6-9-2023

Panel Data Models with Time-Varying Latent Group Structures

Peter C. B. Phillips
Yale University

Liangjun Su
Tsinghua University

Yiren Wang
Singapore Management University

PANEL DATA MODELS WITH TIME-VARYING
LATENT GROUP STRUCTURES

By

Yiren Wang, Peter C. B. Phillips and Liangjun Su

June 2023

COWLES FOUNDATION DISCUSSION PAPER NO. D2364

COWLES FOUNDATION FOR RESEARCH IN ECONOMICS

YALE UNIVERSITY
Box 208281
New Haven, Connecticut 06520-8281

http://cowles.yale.edu/

Panel Data Models with Time-Varying Latent Group Structures*

Yiren Wang[a], Peter C B Phillips[b] and Liangjun Su[c]

[a]School of Economics, Singapore Management University, Singapore

[b]Yale University, University of Auckland, Singapore Management University

[c]School of Economics and Management, Tsinghua University, China

June 9, 2023

Abstract

This paper considers a linear panel model with interactive fixed effects and unobserved individual and time heterogeneities that are captured by some latent group structures and an unknown structural break, respectively. To enhance realism the model may have different numbers of groups and/or different group memberships before and after the break. With the preliminary nuclear-norm-regularized estimation followed by row- and column-wise linear regressions, we estimate the break point based on the idea of binary segmentation and the latent group structures together with the number of groups before and after the break by sequential testing K-means algorithm simultaneously. It is shown that the break point, the number of groups and the group memberships can each be estimated correctly with probability approaching one. Asymptotic distributions of the estimators of the slope coefficients are established. Monte Carlo simulations demonstrate excellent finite sample performance for the proposed estimation algorithm. An empirical application to real house price data across 377 Metropolitan Statistical Areas in the US from 1975 to 2014 suggests the presence both of structural breaks and of changes in group membership.

Key words: Interactive fixed effects, latent group structure, structural break, nuclear norm regularization, sequential testing K-means algorithm.

JEL Classification: C23, C33, C38, C51

*Phillips acknowledges support from a Lee Kong Chian Fellowship at SMU, the Kelly Fund at the University of Auckland, and the NSF under Grant No. SES 18-50860. Su gratefully acknowledges the support from the National Natural Science Foundation of China under Grant No. 72133002. Address correspondence to: Liangjun Su, School of Economics and Management, Tsinghua University, Beijing, 100084, China. E-mail: sulj@sem.tsinghua.edu.cn.

1 Introduction

Heterogeneous panel data models have been widely used in empirical research in economics because they can capture a rich degree of unobserved heterogeneity. But models with complete heterogeneity along either the cross section or time dimension tend to possess too many parameters to be identified, which results in slow convergence and inefficient estimates. For this reason, researchers now frequently advocate the use of panel data models with certain structures imposed along either the cross section or time dimension. On the one hand, the recent burgeoning of panels with latent group structures can be motivated from the observation that different groups of individuals respond differently to exogenous shocks. For instance, Durlauf and Johnson (1995), Berthelemy and Varoudakis (1996), and Ben-David (1998) show economies in different groups of income per capita and/or education level may converge to different steady state equilibria. Klapper and Love (2011), Chu (2012), and Zhang and Cheng (2019) show an exogenous shock like policy implementation has different impacts on different individuals, and Long et al. (2012) argue that the influence of the 2008 financial crisis on economic growth differs for emergent and developed economies. On the other hand, the recent popularity of panels that evidence structural change can be motivated by events such as financial crises, the economic impact of technological progress, and more general economic transitions that occur during the time periods covered by the data. See Qian and Su (2016) for a survey of panel data models and research that consider estimation and inference concerning structural change.

In spite of the now large literature that studies separately individual heterogeneity or time heterogeneity in the slope coefficients of panel models, few works consider both types of heterogeneity simultaneously. Exceptions include Keane and Neal (2020) and Lu and Su (2023) who consider linear panel data models with two-dimensional unobserved heterogeneity in the slope coefficients that are modelled via the usual additive structure, and Chernozhukov et al. (2020) and Wang et al. (2022) who model the slope coefficients via the use of low-rank matrices for conditional mean and quantile regressions, respectively. In addition, Okui and Wang (2021) and Lumsdaine et al. (2023) consider both individual heterogeneity and time heterogeneity by modeling them as a grouped pattern and as structural breaks, respectively. Specifically, Okui and Wang (2021) develop a new panel data model with latent groups where the number of groups and the group memberships do not change over time but the coefficients within each group can change over time and they may have different break-dates; Lumsdaine et al. (2023) consider the panels with a grouped pattern of heterogeneity when the latent group membership structure and/or the values of slope coefficients change at a break point. Both papers provide algorithms to recover the latent group structure based on linear panel models with or without individual fixed effects, but cannot allow for the presence of more complicated fixed effects such as interactive fixed effects (IFEs) to capture strong cross-sectional dependence in the data.

This paper proposes a linear panel data model with IFEs that enable the slope coefficients to exhibit two-way heterogeneity. Following the lead of Okui and Wang (2021) and Lumsdaine et al. (2023) and to encourage the parameter parsimony, we use a latent group structure to capture individual heterogeneity and an unknown structural break to capture time heterogeneity. The latent group structure of the model accommodates different group numbers and different group memberships before and after the break. Given this complicated structure, the approach proposed is to estimate the break point, the number of groups before and after the break, the group membership before and after the break, and the group-specific parameters in multiple steps. The key insight that permits this degree of complication is that the slope coefficients of each of the p regressors in the model are permitted to vary across both cross section and time dimensions by means of a factor structure with a fixed number of factors so that they may be conveniently stacked into a low-rank matrix.

In the first step, the low-rank nature of the slope matrices is explored and initial estimates are obtained by nuclear norm regularization (NNR), a machine learning technique popular in computer science that is increasingly used in econometrics. Such initial matrix estimates are consistent in terms of the Frobenius norm but do not have pointwise or uniform convergence for their elements. Despite this, by applying singular value decomposition (SVD) to these estimates, we can obtain estimates of the associated factors and the factor loadings that are also consistent in terms of the Frobenius norm. In the second step, we use the first-step initial estimates of the factors and factor loadings to run the row- and column-wise linear regressions to update the estimates of the factors and factor loadings which now possess pointwise and uniform consistency and can be used for subsequent analyses. In the third step, we estimate the break point by using the celebrated idea of binary segmentation, as commonly used for break point estimation in the time series literature. Once the break point is estimated, the full-sample is naturally split into two subsamples. In the fourth step, we follow the lead of Lin and Ng (2012) and Jin et al. (2022) to focus on each subsample before and after the estimated break point and propose a sequential testing K-means algorithm to recover the latent group structure and obtain the number of groups simultaneously. In the last step, we use the estimated group structure to estimate the group-specific parameters. Asymptotic analyses show that the break point, the number of groups and the group memberships can be consistently estimated in Steps 3-4, so that the final step estimates for the group-specific coefficients can enjoy the oracle property. This means they have the same asymptotic distributions as the ones obtained by knowing the break point and the latent group structures before and after the break points.

The present paper relates to two branches of literature. First, it contributes to the panel data literature on one-way heterogeneity, especially with either latent group structures or structural breaks. With respect to latent group structures, there are several popular ways to recover the latent groups. The first approach is the K-means algorithm. Lin and Ng (2012) apply the K-means algorithm to linear panel data models with grouped slope coefficients and propose an information criterion and a sequential testing approach to estimate the true number of groups. Sarafidis and Weber (2015) analyze the unknown grouped slopes in the large N and fixed T framework, and Zhang et al. (2019) provide an iterative algorithm based on K-means clustering for a panel quantile regression model. Bonhomme and Manresa (2015) and Ando and Bai (2016) consider panels with grouped fixed effects. The second approach is the Classifier-Lasso (C-Lasso) that has become a popular clustering method since Su et al. (2016). This method is extended by Lu and Su (2017), Su and Ju (2018), Su et al. (2019), Wang et al. (2019), and Huang et al. (2020) to various contexts. In addition, both the clustering algorithm in regression via a data-driven segmentation (CARDS) approach and binary segmentation are also considered in Ke et al. (2015), Wang et al. (2018), Ke et al. (2016) and Wang and Su (2021), among others. As for panel models with structural breaks, binary segmentation has become a common approach to estimate the break point. See Bai (2010), Lin and Hsu (2011), Kim (2011), Kim (2014) and Baltagi et al. (2017), among others. These papers focus on the case of a single break point in the model. In contrast, Qian and Su (2016) and Li et al. (2016) allow for multiple breaks in linear panel models with either classical fixed effects or IFEs, and propose an adaptive grouped fused lasso (AGFL) approach to estimate the break points. Compared to the existing panel literature on one-way heterogeneity, our model allows for two-way heterogeneity. In particular, not only are different membership structures in different time blocks permitted but also changes in the number of groups over time. As a result, our model is more flexible than all existing models that allow only for latent group structures or structural breaks, but not both.

Second, this paper contributes to the recent burgeoning literature that models two-way heterogeneity in the slope coefficients of a panel model. As mentioned above, there are two approaches to model two-way heterogeneity in the slope coefficients. One approach models them in an additive structure so that both individual and time effects enter the slope coefficients additively, as in Keane and Neal (2020) and Lu and Su (2023). The other approach imposes certain low-rank structures on the slope coefficient matrices in which case one models each slope coefficient via the use of IFEs to capture strong cross sectional dependence in the panel. In view of the low-rank structures, we can resort to NNR estimation which has attracted increasing attention recently in panel data analyses. NNR has been used in recent econometric research – see Bai and Ng (2019), Moon and Weidner (2018), Chernozhukov et al. (2020), Belloni et al. (2023), Miao et al. (2023), Feng (2023), and Hong et al. (2023), among others. But none of these papers imposes any latent group structures on the slope coefficients. With latent group structures and structural breaks imposed, Okui and Wang (2021) allows the slope coefficients within each group to have common breaks and the break points to vary across different groups, and they propose to estimate the latent group structures, the structural breaks, and the group-specific regression parameters by the grouped adaptive group fused lasso (GAGFL). But neither the number of groups nor the group memberships is allowed to change over time in Okui and Wang (2021). In a companion paper, Lumsdaine et al. (2023) allows the latent group membership structure and/or the values of slope coefficients to change at a break point, and proposes an estimation algorithm similar to the K-means of Bonhomme and Manresa (2015). Both Okui and Wang (2021) and Lumsdaine et al. (2023) allow for at most one-way heterogeneity (individual fixed effects) in the intercept and neither allows for IFEs to capture strong cross section dependence. In contrast, this paper proposes an algorithm to detect the unknown break point and to recover the group structure based on linear panel model with IFEs, which involves a more general model. In addition, Lumsdaine et al. (2023) first assume the number of groups is known in the estimation algorithm and then estimate the number of groups via an information criterion but they do not establish consistency for such an estimate. Instead, we estimate the number of groups and group membership simultaneously by the sequential testing K-means algorithm and establish the consistency of the number of groups estimator.

The rest of the paper is organized as follows. We first introduce the linear panel model with time-varying latent group structures in Section 2 and provide the estimation algorithm in Section 3. The asymptotic properties are given in Section 4. In Section 5, we propose an alternative approach to detect the break point, provide the test statistics for the null that the slope coefficients exhibit no structure change against the alternative with one break point, and discuss the estimation for the model with multiple breaks. In Sections 6 and 7, we show the finite sample performance of our method by Monte Carlo simulations and an empirical application, respectively. Section 8 concludes. All proofs are provided in the online supplement.

Notation. Let $\|\cdot\|_{\max}$, $\|\cdot\|_{op}$, $\|\cdot\|$, and $\|\cdot\|_*$ denote the (elementwise) maximum norm, operator norm, Frobenius norm, and nuclear norm, respectively. Let $\odot$ denote the element-wise Hadamard product. $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ denote the floor and ceiling functions, respectively. Let $a \vee b = \max(a, b)$ and $a \wedge b = \min(a, b)$. $a_n \lesssim b_n$ means $a_n/b_n = O_p(1)$ and $a_n \gg b_n$ means $b_n a_n^{-1} = o(1)$. Let $A = \{A_{it}\}$ be a matrix with its (i,t)-th entry denoted as A_{it}. Let denote $\{A_j\}_{j\in[p]\cup\{0\}}$ be the collection of matrices A_j, $j \in \{0, 1, \cdots, p\}$. For a specific $A \in \mathbb{R}^{m\times n}$ with rank n, let $P_A = A(A'A)^{-1}A'$ and $M_A = I_m - P_A$. When A is symmetric, $\lambda_{\max}(A)$, $\lambda_{\min}(A)$ and $\lambda_n(A)$ denote its largest, smallest and n-th largest eigenvalues, respectively. The operators $\rightsquigarrow$ and $\xrightarrow{p}$ denote convergence in distribution and in probability, respectively. Let $[n] = \{1, \cdots, n\}$ for any positive integer n, let $\mathbf{1}\{\cdot\}$ be the usual indicator function, and w.p.a.1 and a.s. abbreviate "with probability approaching 1" and "almost surely", respectively.

2 Model Setup

In this paper we consider the following linear panel model with IFEs:

$$Y_{it} = \Theta_{0,it}^0 + X_{it}'\Theta_{it}^0 + e_{it}, \tag{2.1}$$

where $i \in [N]$, $t \in [T]$, Y_{it} is the dependent variable, $X_{it} = (X_{1,it}, \cdots, X_{p,it})'$ is a $p \times 1$ vector of regressors, $\Theta_{it}^0 = (\Theta_{1,it}^0, \cdots, \Theta_{p,it}^0)'$ is a $p \times 1$ vector of slope coefficients, $\Theta_{0,it}^0 = \lambda_i^{0\prime} f_t^0$ is an intercept term that exhibits a factor structure with r_0 factors, and e_{it} is the error term. Here, we assume r_0 is a fixed integer that does not change as $(N,T) \to \infty$. Let $\Lambda^0 = \left(\lambda_1^0, \cdots, \lambda_N^0\right)'$ and $F^0 = \left(f_1^0, \cdots, f_T^0\right)'$. Let $Y = \{Y_{it}\}$, $X_j = \{X_{j,it}\}$, $\Theta_j^0 = \{\Theta_{j,it}^0\}$ and $E = \{e_{it}\}$, all of which are $N \times T$ matrices. Then we can rewrite (2.1) in matrix form as

$$Y = \Theta_0^0 + \sum_{j=1}^{p} X_j \odot \Theta_j^0 + E. \tag{2.2}$$

We assume that the slope coefficients follow time-varying latent group structures, viz.,

$$\Theta_{it}^0 = \sum_{k\in[K_t]} \alpha_{kt}\mathbf{1}\{i \in G_{kt}\},$$

where $\{G_{kt}\}_{k\in K_t}$ forms a partition of $[N]$ for each specific time t with K_t being the number of groups at time t. Moreover, we assume that the group-specific slope coefficients α_{kt} or the memberships change at an unknown time point T_1, i.e.,

$$\alpha_{kt} = \begin{cases} \alpha_k^{(1)}, & \text{for} \quad t = 1, \ldots, T_1, \\ \alpha_k^{(2)}, & \text{for} \quad t = T_1+1, \ldots, T, \end{cases}$$

$$G_{kt} = \begin{cases} G_k^{(1)}, & \text{for} \quad t = 1, \ldots, T_1,\ k = 1, \ldots, K^{(1)}, \\ G_k^{(2)}, & \text{for} \quad t = T_1+1, \ldots, T,\ k = 1, \ldots, K^{(2)}, \end{cases}$$

with $K^{(1)}$ and $K^{(2)}$ being the number of latent groups before and after the break point, respectively. Let $g_i^{(1)}$ and $g_i^{(2)}$ respectively denote the individual group indices before and after the break:

$$g_i^{(1)} = \sum_{k\in K^{(1)}} k\mathbf{1}\{i \in G_k^{(1)}\} \quad \text{and} \quad g_i^{(2)} = \sum_{k\in K^{(2)}} k\mathbf{1}\{i \in G_k^{(2)}\}.$$

Let r_j be the rank of Θ_j^0 for $j \in [p]\cup\{0\}$. It is easy to see that Θ_j^0 exhibits a low-rank structure for all j. By the SVD, we have

$$\Theta_j^0 = \sqrt{NT}\mathcal{U}_j^0\Sigma_j^0\mathcal{V}_j^{0\prime} := U_j^0 V_j^{0\prime}, \quad j \in [p]\cup\{0\},$$

where $\mathcal{U}_j^0 \in \mathbb{R}^{N\times r_j}$, $\mathcal{V}_j^0 \in \mathbb{R}^{T\times r_j}$, $\Sigma_j^0 = \mathrm{diag}(\sigma_{1,j}, \cdots, \sigma_{r_j,j})$, $U_j^0 = \sqrt{N}\mathcal{U}_j^0\Sigma_j^0$ with each row being $u_{i,j}^{0\prime}$, and $V_j^0 = \sqrt{T}\mathcal{V}_j^0$ with each row being $v_{t,j}^{0\prime}$.

Note that we allow $\left\{\Theta_{it}^0\right\}_{i=1}^N$ to exhibit latent group structures before and after the break. For a particular $j \in [p]$, the $N\times T$ matrix Θ_j^0 may have no group structure before or after the break, or no break, or more or fewer groups after the break. Let $K_j^{(1)}$ and $K_j^{(2)}$ denote the number of groups before and after the break, respectively, for $\{\Theta_{j,it}^0\}_{i=1}^N$. Let $\mathcal{G}_j^{(\ell)} = \{G_{1,j}^{(\ell)}, \cdots, G_{K_j^{(\ell)},j}^{(\ell)}\}$, $\ell = 1, 2$, denote the associated latent group structures. Define $N_{k,j}^{(\ell)} = |G_{k,j}^{(\ell)}|$ and $\pi_{k,j}^{(\ell)} = \frac{N_{k,j}^{(\ell)}}{N}$ for $\ell = 1, 2$, where $|A|$ denotes the cardinality of set A. Further define $\tau_T := \frac{T_1}{T}$. We show that Θ_j^0 has a low-rank structure in all of the following cases:

Case 1: Θ_j^0 exhibits neither structural break nor group structure.

In this case, $K_j^{(1)} = K_j^{(2)} = 1$, and $\Theta_{j,it}^0 = \alpha_j\ \forall (i,t) \in [N]\times[T]$. Without loss of generality, assume that $\alpha_j > 0$. Then by the SVD, we have

$$\mathcal{U}_j = \frac{1}{\sqrt{N}}\iota_N \in \mathbb{R}^{N\times 1}, \quad \Sigma_j = \alpha_j, \quad \mathcal{V}_j = \frac{1}{\sqrt{T}}\iota_T \in \mathbb{R}^{T\times 1},$$
$$U_j = \alpha_j\iota_N \in \mathbb{R}^{N\times 1}, \quad V_j = \iota_T \in \mathbb{R}^{T\times 1},$$

where $\iota_d = (1, \cdots, 1)' \in \mathbb{R}^{d\times 1}$ for any natural number d. Obviously, $r_j = 1$ in Case 1.

Case 2: Θ_j^0 exhibits no structural break but a group structure.

In this case, $K_j^{(1)} = K_j^{(2)} = K_j$, $G_{k,j}^{(1)} = G_{k,j}^{(2)} = G_{k,j}$, $N_{k,j}^{(1)} = N_{k,j}^{(2)} = N_{k,j}$, $\pi_{k,j}^{(1)} = \pi_{k,j}^{(2)} = \pi_{k,j}$ $\forall k \in [K_j]$, and $\Theta_{j,it}^0 = \sum_{k\in[K_j]} \alpha_{k,j} \mathbf{1}\{i \in G_{k,j}\}$ for $t \in [T]$. Therefore, we have

$$\mathcal{U}_{j,i} = \frac{\sum_{k\in[K_j]} \alpha_{k,j} \mathbf{1}\{i \in G_{k,j}\}}{\sqrt{\sum_{k\in[K_j]} N_{k,j} (\alpha_{k,j})^2}}, \quad \Sigma_j = \sqrt{\sum_{k\in[K_j]} \pi_{k,j} (\alpha_{k,j})^2}, \quad \mathcal{V}_j = \frac{1}{\sqrt{T}} \iota_T,$$
$$u_{i,j} = \sum_{k\in[K_j]} \alpha_{k,j} \mathbf{1}\{i \in G_{k,j}\}, \quad V_j = \iota_T,$$

where $\mathcal{U}_{j,i}$ is the i-th element in $\mathcal{U}_j$. Obviously, $r_j = 1$ in this case.

Case 3: Θ_j^0 exhibits both a structural break and a group structure.

(i) $K_j^{(1)} \neq K_j^{(2)}$, where we have different numbers of groups before and after the break;

(ii) $K_j^{(1)} = K_j^{(2)} = K_j$ and $G_{k,j}^{(1)} \neq G_{k,j}^{(2)}$, where we have the same number of groups before and after the break, but the group memberships change after the break point;

(iii) $K_j^{(1)} = K_j^{(2)} = K_j$, $G_{k,j}^{(1)} = G_{k,j}^{(2)} = G_{k,j}$ for $\forall k \in [K_j]$, and $\alpha_{k,j}^{(1)} \neq \alpha_{k,j}^{(2)}$ for at least one $k \in [K_j]$, where even though neither the number of groups nor group membership changes after the break, there exists at least one group whose slope coefficients change.

For any positive integer d, we use $\mathbf{0}_d$ to denote a $d \times 1$ vector of zeros. The following lemma lays down the foundation for break point detection in our model.

Lemma 2.1 *For any $j \in [p]$ such that Θ_j^0 lies in Case 3 above, we have rank$(\Theta_j^0) \leq 2$. When rank$(\Theta_j^0) = 2$, we have*

(i) $\Theta_j^0 = \mathcal{U}_j \Sigma_j \mathcal{V}_j' = U_j V_j'$ *where* $U_j = \mathcal{U}_j \Sigma_j / \sqrt{T}$, $V_j = \sqrt{T} \mathcal{V}_j = D_j R_j$, $D_j = \begin{bmatrix} \frac{1}{\sqrt{\tau_T}} \iota_{T_1} & \mathbf{0}_{T_1} \\ \mathbf{0}_{T-T_1} & \frac{1}{\sqrt{1-\tau_T}} \iota_{T-T_1} \end{bmatrix}$ *and* $R_j' R_j = I_2$;

(ii) $\left\| \frac{v_{t,j}^0}{\|v_{t,j}^0\|} - \frac{v_{t^*,j}^0}{\|v_{t^*,j}^0\|} \right\| = \sqrt{2}$ *for any* $t \leq T_1$ *and* $t^* > T_1$.

By Lemma 2.1 for Case 3 and the above analyses for Cases 1 and 2, we conclude that Θ_j^0 is a low-rank matrix with rank equal to or less than 2. In view of the low-rank structure of the slope matrices, we propose to adopt the NNR to obtain the preliminary estimates below. Moreover, under Case 3, Lemma 2.1(ii) indicates that singular vectors of the slope matrix with rank 2 contain the structural break information.

3 Estimation

In this section we provide the estimation algorithm. We first assume that the ranks r_j for $j \in [p] \cup \{0\}$ are known, and then propose a singular value thresholding (SVT) procedure to estimate them. After we recover the break point and the latent group structures, we propose consistent estimates of the group-specific parameters.

3.1 Estimation Algorithm

Given $r_j, \ \forall j \in [p] \cup \{0\}$, we propose the following four-step procedure to estimate the break point and to recover the latent group structures before and after the break.

Step 1: Nuclear Norm Regularization (NNR). We run the nuclear norm regularized regression and obtain the preliminary estimates as follows:

$$\{\tilde{\Theta}_j\}_{j \in [p] \cup \{0\}} = \underset{\{\Theta_j\}_{j=0}^{p}}{\arg\min} \frac{1}{NT} \left\| Y - \sum_{j=1}^{p} X_j \odot \Theta_j - \Theta_0 \right\|^2 + \sum_{j=0}^{p} \nu_j \left\| \Theta_j \right\|_* , \tag{3.1}$$

where ν_j is the tuning parameter for $j \in [p] \cup \{0\}$. For each j, conduct the SVD: $\frac{1}{\sqrt{NT}} \tilde{\Theta}_j = \hat{\tilde{\mathcal{U}}}_j \hat{\tilde{\Sigma}}_j \hat{\tilde{\mathcal{V}}}_j'$, where $\hat{\tilde{\Sigma}}_j$ is a diagonal matrix with the diagonal elements being the descending singular values of $\tilde{\Theta}_j$. Let $\tilde{\mathcal{V}}_j$ consist of the first r_j columns of $\hat{\tilde{\mathcal{V}}}_j$, and $\tilde{V}_j = \sqrt{T} \tilde{\mathcal{V}}_j$. Let $\tilde{v}_{t,j}'$ denote the t-th row of $\tilde{V}_j$ for $t \in [T]$.

Step 2: Row- and Column-Wise Regressions. First run the row-wise regressions of Y_{it} on $\left(\tilde{v}_{t,0}, \{\tilde{v}_{t,j} X_{j,it}\}_{j \in [p]}\right)$ to obtain $\{\dot{u}_{i,j}\}_{j \in [p] \cup \{0\}}$ for $i \in [N]$. Then run the column-wise regressions of Y_{it} on $\left(\dot{u}_{i,0}, \{\dot{u}_{i,j} X_{j,it}\}_{j \in [p]}\right)$ to obtain $\{\dot{v}_{t,j}\}_{j \in [p] \cup \{0\}}$ for $t \in [T]$. Let $\dot{\Theta}_{j,it} = \dot{u}_{i,j}' \dot{v}_{t,j}$ for $(i,t) \in [N] \times [T]$ and $j \in [p] \cup \{0\}$. Specifically, the row- and column-wise regressions are given by

$$\{\dot{u}_{i,j}\}_{j \in [p] \cup \{0\}} = \underset{\{u_{i,j}\}_{j \in [p] \cup \{0\}}}{\arg\min} \frac{1}{T} \sum_{t \in [T]} \left(Y_{it} - u_{i,0}' \tilde{v}_{t,0} - \sum_{j=1}^{p} u_{i,j}' \tilde{v}_{t,j} X_{j,it} \right)^2 , \quad i \in [N], \tag{3.2}$$

$$\{\dot{v}_{t,j}\}_{j \in [p] \cup \{0\}} = \underset{\{v_{t,j}\}_{j \in [p] \cup \{0\}}}{\arg\min} \frac{1}{N} \sum_{i \in [N]} \left(Y_{it} - v_{t,0}' \dot{u}_{i,0} - \sum_{j=1}^{p} v_{t,j}' \dot{u}_{i,j} X_{j,it} \right)^2 , \quad t \in [T]. \tag{3.3}$$

Step 3: Break Point Estimation. We estimate the break point as follows:

$$\hat{T}_1 = \underset{s \in \{2, \cdots, T-1\}}{\arg\min} \frac{1}{pNT} \sum_{j \in [p]} \sum_{i \in [N]} \left\{ \sum_{t=1}^{s} \left(\dot{\Theta}_{j,it} - \bar{\dot{\Theta}}_{j,i}^{(1s)} \right)^2 + \sum_{t=s+1}^{T} \left(\dot{\Theta}_{j,it} - \bar{\dot{\Theta}}_{j,i}^{(2s)} \right)^2 \right\}, \tag{3.4}$$

where $\bar{\dot{\Theta}}_{j,i}^{(1s)} = \frac{1}{s} \sum_{t=1}^{s} \dot{\Theta}_{j,it}$ and $\bar{\dot{\Theta}}_{j,i}^{(2s)} = \frac{1}{T-s} \sum_{t=s+1}^{T} \dot{\Theta}_{j,it}$.

Step 4: Sequential Testing K-means (STK). In this step, we estimate the number of groups and the group membership before and after the break by using the STK algorithm. For each $j \in [p]$, define $\dot{\Theta}_{j,i}^{(1)} = (\dot{\Theta}_{j,i1}, \cdots, \dot{\Theta}_{j,i\hat{T}_1})'$, $\dot{\Theta}_{j,i}^{(2)} = (\dot{\Theta}_{j,i,\hat{T}_1+1}, \cdots, \dot{\Theta}_{j,iT})'$, $\dot{\beta}_i^{(1)} = \frac{1}{\sqrt{\hat{T}_1}}(\dot{\Theta}_{1,i}^{(1)\prime}, \cdots, \dot{\Theta}_{p,i}^{(1)\prime})'$, and $\dot{\beta}_i^{(2)} = \frac{1}{\sqrt{\hat{T}_2}}(\dot{\Theta}_{1,i}^{(2)\prime}, \cdots, \dot{\Theta}_{p,i}^{(2)\prime})'$. Let z_ς be some predetermined value which will be specified in the next subsection. Given the subsample before and after the estimated break point, initialize $m = 1$ and classify each subsample into m groups by the K-means algorithm with group membership obtained as $\hat{\mathcal{G}}_m^{(\ell)} := \{\hat{G}_{k,m}^{(\ell)}\}_{k\in[m]}$. Next, we construct a suitable test statistic $\hat{\Gamma}_m^{(\ell)}$, defined by (3.8) in the next subsection, and compare it to its critical value z_ς at significance level ς under the null hypothesis of m subgroups, setting $m = m + 1$ and moving to the next iteration if $\hat{\Gamma}_m^{(\ell)} > z_\varsigma$ and stopping the STK algorithm otherwise. Lastly, define $\hat{K}^{(\ell)} = m$ and $\hat{\mathcal{G}}^{(\ell)} = \hat{\mathcal{G}}_m^{(\ell)}$. The next subsection provides a detailed breakdown of these steps in the STK algorithm.

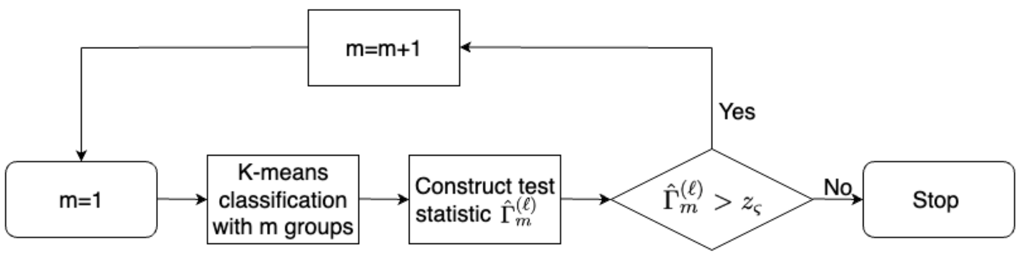


Figure 1: The flow chart of STK algorithm

Several remarks are in order. First, the ranks of the intercept and slope matrices are assumed known in Step 1 but consistent estimates for them are proposed in the SVT below. Second, we obtain preliminary estimates by NNR based on the low-rank structure of the intercept and slope matrices in the model. These estimates are consistent in terms of the Frobenius norm but pointwise or uniform convergence for their elements is not established. Nonetheless, SVD can be employed to obtain preliminary estimates of the factors and factor loadings to be used subsequently. Third, row- and column-wise linear regressions are conducted to obtain updated estimates of the factors and factor loadings for which we can establish pointwise and uniform convergence rates. Fourth, using the consistent estimates obtained in the second step, we can estimate the break point in Step 3 consistently by using a binary segmentation process. Fifth, the STK algorithm in Step 4 then yields the estimated number of groups and the group memberships together.

In the latent group literature, it is standard and popular to assume the number of groups in the K-means algorithm is known and then to estimate the number of groups by using certain information criteria. In this case, one needs to consider not only under- and just-fitting cases, but also over-fitting cases. It is well known that the major difficulty with this approach is showing that the over-fitting case occurs with probability approaching zero. The STK algorithm ensures a focus on the under-and just-fitting cases, which helps to avoid the difficulty caused by K-means classification with a larger than true number of groups. In addition, although this sequential algorithm approach is adopted, the error from the previous iteration does not accumulate in the following iterations owing to fact that the classification in each iteration is new and is not based on the K-means outcomes in previous iterations.

3.2 The STK algorithm

This subsection describes the K-means algorithm and the construction of the test statistics $\hat{\Gamma}_m^{(\ell)}$ that are used in the STK algorithm for $\ell \in \{1, 2\}$.

First, we define the objective function for the K-means algorithm with m clusters at each iteration. Let $a_{k,m}^{(\ell)}$ be a $p\hat{T}_1 \times 1$ and $p(T - \hat{T}_1) \times 1$ vector for $\ell = 1, 2$, respectively. We obtain the group membership with m groups by solving the following minimization problem

$$\left\{\dot{a}_{k,m}^{(\ell)}\right\}_{k\in[m]} = \underset{\left\{a_{k,m}^{(\ell)}\right\}_{k\in[m]}}{\arg\min} \frac{1}{N} \sum_{i\in[N]} \min_{k\in[m]} \left\|\dot{\beta}_i^{(\ell)} - a_{k,m}^{(\ell)}\right\|^2, \tag{3.5}$$

which yields the membership estimates for each individual at the m-th iteration as

$$\hat{g}_{i,m}^{(\ell)} = \underset{k\in[m]}{\arg\min} \left\|\dot{\beta}_i^{(\ell)} - \dot{a}_{k,m}^{(\ell)}\right\| \quad \forall i \in [N]. \tag{3.6}$$

Let $\hat{G}_{k,m}^{(\ell)} := \{i \in [N] : \hat{g}_{i,m}^{(\ell)} = k\}$.

