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Brief paper

Krasovskii and shifted passivity based output consensus☆

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ABSTRACT

Motivated by current sharing in power networks, we consider a class of output consensus (also called agreement) problems for nonlinear systems, where the consensus value is determined by external disturbances, e.g., power demand. This output consensus problem is solved by a simple distributed output feedback controller if a system is either Krasovskii or shifted passive, which is the only essential requirement. The effectiveness of the proposed controller is shown in simulation on an islanded DC power network.

1. Introduction

Steering some variables to a common value is called an agreement problem. Along with massive research attentions of network systems, agreement problems have been studied in various contexts such as distributed optimization (Cenedese, Belgioioso, Kawano, Grammatico, & Cao, 2021; Chang, Nedić, & Scaglione, 2014; Hatanaka, Chopra, Ishizaki, & Li, 2018) and synchronization (Qin, Kawano, Anderson, & Cao, 2022; Qin, Kawano, Portoles, & Cao, 2021) to name a few. Our interest in this paper is output consensus (also called output agreement) under external disturbances. This problem is motivated by current sharing for balancing demand and supply in power networks (Cucuzzella, Trip, et al., 2019; Ferguson, Cucuzzella, & Scherpen, 2020), where currents and demands are modeled as outputs and external disturbances, respectively.

Various physical systems including the aforementioned power networks possess passivity properties. Passivity and its variant concepts have already been witnessed as useful tools for agreement; see, e.g., Arcak (2007), Bürger and De Persis (2015), Bürger, Zelazo, and Allgöwer (2014), Hatanaka et al. (2018), Monshizadeh and De Persis (2017), Scardovi, Arcak, and Sontag (2010), van der Schaft and Maschke (2013). In particular, Bürger and De Persis (2015), Bürger et al. (2014), Monshizadeh and De Persis (2017) have studied output consensus problems based on shifted passivity. The common problem formulation in these papers is that passive node dynamics are interconnected by special edge dynamics such that the networked interconnection naturally possesses an output consensus property, and passivity is used as a tool for the *analysis*. However, many physical systems such as DC microgrids (Cucuzzella, Trip, et al., 2019; Ferguson et al., 2020) do not have such edge dynamics.

Regarding control design, for linear DC microgrids, Cucuzzella, Trip, et al. (2019) provides an output consensus controller without explicitly utilizing passivity. A preliminary version (Feng, Kawano, Cucuzzella, & Scherpen, 2022) of this paper proposes a shifted passivity based output consensus controller for linear port-Hamiltonian systems, but not for nonlinear systems. In summary, a passivity based control framework for output consensus is still missing for general nonlinear network systems, including nonlinear DC microgrids.

Contribution: In this paper, we employ passivity as a tool for output consensus *control* under external disturbances. As passivity concepts, we focus on Krasovskii passivity (Kawano, Kosaraju, & Scherpen, 2021) (also called δ-passivity Schweidel & Arcak, 2021) and shifted passivity (Jayawardhana, Ortega, Garcia-Canseco, & Castanos, 2007; Kawano et al., 2021), which are different properties for nonlinear systems in general (Kawano et al., 2021).

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As the main contributions, we design a simple distributed output feedback controller which achieves output consensus for the following two classes of systems:

(i) Krasovskii passive nonlinear time-invariant systems under unknown constant disturbances without assuming the existence of an equilibrium point;
(ii) Shifted passive nonlinear time-varying systems under unknown time-varying disturbances by assuming the existence of an equilibrium trajectory.

For these systems, the proposed controller can also handle weighted output consensus and partial output consensus problems.

Discussion: The main contributions of our results compared with existing works are summarized as follows.

(I) The main focus of Bürger and De Persis (2015), Bürger et al. (2014) and Monshizadeh and De Persis (2017) is consensus analysis under special edge dynamics. In contrast, we design an output consensus controller based on Krasovskii or shifted passivity, regardless of the structure of the edges dynamics. Namely, we reveal that imposing special edge dynamics is not an essential requirement to achieve output consensus when invoking control design. In fact, the proposed approach can handle a wider class of nonlinear network systems such as the DC microgrids in Cucuzzella, Trip, et al. (2019), Ferguson et al. (2020), and weighted output consensus has not been considered in Bürger and De Persis (2015), Bürger et al. (2014) and Monshizadeh and De Persis (2017).
(II) Krasovskii passivity has not been used before for output consensus control or even analysis. The aforementioned papers (Bürger & De Persis, 2015; Bürger et al., 2014; Monshizadeh & De Persis, 2017) and a preliminary version (Feng et al., 2022) for linear port-Hamiltonian systems utilize shifted passivity, but not Krasovskii passivity. One of the advantages of using Krasovskii passivity is that we do not need to assume the existence of an equilibrium point of the closed-loop system beforehand in contrast to shifted passivity.
(III) Shifted passivity based output consensus control design for nonlinear networks is also a new contribution of this paper. One of the advantages of using shifted passivity is the ease of dealing with time-varying disturbances. This has been partly observed in Bürger and De Persis (2015), Monshizadeh and De Persis (2017) for output consensus analysis, under the assumption that disturbances are generated by Sylvester-type equations for output regulation. In this paper, we do not assume this. Namely, we do not require information of disturbances for control design.

Since Krasovskii and shifted passivity are different properties, there is possibility to enlarge the class of systems for which the results in Bürger and De Persis (2015), Bürger et al. (2014) and Monshizadeh and De Persis (2017) are applicable by revisiting these results from the viewpoint of Krasovskii passivity. As a relevant concept of Krasovskii passivity and shifted passivity, differential passivity, see e.g. Forni, Sepulchre, and van der Schaft (2013) and van der Schaft (2013), and incremental passivity, see e.g. Camlibel and van der Schaft (2013) and Pavlov and Marconi (2008), are known. The proposed output consensus controller is also applicable to differentially (resp. incrementally) passive systems because differential (resp. incremental) passivity implies Krasovskii (resp. shifted) passivity, see e.g. Kawano et al. (2021, Theorem 2.9 (resp. Proposition 2.15)).

Organization: The remainder of this paper is organized as follows. In Section 2, we provide a motivating example. The goal is to show that a current sharing problem for a nonlinear islanded DC power network can be formulated as an output consensus control problem. In Section 3, we propose a distributed output feedback controller to solve the output consensus problem. As the main results, we show that the proposed controller achieves output consensus if a system is either Krasovskii or shifted passive. In Section 4, we revisit the motivating example and confirm that the DC power network is both Krasovskii and shifted passive. Then, we illustrate the effectiveness of the proposed output consensus controller by simulation on the DC network. Finally, Section 5 concludes this paper. All the proofs are given in the Appendix.

