frequency Ω, namely

$$C(t) = \widetilde{C} + \lambda\cos(\Omega t),$$

and we will focus on two particular forms of the torque:

- The gravitational torque for a triaxial rigid body on a Keplerian elliptical orbit around a point perturber:

$$\begin{aligned} N_z^{\text{triaxial}}(\theta,t) &= -\frac{3\nu(B-A)}{2}\frac{\alpha}{r}^3 \sin(2\theta-2f) \\ &= -\frac{3\nu(B-A)}{2}\sum_{m\in\mathbb{Z}\{0\}} W\left(\frac{m}{2},e\right)\sin(2\theta-mt) \end{aligned}$$

  where $A < B < C$ are the moments of inertia in the body frame, α the semi–major axis, ν the orbital frequency, r the distance between the bodies, f the true anomaly and $W(m/2,e)$ are the coefficients of the Fourier expansion w.r.t. the periodic functions r and t. We refer to [61] for a clear explanation of the terminology. Note in particular that the coefficients $W(m/2,e)$, called Cayley coefficients, are power series of the eccentricity: some of their values can be found in [61, Table 2.1] or [62, pp. 271–274]. In the examples we will truncate the series dropping all the powers of the eccentricity that give a contribution smaller than the error.

- The torque from a third body perturbation:

$$N_z^{\text{tidal}}(\theta,t) = \mu + a\dot{\theta}$$

  where $(\mu,a)\in\mathbb{R}_+\times\mathbb{R}_-$.

3.4. Non-projectable systems

We have seen at the beginning of this chapter how we can obtain quadratic Hamiltonians that are projectable on the tangent bundle of the configuration space. This is not always true; but even if the dynamics is not projectable, and we lose in a first instance the physical meaning of the system, the simplicity of the model aids us to figure out some global properties of the systems. Recently they obtained some physical relevance when they appeared in some cosmological models [41, 40] whose Lagrangians are defined on $(\mathbb{R}^{2n+1}, \eta = ds - pdq)$ on the form

$$\mathcal{L}(q_i, \dot{q}_i, s) = \overline{\mathcal{L}}(q_i, \dot{q}_i) + s^2.$$

The simplest example to consider is the traditional free-particle Lagrangian function $\overline{\mathcal{L}} = \sum_i \dot{q}_i^2/2$ or the harmonic oscillator one $(\overline{\mathcal{L}} = \sum_i \dot{q}_i^2/2 + q_i^2/2)$[12, 32].

3.4.1. Free particle quadratic action dependency

An interesting toy example is the free particle, where the Hamiltonian presents a quadratic term in s. The Hamiltonian system is described on $(M, \eta = ds - pdq)$ by the function

$$\mathcal{H}_{QFP} = \frac{p^2}{2m} + \gamma\frac{s^2}{2};$$

whose evolution equations are

$$\begin{cases} \dot{q} = p \\ \dot{p} = -ps \\ \dot{s} = \dfrac{p^2}{2m} - \gamma\dfrac{s^2}{2} \end{cases}.$$

The Hamiltonian is cyclic in the q variable, so in this case we recover the result presented in section 2.2: the momentum is in involution, and the ratio

$$k := \frac{\mathcal{H}}{p},$$

is a conserved quantity.

3.4.2. Harmonic Oscillator with quadratic action dependency

Motivated by the analysis in [63] and [41, 46], in this section we present contact Hamiltonians of the form

$$\mathcal{H}(p, q, s) = \sum_{a=1}^{n} \frac{p_a^2}{2} + \gamma\frac{s^2}{2} + V(q).$$

In particular, we consider the 1-dimensional quadratic contact harmonic oscillator

$$\mathcal{H}_{QHO} = \frac{p^2}{2m} + k\frac{q^2}{2} + \gamma\frac{s^2}{2} - 1 \tag{3.20}$$

defined on $(M = \mathbb{R}^3, \eta = ds - pdq)$. As shown by equation (2.3), the value of the contact Hamiltonian is not preserved unless its initial value is equal to zero [7]. This generally defines an (hyper)surface in the contact manifold that separates two invariant basins for the evolution. In the case at hand, the surface $\mathcal{H} = 0$ is a sphere with radius $\sqrt{2}$. Furthermore, the quadratic contact oscillator presents two equilibrium points of different nature on $\mathcal{H} = 0$: the stable north pole $N = \left(0, 0, \sqrt{2C\gamma^{-1}}\right)$ and the unstable south pole $S = \left(0, 0, -\sqrt{2C\gamma^{-1}}\right)$.