Second, we discuss the construction of the test statistic based on the idea of homogeneity test for several subsamples. At iteration m, we have m potential subgroups $(\hat{G}_{1,m}^{(\ell)}, \cdots, \hat{G}_{m,m}^{(\ell)})$ after the K-means classification for $\ell = 1$ and 2. Let $\hat{\mathcal{T}}_1 = [\hat{T}_1]$, $\hat{\mathcal{T}}_2 = [T]\backslash[\hat{T}_1]$, $\hat{\mathcal{T}}_{1,-1} = \hat{\mathcal{T}}_1\backslash\{\hat{T}_1\}$, $\hat{\mathcal{T}}_{2,-1} = \hat{\mathcal{T}}_2\backslash\{T\}$, $\hat{\mathcal{T}}_{1,j} = \{1 + j, \cdots, \hat{T}_1\}$, and $\hat{\mathcal{T}}_{2,j} = \{\hat{T}_1 + 1 + j, \cdots, T\}$ for some specific $j \in \hat{\mathcal{T}}_{\ell,-1}$. Based on these estimated subgroups, we can obtain the estimates of the coefficients, factors and factor loadings for each subgroup in regime ℓ as follows:

$$\left(\left\{\hat{\theta}_{i,k,m}^{(\ell)}\right\}_{i\in\hat{G}_{k,m}^{(\ell)}}, \hat{\Lambda}_{k,m}^{(\ell)}, \hat{F}_{k,m}^{(\ell)}\right) = \underset{\{\theta_i,\lambda_i,f_t\}_{i\in\hat{G}_{k,m}^{(\ell)},t\in\hat{\mathcal{T}}_\ell}}{\arg\min} \sum_{i\in\hat{G}_{k,m}^{(\ell)}} \sum_{t\in\hat{\mathcal{T}}_\ell} \left(Y_{it} - X_{it}'\theta_i - \lambda_i' f_t\right)^2,$$

where $\hat{\Lambda}_{k,m}^{(\ell)} = \{\hat{\lambda}_{i,k,m}^{(\ell)}\}_{i\in\hat{G}_{k,m}^{(\ell)}}$ and $\hat{F}_{k,m}^{(\ell)} = \{\hat{f}_{t,k,m}^{(\ell)}\}_{t\in\hat{\mathcal{T}}_\ell}$. For all $i \in [N]$ and $t \in [T]$, define the residuals

$$\hat{e}_{it} = \sum_{\ell=1}^{2} \left(Y_{it} - \hat{f}_{t,k,m}^{(\ell)\prime}\hat{\lambda}_{i,k,m}^{(\ell)} - X_{it}'\hat{\theta}_{i,k,m}^{(\ell)}\right) \mathbf{1}\{t \in \hat{\mathcal{T}}_\ell\}.$$

Let $\hat{X}_i^{(1)} = (X_{i1}, \cdots, X_{i\hat{T}_1})'$, $\hat{X}_i^{(2)} = (X_{i,\hat{T}_1+1}, \cdots, X_{iT})'$, and $\hat{T}_2 = T - \hat{T}_1$. Define

$$\hat{\bar{\theta}}_{k,m}^{(\ell)} = \frac{1}{|\hat{G}_{k,m}^{(\ell)}|} \sum_{i\in\hat{G}_{k,m}^{(\ell)}} \hat{\theta}_{i,k,m}^{(\ell)}, \quad M_{\hat{F}_{k,m}^{(\ell)}} = I_{\hat{T}_\ell} - \frac{1}{\hat{T}_\ell}\hat{F}_{k,m}^{(\ell)}\hat{F}_{k,m}^{(\ell)\prime},$$

$$\hat{S}_{ii,k,m}^{(\ell)} = \frac{1}{\hat{T}_\ell}\hat{X}_i^{(\ell)\prime} M_{\hat{F}_{k,m}^{(\ell)}} \hat{X}_i^{(\ell)}, \quad \hat{a}_{ii,k}^{(\ell)} = \hat{\lambda}_{i,k,m}^{(\ell)\prime} \left(|\hat{G}_{k,m}^{(\ell)}|^{-1}\hat{\Lambda}_{k,m}^{(\ell)\prime}\hat{\Lambda}_{k,m}^{(\ell)}\right)^{-1} \hat{\lambda}_{i,k,m}^{(\ell)}.$$

Let $\hat{z}_{it}^{(\ell)\prime}$ be the t-th row of $M_{\hat{F}_{k,m}^{(\ell)}}\hat{X}_i^{(\ell)}$. For each subgroup $\hat{G}_{k,m}^{(\ell)}$ with $k \in [m]$, we follow the lead of Pesaran and Yamagata (2008) and Ando and Bai (2015) and define the following test statistic components

$$\hat{\Gamma}_{k,m}^{(\ell)} = \sqrt{|\hat{G}_{k,m}^{(\ell)}|} \cdot \frac{\frac{1}{|\hat{G}_{k,m}^{(\ell)}|}\sum_{i\in\hat{G}_{k,m}^{(\ell)}} \hat{\mathbb{S}}_{i,k,m}^{(\ell)} - p}{\sqrt{2p}}, \tag{3.7}$$

where

$$\hat{\mathbb{S}}_{i,k,m}^{(\ell)} = \hat{T}_\ell(\hat{\theta}_{i,k,m}^{(\ell)} - \hat{\bar{\theta}}_{k,m}^{(\ell)})'\hat{S}_{ii,k,m}^{(\ell)}(\hat{\Omega}_{i,k,m}^{(\ell)})^{-1}\hat{S}_{ii,k,m}^{(\ell)}(\hat{\theta}_{i,k}^{(\ell)} - \hat{\bar{\theta}}_{k}^{(\ell)})\left(1 - \frac{\hat{a}_{ii,k}^{(\ell)}}{|\hat{G}_{k,m}^{(\ell)}|}\right)^2,$$

$$\hat{\Omega}_{i,k,m}^{(\ell)} = \frac{1}{\hat{T}_\ell}\sum_{t\in\hat{\mathcal{T}}_\ell}\hat{z}_{it}^{(\ell)}\hat{z}_{it}^{(\ell)\prime}\hat{e}_{it}^2 + \frac{1}{\hat{T}_\ell}\sum_{j\in\hat{\mathcal{T}}_{\ell,-1}} k(j/S_T)\sum_{t\in\hat{\mathcal{T}}_{\ell,j}}[\hat{z}_{it}^{(\ell)}\hat{z}_{i,t+j}^{(\ell)\prime}\hat{e}_{it}\hat{e}_{i,t+j} + \hat{z}_{i,t-j}^{(\ell)}\hat{z}_{it}^{(\ell)\prime}\hat{e}_{i,t-j}\hat{e}_{i,t}],$$

$k(\cdot)$ is a kernel function, S_T is a bandwidth/truncation parameter, and $\hat{\Omega}_{i,k,m}^{(\ell)}$ is a traditional HAC estimator. Using the components (3.7) we now define the test statistic

$$\hat{\Gamma}_m^{(\ell)} = \max_{k\in[m]}(\hat{\Gamma}_{k,m}^{(\ell)})^2. \tag{3.8}$$

We will show that $\hat{\Gamma}_m^{(\ell)}$ is asymptotically distributed as the maximum of m independent $\chi^2(1)$ random variables under the null hypothesis that the slope coefficients in each of the m subsamples are homogeneous, whereas it diverges to infinity under the alternative. Let z_ς denote the critical value at significance level ς, which is calculated from the maximum of m independent $\chi^2(1)$ random variables. We reject the null of m subgroups in favor of more groups at level ς if $\hat{\Gamma}_m^{(\ell)} > z_\varsigma$.

3.3 Rank Estimation

To obtain the rank estimator, we use the low-rank estimators from (3.1) and estimate r_j by the singular value thresholding (SVT) criterion

$$\hat{r}_j = \sum_{i=1}^{N\wedge T}\mathbf{1}\left\{\sigma_i\left(\tilde{\Theta}_j\right) \geq 0.5\left(\nu_j\left\|\tilde{\Theta}_j\right\|_{op}\right)^{1/2}\right\} \quad \forall j \in \{0\}\cup[p],$$

where $\sigma_i(A)$ denotes the i-th largest singular value of A and $N \wedge T = \min(N, T)$. By arguments as used in the proof of Proposition D.1 in Chernozhukov et al. (2020) and that of Theorem 3.2 in Hong et al. (2023), we can show that $\mathbb{P}(\hat{r}_j = r_j) \to 1$ for each j as $(N, T) \to \infty$.

3.4 Parameter Estimation

Once we obtain the estimated break point, the number of groups and the group membership before and after the estimated break point, we can estimate the group-specific slope coefficients $\{\alpha_k^{(\ell)}\}_{k\in[\hat{K}^{(\ell)}]}$ along with the factors and factor loadings as follows

$$\left(\hat{\Lambda}^{(\ell)}, \hat{F}^{(\ell)}, \left\{\hat{\alpha}_k^{(\ell)}\right\}_{k\in[\hat{K}^{(\ell)}]}\right) = \arg\min \mathbb{L}\left(\Lambda, F, \left\{a_k^{(\ell)}\right\}_{k\in[\hat{K}^{(\ell)}]}\right), \tag{3.9}$$

where $\mathbb{L}\left(\Lambda, F, \left\{a_k^{(\ell)}\right\}_{k\in[\hat{K}^{(\ell)}]}\right) = \frac{1}{N\hat{T}_\ell}\sum_{k=1}^{\hat{K}^{(\ell)}}\sum_{i\in\hat{G}_k^{(\ell)}}\sum_{t\in\hat{\mathcal{T}}_\ell}\left(Y_{it}-\lambda_i' f_t - X_{it}' a_k^{(\ell)}\right)^2$. Here, we ignore the fact that the prior- and post-break regimes share the same set of factor loadings and estimate the group-specific parameters separately for the two regimes at the cost of sacrificing some efficiency for the factor loading estimates. Alternatively, we can pool the observations before and after the break to estimate the parameters as

$$\left(\hat{\Lambda}, \hat{F}, \left\{\hat{\alpha}_k^{(1)}\right\}_{k\in[\hat{K}^{(1)}]}, \left\{\hat{\alpha}_k^{(2)}\right\}_{k\in[\hat{K}^{(2)}]}\right) = \arg\min \mathbb{L}\left(\Lambda, F, \left\{a_k^{(1)}\right\}_{k\in[\hat{K}^{(1)}]}, \left\{a_k^{(2)}\right\}_{k\in[\hat{K}^{(2)}]}\right)$$

where

$$\mathbb{L}\left(\Lambda, F, \left\{a_k^{(1)}\right\}_{k\in[\hat{K}^{(1)}]}, \left\{a_k^{(2)}\right\}_{k\in[\hat{K}^{(2)}]}\right) = \mathbb{L}\left(\Lambda, F, \left\{a_k^{(1)}\right\}_{k\in[\hat{K}^{(1)}]}\right) + \mathbb{L}\left(\Lambda, F, \left\{a_k^{(2)}\right\}_{k\in[\hat{K}^{(2)}]}\right). \tag{3.10}$$

In either case, as should be clear because of the presence of the group structures, establishment of the asymptotic properties of the post-classification estimators of the group-specific slope coefficients becomes much more involved than in Bai (2009) and Moon and Weidner (2017). For this reason, we will focus on the estimates defined in (3.9).

4 Asymptotic Theory

This section develops the asymptotic properties of the estimators introduced above.

4.1 Basic Assumptions

Define $e_i = (e_{i1}, \cdots, e_{iT})'$ and $X_{j,i} = (X_{j,i1}, \cdots, X_{j,iT})'$. Let V_j^0 be a $T \times r_j$ matrix with its t-th row being $v_{t,j}^{0\prime}$, and U_j^0 be the $N \times r_j$ matrix with its i-th row being $u_{i,j}^{0\prime}$. Throughout the paper, we treat the factors $\{V_j^0\}_{j\in[p]\cup\{0\}}$ as random and their loadings $\{U_j^0\}_{j\in[p]\cup\{0\}}$ as deterministic. Let $\mathscr{D} := \sigma(\{V_j^0\}_{j\in[p]\cup\{0\}})$ denote the minimum σ-field generated by $\{V_j^0\}_{j\in[p]\cup\{0\}}$. Similarly, let $\mathscr{G}_t := \sigma(\mathscr{D}, \{X_{is}\}_{i\in[N],s\le t+1}, \{e_{is}\}_{i\in[N],s\le t})$. Let $\max_i = \max_{i\in[N]}$, $\max_t = \max_{t\in[T]}$ and $\max_{i,t} = \max_{i\in[N],t\in[T]}$. Let M and C be generic bounded positive constants which may vary across lines.

Assumption 1 *(i)* $\{e_{it}, X_{it}\}_{t\in[T]}$ *are conditionally independent across* i *given* $\mathscr{D}$.

(ii) $\mathbb{E}(e_{it}|X_{it}, \mathscr{D}) = 0$.

(iii) For each i, $\{(e_{it}, X_{it}), t \ge 1\}$ *is strong mixing conditional on* $\mathscr{D}$ *with the mixing coefficient* $\alpha_i(\cdot)$ *satisfying* $\max_i \alpha_i(z) \le M\vartheta^z$ *for some constant* $\vartheta \in (0,1)$.

(iv) There exists a constant $C > 0$ *such that* $\max_i \frac{1}{T}\sum_{t\in[T]}\|\xi_{it}\|^2 \le C$ *a.s. and* $\max_t \frac{1}{N}\sum_{i\in[N]}\|\xi_{it}\|^2 \le C$ *a.s. for* $\xi_{it} = e_{it}$, X_{it} *and* $X_{it}e_{it}$.

(v) $\max_{i,t}\mathbb{E}[\|\xi_{it}\|^q\,|\mathscr{D}] \le M$ *a.s. and* $\max_{i,i^*,t}\mathbb{E}[\|X_{it}e_{i^*t}\|^q\,|\mathscr{D}] \le M$ *a.s. for some* $q > 8$ *and* $\xi_{it} = e_{it}$, X_{it} *and* $X_{it}e_{it}$.

(vi) As $(N,T) \to \infty$, $\sqrt{N}(\log N)^2 T^{-1} \to 0$ and $T(\log N)^2 N^{-3/2} \to 0$.

Assumption 1* *(i), (iv) and (v) are same as Assumption 1(i), (iv) and (v). In addition:*

(ii) $\mathbb{E}(e_{it}|\mathscr{G}_{t-1}) = 0 \ \forall (i,t) \in [N] \times [T]$, and $\max_{i,t} \mathbb{E}(e_{it}^2 \big| \mathscr{G}_{t-1}) \leq M$ a.s..

(iii) $\{e_{it}\}_{i\in[N]}$ is conditionally independent across t given $\mathscr{D}$.

Assumption 1(i) imposes conditional independence on $\{e_{it}, X_{it}\}_{t\in[T]}$ across the cross sectional units. Assumption 1(ii) is the conditional moment condition. Assumption 1(iii) imposes conditional strong mixing conditions along the time dimension. See Prakasa Rao (2009) for the definition of conditional strong mixing and Su and Chen (2013) for an application in the panel setup. Assumptions 1(iv)-(v) impose conditions that restrict the tail behavior of ξ_{it}. Note that neither the regressors nor the errors are constrained to be bounded. Assumption 1(vi) imposes restrictions on N and T but does not require N and T to diverge at the the same rate. It is possible to allow N to diverge to infinity faster but not too much faster than T, and vice versa.

Assumption 1* is used for the study of dynamic panel data models. To be specific, Assumption 1* (ii) requires that the error sequence $\{e_{it}, t \geq 1\}$ be a martingale difference sequence (m.d.s.) with respect to the filtration $\mathscr{G}_t$, which allows for lagged dependent variables in X_{it}. Assumption 1* (iii) imposes conditional independence of the errors over t. The presence of serially correlated errors in dynamic panels typically induces endogeneity, which invalidates least-squares-based PCA estimation.

Assumption 2 *$rank(\Theta_j^0) = r_j \leq \bar{r}$ for $j \in [p] \cup \{0\}$ and some fixed $\bar{r}$, and $\max_{j\in[p]\cup\{0\}} ||\Theta_j^0||_{\max} \leq M$.*

Assumption 2 imposes low-rank conditions on the coefficient matrices, which facilitate the use of NNR in obtaining preliminary estimates in the first step. As discussed in the previous section, we see that the low-rank assumption for the slope matrices is satisfied for the model in Section 2. Moreover, we follow Ma et al. (2020) and assume the elements of the coefficient matrices are uniformly bounded to simplify the proofs. The boundedness of the slope coefficients is reasonable given that their cardinality does not grow with the sample size. The boundedness assumption for the intercept coefficient can be relaxed at the cost of more lengthy arguments.

Assumption 3 *Let $\sigma_{l,j}$ denote the l-th largest singular values of Θ_j^0 for $j \in [p] \cup \{0\}$. There exist some constants C_σ and c_σ such that*

$$\infty > C_\sigma \geq \lim \sup_{(N,T)\to\infty} \max_{j\in[p]} \sigma_{1,j} \geq \lim \inf_{(N,T)\to\infty} \min_{j\in[p]} \sigma_{r_j,j} \geq c_\sigma > 0.$$

Assumption 3 imposes some conditions on the singular values of the coefficient matrices. These ensure that only pervasive factors are allowed when the matrices are written as a factor structure. The assumption can be readily verified given the latent group structures of the slope coefficients.

Consider the SVD for Θ_j^0: $\Theta_j^0 = \mathcal{U}_j \Sigma_j \mathcal{V}_j' \ \forall j \in [p] \cup \{0\}$. Decompose $\mathcal{U}_j = (\mathcal{U}_{j,r}, \mathcal{U}_{j,0})$ and $\mathcal{V}_j = (\mathcal{V}_{j,r}, \mathcal{V}_{j,0})$ with $(\mathcal{U}_{j,r}, \mathcal{V}_{j,r})$ being the singular vectors corresponding to nonzero singular values and $(\mathcal{U}_{j,0}, \mathcal{V}_{j,0})$ being the singular vectors corresponding to zero singular values. Hence, for any matrix $W \in \mathbb{R}^{N\times T}$, we define

$$\mathcal{P}_j^{\perp}(W) = \mathcal{U}_{j,0}\mathcal{U}_{j,0}' W \mathcal{V}_{j,0}\mathcal{V}_{j,0}', \quad \mathcal{P}_j(W) = W - \mathcal{P}_j^{\perp}(W),$$

where $\mathcal{P}_j(W)$ can be seen as the linear projection of matrix W into the low-rank space with $\mathcal{P}_j^{\perp}(W)$ being its orthogonal space. Let $\Delta_{\Theta_j} = \Theta_j - \Theta_j^0$ for any Θ_j. Based on the spaces constructed above, with some positive constants C_1 and C_2, we define the restricted set for full-sample parameters as follows:

$$\mathcal{R}(C_1, C_2) := \Bigg\{ (\{\Delta_{\Theta_j}\}_{j\in[p]\cup\{0\}}) : \sum_{j\in[p]\cup\{0\}} \left\| \mathcal{P}_j^{\perp}(\Delta_{\Theta_j}) \right\|_* \leq C_1 \sum_{j\in[p]\cup\{0\}} \left\| \mathcal{P}_j(\Delta_{\Theta_j}) \right\|_*, \\ \sum_{j\in[p]\cup\{0\}} \|\Theta_j\|^2 \geq C_2\sqrt{NT} \Bigg\}. \tag{4.1}$$

Lemma B.4 in the online appendix shows that our nuclear norm estimators are in a restricted set larger than (4.1), which derives from the restriction on the Frobenius norm in the definition of $\mathcal{R}(C_1, C_2)$. Intuitively, the first restriction in (4.1) means the projection onto the orthogonal low-rank space of the estimator error can be controlled by its projection onto the low-rank space. Theorem 4.1 below largely hinges on this property.

Assumption 4 *For any $C_2 > 0$, there are constants C_3 and C_4 such that for any $(\{\Delta_{\Theta_j}\}_{j\in[p]\cup\{0\}}) \in \mathcal{R}(3, C_2)$, we have*

$$\left\| \Delta_{\Theta_0} + \sum_{j=1}^{p} \Delta_{\Theta_j} \odot X_j \right\|^2 \geq C_3 \sum_{j\in[p]\cup\{0\}} \left\|\Delta_{\Theta_j}\right\|^2 - C_4(N+T) \quad \textit{w.p.a.1.}$$

Assumption 4 imposes the restricted strong convexity (RSC) condition, which is similar to Assumption 3.1 in Chernozhukov et al. (2020). The latter authors also provide some sufficient conditions to verify such an assumption.

Let $r = \sum_{j\in[p]\cup\{0\}} r_j$. Define the following $r \times r$ matrices:

$$\Phi_i = \frac{1}{T}\sum_{t=1}^{T} \phi_{it}^0 \phi_{it}^{0\prime} \ \ \forall i \in [N] \text{ and } \Psi_t = \frac{1}{N}\sum_{i\in[N]} \psi_{it}^0 \psi_{it}^{0\prime} \ \ \forall t \in [T],$$

where $\phi_{it}^0 = (v_{t,0}^{0\prime}, v_{t,1}^{0\prime} X_{1,it}, \cdots, v_{t,p}^{0\prime} X_{p,it})'$, and $\psi_{it}^0 = (u_{i,0}^{0\prime}, u_{i,1}^{0\prime} X_{1,it}, \cdots, u_{i,p}^{0\prime} X_{p,it})'$.

Assumption 5 *There exist constants C_ϕ and c_ϕ such that*

$$\infty > C_\phi \geq \limsup_T \max_{t\in[T]} \lambda_{\max}(\Psi_t) \geq \liminf_T \min_{t\in[T]} \lambda_{\min}(\Psi_t) \geq c_\phi > 0,$$

$$\infty > C_\phi \geq \limsup_N \max_{i\in[N]} \lambda_{\max}(\Phi_i) \geq \liminf_N \min_{i\in[N]} \lambda_{\min}(\Phi_i) \geq c_\phi > 0.$$

Assumption 5 is similar to Assumption 8 in Ma et al. (2020) and it imposes some rank conditions.

4.2 Asymptotics of NNR Estimators and Singular Vector Estimators

Let $\eta_{N,1} = \frac{\sqrt{\log T}}{\sqrt{N \wedge T}}$ and $\eta_{N,2} = \frac{\sqrt{\log(N \vee T)}}{\sqrt{N \wedge T}}(NT)^{1/q}$. Let $\tilde{\sigma}_{k,j}$ denote the k-th largest singular value of $\tilde{\Theta}_j$ for $j \in [p] \cup \{0\}$. Our first main result is about the consistency of the first-stage NNR estimators and the second-stage singular vector estimators.

Theorem 4.1 *Suppose that Assumptions 1–4 hold. Then $\forall j \in [p] \cup \{0\}$, we have*

(i) $\frac{1}{\sqrt{NT}}||\tilde{\Theta}_j - \Theta_j^0|| = O_p(\eta_{N,1})$, $\max_{k \in [r_j]} |\tilde{\sigma}_{k,j} - \sigma_{k,j}| = O_p(\eta_{N,1})$, *and* $||V_j^0 - \tilde{V}_j O_j|| = O_p(\sqrt{T}\eta_{N,1})$ *where O_j is an orthogonal matrix defined in the proof.*

If in addition Assumption 5 is also satisfied, then we have

(ii) $\max_{i \in [N]} ||\dot{u}_{i,j} - O_j u_{i,j}^0|| = O_p(\eta_{N,2})$, $\max_{t \in [T]} \left\| \dot{v}_{t,j} - O_j v_{t,j}^0 \right\|_2 = O_p(\eta_{N,2})$,

(iii) $\max_{i \in [N], t \in [T]} |\dot{\Theta}_{j,it} - \Theta_{j,it}^0| = O_p(\eta_{N,2})$.

Theorem 4.1(i) reports the error bounds for $\tilde{\Theta}_j$, $\tilde{\sigma}_{k,j}$, and $\tilde{V}_j$. The $\log T$ term in the numerator of $\eta_{N,1}$ is due to the use of some exponential inequality for (conditional) strong mixing processes. Theorem 4.1(ii)–(iii) report the uniform convergence rate of the factor and factor loading estimators. The extra $(NT)^{1/q}$ term in the definition of $\eta_{N,2}$ is due to the nonboundedness of $X_{j,it}$ in Assumption 1(v), and it disappears when $X_{j,it}$ is assumed to be uniformly bounded.

4.3 Consistency of the Break Point Estimate

Recall that $g_i^{(1)}$ and $g_i^{(2)}$ denote the true group individual i belongs to before and after the break, respectively. To estimate the break point consistently, we add the following condition.

Assumption 6 *(i)* $\sqrt{\frac{1}{N}\sum_{i \in [N]} ||\alpha_{g_i^{(1)}} - \alpha_{g_i^{(2)}}||^2} = C_5 \zeta_{NT}$, *where C_5 is a positive constant and* $\zeta_{NT} \gg \eta_{N,2}$.

(ii) $\tau_T := \frac{T_1}{T} \to \tau \in (0, 1)$ *as* $T \to \infty$.

Assumption 6(i) imposes conditions on the break size in order to identify the break point. Note that we allow the average break size to shrink to zero at a rate slower than $\sqrt{\frac{\log(N \vee T)}{N \wedge T}}(NT)^{1/q}$. This rate is of much bigger magnitude than the optimal $(NT)^{-1/2}$-rate that can be detected in the panel threshold regressions (PTRs) for several reasons. First, in PTRs, the slope coefficients are usually assumed to be homogeneous so that each individual is subject to the same change in the slope coefficients and one can use the cross-sectional information effectively. In contrast, we allow for heterogeneous slope coefficients here and the change can occur only for a subset of cross section units but not all. In addition, in the presence of latent group structures, we not only allow the slope coefficients of some specific groups to change with group membership fixed, but also allow the slope coefficient to remain the same for some groups while the group memberships change after the break.

Second, our break point estimation relies on the binary segmentation idea borrowed from the time series literature where one can allow break sizes of bigger magnitude than $T^{-1/2}$ in order to identify the break ratio consistently but not the break point consistently. As is apparent, even though we require bigger break sizes, we can estimate the break date consistently by using information from both the cross section and time dimensions. Third, as mentioned above, the additional term $\log(N \vee T)$ in the above rate is mainly due to the use of an exponential inequality and the factor $(NT)^{1/q}$ is due to the fact that we only assume the existence of q-th order moments for some random variables.

The following theorem indicates that we can estimate the break date T_1 consistently.

Theorem 4.2 *Suppose Assumptions 1–6 hold, with the true break point being T_1 and the estimator defined in (3.4). Then $\mathbb{P}(\hat{T}_1 = T_1) \to 1$ as $(N, T) \to \infty$.*

Theorem 4.2 shows that we can estimate the true break date consistently w.p.a.1 despite the fact that we allow the break size to shrink to zero at a certain rate.

4.4 Consistency of the Estimates of the Number of Groups and the Latent Group Structures

To study the asymptotic properties of the estimates of the number of groups and the recovery of the latent group structures, we first add the following definition.

Definition 4.1 *Fix $K^{(\ell)} > 1$ and $m \leq K^{(\ell)}$. The estimated group structure $\hat{\mathcal{G}}_m^{(\ell)}$ satisfies the non-splitting property (NSP) if for any pair of individuals in the same true group, the estimated group labels are the same.*

Definition 4.1 describes the non-splitting property introduced by Jin et al. (2022). The latter authors show that the STK algorithm yields the estimated group structures enjoying the NSP.

To proceed, we add the following assumptions.

Assumption 7 *(i) Let k_s and k_{s^*} be different group indices. Assume that $\min_{1 \leq k_s < k_{s^*} \leq K^{(\ell)}} \left\| \alpha_{k_s}^{(\ell)} - \alpha_{k_{s^*}}^{(\ell)} \right\|_2 \geq C_5$ for $\ell \in \{1, 2\}$.*

(ii) Let $N_k^{(\ell)}$ be the number of individuals in group k for $k \in [K^{(\ell)}]$. Define $\pi_k^{(\ell)} = \frac{N_k^{(\ell)}}{N}$ for $\ell = 1, 2$. Assume $K^{(\ell)}$ is fixed and

$$\infty > \bar{C} \geq \limsup_N \sup_{k \in [K^{(\ell)}]} \pi_k^{(\ell)} \geq \liminf_N \inf_{k \in [K^{(\ell)}]} \pi_k^{(\ell)} \geq \underline{c} > 0, \ \ \ell = 1, 2.$$

(iii) For any combination of the collection of n true groups with $n \geq 2$, we have

$$\frac{T_\ell}{\sqrt{N}} \sum_{s=1}^{n} N_{k_s}^{(\ell)} \left\| \sum_{s^* \in [n], s^* \neq s} (\alpha_{k_{s^*}}^{(\ell)} - \alpha_{k_s}^{(\ell)}) \right\|^2 / (\log N)^{1/2} \to \infty, \ \ \ell = 1, 2.$$

Remark 3. Assumption 7(i)–(ii) is the standard assumption for K-means algorithm, which is comparable to Assumption 4 in Su et al. (2020) and greatly facilitates the subsequent analyses. Assumption 7(i) assumes that the minimum distance of two distinct groups is bounded away from 0, and Assumption 7(ii) imposes that each group has asymptotically non-negligible number of units. For Assumption 7(iii), it can be shown to hold under mild conditions. Below we explain this assumption in detail. When $n = 2$, it is clear that

$$\frac{T_\ell}{\sqrt{N}} \sum_{s=1}^{n} N_{k_s}^{(\ell)} \left\| \sum_{s^* \in [n], s^* \neq s} (\alpha_{k_{s^*}}^{(\ell)} - \alpha_{k_s}^{(\ell)}) \right\|^2 = \frac{T_\ell}{\sqrt{N}} \left(N_{k_1}^{(\ell)} \left\| \alpha_{k_2}^{(\ell)} - \alpha_{k_1}^{(\ell)} \right\|^2 + N_{k_2}^{(\ell)} \left\| \alpha_{k_1}^{(\ell)} - \alpha_{k_2}^{(\ell)} \right\|^2 \right)$$

$$\geq \frac{C_5^2 T_\ell (N_{k_1}^{(\ell)} + N_{k_2}^{(\ell)})}{\sqrt{N}} = O(T\sqrt{N})$$

by Assumptions 6(ii) and 7(i)–(ii). When $n > 2$, we consider a special case such that $S_s =: || \sum_{s^* \in [n], s^* \neq s} (\alpha_{k_{s^*}}^{(\ell)} - \alpha_{k_s}^{(\ell)}) || = 0$ for some specific $s = s_0 \in [n]$. Then it is easy to see S_s is non-zero for all $s \in [n] \backslash \{s_0\}$ under Assumption 7(i). Hence, if we assume S_s is lower bounded by a constant c for any $s \in [n] \backslash \{s_0\}$, Assumption 7(iii) will holds naturally. Similar arguments hold for the other general cases.

Assumption 8 *Let $\mathcal{T}_1 = [T_1]$ and $\mathcal{T}_2 = [T] \backslash [T_1]$.*

(i) $\frac{1}{T_\ell} \sum_{t \in \mathcal{T}_\ell} f_t^0 f_t^{0\prime} \xrightarrow{p} \Sigma_F^{(\ell)} > 0$ *as* $T \to \infty$. $\frac{1}{N_k^{(\ell)}} \Lambda_k^{0,(\ell)\prime} \Lambda_k^{0,(\ell)} \xrightarrow{p} \Sigma_{\Lambda,k}^{(\ell)} > 0$ *as* $N \to \infty$, *where* $\Lambda_k^{0,(\ell)}$ *is a stack of* λ_i^0 *for all individuals in group* k *and* $k \in [K^{(\ell)}]$.

(ii) There exists a constant $C > 0$ *such that* $\max_i \frac{1}{T_\ell} \sum_{t \in \mathcal{T}_\ell} \|\xi_{it}\|^2 \leq C$ *a.s. for* $\xi_{it} = e_{it}$, X_{it} *and* $X_{it} e_{it}$.

Assumption 8(i) imposes some standard assumptions on the factors and factor loadings. Assumption 8(ii) is similar as Assumption 1(iv), which strengthens Assumption 1(iv) to hold for two time regimes.

The next theorem reports the asymptotic properties of the STK estimators.

Theorem 4.3 *Let* $\varsigma = \varsigma_N \to 0$ *at rate* N^{-c} *for some* $c > 0$ *as* $N \to \infty$. *Suppose that Assumption* 1* *and Assumptions* 2–8 *hold. Then for* $\ell \in \{1, 2\}$, *we have*

(i) if $m = K^{(\ell)}$,

  (a) $\max_{i \in [N]} \mathbf{1}\{\hat{g}_{i,K^{(\ell)}}^{(\ell)} \neq g_i^{(\ell)}\} = 0$ *w.p.a.1,*

  (b) $\hat{\Gamma}_{K^{(\ell)}}^{(\ell)}$ *is asymptotically distributed as the maximum of* $K^{(\ell)}$ *independent* $\chi^2(1)$ *random variables,*

  (c) $\mathbb{P}(\hat{K}^{(\ell)} \leq K^{(\ell)}) \geq 1 - \alpha + o(1)$,

(ii) if $m < K^{(\ell)}$, $\hat{\Gamma}_m^{(\ell)} / \log N \to \infty$ *w.p.a.1. Thus* $\mathbb{P}(\hat{K}^{(\ell)} \neq K^{(\ell)}) \leq \varsigma + o(1)$.

Theorem 4.3 studies the asymptotic properties of the STK algorithm. Since we allow $\varsigma = \varsigma_N$ to shrink to zero at rate N^{-c}, the critical value z_ς diverges to infinity at rate $\log N$ as $N \to \infty$ by virtue of the tail properties of $\chi^2(1)$ random variables. At iteration m such that $m < K^{(\ell)}$, w.p.a.1, the test statistics $\hat{\Gamma}_m^{(\ell)}$ diverges to infinity at a rate faster than $\log N$, which means the iteration will continue at the $(m+1)$-th iteration. At iteration m with $m = K^{(\ell)}$, however, we can easily find that $z_\varsigma \to \infty$ while the test statistic $\hat{\Gamma}_m^{(\ell)}$ is stochastically bounded. As a result, the iteration stops w.p.a.1 and we have $\mathbb{P}(\hat{K}^{(\ell)} = K^{(\ell)}) \to 1$. As aforementioned, Theorem 4.3 ensures the application of K-means algorithm only for the under-fitting and just-fitting cases and it avoids the theoretical challenge of handling the over-fitting case in the classification.

For dynamic panels, we can focus on Assumption 1* , where the error term is an m.d.s. Under this assumption, the HAC estimator $\hat{\Omega}_{i,k,m}^{(\ell)}$ degenerates to $\frac{1}{\hat{T}_\ell}\sum_{t\in\hat{\mathcal{T}}_\ell} \hat{z}_{it}^{(\ell)} \hat{z}_{it}^{(\ell)\prime} \hat{e}_{it}^2$. For static panels, we typically allow for serially correlated errors and employ the HAC estimator, and the results in Theorem 4.3 continue to hold with Assumption 1* replaced by Assumption 1. For the kernel function and bandwidth, we can follow Andrews (1991) and let $k(\cdot)$ belong to the following class of kernels

$$\mathcal{K} = \Big\{ k(\cdot) : \mathbb{R} \mapsto [-1,1] \mid k(0) = 1,\ k(u) = k(-u),\ \int |k(u)|\, du < \infty,$$
$$k(\cdot) \text{ is continuous at 0 and at all but a finite number of other points} \Big\}.$$

See, e.g., Andrews (1991) and White (2014) for details.

