ON A NONLOCAL ANALOG OF THE KURAMOTO-SIVASHINSKY EQUATION

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ABSTRACT. We study a nonlocal equation, analogous to the Kuramoto-Sivashinsky equation, in which short waves are stabilized by a possibly fractional diffusion of order less than or equal to two, and long waves are destabilized by a backward fractional diffusion of lower order. We prove the global existence, uniqueness, and analyticity of solutions of the nonlocal equation and the existence of a compact attractor. Numerical results show that the equation has chaotic solutions whose spatial structure consists of interacting traveling waves resembling viscous shock profiles.

Keywords: Kuramoto-Sivashinsky equation, spatial chaos, attractor.

1. INTRODUCTION

In this paper, we study a family of nonlinear, nonlocal pseudo-differential equations in one-space dimension for a function $u(x,t)$ given by

$$(1) \qquad \partial_t u + \partial_x \left(\frac{1}{2} u^2 \right) = \Lambda^\gamma u - \epsilon \Lambda^{1+\delta} u,$$

where $\epsilon > 0$ and Λ^s is the fractional derivative

$$\Lambda^s = \left(-\partial_x^2 \right)^{s/2}, \qquad \widehat{\Lambda^s u} = |\xi|^s \hat{u}.$$

We assume that the exponents δ, γ satisfy

$$(2) \qquad 0 < \delta \leq 1, \qquad 0 \leq \gamma < 1 + \delta.$$

Equation (1) consists of an inviscid Burgers equation with a higher-order linear pseudo-differential term that gives long-wave instability and short-wave stability. It is analogous to the well-known Kuramoto-Sivashinsky (KS) equation [28, 36, 37]

$$(3) \qquad \partial_t u + \partial_x \left(\frac{1}{2} u^2 \right) = -\partial_x^2 u - \epsilon \partial_x^4 u,$$

which has negative second-order diffusion stabilized by forth-order diffusion. By contrast, we consider (1) in the parameter regime (2), where the stabilizing diffusion is second-order or less.

A special case of (1), corresponding to $\gamma = \delta = 1$, is

$$(4) \qquad \partial_t u + \partial_x \left(\frac{1}{2} u^2 \right) = \Lambda u + \epsilon \partial_x^2 u,$$

which provides a simple model for the stabilization of a Hadamard instabil-
ity, with growth rate proportional to the absolute value of the wavenumber,
by second-order viscous diffusion. This type of instability occurs in scale-
invariant systems, such as conservation laws (e.g., the Kelvin-Helmholtz
instability for the Euler or MHD equations) and kinetic equations (e.g., the
Vlasov equations), in which the growth rate of long waves is determined by
a parameter with the dimensions of velocity. In particular, (4) provides a
model equation for the negative Landau damping of plasma waves [29, 33].

If $\gamma = 0$ and $\delta = 1$, then (1) is the Burgers-Sivashinsky (BS) equation
introduced by Goodman [20],

$$(5) \qquad \partial_t u + \partial_x \left(\frac{1}{2} u^2 \right) = u + \epsilon \partial_x^2 u.$$

For (5), the growth rate of long waves is bounded independently of the
wavenumber, and its dynamical behavior is much simpler than that of (1)
with $\gamma > 0$.

The KS equation (3) exhibits chaotic behavior and possesses a compact
global attractor [31, 32]. Furthermore, it has an inertial manifold [16] that
appears to contain a chaotic attractor when ϵ is sufficiently small. (See
[4, 7, 20, 19, 34] for further results). The spatial analyticity of solutions
of the KS equation is addressed in [6, 22] and the temporal analyticity in
[23]. More recently, the authors in [1, 15, 41] have used computer-assisted
methods to study the dynamics of the solutions.

In this paper, we prove that (1) possesses a compact global attractor in
the parameter range (2) (see Theorem 6). Moreover, numerical solutions
indicate that if $0 < \gamma < 1 + \delta$, then (1) exhibits chaotic behavior with an in-
teresting spatial structure. Waves that resemble thin viscous shocks appear
spontaneously at different points, after which they propagate toward and
merge with a primary viscous shock. This spatial behavior is qualitatively
different from what one sees in the usual KS equation. (See Section 6.) By
contrast, solutions of the BS equation (5), with $\gamma = 0$, do not behave chaot-
ically; instead, they approach a time-independent viscous sawtooth wave
solution as $t \to \infty$ [20].

The numerical results suggest that (1) with exponents (2) may have an in-
ertial manifold that can be parametrized in some way by the viscous shocks.
We do not investigate this question here, but in Section 5.2 we obtain an up-
per bound on the number of oscillations in solutions of (1) (see Theorem 7).

Nonlocal KS equations similar to (1) have been studied previously by
Frankel and Roytburd [18]. Their results, however, are less detailed than
ours and they apply only in the case when $\delta \geq 1$. A different type of nonlocal
generalization of the KS equation has been studied in [3, 12].

We conclude the introduction by outlining the contents of this paper. In
Section 2, we prove the global existence of smooth solutions of (1), and
in Section 3, we prove that these solutions gain analyticity in a strip. In
Sections 4–5, we prove the existence of an attractor for (1), and in Section 6,
we show some numerical solutions.

2. Global existence of solutions

In this section we use a classical energy method to prove the global existence of solutions of the initial value problem for (1),

$$(6) \qquad \partial_t u + \partial_x \left(\frac{1}{2} u^2 \right) = \Lambda^\gamma u - \epsilon \Lambda^{1+\delta} u, \qquad x \in \Omega,\, t > 0,$$

$$u(x, 0) = u_0(x), \qquad x \in \Omega.$$

We consider either spatially periodic solutions or solutions on the real line, with $\Omega = \mathbb{T}$ or $\Omega = \mathbb{R}$ as appropriate. In the periodic case, we normalize the length of $\mathbb{T}$ to 2π.

To prove the existence result, we first obtain an *a priori* L^∞-estimate, using the ideas in [8] to handle the nonlocal operators (see also [2, 9]). This step of the proof depends on the choice of Ω and δ and is different in each case. In Section 2.1 we obtain the existence of solutions for $0 < \delta < 1$. The gain of derivatives can be as small as $1/2 + \delta/2$, so the well-posedness results are more delicate than for the usual KS or BS equations. In Section 2.2 we treat the simpler case $\delta = 1$. To simplify the notation, we omit the t-dependence of u when convenient and use C to denote a (harmless) constant that can change from one line to another

First, we define what we mean by a weak solution of (6). We denote the usual Sobolev spaces of functions with weak L^2-derivatives of the order less than or equal to s by $H^s(\Omega)$, or H^s, and the real or periodic spatial Hilbert transform, with symbol $-i\,\mathrm{sgn}\,\xi$, by $\mathcal{H}$. In particular, $\Lambda = \mathcal{H}\partial_x$.

Definition 1. *Let $T > 0$. A function $u(x, t)$ with*

$$u(x, t) \in L^2([0, T], H^{\frac{1+\delta}{2}}), \qquad \partial_t u(x, t) \in L^2([0, T], H^{-\frac{1+\delta}{2}})$$

is a weak solution of (6) *if the following equality holds for all test functions* $\phi \in H^{\frac{1+\delta}{2}}(\Omega)$,

$$\int_\Omega \phi\, \partial_t u\, dx - \frac{1}{2} \int_\Omega \Lambda^{\frac{1+\delta}{2}} \phi\, \Lambda^{1 - \frac{1+\delta}{2}} \mathcal{H}(u^2)\, dx$$

$$= \int_\Omega \Lambda^{\gamma/2} \phi\, \Lambda^{\gamma/2} u\, dx - \epsilon \int_\Omega \Lambda^{(1+\delta)/2} \phi\, \Lambda^{(1+\delta)/2} u\, dx \quad a.e.\ 0 < t < T,$$

and $u(x, 0) = u_0(x)$.

We remark that the L^2-boundedness of $\mathcal{H}$, a Moser-type inequality [39], and Sobolev inequalities, imply that

$$\| \Lambda^{1 - \frac{1+\delta}{2}} \mathcal{H}(u^2) \|_{L^2} \le \| \Lambda^{\frac{1-\delta}{2}} (u^2) \|_{L^2} \le C \| u^2 \|_{H^{\frac{1-\delta}{2}}} \le C \| u \|_{L^\infty} \| u \|_{H^{\frac{1-\delta}{2}}}$$

$$\le C \| u \|_{H^{\frac{1+\delta}{2}}} \| u \|_{H^{\frac{1-\delta}{2}}},$$

so the nonlinear term in this weak formulation is well-defined.

2.1. **The case** $0 < \delta < 1$**.** First, we consider spatially periodic solutions. Since the mean of u is preserved by the evolution, we can restrict ourselves to periodic initial data with zero mean,

$$\int_{\mathbb{T}} u_0(x)\, dx = 0.$$

Lemma 1. *If $u(x, t)$ is a spatially periodic, smooth solution of* (6), *then*

$$\|u(t)\|_{L^\infty(\mathbb{R})} \leq \|u_0\|_{L^\infty(\mathbb{R})} \exp\left(C(\epsilon, \gamma, \delta)t\right).$$

Proof. The fractional derivatives can be written as [2]

$$
\begin{aligned}
\Lambda^\alpha u(x) &= \frac{\Gamma(1+\alpha)\cos\left((1-\alpha)\frac{\pi}{2}\right)}{\pi} \text{P.V.} \int_{\mathbb{R}} \frac{u(x) - u(\eta)}{|x - \eta|^{1+\alpha}} d\eta \\
&= \frac{\Gamma(1+\alpha)\cos\left((1-\alpha)\frac{\pi}{2}\right)}{\pi} \sum_{k \in \mathbb{Z}} \text{P.V.} \int_{\mathbb{T}} \frac{u(x) - u(\eta)}{|x - \eta - 2k\pi|^{1+\alpha}} d\eta \\
&= \frac{\Gamma(1+\alpha)\cos\left((1-\alpha)\frac{\pi}{2}\right)}{\pi} \sum_{k \in \mathbb{Z}} \text{P.V.} \int_{\mathbb{T}} \frac{u(x) - u(x - \eta)}{|\eta - 2k\pi|^{1+\alpha}} d\eta
\end{aligned}
$$

(7)

and

$$\Lambda u(x) = \frac{1}{2\pi} \text{P.V.} \int_{\mathbb{T}} \frac{u(x) - u(x - \eta)}{\sin^2\left(\frac{\eta}{2}\right)} d\eta.$$

We start the proof with the case $\gamma = 1$, for which we have a concise expression for the kernel. Let x_t denote the point where $u(\cdot, t)$ attains its maximum, and suppose that the L^∞-norm $\|u(t)\|_{L^\infty(\mathbb{R})} = u(x_t)$ is achieved at the maximum of u. A straightforward calculation shows that $u(x_t)$ is a Lipschitz continuous function of t, so Rademacher's Theorem [13] implies that $\|u(t)\|_{L^\infty(\mathbb{R})}$ is differentiable pointwise almost everywhere. Now we can apply the technique developed in [2, 8, 9, 10], to obtain the evolution of $du(x_t)/dt$. Using the expressions for the kernels, we get

$$
\begin{aligned}
\frac{d}{dt}\|u(t)\|_{L^\infty(\mathbb{R})} &\leq \frac{1}{2\pi} \text{P.V.} \int_{\mathbb{T}} (u(x_t) - u(x_t - \eta)) \left(\frac{1}{\sin^2\left(\frac{\eta}{2}\right)} - \frac{1}{\left(\frac{\eta}{2}\right)^2} \right) d\eta \\
&\quad + \frac{1}{2\pi} \text{P.V.} \int_{\mathbb{T}} (u(x_t) - u(x_t - \eta)) \left(\frac{1}{\left(\frac{\eta}{2}\right)^2} - \frac{2\epsilon\Gamma(2+\delta)\cos\left(\delta\frac{\pi}{2}\right)}{|\eta|^{2+\delta}} \right) d\eta.
\end{aligned}
$$

The first term is not singular and can be estimated as follows:

$$
\begin{aligned}
I_1 &= \frac{1}{2\pi} \text{P.V.} \int_{\mathbb{T}} (u(x_t) - u(x_t - \eta)) \left(\frac{1}{\sin^2\left(\frac{\eta}{2}\right)} - \frac{1}{\left(\frac{\eta}{2}\right)^2} \right) d\eta \\
&\leq \frac{2\|u(t)\|_{L^\infty(\mathbb{T})}}{\pi} \int_0^\pi \left(\frac{1}{\sin^2\left(\frac{\eta}{2}\right)} - \frac{1}{\left(\frac{\eta}{2}\right)^2} \right) d\eta \\
&\leq \frac{8}{\pi^2} \|u(t)\|_{L^\infty(\mathbb{T})}.
\end{aligned}
$$

Notice that there exists $\omega = \omega(\delta, \epsilon)$ such that for $0 < |\eta| \leq \omega$, we have

$$(u(x_t) - u(x_t - \eta)) \left(\frac{1}{\left(\frac{\eta}{2}\right)^2} - \frac{2\epsilon\Gamma(2+\delta)\cos\left(\delta\frac{\pi}{2}\right)}{|\eta|^{2+\delta}} \right) \leq 0$$

We split the second term as

$$
\begin{aligned}
I_2 &= \frac{\text{P.V.}}{2\pi} \int_{\mathbb{T}} (u(x_t) - u(x_t - \eta)) \left(\frac{1}{\left(\frac{\eta}{2}\right)^2} - \frac{2\epsilon\Gamma(2+\delta)\cos\left(\delta\frac{\pi}{2}\right)}{|\eta|^{2+\delta}} \right) d\eta \\
&\leq J_1 + J_2
\end{aligned}
$$

with

$$J_1 = \frac{\text{P.V.}}{2\pi} \int_{B(0,\omega)} (u(x_t) - u(x_t - \eta)) \left(\frac{1}{\left(\frac{\eta}{2}\right)^2} - \frac{2\epsilon\Gamma(2+\delta)\cos\left(\delta\frac{\pi}{2}\right)}{|\eta|^{2+\delta}} \right) d\eta$$
$$\leq 0,$$

and

$$J_2 = \frac{1}{2\pi} \int_{B^c(0,\omega)} (u(x_t) - u(x_t - \eta)) \left(\frac{1}{\left(\frac{\eta}{2}\right)^2} - \frac{2\epsilon\Gamma(2+\delta)\cos\left(\delta\frac{\pi}{2}\right)}{|\eta|^{2+\delta}} \right) d\eta$$
$$\leq C(\epsilon,\delta)\|u(t)\|_{L^\infty(\mathbb{T})},$$

thus,

$$I_2 = J_1 + J_2 \leq J_2 \leq C(\epsilon,\delta)\|u(t)\|_{L^\infty(\mathbb{T})}.$$

The same argument applies if $\|u(t)\|_{L^\infty(\mathbb{R})} = -\min_{x\in\mathbb{T}} u(x,t)$, so

$$\|u(t)\|_{L^\infty(\mathbb{R})} \leq \|u_0\|_{L^\infty(\mathbb{R})} \exp\left(C(\epsilon,\delta)t\right).$$

In the general case $\gamma \neq 1$, some extra terms appear. These terms correspond to $|k| \geq 1$ in (7). Since they are not singular, they can be estimated as follows:

$$\frac{\Gamma(1+\gamma)\cos\left((1-\gamma)\frac{\pi}{2}\right)}{\pi} \sum_{|k|\geq 1} \text{P.V.} \int_{\mathbb{T}} \frac{u(x_t) - u(x_t - \eta)}{|\eta - 2k\pi|^{1+\gamma}} d\eta$$
$$\leq C(\gamma)\|u(t)\|_{L^\infty(\mathbb{T})}.$$

The rest of the proof remains unchanged. $\qquad\square$

Next, we prove our main existence result.