Notation. The set of real numbers is denoted by $\mathbb{R}$. The n-dimensional vector whose all components are 1 is denoted by $\mathbb{1}_n$. The $n \times n$ identity matrix is denoted by I_n. For $P \in \mathbb{R}^{n\times n}$, $P \succ 0$ ($P \succeq 0$) means that P is symmetric and positive (semi) definite. For $x \in \mathbb{R}^n$, its Euclidean norm weighted by $P \succ 0$ is denoted by $|x|_P := \sqrt{x^\top P x}$. If $P = I_n$, this is simply described by $|x|$. A continuous function $\alpha : [0, r) \to [0, \infty)$ is said to be of class $\mathcal{K}$ if $\alpha(0) = 0$ and α is strictly increasing. Moreover, this is said to be of class $\mathcal{K}_\infty$ if $r = \infty$ and $\lim_{r\to\infty} \alpha(r) = \infty$. For a scalar-valued function $V : \mathbb{R}^n \to \mathbb{R}$, the column vector-valued function consisting of its partial derivatives is denoted by $\nabla V(x) := [\partial V/\partial x_1 \quad \cdots \quad \partial V/\partial x_n]^\top(x)$.

2. Motivating example

Consider an islanded DC power network model (Ferguson et al., 2020) with ν nodes and μ edges, described by

$$\dot{x} = f(x) + gu + d \tag{1}$$

$$x := \begin{bmatrix} \varphi^\top & q^\top & \varphi_t^\top \end{bmatrix}^\top$$

$$f(x) := (\mathcal{J} - \mathcal{R})\nabla \mathcal{H}(x) - \begin{bmatrix} 0 \\ I_L^\star + \operatorname{diag}(C^{-1}q)^{-1} P_L^\star \\ 0 \end{bmatrix}$$

$$\mathcal{J} := \begin{bmatrix} 0 & -I_\nu & 0 \\ I_\nu & 0 & D \\ 0 & -D^\top & 0 \end{bmatrix}, \ \mathcal{R} := \begin{bmatrix} R & 0 & 0 \\ 0 & G_L^\star & 0 \\ 0 & 0 & R_t \end{bmatrix}, \ g := \begin{bmatrix} I_\nu \\ 0 \\ 0 \end{bmatrix}$$

$$\mathcal{H}(x) := \left(|\varphi|^2_{L^{-1}} + |q|^2_{C^{-1}} + |\varphi_t|^2_{L_t^{-1}} \right)/2,$$

where $\varphi, q \in \mathbb{R}^\nu$ and $\varphi_t \in \mathbb{R}^\mu$ are state variables denoting, respectively, the flux and charge of the network nodes and the flux associated with the transmission lines interconnecting the nodes, while $u \in \mathbb{R}^\nu$ denotes the control input. The matrices $R, R_t, L, L_t, C \succ 0$ are diagonal and have appropriate dimensions, while $G_L^\star \in \mathbb{R}^{\nu\times\nu}$, $I_L^\star, P_L^\star \in \mathbb{R}^\nu$, and $d \in \mathbb{R}^{2\nu+\mu}$ are unknown; see Fig. 1 and Table 1 for the meaning of the used symbols. The incidence matrix $D \in \mathbb{R}^{\nu\times\mu}$ describes the network topology.

To improve the generation efficiency, it is generally desired in DC microgrids that the total current demand is shared among all the nodes (current sharing) (Cucuzzella, Trip, et al., 2019), i.e., for some $\alpha \in \mathbb{R}$,

$$\lim_{t\to\infty} \frac{\varphi_i(t)}{L_i} = \alpha, \quad \forall i = 1, \dots, \nu. \tag{2}$$

Inspired by Trip, Cucuzzella, Cheng, and Scherpen (2019) and based on the passivity properties of the DC network (1), we will later reveal that (2) is achieved by the following simple distributed output feedback controller:

$$\dot{u} = -EE^\top y \tag{3}$$

$$y = g^\top \nabla \mathcal{H}(x) = \begin{bmatrix} \frac{\varphi_1}{L_1} & \cdots & \frac{\varphi_\nu}{L_\nu} \end{bmatrix}^\top, \tag{4}$$

where $E \in \mathbb{R}^{\nu\times N}$ (with an arbitrary natural number $N \geq \nu - 1$) is such that rank $E = \nu - 1$ and $E^\top \mathbb{1}_\nu = 0$. For example, E can

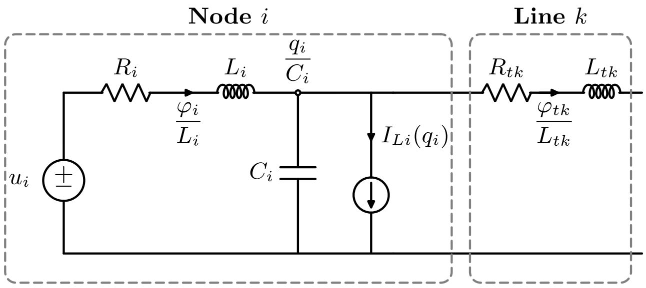

Fig. 1. Electrical scheme of node $i \in \mathcal{V}$ and transmission line $k \in \mathcal{E}$, where $I_{Li}(q_i) := G^\star_{Li}\frac{q_i}{C_i} + I^\star_{Li} + \frac{C_i}{q_i}P^\star_{Li}$.

Table 1
Description of the used symbols.

φ	Flux (node)	$G^\star_L$	Load conductance
q	Charge	$I^\star_L$	Load current
φ_t	Flux (power line)	$P^\star_L$	Load power
u	Control input	R, L, C	Filter parameters
d	Disturbance	R_t, L_t	Line parameters

be the incidence matrix and thus $EE^\top$ can be the Laplacian matrix associated with a connected and undirected communication graph.

We note that the DC network (1) is not a conventional port-Hamiltonian system because of the term $-I^\star_L + \mathrm{diag}(C^{-1}q)^{-1}P^\star_L$ in $f(x)$. However, we will show later that the system (1) possesses two different passivity properties: (I) Krasovskii passivity (Kawano et al., 2021) and (II) shifted passivity (Jayawardhana et al., 2007; Kawano et al., 2021). In general, these two properties are different for nonlinear systems (Kawano et al., 2021). In the rest of this paper, we show that if a system is either Krasovskii or shifted passive, then (3) is a controller achieving output consensus with respect to the passive output. For the DC network (1), the passive output for both Krasovskii and shifted passivity is y in (4).

Remark 2.1. For the DC network (1), the results in Bürger and De Persis (2015), Bürger et al. (2014) and Monshizadeh and De Persis (2017) for output consensus analysis are not directly applicable. These papers impose special edge dynamics such that the interconnected system naturally possesses the output consensus property. However, in the DC network (1), the interconnection structure is determined by the physical couplings of the circuit components, and the interconnected system (1) itself does not have the output consensus property (2) in general. Therefore, we need to design the controller (3) in order to enforce output consensus. ◁

3. Control design for output consensus

In this section, we develop passivity based control techniques for output consensus. As passivity concepts, we employ Krasovskii passivity and shifted passivity. We first provide a distributed output feedback controller to achieve output consensus for Krasovskii passive systems and then show that the same controller works for shifted passive systems. When Krasovskii passivity is used, we do not need to assume the existence of an equilibrium. In contrast, to confirm shifted passivity, we need information of an equilibrium. As an advantage of requiring additional information, the shifted passivity based approach can handle time-varying systems and disturbances.

3.1. Krasovskii passivity based approach

Consider a nonlinear system

$$\begin{cases} \dot{x} = f(x, u, d) \\ y = h(x, d), \end{cases} \tag{5}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^r \to \mathbb{R}^n$ and $h : \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^m$ are of class C^1, and $\partial f(x, u, d)/\partial u$ is of full column rank at each $(x, u, d) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^r$. We recall that $d \in \mathbb{R}^r$ is a constant disturbance, i.e., $\dot{d} = 0$.