3.5. A contact Hamiltonian formulation of Liénard Systems

Liénard systems are a family of planar coupled differential equations on $\mathbb{R}^2$ of the form [64]

$$\begin{cases} \dot{x} = y - F(x) \\ \dot{y} = -g(x) \end{cases}, \tag{3.21}$$

where $F(x) = \int f(x)\, dx$ for an even function $f(x)$ and $g(x)$ is an odd function. Alternatively, (3.21) is equivalent to the second order scalar equation

$$\ddot{x} = -f(x)\dot{x} - g(x).$$

A third equivalent version of (3.21) is given by

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -g(x) - f(x)y\,. \end{cases} \tag{3.22}$$

Example 3.5.1 (The van der Pol oscillator)**.** *Perhaps the most famous example of the family of Liénard systems is the van der Pol oscillator, which can be written using dimensionless variables as follows*

$$\ddot{x} = \epsilon(1 - x^2)\dot{x} - x\,, \tag{3.23}$$

and can be equivalently rewritten in the form (3.22) *as*

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x + \epsilon(1 - x^2)y\,, \end{cases}$$

from which we recognise that in this case $f(x) = -\epsilon(1 - x^2)$ *and* $g(x) = x$*.*

A crucial property of Liénard systems is encoded in the following theorem, guaranteeing the existence and uniqueness of a stable limit cycle for a large class of systems [65].

Theorem 3.5.1 (Liénard's Theorem)**.** *Under the conditions*

- $F, g \in C^1(\mathbb{R})$*,*
- $xg(x) > 0$ *if* $x \neq 0$*,*
- $F(0) = 0$ *and* $f(0) < 0$*,*
- $F(x)$ *has exactly one positive zero at* $x = a$*, is monotone increasing for* $x > a$ *and* $\lim_{x\to+\infty} F(x) = +\infty$*;*

the dynamical system (3.21) *presents a unique, stable limit cycle.*

In particular, the theorem above implies that the van der Pol equation (3.23) with $\epsilon > 0$ has a unique, stable limit cycle. For additional information on the classical approach to the analysis of Liénard systems we refer to [65].

The Hamiltonian formulation can be recovered by a traditional approach [66]. It is well known that any dynamical system on a n-dimensional manifold Q of the form $\dot{x}^i = X^i(x)$ can be extended to a Hamiltonian system defined on the $2n$-dimensional phase–space T^*Q. This can be achieved with the introduction of fictitious conjugate momenta $\tilde{p}_i$ in order to define the Hamiltonian

$$H(x, \tilde{p}) = \tilde{p}_i X^i(x)\,.$$

A direct computation shows that when we consider only the dynamics on the original x-variables, then we recover the original n-dimensional system. For instance, in the case of Liénard systems (3.22), the Hamiltonian reads

$$H(x, y, \tilde{p}_1, \tilde{p}_2) = \tilde{p}_1 y - \tilde{p}_2(g(x) + f(x)y), \quad (\tilde{p}_1, \tilde{p}_2) = (\tilde{p}_x, \tilde{p}_y)\,. \tag{3.24}$$

In [66], such approach has been used to derive a Hamiltonisation of Liénard systems that proved to be useful for perturbation theory. Moreover, in [67] a similar extension, but with a suitably defined new Hamiltonian that non-trivially couples the variables, has been used in order to develop geometric integrators in the extended phase–space and then used e.g. in the case of the van der Pol oscillator.

In principle, one could use the Hamiltonian (3.24) and perform a splitting in order to obtain new geometric integrators that are symplectic in the extended phase–space, we will be back on this in Chapter 5. However, (3.24) is linear in the momenta and, as such, it is naturally associated with a contact Hamiltonian on the $(2n-1)$-dimensional projectivised cotangent bundle PT^*Q, endowed with the contact structure inherited from the canonical symplectic structure of T^*Q [17, 19]. The procedure to perform such reduction is quite simple in this case and it is reviewed e.g. in the recent work [68]. In order to avoid clutter of notation, from now on we focus on the case $Q = \mathbb{R}^2$, which is the relevant case for our study: we start with (3.24) and consider a connected component of the open set in which $\tilde{p}_2 \neq 0$. On such set, we can define the coordinates $(q = x, s = y, p = -\frac{\tilde{p}_1}{\tilde{p}_2})$, which serve as Darboux coordinates on $PT^*\mathbb{R}^2$. Finally, we define the contact Hamiltonian

$$\mathscr{H}(q, p, s) = -\frac{1}{\tilde{p}_2} H(x, y, \tilde{p}_1, \tilde{p}_2) = pX^1(q, s) - X^2(q, s)\,. \tag{3.25}$$

A direct calculation then shows that the restriction of the resulting contact Hamiltonian system to the (q, s) plane recovers the original system.