Theorem 1. *Suppose that $\epsilon > 0$, $0 < \delta < 1$, and $0 \leq \gamma < 1 + \delta$. If*

$$u_0 \in H^\alpha(\mathbb{T}) \cap L^\infty(\mathbb{T}),$$

then the following statements hold:

- *If $\alpha \geq 2 + \delta$, then for every $0 < T < \infty$ the initial value problem (6) has a unique classical solution*

$$u(x,t) \in C([0,T], H^\alpha(\mathbb{T})).$$

- *If $(1-\delta)/2 < \alpha < 2 + \delta$, then for every $0 < T < \infty$ there exists a weak solution of (6) (see Definition 1) such that*

$$u(x,t) \in L^\infty([0,T], H^\alpha(\mathbb{T}) \cap L^\infty(\mathbb{T})) \cap C([0,T], H^s(\mathbb{T}) \cap L^p(\mathbb{T}))$$

  for every $0 \leq s < \alpha$ and $2 \leq p < \infty$.
- *These solutions gain regularity and satisfy*

$$u(x,t) \in L^2([0,T], H^{\alpha + \frac{1+\delta}{2}}(\mathbb{T})).$$

  Moreover, if $3/2 < \alpha + (1+\delta)/2$, then this weak solution is unique.

Proof. **Step 1: L^2 estimate.** We multiply (1) by u and integrate by parts:

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 = -\frac{\epsilon}{2}\|\Lambda^{\frac{1+\delta}{2}}u\|_{L^2}^2 + \int_{\mathbb{T}} u(x)\left(\Lambda^\gamma - \frac{\epsilon}{2}\Lambda^{1+\delta}\right) u\,dx.$$

Using the Fourier transform, we get

$$\int_{\mathbb{T}} u(x) \left(\Lambda^\gamma - \frac{\epsilon}{2}\Lambda^{1+\delta}\right) u \, dx \le \left(\frac{2\gamma}{\epsilon(1+\delta)}\right)^{1/(1+\delta-\gamma)} \|u(t)\|_{L^2(\mathbb{T})}^2.$$

Inserting this into the previous bound we obtain

$$\frac{d}{dt}\|u\|_{L^2}^2 \le -\epsilon\|\Lambda^{\frac{1+\delta}{2}}u\|_{L^2}^2 + 2\left(\frac{2\gamma}{\epsilon(1+\delta)}\right)^{1/(1+\delta-\gamma)} \|u(t)\|_{L^2(\mathbb{T})}^2,$$

and using Gronwall inequality,

(8)

$$\|u(t)\|_{L^2(\mathbb{T})}^2 + \epsilon \int_0^t \exp\left(2\left(\frac{2\gamma}{\epsilon(1+\delta)}\right)^{1/(1+\delta-\gamma)}(t-s)\right)\|\Lambda^{\frac{1+\delta}{2}}u(s)\|_{L^2}^2 \, ds$$

$$\le \|u_0\|_{L^2(\mathbb{R})}^2 \exp\left(2\left(\frac{2\gamma}{\epsilon(1+\delta)}\right)^{1/(1+\delta-\gamma)} t\right).$$

In particular

$$\|u(t)\|_{L^2(\mathbb{T})} + \epsilon \int_0^t \|\Lambda^{\frac{1+\delta}{2}}u(s)\|_{L^2}^2 \, ds$$

$$\le \|u_0\|_{L^2(\mathbb{T})}^2 \exp\left(2\left(\frac{2\gamma}{\epsilon(1+\delta)}\right)^{1/(1+\delta-\gamma)} t\right).$$

Step 2: H^α estimate. We multiply (1) by $\Lambda^{2\alpha}u$ and integrate, which gives

$$\frac{d}{dt}\|\Lambda^\alpha u\|_{L^2(\mathbb{T})}^2 = I_1 + I_2 + I_3,$$

where

$$I_1 = \int_{\mathbb{T}} \Lambda^{\alpha+\frac{1+\delta}{2}}u\Lambda^{\alpha+1-\frac{1-\delta}{2}}\mathcal{H}(u^2) \, dx,$$

$$I_2 = 2\int_{\mathbb{T}} \Lambda^\alpha u\left(\Lambda^\gamma - \frac{\epsilon}{2}\Lambda^{1+\delta}\right)\Lambda^\alpha u \, dx \le C(\epsilon,\gamma,\delta)\|\Lambda^\alpha u\|_{L^2(\mathbb{T})}^2,$$

$$I_3 = -\epsilon\|\Lambda^{\alpha+\frac{1+\delta}{2}}u\|_{L^2(\mathbb{T})}^2.$$

The term I_1 can be handled as follows (see also [26]): We use the Cauchy-Schwarz and Kato-Ponce inequalities (see Lemma 6) and the properties of the Hilbert transform (see [38]) to get

$$I_1 \le \|\Lambda^{\alpha+\frac{1+\delta}{2}}u\|_{L^2(\mathbb{T})}\|\Lambda^{\alpha+1-\frac{1+\delta}{2}}(u^2)\|_{L^2(\mathbb{T})}$$

$$\le C\|\Lambda^{\alpha+\frac{1+\delta}{2}}u\|_{L^2(\mathbb{T})}\|\Lambda^{\alpha+1-\frac{1+\delta}{2}}u\|_{L^2(\mathbb{T})}\|u\|_{L^\infty(\mathbb{T})}.$$

Then, using

$$\alpha + 1 - \frac{1+\delta}{2} = t(\alpha + \frac{1+\delta}{2}) + (1-t)\alpha,$$

for $t = -1 + 2/(1+\delta)$, and Hölder's inequality on the Fourier side (with $p = 1/t$ and $q = 1/(1-t)$), we write

$$\|\Lambda^{\alpha+1-\frac{1+\delta}{2}}u\|_{L^2(\mathbb{T})}^2 \le \|\Lambda^{\alpha+\frac{1+\delta}{2}}u\|_{L^2(\mathbb{T})}^{2t}\|\Lambda^\alpha u\|_{L^2(\mathbb{T})}^{2(1-t)}.$$

Inserting this into the bound for I_1, we obtain

$$I_1 \leq C\|\Lambda^{\alpha+\frac{1+\delta}{2}}u\|_{L^2(\mathbb{T})}^{1+t}\|\Lambda^\alpha u\|_{L^2(\mathbb{T})}^{1-t}\|u\|_{L^\infty(\mathbb{T})}.$$

Using Hölder's inequality again (with $p = 2/(1+t)$ and $q = 2/(1-t)$), we get

$$I_1 \leq C(\epsilon,\delta)\|\Lambda^\alpha u\|_{L^2(\mathbb{T})}^2\|u\|_{L^\infty(\mathbb{T})}^{2/(1-t)} + \frac{\epsilon}{2}\|\Lambda^{\alpha+\frac{1+\delta}{2}}u\|_{L^2(\mathbb{T})}^2.$$

Using the estimate for $\|u(t)\|_{L^\infty(\mathbb{T})}$ and putting all the estimates together, we obtain

$$(9) \quad \frac{d}{dt}\|\Lambda^\alpha u\|_{L^2(\mathbb{T})}^2 \leq \|\Lambda^\alpha u\|_{L^2(\mathbb{T})}^2 \exp\left(C\left(\epsilon,\gamma,\delta,\|u_0\|_{L^\infty(\mathbb{T})}\right)(1+t)\right)$$
$$- \frac{\epsilon}{2}\|\Lambda^{\alpha+\frac{1+\delta}{2}}u\|_{L^2(\mathbb{T})}^2.$$

Finally, from Gronwall inequality, we conclude that

$$\|\Lambda^\alpha u(t)\|_{L^2(\mathbb{T})}^2 + \frac{\epsilon}{2}\int_0^t e^{e^{C\left(\epsilon,\gamma,\delta,\|u_0\|_{L^\infty(\mathbb{T})}\right)(1+t-s)}}\|\Lambda^{\alpha+\frac{1+\delta}{2}}u(s)\|_{L^2(\mathbb{T})}^2 ds$$
$$\leq \|\Lambda^\alpha u_0\|_{L^2(\mathbb{T})}^2 \exp\left(\exp\left(C\left(\epsilon,\gamma,\delta,\|u_0\|_{L^\infty(\mathbb{T})}\right)(1+t)\right)\right).$$

In particular,

$$\int_0^t \|\Lambda^{\alpha+\frac{1+\delta}{2}}u\|_{L^2(\mathbb{T})}^2 ds \leq \frac{2}{\epsilon}\|\Lambda^\alpha u_0\|_{L^2(\mathbb{T})}^2 e^{e^{C\left(\epsilon,\gamma,\delta,\|u_0\|_{L^\infty(\mathbb{T})}\right)(1+t)}}.$$

Step 3: Strong solutions. We denote by $\mathcal{J}_\epsilon$ a positive, symmetric mollifier. Then, in the case $\alpha > 2 + \delta$, we define the regularized problems

$$(10) \qquad \partial_t u_\vartheta + \mathcal{J}_\vartheta * \frac{\partial_x((\mathcal{J}_\vartheta * u_\vartheta)^2)}{2} = \mathcal{J}_\vartheta * \left(\Lambda^\gamma - \epsilon\Lambda^{1+\delta}\right)\mathcal{J}_\vartheta * u_\vartheta,$$

with initial data

$$u_\vartheta(0) = u_0.$$

By Picard's Theorem, these regularized problems have a unique solution $u_\vartheta \in C^1([0,T], H^\alpha(\mathbb{T}))$. Moreover, since the *a priori* estimates remain valid, these solutions are global in time. Thus, for every $T > 0$ there exists

$$u(x,t) \in L^\infty\left([0,T], H^\alpha(\mathbb{T})\right)$$

such that (after picking a subsequence)

$$u_\vartheta \rightharpoonup u \qquad \text{in} \quad L^2\left([0,T], H^{\alpha+\frac{1+\delta}{2}}(\mathbb{T})\right).$$

Next, we want to show that $u_\vartheta \to u$ in $C\left([0,T], L^2(\mathbb{T})\right)$. The method is classical (see e.g., [30]) and we only sketch the proof. We subtract the regularized problems corresponding to labels ϑ and ϖ:

$$\partial_t u_\vartheta - \partial_t u_\varpi + \mathcal{J}_\vartheta * \frac{\partial_x((\mathcal{J}_\vartheta * u_\vartheta)^2)}{2} - \mathcal{J}_\varpi * \frac{\partial_x((\mathcal{J}_\varpi * u_\varpi)^2)}{2}$$
$$= \mathcal{J}_\vartheta * \left(\Lambda^\gamma - \epsilon\Lambda^{1+\delta}\right)\mathcal{J}_\vartheta * u_\vartheta - \mathcal{J}_\varpi * \left(\Lambda^\gamma - \epsilon\Lambda^{1+\delta}\right)\mathcal{J}_\varpi * u_\varpi.$$

From this equation, we obtain

$$\|u_\vartheta - u_\varpi\|_{C([0,T],L^2(\mathbb{T}))} \leq C(T,u_0,\gamma,\epsilon,\delta)\max\{\varpi - \vartheta\},$$

and we get that

$$u_\vartheta \to u \text{ in } C\left([0,T], L^2(\mathbb{T})\right).$$

Using interpolation and the parabolic character of the equation, we have

$$u_\vartheta \to u \text{ in } C\left([0,T], H^\alpha(\mathbb{T})\right),$$

which shows that u is a classical solution. Uniqueness follows by energy estimates.

Step 4: Regularized problems and compactness. We define the regularized problems

$$(11) \qquad \partial_t u_\vartheta + \partial_x \left(\frac{1}{2} u_\vartheta^2\right) = \left(\Lambda^\gamma - \epsilon \Lambda^{1+\delta}\right) u_\vartheta,$$

with initial data

$$u_\vartheta(0) = \mathcal{J}_\vartheta * u_0.$$

These problems have a global in time, smooth solution. Moreover, due to the energy estimates in the previous step, these solutions satisfy a uniform bound in the space

$$u_\vartheta \in L^p([0,T], H^\alpha(\mathbb{R}) \cap L^\infty(\mathbb{R}))$$

for all $1 \le p \le \infty$, and

$$u_\vartheta \in L^2([0,T], H^{\alpha+\frac{1+\delta}{2}}(\mathbb{R})).$$

In particular, we get weak convergence in $L^2([0,T], H^{\alpha+\frac{1+\delta}{2}}(\mathbb{R}))$ and weak-$*$ convergence in $L^\infty([0,T], L^\infty(\mathbb{R}))$ of a subsequence to a function u. Moreover, by the weak lower semi-continuity of the norm, we have

$$\|u\|_{L^2([0,T], H^{\alpha+\frac{1+\delta}{2}}(\mathbb{R}))}, \ \|u\|_{L^\infty([0,T], L^\infty(\mathbb{R}))} \le C(\epsilon, \delta, \gamma, u_0).$$

The dual space of $H^{(1+\delta)/2}(\mathbb{T})$ is $H^{-(1+\delta)/2}(\mathbb{T})$, and the corresponding norm of a function f is given by

$$\|f\|_{H^{-(1+\delta)/2}(\mathbb{T})} = \sup_{\|\psi\|_{H^{(1+\delta)/2}(\mathbb{T})} \le 1} \left|\int_{\mathbb{T}} f\psi dx\right|.$$

We have

$$H^\alpha(\mathbb{T}) \hookrightarrow L^2(\mathbb{T}) \hookrightarrow H^{-(1+\delta)/2}(\mathbb{T}),$$

where the first inclusion is compact and the second inclusion is continuous (see [11]). To invoke the Aubin-Lions compactness Theorem (see Corollary 4, Section 8 in [35]) we need uniform bounds in the Bochner spaces

$$u_\vartheta \in L^\infty([0,T], H^\alpha(\mathbb{T})), \qquad \partial_t u_\vartheta \in L^2([0,T], H^{-(1+\delta)/2}(\mathbb{T})).$$