4.5 Distribution Theory for the Group-specific Slope Estimators

For $\ell \in \{1,2\}$, let $\{\hat{\alpha}_k^{*(\ell)}\}_{k\in K^{(\ell)}}$ be the oracle estimators of the group-specific slope coefficients before and after the break point by using the true break and membership information for all individuals. To study the asymptotic distribution theory for $\{\hat{\alpha}_k^{(\ell)}\}_{k\in K^{(\ell)}}$, $\ell \in \{1,2\}$, we only need to show that for the oracle estimators $\{\hat{\alpha}_k^{*(\ell)}\}_{k\in K^{(\ell)}}$ based on Theorems 4.2 and 4.3 by extending the result of Bai (2009) and Moon and Weidner (2017).

To proceed, we add some notation. For $\ell \in \{1,2\}$, we first define the matrix notation for each subgroup. For $j \in [p]$, let $X_{j,i}^{(1)} = (X_{j,i1}, \cdots, X_{j,iT_1})'$, $X_{j,i}^{(2)} = \left(X_{j,i(T_1+1)}, \cdots, X_{j,iT}\right)'$, $e_i^{(1)} = (e_{i1}, \cdots, e_{iT_1})'$ and $e_i^{(2)} = \left(e_{i(T_1+1)}, \cdots, e_{iT}\right)'$. Then we use $\mathbb{X}_{j,k}^{(\ell)} \in \mathbb{R}^{N_k^{(\ell)} \times T_\ell}$ and $E_k^{(\ell)} \in \mathbb{R}^{N_k^{(\ell)} \times T_\ell}$ to denote the regressor and error matrix for subgroup $k \in [K^{(\ell)}]$ with each row being $X_{j,i}^{(\ell)}$ and $e_i^{(\ell)}$, respectively. Let $\mathcal{X}_{j,k}^{(\ell)} = M_{\Lambda_k^{0,(\ell)}} \mathbb{X}_{j,k}^{(\ell)} M_{F^{0,(\ell)}} \in \mathbb{R}^{N_k^{(\ell)} \times T_\ell}$ with the (i,t)-th entry given by $\mathcal{X}_{j,k,it}^{(\ell)}$. Let $\mathcal{X}_{k,it}^{(\ell)} = (\mathcal{X}_{1,k,it}^{(\ell)}, \cdots, \mathcal{X}_{p,k,it}^{(\ell)})'$. Further define

$$\mathbb{B}_{NT,1,j,k}^{(\ell)} = \frac{1}{N_k^{(\ell)}} tr\left[P_{F^{0,(\ell)}} \mathbb{E}\left(E_k^{(\ell)\prime} \mathbb{X}_{j,k}^{(\ell)} | \mathscr{D}\right)\right],$$

$$\mathbb{B}_{NT,2,j,k}^{(\ell)} = \frac{1}{T_\ell} tr\left[\mathbb{E}\left(E_k^{(\ell)} E_k^{(\ell)\prime} | \mathscr{D}\right) M_{\Lambda_k^{0,(\ell)}} \mathbb{X}_{j,k}^{(\ell)} F^{0,(\ell)} \left(F^{0,(\ell)\prime} F^{0,(\ell)}\right)^{-1} \left(\Lambda_k^{0,(\ell)\prime} \Lambda_k^{0,(\ell)}\right)^{-1} \Lambda_k^{0,(\ell)\prime}\right],$$

$$\mathbb{B}_{NT,3,j,k}^{(\ell)} = \frac{1}{N_k^{(\ell)}} tr\left[\mathbb{E}\left(E_k^{(\ell)} E_k^{(\ell)\prime} | \mathscr{D}\right) M_{F^{0,(\ell)}} \mathbb{X}_{j,k}^{(\ell)} \Lambda_k^{0,(\ell)} \left(\Lambda_k^{0,(\ell)\prime} \Lambda_k^{0,(\ell)}\right)^{-1} \left(F^{0,(\ell)\prime} F^{0,(\ell)}\right)^{-1} F^{0,(\ell)\prime}\right],$$

$$\mathbb{B}^{(\ell)}_{NT,m,k} = \left(\mathbb{B}^{(\ell)}_{NT,m,1,k}, \cdots, \mathbb{B}^{(\ell)}_{NT,m,p,k}\right)', \quad \forall m \in \{1,2,3\},$$
$$\Omega^{(\ell)}_k = \frac{1}{N^{(\ell)}_k T_\ell} \sum_{i \in G^{(\ell)}_k} \sum_{t \in \mathcal{T}_\ell} e^2_{it} \mathcal{X}^{(\ell)}_{k,it} \mathcal{X}^{(\ell)\prime}_{k,it}.$$

Let $\mathbb{W}^{(\ell)}_{NT,k}$ be a $p \times p$ matrix with (j_1, j_2)-th entry $\frac{1}{N^{(\ell)}_k T_\ell} tr\left(M_{F^{0,(\ell)}} \mathbb{X}^{(\ell)\prime}_{j_1,k} M_{\Lambda^{0,(\ell)}_k} \mathbb{X}^{(\ell)}_{j_2,k}\right)$. Then we define the overall bias term for each subgroup as

$$\mathbb{B}^{(\ell)}_{NT,k} = -\rho^{(\ell)}_k \mathbb{B}^{(\ell)}_{NT,1,k} - (\rho^{(\ell)}_k)^{-1} \mathbb{B}^{(\ell)}_{NT,2,k} - \rho^{(\ell)}_k \mathbb{B}^{(\ell)}_{NT,3,k},$$

where $\rho^{(\ell)}_k = \sqrt{\frac{N^{(\ell)}_k}{T_\ell}}$. To state the main result in this subsection, we add the following assumption.

Assumption 9 *(i) As $(N,T) \to \infty$, $T(\log T)N^{-4/3} \to 0$.*

(ii) $\text{plim}_{(N,T)\to\infty} \frac{1}{N^{(\ell)}_k T_\ell} \sum_{i \in G^{(\ell)}_k} \sum_{t \in \mathcal{T}_\ell} X_{it} X'_{it} > 0$ *for $\ell \in \{1,2\}$ and $k \in [K^{(\ell)}]$.*

(iii) For $\ell \in \{1,2\}$ and $k \in [K^{(\ell)}]$, separate the p regressors of each subgroups into p_1 "low-rank regressors" $\mathbb{X}^{(\ell)}_{j,k}$ such that $rank(\mathbb{X}^{(\ell)}_{j,k}) = 1$, $\forall j \in \{1, \cdots, p_1\}$, and "high-rank regressors" $\mathbb{X}^{(\ell)}_{j,k}$ such that $rank(\mathbb{X}^{(\ell)}_{j,k}) > 1$, $\forall j \in \{p_1+1, \cdots, p\}$. Let $p_2 := p - p_1$. These two types of regressors satisfy:

(iii.a) Consider the linear combinations $b \cdot \mathbb{X}^{(\ell)}_{high,k} := \sum_{j=p_1+1}^{p} b_j \mathbb{X}^{(\ell)}_{j,k}$ for high-rank regressors with p_2-vectors b such that $\|b\|_2 = 1$ and $b = (b_{p_1+1}, \cdots, b_p)'$. There exists a positive constant C_b such that

$$\min_{\{\|b\|_2=1\}} \sum_{n=2r_0+p_1+1}^{N} \lambda_n \left[\frac{1}{NT_\ell}\left(b \cdot \mathbb{X}^{(\ell)}_{high,k}\right)\left(b \cdot \mathbb{X}^{(\ell)}_{high,k}\right)'\right] \geq C_b \quad w.p.a.1.$$

(iii.b) For $j \in [p_1]$, write $\mathbb{X}^{(\ell)}_{j,k} = w^{(\ell)}_{j,k} v^{(\ell)\prime}_{j,k}$ with $N^{(\ell)}_k$-vectors $w^{(\ell)}_j$ and T_ℓ-vectors $v^{(\ell)}_j$. Let $w^{(\ell)}_k = (w^{(\ell)}_{1,k}, \cdots, w^{(\ell)}_{p_1,k}) \in \mathbb{R}^{N \times p_1}$, $v^{(\ell)} = (v^{(\ell)}_1, \cdots, v^{(\ell)}_{p_1}) \in \mathbb{R}^{T_\ell \times p_1}$, $M_{w^{(\ell)}_k} = I_{N^{(\ell)}_k} - w^{(\ell)}_k (w^{(\ell)\prime}_k w^{(\ell)}_k)^{-1} w^{(\ell)\prime}_k$ and $M_{v^{(\ell)}} = I_{T_\ell} - v^{(\ell)}\left(v^{(\ell)\prime} v^{(\ell)}\right)^{-1} v^{(\ell)\prime}$. There exists a positive constant C_B such that $(N^{(\ell)}_k)^{-1} \Lambda^{0,(\ell)\prime}_k M_{w^{(\ell)}_k} \Lambda^{0,(\ell)}_k > C_B I_{r_0}$ and $T^{-1}_\ell F^{0,(\ell)\prime} M_{w^{(\ell)}} F^{0,(\ell)} > C_B I_{r_0}$ w.p.a.1.

(iv) For $\forall j \in [p]$, $\ell \in \{1,2\}$, and $k \in K^{(\ell)}$,

$$\frac{1}{N^{(\ell)}_k (T_\ell)^2} \sum_{i \in G^{(\ell)}_k} \sum_{t_1 \in \mathcal{T}_\ell} \sum_{t_2 \in \mathcal{T}_\ell} \sum_{s_1 \in \mathcal{T}_\ell} \sum_{s_2 \in \mathcal{T}_\ell} \left| Cov\left(e_{it_1} \tilde{X}_{j,it_2}, e_{is_1} \tilde{X}_{j,is_2}\right)\right| = O_p(1),$$

where $\tilde{X}_{j,it} = X_{j,it} - \mathbb{E}(X_{j,it}|\mathscr{D})$.

Assumption 9 imposes some conditions to help derive the asymptotic distribution theory for the panel model with IFEs which allows for dynamics. Assumption 9(i) slightly strengthens Assumption

1(vi). Assumption 9(ii) is the standard non-collinearity condition for regressors, which is analogous to Assumption 4(i) in Moon and Weidner (2017). Assumption 9(iii) is the identification assumption which is comparable to Assumption 4 in Moon and Weidner (2017). With the conditional strong mixing condition in Assumption 1(iii), we can verify Assumption 9(iv).

The following theorem establishes the asymptotic distribution of $\{\hat{\alpha}_k^{(\ell)}\}_{k\in K^{(\ell)}}$.

Theorem 4.4 *Suppose that Assumption 1 or 1* and Assumptions 2–9 hold. For $\ell \in \{1,2\}$, the estimators $\{\hat{\alpha}_k^{(\ell)}\}_{k\in K^{(\ell)}}$ are asymptotically equivalent to the oracle estimators $\{\hat{\alpha}_k^{*(\ell)}\}_{k\in K^{(\ell)}}$, and we have*

$$\mathbb{W}_{NT}^{(\ell)}\mathbb{D}_{NT}^{(\ell)}\begin{pmatrix}\hat{\alpha}_1^{(\ell)}-\alpha_1^{(\ell)}\\ \vdots \\ \hat{\alpha}_{K^{(\ell)}}^{(\ell)}-\alpha_{K^{(\ell)}}^{(\ell)}\end{pmatrix}-\mathbb{B}_{NT}^{(\ell)} \rightsquigarrow \mathbb{N}\left(0,\Omega^{(\ell)}\right),$$

such that $\mathbb{D}_{NT}^{(\ell)} = diag\left(\sqrt{N_1^{(\ell)}T_\ell},\cdots,\sqrt{N_{K^{(\ell)}}^{(\ell)}T_\ell}\right)$, $\mathbb{W}_{NT}^{(\ell)} = diag\left(\mathbb{W}_{NT,1}^{(\ell)},\cdots,\mathbb{W}_{NT,K^{(\ell)}}^{(\ell)}\right)$, $\mathbb{B}_{NT}^{(\ell)} = diag\left(\mathbb{B}_{NT,1}^{(\ell)},\cdots,\mathbb{B}_{NT,K^{(\ell)}}^{(\ell)}\right)$ *and* $\Omega^{(\ell)} = diag\left(\Omega_1^{(\ell)},\cdots,\Omega_{K^{(\ell)}}^{(\ell)}\right)$.

Theorem 4.4 establishes the asymptotic distribution for the estimators of the group-specific slope coefficients before and after the break. It shows that the parameter estimators from our algorithm enjoy the oracle property given the results in Theorems 4.2 and 4.3. The proof of the above theorem can be done by following Moon and Weidner (2017) and Lu and Su (2016).

5 Alternatives and Extensions

This section first considers an alternative method to estimate the break point and then discusses several possible extensions.

5.1 Alternative for Break Point Detection

The algorithm proposed in Section 3 uses low-rank estimates of Θ_j^0 to find the break point estimates. However, by Lemma 2.1(ii), we observe that the right singular vector matrix of Θ_j^0, i.e., V_j^0, contains the structural break information when $r_j = 2$. For this reason, we can propose an alternative way to estimate the break point under the case where the maximum rank of the slope matrix in the model is 2. Let $\dot{v}_{t,j}^* := \frac{\dot{v}_{t,j}}{\|\dot{v}_{t,j}\|}$ and $\dot{v}_t^* := \left(\dot{v}_{t,1}^{*\prime},\cdots,\dot{v}_{t,p}^{*\prime}\right)'$, with the true values being $v_{t,j}^* := \frac{O_j v_{t,j}^0}{\|O_j v_{t,j}^0\|}$ and $v_t^* := \left(v_{t,1}^{*\prime},\cdots,v_{t,p}^{*\prime}\right)'$, respectively. Then Step 3 can be replaced by Step 3* below:

Step 3*: Break Point Estimation by Singular Vectors. We estimate the break point as follows:

$$\tilde{T}_1 = \underset{s\in\{2,\cdots,T-1\}}{\arg\min}\frac{1}{T}\left\{\sum_{t=1}^{s}\left\|\dot{v}_t^*-\bar{v}^{*(1)s}\right\|^2+\sum_{t=s+1}^{T}\left\|\dot{v}_t^*-\bar{v}^{*(2)s}\right\|^2\right\}, \tag{5.1}$$

where $\bar{v}^{*(1)s} = \frac{1}{s}\sum_{t=1}^{s}\dot{v}_t^*$ and $\bar{v}^{*(2)s} = \frac{1}{T-s}\sum_{t=s+1}^{T}\dot{v}_t^*$.

The following two theorems establish the consistency of $\dot{v}_t^*$ and $\tilde{T}_1$, respectively.

Theorem 5.1 *Suppose that Assumptions 1–5 hold. Then* $\max_t \|\dot{v}_t^* - v_t^*\| = O_p(\eta_{N,2})$.

Theorem 5.2 *Suppose that Assumptions 1–6 hold. Then* $\mathbb{P}(\tilde{T}_1 = T_1) \to 1$ *as* $(N,T) \to \infty$.

Since the singular vectors of the slope matrices contain the structural change information, Theorem 5.1 indicates that we can consistently estimate the break point by using the factor estimates instead of the slope matrix estimates in (3.4). Given Theorem 5.1 and Lemma 2.1(iii), we can prove Theorem 5.2 with arguments analogous to those used in the proof of Theorem 4.2.

5.2 Test for the Presence of a Structural Break

Section 2 considers time-varying latent group structures with one break point. In this subsection, we propose a test for the null that the slope coefficients are time-invariant against the alternative that there's one structural break as assumed in Section 2.

Since various scenarios can occur once we allow for the presence of a structural break in the latent group structures, the number of groups may or may not change under the alternative and so may some of the group-specific coefficients. First, information on the latent group structures may be ignored and the slope coefficients can be tested for possible time-variation. In this case, we can rewrite Θ_{it}^0 as

$$\Theta_{it}^0 = \Theta_i^0 + c_{it},$$

where $\Theta_i^0 := \frac{1}{T}\sum_{t\in[T]} \Theta_{it}^0$. The null and alternative hypotheses can then be specified as

$$\begin{aligned} H_0 &: \quad c_{it} = 0 \quad \text{for all } i \in [N], \quad \text{and} \\ H_1 &: \quad c_{it} \neq 0 \quad \text{for some } i \in [N]. \end{aligned} \tag{5.2}$$

To construct the test statistics we follow Bai and Perron (1998) and consider a sup-F test. Let $\mathcal{T}_\epsilon := \{T_1 : \epsilon T \le T_1 \le (1-\epsilon)T\}$, where $\epsilon > 0$ is a tuning parameter that avoids breaks at the end of the sample. Define

$$F_{NT}(1|0) := \max_{i\in[N]} \sup_{T_1 \in \mathcal{T}_\epsilon} F_i(T_1),$$

where

$$F_i(T_1) = \frac{T-2p}{p}\left[\tilde{\beta}_i^{(1)}(T_1) - \tilde{\beta}_i^{(2)}(T_1)\right]' \left[\hat{\Sigma}_i(T_1)\right]^{-1} \left[\tilde{\beta}_i^{(1)}(T_1) - \tilde{\beta}_i^{(2)}(T_1)\right],$$

$\tilde{\beta}_i^{(1)}(T_1)$ and $\tilde{\beta}_i^{(2)}(T_1)$ are the PCA slope estimators of Θ_i^0 in the linear panels with IFEs for each individual i with the prior-break observations $\{(i,t) : i \in [N], t \in [T_+]\}$ and post-break observations $\{(i,t) : i \in [N], t \in [T]\backslash[T_+]\}$, respectively,[1] and $\hat{\Sigma}_i(T_+)$ is the consistent estimator for the asymptotic variance of $\tilde{\beta}_i^{(1)}(T_1) - \tilde{\beta}_i^{(2)}(T_1)$. Following Bai and Perron (1998), we conjecture that the asymptotic

[1]See Section C in the appendix for the detail of the PCA estimation in linear panels with IFEs.

distribution of $\sup_{T_1 \in \mathcal{T}_\epsilon} F_i(T_1)$ depends on a p-vector of Wiener processes on $[0,1]$, based on which the corresponding distribution of $F_{NT}(1|0)$ can be obtained.

Alternatively, we can estimate the model with latent group structures by assuming the presence of a break point at T_1. Then we obtain the estimates of the group-specific parameters $\{\alpha_j^{(1)}(T_1)\}_{j \in K^{(1)}}$ prior to the potential break point T_1 and those of the group-specific parameters $\{\alpha_j^{(2)}(T_1)\}_{j \in K^{(2)}}$ after the potential break point T_1. It is possible to construct a test statistic based on the contrast of these two sets of estimates or the corresponding residual sum of squares (RSS) and then take the supremum over $T_1 \in \mathcal{T}_\epsilon$. Evidently, this approach is also quite involved as it is necessary to determine the number of groups before and after the break, $K^{(1)}$ and $K^{(1)}$, at each T_1. It is not clear how the estimation errors from these estimates and those of the factors and factor loadings with slow convergence rates affect the asymptotic properties of the estimators of the group-specific parameters.

Last, it is also possible to estimate the model with latent group structures under the case of no structural changes to obtain the restricted residuals. If there exists a structural change in the latent group structure, it should be reflected in properties of the restricted residuals obtained under the null. We can then consider the regression of the restricted residuals on the regressors and construct an LM-type test statistic to check the goodness of fit of such an auxiliary regression model as in Su and Chen (2013) and Su and Wang (2020). This analysis is left for future research.

5.3 The Case of Multiple Breaks

Section 2 considers only a one-time structural break in the latent group structures. In practice it is possible to have multiple breaks especially if T is large. Here we generalize the model in Section 2 to allow for multiple breaks. In this case, we have

$$\alpha_{kt} = \begin{cases} \alpha_k^{(1)}, & \text{for} \quad t = 1, \ldots, T_1, \\ \alpha_k^{(2)}, & \text{for} \quad t = T_1 + 1, \ldots, T_2, \\ \vdots \\ \alpha_k^{(b+1)}, & \text{for} \quad t = T_b + 1, \ldots, T, \end{cases}$$

where $b \geq 1$ denotes the number of breaks.

To estimate the number of breaks and the break points $T_1, \cdots, T_b$, in principle we can follow the sequential method proposed by Bai and Perron (1998). First, using the full-sample data, we can construct $F_{NT}(1|0)$ defined in the previous subsection and estimate the break point as in (3.4). Second, for each regime before and after the estimated break point, we test the hypothesis in (5.2) and estimate the break point for each regime separately. This sequential method is repeated until the null can not be rejected for all subsamples, leading to the break point estimates $\{\hat{T}_a\}_{a \in [\hat{b}]}$ where $\hat{b}$ is the estimated number of breaks. We conjecture that we can establish the consistency of $\hat{b}$ and $\{\hat{T}_a\}$.

After the estimated number of breaks and break points are obtained for each subsample

$$\left\{(i,t) : i \in [N], t \in \{\hat{T}_{a-1}+1, \cdots, \hat{T}_a\}\right\},$$

where $a \in [\hat{b}+1]$ with $\hat{T}_0 := 0$ and $\hat{T}_{\hat{b}+1} := T$, we can continue Step 4 in the estimation algorithm in Section 3 to obtain the estimated group structure for each subsample.

6 Monte Carlo Simulations

In this section we report simulation results for the low-rank estimates, break point estimates, group membership estimates and the group number estimates based on 1,000 replications, and tuning parameter ν_j chosen by a procedure similar to that described in Chernozhukov et al. (2020). We focus on the linear panel model $Y_{it} = \lambda_i' f_t + X_{it}'\Theta_{it} + e_{it}$, where $X_{it} = (X_{1,it}, X_{2,it})'$ and $\Theta_{it} = (\Theta_{1,it}, \Theta_{2,it})'$.

6.1 Data Generating Processes (DGPs)

The following four main DGPs are employed.

DGP 1: [**Static panel with homoskedasticity**] $X_{1,it} \sim i.i.d.\ U(-2,2)$, $X_{2,it} \sim i.i.d.\ U(-2,2)$, and errors $e_{it} \sim i.i.d.\ \mathbb{N}(0,1)$. For Θ_1, we randomly choose the break point T_1 from $0.4T$ to $0.6T$.

DGP 2: [**Static panel with heteroscedasticity**] Compared to DGP 1, the errors $e_{it} \sim i.i.d.\ \mathbb{N}(0,\sigma_{it}^2)$ with $\sigma_{it}^2 \sim i.i.d.\ U(0.5,1)$. The settings for the regressors and break point are the same as those in DGP 1.

DGP 3: [**Serially correlated error**] Compared to DGP 2, the errors $e_{it} = 0.2e_{i,t-1} + \eta_{it}$, where $\eta_{it} \sim i.i.d.\ \mathbb{N}(0,1)$ and all other settings are the same as in DGP 2.

DGP 4: [**Dynamic panel**] In this case, $X_{1,it} = Y_{i,t-1}$ with $Y_{i,0} \sim i.i.d.\ \mathbb{N}(0,1)$. $X_{2,it} \sim i.i.d.\ U(-2,2)$, and $e_{it} \sim i.i.d.\ \mathbb{N}(0,0.5)$.

For each DGP above, we set $r_0 = 0$ and draw λ_i and f_t from $\mathbb{N}(0,1)$ independently. We define the slope coefficient based on three subcases below.

DGP X.1: In this case, the group membership and the number of groups do not change after the break point and only the value of the slope coefficient changes. We set the number of groups to be 2, the ratio of individuals among the two groups is $N_1 : N_2 = 0.5 : 0.5$, and the group membership G_1 is obtained by a random draw from $[N]$ without replacement. For DGPs 1.1, 2.1, and 3.1,

$$\Theta_{1,it} = \Theta_{2,it} = \begin{cases} 0.1, & i \in G_1,\ t \in \{1, \cdots, T_1\}, \\ 0.9, & i \in G_2,\ t \in \{1, \cdots, T_1\}, \\ 0.05, & i \in G_1,\ t \in \{T_1+1, \cdots, T\}, \\ 0.45, & i \in G_2,\ t \in \{T_1+1, \cdots, T\}. \end{cases}$$

For DGP 4.1, $\Theta_{2,it}$ is same as other DGPs X.1 for X$\in \{1,2,3\}$, and

$$\Theta_{1,it} = \begin{cases} 0.1, & i \in G_1,\ t \in \{1, \cdots, T_1\}, \\ 0.7, & i \in G_2,\ t \in \{1, \cdots, T_1\}, \\ 0.05, & i \in G_1,\ t \in \{T_1+1, \cdots, T\}, \\ 0.35, & i \in G_2,\ t \in \{T_1+1, \cdots, T\}. \end{cases}$$

DGP X.2: Compared to DGP X.1, the values of the slope coefficients for different groups do not change after the break point, but the group membership changes. The number of groups is 2, the ratio of individuals among the groups is still $N_1 : N_2 = 0.5 : 0.5$. Nevertheless, $\{G_1^{(1)}, G_2^{(1)}\}$ is different from $\{G_1^{(2)}, G_2^{(2)}\}$ so that elements in both $G_1^{(1)}$ and $G_1^{(2)}$ are independent draws from $[N]$ without replacement. In addition, for DGPs 1.2, 2.2, and 3.2,

$$\Theta_{1,it} = \Theta_{2,it} = \begin{cases} 0.1, & i \in G_1^{(1)},\ t \in \{1, \cdots, T_1\}, \\ 0.9, & i \in G_2^{(1)},\ t \in \{1, \cdots, T_1\}, \\ 0.1, & i \in G_1^{(2)},\ t \in \{T_1+1, \cdots, T\}, \\ 0.9, & i \in G_2^{(2)},\ t \in \{T_1+1, \cdots, T\}. \end{cases}$$

For DGP 4.2, $\Theta_{2,it}$ is defined same as other DGPs X.2 for $X \in \{1,2,3\}$, and

$$\Theta_{1,it} = \begin{cases} 0.1, & i \in G_1^{(1)},\ t \in \{1, \cdots, T_1\}, \\ 0.7, & i \in G_2^{(1)},\ t \in \{1, \cdots, T_1\}, \\ 0.1, & i \in G_1^{(2)},\ t \in \{T_1+1, \cdots, T\}, \\ 0.7, & i \in G_2^{(2)},\ t \in \{T_1+1, \cdots, T\}. \end{cases}$$

DGP X.3: Under this scenario, the number of groups changes after the break. We set $N_1^{(1)} : N_2^{(1)} = 0.5 : 0.5$ and $N_1^{(2)} : N_2^{(2)} : N_3^{(2)} = 0.4 : 0.3 : 0.3$ before and after the break, respectively. Specifically, for DGPs 1.3, 2.3, and 3.3, we have

$$\Theta_{1,it} = \Theta_{2,it} = \begin{cases} 0.1, & i \in G_1^{(1)},\ t \in \{1, \cdots, T_1\}, \\ 0.9, & i \in G_2^{(1)},\ t \in \{1, \cdots, T_1\}, \\ 0.1, & i \in G_1^{(2)},\ t \in \{T_1+1, \cdots, T\}, \\ 0.5, & i \in G_2^{(2)},\ t \in \{T_1+1, \cdots, T\}, \\ 0.9, & i \in G_3^{(2)},\ t \in \{T_1+1, \cdots, T\}. \end{cases}$$

For DGP 4.3, $\Theta_{2,it}$ is defined as in DGP X.3 for X$\in \{1,2,3\}$, and

$$\Theta_{1,it} = \begin{cases} 0.1, & i \in G_1^{(1)},\ t \in \{1, \cdots, T_1\}, \\ 0.7, & i \in G_2^{(1)},\ t \in \{1, \cdots, T_1\}, \\ 0.1, & i \in G_1^{(2)},\ t \in \{T_1+1, \cdots, T\}, \\ 0.4, & i \in G_2^{(2)},\ t \in \{T_1+1, \cdots, T\}, \\ 0.7, & i \in G_3^{(2)},\ t \in \{T_1+1, \cdots, T\}. \end{cases}$$

6.2 Results

Table 1 reports the proportion of correct rank estimation for the intercept (IFE) and slope matrices based on the SVT in Section 3.3. Note that r_0 denotes the true rank of the intercept matrix and r_1 and r_2 denote those of the two slope matrices. From Table 1, we notice that the true ranks of both the intercept and slope matrices can be almost perfectly estimated for the sample sizes under investigation.

Table 2 reports the results for the break point estimation in Step 3 based on different (N, T) combinations. We summarize some important findings from Table 2. First, when the group membership and the number of groups do not change as in DGP X.1 for X$\in [3]$, the frequency of correct break point estimation may not be 1 especially if N is not large. This suggests that the binary segmentation does not work perfectly in such a scenario. Second, the change of group membership or the number of groups help to identify the break point as reflected in the simulation results for DGP X.2 and X.3 for X$\in [4]$. In general, the binary segmentation works well in our setting.

Table 3 reports the results for the group membership estimation when the number of groups are either known (infeasible in practice) or estimated from the data (feasible). With known number of groups, the STK algorithm degenerates to the traditional K-means algorithm. The "Infeasible" part of Table 3 reports the frequency of correct group membership estimation before and after the estimated break point, G_B and G_A, based on the known true number of groups and K-means algorithm. Evidently, the K-means classification exhibits excellent performance in this case. Nevertheless, without prior information on the true number of groups, the STK algorithm is able to estimate the group membership and the number of groups simultaneously. In this case, the frequency of correct estimation of the group membership and that of the number of groups are shown in the "Feasible" part in Table 3 and in Table 4, respectively. To implement the STK algorithm with unknown number of groups, we set $\varsigma_N = N^{-2}$ to ensure the consistency of the group number estimators. As expected, the performance of the STK algorithm is slightly worse than that of the K-means algorithm with knowledge of the true number of groups. But the performance improves when both N and T increase. Table 4 suggests that the number of groups can be nearly perfectly estimated in DGPs 1.1, 1.2, 1.3 and 2.1. For the more complicated DGPs (e.g., the dynamic case in DGPs 4.1, 4.2, and 4.3 or the static panel with serially correlated errors in DGPs 3.1, 3.2, and 3.3), the performance is not as good as that in the simple DGPs.