To define Krasovskii passivity, we introduce the extended system of (5):

$$\begin{cases} \dot{x} = f(x, u, d) \\ \dfrac{d\dot{x}}{dt} = \dfrac{\partial f(x, u, d)}{\partial x}\dot{x} + \dfrac{\partial f(x, u, d)}{\partial u}\dot{u} \\ \dot{y} = \dfrac{\partial h(x, d)}{\partial x}\dot{x}. \end{cases} \tag{6}$$

This can be understood as a system with the state $(x, \dot{x}, u)$, input $\dot{u}$, and output $\dot{y}$. Focusing on the dynamics of $\dot{x}$, we define strict Krasovskii passivity as a variant of the original definition (Kawano et al., 2021, Definition 2.8).

Definition 3.1. Given $d \in \mathbb{R}^r$, the system (5) is said to be *strictly Krasovskii passive* on $\mathcal{D}_K \subset \mathbb{R}^n \times \mathbb{R}^m$ if for its extended system (6), there exist $S_K : \mathcal{D}_K \times \mathbb{R}^n \to \mathbb{R}$ of class C^1 and $W_K : \mathcal{D}_K \times \mathbb{R}^n \to \mathbb{R}$ of class C^0 such that

$$S_K(x, u, \dot{x}) \geq 0 \tag{7a}$$

$$W_K(x, u, \dot{x}) \geq 0 \tag{7b}$$

$$W_K(x, u, \dot{x}) = 0 \iff \dot{x} = 0 \tag{7c}$$

$$\dot{S}_K(x, u, \dot{x}) \leq -W_K(x, u, \dot{x}) + \dot{y}^\top \dot{u} \tag{7d}$$

for all $(x, u) \in \mathcal{D}_K$ and $(\dot{x}, \dot{u}) \in \mathbb{R}^n \times \mathbb{R}^m$. ◁

The controller (3) in the motivating example is a special case ($M = I$, $K_1 = 0$, $K_2 = 0$) of the following distributed output feedback controller:

$$\begin{cases} \dot{u} = -M^\top E(E^\top My + K_1E^\top M\dot{y}) - K_2(\dot{y} - \dot{\rho}) \\ \dot{\rho} = y - \rho, \end{cases} \tag{8}$$

with the controller state $u, \rho \in \mathbb{R}^m$, where $M \in \mathbb{R}^{m\times m}$ is a weight of the output, and $E \in \mathbb{R}^{m\times N}$ (with an arbitrary natural number $N \geq m - 1$) is such that rank $E = m - 1$ and $E^\top \mathbb{1}_m = 0$, and $0 \preceq K_1 \in \mathbb{R}^{N\times N}$ and $0 \preceq K_2 \in \mathbb{R}^{m\times m}$ are tuning parameters (allowed to be zero as in (3)). The ρ-dynamics play a role of dumping dynamics, which is helpful to improve transient performances, e.g., reduction of settling time and oscillation amplitude.

Now, as the first main result of this paper, we show that the distributed output feedback controller (8) achieves weighted output consensus for Krasovskii passive systems.

Theorem 3.2. *Given $d \in \mathbb{R}^r$, suppose that the closed-loop system consisting of a strictly Krasovskii passive system (5) on $\mathcal{D}_K$ and the distributed output feedback controller (8) is positively invariant on a compact set $\Omega_K \subset \mathcal{D}_K \times \mathbb{R}^m$. Then, for each $(x(0), u(0), \rho(0)) \in \Omega_K$, there exists $\alpha_d : \mathbb{R} \to \mathbb{R}$ such that*

$$\lim_{t\to\infty}(My(t) - \alpha_d(t)\mathbb{1}_m) = 0. \tag{9}$$

Proof. The proof is in Appendix A. □

3.2. Shifted passivity based approach

In this subsection, we confirm that (8) is an output consensus controller also for shifted passive systems. As mentioned before, in the shifted passivity case, we can handle (input-affine) time-varying nonlinear systems:

$$\begin{cases} \dot{x} = f(t, x, d) + g(t, x, d)u \\ y = h(t, x, d), \end{cases} \tag{10}$$

with a bounded uniformly continuous function $d(t)$ on $\mathbb{R}^r$, where $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^n$, $g : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^{n\times m}$, and $h : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^m$ are uniformly continuous in (t, d) and locally Lipschitz in x on $\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^r$, and $g(t, x, d)$ is of full column rank at each $(t, x, d) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^r$.

We assume that the system (10) admits an equilibrium trajectory. Namely, given $d(\cdot)$, there exists a class C^1 bounded trajectory $(x_d^*(\cdot), u_d^*(\cdot))$ such that

$$\dot{x}_d^*(\cdot) = f(\cdot, x_d^*(\cdot), d(\cdot)) + g(\cdot, x_d^*(\cdot), d(\cdot))u_d^*(\cdot).$$

Accordingly, we define

$$y_d^*(\cdot) := h(\cdot, x_d^*(\cdot), d(\cdot)).$$

Note that the boundedness of $(x_d^*(\cdot), u_d^*(\cdot))$ implies that $\dot{x}_d^*(\cdot)$ is bounded also, and thus $x_d^*(\cdot)$ and $y_d^*(\cdot)$ are uniformly continuous. In addition, $\dot{x}_d^*(\cdot)$ is uniformly continuous if $u_d^*(\cdot)$ is uniformly continuous.

Under the assumption for the existence of $(x_d^*(\cdot), u_d^*(\cdot))$, we introduce the dynamics of the error $e_d := x - x_d^*$:

$$\begin{cases} \dot{e}_d = f(t, e_d + x_d^*, d) + g(t, e_d + x_d^*, d)u \\ \qquad - (f(t, x_d^*, d) + g(t, x_d^*, d)u_d^*) \\ y = h(t, e_d + x_d^*, d). \end{cases} \tag{11}$$

Using the error dynamics, we extend the concept of shifted passivity, e.g., Kawano et al. (2021, Definition 2.14) to the time-varying case as follows.

Definition 3.3. The system (10) is said to be *strictly shifted passive* along $(x_d^*(\cdot), u_d^*(\cdot))$ on the error (e_d) space $\mathcal{D}_S \subset \mathbb{R}^n$ if for its error dynamics (11), there exist $S_S : \mathbb{R} \times \mathcal{D}_S \to \mathbb{R}$ of class C^1 and $W_S : \mathcal{D}_S \to \mathbb{R}$ of class C^0 such that

$$S_S(t, e_d) \geq 0 \tag{12a}$$

$$W_S(e_d) \geq 0 \tag{12b}$$

$$W_S(e_d) = 0 \iff e_d = 0 \tag{12c}$$

$$\dot{S}_S(t, e_d) \leq -W_S(e_d) + (y - y_d^*)^\top (u - u_d^*) \tag{12d}$$

for all $(t, e_d) \in \mathbb{R} \times \mathcal{D}_S$ and $u \in \mathbb{R}^m$. When $W_S(\cdot) = 0$, we simply say that the system is *shifted passive*. ◁

We show now that the same distributed output feedback controller (8) achieves weighted output consensus also for shifted passive systems. For the sake of analysis, we rewrite the controller dynamics by introducing an intermediate variable $\xi \in \mathbb{R}^N$:

$$\begin{cases} \dot{\xi} = E^\top My \\ \dot{\rho} = y - \rho \\ u = -M^\top E(\xi + K_1 E^\top My) - K_2(y - \rho), \end{cases} \tag{13}$$

where the controller state is $(\xi, \rho) \in \mathbb{R}^N \times \mathbb{R}^m$. Computing $\dot{u}$, we recover the representation in (8).