Multiplying (11) by $\psi \in H^{(1+\delta)/2}(\mathbb{T})$ and integrating by parts, we obtain

$$\|\partial_t u_\vartheta\|_{H^{-(1+\delta)/2}(\mathbb{T})} \le \|\Lambda^{\frac{1-\delta}{2}} u_\vartheta^2\|_{L^2(\mathbb{T})} + \|\Lambda^{\frac{\gamma}{2}} u_\vartheta\|_{L^2(\mathbb{T})} + \epsilon \|\Lambda^{\frac{1+\delta}{2}} u_\vartheta\|_{L^2(\mathbb{T})}$$

$$\le \|\Lambda^{\frac{1-\delta}{2}} u_\vartheta\|_{L^2(\mathbb{T})} \|u_\vartheta\|_{L^\infty(\mathbb{T})} + \|\Lambda^{\frac{\gamma}{2}} u_\vartheta\|_{L^2(\mathbb{T})} + \epsilon \|\Lambda^{\frac{1+\delta}{2}} u_\vartheta\|_{L^2(\mathbb{T})}$$

Recalling that the energy estimates gives us uniform bounds

$$u_\vartheta \in L^2([0,T], H^{(1+\delta)/2}(\mathbb{T})), \text{ and } u_\vartheta \in L^\infty([0,T], L^\infty(\mathbb{T})),$$

and using Poincaré inequality, we get a uniform bound

$$\partial_t u_\vartheta \in L^2([0,T], H^{-(1+\delta)/2}(\mathbb{T})).$$

Thus, we get

(12a) $$u_\vartheta \rightharpoonup u \in L^2([0,T], H^{\alpha + \frac{1+\delta}{2}}(\mathbb{T})),$$

(12b) $$\partial_t u_\vartheta \rightharpoonup \partial_t u \in L^2([0,T], H^{-(1+\delta)/2}(\mathbb{T})).$$

Applying the Aubin-Lions Lemma, we get that

(13a) $$u_\vartheta \rightarrow u \in C([0,T], L^2(\mathbb{T})),$$

(13b) $$u_\vartheta \rightarrow u \in C([0,T], L^p(\mathbb{T})) \qquad \text{for all } 2 \le p < \infty.$$

Then, using interpolation in Sobolev spaces, we get

(14) $$u_\vartheta \rightarrow u \in C([0,T], H^s(\mathbb{T})), \ 0 \le s < \alpha.$$

Step 5: Convergence of the weak formulation. We need to show that the limit u of the regularized solutions in the previous step is a weak solution in the sense of Definition 1. Let $\phi \in H^{(1+\delta)/2}(\mathbb{T})$ be a test function. Using the properties of mollifiers we obtain $u_\vartheta(0) \rightarrow u_0$ in L^2. To show convergence in the equation, we have to deal with the nonlinear term.

For $0 < \delta < 1/2$, we have $H^\delta \hookrightarrow L^{2/(1-2\delta)}$ and

$$
\begin{aligned}
\int_{\mathbb{T}} &\Lambda^{(1+\delta)/2} \phi \Lambda^{(1-\delta)/2} \mathcal{H}\left(u_\vartheta^2 - u^2\right) \, dx \\
&\le C(\phi) \|\Lambda^{(1-\delta)/2} \mathcal{H}\left(u_\vartheta^2 - u^2\right)\|_{L^2(\mathbb{T})} \\
&\le C(\phi) \left(\|\Lambda^{(1-\delta)/2}(u_\vartheta + u)\|_{L^{2/(1-2\delta)}(\mathbb{T})} \|u_\vartheta - u\|_{L^{1/\delta}(\mathbb{T})} \right. \\
&\qquad\qquad \left. + \|\Lambda^{(1-\delta)/2}(u_\vartheta - u)\|_{L^2(\mathbb{T})} \|u_\vartheta + u\|_{L^\infty(\mathbb{T})} \right) \\
&\le C(\phi) \left(\|\Lambda^{(1+\delta)/2}(u_\vartheta + u)\|_{L^2(\mathbb{T})} \|u_\vartheta - u\|_{L^{1/\delta}(\mathbb{T})} \right. \\
&\qquad\qquad \left. + \|\Lambda^{(1-\delta)/2}(u_\vartheta - u)\|_{L^2(\mathbb{T})} \|u_\vartheta + u\|_{L^\infty(\mathbb{T})} \right) \\
&\le C(\phi, \epsilon, \delta, u_0, \gamma) \left(\|u_\vartheta - u\|_{L^{1/\delta}(\mathbb{T})} \right. \\
&\qquad\qquad \left. + \|\Lambda^{(1-\delta)/2}(u_\vartheta - u)\|_{L^2(\mathbb{T})} \right).
\end{aligned}
$$

For $\delta = 1/2$, we use $H^{1/2} \hookrightarrow L^4$ to get

$$
\begin{aligned}
\|\Lambda^{(1-\delta)/2} &\mathcal{H}\left(u_\vartheta^2 - u^2\right)\|_{L^2(\mathbb{T})} \\
&\le \left(\|\Lambda^{(1-\delta)/2}(u_\vartheta + u)\|_{L^4(\mathbb{T})} \|u_\vartheta - u\|_{L^4(\mathbb{T})} \right. \\
&\qquad\qquad \left. + \|\Lambda^{(1-\delta)/2}(u_\vartheta - u)\|_{L^2(\mathbb{T})} \|u_\vartheta + u\|_{L^\infty(\mathbb{T})} \right) \\
&\le \left(\|\Lambda^{(1+\delta)/2}(u_\vartheta + u)\|_{L^2(\mathbb{T})} \|u_\vartheta - u\|_{L^4(\mathbb{T})} \right. \\
&\qquad\qquad \left. + \|\Lambda^{(1-\delta)/2}(u_\vartheta - u)\|_{L^2(\mathbb{T})} \|u_\vartheta + u\|_{L^\infty(\mathbb{T})} \right).
\end{aligned}
$$

For $1/2 < \delta \leq 1$, we have $H^\delta \hookrightarrow L^\infty$ and

$$\|\Lambda^{(1-\delta)/2}\mathcal{H}\left(u_\vartheta^2 - u^2\right)\|_{L^2(\mathbb{T})}$$
$$\leq \left(\|\Lambda^{(1-\delta)/2}(u_\vartheta + u)\|_{L^\infty(\mathbb{T})}\|u_\vartheta - u\|_{L^2(\mathbb{T})}\right.$$
$$\left. + \|\Lambda^{(1-\delta)/2}(u_\vartheta - u)\|_{L^2(\mathbb{T})}\|u_\vartheta + u\|_{L^\infty(\mathbb{T})}\right)$$
$$\leq \left(\|\Lambda^{(1+\delta)/2}(u_\vartheta + u)\|_{L^2(\mathbb{T})}\|u_\vartheta - u\|_{L^2(\mathbb{T})}\right.$$
$$\left. + \|\Lambda^{(1-\delta)/2}(u_\vartheta - u)\|_{L^2(\mathbb{T})}\|u_\vartheta + u\|_{L^\infty(\mathbb{T})}\right)$$

Using (13b) and (14), we obtain

$$(15) \qquad \sup_t \left|\int_\mathbb{T} \Lambda^{(1+\delta)/2}\phi \Lambda^{(1-\delta)/2}\mathcal{H}\left(u_\vartheta^2 - u^2\right)dx\right| \to 0.$$

Next, we test against $\phi \in C^1([0,T], H^{(1+\delta)/2}(\mathbb{T}))$ and integrate in time. Equation (12a) gives

$$\int_0^T \int_\mathbb{T} \Lambda^s \phi(t)\Lambda^s(u_\vartheta(t) - u(t))\,dxdt \to 0 \qquad 0 \leq s \leq \alpha + \frac{1+\delta}{2},$$

which ensures the convergence of the linear terms with $s = \gamma/2, (1+\delta)/2$, while (15) ensures the convergence of the nonlinear terms. Since

$$C^1([0,T], H^{(1+\delta)/2}(\mathbb{T}))$$

is dense in

$$L^2([0,T], H^{(1+\delta)/2}(\mathbb{T})),$$

it follows that u satisfies the weak formulation for every

$$\phi \in L^2([0,T], H^{(1+\delta)/2}(\mathbb{T})).$$

Taking ϕ independent of t, we find that the weak formulation holds almost everywhere in time, which completes the proof of the existence of weak solutions.

Step 6: Uniqueness of weak solutions. Suppose that u_1, u_2 are weak solutions of (6) with the same initial data and let $w = u_1 - u_2$. Testing against w, we have

$$\frac{1}{2}\frac{d}{dt}\|w\|_{L^2}^2 = -\int_\mathbb{R} w^2\partial_x u_1 + wu_2\partial_x w dx$$
$$+ \int_\mathbb{R} w\Lambda^\gamma w - \epsilon\int_\mathbb{R} w\Lambda^{1+\delta}w dx$$
$$\leq C\|w\|_{L^2}^2(\|u_1\|_{H^{1.5+\varepsilon}} + \|u_2\|_{H^{1.5+\varepsilon}} + 1),$$

and Gronwall's inequality implies that $w = 0$. $\qquad\square$

The proof of global existence for $\Omega = \mathbb{R}$ is similar to the one for $\Omega = \mathbb{T}$, but we need to modify the proof of the L^∞-estimate to account for the difference in the kernel of the fractional derivatives.

Lemma 2. *If $u(x,t)$ is a smooth solution of* (6) *on $\Omega = \mathbb{R}$, then*

$$\|u(t)\|_{L^\infty(\mathbb{R})} \leq \|u_0\|_{L^\infty(\mathbb{R})}\exp\left(C(\epsilon, \gamma, \delta)t\right).$$

Proof. The fractional derivative on $\mathbb{R}$ can be written as

$$\Lambda^\alpha u(x) = \frac{c(\alpha)}{\pi} \text{P.V.} \int_\mathbb{R} \frac{u(x) - u(x-\eta)}{|\eta|^{1+\alpha}} d\eta$$

where

$$c(\alpha) = \Gamma(1+\alpha) \cos\left((1-\alpha)\frac{\pi}{2}\right).$$

Let x_t denote the point where u reaches its maximum (this point is contained in a compact set in the real line since $u \in H^\alpha$ where α is certainly greater than $1/2$) and assume that $\|u(t)\|_{L^\infty(\mathbb{R})} = u(x_t)$. Then, using Rademacher's Theorem as before, we get

$$\frac{d}{dt}\|u(t)\|_{L^\infty(\mathbb{R})} = \partial_t u(x_t)$$

$$= \frac{1}{\pi}\text{P.V.}\int_\mathbb{R} \frac{(u(x_t) - u(x_t - \eta))\left(c(\gamma)|\eta|^{1+\delta-\gamma} - c(1+\delta)\epsilon\right)}{|\eta|^{2+\delta}} d\eta$$

$$\leq \frac{1}{\pi}\int_{|\eta|>C(\epsilon,\gamma,\delta)} \frac{(u(x_t) - u(x_t - \eta))}{|\eta|^{1+\gamma}} d\eta$$

$$\leq C(\epsilon,\gamma,\delta)\|u(t)\|_{L^\infty(\mathbb{R})}.$$

Similarly, if $\|u(t)\|_{L^\infty(\mathbb{R})} = -u(x_t)$ where x_t for the point where u attains its minimum, we have

$$\frac{d}{dt}\|u(t)\|_{L^\infty(\mathbb{R})} = -\partial_t u(x_t)$$

$$= -\frac{1}{\pi}\text{P.V.}\int_\mathbb{R} \frac{(\|u(t)\|_{L^\infty(\mathbb{R})} + u(x_t - \eta))\left(c(1+\delta)\epsilon - c(\gamma)|\eta|^{1+\delta-\gamma}\right)}{|\eta|^{2+\delta}}$$

$$\leq C(\epsilon,\gamma,\delta)\|u(t)\|_{L^\infty(\mathbb{R})},$$

and it follows that

$$\|u(t)\|_{L^\infty(\mathbb{R})} \leq \|u_0\|_{L^\infty(\mathbb{R})} \exp\left(C(\epsilon,\gamma,\delta)t\right).$$

$\square$

Using Lemma 2 and the same ideas as in Theorem 1, we then get the following result.

Theorem 2. *Let $0 < \delta < 1$, $0 \leq \gamma < 1 + \delta$, and $\epsilon > 0$. If*

$$u_0 \in H^\alpha(\mathbb{R}) \cap L^\infty(\mathbb{R})$$

with $\alpha \geq 2 + \delta$, then for every $0 < T < \infty$ there exists a unique classical solution of (6) such that

$$u(x,t) \in C([0,T], H^\alpha(\mathbb{R})).$$

Moreover, the solution gains regularity and satisfies

$$u(x,t) \in L^2([0,T], H^{\alpha + \frac{1+\delta}{2}}(\mathbb{R})).$$

2.2. **The case $\delta = 1$.** In this case, equation (1) becomes

$$(16) \qquad \partial_t u + \partial_x \left(\frac{1}{2} u^2 \right) = \Lambda^\gamma u + \epsilon \partial_x^2 u, \ x \in \Omega, t > 0,$$

The previous proofs do not apply directly since they use a kernel representation of $\Lambda^{1+\delta}$ which is not valid if $\delta = 1$. Nevertheless, we have an analogous existence result.

Theorem 3. *Let $u_0 \in H^\alpha(\Omega)$ with $\alpha \geq 1$ be the initial data for equation (16), where $\epsilon > 0$, $0 \leq \gamma < 2$, and Ω is $\mathbb{T}$ or $\mathbb{R}$. Then the following statements hold.*

- *If $\alpha \geq 3$, then for every $0 < T < \infty$ there exists a unique classical solution*

$$u(x,t) \in C([0,T], H^\alpha(\Omega)).$$

- *If $1 \leq \alpha < 3$, then for every $0 < T < \infty$ there exists a weak solution*

$$u(x,t) \in L^\infty([0,T], H^\alpha(\Omega)) \cap C([0,T], L^2(\Omega)).$$

- *Moreover, the solution gains regularity and satisfies*

$$u(x,t) \in L^2([0,T], H^{\alpha+1}(\Omega)).$$

Proof. We give only the *a priori* estimates. The proof then follows from the one for $0 \leq \delta < 1$ with minor changes.