Table 1: Frequency of correct rank estimation

N		100		200		N		100		200	
T		100	200	100	200	T		100	200	100	200
DGP 1.1	$r_0 = 1$	1.00	1.00	1.00	1.00	DGP 3.1	$r_0 = 1$	1.00	1.00	1.00	1.00
	$r_1 = 1$	1.00	1.00	1.00	1.00		$r_1 = 1$	1.00	1.00	1.00	1.00
	$r_2 = 1$	1.00	1.00	1.00	1.00		$r_2 = 1$	1.00	1.00	1.00	1.00
DGP 1.2	$r_0 = 1$	1.00	1.00	1.00	1.00	DGP 3.2	$r_0 = 1$	1.00	1.00	1.00	1.00
	$r_1 = 2$	1.00	1.00	1.00	1.00		$r_1 = 2$	1.00	1.00	1.00	1.00
	$r_2 = 2$	1.00	1.00	1.00	1.00		$r_2 = 2$	1.00	1.00	1.00	1.00
DGP 1.3	$r_0 = 1$	1.00	1.00	1.00	1.00	DGP 3.3	$r_0 = 1$	1.00	1.00	1.00	1.00
	$r_1 = 2$	1.00	1.00	1.00	1.00		$r_1 = 2$	1.00	1.00	1.00	1.00
	$r_2 = 2$	1.00	1.00	1.00	1.00		$r_2 = 2$	0.998	1.00	1.00	1.00
DGP 2.1	$r_0 = 1$	1.00	1.00	1.00	1.00	DGP 4.1	$r_0 = 1$	1.00	1.00	1.00	1.00
	$r_1 = 1$	1.00	1.00	1.00	1.00		$r_1 = 1$	1.00	1.00	1.00	1.00
	$r_2 = 1$	1.00	1.00	1.00	1.00		$r_2 = 1$	1.00	1.00	1.00	1.00
DGP 2.2	$r_0 = 1$	1.00	1.00	1.00	1.00	DGP 4.2	$r_0 = 1$	1.00	1.00	1.00	1.00
	$r_1 = 2$	1.00	1.00	1.00	1.00		$r_1 = 1$	1.00	1.00	1.00	1.00
	$r_2 = 2$	1.00	1.00	1.00	1.00		$r_2 = 2$	1.00	1.00	1.00	1.00
DGP 2.3	$r_0 = 1$	1.00	1.00	1.00	1.00	DGP 4.3	$r_0 = 1$	1.00	1.00	1.00	1.00
	$r_1 = 2$	1.00	1.00	1.00	1.00		$r_1 = 1$	1.00	1.00	1.00	1.00
	$r_2 = 2$	1.00	1.00	1.00	1.00		$r_2 = 2$	1.00	1.00	1.00	1.00

Table 2: Frequency of correct break point estimation

N	100		200		N	100		200	
T	100	200	100	200	T	100	200	100	200
DGP 1.1	0.980	0.993	1.00	1.00	DGP 3.1	0.985	0.972	1.00	0.999
DGP 1.2	0.999	1.00	1.00	1.00	DGP 3.2	1.00	1.00	1.00	1.00
DGP 1.3	1.00	1.00	1.00	1.00	DGP 3.3	1.00	1.00	1.00	1.00
DGP 2.1	0.998	0.999	1.00	1.00	DGP 4.1	1.00	1.00	1.00	1.00
DGP 2.2	1.00	1.00	1.00	1.00	DGP 4.2	1.00	1.00	1.00	1.00
DGP 2.3	1.00	1.00	1.00	1.00	DGP 4.3	1.00	1.00	1.00	1.00

Table 3: Frequency of correct group membership estimation

	N		100		200			N		100		200	
	T		100	200	100	200		T		100	200	100	200
Infeasible	DGP 1.1	G_B	1.00	1.00	1.00	1.00	Feasible	DGP 1.1	G_B	1.00	1.00	1.00	1.00
		G_A	1.00	1.00	1.00	1.00			G_A	1.00	1.00	1.00	1.00
	DGP 1.2	G_B	1.00	1.00	1.00	1.00		DGP 1.2	G_B	1.00	1.00	1.00	1.00
		G_A	1.00	1.00	1.00	1.00			G_A	1.00	1.00	1.00	1.00
	DGP 1.3	G_B	1.00	1.00	1.00	1.00		DGP 1.3	G_B	1.00	1.00	1.00	1.00
		G_A	0.989	0.999	0.978	0.999			G_A	0.989	0.999	0.978	0.999
	DGP 2.1	G_B	1.00	1.00	1.00	1.00		DGP 2.1	G_B	1.00	1.00	1.00	1.00
		G_A	1.00	1.00	1.00	1.00			G_A	1.00	1.00	1.00	1.00
	DGP 2.2	G_B	1.00	1.00	1.00	1.00		DGP 2.2	G_B	0.989	0.999	0.992	0.999
		G_A	1.00	1.00	1.00	1.00			G_A	0.992	0.999	0.977	0.998
	DGP 2.3	G_B	1.00	1.00	1.00	1.00		DGP 2.3	G_B	0.992	0.999	0.961	0.999
		G_A	0.998	1.00	0.999	1.00			G_A	0.989	0.999	0.992	0.999
	DGP 3.1	G_B	1.00	1.00	1.00	1.00		DGP 3.1	G_B	0.981	0.999	0.949	0.999
		G_A	1.00	1.00	1.00	1.00			G_A	0.981	0.993	0.979	0.996
	DGP 3.2	G_B	1.00	1.00	1.00	1.00		DGP 3.2	G_B	0.985	0.996	0.962	0.993
		G_A	1.00	1.00	1.00	1.00			G_A	0.985	0.994	0.973	0.998
	DGP 3.3	G_B	1.00	1.00	1.00	1.00		DGP 3.3	G_B	0.985	0.998	0.973	0.995
		G_A	0.981	0.997	0.982	0.999			G_A	0.971	0.994	0.968	0.998
	DGP 4.1	G_B	1.00	1.00	1.00	1.00		DGP 4.1	G_B	0.975	0.999	0.984	0.999
		G_A	1.00	1.00	1.00	1.00			G_A	0.985	0.998	0.949	0.997
	DGP 4.2	G_B	1.00	1.00	1.00	1.00		DGP 4.2	G_B	0.994	0.998	0.952	0.997
		G_A	1.00	1.00	1.00	1.00			G_A	0.977	0.999	0.985	0.999
	DGP 4.3	G_B	1.00	1.00	1.00	1.00		DGP 4.3	G_B	0.983	0.998	0.948	0.999
		G_A	1.00	1.00	1.00	1.00			G_A	0.982	0.998	0.983	0.998

Table 4: Frequency of correct estimation of the number of groups

N		100		200		N		100		200	
T		100	200	100	200	T		100	200	100	200
DGP 1.1	$K^{(1)} = 2$	1.00	1.00	0.999	1.00	DGP 3.1	$K^{(1)} = 2$	0.880	0.993	0.675	0.985
	$K^{(2)} = 2$	1.00	1.00	1.00	1.00		$K^{(2)} = 2$	0.890	0.960	0.873	0.971
DGP 1.2	$K^{(1)} = 2$	1.00	1.00	1.00	1.00	DGP 3.2	$K^{(1)} = 2$	0.868	0.985	0.759	0.940
	$K^{(2)} = 2$	1.00	1.00	1.00	0.999		$K^{(2)} = 2$	0.897	0.971	0.829	0.987
DGP 1.3	$K^{(1)} = 2$	0.999	1.00	1.00	1.00	DGP 3.3	$K^{(1)} = 2$	0.889	0.988	0.802	0.965
	$K^{(2)} = 3$	1.00	0.999	1.00	1.00		$K^{(2)} = 3$	0.932	0.977	0.907	0.988
DGP 2.1	$K^{(1)} = 2$	1.00	1.00	1.00	1.00	DGP 4.1	$K^{(1)} = 2$	0.807	0.981	0.825	0.982
	$K^{(2)} = 2$	1.00	1.00	1.00	1.00		$K^{(2)} = 2$	0.919	0.988	0.714	0.980
DGP 2.2	$K^{(1)} = 2$	0.919	0.995	0.940	0.994	DGP 4.2	$K^{(1)} = 2$	0.933	0.988	0.630	0.975
	$K^{(2)} = 2$	0.930	0.993	0.809	0.982		$K^{(2)} = 2$	0.758	0.988	0.870	0.989
DGP 2.3	$K^{(1)} = 2$	0.940	0.989	0.724	0.990	DGP 4.3	$K^{(1)} = 2$	0.877	0.991	0.657	0.991
	$K^{(2)} = 3$	0.946	0.995	0.952	0.992		$K^{(2)} = 3$	0.900	0.987	0.874	0.980

Table 5 presents more detailed results for the estimation of the number of groups. For DGPs 1.X and DGP 2.X where we have static panels with independent errors, the results show that the group membership and the number of groups can be well estimated with nearly 100% accuracy under different (N, T) combinations. For DGPs 3.X and 4.X where we have static panels with serially correlated errors and dynamic panels, respectively, the frequency of correct estimation of both the group membership and the number of groups is not great when T is small, but gradually approaches unity as T increases. One reason for this is that we need to use HAC estimates of certain long-run variance objects in the STK algorithm and it is well known that a relatively large value of T is required in order for the HAC estimates to be reasonably well behaved in finite samples.

Table 6 shows results for the post-classification estimator for the first slope coefficient. We follow Su et al. (2016) to define the evaluation criteria as bias and coverage. Specifically, we define the bias to be the weighted versions of bias for slope estimator from all estimated groups, i.e. Bias = $\sum_{k=1}^{K^{(1)}} \text{Bias}(\alpha_{k,1}^{(\ell)})$ for $\ell \in \{1, 2\}$. Similarly, we define the weighted version of the coverage ratio of the 95% confidence interval estimators. The "Infeasible" panel shows the result assuming the number of groups information is known, and the "Feasible" panel shows the result without knowing the number of groups information by the STK algorithm. From Table 6, we notice that the coverage ratio for DGP 1 and 2 is close to 95% under different combinations of N and T for both the "Infeasible" and "Feasible" panels, which is due to the higher correct classification ratio. For DGPs 3 and 4, by using the STK algorithm, the coverage ratio is a bit lower for $T = 100$, which is due to the inaccuracy of the group number and membership estimators; but coverage approaches 95% quickly when T doubles.

7 Empirical Study

The estimation methods were applied to analyze the time-varying latent group structure of real house price changes in Metropolitan Statistical Areas (MSAs) in the United States. Studies of U.S. house price changes are plentiful in the literature. Malpezzi (1999), Capozza et al. (2002), Gallin (2006), and Ortalo-Magne and Rady (2006) all show that the house price changes are closely correlated with real income in the long run. Su et al. (2023) consider a heterogenous spatial panel and show that real income growth affects the U.S. house prices in different ways for different MSAs. In this application, we examine whether there exist latent group structures for the real income growth elasticity of house price changes and whether these structures change over the time dimension.

7.1 Model

We consider the following panel data model with IFEs and two-way slope heterogeneity

$$\pi_{it} = \lambda_i' f_t + \Theta_{1,it} ginc_{it} + \Theta_{2,it} ginc_{i,t-1} + e_{it}, \tag{7.1}$$

where the dependent variable π_{it} measures the percentage of real house price growth for MSA i at time period t. The λ_i and f_t are the individual fixed effects and time fixed effects, the covariate $ginc_{it}$

Table 5: Determination of the number of groups

DGP	N	T	$\hat{K}^{(1)}$ 2	3	4	≥5	$\hat{K}^{(2)}$ 2	3	4	≥5
DGP 1.1	100	100	**1.00**	0	0	0	**1.00**	0	0	0
		200	**1.00**	0	0	0	**1.00**	0	0	0
	200	100	**0.999**	0.001	0	0	**1.00**	0	0	0
		200	**1.00**	0	0	0	**1.00**	0	0	0
DGP 1.2	100	100	**1.00**	0.00	0.00	0.00	**1.00**	0.00	0.00	0.00
		200	**1.00**	0.00	0.00	0.00	**1.00**	0.00	0.00	0.00
	200	100	**1.00**	0.00	0.00	0.00	**1.00**	0.00	0.00	0.00
		200	**1.00**	0.00	0.00	0.00	**1.00**	0.00	0.00	0.00
DGP 1.3	100	100	**0.999**	0.001	0.00	0.00	0.00	**1.00**	0.00	0.00
		200	**1.00**	0.00	0.00	0.00	0.00	**0.999**	0.001	0.00
	200	100	**1.00**	0.00	0.00	0.00	0.00	**1.00**	0.00	0.00
		200	**1.00**	0.00	0.00	0.00	0.00	**1.00**	0.00	0.00
DGP 2.1	100	100	**0.933**	0.058	0.009	0.00	**0.936**	0.060	0.003	0.001
		200	**0.990**	0.010	0.00	0.00	**0.987**	0.013	0.00	0.00
	200	100	**0.864**	0.126	0.010	0.00	**0.901**	0.090	0.009	0.00
		200	**0.989**	0.011	0.00	0.00	**0.990**	0.010	0.000	0.00
DGP 2.2	100	100	**0.919**	0.074	0.007	0.00	**0.930**	0.067	0.003	0.00
		200	**0.995**	0.003	0.00	0.002	**0.993**	0.006	0.00	0.001
	200	100	**0.940**	0.056	0.004	0.00	**0.809**	0.164	0.027	0.00
		200	**0.994**	0.006	0.00	0.00	**0.982**	0.018	0.00	0.00
DGP 2.3	100	100	**0.940**	0.055	0.005	0.00	0.00	**0.946**	0.039	0.015
		200	**0.989**	0.011	0.00	0.00	0.00	**0.995**	0.002	0.003
	200	100	**0.724**	0.230	0.046	0.00	0.00	**0.952**	0.031	0.017
		200	**0.990**	0.010	0.00	0.00	0.00	**0.992**	0.006	0.002
DGP 3.1	100	100	**0.880**	0.097	0.022	0.001	**0.890**	0.062	0.031	0.017
		200	**0.993**	0.007	0	0	**0.960**	0.019	0.012	0.009
	200	100	**0.675**	0.224	0.099	0.002	**0.873**	0.081	0.041	0.005
		200	**0.985**	0.015	0	0	**0.971**	0.023	0.005	0.001
DGP 3.2	100	100	**0.868**	0.109	0.023	0.00	**0.897**	0.099	0.004	0.00
		200	**0.985**	0.008	0.003	0.004	**0.971**	0.021	0.006	0.002
	200	100	**0.759**	0.198	0.042	0.001	**0.829**	0.147	0.024	0.00
		200	**0.940**	0.055	0.005	0.00	**0.987**	0.013	0.000	0.00
DGP 3.3	100	100	**0.889**	0.100	0.011	0.00	0.00	**0.932**	0.055	0.013
		200	**0.988**	0.009	0.003	0.00	0.00	**0.977**	0.013	0.010
	200	100	**0.802**	0.175	0.023	0.00	0.00	**0.907**	0.073	0.020
		200	**0.965**	0.035	0.000	0.000	0.000	**0.988**	0.010	0.002
DGP 4.1	100	100	**0.807**	0.084	0.089	0.02	**0.919**	0.041	0.019	0.021
		200	**0.981**	0.013	0.004	0.002	**0.988**	0.004	0.005	0.003
	200	100	**0.825**	0.107	0.061	0.007	**0.714**	0.118	0.084	0.084
		200	**0.982**	0.011	0.006	0.001	**0.98**	0.010	0.004	0.006
DGP 4.2	100	100	**0.933**	0.051	0.012	0.004	**0.758**	0.141	0.089	0.012
		200	**0.988**	0.006	0.004	0.002	**0.988**	0.005	0.006	0.001
	200	100	**0.630**	0.158	0.196	0.016	**0.870**	0.080	0.048	0.002
		200	**0.975**	0.013	0.012	0.000	**0.989**	0.009	0.002	0.000
DGP 4.3	100	100	**0.877**	0.076	0.042	0.005	0.000	**0.900**	0.055	0.045
		200	**0.991**	0.006	0.002	0.001	0.000	**0.987**	0.010	0.003
	200	100	**0.657**	0.191	0.129	0.023	0.000	**0.874**	0.072	0.054
		200	**0.991**	0.005	0.004	0.000	0.000	**0.980**	0.012	0.008

Table 6: Point estimation of $\alpha^{(1)}_{\cdot,1}$ and $\alpha^{(2)}_{\cdot,1}$

DGP	N	T	Infeasible				Feasible			
			Before the break		After the break		Before the break		After the break	
			Bias($\times 10^{-6}$)	Coverage	Bias($\times 10^{-6}$)	Coverage	Bias($\times 10^{-6}$)	Coverage	Bias($\times 10^{-6}$)	Coverage
1.1	100	100	2.585	0.951	-2.869	0.946	2.585	0.951	-2.869	0.946
		200	-1.944	0.944	-8.920	0.945	-1.958	0.944	-8.920	0.945
	200	100	-1.096	0.943	1.407	0.947	-1.096	0.943	1.407	0.947
		200	-1.910	0.945	0.960	0.947	-1.910	0.945	0.960	0.947
1.2	100	100	-1.050	0.949	-27.398	0.941	-1.050	0.949	-27.398	0.941
		200	-5.449	0.930	7.616	0.953	-5.449	0.930	7.655	0.953
	200	100	4.770	0.949	1.866	0.951	4.770	0.949	1.866	0.951
		200	1.317	0.941	1.874	0.945	1.317	0.941	1.874	0.945
1.3	100	100	-0.961	0.943	11.417	0.944	-1.050	0.949	-27.398	0.941
		200	-4.213	0.951	-5.002	0.941	-5.449	0.930	7.655	0.953
	200	100	-1.571	0.938	-3.756	0.938	4.770	0.949	1.866	0.951
		200	0.403	0.941	-4.159	0.945	1.317	0.941	1.874	0.945
2.1	100	100	14.840	0.944	9.410	0.950	14.816	0.943	9.406	0.950
		200	-7.222	0.951	1.795	0.951	-7.222	0.951	1.795	0.951
	200	100	0.916	0.940	3.575	0.948	0.916	0.940	3.575	0.948
		200	0.452	0.948	-0.797	0.947	0.452	0.948	-0.797	0.947
2.2	100	100	-21.379	0.946	0.234	0.937	-21.379	0.946	0.234	0.937
		200	0.264	0.942	-15.542	0.953	0.264	0.942	-15.542	0.953
	200	100	-1.379	0.945	-1.489	0.951	-1.379	0.944	-1.489	0.951
		200	-1.101	0.950	1.127	0.949	-1.101	0.950	1.127	0.949
2.3	100	100	-8.610	0.945	5.254	0.952	-8.610	0.945	5.261	0.952
		200	0.927	0.949	5.840	0.949	0.927	0.949	5.840	0.949
	200	100	-1.560	0.943	-2.569	0.941	-1.560	0.943	-2.569	0.941
		200	-0.775	0.947	4.408	0.947	-0.775	0.947	4.386	0.947
3.1	100	100	-20.928	0.955	-73.947	0.945	-26.250	0.927	-77.613	0.920
		200	3.066	0.949	-12.443	0.937	2.884	0.940	-13.116	0.934
	200	100	-2.663	0.951	-8.742	0.944	-3.517	0.857	-7.730	0.888
		200	-3.747	0.949	-2.107	0.945	-3.642	0.939	-1.971	0.938
3.2	100	100	-55.980	0.952	-10.846	0.943	-58.714	0.926	-15.109	0.863
		200	-2.774	0.950	4.690	0.946	-3.218	0.945	4.913	0.942
	200	100	6.979	0.951	8.879	0.945	6.287	0.858	6.894	0.848
		200	-1.704	0.947	0.438	0.945	-2.122	0.928	0.381	0.940
3.3	100	100	-25.340	0.950	37.639	0.907	-29.905	0.924	37.016	0.890
		200	2.042	0.947	-6.431	0.960	1.667	0.940	-6.245	0.960
	200	100	-2.391	0.946	14.364	0.892	-2.735	0.891	13.680	0.840
		200	4.339	0.943	4.890	0.942	4.493	0.932	5.113	0.938
4.1	100	100	800.620	0.930	-466.590	0.929	777.650	0.928	-454.980	0.924
		200	126.760	0.931	550.210	0.942	126.220	0.942	548.160	0.943
	200	100	-224.960	0.931	-339.020	0.939	-214.900	0.904	-313.430	0.876
		200	417.320	0.938	412.500	0.947	415.110	0.941	410.700	0.944
4.2	100	100	1246.000	0.921	726.940	0.943	1205.600	0.918	709.260	0.903
		200	-440.880	0.943	83.433	0.944	-436.670	0.943	81.585	0.951
	200	100	-1600.500	0.930	1937.500	0.927	-1538.300	0.901	1781.600	0.819
		200	-1513.300	0.950	-272.500	0.946	-1502.000	0.935	-271.420	0.953
4.3	100	100	-2067.900	0.931	-505.100	0.940	-1951.700	0.866	-491.200	0.929
		200	317.360	0.946	411.770	0.945	316.550	0.951	407.820	0.935
	200	100	1279.900	0.930	3660.100	0.888	1246.900	0.906	3355.100	0.874
		200	-772.380	0.940	-335.250	0.948	-768.870	0.932	-334.190	0.940

denotes the percentage of income growth for MSA i at time period t, and $ginc_{i,t-1}$ is the lagged value of $ginc_{it}$. Unlike Aquaro et al. (2021) and Su et al. (2023) who consider individual fixed effects and additive two-way fixed effects, respectively, we allow the model to have IFEs. In the above model, we allow the slope parameters $(\Theta_{1,it}, \Theta_{2,it})$ to exhibit latent group structures along the cross-sectional dimension and an unknown break along the time dimension.

7.2 Data

The dataset we use is obtained from Aquaro et al. (2021), which is the quarterly data for 377 MSAs over 1975 to 2014. To construct the growth rate and the lagged term, we lose two periods of observations, which yields $T = 158$. Similar to Su et al. (2023), we deseasonalize the growth rate of real house price and real income. We do not de-factor the variables since our model contains IFEs to control the common shocks.

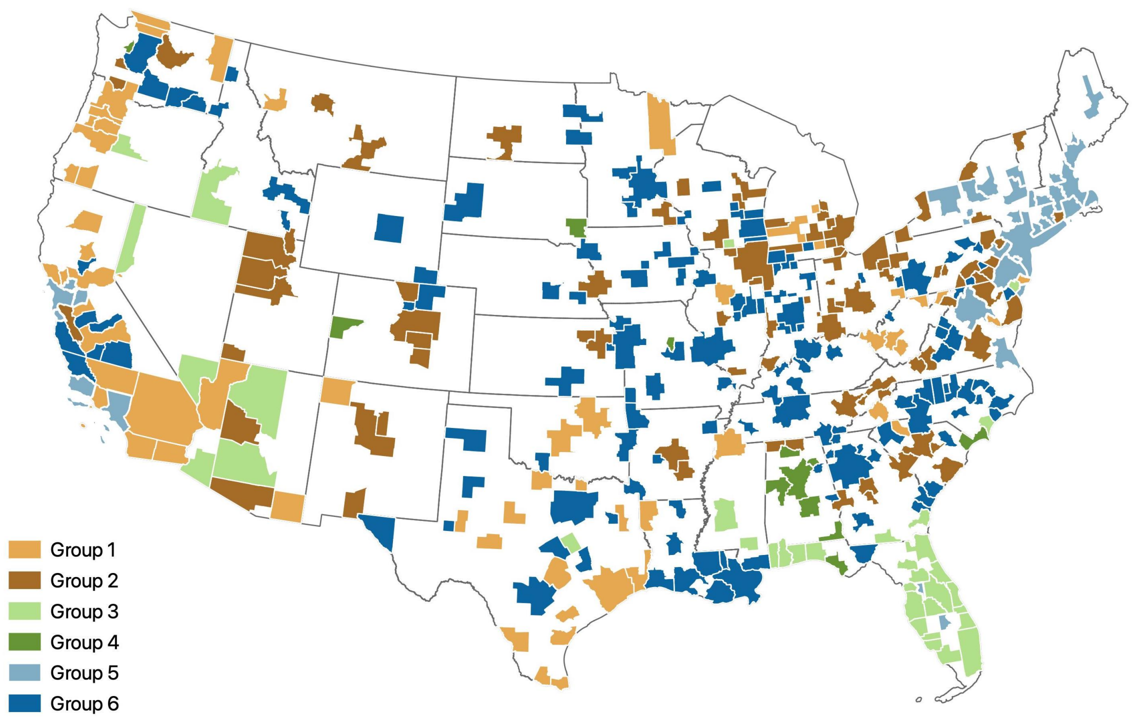
7.3 Empirical Results

We first apply the singular value thresholding to estimate the ranks of $\Theta_0 = \{\lambda_i' f_t\}$, $\Theta_1 = \{\Theta_{1,it}\}$ and $\Theta_2 = \{\Theta_{2,it}\}$. The estimates are: $\hat{r}_0 = 1$, $\hat{r}_1 = 2$, and $\hat{r}_2 = 2$. Before applying the proposed estimation algorithm in Section 3, we first test the presence of a structural break as in Section 5.2. As given in Bai and Perron (2003), for each individual i, the critical value of test statistic $\sup_{T_1 \in \mathcal{T}_\epsilon} F_i(T_1)$ is 15.37. We then construct the sup-F test statistic for each MSA. Results show that $\min_{i \in [N]} \sup_{T_1 \in \mathcal{T}_\epsilon} F_i(T_1) = 0.0195$ and the final test statistic is $F_{NT}(1|0) = \max_{i \in [N]} \sup_{T_1 \in \mathcal{T}_\epsilon} F_i(T_1) = 2161.65$. Based on this outcome, we reject the null that there is no structural break for slope coefficients $\Theta_{1,it}$ and $\Theta_{2,it}$ at the 1% significance level.

With the presence of a structural break, we apply the proposed multi-stage estimation result in Section 3 to estimate the break date and numbers of groups before and after the break. The estimated break date is given by $\hat{T}_1 = 51$, which suggests that the structural break happens at the first quarter in 1988. We conjecture that this break may be related to the catastrophic stock market crash that occurred on October 1987, which is considered to be the first contemporary global financial crisis event.

By setting $\varsigma_N = N^{-2}$ for the STK algorithm as in the simulations, we obtain the estimated prior- and post-break numbers of groups given by $\hat{K}^{(1)} = 6$ and $\hat{K}^{(2)} = 2$, respectively. As for the group structure, Figures 2 and 3 use six and two colors to show the classification results for the 377 MSAs during 1975Q3 to 1987Q4 and 1988Q1 to 2014Q4, respectively. Table 7 reports the pooled regression results for the full sample in column (1), the subsample before the estimated break point in column (2), and the subsample after the estimated break point in column (3). All the slope estimators are bias-corrected. The pooled regression results in Table 7 show that real income growth has positive and significant effect on house prices. Comparing the two subsamples before and after the estimated break, we observe that, with a 1 percentage increase in real income growth, the real house price

Figure 2: Group classification result 1975Q3-1987Q4



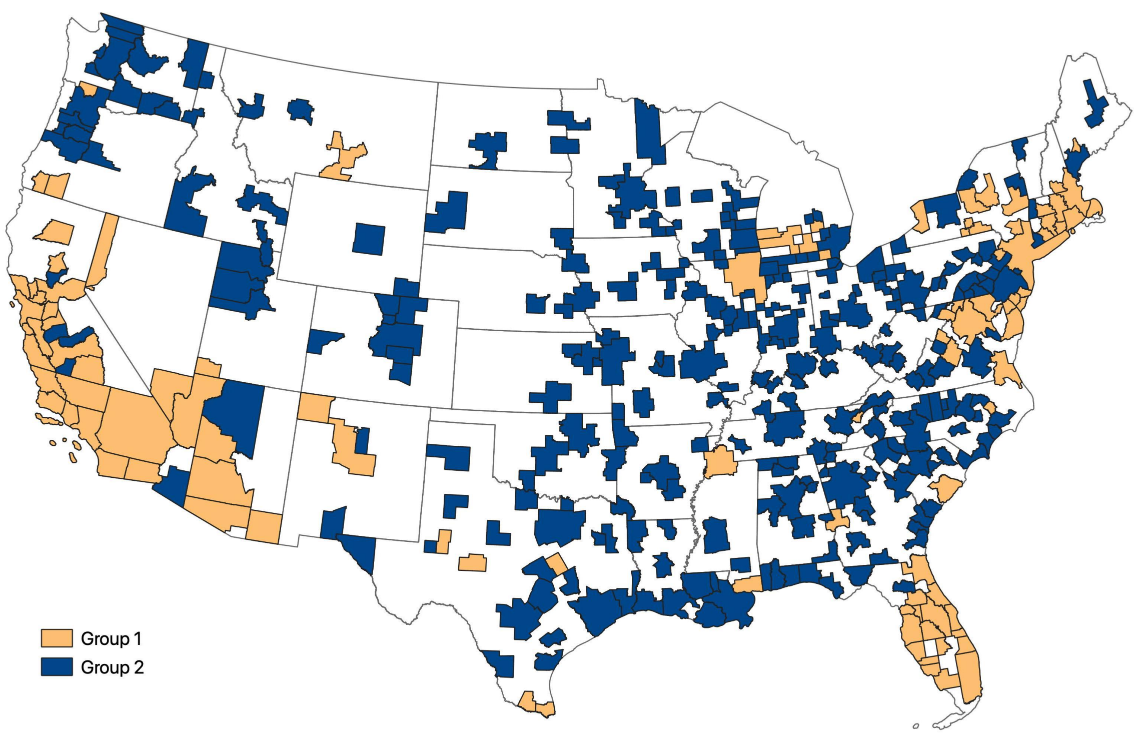


Figure 3: Group classification result 1988Q1-2014Q4

growth rate will increase 0.09 percentage before 1988, which is 0.02 percentage points higher than that after 1988. The slope estimates for the lagged term are similar for the two subsamples.

Table 7: Results for the pooled regressions

	Pooled (full sample) (1)	Pooled $(1975Q3-1987Q4)$ (2)	Pooled $(1988Q1-2014Q4)$ (3)
$ginc_{it}$	0.1021***	0.0904***	0.0702***
	(0.0067)	(0.0119)	(0.0065)
$ginc_{i,t-1}$	0.0590***	0.0392***	0.0401***
	(0.0067)	(0.0117)	(0.0066)
#individuals	377	377	377

Note: Column (1) reports the pooled regression results for the full sample. Columns (2) and (3) report the pooled regression results for the subsamples before and after the estimated break point, respectively. Slope estimators are all bias-corrected. Values in parentheses are standard errors and *** indicates significance at 1% level.

To examine the difference for each of the 6 estimated groups before the break, Table 8 reports the post-classification regression results for each estimated group before the estimated break. Even though the effects of real income growth are positive for all estimated groups, they differ vastly across groups. The effect of real income growth for Group 2 is the highest, followed by Groups 5 and 3, and the effects of real income growth on real house prices in all of these three groups are higher than 0.15 percent. In contrast, the effects of real income growth for the remaining three groups, viz., Groups 1, 4, and 6, are less than 0.07 percent. Similarly, Table 9 reports the post-classification regression results for each estimated group after the estimated break. The estimated slope coefficients for both groups are statistically significant. Especially, during 1988Q1-2014Q1, the slope estimator for the lagged term in the first group is much higher than that for the second group.

We also applied the C-Lasso algorithm of Su et al. (2016) to estimate the group structure before and after the estimated break point. The C-Lasso approach in conjuction with IC detects 2 groups both before and after the break. In view of the six groups detected by our present algorithm, we conjecture that the difference may due to the smaller time periods before the break.

8 Conclusion

This paper considers a linear panel data model with interactive fixed effects and two-way heterogeneity such that the heterogeneity across individuals is captured by latent group structures and the heterogeneity across time is captured by an unknown structural break. We allow the model to have different group numbers, or different group memberships, or just changes in the slope coefficients

Table 8: Results for the post-classification regressions before the break

	Group 1 (1)	Group 2 (2)	Group 3 (3)	Group 4 (4)	Group 5 (5)	Group 6 (6)
$ginc_{it}$	0.0301	0.3169***	0.1522***	0.0168	0.1877**	0.0661***
	(0.0345)	(0.0292)	(0.0408)	(0.0561)	(0.0775)	(0.0153)
$ginc_{i,t-1}$	0.1217***	−0.0191	−0.0089	−0.0298	−0.1269*	0.0331***
	(0.0348)	(0.0288)	(0.0407)	(0.0560)	(0.0754)	(0.0151)
#individuals	60	92	35	12	36	142

Note: Each column reports the regression results for each estimated group during 1977Q3-1987Q4. Slope estimators are all bias-corrected. Values in parentheses are standard errors. ***, **, and * indicate significance at 1% level, 5% level, 10% level, respectively.

for some specific groups before and after the break. To estimate the unknown structural break, the number of groups and group memberships before and after the break point, we propose an estimation algorithm with initial nuclear-norm-regularized estimates, followed by row- and column-wise linear regressions. Then, the break point estimator is obtained by binary segmentation and the group structure together with the number of groups are estimated simultaneously using a sequential testing K-means algorithm. We show that the structural break estimator, the group number estimators, and the group membership estimators before and after the break point are all consistent, and the final post-classification slope coefficient estimators enjoy the oracle property.

There are several interesting topics for further research. First, even though we discuss a possible test for the existence of a break in the panel data models with latent group structures, we have not fully worked out the asymptotic theory, a challenge that deserves separate treatment. Second, we assume the presence of a single break in the data and it is interesting to extend our theory to allow for multiple breaks. Third, the present treatment rules out both unit-root-type nonstationarity and nonstochastic trending nonstationarity and it is interesting to extend our theory to allow for nonstationarity. We will pursue these topics in future research.

References

Ando, T. and Bai, J. (2015). A simple new test for slope homogeneity in panel data models with interactive effects. *Economics Letters*, 136:112–117.

Ando, T. and Bai, J. (2016). Panel data models with grouped factor structure under unknown group membership. *Journal of Applied Econometrics*, 31(1):163–191.

Table 9: Results for the post-classification regressions after the break

	Group 1 (1)	Group 2 (2)
$ginc_{it}$	0.0670***	0.0714***
	(0.0130)	(0.0079)
$ginc_{i,t-1}$	0.0870***	0.0275***
	(0.0135)	(0.0079)
#individuals	103	274

Note: Each column reports the regression results for each estimated group during 1988Q1-2014Q4. Slope estimators are all bias-corrected. Values in parentheses are standard errors. *** indicates significance at 1% level.

Andrews, D. W. (1991). Heteroskedasticity and autocorrelation consistent covariance matrix estimation. *Econometrica: Journal of the Econometric Society*, pages 817–858.

Aquaro, M., Bailey, N., and Pesaran, M. H. (2021). Estimation and inference for spatial models with heterogeneous coefficients: An application to us house prices. *Journal of Applied Econometrics*, 36(1):18–44.

Bai, J. (2009). Panel data models with interactive fixed effects. *Econometrica*, 77(4):1229–1279.

Bai, J. (2010). Common breaks in means and variances for panel data. *Journal of Econometrics*, 157(1):78–92.

Bai, J. and Ng, S. (2019). Rank regularized estimation of approximate factor models. *Journal of Econometrics*, 212(1):78–96.

Bai, J. and Perron, P. (1998). Estimating and testing linear models with multiple structural changes. *Econometrica*, pages 47–78.

Bai, J. and Perron, P. (2003). Critical values for multiple structural change tests. *The Econometrics Journal*, 6(1):72–78.

Baltagi, B. H., Kao, C., and Liu, L. (2017). Estimation and identification of change points in panel models with nonstationary or stationary regressors and error term. *Econometric Reviews*, 36(1-3):85–102.

Belloni, A., Chen, M., Madrid Padilla, O. H., and Wang, Z. (2023). High-dimensional latent panel quantile regression with an application to asset pricing. *The Annals of Statistics*, 51(1):96–121.

Ben-David, D. (1998). Convergence clubs and subsistence economies. *Journal of Development Economics*, 55(1):155–171.

Berthelemy, J.-C. and Varoudakis, A. (1996). Economic growth, convergence clubs, and the role of financial development. *Oxford economic papers*, 48(2):300–328.

Bonhomme, S. and Manresa, E. (2015). Grouped patterns of heterogeneity in panel data. *Econometrica*, 83(3):1147–1184.

Capozza, D. R., Hendershott, P. H., Mack, C., and Mayer, C. J. (2002). Determinants of real house price dynamics.

Chernozhukov, V., Hansen, C. B., Liao, Y., and Zhu, Y. (2020). Inference for heterogeneous effects using low-rank estimations. Technical report, CEMMAP working paper.

Chu, Z. (2012). Logistics and economic growth: a panel data approach. *The Annals of regional science*, 49(1):87–102.

Durlauf, S. N. and Johnson, P. A. (1995). Multiple regimes and cross-country growth behaviour. *Journal of applied econometrics*, 10(4):365–384.

Feng, J. (2023). Regularized quantile regression with interactive fixed effects. *Econometric Theory*, forthcoming.

Gallin, J. (2006). The long-run relationship between house prices and income: evidence from local housing markets. *Real Estate Economics*, 34(3):417–438.

Golub, G. H. and Van Loan, C. F. (1996). Matrix computations. edition.

Hong, S., Su, L., and Jiang, T. (2023). Profile gmm estimation of panel data models with interactive fixed effects. *Journal of Econometrics*, forthcoming.

Huang, W., Jin, S., and Su, L. (2020). Identifying latent grouped patterns in cointegrated panels. *Econometric Theory*, 36(3):410–456.

Jin, J., Ke, Z. T., Luo, S., and Wang, M. (2022). Optimal estimation of the number of network communities. *Journal of the American Statistical Association*, 0(0):1–16.

Ke, Y., Li, J., and Zhang, W. (2016). Structure identification in panel data analysis. *The Annals of Statistics*, 44(3):1193–1233.

Ke, Z. T., Fan, J., and Wu, Y. (2015). Homogeneity pursuit. *Journal of the American Statistical Association*, 110(509):175–194.

Keane, M. and Neal, T. (2020). Climate change and US agriculture: Accounting for multidimensional slope heterogeneity in panel data. *Quantitative Economics*, 11(4):1391–1429.

Kim, D. (2011). Estimating a common deterministic time trend break in large panels with cross sectional dependence. *Journal of Econometrics*, 164(2):310–330.

Kim, D. (2014). Common breaks in time trends for large panel data with a factor structure. *The Econometrics Journal*, 17(3):301–337.

Klapper, L. and Love, I. (2011). The impact of the financial crisis on new firm registration. *Economics Letters*, 113(1):1–4.

Li, D., Qian, J., and Su, L. (2016). Panel data models with interactive fixed effects and multiple structural breaks. *Journal of the American Statistical Association*, 111(516):1804–1819.