In the shifted passivity case, we consider two scenarios: (i) disturbance d has a convergence property, and $K_1 \succeq 0$ (see item (IV) in Theorem 3.4); (ii) d does not have such a property, and $K_1 \succ 0$ (see Corollary 3.5). In both cases, the distributed output feedback controller (13) achieves output consensus.

Theorem 3.4. *Given $d(\cdot)$, consider the closed-loop system consisting of system (10) and the distributed output feedback controller (13). Suppose that*

(I) the closed-loop system admits a class C^1 bounded trajectory $(x_d^*(\cdot), \xi_d^*(\cdot), \rho_d^*(\cdot))$ such that $E^\top My_d^*(\cdot) = 0$;

(II) the system (10) is strictly shifted passive along $(x_d^*(\cdot), u_d^*(\cdot))$ on $\mathcal{D}_S$, where

$$u_d^*(\cdot) := -M^\top E(\xi_d^*(\cdot) + K_1 E^\top My_d^*(\cdot)) - K_2\rho_d^*(\cdot);$$

(III) when rewriting the system (10) as the error dynamics (11), the closed-loop system is positively invariant on a compact set $\Omega_S \subset \mathcal{D}_S \times \mathbb{R}^N \times \mathbb{R}^m$ (for any initial time $t_0 \in \mathbb{R}$), where the projection of Ω_S onto $\mathcal{D}_S$ contains the origin;

(IV) $\lim_{t\to\infty} f(t, x_d^*(t), d(t))$ and $\lim_{t\to\infty} g(t, x_d^*(t), d(t))$ exist and are finite.

Then, for each $(e_d(t_0), \xi(t_0), \rho(t_0)) \in \Omega_S$ and every $t_0 \in \mathbb{R}$, there exists $\alpha_d : \mathbb{R} \to \mathbb{R}$ such that (9) holds.

Proof. The proof is in Appendix B. □

Corollary 3.5. *Given $d(\cdot)$, consider the closed-loop system consisting of a system (10) and controller (13). If $K_1 \succ 0$ and items (I) – (III) in Theorem 3.4 hold, then for each $(e_d(t_0), \xi(t_0), \rho(t_0)) \in \Omega_S$ and every $t_0 \in \mathbb{R}$, there exists $\alpha_d : \mathbb{R} \to \mathbb{R}$ such that (9) holds. Moreover, this statement holds even if strict shifted passivity in item (II) is relaxed into shifted passivity.*

Proof. The proof is in Appendix C. □

Corollary 3.5 holds even for non-necessarily input-affine systems, which can be confirmed from its proof in Appendix C. In the proofs of Theorem 3.4 and Corollary 3.5, we show $\lim_{t\to\infty} e_d(t) = 0$, i.e., $\lim_{t\to\infty}(x(t) - x_d^*(t)) = 0$. This implies that the consensus value (more precisely, the valued-function):

$$\alpha_d(t) = \frac{\mathbb{1}_m^\top Mh(t, x_d^*(t), d(t))}{\mathbb{1}_m^\top \mathbb{1}_m}$$

does not depend on the initial state $(x(t_0), \xi(t_0), \rho(t_0))$ (or initial time $t_0 \in \mathbb{R}$). This further implies that $\alpha_d(\cdot)$ is constant if $(x_d^*(\cdot), d(\cdot))$ is constant, and h is time-invariant.

3.3. Remarks

In this subsection, we provide several remarks for Krasovskii and shifted passivity based approaches.

First, the controller dynamics (8) (or equivalently (13)) look similar to the edge dynamics in Bürger and De Persis (2015), Bürger et al. (2014) and Monshizadeh and De Persis (2017). These references analyze networked interconnections of decoupled shifted passive node dynamics

$$\begin{cases} \dot{x}_i = f_i(x_i, w_i, d_i) \\ y_i = h_i(x_i, d_i) \end{cases}$$

interconnected by the edge dynamics $\dot{w} = -EE^\top y$. In this paper, we study a compact form of a network system (5) or (10) without imposing assumptions for node or edge dynamics and show that if a compact form is Krasovskii or shifted passive, (8) is an output consensus controller. By viewing the edge dynamics as the controller dynamics, the results in Bürger and De Persis (2015), Bürger et al. (2014) and Monshizadeh and De Persis (2017) can be understood as a special case of our results where node dynamics are decoupled. However, some physical systems such as the DC network (1), the motivating example in Section 2 does not possess such a structure, since the edge dynamics are determined by the physical coupling. As shown in the next section, we can achieve output consensus for the DC network by control design, which also improves transient performances. In Bürger et al. (2014), the disturbances are assumed to be constant, while in Bürger and De Persis (2015) and Monshizadeh and De Persis (2017), time-varying disturbances are assumed to be generated by exosystems. In contrast, we do not assume this in Theorem 3.4 and Corollary 3.5, which is another aspect that our results are more general than those in Bürger and De Persis (2015), Bürger et al. (2014) and Monshizadeh and De Persis (2017). In other words, we reveal that some restrictions in Bürger and De Persis (2015), Bürger et al. (2014) and Monshizadeh and De Persis (2017) can be relaxed by invoking control design.

Next, we are able to shift the consensus valued function $\alpha_d(t)$ in (9) by a simple modification of the controller (8):

$$\begin{cases} \dot{v} = -M^\top E(E^\top M y + K_1 E^\top M \dot{y}) - K_2(\dot{y} - \dot{\rho}) \\ \dot{\rho} = y - \rho \\ u = v + \bar{u}, \end{cases} \tag{14}$$

where $\bar{u}(t) \in \mathbb{R}^m$ is an external input to shift $\alpha_d(t)$. Similarly, (13) can also be modified.

Also, our results can be generalized to achieve partial weighted output consensus. To achieve consensus among y_j, $j = i_1, \dots, i_{\hat{m}}$, $\hat{m} \le m$, it suffices to replace (8) by

$$\begin{cases} \dot{\hat{u}} = -\hat{M}^\top \hat{E}(\hat{E}^\top \hat{M}\hat{y} + \hat{K}_1 \hat{E}^\top \hat{M}\dot{\hat{y}}) - \hat{K}_2(\dot{\hat{y}} - \dot{\hat{\rho}}) \\ \dot{\hat{\rho}} = \hat{y} - \hat{\rho} \end{cases}$$
$$\hat{y} := \begin{bmatrix} y_{i_1} & \cdots & y_{i_{\hat{m}}} \end{bmatrix}^\top, \quad \hat{u} := \begin{bmatrix} u_{i_1} & \cdots & u_{i_{\hat{m}}} \end{bmatrix}^\top,$$

with $\hat{u}, \hat{y}, \hat{\rho} \in \mathbb{R}^{\hat{m}}$, where $\hat{M} \in \mathbb{R}^{\hat{m}\times\hat{m}}$, $\hat{E} \in \mathbb{R}^{\hat{m}\times N}$ (with an arbitrary natural number $N \ge \hat{m} - 1$) is such that rank $\hat{E} = \hat{m} - 1$ and $\hat{E}^\top \mathbb{1}_{\hat{m}} = 0$, and $0 \preceq \hat{K}_1 \in \mathbb{R}^{N\times N}$ and $0 \preceq \hat{K}_2 \in \mathbb{R}^{\hat{m}\times\hat{m}}$ are tuning parameters (allowed to be zero). It is further possible to achieve weighted consensus among some of the outputs y_j, $j \ne i_1, \dots, i_{\hat{m}}$.