The L^2 energy estimate is

$$\frac{1}{2} \frac{d}{dt} \|u(t)\|_{L^2(\Omega)}^2 + \frac{\epsilon}{2} \|\partial_x u\|_{L^2(\Omega)}^2 = \|\Lambda^{\gamma/2} u\|_{L^2(\Omega)}^2 - \frac{\epsilon}{2} \|\partial_x u\|_{L^2(\Omega)}^2.$$

Using Fourier estimates and Gronwall's inequality, we obtain

$$\|u(t)\|_{L^2(\Omega)} + \epsilon \int_0^t \|\partial_x u(s)\|_{L^2(\Omega)}^2 ds \leq \|u_0\|_{L^2(\Omega)}^2 \exp\left(c(\epsilon,\gamma)t\right).$$

In particular

$$\int_0^T \|u(s)\|_{L^\infty(\Omega)}^2 ds \leq c \int_0^T \|\partial_x u(s)\|_{L^2(\Omega)}^2 ds \leq C(T, u_0, \gamma, \epsilon).$$

The H^1 energy estimate is

$$\frac{1}{2} \frac{d}{dt} \|\partial_x u(t)\|_{L^2(\Omega)}^2 \leq \frac{c}{\epsilon} \|u(t)\|_{L^\infty(\Omega)}^2 \|\partial_x u(t)\|_{L^2(\Omega)}^2 + \|\Lambda^{\gamma/2} u\|_{L^2(\Omega)}^2$$

$$- \frac{\epsilon}{2} \|\partial_x u\|_{L^2(\Omega)}^2 \leq C(\epsilon,\gamma)(\|u(t)\|_{L^\infty(\Omega)}^2 + 1)\|\partial_x u(t)\|_{L^2(\Omega)}^2.$$

Also, using Sobolev and Gronwall inequalities we obtain

$$\sup_{t \in [0,T]} \|u(t)\|_{L^\infty(\Omega)}^2 \leq c \|\partial_x u(t)\|_{L^2(\Omega)}^2 \leq C(\epsilon, \gamma, u_0, T).$$

With these global estimate in H^1 and L^∞, we can mimic the previous proof that used H^α norms. $\qquad \square$

3. Instant Analyticity

In this section, we prove that solutions of (1) immediately gain some analyticity. As in [5] (see also [2, 10, 22]), our proof is based on *a priori* estimates in Hardy-Sobolev spaces for the complex extension of the function u in a (growing) complex strip

$$\mathbb{B}_k(t) = \{x + i\xi : x \in \Omega, |\xi| < kt\},$$

where k is a positive constant. We also consider a (shrinking) complex strip

$$\mathbb{V}_h(t) = \{x + i\xi : x \in \Omega, |\xi| < h(t)\},$$

where $h(t)$ is a positive, decreasing function. When convenient, we do not display the t-dependence of these strips explicitly.

We define the norms

$$\|u\|_{L^2(\mathbb{B}_k)}^2 = \sum_{\pm} \int_{\Omega} |u(x \pm ikt)|^2 dx,$$

$$\|u\|_{H^n(\mathbb{B}_k)}^2 = \|u\|_{L^2(\mathbb{B}_k)}^2 + \|\partial_x^n u\|_{L^2(\mathbb{B}_k)}^2,$$

with their analogous counterparts for the strip $\mathbb{V}_h$. The corresponding function spaces have the same flavour as the Gevrey classes used in [14, 17]. In particular, the tools in [14] may be adapted to get $u(x, t) \in G_t^1(\Omega)$, which implies the analyticity for real spatial arguments x.

Theorem 4. *Let u be a classical solution of* (6) *with (real-valued) initial data u_0, where $\epsilon > 0$ and γ, δ satisfy* (2). *Then the following statements hold.*

- *If $u_0 \in H^3(\Omega)$ and $k > 0$, then there exists a time $T(k, u_0, \epsilon, \delta, \gamma) > 0$ such that u continues analytically into the strip $\mathbb{B}_k(t)$ for $0 < t < T(k, u_0, \epsilon, \delta, \gamma)$.*
- *If $u_0 \in H^3(\Omega)$ continues to an analytic function in a complex strip of width $h_0 > 0$, then there exists a time $T(u_0, \epsilon, \delta, \gamma)$ and a positive decreasing function $h : [0, T) \to (0, \infty)$ such that $h(0) = h_0$ and u continues analytically into the strip $\mathbb{V}_h(t)$ for $0 < t < T(u_0, \epsilon, \delta, \gamma)$ with finite $H^3(\mathbb{V}_h)$-norm.*

Proof. **Step 1: Growing strip.** We prove the result in the case $\Omega = \mathbb{T}$; the case $\Omega = \mathbb{R}$ is similar. We write $z = x \pm ikt$. Then the extended equation is

$$(17) \qquad \partial_t u(z, t) + u(z, t) \partial_x u(z, t) = \left(\Lambda^\gamma - \epsilon \Lambda^{1+\delta} \right) u(z, t), \ x \in \Omega, t > 0.$$

First, we study the evolution of $\|u\|_{H^3(\mathbb{B}_k)}$. Since we consider periodic solutions with zero mean, it follows from Poincaré inequalities that we only need to estimate the L^2 norm of the third derivative.

Using Plancherel's theorem, we have

$$\frac{d}{dt} \|\partial_x^3 u\|_{L^2(\mathbb{B}_k)}^2 = 2\Re \int_{\mathbb{T}} \partial_x^3 \bar{u}(z) \left(\partial_t \partial_x^3 u(z) \pm ik \partial_x^4 u(z) \right) dx,$$

and from (17), we get that

$$(18) \qquad \partial_t \partial_x^3 u = -3(\partial_x^2 u)^2 - 4\partial_x u \partial_x^3 u - u \partial_x^4 u + \Lambda^\gamma \partial_x^3 u - \Lambda^{1+\delta} \partial_x^3 u.$$

We have the following estimates:

$$A_1 = -3 \int_{\mathbb{T}} (\partial_x^2 u(z))^2 \partial_x^3 \bar{u}(z) dx \leq C \|\partial_x^3 u\|_{L^2(\mathbb{B}_k)} \|\partial_x^2 u\|_{L^2(\mathbb{B}_k)} \|\partial_x^2 u\|_{L^\infty(\mathbb{B}_k)}$$

$$\leq C \|\partial_x^3 u\|_{L^2(\mathbb{B}_k)}^3,$$

$$A_2 = -4 \int_{\mathbb{T}} \partial_x u(z) |\partial_x^3 u(z)|^2 dx \leq C \|\partial_x^3 u\|_{L^2(\mathbb{B}_k)}^2 \|\partial_x u\|_{L^\infty(\mathbb{B}_k)}$$

$$\leq C \|\partial_x^3 u\|_{L^2(\mathbb{B}_k)}^3,$$

$$A_3 = \pm ik \int_{\mathbb{T}} \partial_x^3 \bar{u}(z) \partial_x^4 u(z) dx$$

$$= \mp ik \int_{\mathbb{T}} \partial_x^3 \bar{u}(z) \Lambda \mathcal{H} \partial_x^3 u(z) dx$$

$$= \mp ik \int_{\mathbb{T}} \Lambda^{1/2} \partial_x^3 \bar{u}(z) \Lambda^{1/2} \mathcal{H} \partial_x^3 u(z) dx \leq 2k \|\Lambda^{1/2} \partial_x^3 u\|_{L^2(\mathbb{B}_k)}^2.$$

Moreover, we have

$$A_4 = \Re \int_{\mathbb{T}} \partial_x^3 \bar{u}(z) u(z) \partial_x^4 u(z) dx$$

$$= \int_{\mathbb{T}} \Re \partial_x^3 u \Re \partial_x^4 u \Re u + \Im \partial_x^3 u \Im \partial_x^4 u \Re u \, dx$$

$$+ \int_{\mathbb{T}} -\Re \partial_x^3 u \Im \partial_x^4 u \Im u + \Re \partial_x^4 u \Im \partial_x^3 u \Im u \, dx$$

$$= -\frac{1}{2} \int_{\mathbb{T}} |\partial_x^3 u|^2 \Re \partial_x u \, dx$$

$$- 2 \int_{\mathbb{T}} \Re \partial_x^3 u \Im \partial_x^4 u \Im u + \Re \partial_x^3 u \Im \partial_x^3 u \Im \partial_x u \, dx$$

$$= -\frac{1}{2} \int_{\mathbb{T}} |\partial_x^3 u|^2 \Re \partial_x u \, dx + \int_{\mathbb{T}} \Re \partial_x^3 u \Im \partial_x^3 u \Im \partial_x u \, dx$$

$$- 2 \int_{\mathbb{T}} \left[\Lambda^{1/2}, \Im u \right] \Re \partial_x^3 u \Lambda^{1/2} \mathcal{H} \Im \partial_x^3 u \, dx$$

$$- 2 \int_{\mathbb{T}} \Im u \Lambda^{1/2} \Re \partial_x^3 u \Lambda^{1/2} \mathcal{H} \Im \partial_x^3 u \, dx,$$

so, using the commutator estimate (see Lemma 6)

$$\left\| \left[\Lambda^{1/2}, F \right] G \right\|_{L^2} \leq c \|\partial_x F\|_{L^\infty} \|G\|_{L^2},$$

we get that

$$A_4 \leq C \|\partial_x^3 u\|_{L^2(\mathbb{B}_k)}^2 \|\partial_x u\|_{L^\infty(\mathbb{B}_k)}$$

$$+ C \|\partial_x^3 u\|_{L^2(\mathbb{B}_k)} \|\partial_x u\|_{L^\infty(\mathbb{B}_k)} \|\Lambda^{1/2} \Im \partial_x^3 u\|_{L^2(\mathbb{B}_k)}$$

$$+ 2 \|\Lambda^{1/2} \partial_x^3 u\|_{L^2(\mathbb{B}_k)}^2 \|\Im u\|_{L^\infty(\mathbb{B}_k)}$$

$$\leq C (\|\partial_x^3 u\|_{L^2(\mathbb{B}_k)}^4 + 1)$$

$$+ \|\Lambda^{1/2} \partial_x^3 u\|_{L^2(\mathbb{B}_k)}^2 \left(2 \|\Im u\|_{L^\infty(\mathbb{B}_k)} + 1 \right).$$

Let $\lambda > \|u_0\|_{L^\infty}$ be a positive constant. Putting these results together and using Poincaré's inequality, we get

$$\frac{d}{dt}\|\partial_x^3 u\|_{L^2(\mathbb{B}_k)}^2 \leq C(\|\partial_x^3 u\|_{L^2(\mathbb{B}_k)}^4 + 1)$$
$$+ \|\Lambda^{1/2}\partial_x^3 u\|_{L^2(\mathbb{B}_k)}^2 \left(2\|\Im u\|_{L^\infty(\mathbb{B}_k)} - 2\lambda + 2\lambda + 2k + 1\right)$$
$$+ \|\Lambda^{\gamma/2}\partial_x^3 u\|_{L^2(\mathbb{B}_k)}^2 - \epsilon\|\Lambda^{(1+\delta)/2}\partial_x^3 u\|_{L^2(\mathbb{B}_k)}^2$$
$$\leq C(\|\partial_x^3 u\|_{L^2(\mathbb{B}_k)} + 1)^4 + \|\Lambda^{1/2}\partial_x^3 u\|_{L^2(\mathbb{B}_k)}^2 \left(2\|\Im u\|_{L^\infty(\mathbb{B}_k)} - 2\lambda\right)$$
$$+ 2\left(\lambda + k + 1\right)\|\Lambda^{\max\{1,\gamma\}/2}\partial_x^3 u\|_{L^2(\mathbb{B}_k)}^2 - \epsilon\|\Lambda^{(1+\delta)/2}\partial_x^3 u\|_{L^2(\mathbb{B}_k)}^2.$$

Define a constant $C(\lambda, k, \epsilon, \delta, \gamma) > 0$ by

$$C(\lambda, k, \epsilon, \delta, \gamma) = \max_{\xi \in \mathbb{R}} \left[2\left(\lambda + k + 1\right)|\xi|^{\max\{1,\gamma\}} - \epsilon|\xi|^{1+\delta}\right]$$

$$(19) \qquad = 2\left(\lambda + k + 1\right)\left(\frac{\max\{1,\gamma\}2\left(\lambda + k + 1\right)}{\epsilon(1+\delta)}\right)^{\frac{\max\{1,\gamma\}}{1+\delta-\max\{1,\gamma\}}}$$

$$- \epsilon\left(\frac{\max\{1,\gamma\}2\left(\lambda + k + 1\right)}{\epsilon(1+\delta)}\right)^{\frac{1+\delta}{1+\delta-\max\{1,\gamma\}}}.$$

Then, using Plancherel's theorem, we get that

$$2\left(\lambda + k + 1\right)\|\Lambda^{\max\{1,\gamma\}/2}\partial_x^3 u\|_{L^2(\mathbb{B}_k)}^2 - \epsilon\|\Lambda^{(1+\delta)/2}\partial_x^3 u\|_{L^2(\mathbb{B}_k)}^2$$
$$\leq C(\lambda, k, \epsilon, \delta, \gamma)\|\partial_x^3 u\|_{L^2(\mathbb{B}_k)}^2,$$

and therefore

$$\frac{d}{dt}\|\partial_x^3 u\|_{L^2(\mathbb{B}_k)}^2 \leq C(\|\partial_x^3 u\|_{L^2(\mathbb{B}_k)} + 1)^4 + 2\|\Lambda^{1/2}\partial_x^3 u\|_{L^2(\mathbb{B}_k)}^2 \left(\|\Im u\|_{L^\infty(\mathbb{B}_k)} - \lambda\right)$$
$$+ C(\lambda, k, \epsilon, \delta, \gamma)\|\partial_x^3 u\|_{L^2(\mathbb{B}_k)}^2.$$

We define a new energy by

$$\|u\|_{\mathbb{B}_k} = \|\partial_x^3 u\|_{L^2(\mathbb{B}_k)}^2 + \|d^\lambda[u]\|_{L^\infty(\mathbb{B}_k)}$$

where

$$d^\lambda[u](z) = \frac{1}{\lambda^2 - |u(z)|^2}.$$

Note that $|u(z)| < \lambda$ as long as $\|u\|_{\mathbb{B}_k}$ remains finite. We need a bound for the remaining term in the energy $\|u\|_{\mathbb{B}_k}$. Using (17) and Sobolev embedding to estimate $\partial_t u$, we have

$$\frac{d}{dt}d^\lambda[u] \leq 4d^\lambda[u]^2\|u\|_{L^\infty(\mathbb{B}_k)}\|\partial_t u\|_{L^\infty(\mathbb{B}_k)} \leq C(\|u\|_{\mathbb{B}_k} + 1)^3 d^\lambda[u]$$

Thus, we obtain

$$d^\lambda[u](t+h) \leq d^\lambda[u](t)\exp\left(\int_t^{t+h} C(\|u\|_{\mathbb{B}_k} + 1)^3 ds\right).$$

Finally, we have

$$\frac{d}{dt}\|d^\lambda[u]\|_{L^\infty(\mathbb{T})} = \lim_{h \to 0}\frac{\|d^\lambda[u](t+h)\|_{L^\infty(\mathbb{T})} - \|d^\lambda[u](t)\|_{L^\infty(\mathbb{T})}}{h}$$
$$\leq C(\|u\|_{\mathbb{B}_k} + 1)^4.$$