Lin, C.-C. and Hsu, C.-C. (2011). Change-point estimation for nonstationary panel data. Technical report, Working Paper, Department of Economics, National Central University.

Lin, C.-C. and Ng, S. (2012). Estimation of panel data models with parameter heterogeneity when group membership is unknown. *Journal of Econometric Methods*, 1(1):42–55.

Long, W., Li, N., Wang, H., and Cheng, S. (2012). Impact of us financial crisis on different countries: based on the method of functional analysis of variance. *Procedia Computer Science*, 9:1292–1298.

Lu, X. and Su, L. (2016). Shrinkage estimation of dynamic panel data models with interactive fixed effects. *Journal of Econometrics*, 190(1):148–175.

Lu, X. and Su, L. (2017). Determining the number of groups in latent panel structures with an application to income and democracy. *Quantitative Economics*, 8(3):729–760.

Lu, X. and Su, L. (2023). Uniform inference in linear panel data models with two-dimensional heterogeneity. *Journal of Econometrics*, forthcoming.

Lumsdaine, R. L., Okui, R., and Wang, W. (2023). Estimation of panel group structure models with structural breaks in group memberships and coefficients. *Journal of Econometrics*, 233(1):45–65.

Ma, S., Su, L., and Zhang, Y. (2020). Detecting latent communities in network formation models. *arXiv preprint arXiv:2005.03226*.

Malpezzi, S. (1999). A simple error correction model of house prices. *Journal of housing economics*, 8(1):27–62.

Merlevède, F., Peligrad, M., and Rio, E. (2009). Bernstein inequality and moderate deviations under strong mixing conditions. In *High dimensional probability V: the Luminy volume*, pages 273–292. Institute of Mathematical Statistics.

Miao, K., Phillips, P. C. B., and Su, L. (2023). High-dimensional VARs with common factors. *Journal of Econometrics*, 233(1):155–183.

Moon, H. R. and Weidner, M. (2017). Dynamic linear panel regression models with interactive fixed effects. *Econometric Theory*, 33(1):158–195.

Moon, H. R. and Weidner, M. (2018). Nuclear norm regularized estimation of panel regression models. *arXiv preprint arXiv:1810.10987*.

Okui, R. and Wang, W. (2021). Heterogeneous structural breaks in panel data models. *Journal of Econometrics*, 220(2):447–473.

Ortalo-Magne, F. and Rady, S. (2006). Housing market dynamics: On the contribution of income shocks and credit constraints. *The Review of Economic Studies*, 73(2):459–485.

Pesaran, M. H. and Yamagata, T. (2008). Testing slope homogeneity in large panels. *Journal of Econometrics*, 142(1):50–93.

Prakasa Rao, B. L. (2009). Conditional independence, conditional mixing and conditional association. *Annals of the Institute of Statistical Mathematics*, 61(2):441–460.

Qian, J. and Su, L. (2016). Shrinkage estimation of common breaks in panel data models via adaptive group fused lasso. *Journal of Econometrics*, 191(1):86–109.

Sarafidis, V. and Weber, N. (2015). A partially heterogeneous framework for analyzing panel data. *Oxford Bulletin of Economics and Statistics*, 77(2):274–296.

Song, M. (2013). *Essays on Large Panel Data Analysis*. Columbia University.

Su, L. and Chen, Q. (2013). Testing homogeneity in panel data models with interactive fixed effects. *Econometric Theory*, 29(6):1079–1135.

Su, L. and Ju, G. (2018). Identifying latent grouped patterns in panel data models with interactive fixed effects. *Journal of Econometrics*, 206(2):554–573.

Su, L., Shi, Z., and Phillips, P. C. B. (2016). Identifying latent structures in panel data. *Econometrica*, 84(6):2215–2264.

Su, L., Wang, W., and Xu, X. (2023). Identifying latent group structures in spatial dynamic panels. *Journal of Econometrics*, forthcoming.

Su, L., Wang, W., and Zhang, Y. (2020). Strong consistency of spectral clustering for stochastic block models. *IEEE Transactions on Information Theory*, 66(1):324–338.

Su, L. and Wang, X. (2020). Testing for structural changes in factor models via a nonparametric regression. *Econometric Theory*, 36(6):1127–1158.

Su, L., Wang, X., and Jin, S. (2019). Sieve estimation of time-varying panel data models with latent structures. *Journal of Business & Economic Statistics*, 37(2):334–349.

Tropp, J. A. (2011). User-friendly tail bounds for matrix martingales. Technical report, California Institute of Technology.

Vaart, A. W. and Wellner, J. A. (1996). Weak convergence. In *Weak convergence and empirical processes*, pages 16–28. Springer.

Vershynin, R. (2010). Introduction to the non-asymptotic analysis of random matrices. *arXiv preprint arXiv:1011.3027*.

Wang, W., Phillips, P. C. B., and Su, L. (2018). Homogeneity pursuit in panel data models: Theory and application. *Journal of Applied Econometrics*, 33(6):797–815.

Wang, W., Phillips, P. C. B., and Su, L. (2019). The heterogeneous effects of the minimum wage on employment across states. *Economics Letters*, 174:179–185.

Wang, W. and Su, L. (2021). Identifying latent group structures in nonlinear panels. *Journal of Econometrics*, 220(2):272–295.

Wang, Y., Su, L., and Zhang, Y. (2022). Low-rank panel quantile regression: Estimation and inference. *https://arxiv.org/abs/2210.11062*.

White, H. (2014). *Asymptotic Theory for Econometricians*. Academic press.

Yu, Y., Wang, T., and Samworth, R. J. (2015). A useful variant of the davis–kahan theorem for statisticians. *Biometrika*, 102(2):315–323.

Zhang, J. and Cheng, L. (2019). Threshold effect of tourism development on economic growth following a disaster shock: Evidence from the wenchuan earthquake, pr china. *Sustainability*, 11(2):371.

Zhang, Y., Wang, H. J., and Zhu, Z. (2019). Quantile-regression-based clustering for panel data. *Journal of Econometrics*, 213(1):54–67.

Online Supplement for
"Panel Data Models with Time-Varying Latent Group Structures"

Yiren Wang[a], Peter C.B. Phillips[b] and Liangjun Su[c]

[a]School of Economics, Singapore Management University, Singapore

[b]Yale University, University of Auckland, Singapore Management University

[c]School of Economics and Management, Tsinghua University, China

This online supplement has five sections. Section A contains the proofs of the main results by calling upon some technical lemmas in Sections B and D. Section B states and proves the technical lemmas used in Section A. Section C contains the estimation procedure for the panel model with interactive fixed effects (IFEs) and slope heterogeneity, and proposes the test statistics for the slope homogeneity. Section D shows the uniform asymptotic theories for the slope estimators and test statistics proposed in Section C. Section E provides details of the algorithm for the nuclear norm regularized regressions.

A Proofs of the Main Results

A.1 Proof of Lemma 2.1

(i) Recall that $\mathcal{G}_j^{(\ell)} = \{G_{1,j}^{(\ell)}, \cdots, G_{K_\ell,j}^{(\ell)}\}$. For the special case when $\mathcal{G}_j^{(1)} = \mathcal{G}_j^{(2)}$ and $\alpha_{k,j}^{(1)} = \mu \alpha_{k,j}^{(2)}$ such that μ is a constant, the group structure does not change, the relative break size is the same for all groups, and $r_j = 1$. Except for this case, below we will show that $r_j = 2$.

Let $A_{j,i}^{(\ell)} = \sum_{k=1}^{K_\ell} \alpha_{k,j}^{(\ell)} \mathbf{1}\{i \in G_{k,j}^{(\ell)}\}$, $A_{j,i} = (A_{j,i}^{(1)}, A_{j,i}^{(2)})'$ and $A_j = (A_{j,1}, \cdots, A_{j,N})' \in \mathbb{R}^{N\times 2}$. Define the 2×2 symmetric matrix $B_j = A_j' A_j$. Let $B_j^{\frac{1}{2}}$ be the symmetric square root of B_j. By the singular value decomposition (SVD), $B_j^{\frac{1}{2}} \begin{bmatrix} \sqrt{\tau_T} & 0 \\ 0 & \sqrt{1-\tau_T} \end{bmatrix} = L_j S_j R_j'$, where $L_j' L_j = R_j' R_j = I_2$ and S_j is diagonal. Then

$$
\begin{aligned}
\Theta_j^0 &= \begin{bmatrix} A_{j,1}^{(1)} \iota_{T_1}' & A_{j,1}^{(2)} \iota_{T-T_1}' \\ \vdots & \vdots \\ A_{j,N}^{(1)} \iota_{T_1}' & A_{j,N}^{(2)} \iota_{T-T_1}' \end{bmatrix} = A_j \begin{bmatrix} \iota_{T_1} & \mathbf{0}_{T_1} \\ \mathbf{0}_{T-T_1} & \iota_{T-T_1} \end{bmatrix}' \\
&= A_j B_j^{-1/2} L_j S_j R_j' \begin{bmatrix} \frac{1}{\sqrt{\tau_T}} & 0 \\ 0 & \frac{1}{\sqrt{1-\tau_T}} \end{bmatrix} \begin{bmatrix} \iota_{T_1} & \mathbf{0}_{T_1} \\ \mathbf{0}_{T-T_1} & \iota_{T-T_1} \end{bmatrix}' \\
&= A_j B_j^{-\frac{1}{2}} L_j S_j R_j' \begin{bmatrix} \frac{1}{\sqrt{\tau_T}} \iota_{T_1} & \mathbf{0}_{T_1} \\ \mathbf{0}_{T-T_1} & \frac{1}{\sqrt{1-\tau_T}} \iota_{T-T_1} \end{bmatrix}' \\
&= \left(A_j B_j^{-\frac{1}{2}} L_j\right) \left(\sqrt{T} S_j\right) \left\{ \frac{1}{\sqrt{T}} R_j' \begin{bmatrix} \frac{1}{\sqrt{\tau_T}} \iota_{T_1} & \mathbf{0}_{T_1} \\ \mathbf{0}_{T-T_1} & \frac{1}{\sqrt{1-\tau_T}} \iota_{T-T_1} \end{bmatrix}' \right\} := \mathcal{U}_j \Sigma_j \mathcal{V}_j',
\end{aligned}
$$

where $\mathcal{U}_j = A_j B_j^{-\frac{1}{2}} L_j \in \mathbb{R}^{N\times 2}$, $\mathcal{V}_j = \frac{1}{\sqrt{T}} \begin{bmatrix} \frac{1}{\sqrt{\tau_T}} \iota_{T_1} & \mathbf{0}_{T_1} \\ \mathbf{0}_{T-T_1} & \frac{1}{\sqrt{1-\tau_T}} \iota_{T-T_1} \end{bmatrix} R_j \in \mathbb{R}^{T\times 2}$, and $\Sigma_j = \sqrt{T} S_j \in \mathbb{R}^{2\times 2}$. It is easy to verify that

$$\begin{aligned}
\mathcal{U}_j'\mathcal{U}_j &= L_j'B_j^{-\frac{1}{2}}A_j'A_jB_j^{-\frac{1}{2}}L_j = L_j'B_j^{-\frac{1}{2}}B_jB_j^{-\frac{1}{2}}L_j = L_j'L_j = I_2 \quad \text{and} \\
\mathcal{V}_j'\mathcal{V}_j &= R_j'R_j = I_2.
\end{aligned}$$

Now, let $U_j = \mathcal{U}_j\Sigma_j/\sqrt{T}$ and $V_j = \sqrt{T}\mathcal{V}_j$. We have $\Theta_j^0 = U_jV_j^\top$ and $V_j = \begin{bmatrix} \frac{1}{\sqrt{\tau_T}}\iota_{T_1} & \mathbf{0}_{T_1} \\ \mathbf{0}_{T-T_1} & \frac{1}{\sqrt{1-\tau_T}}\iota_{T-T_1} \end{bmatrix} R_j = D_jR_j$. This proves (i).

(ii) Given R_j is an orthonormal matrix, this follows from (i) automatically. ■

A.2 Proof of Theorem 4.1

A.2.1 Proof of Statement (i).

Let $\mathcal{R}(C_1) := \left\{\{\Delta_{\Theta_j}\}_{j\in[p]\cup\{0\}} \in \mathbb{R}^{N\times T} : \sum_{j\in[p]\cup\{0\}} \left\|\mathcal{P}_j^\perp(\Delta_{\Theta_j})\right\|_* \le C_1 \sum_{j\in[p]\cup\{0\}} \left\|\mathcal{P}_j(\Delta_{\Theta_j})\right\|_*\right\}$. By Lemma B.4, we notice that

$$\mathbb{P}\left\{\left\{\tilde{\Delta}_{\Theta_j}\right\}_{j\in[p]\cup\{0\}} \in \mathcal{R}(3)\right\} \to 1.$$

Recall from (4.1) that

$$\mathcal{R}(C_1, C_2) := \left\{(\{\Delta_{\Theta_j}\}_{j\in[p]\cup\{0\}}) : \sum_{j\in[p]\cup\{0\}} \left\|\mathcal{P}_j^\perp(\Delta_{\Theta_j})\right\|_* \le C_1 \sum_{j\in[p]\cup\{0\}} \left\|\mathcal{P}_j(\Delta_{\Theta_j})\right\|_*, \sum_{j\in[p]\cup\{0\}} \|\Theta_j\|^2 \ge C_2\sqrt{NT}\right\}.$$

When $\{\tilde{\Delta}_{\Theta_j}\}_{j\in[p]\cup\{0\}} \in \mathcal{R}(3)$ and $\{\tilde{\Delta}_{\Theta_j}\}_{j\in[p]\cup\{0\}} \notin \mathcal{R}(3, C_2)$, we have $\sum_{j\in[p]\cup\{0\}} \left\|\tilde{\Delta}_{\Theta_j}\right\|^2 < C_2\sqrt{NT}$, which gives

$$\frac{1}{\sqrt{NT}}\left\|\tilde{\Delta}_{\Theta_j}\right\| < \frac{C_2}{\sqrt{N\wedge T}}, \quad \forall j \in [p]\cup\{0\}.$$

So it suffices to focus on the case that $\{\tilde{\Delta}_{\Theta_j}\}_{j\in[p]\cup\{0\}} \in \mathcal{R}(3, C_2)$.

Define the event

$$\mathscr{A}_{1,N}(c_3) = \left\{\|E\|_{op} \le c_3(\sqrt{N} \vee \sqrt{T\log T}), \|X_j \odot E\|_{op} \le c_3(\sqrt{N} \vee \sqrt{T\log T}), \forall j \in [p]\right\}$$

for some positive constant c_3. By Lemma B.3, we have $\mathbb{P}(\mathscr{A}_{1,N}^c(c_3)) \le \epsilon$ for any $\epsilon > 0$. By the definition of $\{\tilde{\Theta}_j\}_{j\in[p]\cup\{0\}}$ in (3.1), we have

$$\begin{aligned}
\sum_{j\in[p]\cup\{0\}} \nu_j\left(\|\Theta_j^0\|_* - \left\|\tilde{\Theta}_j\right\|_*\right) &\ge \frac{1}{NT}\left\|Y - \tilde{\Theta}_0 - \sum_{j\in[p]} X_j \odot \tilde{\Theta}_j\right\|^2 - \frac{1}{NT}\left\|Y - \Theta_0^0 - \sum_{j\in[p]} X_j \odot \Theta_j^0\right\|^2 \\
&= \frac{1}{NT}\left\|\tilde{\Delta}_{\Theta_0} + \sum_{j\in[p]} \tilde{\Delta}_{\Theta_j} \odot X_j\right\|^2 - \frac{2}{NT}tr(\tilde{\Delta}_{\Theta_0}'E) - \frac{2}{NT}\sum_{j\in[p]} tr\left[\tilde{\Delta}_{\Theta_0}'(E \odot X_j)\right].
\end{aligned}$$

Then conditioning on the event $\mathscr{A}_{1,N}(c_3)$, we have

$$\begin{aligned}
&\frac{1}{NT}\left\|\tilde{\Delta}_{\Theta_0} + \sum_{j\in[p]} \tilde{\Delta}_{\Theta_j} \odot X_j\right\|^2 \\
&\le \frac{2}{NT}tr\left(\tilde{\Delta}_{\Theta_0}'E\right) + \frac{2}{NT}\sum_{j\in[p]} tr\left[\tilde{\Delta}_{\Theta_0}'(E \odot X_j)\right] + \sum_{j\in[p]\cup\{0\}} \nu_j\left(\|\Theta_j^0\|_* - \left\|\tilde{\Theta}_j\right\|_*\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq 2c_3 \frac{\sqrt{N} \vee \sqrt{T \log T}}{NT} \sum_{j \in [p] \cup \{0\}} \left\| \tilde{\Delta}_{\Theta_j} \right\|_* + \sum_{j \in [p] \cup \{0\}} \nu_j \left(\left\| \mathcal{P}_j(\tilde{\Delta}_{\Theta_j}) \right\|_* - \left\| \mathcal{P}_j^{\perp}(\tilde{\Delta}_{\Theta_j}) \right\|_* \right) \\
&= 2c_3 \frac{\sqrt{N} \vee \sqrt{T \log T}}{NT} \sum_{j \in [p] \cup \{0\}} \left(\left\| \mathcal{P}_j(\tilde{\Delta}_{\Theta_j}) \right\|_* + \left\| \mathcal{P}_j^{\perp}(\tilde{\Delta}_{\Theta_j}) \right\|_* \right) \\
&+ 4c_3 \frac{\sqrt{N} \vee \sqrt{T \log T}}{NT} \sum_{j \in [p] \cup \{0\}} \left(\left\| \mathcal{P}_j(\tilde{\Delta}_{\Theta_j}) \right\|_* - \left\| \mathcal{P}_j^{\perp}(\tilde{\Delta}_{\Theta_j}) \right\|_* \right) \\
&= 6c_3 \frac{\sqrt{N} \vee \sqrt{T \log T}}{NT} \sum_{j \in [p] \cup \{0\}} \left\| \mathcal{P}_j(\tilde{\Delta}_{\Theta_j}) \right\|_* - 2c_3 \frac{\sqrt{N} \vee \sqrt{T \log T}}{NT} \sum_{j \in [p] \cup \{0\}} \left\| \mathcal{P}_j^{\perp}(\tilde{\Delta}_{\Theta_j}) \right\|_* \\
&\leq 6c_3 \frac{\sqrt{N} \vee \sqrt{T \log T}}{NT} \sum_{j \in [p] \cup \{0\}} \left\| \mathcal{P}_j(\tilde{\Delta}_{\Theta_j}) \right\|_*, \qquad \text{(A.1)}
\end{aligned}
$$

where the second inequality holds by the definition of event $\mathscr{A}_1(c_3)$, the fact that $|\mathrm{tr}\,(AB)| \leq \|A\|_* \|B\|_{op}$, and (B.9), the first equality holds by the fact that $\left\| \tilde{\Delta}_{\Theta_j} \right\|_* = \left\| \mathcal{P}_j(\tilde{\Delta}_{\Theta_j}) \right\|_* + \left\| \mathcal{P}_j^{\perp}(\tilde{\Delta}_{\Theta_j}) \right\|_*$ (see, e.g., Lemma D.2(i) in Chernozhukov et al. (2020)) and that $\nu_j = \frac{4c_3(\sqrt{N} \vee \sqrt{T \log T})}{NT}$. It follows that

$$
\begin{aligned}
C_3 \sum_{j \in [p] \cup \{0\}} \left\| \tilde{\Delta}_{\Theta_j} \right\|^2 &\leq \left\| \tilde{\Delta}_{\Theta_0} + \sum_{j \in [p]} \tilde{\Delta}_{\Theta_j} \odot X_j \right\|^2 + C_4 (N+T) \\
&\leq 6c_3 \left(\sqrt{N} \vee \sqrt{T \log T} \right) \sum_{j \in [p] \cup \{0\}} \left\| \mathcal{P}_j(\tilde{\Delta}_{\Theta_j}) \right\|_* + C_4 (N+T) \\
&\leq 12\bar{r}c_3 \left(\sqrt{N} \vee \sqrt{T \log T} \right) \sum_{j \in [p] \cup \{0\}} \left\| \mathcal{P}_j(\tilde{\Delta}_{\Theta_j}) \right\| + C_4 (N+T) \\
&\leq 12\bar{r}c_3 \left(\sqrt{N} \vee \sqrt{T \log T} \right) \sum_{j \in [p] \cup \{0\}} \left\| \tilde{\Delta}_{\Theta_j} \right\| + C_4 (N+T) \\
&\leq 12\bar{r}c_3 \left(\sqrt{N} \vee \sqrt{T \log T} \right) \sqrt{p+1} \sqrt{\sum_{j \in [p] \cup \{0\}} \left\| \tilde{\Delta}_{\Theta_j} \right\|^2} + C_4 (N+T),
\end{aligned}
$$

where the first inequality holds by Assumption 4, the second inequality follows by (A.1), the third inequality is by the fact that $\left\| \mathcal{P}_j(\tilde{\Delta}_{\Theta_j}) \right\|_* \leq rank(\mathcal{P}_j(\tilde{\Delta}_{\Theta_j})) \left\| \mathcal{P}_j(\tilde{\Delta}_{\Theta_j}) \right\|$ with $rank(\mathcal{P}_j(\tilde{\Delta}_{\Theta_j})) \leq 2\bar{r}$ by Lemma D.2(ii) in Chernozhukov et al. (2020), the fourth inequality is by the fact that $\left\| \tilde{\Delta}_{\Theta_j} \right\| = \left\| \mathcal{P}_j(\tilde{\Delta}_{\Theta_j}) \right\| + \left\| \mathcal{P}_j^{\perp}(\tilde{\Delta}_{\Theta_j}) \right\|$ (see, e.g., Lemma D.2(ii) in Chernozhukov et al. (2020)), and the last inequality holds by Jensen inequality. Consequently, we can conclude that

$$
\sum_{j \in [p] \cup \{0\}} \left\| \tilde{\Delta}_{\Theta_j} \right\|^2 = O_p \left(N \vee (T \log T) \right).
$$

In addition, $\max_{k \in [r_j]} |\tilde{\sigma}_{k,j} - \sigma_{k,j}| = O_p(\eta_{N,1})$ by the Weyl's inequality with $\eta_{N,1} = \frac{\sqrt{\log T}}{\sqrt{N \wedge T}}$.

Now, we show the convergence rate for the singular vector estimates. For $\forall j \in [p] \cup \{0\}$, let $\tilde{D}_j = \frac{1}{NT} \tilde{\Theta}_j' \tilde{\Theta}_j = \hat{\tilde{\mathcal{V}}}_j \hat{\tilde{\Sigma}}_j \hat{\tilde{\mathcal{V}}}_j'$, and $D_j^0 = \frac{1}{NT} \Theta_j^{0\prime} \Theta_j^0 = \mathcal{V}_j^0 \Sigma_j^0 \mathcal{V}_j^{0\prime}$. Define the event

$$
\mathscr{A}_{2,N}(M) = \left\{ \frac{1}{\sqrt{NT}} \left\| \tilde{\Theta}_j - \Theta_j^0 \right\| \leq M \eta_{N,1}, \quad \forall j \in \{0, \cdots, p\} \right\}
$$

for a large enough constant M. By the above analyses, we have $\mathbb{P}\left(\mathscr{A}_{2,N}^c(M) \right) \leq \epsilon$ for any $\epsilon > 0$. On the event

$\mathscr{A}_{2,N}(M)$, we observe that

$$\left\|\tilde{D}_j - D_j^0\right\| \leq \frac{1}{NT}\left(\left\|\tilde{\Theta}_j\right\| + \|\Theta_j^0\|\right)\left\|\tilde{\Theta}_j - \Theta_j^0\right\| \leq 2M^2\eta_{N,1}.$$

With Lemma C.1 of Su et al. (2020) and Davis-Kahan sinΘ theorem in Yu et al. (2015), we are ready to show that for some orthogonal matrix O_j,

$$\begin{aligned}\left\|\mathcal{V}_j^0 - \hat{\tilde{\mathcal{V}}}_j O_j\right\| &\leq \sqrt{r_j}\left\|\mathcal{V}_j^0 - \hat{\tilde{\mathcal{V}}}_j O_j\right\|_{op} \leq \sqrt{r_j}\frac{2\sqrt{2}M^2\eta_{N,1}}{\sigma_{K_j,1}^2 - 2M^2\eta_{N,1}} \\ &\leq \sqrt{r_j}\frac{2\sqrt{2}M^2\eta_{N,1}}{c_\sigma^2 - 2M^2\eta_{N,1}} \leq \sqrt{r_j}\frac{2\sqrt{2}M^2\eta_{N,1}}{C_6 c_\sigma^2} \leq C_7\eta_{N,1}\end{aligned} \tag{A.2}$$

for $C_7 = \frac{2\sqrt{2}M^2\sqrt{\bar{r}}}{C_6 c_\sigma^2}$, where the second inequality in line two is due to the fact that $\eta_{N,1}$ is sufficiently small and C_6 is some positive constant.

Then $\left\|V_j^0 - \tilde{V}_j O_j\right\| \leq C_7\sqrt{T}\eta_{N,1}$ by the definition of $\tilde{V}_j$ and V_j. Together with the fact that $\mathbb{P}(\mathscr{A}_{2,N}^c(M)) \to 0$ by Theorem 4.1(i), it implies $\frac{1}{\sqrt{T}}\left\|V_j^0 - \tilde{V}_j O_j\right\| = O_p\left(T\eta_{N,1}\right)$.

A.2.2 Proof of Statement (ii).

Define

$$\begin{aligned}&u_i^0 = [u_{i,0}^{0\prime}, \cdots, u_{i,p}^{0\prime}]', \quad \dot{\Delta}_{i,j} = O_j'\dot{u}_{i,j} - u_{i,j}^0, \quad \dot{\Delta}_{i,u} = [\dot{\Delta}_{i,0}', \cdots, \dot{\Delta}_{i,p}']', \\ &\tilde{\phi}_{it} = \left[\left(O_0'\tilde{v}_{t,0}\right)', \left(O_1'\tilde{v}_{t,1}X_{1,it}\right)', \cdots, \left(O_p'\tilde{v}_{t,p}X_{p,it}\right)'\right]', \quad \text{and} \quad \tilde{\Phi}_i = \frac{1}{T}\sum_{t=1}^T \tilde{\phi}_{it}\tilde{\phi}_{it}'.\end{aligned}$$

Let $\tilde{Y}_{it} := Y_{it} - \left(O_0 u_{i,0}^0\right)'\tilde{v}_{t,0} - \sum_{j=1}^p \left(O_j u_{i,j}^0\right)'\tilde{v}_{t,j}X_{j,it}$. By the definition of $\{\dot{u}_{i,j}\}$ in (3.2), we have

$$\begin{aligned}0 &\geq \frac{1}{T}\sum_{t\in[T]}\left(Y_{it} - \dot{u}_{i,0}'\tilde{v}_{t,0} - \sum_{j=1}^p \dot{u}_{i,j}'\tilde{v}_{t,j}X_{j,it}\right)^2 - \frac{1}{T}\sum_{t\in[T]}\tilde{Y}_{it}^2 \\ &= \frac{1}{T}\sum_{t\in[T]}\left(\tilde{Y}_{it} - \left(\dot{u}_{i,0} - O_0 u_{i,0}^0\right)'\tilde{v}_{t,0} - \sum_{j\in[p]}\left(\dot{u}_{i,j} - O_j u_{i,j}^0\right)'\tilde{v}_{t,j}X_{j,it}\right)^2 - \frac{1}{T}\sum_{t\in[T]}\tilde{Y}_{it}^2 \\ &= \frac{1}{T}\sum_{t\in[T]}\left[\left(\dot{\Delta}_{i,u}'\tilde{\phi}_{it}\right)^2 - 2\left(\dot{\Delta}_{i,u}'\tilde{\phi}_{it}\right)\left(Y_{it} - u_i^{0\prime}\tilde{\phi}_{it}\right)\right],\end{aligned}$$

which implies

$$\begin{aligned}&\left\|\dot{\Delta}_{i,u}\right\|_2^2 \lambda_{\min}\left(\frac{1}{T}\sum_{t\in[T]}\tilde{\phi}_{it}\tilde{\phi}_{it}'\right) \leq \frac{1}{T}\sum_{t\in[T]}\left(\dot{\Delta}_{i,u}'\tilde{\phi}_{it}\right)^2 \leq \frac{2}{T}\sum_{t\in[T]}\dot{\Delta}_{i,u}'\tilde{\phi}_{it}\left(Y_{it} - u_i^{0\prime}\tilde{\phi}_{it}\right) \\ &= \frac{2}{T}\sum_{t\in[T]}\dot{\Delta}_{i,u}'\tilde{\phi}_{it}\left[e_{it} - u_i^{0\prime}\left(\tilde{\phi}_{it} - \phi_{it}^0\right)\right] \\ &= 2\left\{\frac{1}{T}\sum_{t\in[T]}\phi_{it}^0 e_{it}\right\}'\dot{\Delta}_{i,u} + 2\left\{\frac{1}{T}\sum_{t\in[T]}\left(\tilde{\phi}_{it} - \phi_{it}^0\right)e_{it}\right\}'\dot{\Delta}_{i,u} - \frac{2}{T}\sum_{t\in[T]}\tilde{\phi}_{it}'\dot{\Delta}_{i,u}\left[u_i^{0\prime}\left(\tilde{\phi}_{it} - \phi_{it}^0\right)\right] \\ &=: 2G_{i,1}'\dot{\Delta}_{i,u} + 2G_{i,2} - 2G_{i,3}.\end{aligned} \tag{A.3}$$

We first deal with $G_{1,i}$. Conditional on $\mathscr{D}$, the randomness in $G_{1,i}$ comes from $\{e_{it}, X_{it}\}_{t\in[T]}$, which is the (conditional) strong mixing sequence by Assumption 1(iii). Besides, we observe that

$$\max_{i],t\in[T]} \left\|Var\left(\phi_{it}^0 e_{it}\middle|\mathscr{D}\right)\right\| \lesssim \max_{i,t}\left[\mathbb{E}\left(e_{it}^2\middle|\mathscr{D}\right)+\sum_{j\in[p]}\mathbb{E}\left(X_{j,it}^2 e_{it}^2\middle|\mathscr{D}\right)\right]=O_{a.s.}(1)$$

by Lemma B.7(ii) and Assumption 1(v). Following similar arguments, we have

$$\begin{aligned}
&\max_{i,t}\sum_{s=t+1}^{T}\left\|Cov\left(\phi_{it}^0 e_{it},\phi_{is}^0 e_{is}\middle|\mathscr{D}\right)\right\| \\
&\leq 8\max_{t}\sum_{s=t+1}^{T}\left[\mathbb{E}\left(\left\|\phi_{it}^0 e_{it}\right\|_2^q\middle|\mathscr{D}\right)\right]^{1/q}\left[\left\|\mathbb{E}\left(\phi_{is}^0 e_{is}\right\|_2^q\middle|\mathscr{D}\right)\right]^{1/q}(\alpha(t-s))^{1-2/q}=O_{a.s.}(1),
\end{aligned}$$

where the first inequality is by the conditional Davydov's inequality that says

$$\left|Cov\left[a\left(x_t\right),a\left(x_s\right)|\mathscr{D}\right]\right|\leq 8\left[\mathbb{E}[\|a\left(x_t\right)\|^p|\mathscr{D}]\right]^{\frac{1}{p}}\left[\mathbb{E}[\|a\left(x_s\right)\|^q|\mathscr{D}]\right]^{\frac{1}{q}}\alpha(t-s)^{\frac{1}{r}}$$

for any conditional strong mixing sequence $(x_t, t\in[T])$ with mixing coefficient $\alpha(\cdot)$ and $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1$. See Lemma A.4 in Su and Chen (2013).