Finally, for shifted passivity, the positive invariance assumption, i.e., item (III), can be removed from Theorem 3.4 and Corollary 3.5 if there exist class $\mathcal{K}$ functions α_1, α_2 such that

$$\alpha_1(|e_d|) \le S_S(t, e_d) \le \alpha_2(|e_d|), \quad \forall t \in \mathbb{R},\ \forall e_d \in \mathcal{D}_S.$$

Furthermore, global output consensus can be achieved if $\mathcal{D}_S = \mathbb{R}^n$, and α_1, α_2 are class $\mathcal{K}_\infty$ functions. These can be confirmed from the proofs in Appendices B–C. Also for Krasovskii passivity, the positive invariance can be verified by studying a sort of Lyapunov candidate $V_K(x, u, \dot{x}, \rho)$ in Appendix A. In particular, if $V_K(x, u, \dot{x}, \rho)$ is lower and upper bounded by class $\mathcal{K}$ functions of $(|x|, |u|, |\rho|)$, the positive invariance is guaranteed.

4. Motivating example revisited

In this section, we revisit the motivating example in Section 2 and show that the considered DC microgrid is both Krasovskii and shifted passive. Then, numerical simulations illustrate the effectiveness of the proposed output consensus controller.

4.1. Krasovskii passivity

We confirm that the DC microgrid (1) is Krasovskii passive. Compute

$$\frac{d\dot{x}}{dt} = (\mathcal{J} - \mathcal{R})\nabla^2\mathcal{H}\dot{x} + \begin{bmatrix} 0 \\ \mathrm{diag}\{C_1\frac{\dot{q}_1}{q_1^2}, \dots, C_n\frac{\dot{q}_n}{q_n^2}\}P_L^\star \\ 0 \end{bmatrix} + g\dot{u},$$

where note that the Hessian matrix $\nabla^2\mathcal{H}$ is constant. Strict Krasovskii passivity can be shown with respect to the storage function $S_K(\dot{x}) = |\dot{x}|^2_{\nabla^2\mathcal{H}}/2$. Indeed, it follows that

$$\begin{aligned} \dot{S}_K(\dot{x}) =& \dot{x}^\top\nabla^2\mathcal{H}(\mathcal{J} - \mathcal{R})\nabla^2\mathcal{H}\dot{x} \\ &+ \dot{x}^\top\nabla^2\mathcal{H}\begin{bmatrix} 0 \\ \mathrm{diag}\{C_1\frac{\dot{q}_1}{q_1^2}, \dots, C_n\frac{\dot{q}_n}{q_n^2}\}P_L^\star \\ 0 \end{bmatrix} \\ &+ \dot{x}^\top\nabla^2\mathcal{H}g\dot{u} \\ =& -W_K(\dot{x}, q) + \dot{y}^\top\dot{u}, \end{aligned}$$

where y is defined in (4), and

$$\begin{aligned} W_K(\dot{x}, q) :=& \dot{x}^\top\nabla^2\mathcal{H}\mathcal{R}\nabla^2\mathcal{H}\dot{x} \\ &- \dot{x}^\top\nabla^2\mathcal{H}\begin{bmatrix} 0 \\ \mathrm{diag}\{C_1\frac{\dot{q}_1}{q_1^2}, \dots, C_n\frac{\dot{q}_n}{q_n^2}\}P_L^\star \\ 0 \end{bmatrix}. \end{aligned}$$

Let $\mathcal{D}_{K,q} \subset \mathbb{R}^\nu$ and $\Gamma \succ 0$ be such that

$$G_L^\star - \mathrm{diag}\left(\frac{C_1^2 P_{L,1}^\star}{q_1^2}, \dots, \frac{C_n^2 P_{L,n}^\star}{q_n^2}\right) \succeq \Gamma, \quad \forall q \in \mathcal{D}_{K,q}.$$

Then, the DC microgrid (1) is strictly Krasovskii passive on $\mathcal{D}_K = \mathbb{R}^\nu \times \mathcal{D}_{K,q} \times \mathbb{R}^\mu \times \mathbb{R}^\nu$.

From Theorem 3.2, the proposed distributed output feedback controller (8) achieves weighted current sharing (9) under the positive invariance assumption that is common in the literature on DC microgrids with constant power loads; see, e.g. Ferguson et al. (2020) and the references therein. We note that Krasovskii-like passivity has been already used for the design and analysis of voltage controllers for electric circuits and grids, e.g. Cucuzzella, Lazzari, Kawano, Kosaraju, and Scherpen (2019), Kosaraju, Cucuzzella, Scherpen, and Pasumarthy (2021) and Kawano et al. (2021). However, to the best of our knowledge, Krasovskii passivity has never been exploited before for achieving current sharing and, more generally, output consensus.

4.2. Shifted passivity

We confirm that the DC microgrid (1) is also shifted passive. From the convexity of $\mathcal{H}(x)$ in (1), it follows that

$$\mathcal{H}_S(e_d, x_d^*) := \mathcal{H}(e_d + x_d^*) - \mathcal{H}(x_d^*) - \nabla^\top\mathcal{H}e_d \ge 0$$

for any $e_d, x_d^* \in \mathbb{R}^{2\nu+\mu}$; this is a standard technique for shifting a storage function as found in Jayawardhana et al. (2007). Also, $\mathcal{H}_S(e_d, x_d^*) = 0$ if and only if $e_d = 0$. Shifted passivity can be shown with respect to the storage function $\mathcal{H}_S(e_d, x_d^*)$. Indeed, it follows that

$$\begin{aligned} \dot{\mathcal{H}}_S(e_d, x_d^*) &= (\nabla^\top\mathcal{H}(e_d + x_d^*) - \nabla^\top\mathcal{H}(x_d^*))(\dot{e}_d + \dot{x}_d^*) \\ &\quad - e_d^\top\nabla^2\mathcal{H}\dot{x}_d^* \\ &= e_d^\top\nabla^2\mathcal{H}\dot{e}_d \\ &= e_d^\top\nabla^2\mathcal{H}(f(e_d + x_d^*) - f(x_d^*) + g(u - u_d^*)) \\ &= -W_S(e_d, x_d^*) + (y - y_d^*)^\top(u - u_d^*), \end{aligned}$$

where y is defined in (4), and

$$\begin{aligned} &W_S(e_d, x_d^*) \\ &:= -e_d^\top\nabla^2\mathcal{H}(f(e_d + x_d^*) - f(x_d^*)) \\ &= e_d^\top\nabla^2\mathcal{H}R\nabla^2\mathcal{H}e_d \\ &\quad + e_d^\top\nabla^2\mathcal{H}\begin{bmatrix} 0 \\ (\mathrm{diag}(C^{-1}q)^{-1} - \mathrm{diag}(C^{-1}q^*)^{-1})P_L^\star \\ 0 \end{bmatrix} \end{aligned}$$