By means of the above prescription, we arrive at the following result for Liénard systems.

Theorem 3.5.2 (Hamiltonisation of Liénard systems)**.** *Liénard systems are contact Hamiltonian systems on* $(\mathbb{R}^3, \eta)$*, with a Hamiltonian of the form*

$$\mathscr{H} = ps + f(q)s + g(q). \tag{3.26}$$

The associated contact Hamiltonian system is

$$\dot{q} = s \tag{3.27}$$
$$\dot{s} = -f(q)s - g(q),$$
$$\dot{p} = -p^2 - f(q)p - f'(q)s - g'(q)\,. \tag{3.28}$$

From the first two equations we recover the original Liénard system in the (q, s)*-space, while the third equation is decoupled.*

Example 3.5.2 (The van der Pol oscillator revisited)**.** *As we have already seen in Section 3.5 the van der Pol equation is a particular case of a Liénard system, which is obtained by choosing* $f(x)$ *and* $g(x)$ *as*

$$f(x) = -\epsilon(1 - x^2), \qquad g(x) = x\,.$$

Consequently the contact Hamiltonian in this case reads

$$\mathcal{H} = ps - \epsilon(1 - q^2)s + q\,, \tag{3.29}$$

and the corresponding contact Hamiltonian systems is

$$\begin{cases} \dot{q} = s \\ \dot{s} = \epsilon(1 - q^2)s - q \\ \dot{p} = -1 - p^2 + \epsilon\left[(1 - q^2)p - 2qs\right]\,. \end{cases} \tag{3.30}$$

As expected, from the first two equations we recover the original van der Pol equation (3.23)*.*

Remark 15. *For* $s \neq 0$ *and setting the appropriate initial condition* $p_0 = -f(q_0) - g(q_0)/s_0$*,* $p(t)$ *derived from* (3.28) *turns out to be the slope of the tangent* $\frac{ds}{dq}$ *to the orbit of the system at each point* $(q(t), s(t))$ *of its evolution. This stems from the fact that* (3.27)*-*(3.28) *are the characteristic equations of the Hamilton-Jacobi equation for* (3.26)*. Details of this derivation are in preparation by [69].*

Remark 16. *The reduction procedure that led us to* (3.25) *is not unique. Indeed, we could have selected the connected component in which* $\tilde{p}_1 \neq 0$ *and set* $(q = y, s = x, p = -\frac{\tilde{p}_2}{\tilde{p}_1})$*. The corresponding contact Hamiltonian for Liénard systems is*

$$\begin{aligned} \mathcal{K}(q, p, s) &= -\frac{1}{\tilde{p}_1} H(x, y, \tilde{p}_1, \tilde{p}_2) = pX^2(q, s) - X^1(q, s) \\ &= -p(f(s)q + g(s)) - q\,. \end{aligned}$$

Beware that in this case $X^1(q, s) = q$ *and* $X^2(q, s) = -f(s)q - g(s)$*, that is, the roles of* q *and* s *are switched, and the resulting system is*

$$\begin{cases} \dot{q} = -f(s)q - g(s) \\ \dot{s} = q \\ \dot{p} = 1 + pf(s) + p\left(pqf'(s) + g'(s)\right)\,, \end{cases}$$

which is equivalent to (3.27)*-*(3.28) *for the* (q, s) *part, but not so much for* p*.*

The choice of reduction is in general dictated by convenience: the Hamiltonian $\mathcal{H}$ *from* (3.25) *results in a simpler form of the algorithms that we will discuss later in this thesis.*

3.6. Thermodynamic Processes

In thermodynamics, the arena is an odd dimensional space in which the variables are the macroscopic quantities as temperature, pressure and number of the particles. The contact structure arises when one begins with Gibbs thermodynamic relation or Gibbs one-form [70], which looks like

$$\eta_{TD} = dU - TdS + pdV - \mu dN, \tag{3.31}$$

where U, S, V and N are the extensive variables, respectively, internal energy, entropy, volume and the number of particles, while T, p and μ are the intensive variables, i.e., temperature, pressure and chemical potential. We refer to the contact manifold (M_{TD}, η_{TD}), where the contact form η_{TD} takes the form of the left side of equation (3.31), as thermodynamic space.