Following this, for some constant C_8, we have

$$\max_{i,t}\left[\left\|Var\left(\phi_{it}^0 e_{it}\middle|\mathscr{D}\right)\right\|+2\sum_{s=t+1}^{T}\left\|Cov\left(\phi_{it}^0 e_{it},\phi_{is}^0 e_{is}\middle|\mathscr{D}\right)\right\|\right]\leq C_8 \text{ a.s.},$$

and $\max_{i,t}\left\|\phi_{it}^0 e_{it}\right\|_{\max}\leq C_8(NT)^{1/q}$ by Lemma B.7(i) and Assumption 1(iv). Define $\mathscr{A}_{3,N}(M)=\{\max_{i,t}\left\|\phi_{it}^0 e_{it}\right\|\leq M(NT)^{1/q}\}$ and $\mathscr{A}_{3,N,i}(M)=\{\max_{t}\left\|\phi_{it}^0 e_{it}\right\|\leq M(NT)^{1/q}\}$ for a large enough constant M. For a positive constant C_9, it yields that

$$\begin{aligned}
&\mathbb{P}\left(\max_i\frac{1}{T}\left\|\sum_{t\in[T]}\phi_{it}^0 e_{it}\right\|>C_9\sqrt{\frac{\log N}{T}}(NT)^{\frac{1}{q}}\right) \\
&\leq \mathbb{P}\left(\max_i\frac{1}{T}\left\|\sum_{t\in[T]}\phi_{it}^0 e_{it}\right\|>C_9\sqrt{\frac{\log N}{T}}(NT)^{\frac{1}{q}},\mathscr{A}_{3,N}(M)\right)+\mathbb{P}\left(\mathscr{A}_{3,N}^c(M)\right) \\
&\leq \sum_{i\in[N]}\mathbb{P}\left(\frac{1}{T}\left\|\sum_{t\in[T]}\phi_{it}^0 e_{it}\right\|>C_9\sqrt{\frac{\log N}{T}}(NT)^{\frac{1}{q}},\mathscr{A}_{3,N}(M)\right)+\mathbb{P}\left(\mathscr{A}_{3,N}^c(M)\right) \\
&\leq \sum_{i\in[N]}\mathbb{P}\left(\frac{1}{T}\left\|\sum_{t\in[T]}\phi_{it}^0 e_{it}\right\|>C_9\sqrt{\frac{\log N}{T}}(NT)^{\frac{1}{q}},\mathscr{A}_{3,N,i}(M)\right)+\mathbb{P}\left(\mathscr{A}_{3,N}^c(M)\right) \\
&\leq \sum_{i\in[N]}\exp\left(-\frac{c_4 C_9^2 T\log N(NT)^{2/q}}{TC_8+C_8^2(NT)^{2/q}+C_8C_9(NT)^{2/q}\sqrt{T\log N}(\log T)^2}\right)+o(1)=o(1)
\end{aligned}$$

where the last inequality holds by Bernstein's inequality in Lemma B.6(ii) and the fact that $\mathbb{P}\left(\mathscr{A}_{3,N}^c(M)\right)=o(1)$. It follows that

$$\max_i\frac{\left|G_{i,1}'\dot{\Delta}_{i,u}\right|}{\left\|\dot{\Delta}_{i,u}\right\|}\leq\max_i\|G_{i,1}\|=O_p(\sqrt{(\log N)/T}(NT)^{\frac{1}{q}}). \tag{A.4}$$

For $G_{i,2}$, we notice that

$$\max_i \frac{|G_{i,2}|}{\left\|\dot{\Delta}_{i,u}\right\|} = \max_i \frac{\left|\left\{\frac{1}{T}\sum_{t\in[T]}\left(\tilde{\phi}_{it}-\phi_{it}^0\right)e_{it}\right\}'\dot{\Delta}_{i,u}\right|}{\left\|\dot{\Delta}_{i,u}\right\|} \leq \max_i \left\|\frac{1}{T}\sum_{t\in[T]}\left(\tilde{\phi}_{it}-\phi_{it}^0\right)e_{it}\right\|_2$$

$$\leq \max_i \sqrt{\frac{1}{T}\sum_{t\in[T]}\left\|\tilde{\phi}_{it}-\phi_{it}^0\right\|_2^2}\max_i \sqrt{\frac{1}{T}\sum_{t\in[T]}|e_{it}|^2} = O_p(\eta_{N,1}(NT)^{1/q}), \tag{A.5}$$

where the second inequality holds by Cauchy's inequality and the last equality is by Lemma B.7(iv) and Assumption 1(iv)

For $G_{i,3}$, we have

$$\max_i \frac{|G_{i,3}|}{\left\|\dot{\Delta}_{i,u}\right\|} = \max_i \frac{\left|\frac{1}{T}\sum_{t\in[T]}\tilde{\phi}_{it}'\dot{\Delta}_{i,u}\left[u_i^{0\prime}\left(\tilde{\phi}_{it}-\phi_{it}^0\right)\right]\right|}{\left\|\dot{\Delta}_{i,u}\right\|}$$

$$\leq \max_i \sqrt{\frac{1}{T}\sum_{t\in[T]}\left\|\tilde{\phi}_{it}\right\|^2}\max_i \left\|u_i^0\right\| \max_{i,t}\sqrt{\frac{1}{T}\sum_{t\in[T]}\left\|\tilde{\phi}_{it}-\phi_{it}^0\right\|^2} = O_p(\eta_{N,1}(NT)^{1/q}), \tag{A.6}$$

where the inequality holds by Cauchy's inequality and the last line is by Lemma B.7(i) and (iv).

Combining (A.3)-(A.6) and Lemma B.8 yields

$$\max_i \left\|\dot{u}_{i,j} - O_{i,j}^{(1)}u_{i,j}^0\right\| \leq \max_i \left\|\dot{\Delta}_{i,u}\right\| = O_p\left(\sqrt{\frac{\log N \vee T}{N\wedge T}}(NT)^{1/q}\right).$$

The union bound of $\dot{v}_{t,j}$ can be obtained in the same manner and we sketch the proof here. Define

$$v_t^0 = [v_{t,0}^{0\prime},\cdots,v_{t,p}^{0\prime}]', \quad \dot{\Delta}_{t,j} = O_j'\dot{v}_{t,j} - v_{t,j}^0, \quad \dot{\Delta}_{t,v} = [\dot{\Delta}_{t,0}',\cdots,\dot{\Delta}_{t,p}']',$$
$$\dot{\psi}_{it} = \left[(O_0'\dot{u}_{i,0})', (O_1'\dot{u}_{i,1}X_{1,it})', \cdots, (O_p'\dot{u}_{i,p}X_{p,it})'\right]', \quad \text{and} \quad \dot{\Psi}_t = \frac{1}{N}\sum_{i\in[N]}\dot{\psi}_{it}\dot{\psi}_{it}'.$$

Following the steps to derive (A.3), we can also obtain

$$\left\|\dot{\Delta}_{t,v}\right\|^2 \lambda_{\min}\left(\frac{1}{N}\sum_{i\in[N]}\dot{\psi}_{it}\dot{\psi}_{it}'\right)$$
$$= 2\left\{\frac{1}{N}\sum_{i\in[N]}\psi_{it}^0 e_{it}\right\}'\dot{\Delta}_{t,v} + \frac{2}{N}\sum_{i\in[N]}\left(\dot{\psi}_{it}-\psi_{it}^0\right)'\dot{\Delta}_{t,v}e_{it} - \frac{2}{N}\sum_{i\in[N]}\dot{\psi}_{it}'\dot{\Delta}_{t,v}\left[v_t^{0\prime}\left(\dot{\psi}_{it}-\psi_{it}^0\right)\right]. \tag{A.7}$$

By the fact that

$$\max_t \frac{1}{N}\sum_{i\in[N]}\left\|\dot{\psi}_{it}\right\|^2 = \max_t \frac{1}{N}\sum_{i\in[N]}\left(\|\dot{u}_{i,0}\|^2 + \sum_{j\in[p]}\|\dot{u}_{i,j}\|^2 X_{j,it}^2\right)$$
$$\leq \max_i \|\dot{u}_{i,0}\|^2 + \max_{i\in[N],j\in[p]}\|\dot{u}_{i,j}\|^2\sum_{j\in[p]}\max_t \frac{1}{N}\sum_{i\in[N]}X_{j,it}^2 = O_p(1),$$

$$\max_t \frac{1}{N}\sum_{i\in[N]}\left\|\dot\psi_{it}-\psi_{it}^0\right\|^2 \le \max_i \left\|\dot u_{i,0}-u_{i,0}^0\right\|^2 + \max_{i\in[N],j\in[p]}\left\|\dot u_{i,j}-u_{i,j}^0\right\|^2 \sum_{j\in[p]}\max_t \frac{1}{N}\sum_{i\in[N]} X_{j,it}^2 = O_p(\eta_{N,2}),$$

where $\eta_{N,2}=\sqrt{\frac{\log N\vee T}{N\wedge T}}(NT)^{1/q}$ and the first inequality holds by Lemma B.7(i), we obtain that

$$\max_t \frac{\left|\left\{\frac{1}{N}\sum_{i\in[N]}\psi_{it}^0 e_{it}\right\}'\dot\Delta_{t,v}\right|}{\left\|\dot\Delta_{t,v}\right\|} = O_p\left(\sqrt{\frac{\log T}{N}}(NT)^{\frac{1}{q}}\right),$$

$$\max_t \frac{\left|\frac{1}{N}\sum_{i\in[N]}\dot\psi_{it}'\dot\Delta_{t,v}\left[v_t^{0\prime}\left(\dot\psi_{it}-\psi_{it}^0\right)\right]\right|}{\left\|\dot\Delta_{t,v}\right\|} = O_p(\eta_{N,2}), \text{ and}$$

$$\max_t \frac{\left|\frac{1}{N}\sum_{i\in[N]}\left(\dot\psi_{it}-\psi_{it}^0\right)'\dot\Delta_{t,v}e_{it}\right|}{\left\|\dot\Delta_{t,v}\right\|} = O_p(\eta_{N,2}),$$

where the first line is by conditional Bernstein's inequality for i.i.d. sequence and the last two lines are by the analogous arguments in (A.5) and (A.6). It follows that

$$\max_t \left\|\dot v_{t,j}-O_j v_{t,j}^0\right\| = O_p\left(\sqrt{\frac{\log N\vee T}{N\wedge T}}(NT)^{1/q}\right). \quad \blacksquare$$

A.2.3 Proof of Statement (iii).

For $\forall j\in[p]\cup\{0\}$, $i\in[N]$ and $t\in[T]$, we can show that

$$\begin{aligned}\dot\Theta_{j,it}-\Theta_{j,it}^0 &= \dot u_{i,j}'\dot v_{t,j} - u_{i,j}^{0\prime}v_{t,j}^0\\ &= (\dot u_{i,j}-O_j u_{i,j}^0)'(\dot v_{t,j}-O_j v_{t,j}^0) + O_j u_{i,j}^{0\prime}(\dot v_{t,j}-O_j v_{t,j}^0) + O_j v_{t,j}^{0\prime}(\dot u_{i,j}-O_j u_{i,j}^0),\end{aligned}$$

which implies

$$\begin{aligned}\max_{i,t}\left|\dot\Theta_{j,it}-\Theta_{j,it}^0\right| &\le \max_i\left\|\dot u_{i,j}-O_j u_{i,j}^0\right\|\max_t\left\|\dot v_{t,j}-O_j v_{t,j}^0\right\|\\ &\quad + \max_i\left\|O_j u_{i,j}^0\right\|\max_t\left\|\dot v_{t,j}-O_j v_{t,j}^0\right\| + \max_i\left\|\dot u_{i,j}-O_j u_{i,j}^0\right\|\max_t\left\|O_j v_{t,j}^0\right\| = O_p(\eta_{N,2}),\end{aligned}$$

where the last equality combines results from Theorem 4.1(ii) and Lemma B.7(i). $\blacksquare$

A.3 Proof of Theorem 4.2

To prove $\mathbb{P}(\hat T_1 = T_1)\to 1$, it suffices to show: (i) $\mathbb{P}(\hat T_1 < T_1)\to 0$ and (ii) $\mathbb{P}(\hat T_1 > T_1)\to 0$.

First, we focus on (i). Let $\Delta_{it}(j) = \dot\Theta_{j,it}-\Theta_{j,it}^0$, $\bar\Delta_{s,i}(j) = \frac{1}{s}\sum_{t=1}^{s}(\dot\Theta_{j,it}-\Theta_{j,it}^0)$ and $\bar\Delta_{s+,i}(j) = \frac{1}{T-s}\sum_{t=s+1}^{T}(\dot\Theta_{j,it}-\Theta_{j,it}^0)$. When $s<T_1$, we have

$$\bar{\dot\Theta}_{j,i}^{(1s)} = \frac{1}{s}\sum_{t=1}^{s}\dot\Theta_{j,it} = \frac{1}{s}\sum_{t=1}^{s}\left[\Theta_{j,it}^0 + (\dot\Theta_{j,it}-\Theta_{j,it}^0)\right] = \alpha_{g_i^{(1)},j}^{(1)} + \bar\Delta_{s,i}(j),$$

$$\bar{\dot\Theta}_{j,i}^{(2s)} = \frac{1}{T-s}\sum_{t=s+1}^{T}\dot\Theta_{j,it} = \frac{1}{T-s}\sum_{t=s+1}^{T}\left[\Theta_{j,it}^0 + (\dot\Theta_{j,it}-\Theta_{j,it}^0)\right]$$

$$= \frac{T_1 - s}{T - s}\alpha^{(1)}_{g_i^{(1)},j} + \frac{T - T_1}{T - s}\alpha^{(2)}_{g_i^{(2)},j} + \bar{\Delta}_{s_+,i}(j),$$

with $\alpha^{(1)}_{g_i^{(1)},j}$ and $\alpha^{(2)}_{g_i^{(2)},j}$ being the j-th element of $\alpha^{(1)}_{g_i^{(1)}}$ and $\alpha^{(2)}_{g_i^{(2)}}$, respectively. It yields

$$\begin{aligned}
\dot{\Theta}_{j,it} - \bar{\dot{\Theta}}^{(1s)}_{j,i} =& \dot{\Theta}_{j,it} - \alpha^{(1)}_{g_i^{(1)},j} - \bar{\Delta}_{s,i}(j) = \Delta_{it}(j) - \bar{\Delta}_{s,i}(j), \quad t \le s, \text{ and} \\
\dot{\Theta}_{j,it} - \bar{\dot{\Theta}}^{(2s)}_{j,i} =& \dot{\Theta}_{j,it} - \frac{T_1 - s}{T - s}\alpha^{(1)}_{g_i^{(1)},j} - \frac{T - T_1}{T - s}\alpha^{(2)}_{g_i^{(2)},j} - \bar{\Delta}_{s_+,i}(j) \\
=& \begin{cases} \frac{T-T_1}{T-s}(\alpha^{(1)}_{g_i^{(1)},j} - \alpha^{(2)}_{g_i^{(2)},j}) + \Delta_{it}(j) - \bar{\Delta}_{s_+,i}(j) & \text{if } s+1 \le t \le T_1 \\ \frac{T_1-s}{T-s}(\alpha^{(2)}_{g_i^{(2)},j} - \alpha^{(1)}_{g_i^{(1)},j}) + \Delta_{it}(j) - \bar{\Delta}_{s_+,i}(j) & \text{if } T_1+1 \le t \le T \end{cases}.
\end{aligned}$$

Then, we have $\sum_{t=1}^{s}\left[\dot{\Theta}_{j,it} - \bar{\dot{\Theta}}^{(1s)}_{j,i}\right]^2 = \sum_{t=1}^{s}\left[\Delta_{it}(j) - \bar{\Delta}_{s,i}(j)\right]^2$, and

$$\begin{aligned}
\sum_{t=s+1}^{T}\left[\dot{\Theta}_{j,it} - \bar{\dot{\Theta}}^{(2s)}_{j,i}\right]^2 =& \sum_{t=s+1}^{T_1}\left[\frac{T - T_1}{T - s}(\alpha^{(1)}_{g_i^{(1)},j} - \alpha^{(2)}_{g_i^{(2)},j}) + \Delta_{it}(j) - \bar{\Delta}_{s_+,i}(j)\right]^2 \\
&+ \sum_{t=T_1+1}^{T}\left[\frac{T_1 - s}{T - s}(\alpha^{(2)}_{g_i^{(2)},j} - \alpha^{(1)}_{g_i^{(1)},j}) + \Delta_{it}(j) - \bar{\Delta}_{s_+,i}(j)\right]^2 \\
=& \frac{(T_1 - s)(T - T_1)^2}{(T - s)^2}(\alpha^{(1)}_{g_i^{(1)},j} - \alpha^{(2)}_{g_i^{(2)},j})^2 + \sum_{t=s+1}^{T_1}\left[\Delta_{it}(j) - \bar{\Delta}_{s_+,i}(j)\right]^2 \\
&+ 2\frac{T - T_1}{T - s}(\alpha^{(1)}_{g_i^{(1)},j} - \alpha^{(2)}_{g_i^{(2)},j})\sum_{t=s+1}^{T_1}\left[\Delta_{it}(j) - \bar{\Delta}_{s_+,i}(j)\right] \\
&+ \frac{(T - T_1)(T_1 - s)^2}{(T - s)^2}(\alpha^{(2)}_{g_i^{(2)},j} - \alpha^{(1)}_{g_i^{(1)},j})^2 + \sum_{t=T_1+1}^{T}\left[\Delta_{it}(j) - \bar{\Delta}_{s_+,i}(j)\right]^2 \\
&+ 2\frac{T_1 - s}{T - s}(\alpha^{(2)}_{g_i^{(2)},j} - \alpha^{(1)}_{g_i^{(1)},j})\sum_{t=T_1+1}^{T}\left[\Delta_{it}(j) - \bar{\Delta}_{s_+,i}(j)\right] \\
=& \frac{(T_1 - s)(T - T_1)}{T - s}(\alpha^{(1)}_{g_i^{(1)},j} - \alpha^{(2)}_{g_i^{(2)},j})^2 + \sum_{t=s+1}^{T}\left[\Delta_{it}(j) - \bar{\Delta}_{s_+,i}(j)\right]^2 \\
&+ 2\frac{T - T_1}{T - s}(\alpha^{(1)}_{g_i^{(1)},j} - \alpha^{(2)}_{g_i^{(2)},j})\sum_{t=s+1}^{T_1}\left[\Delta_{it}(j) - \bar{\Delta}_{s_+,i}(j)\right] \\
&+ 2\frac{T_1 - s}{T - s}(\alpha^{(2)}_{g_i^{(2)},j} - \alpha^{(1)}_{g_i^{(1)},j})\sum_{t=T_1+1}^{T}\left[\Delta_{it}(j) - \bar{\Delta}_{s_+,i}(j)\right]. \qquad \text{(A.8)}
\end{aligned}$$

Define $L(s) = \frac{1}{pNT}\sum_{j\in[p]}\sum_{i\in[N]}\{\sum_{t=1}^{s}[\dot{\Theta}_{j,it} - \bar{\dot{\Theta}}^{(1s)}_{j,i}]^2 + \sum_{t=s+1}^{T}[\dot{\Theta}_{j,it} - \bar{\dot{\Theta}}^{(2s)}_{j,i}]^2\}$. Then we have

$$\begin{aligned}
L(s) =& \frac{1}{pNT}\sum_{j\in[p]}\sum_{i\in[N]}\frac{(T_1 - s)(T - T_1)}{T - s}(\alpha^{(1)}_{g_i^{(1)},j} - \alpha^{(2)}_{g_i^{(2)},j})^2 \\
&+ \frac{1}{pNT}\sum_{j\in[p]}\sum_{i\in[N]}\sum_{t=1}^{s}\left[\Delta_{it}(j) - \bar{\Delta}_{s,i}(j)\right]^2 + \frac{1}{pNT}\sum_{j\in[p]}\sum_{i\in[N]}\sum_{t=s+1}^{T}\left[\Delta_{it}(j) - \bar{\Delta}_{s_+,i}(j)\right]^2 \\
&+ \frac{2}{pNT}\sum_{j\in[p]}\sum_{i\in[N]}\frac{T - T_1}{T - s}(\alpha^{(1)}_{g_i^{(1)},j} - \alpha^{(2)}_{g_i^{(2)},j})\sum_{t=s+1}^{T_1}\left[\Delta_{it}(j) - \bar{\Delta}_{s_+,i}(j)\right]
\end{aligned}$$

$$+\frac{2}{pNT}\sum_{j\in[p]}\sum_{i\in[N]}\frac{T_1-s}{T-s}(\alpha^{(2)}_{g_i^{(2)},j}-\alpha^{(1)}_{g_i^{(1)},j})\sum_{t=T_1+1}^{T}\left[\Delta_{it}(j)-\bar{\Delta}_{s_+,i}(j)\right]:=\sum_{\ell=1}^{5}L_\ell(s). \tag{A.9}$$

Obviously,

$$L(T_1)=L_2(T_1)+L_3(T_1). \tag{A.10}$$

Note that the event $\hat{T}_1<T$ implies that there exists an $s<T_1$ such that $L(s)-L(T_1)<0$, which means we can prove (i) by showing that $\mathbb{P}\left(\exists s<T_1,L(s)-L(T_1)<0\right)\to 0$. By (A.9) and (A.10), we observe that

$$\begin{aligned}L(s)-L(T_1)&=L_1(s)+[L_2(s)-L_2(T_1)]+[L_3(s)-L_3(T_1)]+L_4(s)+L_5(s)\\&:=A_1\left(s\right)+A_2\left(s\right)+A_3\left(s\right)+A_4\left(s\right)+A_5\left(s\right).\end{aligned} \tag{A.11}$$

Recall that $\eta_{N,2}=\sqrt{\frac{\log N\vee T}{N\wedge T}}(NT)^{1/q}$. Let $\frac{T_1-s}{T}=\kappa_s$ and note that $0<\frac{1}{T}\leq\kappa_s\leq\frac{T_1-2}{T}\asymp 1$. We analyze the five terms in (A.11) in turn.

For $A_1\left(s\right)$, we have

$$\begin{aligned}A_1\left(s\right)&=\frac{1}{pNT}\sum_{j\in[p]}\sum_{i\in[N]}\frac{\left(T_1-s\right)\left(T-T_1\right)}{T-s}(\alpha^{(1)}_{g_i^{(1)},j}-\alpha^{(2)}_{g_i^{(2)},j})^2\\&=\frac{T_1-s}{T-s}\left(1-\tau_T\right)\frac{1}{pN}\sum_{j\in[p]}\sum_{i\in[N]}(\alpha^{(1)}_{g_i^{(1)},j}-\alpha^{(2)}_{g_i^{(2)},j})^2\\&=\frac{T_1-s}{T-s}\left(1-\tau_T\right)\frac{1}{pN}\sum_{i\in[N]}\left\|\alpha^{(1)}_{g_i^{(1)}}-\alpha^{(2)}_{g_i^{(2)}}\right\|^2\\&=\kappa_s\frac{\left(1-\tau_T\right)}{1-\frac{s}{T}}D_{N\alpha}=\kappa_s O(\zeta^2_{NT}),\end{aligned} \tag{A.12}$$

where $D_{N\alpha}:=\frac{1}{pN}\sum_{i\in[N]}\left\|\alpha^{(1)}_{g_i^{(1)}}-\alpha^{(2)}_{g_i^{(2)}}\right\|^2$ and the last equality holds by Assumption 6.

For $A_2\left(s\right)$, noting that

$$\bar{\Delta}_{T_1,i}\left(j\right)-\bar{\Delta}_{s,i}(j)=\frac{1}{T_1}\sum_{t=1}^{T_1}\Delta_{it}(j)-\frac{1}{s}\left[\sum_{t=1}^{T_1}\Delta_{it}(j)-\sum_{t=s+1}^{T_1}\Delta_{it}(j)\right]=\frac{s-T_1}{T_1 s}\sum_{t=1}^{T_1}\Delta_{it}(j)+\frac{1}{s}\sum_{t=s+1}^{T_1}\Delta_{it}(j),$$

we have

$$\begin{aligned}T_1\bar{\Delta}^2_{T_1,i}\left(j\right)-s\bar{\Delta}^2_{s,i}\left(j\right)&=\left(T_1-s\right)\bar{\Delta}^2_{T_1,i}\left(j\right)+s\left[\bar{\Delta}^2_{T_1,i}\left(j\right)-\bar{\Delta}^2_{s,i}\left(j\right)\right]\\&=\left(T_1-s\right)\bar{\Delta}^2_{T_1,i}\left(j\right)+s\left[\bar{\Delta}_{T_1,i}\left(j\right)+\bar{\Delta}_{s,i}(j)\right]\left[\bar{\Delta}_{T_1,i}\left(j\right)-\bar{\Delta}_{s,i}(j)\right]\\&=\left(T_1-s\right)\bar{\Delta}^2_{T_1,i}\left(j\right)+\left[\bar{\Delta}_{T_1,i}\left(j\right)+\bar{\Delta}_{s,i}(j)\right]\left[\sum_{t=s+1}^{T_1}\Delta_{it}(j)-\frac{T_1-s}{T_1}\sum_{t=1}^{T_1}\Delta_{it}(j)\right].\end{aligned}$$

It follows that

$$\begin{aligned}A_2\left(s\right)&=\frac{1}{pNT}\sum_{j\in[p]}\sum_{i\in[N]}\left\{\sum_{t=1}^{s}\left[\Delta_{it}(j)-\bar{\Delta}_{s,i}(j)\right]^2-\sum_{t=1}^{T_1}\left[\Delta_{it}(j)-\bar{\Delta}_{T_1,i}\left(j\right)\right]^2\right\}\\&=\frac{1}{pNT}\sum_{j\in[p]}\sum_{i\in[N]}\left\{\sum_{t=1}^{s}\Delta^2_{it}\left(j\right)-s\bar{\Delta}^2_{s,i}\left(j\right)-\sum_{t=1}^{T_1}\Delta^2_{it}\left(j\right)+T_1\bar{\Delta}^2_{T_1,i}\left(j\right)\right\}\\&=\frac{1}{pNT}\sum_{j\in[p]}\sum_{i\in[N]}\left\{-\sum_{t=s+1}^{T_1}\Delta^2_{it}\left(j\right)+T_1\bar{\Delta}^2_{T_1,i}\left(j\right)-s\bar{\Delta}^2_{s,i}\left(j\right)\right\}\end{aligned}$$

$$
\begin{aligned}
&= -\kappa_s \frac{1}{pN\left(T_1-s\right)} \sum_{j\in[p]} \sum_{i\in[N]} \sum_{t=s+1}^{T_1} \Delta_{it}^2\left(j\right) + \kappa_s \frac{1}{pN} \sum_{j\in[p]} \sum_{i\in[N]} \bar{\Delta}_{T_1,i}^2\left(j\right) \\
&+ \kappa_s \frac{1}{pN} \sum_{j\in[p]} \sum_{i\in[N]} \left[\bar{\Delta}_{T_1,i}\left(j\right) + \bar{\Delta}_{s,i}(j)\right] \frac{1}{T_1-s} \sum_{t=s+1}^{T_1} \Delta_{it}(j) \\
&- \kappa_s \frac{1}{pN} \sum_{j\in[p]} \sum_{i\in[N]} \left[\bar{\Delta}_{T_1,i}\left(j\right) + \bar{\Delta}_{s,i}(j)\right] \frac{1}{T_1} \sum_{t=1}^{T_1} \Delta_{it}(j) \\
&= \kappa_s O_p\left(\eta_{N,2}^2\right) = \kappa_s o_p(\zeta_{NT}^2), \qquad \text{(A.13)}
\end{aligned}
$$

where the second last equality holds by the fact that

$$
\max_{i\in[N],t\in[T],j\in[p]} \left|\Delta_{it}(j)\right| = O_p\left(\eta_{N,2}\right) \qquad \text{(A.14)}
$$

from Theorem 4.1(iii) and

$$
\begin{aligned}
\max_{i\in[N],j\in[p]} \left|\bar{\Delta}_{s,i}(j)\right| &= \max_{i\in[N],j\in[p]} \left|\frac{1}{s}\sum_{t=1}^{s}\Delta_{it}(j)\right| \leq \max_{i\in[N],t\in[T],j\in[p]} \left|\Delta_{it}(j)\right| = O_p\left(\eta_{N,2}\right), \\
\max_{i\in[N],j\in[p]} \left|\bar{\Delta}_{T_1,i}\left(j\right)\right| &= \max_{i\in[N],j\in[p]} \left|\frac{1}{T_1}\sum_{t=1}^{T_1}\Delta_{it}(j)\right| = O_p\left(\eta_{N,2}\right). \qquad \text{(A.15)}
\end{aligned}
$$

Similarly, noting that

$$
\begin{aligned}
\bar{\Delta}_{T_{1+},i}\left(j\right) - \bar{\Delta}_{s_+,i}(j) &= \frac{1}{T-T_1}\sum_{t=T_1+1}^{T}\Delta_{it}(j) - \frac{1}{T-s}\sum_{t=s+1}^{T}\Delta_{it}(j) \\
&= \left(\frac{1}{T-T_1} - \frac{1}{T-s}\right)\sum_{t=T_1+1}^{T}\Delta_{it}(j) + \frac{1}{T-s}\left[\sum_{t=T_1+1}^{T}\Delta_{it}(j) - \sum_{t=s+1}^{T}\Delta_{it}(j)\right] \\
&= \frac{T_1-s}{\left(T-T_1\right)\left(T-s\right)}\sum_{t=T_1+1}^{T}\Delta_{it}(j) - \frac{1}{T-s}\sum_{t=s+1}^{T_1}\Delta_{it}(j)
\end{aligned}
$$

and

$$
\begin{aligned}
&\left(T-T_1\right)\bar{\Delta}_{T_{1+},i}^2\left(j\right) - \left(T-s\right)\bar{\Delta}_{s_+,i}^2\left(j\right) \\
&= \left(s-T_1\right)\bar{\Delta}_{T_{1+},i}^2\left(j\right) + \left(T-s\right)\left[\bar{\Delta}_{T_{1+},i}^2\left(j\right) - \bar{\Delta}_{s_+,i}^2\left(j\right)\right] \\
&= \left(s-T_1\right)\bar{\Delta}_{T_{1+},i}^2\left(j\right) + \left(T-s\right)\left[\bar{\Delta}_{T_{1+},i}\left(j\right) + \bar{\Delta}_{s_+,i}(j)\right]\left[\bar{\Delta}_{T_{1+},i}\left(j\right) - \bar{\Delta}_{s_+,i}(j)\right] \\
&= \left(s-T_1\right)\bar{\Delta}_{T_{1+},i}^2\left(j\right) + \left[\bar{\Delta}_{T_{1+},i}\left(j\right) + \bar{\Delta}_{s_+,i}(j)\right]\left[\frac{T_1-s}{T-T_1}\sum_{t=T_1+1}^{T}\Delta_{it}(j) - \sum_{t=s+1}^{T_1}\Delta_{it}(j)\right],
\end{aligned}
$$

we have

$$
\begin{aligned}
A_3\left(s\right) &= \frac{1}{pNT}\sum_{j\in[p]}\sum_{i\in[N]}\left\{\sum_{t=s+1}^{T}\left[\Delta_{it}(j) - \bar{\Delta}_{s_+,i}(j)\right]^2 - \sum_{t=T_1+1}^{T}\left[\Delta_{it}(j) - \bar{\Delta}_{T_{1+},i}\left(j\right)\right]^2\right\} \\
&= \frac{1}{pNT}\sum_{j\in[p]}\sum_{i\in[N]}\left\{\sum_{t=s+1}^{T}\Delta_{it}^2\left(j\right) - \left(T-s\right)\bar{\Delta}_{s_+,i}^2(j) - \sum_{t=T_1+1}^{T}\Delta_{it}^2\left(j\right) + \left(T-T_1\right)\bar{\Delta}_{T_{1+},i}\left(j\right)\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{pNT}\sum_{j\in[p]}\sum_{i\in[N]}\left\{\sum_{t=s+1}^{T_1}\Delta_{it}^2(j) + (T-T_1)\bar{\Delta}_{T_1+,i}^2(j) - (T-s)\bar{\Delta}_{s+,i}(j)\right\} \\
&= \kappa_s\frac{1}{pN(T_1-s)}\sum_{j\in[p]}\sum_{i\in[N]}\sum_{t=s+1}^{T_1}\Delta_{it}^2(j) - \kappa_s\frac{1}{pN}\sum_{j\in[p]}\sum_{i\in[N]}\bar{\Delta}_{T_1+,i}^2(j) \\
&+ \kappa_s\frac{1}{pN}\sum_{j\in[p]}\sum_{i\in[N]}\left[\bar{\Delta}_{T_1+,i}(j) + \bar{\Delta}_{s+,i}(j)\right]\frac{1}{T-T_1}\sum_{t=T_1+1}^{T}\Delta_{it}(j) \\
&- \kappa_s\frac{1}{pN}\sum_{j\in[p]}\sum_{i\in[N]}\left[\bar{\Delta}_{T_1+,i}(j) + \bar{\Delta}_{s+,i}(j)\right]\frac{1}{T_1-s}\sum_{t=s+1}^{T_1}\Delta_{it}(j) \\
&= \kappa_s O_p\left(\eta_{N,2}^2\right) = \kappa_s o_p(\zeta_{NT}^2), \qquad (A.16)
\end{aligned}
$$

where the second last equality holds by (A.14) and the fact that

$$
\begin{aligned}
\max_{i\in[N],j\in[p]}\left|\bar{\Delta}_{s+,i}(j)\right| &= \max_{i\in[N],j\in[p]}\left|\frac{1}{T-s}\sum_{t=s+1}^{T}\Delta_{it}(j)\right| = O_p\left(\eta_{N,2}\right), \\
\max_{i\in[N],j\in[p]}\left|\bar{\Delta}_{T_1+,i}(j)\right| &= \max_{i\in[N],j\in[p]}\left|\frac{1}{T-T_1}\sum_{t=T_1+1}^{T}\Delta_{it}(j)\right| = O_p\left(\eta_{N,2}\right). \qquad (A.17)
\end{aligned}
$$

Finally, we observe that

$$
\begin{aligned}
&A_4(s) + A_5(s) \\
&= \frac{2}{pNT}\sum_{j\in[p]}\sum_{i\in[N]}(\alpha^{(1)}_{g_i^{(1)},j} - \alpha^{(2)}_{g_i^{(2)},j})\left\{\frac{T-T_1}{T-s}\sum_{t=s+1}^{T_1}\left[\Delta_{it}(j) - \bar{\Delta}_{s+,i}(j)\right] - \frac{T_1-s}{T-s}\sum_{t=T_1+1}^{T}\left[\Delta_{it}(j) - \bar{\Delta}_{s+,i}(j)\right]\right\} \\
&= \frac{2}{pNT}\sum_{j\in[p]}\sum_{i\in[N]}(\alpha^{(1)}_{g_i^{(1)},j} - \alpha^{(2)}_{g_i^{(2)},j})\left[\frac{T-T_1}{T-s}\sum_{t=s+1}^{T_1}\Delta_{it}(j) - \frac{T_1-s}{T-s}\sum_{t=T_1+1}^{T}\Delta_{it}(j)\right] \\
&= 2\kappa_s\frac{1-\tau_T}{1-\frac{s}{T}}\frac{1}{pN}\sum_{j\in[p]}\sum_{i\in[N]}(\alpha^{(1)}_{g_i^{(1)},j} - \alpha^{(2)}_{g_i^{(2)},j})\frac{1}{T_1-s}\sum_{t=s+1}^{T_1}\Delta_{it}(j) \\
&- 2\kappa_s\frac{1-\tau_T}{1-\frac{s}{T}}\frac{1}{pN}\sum_{j\in[p]}\sum_{i\in[N]}(\alpha^{(1)}_{g_i^{(1)},j} - \alpha^{(2)}_{g_i^{(2)},j})\frac{1}{T-T_1}\sum_{t=T_1+1}^{T}\Delta_{it}(j) \\
&\leq 2\kappa_s\frac{1-\tau_T}{1-\frac{s}{T}}\sqrt{\frac{1}{N}\sum_{i\in[N]}\left\|\alpha^{(1)}_{g_i^{(1)}} - \alpha^{(2)}_{g_i^{(2)}}\right\|^2}\sqrt{\frac{1}{N}\sum_{i\in[N]}\left[\frac{1}{p(T_1-s)}\sum_{j\in[p]}\sum_{t=s+1}^{T_1}\Delta_{it}(j)\right]^2} \\
&+ 2\kappa_s\frac{1-\tau_T}{1-\frac{s}{T}}\sqrt{\frac{1}{N}\sum_{i\in[N]}\left\|\alpha^{(1)}_{g_i^{(1)}} - \alpha^{(2)}_{g_i^{(2)}}\right\|^2}\sqrt{\frac{1}{N}\sum_{i\in[N]}\left[\frac{1}{p(T-T_1)}\sum_{j\in[p]}\sum_{t=T_1+1}^{T}\Delta_{it}(j)\right]^2} \\
&= \kappa_s\frac{1-\tau_T}{1-\frac{s}{T}}\zeta_{NT}O_p\left(\eta_{N,2}\right) = \kappa_s o_p(\zeta_{NT}^2), \qquad (A.18)
\end{aligned}
$$

where the first inequality holds by Cauchy-Schwarz inequality.