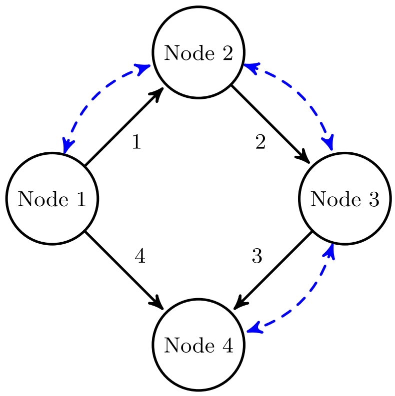

Fig. 2. Scheme of the considered microgrid composed of 4 nodes. The black solid arrows indicate the positive direction of the currents through the power lines. The dashed blue lines represent the communication network.

Table 2
Load parameters.

Node		1	2	3	4
$P_{Li}^{\star}$	(kW)	1	2.5	1.5	5
$G_{Li}^{\star}$	(S)	0.08	0.04	0.02	0.08
$I_{Li}^{\star}$	(A)	12.5	7.5	5.0	15.0
$\Delta P_{Li}^{\star}$	(kW)	4	1	1	−4

Let $\mathcal{D}_{S,q} \subset \mathbb{R}^{\nu}$ and $\Gamma \succ 0$ be such that

$$G_L^{\star} - \mathrm{diag}\left(\frac{C_1^2 P_{L,1}^{\star}}{(e_{d,q_1} + q_1^*)q_1^*}, \dots, \frac{C_\nu^2 P_{L,\nu}^{\star}}{(e_{d,q_\nu} + q_\nu^*)q_\nu^*}\right) \succeq \Gamma$$

$$\forall e_{d,q} = q - q_d^* \in \mathcal{D}_{S,q},\ \forall t \geq t_0,\ \forall t_0 \in \mathbb{R}.$$

Then, the DC microgrid (1) is strictly shifted passive on $\mathcal{D}_S = \mathbb{R}^{\nu} \times \mathcal{D}_{S,q} \times \mathbb{R}^{\mu}$.

From Theorem 3.4, the proposed controller (13) achieves weighted current sharing under the positive invariance assumption. When x_d^* is constant, output consensus is achieved on a positively invariant set contained in $\mathcal{D}_K \cup \mathcal{D}_S$. In general, there is no inclusion relation between $\mathcal{D}_K$ and $\mathcal{D}_S$. Thus, a larger region of output consensus can be estimated by combining both Krasovskii and shifted passivity analysis.

4.3. Simulations

In this section, the proposed distributed output feedback controller (14) is verified in simulation. We consider an islanded DC microgrid composed of 4 nodes in ring topology as shown in Fig. 2, where the dashed blue lines represent the communication network. As shown above, this system is both Krasovskii and shifted passive. The values of the parameters of each node and line are mainly taken from Cucuzzella, Trip, et al. (2019, Tables II, III), while those of the nominal loads are reported in Table 2. Note that we consider also load variations, which are gathered into the disturbance d. For the sake of notational simplicity, let $V_i := q_i/C_i$ and $I_i := \varphi_i/L_i$ denote respectively the voltage and the generated current associated with node $i = 1, \dots, 4$. The desired voltage value at each node is chosen equal to $V_i^* = 380\,[\mathrm{V}]$ for all i. The controller parameters in (14) are chosen as $M = 100I_4$, $K_1 = I_3$, and $K_2 = 0.2I_4$. Moreover, for the considered application, we select $\bar{u}$ in (14) as $\bar{u} = C^{-1}q^* + RL^{-1}\varphi$. This simply allows us to shift the system equilibrium such that the voltage average (V_{av}) is equal to the voltage reference; see, e.g., Trip et al. (2019). Below, we investigate three different scenarios: (1) output consensus (i.e., current sharing) and (2) weighted output consensus under constant disturbance; (3) output consensus under time-varying disturbance.

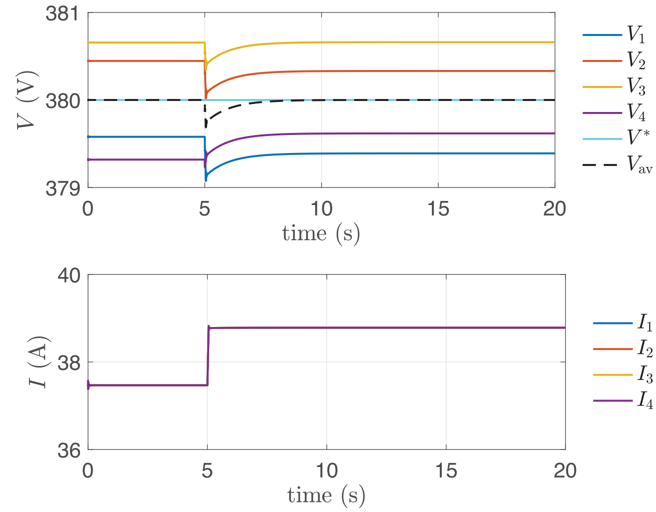

Fig. 3. Scenario 1: current sharing with constant loads. **(Top)** Time evolution of the voltages and their average (dashed line) together with the corresponding reference (cyan line). **(Bottom)** Time evolution of the currents. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Scenario 1 (output consensus under constant disturbance) In this scenario, we show current sharing with constant loads. Let the system initially be at the equilibrium. Then, at the time instant $t = 5\,[\mathrm{s}]$ a step variation equal to $\Delta P_L^{\star}$ (see Table 2) occurs in the P loads. From Fig. 3, we can observe that both voltages and currents converge to a constant equilibrium, where the voltage average converges to the voltage reference, and output consensus (i.e., current sharing) is achieved. Specifically, we can observe that since the loads are constant, then the consensus value is also constant.

Scenario 2 (weighted output consensus under constant disturbance) In this scenario, we show weighted current sharing with constant loads. Consider Scenario I) with $M_{33} = 80$, which implies that node 3 is required to generate a current that is 25% higher than the current generated by each of the other nodes. From Fig. 4, we can observe that both voltages and currents converge to a constant equilibrium, where the voltage average converges to the voltage reference, and weighted output consensus (i.e., proportional current sharing) is achieved. Specifically, we can observe that $I_1 = I_2 = I_4$ and $I_3 = 1.25I_1$.

Scenario 3 (output consensus under time-varying disturbance) In this scenario, we show current sharing with time-varying loads. Consider Scenario 1. At the time instant $t = 5\,[\mathrm{s}]$, we add to the P load of node 3, a non-converging time-varying component equal to $0.1\sin(4t)$ [kW]. From Fig. 5, we can observe that both voltages and currents converge to an equilibrium trajectory, where the voltage average is stabilized around the voltage reference, and output consensus (i.e., current sharing) is achieved. Specifically, we can observe that since the loads are time-varying, the consensus value depends on time.