3.7. Hamiltonian systems on specific manifolds

For now, we focused on contact Hamiltonian systems defined on $\mathbb{R}^{2n+1}$, but for Martinet's Theorem [1] all 3-manifolds admit a contact structure. We focus on two specific examples of 3-contact manifolds endowed by what is called their standard contact structure, these examples are already introduced in section 1.4 are the 3-sphere, and the 3-torus.

3.7.1. The 3-sphere: $\mathbb{S}^3$

On the contact manifold $(\mathbb{S}^3, \eta_{std})$, we can construct in the different coordinate systems the Hamiltonian dynamics.

Spherical Coordinates

The Hamiltonian vector field in spherical coordinates of an Hamiltonian $\mathcal{H}(\psi, \theta, \phi)$ is

$$X_{\mathcal{H}} = \dot{\psi}\frac{\partial}{\partial \psi} + \dot{\theta}\frac{\partial}{\partial \theta} + \dot{\phi}\frac{\partial}{\partial \phi}.$$

where $(\dot{\psi}, \dot{\theta}, \dot{phi})$ are obtained from equation (2.2):

$$\begin{cases} \dot{\psi} = -\frac{1}{2}\sin(\theta)\frac{\partial \mathcal{H}}{\partial \theta}(\psi,\theta,\phi) - \frac{1}{2}\cot(\psi)\frac{\partial \mathcal{H}}{\partial \phi}(\psi,\theta,\phi) + \cos(\theta)\mathcal{H}(\psi,\theta,\phi) \\ \dot{\theta} = \frac{1}{2}\left(\sin(\theta)\left(\frac{\partial \mathcal{H}}{\partial \psi}(\psi,\theta,\phi) - 2\cot(\psi)\mathcal{H}(\psi,\theta,\phi)\right) - \csc^2(\psi)\cot(\theta)\frac{\partial \mathcal{H}}{\partial \phi}(\psi,\theta,\phi)\right) \\ \dot{\phi} = \frac{1}{2}\left(\cot(\psi)\frac{\partial \mathcal{H}}{\partial \psi}(\psi,\theta,\phi) + \csc^2(\psi)\cot(\theta)\frac{\partial \mathcal{H}}{\partial \theta}(\psi,\theta,\phi) + 2\mathcal{H}(\psi,\theta,\phi)\right) \end{cases}.$$

Moreover, the Jacobi brackets look like

$$\begin{aligned}\{f,g\}_\eta = {} & f(\psi,\theta,\phi)\left(-\frac{\partial g}{\partial\phi}-\cos(\theta)\frac{\partial g}{\partial\psi}+\cot(\psi)\sin(\theta)\frac{\partial g}{\partial\theta}\right)\\ & + g(\psi,\theta,\phi)\left(\frac{\partial f}{\partial\phi}+\cos(\theta)\frac{\partial f}{\partial\psi}-\cot(\psi)\sin(\theta)\frac{\partial f}{\partial\theta}\right)\\ & -\frac{1}{2}\cot(\psi)\frac{\partial f}{\partial\psi}\frac{\partial g}{\partial\phi}-\frac{1}{2}\sin(\theta)\frac{\partial f}{\partial\psi}\frac{\partial g}{\partial\theta}+\\ & +\frac{1}{2}\cot(\psi)\frac{\partial f}{\partial\phi}\frac{\partial g}{\partial\psi}+\frac{1}{2}\sin(\theta)\frac{\partial f}{\partial\theta}\frac{\partial g}{\partial\psi}+\\ & +\frac{1}{2}\csc^2(\psi)\cot(\theta)\frac{\partial f}{\partial\phi}\frac{\partial g}{\partial\theta}-\frac{1}{2}\csc^2(\psi)\cot(\theta)\frac{\partial f}{\partial\theta}\frac{\partial g}{\partial\phi}.\end{aligned}$$