It follows that

$$
\begin{aligned}
\frac{d}{dt}\|u\|_{\mathbb{B}_k} &= \frac{d}{dt}\|\partial_x^3 u\|_{L^2(\mathbb{B}_k)}^2 + \frac{d}{dt}\|d^\lambda[u]\|_{L^\infty(\mathbb{T})} \\
&\leq c(\|\partial_x^3 u\|_{L^2(\mathbb{B}_k)} + 1)^4 + C(\lambda, k, \epsilon, \delta, \gamma)\|\partial_x^3 u\|_{L^2(\mathbb{B}_k)}^2 + c(\|u\|_{\mathbb{B}_k} + 1)^4 \\
&\leq c(\|u\|_{\mathbb{B}_k} + 1)^4 + C(\lambda, k, \epsilon, \delta, \gamma)\|u\|_{\mathbb{B}_k}.
\end{aligned}
$$

Thus,

$$
\|u(t)\|_{\mathbb{B}_k} \leq \frac{\sqrt[3]{C(\lambda, k, \epsilon, \delta, \gamma)} \exp\left(C(\lambda, k, \epsilon, \delta, \gamma)\left[\frac{\log\left(\frac{\|u(0)\|_{\mathbb{B}_k}}{\sqrt[3]{c\|u(0)\|_{\mathbb{B}_k}^3 + C(\lambda, k, \epsilon, \delta, \gamma)}}\right)}{C(\lambda, k, \epsilon, \delta, \gamma)} + t\right]\right)}{\sqrt[3]{1 - c\exp\left(3C(\lambda, k, \epsilon, \delta, \gamma)\left[\frac{\log\left(\frac{\|u(0)\|_{\mathbb{B}_k}}{\sqrt[3]{c\|u(0)\|_{\mathbb{B}_k}^3 + C(\lambda, k, \epsilon, \delta, \gamma)}}\right)}{C(\lambda, k, \epsilon, \delta, \gamma)} + t\right]\right)}}.
$$

The time of existence of analytic solutions is then at least

$$
(20) \qquad T(k, u_0, \epsilon, \delta, \gamma) = \frac{\log\left(\frac{C(\lambda, k, \epsilon, \delta, \gamma)}{\left(\|\partial_x^3 u_0\|_{L^2}^2 + \frac{1}{\lambda^2 - \|u_0\|_{L^\infty}^2}\right)^3 c} + 1\right)}{3C(\lambda, k, \epsilon, \delta, \gamma)},
$$

where $C(\lambda, k, \epsilon, \delta, \gamma)$ is given by (19), and we may choose $\lambda = \sqrt{2}\|u_0\|_\infty$, for example.

Now we approximate this problem using an analytic mollifier such as the heat kernel. The regularized problems have entire solutions and satisfy the same *a priori* bounds. Using the uniqueness of classical solutions, we obtain the first part of the result.

Step 2: Shrinking strip As before, we consider the evolution in the Hardy-Sobolev spaces in the strip $\mathbb{V}_h$. We write $z = x \pm ih(t)$. Notice that since the solution is real for real z we have

$$
\partial_x^k u(x \pm ih(t)) - \partial_x^k u(x \pm i0) = \int_\Gamma \partial_x^{k+1} u(x \pm \zeta) d\zeta = \int_0^{h(t)} i\partial_x^{k+1} u(x \pm i\theta) d\theta.
$$

Thus, using the Hadamard Three Lines Theorem, we get

$$
\begin{aligned}
\left|\partial_x^k u(x \pm ih(t)) - \partial_x^k u(x \pm i0)\right| &\leq h(t) \sup_{x \in \mathbb{T}} \sup_{|\theta| < h(t)} |\partial_x^{k+1} u(x \pm i\theta)| \\
&\leq h(t)\|\partial_x^{k+1} u\|_{L^\infty(\mathbb{V}_h)}.
\end{aligned}
$$

Using Lemma 8 and equation (18) for $\partial_t \partial_x^3 u$, we have

$$
\begin{aligned}
\frac{d}{dt} \|\partial_x^3(t)\|_{L^2(\mathbb{V}_h)}^2 \leq & \frac{h'(t)}{10} \sum_{\pm} \int_{\mathbb{T}} \Lambda \partial_x^3 u(z) \overline{\partial_x^3 u(z)} dx \\
& - 10 h'(t) \sum_{\pm} \int_{\mathbb{T}} \Lambda \partial_x^3 u(x) \overline{\partial_x^3 u(x)} dx \\
& + 2\Re \sum_{\pm} \int_{\mathbb{T}} \partial_t \partial_x^3 u(z) \overline{\partial_x^3 u(z)} dx \\
= & J_1 + J_2 + J_3 + J_4,
\end{aligned}
$$

where

$$
\begin{aligned}
J_1 =& \frac{h'(t)}{10} \sum_{\pm} \int_{\mathbb{T}} \Lambda \partial_x^3 u(z) \overline{\partial_x^3 u(z)} dx, \\
J_2 =& -10 h'(t) \sum_{\pm} \int_{\mathbb{T}} \Lambda \partial_x^3 u(x) \overline{\partial_x^3 u(x)} dx \\
J_3 =& 2\Re \int_{\mathbb{T}} \left[-3(\partial_x^2 u)^2 - 4\partial_x u \partial_x^3 u - u \partial_x^4 u \right] \overline{\partial_x^3 u(z)} dx \\
=& K_1 + K_2 + K_3, \\
J_4 =& 2\Re \sum_{\pm} \int_{\mathbb{T}} \left(\Lambda^\gamma - \epsilon \Lambda^{1+\delta} \right) \partial_x^3 u(x) \overline{\partial_x^3 u(x)} dx.
\end{aligned}
$$

We have the estimates

$$
J_2 \leq 20 |h'(t)| \|u_0\|_{H^{3.5}}^2 \exp\left(\exp\left(C(\epsilon, \delta, \gamma, \|u_0\|_{L^\infty}(1+t)) \right) \right),
$$

$$
J_4 \leq 2 \left(\frac{\gamma}{\epsilon(1+\delta)} \right)^{\frac{1}{1+\delta-\gamma}} \|\partial_x^3 u\|_{L^2(\mathbb{V}_h)}^2.
$$

Moreover, following the previous ideas, and using Gagliardo-Nirenberg and Sobolev inequalities, we find that

$$
K_1 + K_2 \leq C \|\partial_x^3 u\|_{L^2(\mathbb{V}_h)}^2 \|\partial_x u\|_{L^\infty(\mathbb{V}_h)} \leq C \|\partial_x^3 u\|_{L^2(\mathbb{V}_h)}^3.
$$

We also have

$$
\begin{aligned}
K_3 =& 2 \int_{\mathbb{T}} \Re u \Re \partial_x^4 u \Re \partial_x^3 u + \Re u \Im \partial_x^4 u \Im \partial_x^3 u \, dx \\
& + 2 \int_{\mathbb{T}} -\Im u \Re \partial_x^4 u \Im \partial_x^3 u + \Im u \Im \partial_x^4 u \Re \partial_x^3 u \, dx \\
\leq & C \|\partial_x^3 u\|_{L^2(\mathbb{V}_h)}^2 \|\partial_x u\|_{L^\infty(\mathbb{V}_h)} - 4 \int_{\mathbb{T}} \Im u \Re \partial_x^4 u \Im \partial_x^3 u \, dx \\
\leq & C \|\partial_x^3 u\|_{L^2(\mathbb{V}_h)}^3 - 4 \int_{\mathbb{T}} \Lambda^{1/2} \mathcal{H} \Re \partial_x^3 u \Lambda^{1/2} \left(\Im u \Im \partial_x^3 u \right) dx.
\end{aligned}
$$

The last integral can be written in terms of a commutator as

$$
\int_{\mathbb{T}} \Lambda^{1/2} \mathcal{H} \Re \partial_x^3 u \left[\Lambda^{1/2}, \Im u \right] \Im \partial_x^3 u \, dx + \int_{\mathbb{T}} \Lambda^{1/2} \mathcal{H} \Re \partial_x^3 u \Im u \Lambda^{1/2} \Im \partial_x^3 u \, dx,
$$

and using Lemma 6, we get

$$
\begin{aligned}
K_3 \;\leq\; & C\|\partial_x^3 u\|_{L^2(\mathbb{V}_h)}^3 + C\|\Lambda^{1/2}\partial_x^3 u\|_{L^2(\mathbb{V}_h)}\|\partial_x \Im u\|_{L^\infty(\mathbb{V}_h)}\|\partial_x^3 u\|_{L^2(\mathbb{V}_h)} \\
& -4\int_{\mathbb{T}} \Lambda^{1/2}\mathcal{H}\Re\partial_x^3 u\,\Im u\,\Lambda^{1/2}\Im\partial_x^3 u\,dx \\
\leq\; & C\left(\|\partial_x^3 u\|_{L^2(\mathbb{V}_h)}+1\right)^3 \\
& +C\left(\|\partial_x \Im u\|_{L^\infty(\mathbb{V}_h)}^2 + \|\Im u\|_{L^\infty(\mathbb{V}_h)}\right)\|\Lambda^{1/2}\partial_x^3 u\|_{L^2(\mathbb{V}_h)}^2 \\
\leq\; & C\left(\|\partial_x^3 u\|_{L^2(\mathbb{V}_h)}+1\right)^3 \\
& +Ch(t)\left(\|\partial_x^3 u\|_{L^2(\mathbb{V}_h)}+1\right)^2\|\Lambda^{1/2}\partial_x^3 u\|_{L^2(\mathbb{V}_h)}^2.
\end{aligned}
$$

Collecting the bounds for K_3 and for J_1, we have

$$
\begin{aligned}
K_3 + J_1 \;\leq\; & C\left(\|\partial_x^3 u\|_{L^2(\mathbb{V}_h)}+1\right)^3 \\
& +\left(Ch(t)\left(\|\partial_x^3 u\|_{L^2(\mathbb{V}_h)}+1\right)^2 + 10h'(t)\right)\|\Lambda^{1/2}\partial_x^3 u\|_{L^2(\mathbb{V}_h)}^2,
\end{aligned}
$$

and, choosing

$$
(21) \qquad h(t) = h(0)\exp\left(-10C\int_0^t \left(\|\partial_x^3 u(s)\|_{L^2(\mathbb{V}_h)}+1\right)^2 ds\right),
$$

we obtain

$$
\begin{aligned}
\frac{d}{dt}\|\partial_x^3 u(t)\|_{L^2(\mathbb{V}_h)}^2 \leq\; & C\left(\|\partial_x^3 u\|_{L^2(\mathbb{V}_h)}+1\right)^3 \\
& + C\left(\|\partial_x^3 u\|_{L^2(\mathbb{V}_h)}+1\right)^2\|u_0\|_{H^{3.5}}^2 \exp\left(\exp\left(C(\epsilon,\delta,\gamma,\|u_0\|_{L^\infty}(1+t))\right)\right).
\end{aligned}
$$

Finally, we use a standard Galerkin approximation method to obtain a local solution that satisfies these estimates, which completes the proof.   □

In the previous proof, we can choose the parameter $k > 0$ that determines the strips of analyticity in any way we wish, but we get shorter existence times for larger values of k, so we cannot conclude that the solution is entire for $t > 0$.

To obtain an explicit estimate for the width of a strip that depends only on the initial data (and the parameters in the equation), we choose

$$
(22) \qquad k = \left(\|\partial_x^3 u_0\|_{L^2}^2 + \frac{1}{\lambda^2 - \|u_0\|_{L^\infty}^2}\right)^3, \qquad \lambda = \sqrt{2}\|u_0\|_\infty
$$

in the proof of Theorem 4. Then the corresponding time T of analyticity is given by (20), and the width of the strip of analyticity at time T is at least kT. Using the preceding equations, we find that

$$
(23) \qquad\qquad kT = \frac{\log\left(\mathcal{E}/c + 1\right)}{3\mathcal{E}},
$$

where c is a constant, and $\mathcal{E}$ is given by

(24)

$$
\mathcal{E} = \frac{2\left(\sqrt{2}\|u_0\|_{L^\infty(\mathbb{T})} + k + 1\right)\left(\frac{\max\{1,\gamma\}2\left(\sqrt{2}\|u_0\|_{L^\infty(\mathbb{T})}+k+1\right)}{\epsilon(1+\delta)}\right)^{\frac{\max\{1,\gamma\}}{1+\delta-\max\{1,\gamma\}}}}{\left(\|\partial_x^3 u_0\|_{L^2}^2 + \frac{1}{\|u_0\|_{L^\infty}^2}\right)^3}
$$

$$
- \frac{\epsilon\left(\frac{\max\{1,\gamma\}2\left(\sqrt{2}\|u_0\|_{L^\infty(\mathbb{T})}+k+1\right)}{\epsilon(1+\delta)}\right)^{\frac{1+\delta}{1+\delta-\max\{1,\gamma\}}}}{\left(\|\partial_x^3 u_0\|_{L^2}^2 + \frac{1}{\|u_0\|_{L^\infty}^2}\right)^3}.
$$

Finally, we remark that by using this smoothing effect, one can prove the ill-posedness in Sobolev spaces of the evolution problem backward in time.