5. Conclusion

In this paper, we have studied an output consensus problem for nonlinear systems under external disturbances. As the main contribution, we have proposed a simple distributed output feedback controller that achieves output consensus for Krasovskii or shifted passive systems. An advantage of the Krasovskii passivity based approach is that we do not need to assume the existence of

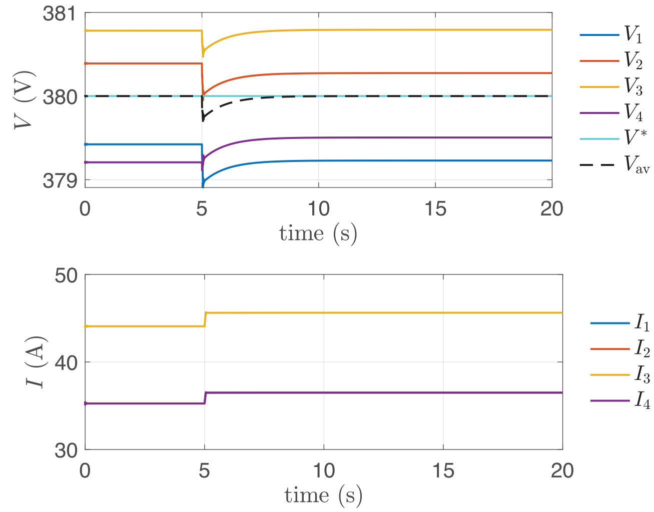

Fig. 4. Scenario 2: weighted current sharing with constant loads. **(Top)** Time evolution of the voltages and their average (dashed line) together with the corresponding reference (cyan line). **(Bottom)** Time evolution of the currents. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

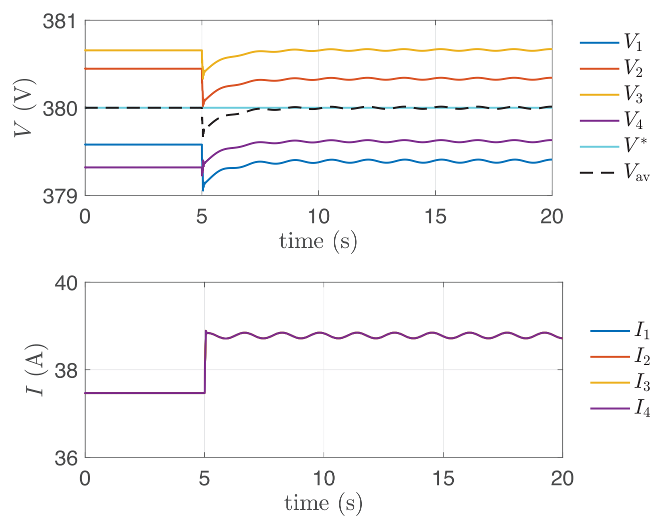

Fig. 5. Scenario 3: current sharing with time-varying loads. **(Top)** Time evolution of the voltages and their average (dashed line) together with the corresponding reference (cyan line). **(Bottom)** Time evolution of the currents. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

an equilibrium point for the closed-loop system. An advantage of the shifted passivity based approach is the ease of dealing with time-varying cases. The proposed controller has been validated in simulation by achieving current sharing in an islanded DC microgrid, which is both Krasovskii and shifted passive.

Appendix A. Proof of Theorem 3.2

Define

$$V_K(x,u,\dot{x},\rho) := S_K(x,u,\dot{x}) + \frac{1}{2}(|E^\top My|^2 + |y-\rho|^2_{K_2}).$$

From (7d) and (8), its time-derivative along the closed-loop trajectory satisfies

$$\begin{aligned}\dot{V}_K(x,u,\dot{x},\rho) &\le -W_K(x,u,\dot{x}) + \dot{y}^\top \dot{u} + \dot{y}^\top M^\top E E^\top M y \\ &\quad + (y-\rho)^\top K_2(\dot{y}-\dot{\rho}) \\ &= -W_K(x,u,\dot{x}) - |E^\top M\dot{y}|^2_{K_1} \\ &\quad - \dot{y}^\top K_2(\dot{y}-\dot{\rho}) + \dot{\rho}^\top K_2(\dot{y}-\dot{\rho}) \\ &\le -W_K(x,u,\dot{x}) - |E^\top M\dot{y}|^2_{K_1} - |\dot{y}-\dot{\rho}|^2_{K_2}\end{aligned}$$

for all $(x,u,\rho) \in \mathcal{D}_K \times \mathbb{R}^m$. Taking the time integration yields

$$\begin{aligned}&V_K(x(t),u(t),\dot{x}(t),\rho(t)) \\ &\quad + \int_0^t (W_K(x(\tau),u(\tau),\dot{x}(\tau)) \\ &\qquad + |E^\top M\dot{y}(\tau)|^2_{K_1} + |\dot{y}(\tau)-\dot{\rho}(\tau)|^2_{K_2})d\tau \\ &\le V_K(x(0),u(0),\dot{x}(0),\rho(0)),\end{aligned} \tag{A.1}$$

where $\dot{x}(0) = f(x(0),u(0),d)$. Since the closed-loop system is positively invariant on the compact set Ω_K, the integral term exists for each $(x(0),u(0),\rho(0)) \in \Omega_K$. Also, this is upper bounded and increasing with respect to $t \ge 0$, which implies that the limit of the integral term at $t \to \infty$ exists and is finite for each $(x(0),u(0),\rho(0)) \in \Omega_K$.

To use Barbalat's lemma (Khalil, 1996, Lemma 8.2), we show the uniform continuity of $(x(\cdot),u(\cdot),\rho(\cdot))$, $(\dot{x}(\cdot),\dot{u}(\cdot),\dot{\rho}(\cdot))$, and $\ddot{x}(\cdot)$ for each $(x(0),u(0),\rho(0)) \in \Omega_K$. Recall that Ω_K is compact and positively invariant, and f and h in (5) are class C^1 functions of $(x,u) \in \mathbb{R}^n \times \mathbb{R}^m$ at each $d \in \mathbb{R}^r$. This implies that $(x(\cdot),u(\cdot),\rho(\cdot))$ exists and is a bounded class C^2 function of $t \ge 0$ for each $(x(0),u(0),\rho(0)) \in \Omega_K$. Consequently, $f(x(\cdot),u(\cdot),d)$, $h(x(\cdot),d)$, and $\partial h(x(\cdot),d)/\partial x$ are bounded, and also $(\dot{x}(\cdot),\dot{u}(\cdot),\dot{\rho}(\cdot))$ is bounded from (5) and (8). Namely, $(x(\cdot),u(\cdot),\rho(\cdot))$ has bounded derivative and thus is uniformly continuous for each $(x(0),u(0),\rho(0)) \in \Omega_K$. Again from (5) and (8), $(\dot{x}(\cdot),\dot{u}(\cdot),\dot{\rho}(\cdot))$ is uniformly continuous for each $(x(0),u(0),\rho(0)) \in \Omega_K$.[1] Similarly, the uniform continuity of $\ddot{x}(\cdot)$ for each $(x(0),u(0),\rho(0)) \in \Omega_K$ can be shown from the dynamics of $\dot{x}$ in (6) with the continuity of $\partial f/\partial x$ and $\partial f/\partial u$ and the uniform continuity and boundedness of $(x(\cdot),u(\cdot))$ and $(\dot{x}(\cdot),\dot{u}(\cdot))$.