Combining (A.11), (A.12), (A.13), (A.16), (A.18) and Assumption 6(i) yields that

$$
L(s) - L(T_1) = \kappa_s\frac{(1-\tau_T)}{1-\frac{s}{T}}D_{N\alpha} + \kappa_s o_p(\zeta_{NT}^2).
$$

Then for any $s < T_1$,

$$\text{plim}_{(N,T)\to\infty}\frac{1}{\kappa_s\zeta_{NT}^2}\left[L(s)-L(T_1)\right]=\text{plim}_{(N,T)\to\infty}\frac{1-\tau_T}{1-\frac{s}{T}}\frac{1}{\zeta_{NT}^2}D_{N\alpha}\geq(1-\tau)\,D_\alpha>0,$$

where $D_\alpha :=\text{plim}_{(N,T)\to\infty}\frac{1}{\zeta_{NT}^2}D_{N\alpha}>0$ by Assumption 6(i). This implies that

$$\mathbb{P}\left(\hat{T}_1<T_1\right)\leq\mathbb{P}\left(\exists s<T_1, L(s)-L(T_1)<0\right)\to 0. \tag{A.19}$$

By analogous arguments, we prove (ii) in the following part. When $s > T_1$, we have

$$\begin{aligned}
\bar{\dot{\Theta}}_{j,i}^{(1s)}&=\frac{1}{s}\sum_{t=1}^{s}\dot{\Theta}_{j,it}=\frac{1}{s}\sum_{t=1}^{s}\left[\Theta_{j,it}^0+\left(\dot{\Theta}_{j,it}-\Theta_{j,it}^0\right)\right]=\frac{T_1}{s}\alpha_{g_i^{(1)},j}^{(1)}+\frac{s-T_1}{s}\alpha_{g_i^{(2)},j}^{(2)}+\bar{\Delta}_{s,i}(j),\\
\bar{\dot{\Theta}}_{j,i}^{(2s)}&=\frac{1}{T-s}\sum_{t=s+1}^{T}\dot{\Theta}_{j,it}=\frac{1}{T-s}\sum_{t=s+1}^{T}\left[\Theta_{j,it}^0+\left(\dot{\Theta}_{j,it}-\Theta_{j,it}^0\right)\right]=\alpha_{g_i^{(2)},j}^{(2)}+\bar{\Delta}_{s_+,i}(j).
\end{aligned}$$

It follows that

$$\begin{aligned}
\dot{\Theta}_{j,it}-\bar{\dot{\Theta}}_{j,i}^{(1s)}&=\begin{cases}\frac{s-T_1}{s}(\alpha_{g_i^{(1)},j}^{(1)}-\alpha_{g_i^{(2)},j}^{(2)})+\Delta_{it}(j)-\bar{\Delta}_{s,i}(j) & \text{if } 1\leq t\leq T_1\\ \frac{T_1}{s}(\alpha_{g_i^{(2)},j}^{(2)}-\alpha_{g_i^{(1)},j}^{(1)})+\Delta_{it}(j)-\bar{\Delta}_{s,i}(j) & \text{if } T_1+1\leq t\leq s\end{cases},\ \text{and}\\
\dot{\Theta}_{j,it}-\bar{\dot{\Theta}}_{j,i}^{(2s)}&=\Delta_{it}(j)-\bar{\Delta}_{s_+,i}(j),\quad s<t\leq T.
\end{aligned}$$

As in (A.8), we obtain that

$$\begin{aligned}
\sum_{t=1}^{s}\left[\dot{\Theta}_{j,it}-\bar{\dot{\Theta}}_{j,i}^{(1s)}\right]^2&=\sum_{t=1}^{T_1}\left[\frac{s-T_1}{s}(\alpha_{g_i^{(1)},j}^{(1)}-\alpha_{g_i^{(2)},j}^{(2)})+\Delta_{it}(j)-\bar{\Delta}_{s,i}(j)\right]^2\\
&+\sum_{t=T_1+1}^{s}\left[\frac{T_1}{s}(\alpha_{g_i^{(2)},j}^{(2)}-\alpha_{g_i^{(1)},j}^{(1)})+\Delta_{it}(j)-\bar{\Delta}_{s,i}(j)\right]^2\\
&=\frac{T_1\left(s-T_1\right)^2}{s^2}(\alpha_{g_i^{(1)},j}^{(1)}-\alpha_{g_i^{(2)},j}^{(2)})^2+\sum_{t=1}^{T_1}\left[\Delta_{it}(j)-\bar{\Delta}_{s,i}(j)\right]^2\\
&+2\frac{s-T_1}{s}(\alpha_{g_i^{(1)},j}^{(1)}-\alpha_{g_i^{(2)},j}^{(2)})\sum_{t=1}^{T_1}\left[\Delta_{it}(j)-\bar{\Delta}_{s,i}(j)\right]\\
&+\frac{\left(s-T_1\right)T_1^2}{s^2}(\alpha_{g_i^{(2)},j}^{(2)}-\alpha_{g_i^{(1)},j}^{(1)})^2+\sum_{t=T_1+1}^{s}\left[\Delta_{it}(j)-\bar{\Delta}_{s,i}(j)\right]^2\\
&+2\frac{T_1}{s}(\alpha_{g_i^{(2)},j}^{(2)}-\alpha_{g_i^{(1)},j}^{(1)})\sum_{t=T_1+1}^{s}\left[\Delta_{it}(j)-\bar{\Delta}_{s,i}(j)\right]\\
&=\frac{\left(s-T_1\right)T_1}{s}(\alpha_{g_i^{(1)},j}^{(1)}-\alpha_{g_i^{(2)},j}^{(2)})^2+\sum_{t=1}^{s}\left[\Delta_{it}(j)-\bar{\Delta}_{s,i}(j)\right]^2\\
&+2\frac{T_1\left(s-T_1\right)}{s}(\alpha_{g_i^{(1)},j}^{(1)}-\alpha_{g_i^{(2)},j}^{(2)})\left[\frac{1}{T_1}\sum_{t=s+1}^{T_1}\Delta_{it}(j)-\frac{1}{s-T_1}\sum_{t=T_1+1}^{T}\Delta_{it}(j)\right]
\end{aligned}$$

and $\sum_{t=s+1}^{T}[\dot{\Theta}_{j,it}-\bar{\dot{\Theta}}_{j,i}^{(2s)}]^2=\sum_{t=s+1}^{T}\left[\Delta_{it}(j)-\bar{\Delta}_{s_+,i}(j)\right]^2$. It follows that

$$L(s)-L(T_1)=\frac{T_1\left(s-T_1\right)}{sT}\frac{1}{pN}\sum_{i\in[N]}\left\|\alpha_{g_i^{(1)}}^{(1)}-\alpha_{g_i^{(2)}}^{(2)}\right\|^2$$

$$+\frac{1}{pNT}\sum_{j\in[p]}\sum_{i\in[N]}\left\{\sum_{t=1}^{s}\left[\Delta_{it}(j)-\bar{\Delta}_{s,i}(j)\right]^2-\sum_{t=1}^{T_1}\left[\Delta_{it}(j)-\bar{\Delta}_{T_1,i}(j)\right]^2\right\}$$
$$+\frac{1}{pNT}\sum_{j\in[p]}\sum_{i\in[N]}\left\{\sum_{t=s+1}^{T}\left[\Delta_{it}(j)-\bar{\Delta}_{s_+,i}(j)\right]^2-\sum_{t=T_1+1}^{T}\left[\Delta_{it}(j)-\bar{\Delta}_{T_{1+},i}(j)\right]^2\right\}$$
$$+2\frac{T_1(s-T_1)}{sT}\frac{1}{pN}\sum_{j\in[p]}\sum_{i\in[N]}(\alpha^{(1)}_{g_i^{(1)},j}-\alpha^{(2)}_{g_i^{(2)},j})\left[\frac{1}{T_1}\sum_{t=s+1}^{T_1}\Delta_{it}(j)-\frac{1}{s-T_1}\sum_{t=T_1+1}^{T}\Delta_{it}(j)\right]$$
$$:=B_1(s)+B_2(s)+B_3(s)+B_4(s),$$

where $B_4(s)$ parallels $A_4(s)+A_5(s)$ in (A.11). Let $\bar{\kappa}_s=\frac{s-T_1}{T}\in\left[\frac{1}{T},1-\tau_T\right]$. Following the analyses of $A_\ell(s)$'s, we can readily show that

$$B_1(s)=\bar{\kappa}_s\frac{T}{s}\frac{T_1}{T}D_{N\alpha}=\bar{\kappa}_sO_p(\zeta_{NT}^2),\ B_\ell(s)=\bar{\kappa}_sO_p\left(\eta_{N,2}^2\right)=\bar{\kappa}_so_p(\zeta_{NT}^2)\text{ for }\ell=2,3,$$

and $B_4(s)=\bar{\kappa}_sO_p\left(\eta_{N,2}\zeta_{NT}\right)=\bar{\kappa}_so_p(\zeta_{NT}^2)$. It follows that for any $s>T_1$,

$$\text{plim}_{(N,T)\to\infty}\frac{1}{\bar{\kappa}_s\zeta_{NT}^2}\left[L(s)-L(T_1)\right]=\text{plim}_{(N,T)\to\infty}\frac{T_1T}{Ts}\frac{1}{\zeta_{NT}^2}D_{N\alpha}\geq\tau D_\alpha>0.$$

This implies that

$$\mathbb{P}(\hat{T}_1>T_1)\leq\mathbb{P}\left(\exists s>T_1,L(s)-L(T_1)<0\right)\to 0. \tag{A.20}$$

Combining (A.19) and (A.20), we conclude that $\mathbb{P}(\hat{T}_1=T_1)\to 1$. ■

A.4 Proof of Theorem 4.3

By Theorem 4.2, $\mathbb{P}(\hat{T}_1=T_1)\to 1$. It follows that we can prove Theorem 4.3 by conditioning on the event that $\{\hat{T}_1=T_1\}$. Below we prove the theorem under the event that $\{\hat{T}_1=T_1\}$.

Define $\dot{\Theta}^{0,(1)}_{j,i}=(\dot{\Theta}_{j,i1},\cdots,\dot{\Theta}_{j,iT_1})'$, $\dot{\Theta}^{0,(2)}_{j,i}=(\dot{\Theta}_{j,i,T_1+1},\cdots,\dot{\Theta}_{j,iT})'$, $\dot{\beta}_i^{0,(1)}=\frac{1}{\sqrt{T_1}}(\dot{\Theta}^{0,(1)\prime}_{1,i},\cdots,\dot{\Theta}^{0,(1)\prime}_{p,i})'$, and $\dot{\beta}_i^{0,(2)}=\frac{1}{\sqrt{T_2}}(\dot{\Theta}^{0,(2)\prime}_{1,i},\cdots,\dot{\Theta}^{0,(2)\prime}_{p,i})'$. Noted that in the definitions of $\dot{\beta}_i^{0,(1)}$ and $\dot{\beta}_i^{0,(2)}$ we use the true break date T_1 rather than the estimated one compared to $\dot{\beta}_i^{(1)}$ and $\dot{\beta}_i^{(2)}$ defined in Step 4. As in (3.5) and (3.6), we further define

$$\left\{\dot{a}^{0,(\ell)}_{k,m}\right\}_{k\in[m]}=\underset{\left\{a_k^{(\ell)}\right\}_{k\in[m]}}{\arg\min}\frac{1}{N}\sum_{i\in[N]}\min_{k\in[m]}\left\|\dot{\beta}_i^{0,(\ell)}-a_k^{(\ell)}\right\|^2, \tag{A.21}$$
$$\hat{g}^{0,(\ell)}_{i,m}=\underset{k\in[m]}{\arg\min}\left\|\dot{\beta}_i^{(\ell)}-\dot{a}^{0,(\ell)}_{k,m}\right\|,\quad\forall i\in[N]. \tag{A.22}$$

(i) In the case of $m=K^{(\ell)}$, Theorem 4.3(i.a) is from the combination of Lemma B.9 for the consistency of the membership estimates via K-means algorithm and Theorem 4.2 for the consistency of the break point estimator.

Next, we show (i.c). Recall that z_ς is the critical value at significance level ς calculated from the maximum of m independent $\chi^2(1)$ random variables. By the definition of the STK algorithm, we observe that

$$\mathbb{P}(\hat{K}^{(\ell)}\leq K^{(\ell)})\geq\mathbb{P}(\hat{\Gamma}^{(\ell)}_{K^{(\ell)}}\leq z_\varsigma),$$

which leads to the fact that (i.c) holds as long as we can show (i.b). This is because, under (i.b), we have

$$\mathbb{P}(\hat{\Gamma}^{(\ell)}_{K^{(\ell)}} \leq z_\varsigma) \geq 1 - \varsigma + o(1).$$

Now, we focus on (i.b). Notice that $\hat{\Gamma}^{(\ell)}_{k,K^{(\ell)}}$ depends on the K-means classification result, i.e., the estimated group membership $\hat{G}^{(\ell)}_{k,K^{(\ell)}}$ for $k \in [K^{(\ell)}]$. From Theorem 4.3(i.1), we notice that we can change the estimated group membership $\hat{G}^{(\ell)}_{k,K^{(\ell)}}$ to the true group membership $G^{(\ell)}_k$, and the replacement has only an asymptotically negligible effect. Recall that $\mathcal{T}_1 = [T_1]$ and $\mathcal{T}_2 = [T]\backslash[T_1]$. Define $\mathcal{T}_{1,-1} = \mathcal{T}_1\backslash\{T_1\}$, $\mathcal{T}_{2,-1} = \mathcal{T}_2\backslash\{T\}$, $\mathcal{T}_{1,j} = \{1+j, \cdots, T_1\}$ and $\mathcal{T}_{2,j} = \{T_1+1+j, \cdots, T\}$ for some specific $j \in \mathcal{T}_{\ell,-1}$. Let

$$\left(\left\{\hat{\theta}^{0,(\ell)}_{i,k,K^{(\ell)}}\right\}_{i\in G^{(\ell)}_k}, \hat{\Lambda}^{0,(\ell)}_{k,K^{(\ell)}}, \left\{\hat{f}^{0,(\ell)}_{t,k,K^{(\ell)}}\right\}_{t\in\mathcal{T}_\ell}\right) = \underset{\{\theta_i,\lambda_i\}_{i\in G^{(\ell)}_k}, \{f_t\}_{t\in\mathcal{T}_\ell}}{\arg\min} \sum_{i\in G^{(\ell)}_k}\sum_{t\in\mathcal{T}_\ell}\left(Y_{it} - X'_{it}\theta_i - \lambda'_i f_t\right)^2,$$

$\hat{F}^{0,(1)}_{k,K^{(1)}} = (\hat{f}_{1,k,K^{(1)}}, \cdots, \hat{f}_{T_1,k,K^{(1)}})'$, $\hat{F}^{0,(2)}_{k,K^{(2)}} = (\hat{f}_{T_1+1,k,K^{(1)}}, \cdots, \hat{f}_{T,k,K^{(2)}})'$, $\hat{\Lambda}^{(\ell)}_{k,K^{(\ell)}} = \{\hat{\lambda}^{(\ell)}_{i,k,K^{(\ell)}}\}_{i\in G^{(\ell)}_k}$, and $(\hat{z}^{0,(\ell)}_{it})'$ denote the t-th row of $M_{\hat{F}^{0,(\ell)}_{k,K^{(\ell)}}} X^{(\ell)}_i$. Further define

$$\begin{aligned}
\hat{\bar{\theta}}^{0,(\ell)}_{k,K^{(\ell)}} &= \frac{1}{|G^{(\ell)}_k|}\sum_{i\in G^{(\ell)}_k}\hat{\theta}^{0,(\ell)}_{i,k,K^{(\ell)}}, \qquad \hat{S}^{0,(\ell)}_{ii,k,K^{(\ell)}} = \frac{1}{T_\ell}(X^{(\ell)}_i)' M_{\hat{F}^{0,(\ell)}_{k,K^{(\ell)}}} X^{(\ell)}_i, \\
\hat{\Omega}^{0,(\ell)}_{i,k,K^{(\ell)}} &= \frac{1}{T_\ell}\sum_{t\in\mathcal{T}_\ell}\hat{z}^{0,(\ell)}_{it}\hat{z}^{0,(\ell)\prime}_{it}\hat{e}^2_{it} + \frac{1}{T_\ell}\sum_{j\in\mathcal{T}_{\ell,-1}} k(j,L)\sum_{t\in\mathcal{T}_{\ell,j}}[\hat{z}^{0,(\ell)}_{it}\hat{z}^{0,(\ell)\prime}_{i,t+j}\hat{e}_{it}\hat{e}_{i,t+j} + \hat{z}^{0,(\ell)}_{i,t-j}\hat{z}^{0,(\ell)\prime}_{it}\hat{e}_{i,t-j}\hat{e}_{it}], \\
\hat{a}^{0,(\ell)}_{ii,k,K^{(\ell)}} &= \hat{\lambda}^{(\ell)\prime}_{i,k,K^{(\ell)}}\left(\frac{1}{|G^{(\ell)}_k|}\hat{\Lambda}^{(\ell)\prime}_{k,K^{(\ell)}}\hat{\Lambda}^{(\ell)}_{k,K^{(\ell)}}\right)^{-1}\hat{\lambda}^{(\ell)}_{i,k,K^{(\ell)}}.
\end{aligned}$$

Then $\forall k \in [K^{(\ell)}]$, we can define

$$\hat{\Gamma}^{0,(\ell)}_{k,K^{(\ell)}} = \sqrt{\left|G^{(\ell)}_k\right|}\frac{\frac{1}{\left|G^{(\ell)}_k\right|}\sum_{i\in G^{(\ell)}_k}\hat{\mathbb{S}}^{0,(\ell)}_{i,k,K^{(\ell)}} - p}{\sqrt{2p}},$$

where

$$\hat{\mathbb{S}}^{0,(\ell)}_{i,k,K^{(\ell)}} = T_\ell(\hat{\theta}^{0,(\ell)}_{i,k,K^{(\ell)}} - \hat{\bar{\theta}}^{0,(\ell)}_{k,K^{(\ell)}})'\hat{S}^{0,(\ell)}_{ii,k,K^{(\ell)}}(\hat{\Omega}^{0,(\ell)}_{i,k,K^{(\ell)}})^{-1}\hat{S}^{0,(\ell)}_{ii,k,K^{(\ell)}}(\hat{\theta}^{0,(\ell)}_{i,k,K^{(\ell)}} - \hat{\bar{\theta}}^{0,(\ell)}_{k,K^{(\ell)}})\left(1 - \hat{a}^{0,(\ell)}_{ii,k,K^{(\ell)}}/|G^{(\ell)}_k|\right)^2.$$

By Lemma D.8, we notice that $\hat{\Gamma}^{0,(\ell)}_{k,K^{(\ell)}} \rightsquigarrow \mathbb{N}(0,1)$ owing to the fact that the slope coefficient $\alpha^{(\ell)}_k$ is homogeneous across $i \in G^{(\ell)}_k$ $\forall k \in [K^{(\ell)}]$. Furthermore, $\{\hat{\Gamma}^{0,(\ell)}_{k,K^{(\ell)}},\ k \in [K^{(\ell)}]\}$ are asymptotically independent under Assumption 1(i). It follows that

$$\hat{\Gamma}^{(\ell)}_{K^{(\ell)}} = \max_{k\in[K^{(\ell)}]}\left(\hat{\Gamma}^{(\ell)}_{k,K^{(\ell)}}\right)^2 = \max_{k\in[m]}\left(\hat{\Gamma}^{0,(\ell)}_{k,K^{(\ell)}}\right)^2 + o_p(1) \rightsquigarrow \mathcal{Z},$$

where $\mathcal{Z}$ is the maximum of m independent $\chi^2(1)$ random variables. Then Theorem 4.3(i.b) follows.

(ii) When $m < K^{(\ell)}$, Theorem 4.3(i.1) does not hold and we can not change the estimated group membership $\hat{\mathcal{G}}^{(\ell)}_{K^{(\ell)}}$ to the true group membership $\mathcal{G}^{(\ell)}$. To get around of this issue, we define the "pseudo groups". For $m < K^{(\ell)}$, let $\mathbb{G}^{(\ell)}_m := \{G^{(\ell)}_{1,m}, \cdots, G^{(\ell)}_{m,m}\}$ such that $[N] = G^{(\ell)}_{1,m} \cup \cdots \cup G^{(\ell)}_{m,m}$, which indicates one possible partition of the set $[N]$. We further define $\mathcal{G}^{(\ell)}_m$ to be the collection of all possible $\mathbb{G}^{(\ell)}_m$.

By Theorem 4.3(i.c), we can conclude that $\mathbb{P}\left(\hat{K}^{(\ell)} \neq K^{(\ell)}\right) \leq \varsigma + o(1)$ provided we can show that $\hat{\Gamma}_m^{(\ell)} \to \infty$ when $m < K^{(\ell)}$. By Lemma B.10, we notice that $\hat{\mathcal{G}}_m^{(\ell)} \in \mathbb{G}_m^{(\ell)}$ w.p.a.1. Conditioning on the event $\{\hat{\mathcal{G}}_m^{(\ell)} \in \mathbb{G}_m^{(\ell)}\} \cap \{\hat{T}_1 = T_1\}$, we have

$$\hat{\Gamma}_m^{(\ell)} > \min_{\mathcal{G}_m^{(\ell)} \in \mathbb{G}_m^{(\ell)}} \hat{\Gamma}_m^{0,(\ell)}(\mathcal{G}_m^{(\ell)}) := \min_{\mathcal{G}_m^{(\ell)} \in \mathbb{G}_m^{(\ell)}} \left\{ \max_{k \in [m]} \left[\hat{\Gamma}_{k,m}^{0,(\ell)}(G_{k,m}^{(\ell)}) \right]^2 \right\},$$

where

$$\hat{\Gamma}_{k,m}^{0,(\ell)}(G_{k,m}^{(\ell)}) = \sqrt{\left|G_{k,m}^{(\ell)}\right|} \frac{\frac{1}{\left|G_{k,m}^{(\ell)}\right|} \sum_{i \in G_{k,m}^{(\ell)}} \hat{\mathbb{S}}_{i,k,m}^{0,(\ell)} - p}{\sqrt{2p}},$$

and $\hat{\mathbb{S}}_{i,k,m}^{0,(\ell)}$ is defined similarly to $\hat{\mathbb{S}}_{i,k,K^{(\ell)}}^{0,(\ell)}$ in the proof of (i).

Owing to the fact that $|\mathbb{G}_m^{(\ell)}| = m^{K^{(\ell)}}$ which is a constant since $K^{(\ell)}$ is a constant, we can show that $\hat{\Gamma}_m^{(\ell)} \to \infty$ by showing that $\hat{\Gamma}_m^{0,(\ell)}(\mathcal{G}_m^{(\ell)}) \to \infty$ for any possible realization $\mathcal{G}_m^{(\ell)}$. Under the case when $m < K^{(\ell)}$, there exists at least one $k \in [m]$ such that the slope coefficient is not homogeneous across $i \in G_{k,m}^{(\ell)}$. Assume that $G_{k,m}^{(\ell)}$ contains n true groups, i.e., $G_{k,m}^{(\ell)} = G_{k_1}^{(\ell)} \cup \cdots \cup G_{k_n}^{(\ell)}$ for $k_1, \cdots, k_n \in [K^{(\ell)}]$ and $k_1 \neq \cdots \neq k_n$. Then for $i \in G_{k,m}^{(\ell)}$, we have

$$\begin{aligned}
\theta_i^{0,(\ell)} &= \sum_{s=1}^n \alpha_{k_s}^{(\ell)} \mathbf{1}\{i \in G_{k_s}^{(\ell)}\} = \frac{1}{n}\sum_{s^*=1}^n \alpha_{k_{s^*}}^{(\ell)} + \sum_{s=1}^n \left(\frac{n-1}{n}\alpha_{k_s}^{(\ell)} - \frac{1}{n} \sum_{s^* \in [n], s^* \neq s} \alpha_{k_{s^*}}^{(\ell)} \right) \mathbf{1}\{i \in G_{k_s}^{(\ell)}\} \\
&= \frac{1}{n}\sum_{s^*=1}^n \alpha_{k_{s^*}}^{(\ell)} + \sum_{s=1}^n \frac{1}{n} \sum_{s^* \in [n], s^* \neq s} (\alpha_{k_s}^{(\ell)} - \alpha_{k_{s^*}}^{(\ell)}) \mathbf{1}\{i \in G_{k_s}^{(\ell)}\} := \bar{\theta}_n^{0,(\ell)} + c_i^{(\ell)}
\end{aligned}$$

such that

$$\begin{aligned}
\frac{T_\ell}{\sqrt{N}} \sum_{i \in [N]} \left\| c_i^{(\ell)} \right\|^2 / (\log N)^{1/2} &= \frac{T_\ell}{\sqrt{N}} \sum_{s=1}^n \frac{N_{k_s}^{(\ell)}}{n} \left\| \sum_{s^* \in [n], s^* \neq s} (\alpha_{k_s}^{(\ell)} - \alpha_{k_{s^*}}^{(\ell)}) \right\|^2 / (\log N)^{1/2} \\
&= \frac{T_\ell}{\sqrt{N} n} \sum_{s=1}^n N_{k_s}^{(\ell)} \left\| \frac{\sum_{s^* \in [n], s^* \neq s} \alpha_{k_{s^*}}^{(\ell)}}{n-1} - \alpha_{k_s}^{(\ell)} \right\|^2 / (\log N)^{1/2} \to \infty,
\end{aligned}$$

by Assumption 7(iii). Then $|\hat{\Gamma}_{k,m}^{0,(\ell)}(G_{k,m}^{(\ell)})|/(\log N)^{1/2} \to \infty$ for some $k \in [m]$ by Lemma D.9. By the definition of $\hat{\Gamma}_m^{0,(\ell)}(\mathcal{G}_m^{(\ell)})$, we have $\hat{\Gamma}_m^{0,(\ell)}(\mathcal{G}_m^{(\ell)})/(\log N)^{1/2} \to \infty$, which yields $\hat{\Gamma}_m^{(\ell)}/\log N \to \infty$ w.p.a.1 for $m < K^{(\ell)}$ and $\mathbb{P}(\hat{K}^{(\ell)} \neq K^{(\ell)}) \leq \varsigma + o(1)$ as z_ς diverges to infinity at rate $\log N$ as $\varsigma = \varsigma_N \to 0$ at some rate N^{-c} for some $c > 0$. ■

A.5 Proof of Theorem 4.4

To show Theorem 4.4, we can directly derive the asymptotic distribution for the oracle estimator $\hat{\alpha}_k^{*(\ell)}$ by combining Theorems 4.2 and 4.3.

The asymptotic distribution theory for the linear panel model with IFEs has already been studied in the literature; see Bai (2009), Moon and Weidner (2017) and Lu and Su (2016) for instance. However, Bai (2009) rules out dynamic panels. Moon and Weidner (2017) allow dynamic panels and assume the independence over both i and t for the error term. For the dynamic linear panel model under Assumptions 1* and 2–9, Theorem 4.4 extends Theorem 4.3 in Moon and Weidner (2017) to allow for multiple groups.

Below, we follow the arguments in Moon and Weidner (2017) and sketch the proof to allow the serial correlation of error terms in non-dynamic panels.[1] To proceed, let $\mathbb{C}_{NT,k}^{(\ell)}$ be the p-vector with j-th entry being $\mathbb{C}_{NT,k,j}^{(\ell)} = \mathbb{C}_1(\Lambda_k^{0,(\ell)}, F^{0,(\ell)}, \mathbb{X}_{j,k}^{(\ell)}, E_k^{(\ell)}) + \mathbb{C}_2(\Lambda_k^{0,(\ell)}, F^{0,(\ell)}, \mathbb{X}_{j,k}^{(\ell)}, E_k^{(\ell)})$, where

$$\mathbb{C}_1(\Lambda_k^{0,(\ell)}, F^{0,(\ell)}, \mathbb{X}_{j,k}^{(\ell)}, E_k^{(\ell)}) = \frac{1}{\sqrt{N_k^{(\ell)}T_\ell}} tr\left(M_{F^{0,(\ell)}} E_k^{(\ell)\prime} M_{\Lambda_k^{0,(\ell)}} \mathbb{X}_{j,k}^{(\ell)}\right), \text{ and}$$

$$\begin{aligned}
&\mathbb{C}_2(\Lambda_k^{0,(\ell)}, F^{0,(\ell)}, \mathbb{X}_{j,k}^{(\ell)}, E_k^{(\ell)}) \\
&= -\frac{1}{\sqrt{N_k^{(\ell)}T_\ell}} tr\left(E_k^{(\ell)} M_{F^{0,(\ell)}} E_k^{(\ell)\prime} M_{\Lambda_k^{0,(\ell)}} \mathbb{X}_{j,k}^{(\ell)} F^{0,(\ell)} \left(F^{0,(\ell)\prime}F^{0,(\ell)}\right)^{-1} \left(\Lambda_k^{0,(\ell)\prime}\Lambda_k^{0,(\ell)}\right)^{-1} \Lambda_k^{0,(\ell)\prime}\right) \\
&- \frac{1}{\sqrt{N_k^{(\ell)}T_\ell}} tr\left(E_k^{(\ell)\prime} M_{\Lambda_k^{0,(\ell)}} E_k^{(\ell)} M_{F^{0,(\ell)}} \mathbb{X}_{j,k}^{(\ell)\prime} \Lambda_k^{0,(\ell)} \left(\Lambda_k^{0,(\ell)\prime}\Lambda_k^{0,(\ell)}\right)^{-1} \left(F^{0,(\ell)\prime}F^{0,(\ell)}\right)^{-1} \left(\Lambda_k^{0,(\ell)\prime}\Lambda_k^{0,(\ell)}\right)^{-1} F^{0,(\ell)\prime}\right) \\
&- \frac{1}{\sqrt{N_k^{(\ell)}T_\ell}} tr\left(E_k^{(\ell)\prime} M_{\Lambda_k^{0,(\ell)}} \mathbb{X}_{j,k}^{(\ell)} M_{F^{0,(\ell)}} E_k^{(\ell)\prime} \Lambda_k^{0,(\ell)} \left(\Lambda_k^{0,(\ell)\prime}\Lambda_k^{0,(\ell)}\right)^{-1} \left(F^{0,(\ell)\prime}F^{0,(\ell)}\right)^{-1} \left(\Lambda_k^{0,(\ell)\prime}\Lambda_k^{0,(\ell)}\right)^{-1} F^{0,(\ell)\prime}\right).
\end{aligned}$$

By Lemma B.11, we have $\sqrt{N_k^{(\ell)}T_\ell}(\hat{\alpha}_k^{(\ell)} - \alpha_k^{(\ell)}) = \mathbb{W}_{NT,k}^{(\ell)-1}\mathbb{C}_{NT,k}^{(\ell)} + o_p(1)$.

In Moon and Weidner (2017), the asymptotic distribution is derived mainly relying on their Lemmas B.1 and B.2. Lemma B.2 is the standard central limit theorem, which also holds under our Assumption 1. For Lemma B.1, we need to extend it to allow for serially correlated errors in non-dynamic panels in Lemma B.12. Hence, by Lemma B.12 and following arguments analogous to those in the proof of Theorem 4.3 (Moon and Weidner (2017)), for a specific $\ell \in \{1,2\}$ and $k \in [K^{(\ell)}]$, we can readily show that

$$\mathbb{W}_{NT,k}^{(\ell)}\sqrt{N_k^{(\ell)}T_\ell}(\hat{\alpha}_k^{(\ell)} - \alpha_k^{(\ell)}) - \mathbb{B}_{NT,k}^{(\ell)} \rightsquigarrow \mathcal{N}(0, \Omega_k^{(\ell)}),$$

which yields the final distributional results in Theorem 4.4 by stacking all subgroups of parameter estimators into a large vector and resorting to the Cramér-Wold device.

A.6 Proof of Theorem 5.1

Recall that $\dot{v}_{t,j}^* := \frac{\dot{v}_{t,j}}{\|\dot{v}_{t,j}\|}$, $\dot{v}_t^* = \left(\dot{v}_{t,1}^{*\prime}, \cdots, \dot{v}_{t,p}^{*\prime}\right)'$, $v_{t,j}^* = \frac{O_j v_{t,j}^0}{\|O_j v_{t,j}^0\|}$ and $v_t^* = \left(v_{t,1}^{*\prime}, \cdots, v_{t,p}^{*\prime}\right)'$. With the fact that

$$\begin{aligned}
\frac{\dot{v}_{t,j}}{\|\dot{v}_{t,j}\|} - \frac{O_j v_{t,j}^0}{\|O_j v_{t,j}^0\|} &= \frac{\dot{v}_{t,j}\left\|O_j v_{t,j}^0\right\| - O_j v_{t,j}^0 \left\|\dot{v}_{t,j}\right\|}{\|\dot{v}_{t,j}\| \left\|O_j v_{t,j}^0\right\|} \\
&= \frac{\left(\dot{v}_{t,j} - O_j v_{t,j}^0\right)\left\|O_j v_{t,j}^0\right\| + O_j v_{t,j}^0 \left(\left\|O_j v_{t,j}^0\right\| - \|\dot{v}_{t,j}\|\right)}{\|\dot{v}_{t,j}\| \left\|O_j v_{t,j}^0\right\|},
\end{aligned}$$

it follows that

$$\max_t \|\dot{v}_t^* - v_t^*\| \leq p \max_{j\in[p],t\in[T]} \left\|\dot{v}_{t,j}^* - v_{t,j}^*\right\| \leq 2p \max_{j\in[p],t\in[T]} \frac{\left\|\dot{v}_{t,j} - O_j v_{t,j}^0\right\|}{\|\dot{v}_{t,j}\|} = O_p(\eta_{N,2}),$$

where the last line is by Lemma B.7(i) and Theorem 4.1(ii).

[1] It is well known that one cannot allow for serially correlated errors in dynamic panels in general in order to avoid issues with endogeneity.