Corollary 1. *There are solutions $\tilde{u}$ to the backward in time equation (1), such that $\|\tilde{u}\|_{H^4}(0) < \epsilon$ and $\|\tilde{u}\|_{H^4}(\mu) = \infty$ for all $\epsilon > 0$ and sufficiently small $\mu > 0$.*

Proof. The proof follows the idea in [2, 10]. We consider the solution (forward in time) u^ν to the equation (1) with initial data $u(x,0) = \nu v(x)$ where $v \in H^3$, $v \notin H^4$ $0 < \nu < 1$. Now define $\tilde{u}^{\nu,\mu}(x,t) = u^\nu(x, -t + \mu)$ for fixed, small enough $0 < \mu(v) \ll 1$. This function is analytic at time 0 but it does not belong to H^4 at time μ. Taking $0 < \nu \ll 1$ we conclude the proof. $\square$

4. Large time dynamics

In this section we prove the existence of an absorbing ball in L^p for the problem (1) in the periodic case $\Omega = \mathbb{T}$. We will require a Lemma similar to the results in [20, 31, 40]:

Lemma 3. *Let $M \in \mathbb{N}$, $\delta > 0$, and $x_0 \in \mathbb{T}$. Then there exists a smooth, periodic function $b_M^{x_0} \in C^\infty(\mathbb{T})$ and a constant*

$$
C_1(\delta, M) = c_1(\delta)\left(\frac{1}{M^{1+\delta}} + \frac{1}{\delta M^\delta}\right)^{1/2}
$$

such that the following inequality holds: for every $u \in C^\infty(\mathbb{T})$ with $u(x_0) = 0$,

$$
\left|\int_{\mathbb{T}} b_M^{x_0}(x)u^2(x,t)dx\right| \le C_1(\delta, M)\|\Lambda^{\frac{1+\delta}{2}}u\|_{L^2(\mathbb{T})}^2.
$$

Proof. We define

$$
b_M^{x_0}(x) = \sum_{|\xi| \le M} e^{-i\xi(x-x_0)}.
$$

We have

$$
\int_{\mathbb{T}} b_M^{x_0}(x)u^2(x,t)dx = \sum_{|\xi| \le M} \int_{\mathbb{T}} u^2(x,t)e^{-i\xi(x-x_0)}dx
$$

$$
= \sum_{|\xi| \le M} \int_{\mathbb{T}} u^2(x+x_0,t)e^{-i\xi x}dx
$$

$$
= 2\pi \sum_{|\xi| \le M} \widehat{g}(\xi),
$$

where $g(x) = u^2(x + x_0)$. Since $\sum \widehat{g}(\xi) = g(0)$, it follows from the definition of x_0 that $\sum \widehat{g}(\xi) = 0$, and therefore

$$\left| \sum_{|\xi| \le M} \widehat{g}(\xi) \right| \le \left| \sum_{|\xi| > M} \widehat{g}(\xi) \right|$$

$$\le \left(\sum_{|\xi| > M} |\xi|^{1+\delta} (\widehat{g}(\xi))^2 \right)^{1/2} \left(\sum_{|\xi| > M} \frac{1}{|\xi|^{1+\delta}} \right)^{1/2}$$

$$\le \frac{1}{\sqrt{2\pi}} \|\Lambda^{\frac{1+\delta}{2}} g\|_{L^2(\mathbb{T})} \left(\frac{1}{M^{1+\delta}} + \frac{1}{\delta M^\delta} \right)^{1/2}.$$

The Kato-Ponce inequality then implies that there is a constant $c_1(\delta)$ such that

$$\left| \int_{\mathbb{T}} b_M^{x_0}(x) u^2(x, t) dx \right| \le c_1(\delta) \|\Lambda^{\frac{1+\delta}{2}} u\|_{L^2(\mathbb{T})}^2 \left(\frac{1}{M^{1+\delta}} + \frac{1}{\delta M^\delta} \right)^{1/2},$$

which proves the result. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Next, we prove that solutions of (1) remain uniformly bounded in L^p. The key step is to prove the existence of an absorbing set in L^2, and we do this following the ideas of [20, 31].

Theorem 5. *Suppose that $u_0 \in H^\alpha(\mathbb{T})$, where $\alpha > 1$, has zero mean. Then the solution u of the initial-value problem (6) in the periodic case satisfies*

$$\limsup_{t \to \infty} \|u(t)\|_{L^2(\mathbb{T})} \le r_2(\epsilon, \delta, \gamma),$$

$$\|u(t)\|_{L^2(\mathbb{T})} \le \max\{\|u_0\|_{L^2(\mathbb{T})}, r_2\} = R(\epsilon, \delta, \gamma).$$

Moreover, for $2 < p \le \infty$ and $0 < \delta < 1$, we have

$$\limsup_{t \to \infty} \|u(t)\|_{L^p(\mathbb{T})} \le r_2^{2/p} \left(\max \left\{ \sqrt{\frac{3}{\pi}} R, C(\delta) R \right\} \right)^{1-2/p}.$$

Proof. We start by assuming that the initial data is odd.

Step 1: Absorbing set in L^2 Let s be a smooth, periodic function, which we will choose later. We compute that

$$\frac{1}{2} \frac{d}{dt} \|u(t) - s\|_{L^2(\mathbb{T})}^2 = \|\Lambda^{\gamma/2} u\|_{L^2(\mathbb{T})}^2 - \epsilon \|\Lambda^{(1+\delta)/2} u\|_{L^2(\mathbb{T})}^2 - \int_{\mathbb{T}} \partial_x s \frac{u^2}{2} dx$$

$$- \int_{\mathbb{T}} \Lambda^{(1+\delta)/2} u \left(\epsilon \Lambda^{(1+\delta)/2} s + \Lambda^{\gamma - (1+\delta)/2} s \right) dx.$$

Using the inequality

$$2|\xi|^\gamma \le \frac{\epsilon}{3} |\xi|^{1+\delta} + \left(\frac{6\gamma}{(1+\delta)\epsilon} \right)^{\frac{1}{1+\delta-\gamma}}, \text{ for all } \xi \in \mathbb{R}$$

and the Plancherel theorem, we get

$$\frac{1}{2}\frac{d}{dt}\|u(t) - s\|_{L^2(\mathbb{T})}^2 \leq -\|\Lambda^{\gamma/2}u\|_{L^2(\mathbb{T})}^2 - \frac{2\epsilon}{3}\|\Lambda^{(1+\delta)/2}u\|_{L^2(\mathbb{T})}^2 - \|u\|_{L^2(\mathbb{T})}^2$$
$$+ \int_{\mathbb{T}}\left(\lambda - \frac{\partial_x s}{2}\right)u^2 dx$$
$$+ \int_{\mathbb{T}}\Lambda^{(1+\delta)/2}u\left(-\epsilon\Lambda^{(1+\delta)/2}s + \Lambda^{\gamma-(1+\delta)/2}s\right)dx,$$

where

$$(25) \qquad \lambda = \left(\frac{6\gamma}{(1+\delta)\epsilon}\right)^{\frac{1}{1+\delta-\gamma}} + 1.$$

Then, using the Young and Cauchy-Schwarz inequalities, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u(t) - s\|_{L^2(\mathbb{T})}^2 \leq -\|\Lambda^{\gamma/2}u\|_{L^2(\mathbb{T})}^2 - \|u\|_{L^2(\mathbb{T})}^2 - \frac{\epsilon}{3}\|\Lambda^{(1+\delta)/2}u\|_{L^2(\mathbb{T})}^2$$
$$+ \int_{\mathbb{T}}\left(\lambda - \frac{\partial_x s}{2}\right)u^2 dx$$
$$+ \frac{3}{\epsilon}\int_{\mathbb{T}}\left(\left(-\epsilon\Lambda^{(1+\delta)/2} + \Lambda^{\gamma-(1+\delta)/2}\right)s\right)^2 dx.$$

Since the odd symmetry is preserved by (1) and u_0 is odd, we have $u(0,t) = 0$. For $M \in \mathbb{N}$, we choose s such that

$$(26) \qquad \partial_x s(x) = -2\lambda\sum_{0 < |\xi| \leq M}e^{-i\xi x} = -2\lambda\left[b_M^0(x) - 1\right].$$

Then from the preceding inequality and Lemma 3, we get

$$\frac{1}{2}\frac{d}{dt}\|u(t) - s\|_{L^2(\mathbb{T})}^2$$
$$\leq -\|\Lambda^{\gamma/2}u\|_{L^2(\mathbb{T})}^2 - \|u\|_{L^2(\mathbb{T})}^2 - \frac{\epsilon}{3}\|\Lambda^{(1+\delta)/2}u\|_{L^2(\mathbb{T})}^2$$
$$+ \int_{\mathbb{T}}b_M^{\lambda,0}u^2 dx + \frac{3}{\epsilon}\int_{\mathbb{T}}\left(\left(-\epsilon\Lambda^{(1+\delta)/2} + \Lambda^{\gamma-(1+\delta)/2}\right)s\right)^2 dx$$
$$\leq -\|\Lambda^{\gamma/2}u\|_{L^2(\mathbb{T})}^2 - \|u\|_{L^2(\mathbb{T})}^2 - \frac{\epsilon}{3}\|\Lambda^{(1+\delta)/2}u\|_{L^2(\mathbb{T})}^2$$
$$+ c_1\lambda\|\Lambda^{(1+\delta)/2}u\|_{L^2(\mathbb{T})}^2\left(\frac{1}{M^{1+\delta}} + \frac{1}{\delta M^\delta}\right)^{1/2} + \frac{6}{\epsilon}\|\Lambda s\|_{L^2(\mathbb{T})}^2.$$

We take $M = M(\epsilon, \delta, \gamma)$ such that

$$c_1\left(\left(\frac{6\gamma}{(1+\delta)\epsilon}\right)^{\frac{1}{1+\delta-\gamma}} + 1\right)\left(\frac{1}{M^{1+\delta}} + \frac{1}{\delta M^\delta}\right)^{1/2} \leq \frac{\epsilon}{3},$$

and we obtain

$$\frac{1}{2}\frac{d}{dt}\|u(t) - s\|_{L^2(\mathbb{T})}^2 \leq -2\|u(t) - s\|_{L^2(\mathbb{T})}^2 + 2\|s\|_{L^2(\mathbb{T})}^2 + \frac{6}{\epsilon}\|\Lambda s\|_{L^2(\mathbb{T})}^2.$$

Using Gronwall inequality, we conclude that

$$\|u(t) - s\|_{L^2(\mathbb{T})}^2 \leq \left(\|u_0 - s\|_{L^2(\mathbb{T})}^2 + \|s\|_{L^2(\mathbb{T})}^2 + \frac{3}{\epsilon}\|\Lambda s\|_{L^2(\mathbb{T})}^2 \right) e^{-4t}$$
$$+ \|s\|_{L^2(\mathbb{T})}^2 + \frac{3}{\epsilon}\|\Lambda s\|_{L^2(\mathbb{T})}^2.$$

The existence of an absorbing set in L^2 is now straightforward. Thus we have the existence of a constant $R = R(\epsilon, \delta, \gamma)$ such that

$$\|u(t)\|_{L^2(\mathbb{T})} \leq R(\epsilon, \delta, \gamma).$$

Step 2: Absorbing set in L^∞ We assume $u(x_t) = \|u(t)\|_{L^\infty(\mathbb{T})}$. We take $\nu > 0$ a positive number and define

$$\mathcal{U}_1 = \{\eta \in [-\nu, \nu] \text{ s.t. } u(x_t) - u(x_t - \eta) > u(x_t)/2\},$$

and $\mathcal{U}_2 = [-\nu, \nu] - \mathcal{U}_1$. We have

$$R^2(\epsilon, \delta, \gamma) \geq \|u(t)\|_{L^2(\mathbb{T})}^2$$
$$\geq \int_{\mathbb{R}} (u(x_t - \eta))^2 \, d\eta$$
$$\geq \int_{\mathcal{U}_2} (u(x_t - \eta))^2 \, d\eta$$
$$\geq \left(\frac{u(x_t)}{2} \right)^2 |\mathcal{U}_2|.$$

Equivalently,

$$2\nu - \frac{4R^2}{\|u(t)\|_{L^\infty(\mathbb{T})}^2} \leq 2\nu - |\mathcal{U}_2| = |\mathcal{U}_1|.$$

Using the fact that the initial data has zero mean, we get

$$\Lambda^{1+\delta} u(x_t) = \sum_{k \in \mathbb{Z}} \int_{\mathbb{T}} \frac{u(x_t) - u(x_t - \eta)}{|\eta - 2k\pi|^{2+\delta}} d\eta$$
$$\geq \sum_{|k|>0} \int_{\mathbb{T}} \frac{u(x_t) - u(x_t - \eta)}{|\eta - 2k\pi|^{2+\delta}} d\eta + \int_{\mathcal{U}_1} \frac{u(x_t) - u(x_t - \eta)}{|\eta|^{2+\delta}} d\eta$$
$$\geq \sum_{|k|>1} \int_{\mathbb{T}} \frac{u(x_t) - u(x_t - \eta)}{|2(k-1)\pi|^{2+\delta}} d\eta + \frac{\frac{u(x_t)}{2}}{\nu^2} |\mathcal{U}_1|$$
$$\geq \frac{u(x_t)}{\nu^{2+\delta}} \left(\nu - 2 \left(\frac{R}{u(x_t)} \right)^2 \right) + \frac{2\zeta(2+\delta)u(x_t)}{(2\pi)^{1+\delta}}.$$

We define

$$\nu = 3 \left(\frac{R}{u(x_t)} \right)^2,$$

and we obtain

$$\Lambda^{1+\delta} u(x_t) \geq \frac{(u(x_t))^{3+2\delta}}{3^{2+\delta} R^{2(1+\delta)}} + \frac{2\zeta(2+\delta)u(x_t)}{(2\pi)^{1+\delta}}.$$

As $\nu \leq \pi$ this choice implies

$$\sqrt{\frac{3}{\pi}} R \leq u(x_t).$$

We have

$$\frac{d}{dt}\|u(t)\|_{L^\infty(\mathbb{T})} \leq \Lambda^\gamma u(x_t) - \frac{1}{2}\Lambda^{1+\delta}u(x_t) - \frac{1}{2}\Lambda^{1+\delta}u(x_t)$$

$$\leq C(\gamma,\delta)\|u(t)\|_{L^\infty(\mathbb{T})} - \frac{1}{2}\left(\frac{(u(x_t))^{3+2\delta}}{3^{2+\delta}R^{2(1+\delta)}} + \frac{2\zeta(2+\delta)u(x_t)}{(2\pi)^{1+\delta}}\right)$$

$$\leq C(\gamma,\delta)\|u(t)\|_{L^\infty(\mathbb{T})} - \frac{\|u(t)\|_{L^\infty(\mathbb{T})}^{3+2\delta}}{2\cdot 3^{2+\delta}R^{2(1+\delta)}}.$$

On the other hand, if $\|u(t)\|_{L^\infty(\mathbb{T})} = -\min_x u(x,t)$, we define

$$\mathcal{U}_1 = \{\eta \in [-\nu,\nu] \text{ s.t. } -u(x_t) + u(x_t-\eta) > -u(x_t)/2\},$$

and $\mathcal{U}_2 = [-\nu,\nu] - \mathcal{U}_1$. We get

$$\frac{d}{dt}\|u(t)\|_{L^\infty(\mathbb{T})} = -\Lambda^\gamma u(x_t) + \Lambda^{1+\delta}u(x_t) = \Lambda^\gamma(-u(x_t)) - \Lambda^{1+\delta}(-u(x_t))$$

$$\leq C(\gamma,\delta)\|u(t)\|_{L^\infty(\mathbb{T})} - \frac{\|u(t)\|_{L^\infty(\mathbb{T})}^{3+2\delta}}{2\cdot 3^{2+\delta}R^{2(1+\delta)}}.$$

Collecting these inequalities, we obtain the existence of an absorbing ball in L^∞ with radius

$$r_\infty = \max\left\{\sqrt{\frac{3}{\pi}}R, C(\gamma,\delta)R\right\}.$$

Step 3: Absorbing set in L^p For the case $2 < p < \infty$, we use interpolation. We get

$$\|u(t)\|_{L^p(\mathbb{T})} \leq \|u(t)\|_{L^2(\mathbb{T})}^{2/p}\|u(t)\|_{L^\infty(\mathbb{T})}^{1-2/p}$$

$$\leq R^{2/p}\max\left\{\sqrt{\frac{3}{\pi}}R, C(\delta)R, \|u_0\|_{L^\infty(\mathbb{T})}\right\}^{1-2/p}.$$

The radius for this case can be obtained in a similar way.