Now, we are ready to apply Barbalat's lemma to (A.1). It follows from (7c) with the continuity of W_K and the uniform continuity of (x,u,ρ), $(\dot{x},\dot{u},\dot{\rho})$, and $\ddot{x}$ on Ω_K that

$$\begin{aligned}&\lim_{t\to\infty} W_K(x(\tau),u(\tau),\dot{x}(\tau)) = 0 \\ &\quad \iff \lim_{t\to\infty} \dot{x}(t) = 0 \implies \lim_{t\to\infty} \ddot{x}(t) = 0\end{aligned}$$

and

$$\lim_{t\to\infty} K_1 E^\top M\dot{y}(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} K_2(\dot{y}(t)-\dot{\rho}(t)) = 0$$

for each $(x(0),u(0),\rho(0)) \in \Omega_K$. Therefore, it holds from (6) with $\partial f/\partial u$ of full column rank and (8) that

$$\lim_{t\to\infty} \dot{u}(t) = 0 \iff \lim_{t\to\infty} M^\top E E^\top M y(t) = 0$$

for each $(x(0),u(0),\rho(0)) \in \Omega_K$. From the property of E, we have weighted output consensus (9). □

Appendix B. Proof of Theorem 3.4

Define

$$V_S(t,e_d,\xi,\rho) := S_S(t,e_d) + \frac{1}{2}(|\xi-\xi_d^*|^2 + |\rho-\rho_d^*|^2_{K_2}).$$

From (12d) and (13), its time-derivative along the closed-loop trajectory satisfies

$$\begin{aligned}&\dot{V}_S(t,e_d,\xi,\rho) \\ &\le -W_S(e_d) + (y-y_d^*)^\top(u-u_d^*)\end{aligned}$$

[1] A continuous function is uniformly continuous on a compact subset, and the composition of uniformly continuous functions is again uniformly continuous.

$$
\begin{aligned}
&+ (y - y_d^*)^\top M^\top E(\xi - \xi_d^*) + (\rho - \rho_d^*)^\top K_2(\dot\rho - \dot\rho_d^*) \\
&= -W_S(e_d) - |E^\top M(y - y_d^*)|_{K_1}^2 \\
&\quad - (y - y_d^*)^\top K_2(y - \rho - y_d^* + \rho_d^*) \\
&\quad + (\rho - \rho_d^*)^\top K_2(y - \rho - y_d^* + \rho_d^*) \\
&= -W_S(e_d) - |E^\top M(y - y_d^*)|_{K_1}^2 - |y - \rho - y_d^* + \rho_d^*|_{K_2}^2
\end{aligned}
$$

for all $(e_d, \xi, \rho) \in \mathcal{D}_S$. Taking the time integration yields

$$
\begin{aligned}
&V_S(t, e_d(t), \xi(t), \rho(t)) \\
&\quad + \int_{t_0}^{t} (W_S(e_d(\tau)) + |E^\top M(y(\tau) - y_d^*(\tau))|_{K_1}^2 \\
&\qquad\qquad + |y(\tau) - \rho(\tau) - y_d^*(\tau) + \rho_d^*(\tau)|_{K_2}^2) d\tau \\
&\le V_S(t_0, e_d(t_0), \xi(t_0), \rho(t_0)).
\end{aligned}
$$

As in the proof of Theorem 3.2, it is possible to show that the limit of the integral term at $t \to \infty$ exists and is finite for each $(e_d(t_0), \xi_d(t_0), \rho(t_0)) \in \Omega_S$ and every $t_0 \in \mathbb{R}$. Also, the uniform continuity of $(e_d(\cdot), \xi(\cdot), \rho(\cdot))$ and $(\dot e_d(\cdot), \dot\xi(\cdot), \dot\rho(\cdot))$ can be shown similarly starting from the positively invariance of the closed-loop system on compact Ω_S with the uniform continuity of $d(\cdot)$ and the properties of f, g, and h in (10).

Applying Barbalat's lemma, it follows from (12c) with the continuity of W_S and the uniform continuity of $(e_d(\cdot), \xi(\cdot), \rho(\cdot))$ and $\dot e_d(\cdot)$ on Ω_S that

$$
\begin{aligned}
&\lim_{t\to\infty} W_S(e_d(t)) = 0 \\
&\iff \lim_{t\to\infty} e_d(t) = 0 \implies \lim_{t\to\infty} \dot e_d(t) = 0 \qquad \text{(B.1)}
\end{aligned}
$$

and

$$
\lim_{t\to\infty} K_1 E^\top M(y(t) - y_d^*(t)) = 0 \qquad \text{(B.2)}
$$

$$
\lim_{t\to\infty} K_2(y(t) - \rho(t) - y_d^*(t) + \rho_d^*(t)) = 0 \qquad \text{(B.3)}
$$

for each $(e_d(t_0), \xi(t_0), \rho(t_0)) \in \Omega_S$ and every $t_0 \in \mathbb{R}$, where recall that the projection of Ω_S onto the error space $\mathcal{D}_S$ contains the origin by item (III). Rewriting $\dot e_d$ with (11) and applying (B.1) lead to

$$
\begin{aligned}
&\lim_{t\to\infty} (f(t, e_d(t) + x_d^*(t), d(t)) \\
&\qquad + g(t, e_d(t) + x_d^*(t), d(t))u(t) \\
&\qquad - (f(t, x_d^*(t), d(t)) + g(t, x_d^*(t), d(t))u_d^*(t))) \\
&= \lim_{t\to\infty} g(t, x_d^*(t), d(t))(u(t) - u_d^*(t)) = 0,
\end{aligned}
$$

where item (IV) and the continuity of f and g are utilized. Note that g is of full column rank. Thus, it follows from (13), (B.2), (B.3), and the uniform continuity of $\dot\xi(\cdot)$ and $\dot\xi_d^*(\cdot)$ that

$$
\begin{aligned}
&\lim_{t\to\infty} (u(t) - u_d^*(t)) = 0 \\
&\iff \lim_{t\to\infty} M^\top E(\xi(t) - \xi_d^*(t)) = 0 \\
&\implies \lim_{t\to\infty} M^\top E(\dot\xi(t) - \dot\xi_d^*(t)) = 0.
\end{aligned}
$$

Finally, (8) and $E^\top M y_d^*(\cdot) = 0$ in item (I) imply

$$
\begin{aligned}
&\lim_{t\to\infty} M^\top E E^\top M(y(t) - y_d^*(t)) \\
&= \lim_{t\to\infty} M^\top E E^\top M y(t) = 0
\end{aligned}
$$

for each $(e_d(t_0), \xi_d(t_0), \rho(t_0)) \in \Omega_S$ and every $t_0 \in \mathbb{R}$. From the property of E, we have weighted output consensus (9). □

Appendix C. Proof of Corollary 3.5

If $K_1 \succ 0$, then (B.2) and $E^\top M y_d^*(\cdot) = 0$ in item (I) imply (9). Moreover, this discussion holds even if $W_S(\cdot) = 0$. Finally, we remark that we do not use the input-affine structure of the system (10) in this proof in contrast to the proof of Theorem 3.4 above. □

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