Hopf Coordinates

With these new coordinates the Hamiltonian vector field becomes

$$h(\psi_1,\psi_2,\theta)\Rightarrow X_h=\dot\psi_1\frac{\partial}{\partial\psi_1}+\dot\psi_2\frac{\partial}{\partial\psi_2}+\dot\phi\frac{\partial}{\partial\phi},$$

where $(\dot\psi_1,\dot\psi_2,\dot\phi)$ can be read off Hamiltoni's equations

$$\begin{cases}\dot\psi_1=-h(\psi_1,\psi_2,\phi)-\dfrac{1}{4}\dfrac{\partial h}{\partial\phi}\cot\phi\\ \dot\psi_2=-h(\psi_1,\psi_2,\phi)+\dfrac{1}{4}\dfrac{\partial h}{\partial\phi}\tan\phi\\ \dot\phi=\dfrac{1}{4}\left(\dfrac{\partial h}{\partial\psi_1}\cot\phi-\dfrac{\partial h}{\partial\psi_2}\tan\phi\right)\end{cases}$$

From this expression we can infer the shape of the Jacobi brackets, obtaining

$$\begin{aligned}\{h,g\}_{(\psi_1,\psi_2,\phi)} = {} & h\left(\frac{\partial g}{\partial\psi_1}+\frac{\partial g}{\partial\psi_2}\right)-g\left(\frac{\partial h}{\partial\psi_1}+\frac{\partial h}{\partial\psi_2}\right)\\ & -\frac{1}{4}\left(\frac{\partial h}{\partial\psi_1}\cot\phi-\frac{\partial h}{\partial\psi_2}\tan\phi\right)\frac{\partial g}{\partial\phi}+\frac{1}{4}\left(\frac{\partial g}{\partial\psi_1}\cot\phi-\frac{\partial g}{\partial\psi_2}\tan\phi\right)\frac{\partial h}{\partial\phi}\end{aligned}$$

3.7.2. The 3-torus: $\mathbb{T}^3$

In section 1.4 we introduced the standard contact structure on a 3 - torus. This structure is represented by the 1-form:

$$\eta^{\mathbb{T}^3}=\cos(\theta)d\phi+\sin(\theta)d\xi.$$

From this, we can recover the expression of the contact Hamiltonian vector field for a Hamiltonian function $\mathcal{H}(\theta,\phi,\xi)$:

$$\begin{aligned}X_{\mathcal{H}} = {} & \left(-\frac{\partial\mathcal{H}}{\partial\theta}\cos(\theta)-\mathcal{H}\sin(\theta)\right)\frac{\partial}{\partial\xi}+\left(-\mathcal{H}\cos(\theta)+\frac{\partial\mathcal{H}}{\partial\theta}\sin(\theta)\right)\frac{\partial}{\partial\phi}+\\ & +\left(\frac{\partial\mathcal{H}}{\partial\xi}\cos(\theta)-\frac{\partial\mathcal{H}}{\partial\phi}\sin(\theta)\right)\frac{\partial}{\partial\theta},\end{aligned}$$

from which we identify the correspondent contact Hamiltonian equations:

$$\begin{cases} \dot{\xi} = -\dfrac{\partial \mathcal{H}}{\partial \theta}\cos(\theta) - \mathcal{H}\sin(\theta) \\ \dot{\phi} = -\mathcal{H}\cos(\theta) + \dfrac{\partial \mathcal{H}}{\partial \theta}\sin(\theta) \\ \dot{\theta} = \dfrac{\partial \mathcal{H}}{\partial \xi}\cos(\theta) - \dfrac{\partial \mathcal{H}}{\partial \phi}\sin(\theta) \end{cases} .$$

The Jacobi brackets then takes the form:

$$\begin{aligned} \{f, g\}_{\eta^{\mathbb{T}^3}} = \left(-\frac{\partial g}{\partial \theta}\cos(\theta) - g\sin(\theta)\right)\frac{\partial f}{\partial \xi} + \left(-g\cos(\theta) + \frac{\partial g}{\partial \theta}\sin(\theta)\right)\frac{\partial f}{\partial \phi} + \\ + \left(\frac{\partial g}{\partial \xi}\cos(\theta) - \frac{\partial g}{\partial \phi}\sin(\theta)\right)\frac{\partial f}{\partial \theta} + f\left(\sin(\theta)\frac{\partial g}{\partial \xi} + \cos(\theta)\frac{\partial g}{\partial \phi}\right) . \end{aligned}$$

Part II

Numerical Methods