Step 4: Initial data without odd symmetry Following the same ideas as in [20] (see also [7, 18]), we introduce the set of translations of the function s defined in (26):

$$\mathcal{S} = \{\tilde{s} : \tilde{s}(x) = s(x+\chi) \text{ with } |\chi| \leq \pi\}.$$

Since the function u_0 has zero mean, the solution $u(t)$ has zero mean for all time, so there exists at least one point $x_0(t)$ such that $u(x_0(t),t) = 0$. Then, for any particular time t, we consider, as in the step 1 above, the function $b_M^{x_0(t)}(x)$ defined in Lemma 3 where λ was defined in (25), and let

$$\partial_x \tilde{s}(x,t) = -2\lambda \sum_{0 < |\xi| \leq M} e^{-i\xi(x-x_0(t))} = -2\lambda\left[b_M^{x_0(t)}(x) - 1\right]$$

Notice that $\tilde{s}(x) = s(x + x_0(t))$, with s defined in (26). As before, we obtain

$$\frac{d}{dt'}\|u(t + t') - \tilde{s}(t)\|^2_{L^2(\mathbb{T})}$$
$$\leq -4\|u(t + t') - \tilde{s}(t)\|^2_{L^2(\mathbb{T})} + 4\|s(t)\|^2_{L^2(\mathbb{T})} + \frac{12}{\epsilon}\|\Lambda s(t)\|^2_{L^2(\mathbb{T})}.$$

If follows that

$$\frac{d}{dt'}\|u(t + t') - \tilde{s}(t)\|^2_{L^2(\mathbb{T})}\bigg|_{t'=0} \leq 0$$

if

$$d(u(t), \tilde{s}(t)) = \|u(t) - \tilde{s}(t)\|_{L^2(\mathbb{T})} \gg 1.$$

As a consequence, we find that

$$d(u(t), \tilde{s}(t)) = \|u(t) - \tilde{s}(t)\|_{L^2(\mathbb{T})}$$

is a bounded function of time. Since $d(u(t), \mathcal{S}) \leq d(u(t), \tilde{s}(t))$, this completes the proof. $\qquad\square$

Corollary 2. *Let $u_0 \in H^\alpha(\mathbb{T}), \alpha > 1$ be the mean-zero initial data for the problem* (1) *with $\epsilon \geq 1 > \delta$ in the periodic case. Then we have*

$$\|u(t)\|_{L^p(\mathbb{T})} \leq \|u_0\|^{2/p}_{L^2(\mathbb{T})} \max\left\{\sqrt{\frac{3}{\pi}}\|u_0\|_{L^2(\mathbb{T})}, C\left(\delta\right)\|u_0\|_{L^2(\mathbb{T})}, \|u_0\|_{L^\infty(\mathbb{T})}\right\}^{1-2/p},$$

and

$$\limsup_{t\to\infty}\|u(t)\|_{L^p(\mathbb{T})} = \|u_0\|^{2/p}_{L^2(\mathbb{T})}\left(\max\left\{\sqrt{\frac{3}{\pi}}\|u_0\|_{L^2(\mathbb{T})}, C\left(\delta\right)\|u_0\|_{L^2(\mathbb{T})}\right\}\right)^{1-2/p}.$$

Proof. The result follows from Poincaré's inequality. $\qquad\square$

The existence of an absorbing set in the L^2-norm and the regularity results from Section 2 imply the existence of an absorbing set in higher Sobolev norms. The proof is straightforward, and we just state the result.

Lemma 4. *Suppose that $\alpha > 1$ and $u_0 \in H^\alpha(\mathbb{T})$ has zero mean. Then for every $0 < s \leq \alpha$ the solution u of the initial-value problem* (6) *in the periodic case satisfies*

$$\limsup_{t\to\infty}\|u(t)\|_{H^s} \leq C\left(s, \epsilon, \delta, \gamma, \|u_0\|_{L^2(\mathbb{T})}\right).$$

5. THE ATTRACTOR

In this section we prove the existence of an attractor for spatially periodic solutions $(\Omega = \mathbb{T})$ and derive some of its properties.

5.1. **Existence.** We denote the solution operators for (6) by $S(t)$, where $S(t)u_0 = u(x, t)$. The compactness of a nonlinear semigroup, or semiflow, is defined as follows [40].

Definition 2. *The solution operator $S(t)u_0 = u(t, x)$ defines a compact semiflow in H^s if, for every $u_0 \in H^s$ the following statements hold:*

- $S(0)u_0 = u_0$.

- *for all t, s, u_0, the semigroup property hold,* i.e.,

$$S(t + s)u_0 = S(t)S(s)u_0 = S(s)S(t)u_0.$$

- *For every $t > 0$, $S(t)$ is continuous (as an operator from H^s to H^s).*
- *There exists $t_1 > 0$ such that $S(t_1)$ is a compact operator,* i.e. *for every bounded set $B \subset H^s$, $S(t_1)B \subset H^s$ is a compact set.*

It is then is straightforward to use our existence results to prove the following lemma.

Lemma 5. *Let $u_0 \in H^\alpha(\mathbb{T})$ for $\alpha \geq 3$ be the initial data for the problem (1). Then $S(t)u_0 = u(\cdot, t)$ defines a compact semiflow in $H^\alpha(\mathbb{T})$. Moreover $S(t)u_0$ is a continuous map from $[0, T]$ to $H^\alpha(\mathbb{T})$ for every initial data u_0, i.e., $S(\cdot)u_0 \in C([0, T], H^\alpha)$.*

Now we can apply Theorem 1.1 in [40] to obtain the existence of the attractor

Theorem 6. *In the spatially periodic case with $\Omega = \mathbb{T}$, equation (1) has a maximal, connected, compact attractor in the space $H^\alpha(\mathbb{T})$ for every $\alpha \geq 3$.*

Proof. The result follows from Lemma 4, where the existence of an absorbing set is proved, and Lemma 5, where the properties of the semigroup are proved. □

5.2. **Number of wild oscillations.** In this section we obtain a bound for the number of wild oscillations that a solution u can develop. This bound is similar to the bound in [22] for the standard KS equation (see also [27]), and splits $\mathbb{T}$ into a set I_M where $\partial_x u$ is uniformly bounded and a set R_M where $\partial_x u$ may be large but u cannot have too many critical points. However, our bound is valid for arbitrary initial data while the bound in [22] only works for initial data in a neighborhood of a stationary solution.

Theorem 7. *Let u be the solution of (6) for initial data $u_0 \in H^3(\mathbb{T})$ and define $T > 0$ as in (20), (22). Then for every $M > 1$, there exist $\tau_M > 0$ and $I_M, R_M \subset \mathbb{T}$, where I_M a union of at most $[4\pi/\tau_M]$ open intervals, such that $\mathbb{T} = I_M \cup R_M$ and the following estimates hold for $T/M < t < T$:*

$$|\partial_x u(x, t)| \leq \frac{\sqrt{2}\|u_0\|_{L^\infty(\mathbb{T})}}{M} \qquad \text{for all } x \in I_M,$$

$$\operatorname{card}\{x \in R_M : \partial_x u(x, t) = 0\} \leq \frac{4\pi}{\log 2} \frac{\log(M/\tau_M)}{\tau_M}.$$

An explicit choice for τ_M is

$$\tau_M = \frac{1}{M}\left[\frac{\log(\mathcal{E}/c + 1)}{3\mathcal{E}}\right],$$

where $\mathcal{E}$ is given by (24).

Proof. From Theorem 4, after time $t > 0$ the solution becomes analytic in a complex strip $\mathbb{B}_k(t)$. In particular, choosing the parameters k, λ as in (22), we get from (23) that the width of the strip after time T/M is at least

$$\tau_M = \frac{1}{M}\left[\frac{\log(\mathcal{E}/c + 1)}{3\mathcal{E}}\right].$$

Using Cauchy's integral formula and the definition of $d^\lambda[u]$ in Theorem 4, we find that

$$\|\partial_x u(t)\|_{L^\infty(\mathbb{B}_k)} \leq \frac{\|u(t)\|_{L^\infty(\mathbb{B}_k)}}{\tau_M} \leq \frac{\lambda}{\tau_M},$$

and an application of Lemma 9 with $\mu = \lambda/M$ then gives the result. $\square$

Theorem 7 is local in time, but we can apply the result repeatedly to get bounds on the number of oscillations on successive time intervals

$$[T/M, T] \cup [T + T_1/M, T + T_1] \cup \ldots,$$

where T_1 is given by (20) with u_0 replaced by $u(T)$. In view of the uniform H^3-bounds on $u(t)$, we can extend the estimates to arbitrarily large times, but there are small gaps between successive time intervals in which the estimates may not apply.

6. NUMERICAL SIMULATIONS

In this section, we show some numerical solutions of (1), which we repeat here for convenience

$$(27) \qquad \partial_t u + \partial_x \left(\frac{1}{2}u^2\right) = \Lambda^\gamma u - \epsilon \Lambda^{1+\delta} u,$$

with 2π-periodic boundary conditions. We approximate the spatial part by a pseudo-spectral scheme, typically using 2^{12}–2^{14} Fourier modes, and advance in time with an explicit method such as the `ode45` function in MATLAB.

In Figures 1–2, we show a numerical solution of (27) with $\delta = \gamma = 1$ in $-\pi < x < \pi$ for initial data

$$(28) \qquad u_0(x) = \cos x + e^{-x^2} \sin x.$$

A primary "viscous shock" forms from the initial data, after which smaller "viscous sub-shocks" develop spontaneously throughout the interval. These sub-shocks grow, propagate toward the primary shock, and merge with it. The number of sub-shocks and their rate of formation increases as ϵ decreases. Some movies of the numerical simulations are available at

http://youtu.be/8r0QMgxZJMk?list=PLUwnEWNEnlmhroc7JS_cZ2PLN6pe-HiX7

In Figure 3, we show a solution of the usual KS equation (3) with the same initial data as in Figure 1. The spatial "shock-like" structure of chaotic solutions of (27) is qualitatively different from the "worm-like" structure of solutions of (3).

Similar behavior is observed for (27) with other values of $0 < \delta < 1$, $0 < \gamma < 1+\delta$, and $\epsilon > 0$. In Figure 4 we show a solution for $\delta = 0.5$, $\gamma = 1.45$, and $\epsilon = 0.8$, with the initial data

$$(29) \qquad u_0(x) = \cos x.$$

Chaotic behaviour occurs for larger values of ϵ as γ gets closer to $1+\delta$. This is consistent with the fact that the band of unstable wavenumbers k for the linearization of (27) at $u = 0$ is given by

$$0 < k < k_*(\delta, \gamma, \epsilon) \qquad \text{where} \qquad \epsilon k_*^{1+\delta-\gamma} = 1.$$

Thus, for a fixed value of ϵ, the unstable band gets wider as γ increases toward $1 + \delta$. (We have $k_* = 100$ in Figure 2 and $k_* \approx 87$ in Figure 4.)

Figures 5–7 show the transition to chaos for $\epsilon = 0.5$, $\delta = 0.5$ as γ increases toward 1.5. For each value of γ, we plot the L^∞ and L^2 norms of u at a number
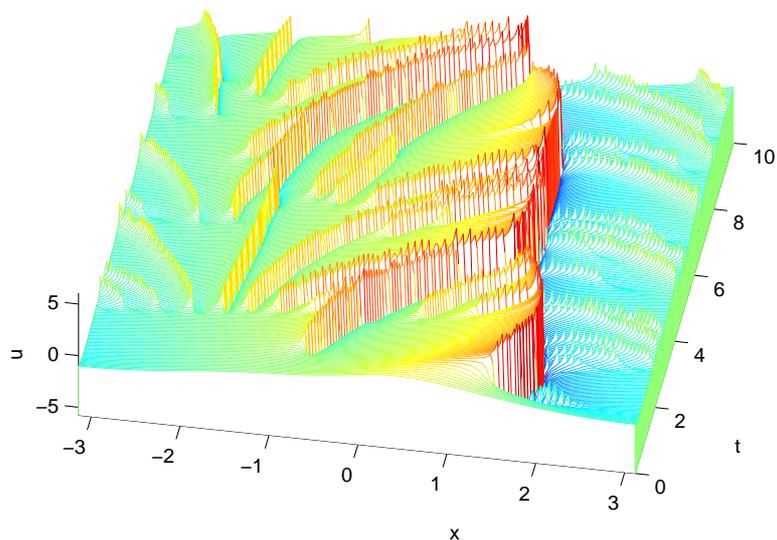
FIGURE 1. A spatially periodic numerical solution of the nonlocal KS equation (27) with $\delta = 1$, $\gamma = 1$, $\epsilon = 0.01$, and initial data (28).
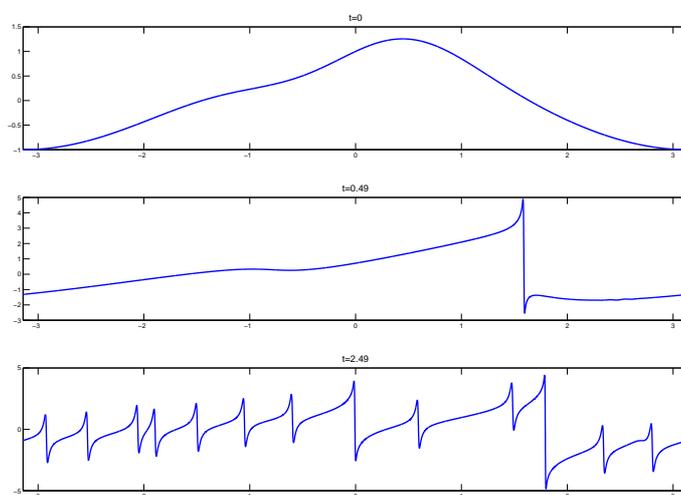

FIGURE 2. A numerical solution of (27) with $\delta = 1$, $\gamma = 1$, $\epsilon = 0.01$ and the same initial data as in Figure 1 at $t = 0, 0.49, 2.49$.

of different times after the solution has approached its time-asymptotic state. For $\gamma \lesssim 1.3$ the solution is steady, but for $\gamma \gtrsim 1.3$ its norms fluctuate wildly in time. We have $k_* \approx 32$ at transition.
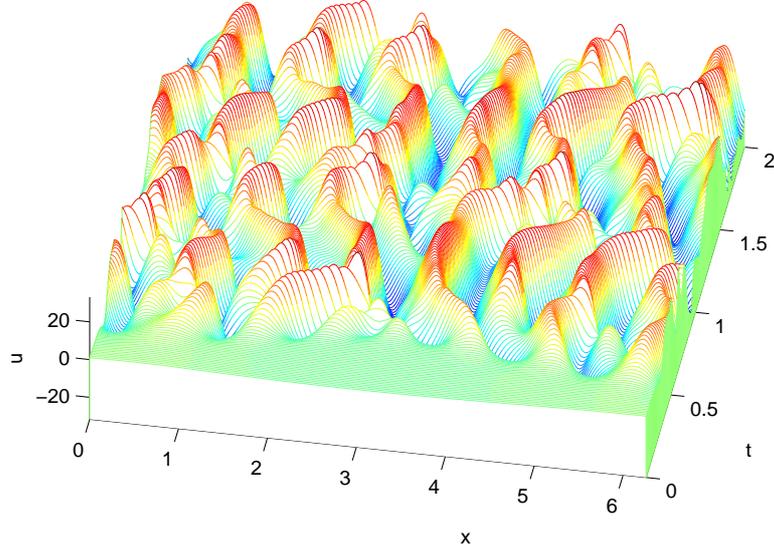
FIGURE 3. A numerical solution of the usual KS equation
(3) with $\epsilon = 0.01$ and initial data (28).
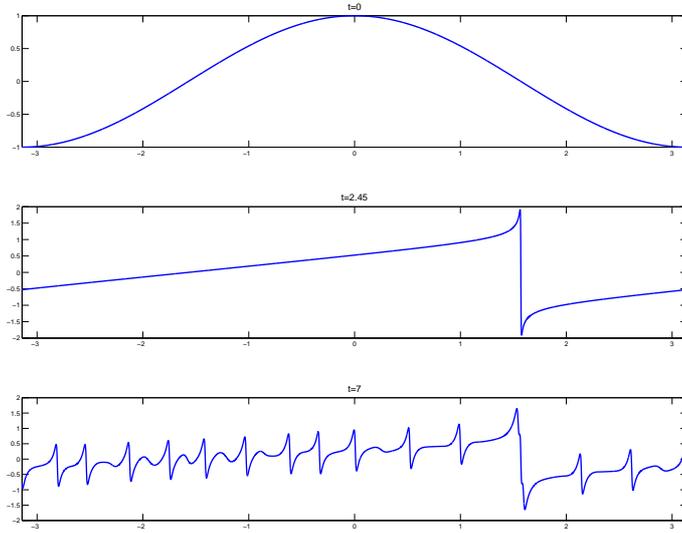
FIGURE 4. A solution of (27) with $\delta = 0.5$, $\gamma = 1.45$, $\epsilon = 0.8$,
and initial data (29) at $t = 0, 2.45, 7$.

Similarly, in Figures 8–10, we show the transition to chaos for $\delta = 1$, $\gamma = 1$ as
ϵ decreases toward 0. The solution is steady for $\epsilon \gtrsim 0.04$ and chaotic for $\epsilon \lesssim 0.04$,
with $k_* \approx 25$ at transition.

APPENDIX A. AUXILIARY RESULTS

In this appendix, we state without proof several results used in the paper.
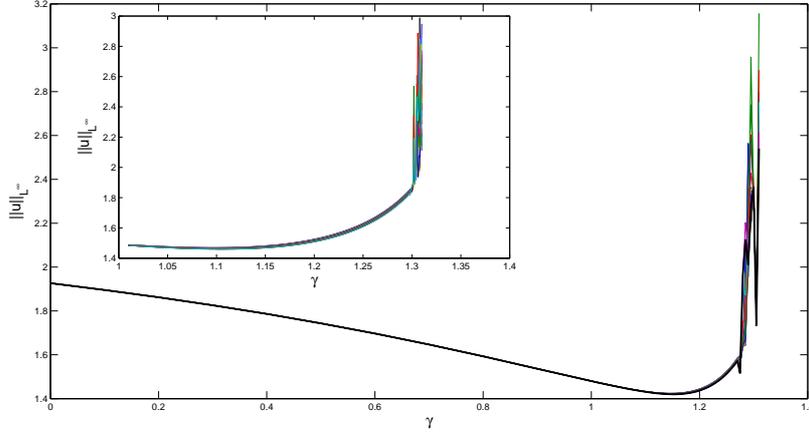
FIGURE 5. The large time behavior of $\|u\|_{L^\infty}$ for different values of $\gamma \in (1, 1.4)$ with $\delta = 0.5$, $\epsilon = 0.5$.
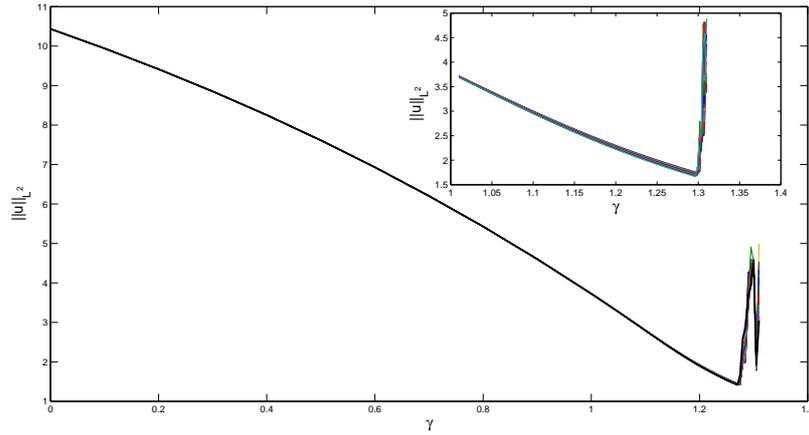

FIGURE 6. The large time behavior of $\|u\|_{L^2}$ for different values of $\gamma \in (1, 1.4)$ with $\delta = 0.5$, $\epsilon = 0.5$.

We start with the Kato-Ponce inequality and the Kenig-Ponce-Vega commutator estimate for $[\Lambda^s, F] = \Lambda^s F - F \Lambda^s$, where $\Lambda = \sqrt{-\partial_x^2}$ (see [21, 24, 25]).

Lemma 6. *Let* F, G *be two smooth functions that decay at infinity. Then, for* $0 < s \leq 1$, *we have*

$$\|\Lambda^s(FG) - F\Lambda^s G\|_{L^p} \leq C \left(\|F\|_{W^{s,p_1}} \|G\|_{L^{p_2}} \right.$$
$$\left. + \|G\|_{W^{s-1,p_3}} \|\partial_x F\|_{L^{p_4}} \right),$$

with

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4} \qquad \text{where } 1 \leq p_2, p_4 \leq \infty, \ 1 < p, p_1, p_3 < \infty.$$

Furthermore, if $s > \max\{0, 1/p - 1\}$, *then*

$$\|\Lambda^s(FG)\|_{L^p(\mathbb{R})} \leq C \left(\|\Lambda^s F\|_{L^{p_1}(\mathbb{R})} \|G\|_{L^{p_2}(\mathbb{R})} + \|\Lambda^s G\|_{L^{p_3}(\mathbb{R})} \|F\|_{L^{p_4}(\mathbb{R})} \right),$$
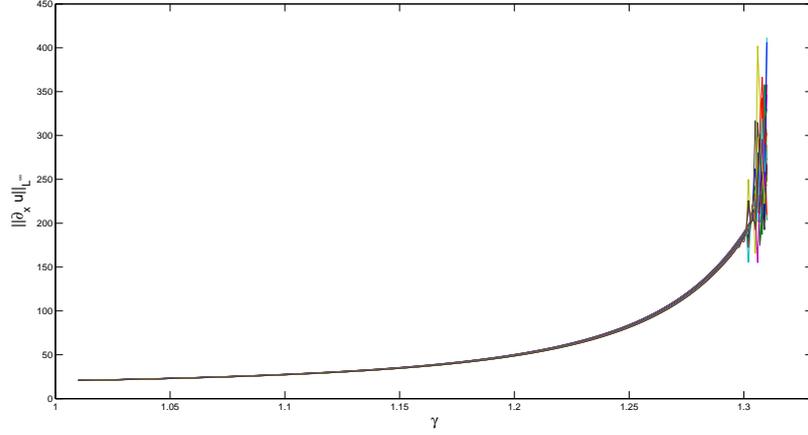
FIGURE 7. The large time behavior of $\|\partial_x u\|_{L^\infty}$ for different values of $\gamma \in (1, 1.4)$ with $\delta = 0.5$, $\epsilon = 0.5$.
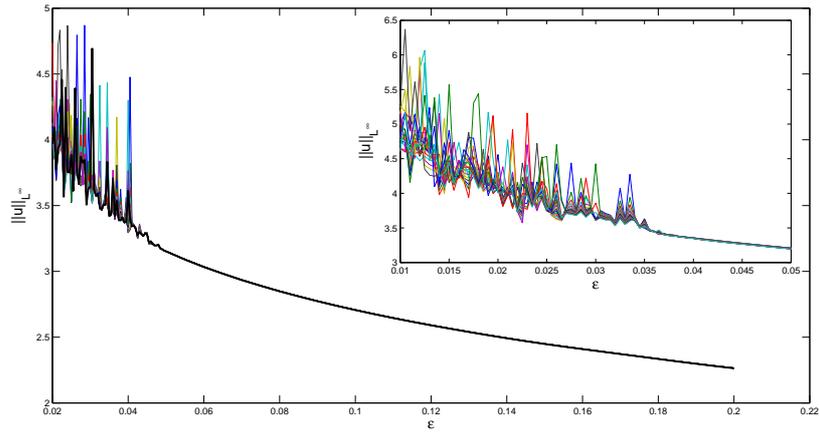

FIGURE 8. The large time behaviour of $\|u\|_{L^\infty}$ for different values of $\epsilon \in (0.02, 0.2)$ with $\delta = 1$, $\gamma = 1$.

with

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4} \qquad \text{where } 1/2 < p < \infty, 1 < p_i \leq \infty.$$

We require the following uniform Gronwall lemma (see [40]).

Lemma 7. *Suppose that g, h, y are non-negative, locally integrable functions on $(0, \infty)$ and dy/dt is locally integrable. If there are positive constants a_1, a_2, a_3, r such that*

$$\frac{dy}{dt} \leq gy + h, \quad \int_t^{t+r} g(s)ds \leq a_1, \ \int_t^{t+r} h(s)ds \leq a_2, \ \int_t^{t+r} y(s)ds \leq a_3$$

for $t \geq 0$, then

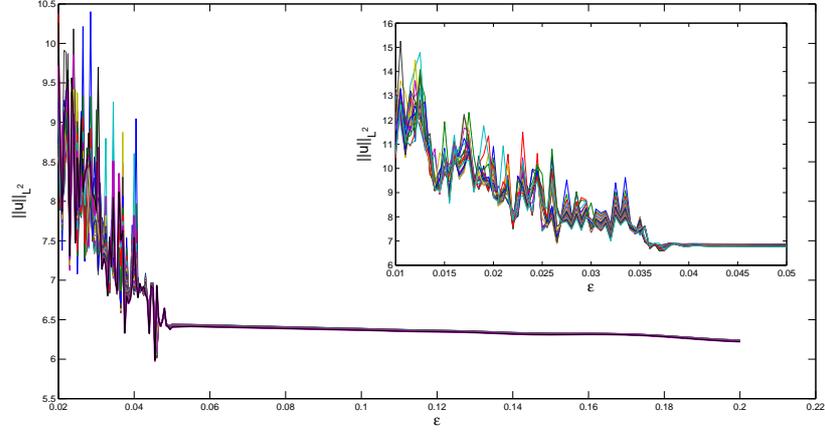$$y(t + r) \leq \left(\frac{a_3}{r} + a_2\right) e^{a_1}.$$

FIGURE 9. The large time behaviour of $\|u\|_{L^2}$ for different values of $\epsilon \in (0.02, 0.2)$ with $\delta = 1$, $\gamma = 1$.
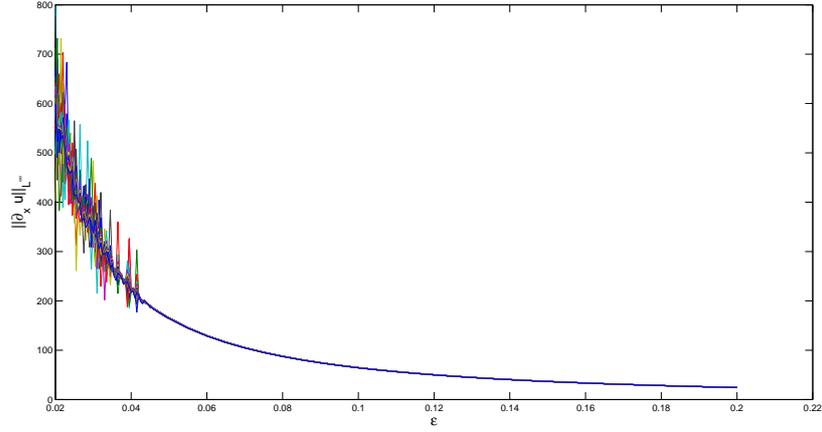

FIGURE 10. The large time behaviour of $\|\partial_x u\|_{L^\infty}$ for different values of $\epsilon \in (0.02, 0.2)$ with $\delta = 1$, $\gamma = 1$.

We also use the following result on the time derivative of a complex function (see [5]).

Lemma 8. *Suppose that $h(t) > 0$ is a decreasing, smooth function of t, and*

$$\phi(x \pm i\zeta, t) = \sum_{|\xi| \leq N} A_\xi(t) e^{i\xi(x \pm i\zeta)}.$$

Then

$$\partial_t \sum_{\pm} \int_{\mathbb{T}} |\phi(x \pm i\zeta, t)|^2 dx$$

$$\leq \frac{h'(t)}{10} \sum_{\pm} \int_{\mathbb{T}} \Lambda \phi(x \pm i\zeta, t) \overline{\phi(x \pm i\zeta, t)} dx$$

$$- 10 h'(t) \sum_{\pm} \int_{\mathbb{T}} \Lambda \phi(x, t) \overline{\phi(x, t)} dx$$

$$+ 2\Re \sum_{\pm} \int_{\mathbb{T}} \partial_t \phi(x \pm i\zeta, t) \overline{\phi(x \pm i\zeta, t)} dx$$

The last Lemma concerns the number of wild spatial oscillations of an analytic function (see [22] and the references therein)

Lemma 9. *Let $L, \tau > 0$, and let u be analytic in the neighborhood of $\{z : |\Im z| \leq \tau\}$ and L-periodic in the x-direction. Then, for any $\mu > 0$, $[0, L] = I_\mu \cup R_\mu$, where I_μ is an union of at most $[\frac{2L}{\tau}]$ intervals open in $[0, L]$, and*

- $|\partial_x u(x)| \leq \mu$, *for all $x \in I_\mu$,*
- $\text{card}\{x \in R_\mu : \partial_x u(x) = 0\} \leq \frac{2}{\log 2} \frac{L}{\tau} \log\left(\frac{\max_{|\Im z| \leq \tau} |\partial_x u(z)|}{\mu} \right).$

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