B Technical Lemmas

Lemma B.1 *Consider a matrix sequence $\{A_i, i \in [N]\}$ whose elements are symmetric matrices with dimension d. Suppose $\{A_i, i \in [N]\}$ is independent with $\mathbb{E}(A_i) = 0$ and $\|A_i\|_{op} \le M$ a.s.. Let $\sigma^2 = \left\|\sum_{i\in[N]} \mathbb{E}\left(A_i^2\right)\right\|_{op}$. Then for all $t > 0$, we have*

$$\mathbb{P}\left(\left\|\sum_{i\in[N]} A_i\right\|_{op} > t\right) \le d \cdot \exp\left\{-\frac{t^2/2}{\sigma^2 + Mt/3}\right\}.$$

Proof. Lemma B.1 states a matrix Bernstein inequality; see Theorem 1.3 in Tropp (2011). ■

Lemma B.2 *Consider a specific matrix $A \in \mathbb{R}^{N\times T}$ whose rows (denoted as A_i') are independent random vectors in $\mathbb{R}^T$ with $\mathbb{E}A_i = 0$ and $\Sigma_i = \mathbb{E}(A_iA_i')$. Suppose $\max_i \|A_i\| \le \sqrt{m}$ almost surely and $\max_i \|\Sigma_i\|_{op} \le M$ for some positive constant M. Then for every $t > 0$, with probability $1 - 2T\exp\left(-c_1t^2\right)$, we have*

$$\|A\|_{op} \le \sqrt{NM} + t\sqrt{m+M},$$

where c_1 is an absolute constant.

Proof. The proof follows arguments like those used in the proof of Theorem 5.41 in Vershynin (2010). Define $Z_i := \frac{1}{N}(A_iA_i' - \Sigma_i) \in \mathbb{R}^{T\times T}$, and we notice that $(Z_1, \cdots, Z_N)$ is an independent sequence with $\mathbb{E}(Z_i) = 0$. To use the matrix Bernstein's inequality, we analyze $\|Z_i\|_{op}$ and $\left\|\sum_{i\in[N]} \mathbb{E}\left(X_i^2\right)\right\|_{op}$ as follows:

$$\|Z_i\|_{op} \le \frac{1}{N}\left(\|A_iA_i'\|_{op} + \|\Sigma_i\|_{op}\right) \le \frac{1}{N}\left(\|A_i\|_2^2 + \|\Sigma_i\|_{op}\right) \le \frac{m+M}{N} \quad a.s. \tag{B.1}$$

uniformly over i. Moreover, note that

$$\mathbb{E}\left[\left(A_iA_i'\right)^2\right] = \mathbb{E}\left[\|A_i\|_2 A_iA_i'\right] \le m\Sigma_i$$

and

$$Z_i^2 = \frac{1}{N^2}\left[\left(A_iA_i'\right)^2 - A_iA_i'\Sigma_i - \Sigma_iA_iA_i' + \Sigma_i^2\right].$$

We then obtain that

$$\begin{aligned}\left\|\mathbb{E}\left(Z_i^2\right)\right\|_{op} &= \left\|\mathbb{E}\left\{\frac{1}{N^2}\left[\left(A_iA_i'\right)^2 - \Sigma_i^2\right]\right\}\right\|_{op} \le \frac{1}{N^2}\left\{\left\|\mathbb{E}\left[\left(A_iA_i'\right)^2\right]\right\|_{op} + \|\Sigma_i\|_{op}^2\right\} \\ &\le \frac{1}{N^2}\left(m\|\Sigma_i\|_{op} + \|\Sigma_i\|_{op}^2\right) \le \frac{mM+M^2}{N^2} \quad a.s.\end{aligned}$$

uniformly over i, and

$$\left\|\sum_{i\in[N]} \mathbb{E}\left(Z_i^2\right)\right\|_{op} \le N\max_i \left\|\mathbb{E}\left(Z_i^2\right)\right\|_{op} \le \frac{mM+M^2}{N} \quad a.s. \tag{B.2}$$

Define $\varepsilon = \max(\sqrt{M}\delta, \delta^2)$ with $\delta = t\sqrt{\frac{m+M}{N}}$. Combining (B.1) and (B.2), by matrix Bernstein's inequality, we have

$$\mathbb{P}\left\{\left\|\frac{1}{N}\left(A'A - \sum_{i\in[N]}\Sigma_i\right)\right\|_{op} \ge \varepsilon\right\} = \mathbb{P}\left(\left\|\sum_{i\in[N]} Z_i\right\|_{op} \ge \varepsilon\right)$$

$$\leq 2T\exp\left\{-c\min\left(\frac{\varepsilon^2}{\frac{mM+M^2}{N}},\frac{\varepsilon}{\frac{m+M}{N}}\right)\right\}\leq 2T\exp\left\{-c\min\left(\frac{\varepsilon^2}{M},\varepsilon\right)\frac{N}{m+M}\right\}$$
$$\leq 2T\exp\left\{-\frac{c\delta^2 N}{m+M}\right\}=2T\exp\left\{-c_1t^2\right\},$$

for some positive constant c, where the third inequality is due to the fact that

$$\min\left(\frac{\varepsilon^2}{M},\varepsilon\right)=\min\left(\max\left(\delta^2,\delta^4/M\right),\max\left(\sqrt{M}\delta,\delta\right)\right)$$
$$=\begin{cases}\min\left(\delta^2,\sqrt{M}\delta\right)=\delta^2, & \text{if}\quad \delta^2\geq\frac{\delta^4}{M},\\ \min\left(\delta^4/M,\delta^2\right)=\delta^2, & \text{if}\quad \delta^2<\frac{\delta^4}{M}.\end{cases}$$

It implies that

$$\left\|\frac{1}{N}A'A-\frac{1}{N}\sum_{i\in[N]}\Sigma_i\right\|_{op}\leq\max\left(\sqrt{M}\delta,\delta^2\right) \tag{B.3}$$

with probability $1-\exp\left(-ct^2\right)$. Combining the fact that $\|\Sigma_i\|_{op}\leq M$ uniformly over i and (B.3), we show that

$$\frac{1}{N}\|A\|_{op}^2=\left\|\frac{1}{N}A'A\right\|_{op}\leq\left\|\frac{1}{N}\sum_{i\in[N]}\Sigma_i\right\|_{op}+\left\|\frac{1}{N}A'A-\frac{1}{N}\sum_{i\in[N]}\Sigma_i\right\|_{op}$$
$$\leq\max_i\|\Sigma_i\|_{op}+\sqrt{M}\delta+\delta^2\leq M+\sqrt{M}t\sqrt{\frac{m+M}{N}}+t^2\frac{m+M}{N}$$
$$\leq\left(\sqrt{M}+t\sqrt{\frac{m+M}{N}}\right)^2,$$

and the result follows: $\|A\|_{op}\leq\sqrt{NM}+t\sqrt{m+M}$. ■

Lemma B.3 *Recall that $X_j=\{X_{j,it}\}$ and $E=\{e_{it}\}$. Under Assumption 1, $\forall j\in[p]$, we have $\|X_j\odot E\|_{op}=O_p(\sqrt{N}+\sqrt{T\log T})$ and $\|E\|_{op}=O_p(\sqrt{N}+\sqrt{T\log T})$.*

Proof. We focus on $\|X_j\odot E\|_{op}$ as the result for $\|E\|_{op}$ can be derived in the same manner. We first note that, conditional on $\left\{V_j^0\right\}_{j\in[p]\cup\{0\}}$, the rows of $X_j\odot E$ are independent across i. Denote the i-th row of $X_j\odot E$ as $A_i'=X_{j,i}'\odot E_i'$, where $X_{j,i}'$ and E_i' being the i-th row of matrix X_j and E, respectively. Recall that $\mathscr{D}$ is the minimum σ-field generated by $\left\{V_j^0\right\}_{j\in[p]\cup\{0\}}$. In addition, for the t-th element of A_i, we have

$$\mathbb{E}\left[X_{j,it}e_{it}|\mathscr{D}\right]=\mathbb{E}\left\{X_{j,it}\mathbb{E}\left[e_{it}|\mathscr{D},X_{it}\}\right]|\mathscr{D}\right\}=0,$$

where the second equality holds by Assumption 1(ii). Therefore, to apply Lemma B.2 conditionally on $\mathscr{D}$, we only need to upper bound $\|A_i\|$ and $||\mathbb{E}[A_iA_i'|\mathscr{D}]||_{op}$.

First, under Assumption 1, we have $\frac{1}{T}\sum_{t\in[T]}\left(X_{j,it}e_{it}\right)^2\leq C$ *a.s.* by Assumption 1(iv), which implies

$$||A_i||=\|X_{j,i}\odot E_i\|\leq C\sqrt{T}\ a.s. \tag{B.4}$$

Second, let $\Sigma_i = \mathbb{E}\left\{\left[(X_{j,i} \odot E_i)(X_{j,i} \odot E_i)'\right] \middle| \mathscr{D}\right\}$ with (t,s) element being $\mathbb{E}\left(X_{j,it}X_{j,is}e_{it}e_{is} \middle| \mathscr{D}\right)$. Let $\|\cdot\|_1$ and $\|\cdot\|_\infty$ denote the norms induced by the 1-norms and ∞-norms, respectively: ,

$$\|\Sigma_i\|_1 = \max_{s\in[T]} \sum_{t\in[T]} \left|\mathbb{E}\left(X_{j,it}X_{j,is}e_{it}e_{is} \middle| \mathscr{D}\right)\right| \text{ and } \|\Sigma_i\|_\infty = \max_{t} \sum_{s\in[T]} \left|\mathbb{E}\left(X_{j,it}X_{j,is}e_{it}e_{is} \middle| \mathscr{D}\right)\right|.$$

By Davydov's inequality for conditional strong mixing sequence (e.g., Lemma 4.3 in Su and Chen (2013)), we can show that

$$\begin{aligned}
&\max_{s\in[T]} \sum_{t\in[T]} \left|\mathbb{E}\left(X_{j,it}X_{j,is}e_{it}e_{is} \middle| \mathscr{D}\right)\right| = \max_{s\in[T]} \sum_{t\in[T]} \left|Cov\left(X_{j,it}e_{it}, X_{j,is}e_{is} \middle| \mathscr{D}\right)\right| \\
&\lesssim \max_{s\in[T]} \sum_{t\in[T]} \left\{\mathbb{E}\left[|X_{j,it}e_{it}|^q \middle| \mathscr{D}\right]\right\}^{1/q} \left\{\mathbb{E}\left[|X_{j,is}e_{is}\}|^q \middle| \mathscr{D}\right]\right\}^{1/q} \alpha(t-s)^{(q-2)/q} \\
&\le \max_{i,t} \left\{\mathbb{E}\left[|X_{j,it}e_{j,it}|^q \middle| \mathscr{D}\right]\right\}^{2/q} \max_{s\in[T]} \sum_{t\in[T]} [\alpha(t-s)]^{(q-2)/q} \le c_2 \ a.s.,
\end{aligned}$$

where c_2 is a positive constant which does not depend on i. Similarly, we have

$$\max_{t} \sum_{s\in[T]} \left|\mathbb{E}\left(X_{j,it}X_{j,is}e_{it}e_{is} \middle| \mathscr{D}\right)\right| \le c_2 \ a.s.$$

Therefore, by Corollary 2.3.2 in Golub and Van Loan (1996), we have

$$\max_{i} \|\Sigma_i\|_{op} \le \sqrt{\|\Sigma_i\|_1 \|\Sigma_i\|_\infty} \le c_2 \ a.s. \tag{B.5}$$

Combining (B.4)-(B.5) and using Lemma B.2 with $t = \sqrt{\log T}$, we obtain the desired result. ■

Recall that $\mathcal{R}(C_1) := \left\{\left\{\Delta_{\Theta_j}\right\}_{j\in[p]\cup\{0\}} \in \mathbb{R}^{N\times T\times(p+1)} : \sum_{j\in[p]\cup\{0\}} \left\|\mathcal{P}_j^\perp(\Delta_{\Theta_j})\right\|_* \le C_1 \sum_{j\in[p]\cup\{0\}} \left\|\mathcal{P}_j(\Delta_{\Theta_j})\right\|_*\right\}$.

Lemma B.4 *Suppose Assumptions 1–3 hold. Then* $\{\tilde{\Delta}_{\Theta_j}\}_{j\in[p]\cup\{0\}} \in \mathcal{R}(3)$ *w.p.a.1.*

Proof. Let A^c denote the complement of A. Define event

$$\mathscr{A}_{1,N}(c_3) = \left\{\|E\|_{op} \le c_3(\sqrt{N} \vee \sqrt{T\log T}), \|X_j \odot E\|_{op} \le c_3(\sqrt{N} \vee \sqrt{T\log T}), \forall j \in [p]\right\}.$$

Then there exists a positive constant c_3 such that $\mathbb{P}(\mathscr{A}_{1,N}^c(c_3)) \le \epsilon$ for any $\epsilon > 0$ by Lemma B.3. Under event $\mathscr{A}_{1,N}(c_3)$, by the definition of $\tilde{\Theta}_j$ in (3.1), we notice that

$$0 \le \frac{1}{NT}\left\|Y - \Theta_0^0 - \sum_{j\in[p]} X_j \odot \Theta_j^0\right\|^2 - \frac{1}{NT}\left\|Y - \tilde{\Theta}_0 - \sum_{j\in[p]} X_j \odot \tilde{\Theta}_j\right\|^2 + \sum_{j\in[p]\cup\{0\}} \nu_j \left(\left\|\Theta_j^0\right\|_* - \left\|\tilde{\Theta}_j\right\|_*\right) \tag{B.6}$$

and

$$\begin{aligned}
&\frac{1}{NT}\left\|Y - \Theta_0^0 - \sum_{j\in[p]} X_j \odot \Theta_j^0\right\|^2 - \frac{1}{NT}\left\|Y - \tilde{\Theta}_0 - \sum_{j\in[p]} X_j \odot \tilde{\Theta}_j\right\|^2 \\
&= \frac{1}{NT}\sum_{i\in[N]}\sum_{t\in[T]}\left\{e_{it}^2 - \left[e_{it} - \left(\tilde{\Delta}_{\Theta_0,it} + \sum_{j\in[p]} X_{j,it}\tilde{\Delta}_{\Theta_j,it}\right)\right]^2\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2}{NT} tr\left(E'\tilde{\Delta}_{\Theta_0}\right) + \sum_{j\in[p]} \frac{2}{NT} tr\left((E\odot X_j)'\,\tilde{\Delta}_{\Theta_j}\right) - \frac{1}{NT}\sum_{i\in[N]}\sum_{t\in[T]}\left(\tilde{\Delta}_{\Theta_0,it} + \sum_{j\in[p]} X_{j,it}\tilde{\Delta}_{\Theta_j,it}\right)^2 \\
&\le \frac{2}{NT}\left|tr\left(E'\tilde{\Delta}_{\Theta_0}\right)\right| + \sum_{j\in[p]}\frac{2}{NT}\left|tr\left((E\odot X_j)'\,\tilde{\Delta}_{\Theta_j}\right)\right| \\
&\le \frac{2}{NT}\|E\|_{op}\left\|\tilde{\Delta}_{\Theta_0}\right\|_* + \sum_{j\in[p]}\frac{2}{NT}\|E\odot X_j\|_{op}\left\|\tilde{\Delta}_{\Theta_j}\right\|_* \\
&\le 2c_3\sum_{j\in[p]\cup\{0\}}\frac{\sqrt{N}\vee\sqrt{T\log T}}{NT}\left\|\tilde{\Delta}_{\Theta_j}\right\|_*, \qquad \text{(B.7)}
\end{aligned}
$$

where the second inequality holds by the fact that $tr(AB) \le ||A||_{op}||B||_*$, and the last inequality is by the definition of event $\mathscr{A}_{1,N}$.

Combining (B.6) and (B.7), we have

$$
0 \le \sum_{j\in[p]\cup\{0\}}\left\{\frac{2c_3(\sqrt{N}\vee\sqrt{T\log T})}{NT}\left\|\tilde{\Delta}_{\Theta_j}\right\|_* + \nu_j\left(\|\Theta_j^0\|_* - \left\|\tilde{\Theta}_j\right\|_*\right)\right\}\ w.p.a.1. \qquad \text{(B.8)}
$$

Besides, we can show that

$$
\begin{aligned}
\left\|\tilde{\Theta}_j\right\|_* &= \left\|\tilde{\Delta}_{\Theta_j} + \Theta_j^0\right\|_* = \left\|\Theta_j^0 + \mathcal{P}_j^{\perp}(\tilde{\Delta}_{\Theta_j}) + \mathcal{P}_j(\tilde{\Delta}_{\Theta_j})\right\|_* \\
&\ge \left\|\Theta_j^0 + \mathcal{P}_j^{\perp}(\tilde{\Delta}_{\Theta_j})\right\|_* - \left\|\mathcal{P}_j(\tilde{\Delta}_{\Theta_j})\right\|_* = \|\Theta_j^0\|_* + \left\|\mathcal{P}_j^{\perp}(\tilde{\Delta}_{\Theta_j})\right\|_* - \left\|\mathcal{P}_j(\tilde{\Delta}_{\Theta_j})\right\|_*, \qquad \text{(B.9)}
\end{aligned}
$$

where the second equality holds by Lemma D.2(i) in Chernozhukov et al. (2020), the first inequality is by the triangle inequality and the last equality is by the construction of the spaces $\mathcal{P}_j^{\perp}$ and $\mathcal{P}_j$. Then combining (B.8) and (B.9), w.p.a.1, we have

$$
\sum_{j\in[p]\cup\{0\}}\nu_j\left\|\tilde{\Theta}_j\right\|_* \le \sum_{j\in[p]\cup\{0\}}\left\{\nu_j\|\Theta_j^0\|_* + 2c_3\sum_{j\in[p]\cup\{0\}}\frac{(\sqrt{N}\vee\sqrt{T})}{NT}\left\|\tilde{\Delta}_{\Theta_j}\right\|_*\right\}
$$

and

$$
\begin{aligned}
\sum_{j\in[p]\cup\{0\}}\nu_j\left\{\left\|\mathcal{P}_j^{\perp}\left(\tilde{\Delta}_{\Theta_j}\right)\right\|_* - \left\|\mathcal{P}_j\left(\tilde{\Delta}_{\Theta_j}\right)\right\|_*\right\} &\le 2c_3\sum_{j\in[p]\cup\{0\}}\frac{(\sqrt{N}\vee\sqrt{T\log T})}{NT}\left\|\tilde{\Delta}_{\Theta_j}\right\|_* \\
&= 2c_3\sum_{j\in[p]\cup\{0\}}\frac{(\sqrt{N}\vee\sqrt{T\log T})}{NT}\left\{\left\|\mathcal{P}_j\left(\tilde{\Delta}_{\Theta_j}\right)\right\|_* + \left\|\mathcal{P}_j^{\perp}\left(\tilde{\Delta}_{\Theta_j}\right)\right\|_*\right\},
\end{aligned}
$$

If we set $\nu_j = \frac{4c_3(\sqrt{N}\vee\sqrt{T\log T})}{NT}$, we obtain the final result $\sum_{j\in[p]\cup\{0\}}\left\|\mathcal{P}_j^{\perp}(\tilde{\Delta}_{\Theta_j})\right\|_* \le 3\sum_{j\in[p]\cup\{0\}}\left\|\mathcal{P}_j(\tilde{\Delta}_{\Theta_j})\right\|_*$.
■

Lemma B.5 *Consider a sequence of random variables $\{B_i, i\in[n]\}$.*

(i) Suppose B_i, $i\in[n]$, are independent with $\mathbb{E}(B_i) = 0$ and $\max_{i\in[n]}|B_i| \le M$ a.s. Let $\sigma^2 = \sum_{i\in[n]}\mathbb{E}(B_i^2)$. Then for all $t > 0$, we have

$$
\mathbb{P}\left(\left|\sum_{i\in[n]} B_i\right| > t\right) \le \exp\left\{-\frac{t^2/2}{\sigma^2 + Mt/3}\right\}.
$$

(ii) *Suppose* $\{B_i, i\in[n]\}$ *is an m.d.s. with* $\mathbb{E}_{i-1}(B_i)=0$ *and* $\max_{i\in[n]}|B_i|\le M$ *a.s., where* $\mathbb{E}_{i-1}$ *denotes* $\mathbb{E}(\cdot|\mathscr{F}_{i-1})$*, where* $\{\mathscr{F}_i : i\le n\}$ *denotes the filtration that is clear from the context. Let* $\left|\sum_{i\in[n]}\mathbb{E}_{i-1}\left(B_i^2\right)\right|\le\sigma^2$*. Then for all* $t>0$*, we have*

$$\mathbb{P}\left(\left|\sum_{i\in[n]}B_i\right|>t\right)\le\exp\left\{-\frac{t^2/2}{\sigma^2+Mt/3}\right\}.$$

Proof. Lemma B.5(i) and (ii) are Bernstein inequality for the partial sum of an independent sequence and the Freedman inequality for the partial sum of an m.d.s., which are respectively stated in Lemma 2.2.9 Vaart and Wellner (1996) and Theorem 1.1 Tropp (2011). ■

Lemma B.6 *Let* $\{\Upsilon_t, t\ge1\}$ *be a zero-mean strong mixing process, not necessarily stationary, with the mixing coefficients satisfying* $\alpha(z)\le c_\alpha\vartheta^z$ *for some* $c_\alpha>0$ *and* $\vartheta\in(0,1)$*. If* $\sup_{t\in[T]}|\Upsilon_t|\le M_T$*, then there exists a constant* c_4 *depending on* c_α *and* ϑ *such that for any* $T\ge2$ *and* $\varepsilon>0$*,*

(i) $\mathbb{P}\left\{\left|\sum_{t=1}^T\Upsilon_t\right|>\varepsilon\right\}\le\exp\left\{-\frac{c_4\varepsilon^2}{M_T^2T+\varepsilon M_T(\log T)(\log\log T)}\right\}$,

(ii) $\mathbb{P}\left\{\left|\sum_{t=1}^T\Upsilon_t\right|>\varepsilon\right\}\le\exp\left\{-\frac{c_4\varepsilon^2}{v_0^2T+M_T^2+\varepsilon M_T(\log T)^2}\right\}$,

where $v_0^2=\sup_{t\in[T]}\left[Var(\Upsilon_t)+2\sum_{s>t}|Cov(\Upsilon_t,\Upsilon_s)|\right]$.

Proof. The proof is the same as that of Theorems 1 and 2 in Merlevède et al. (2009) with the condition $\alpha(a)\le\exp\{-2ca\}$ for some $c>0$. Here we can set $c=-\log\gamma$ if $c_\alpha\ge1$ and $c=-\log(\gamma/c_\alpha)$ otherwise. ■

Lemma B.7 *Suppose Assumptions 1–4 hold, for* $j\in\{0,\cdots,p\}$*, we have*

(i) $\max_i\left\|u_{i,j}^0\right\|\le M$ *and* $\max_t\left\|v_{t,j}^0\right\|\le\frac{M}{\sigma_{K_j,j}}\le\frac{M}{c_\sigma}$,

(ii) $\max_t\left\|O_j'\tilde{v}_{t,j}\right\|\le\frac{2M}{\sigma_{K_j,j}}\le\frac{2M}{c_\sigma}$ *w.p.a.1,*

(iii) $\max_i\frac{1}{T}\sum_{t\in[T]}\left\|\tilde{\phi}_{it}\right\|^2\le\frac{4M^2}{c_\sigma^2}(1+pC)$ *w.p.a.1,*

(iv) $\max_i\frac{1}{T}\sum_{t\in[T]}\left\|\tilde{\phi}_{it}-\phi_{it}^0\right\|^2=O_p(\eta_{N,1}^2(NT)^{2/q})$.

Proof. (i) Recall that $\frac{1}{\sqrt{NT}}\Theta_j^0=\mathcal{U}_j^0\Sigma_j^0\mathcal{V}_j^{0\prime}$, $U_j^0=\sqrt{N}\mathcal{U}_j^0\Sigma_j^0$ and $V_j=\sqrt{T}\mathcal{V}_j^0$. Let $[A]_{i\cdot}$ and $[A]_{\cdot t}$ denote the i-th row and t-th column of A, respectively. Note that

$$\frac{1}{\sqrt{T}}\Theta_j^0\mathcal{V}_j^0=\sqrt{N}\mathcal{U}_j^0\Sigma_j^0=U_j^0,\text{ and }\frac{1}{\sqrt{N}}\mathcal{U}_j^{0\prime}\Theta_j^0=\sqrt{T}\Sigma_j^0\mathcal{V}_j^{0\prime}=\Sigma_j^0V_j^{0\prime}. \tag{B.10}$$

Hence, it's easy to see that

$$\left\|u_{i,j}^0\right\|=\frac{1}{\sqrt{T}}\left\|\left[\Theta_j^0\mathcal{V}_j^0\right]_{i\cdot}\right\|\le\frac{1}{\sqrt{T}}\left\|[\Theta_j]_{i\cdot}\right\|\le M,$$

where the first inequality is due to the fact that $\mathcal{V}_j$ is the unitary matrix and the last inequality holds by Assumption 2. Since the upper bound M is not dependent on i, this result holds uniformly. Analogously, we see that

$$\left\|v_{t,j}^0\right\|\le\frac{1}{\sqrt{N}}c_\sigma^{-1}\left\|\left[\mathcal{U}_j^{0\prime}\Theta_j^0\right]_{\cdot t}\right\|\le\frac{1}{\sqrt{N}}c_\sigma^{-1}\left\|\left[\Theta_j^0\right]_{\cdot t}\right\|\le\frac{M}{c_\sigma}.$$

(ii) As in (B.10), we have

$$\frac{1}{\sqrt{N}}\tilde{\mathcal{U}}_j^{(1)\prime}\tilde{\Theta}_j = \sqrt{T}\tilde{\Sigma}_j^{(1)}\tilde{\mathcal{V}}_j^{(1)\prime} = \tilde{\Sigma}_j^{(1)}\tilde{V}_j^{(1)\prime},$$

and

$$\left\|O_j'\tilde{v}_{t,j}\right\| \leq \frac{1}{\sqrt{N}}\frac{1}{\tilde{\sigma}_{K_j,j}^{(1)}}\left\|\left[\tilde{\mathcal{U}}_j^{(1)\prime}\tilde{\Theta}_j\right]_{\cdot t}\right\| \leq \frac{1}{\sqrt{N}}\frac{1}{\tilde{\sigma}_{K_j,j}^{(1)}}\left\|\left[\tilde{\Theta}_j\right]_{\cdot t}\right\| \leq \frac{2M}{c_\sigma},$$

where the last inequality holds due to the constrained optimization in (3.1) and the fact that $\max_{k\in[K_j]}|\tilde{\sigma}_{k,j}^{-1} - \sigma_{k,j}^{-1}| \leq \sigma_{K_j,j}^{-1}$ w.p.a.1.

(iii) Note that

$$\begin{aligned}
\max_i \frac{1}{T}\sum_{t\in[T]}\left\|\tilde{\phi}_{it}\right\|^2 &\leq \max_i\left\{\frac{1}{T}\sum_{t\in[T]}\|O_0'\tilde{v}_{t,0}\|^2 + \sum_{j\in[p]}\frac{1}{T}\sum_{t\in[T]}\left\|O_j'\tilde{v}_{t,j}\right\|^2|X_{j,it}|^2\right\} \\
&\leq \max_{t\in[T],j\in[p]\cup\{0\}}\left\|O_j'\tilde{v}_{t,j}\right\|^2\left\{1+\max_i\sum_{j\in[p]}\frac{1}{T}\sum_{t\in[T]}|X_{j,it}|^2\right\} \leq \frac{4M^2}{c_\sigma^2}(1+pC)
\end{aligned}$$

where the last inequality holds by Lemma B.7(ii).

(iv) Note that

$$\begin{aligned}
&\max_i \frac{1}{T}\sum_{t\in[T]}\left\|\tilde{\phi}_{it}-\phi_{it}^0\right\|^2 \leq \frac{1}{T}\sum_{t\in[T]}\left\|\tilde{O}_0^{(1)\prime}\tilde{v}_{t,0}-v_{t,0}^0\right\|^2 + p\max_{t\in[T],j\in[p]}\frac{1}{T}\sum_{t\in[T]}|X_{j,it}|^2\left\|\tilde{O}_j^{(1)\prime}\tilde{v}_{t,j}-v_{t,j}^0\right\|^2 \\
&\lesssim \frac{1}{T}\sum_{t\in[T]}\left\|\tilde{O}_0^{(1)\prime}\tilde{v}_{t,0}-v_{t,0}^0\right\|^2 + p(NT)^{2/q}\max_{j\in[p]}\frac{1}{T}\sum_{t\in[T]}\left\|\tilde{O}_j^{(1)\prime}\tilde{v}_{t,j}-v_{t,j}^0\right\|^2 \\
&= \frac{1}{T}\left\|O_0\tilde{V}_0-V_0^0\right\|^2 + p(NT)^{2/q}\max_{j\in[p]}\frac{1}{T}\left\|O_j\tilde{V}_j-V_j^0\right\|^2 = O_p(\eta_{N,1}^2(NT)^{2/q}),
\end{aligned}$$

where the second inequality is by Assumption 1(v) and the last equality holds by Theorem 4.1(ii). ■

Lemma B.8 *Under Assumptions 1–5, we have* $\min_{i\in[N]}\lambda_{\min}(\tilde{\Phi}_i) \geq \frac{c_\phi}{2}$ *w.p.a.1, and* $\min_{t\in[T]}\lambda_{\min}(\tilde{\Psi}_t) \geq \frac{c_\phi}{2}$ *w.p.a.1.*

Proof. Recall that $\Phi_i = \frac{1}{T}\sum_{t=1}^T \phi_{it}^0\phi_{it}^{0\prime}$ and $\tilde{\Phi}_i = \frac{1}{T}\sum_{t=1}^T\tilde{\phi}_{it}\tilde{\phi}_{it}'$, where

$$\phi_{it}^0 = [v_{t,0}^{0\prime}, v_{t,1}^{0\prime}X_{1,it},\cdots,v_{t,p}^{0\prime}X_{p,it}]' \text{ and } \tilde{\phi}_{it} = \left[(O_0'\tilde{v}_{t,0})', (O_1'\tilde{v}_{t,1}X_{1,it})',\cdots,(O_p'\tilde{v}_{t,p}X_{p,it})'\right]'.$$

Uniformly over $i\in[N]$, it is clear that

$$\begin{aligned}
\left\|\tilde{\Phi}_i-\Phi_i\right\| &\lesssim \frac{4M}{c_\sigma T}\sum_{t=1}^T\left\|O_0'\tilde{v}_{t,0}-v_{t,0}^0\right\| + \frac{4M}{c_\sigma T}\sum_{j=1}^p\sum_{t=1}^T\left\|O_j'\tilde{v}_{t,j}-v_{t,j}^0\right\||X_{j,it}| \\
&\leq \frac{4M}{c_\sigma}\frac{1}{\sqrt{T}}\left\|O_0'\tilde{V}_0-V_0^0\right\| + \frac{4M^2}{c_\sigma}\sum_{j=1}^p\frac{1}{\sqrt{T}}\left\|O_j'\tilde{V}_j-V_j^0\right\|\left(\frac{1}{T}\sum_{t\in[T]}|X_{j,it}|^2\right)^{1/2} = O_p\left(\eta_{N,1}\right),
\end{aligned}$$

where the last equality holds by Lemma B.7(i) and Assumption 1(iv). It follows that

$$\min_{i\in[N]}\lambda_{\min}(\tilde{\Phi}_i) \geq \min_{i\in[N]}\lambda_{\min}(\Phi_i) - O\left(\eta_{N,1}\right) \geq \frac{c_\phi}{2}, \quad \text{w.p.a.1.}$$

Analogously, we can establish the lower bound of $\lambda_{\min}(\tilde{\Psi}_t)$. ■

Lemma B.9 *Under Assumptions 1–7, we have* $\max_i \mathbf{1}\{\hat{g}^{0,(\ell)}_{i,K^{(\ell)}} \neq g^{(\ell)}_i\} = 0$ *w.p.a.1, where* $\hat{g}^{0,(\ell)}_{i,K^{(\ell)}}$ *is defined in* (A.21).

Proof. The above lemma holds by Theorem 2.3 in Su et al. (2020) provided we can verify the conditions in their Assumption 4. Let $\alpha^{(\ell)}_k = (\alpha^{(\ell)}_{k,1}, \cdots, \alpha^{(\ell)}_{k,p})'$. Then we have

$$\beta^{0,(\ell)}_i = \frac{1}{\sqrt{T_\ell}} \sum_{k \in [K^{(\ell)}]} \alpha^{(\ell)}_k \otimes \iota_{T_\ell} \mathbf{1}\{g^{(\ell)}_i = k\}$$

and

$$\max_{k\in[K^{(\ell)}]} \left\| \frac{1}{\sqrt{T_\ell}} \alpha^{(\ell)}_k \otimes \iota_{T_\ell} \right\| = \max_{k\in[K^{(\ell)}]} \frac{1}{\sqrt{T_\ell}} \sqrt{T_\ell \sum_{j=1}^{p} (\alpha^{(\ell)}_{k,j})^2} \leq \sqrt{p} \max_{k\in[K^{(\ell)}], j\in[p]} |\alpha^{(\ell)}_{k,j}| \leq \sqrt{p} M, \tag{B.11}$$

where the last inequality is due to Assumption 2.

Second, with $\Theta^{0,(1)}_{j,i} = \left(\Theta^0_{j,i1}, \cdots, \Theta^0_{j,iT_1}\right)'$ and $\Theta^{0,(2)}_{j,i} = \left(\Theta^0_{j,i,T_1+1}, \cdots, \Theta^0_{j,iT}\right)'$, we observe that

$$\begin{aligned} \max_i \left\| \dot{\beta}^{0,(\ell)}_i - \beta^{0,(\ell)}_i \right\| &= \frac{1}{\sqrt{T_\ell}} \max_i \left\| \dot{\Theta}^{(\ell)}_i - \Theta^{0,(\ell)}_i \right\| = \frac{1}{\sqrt{T_\ell}} \max_i \sqrt{\sum_{j=1}^{p} \sum_{t\in\mathcal{T}_\ell} (\dot{\Theta}^{(\ell)}_{j,it} - \Theta^{0,(\ell)}_{j,it})^2} \\ &\leq \sqrt{p} \max_{j\in[p], i\in[N], t\in[T]} \left| \dot{\Theta}_{j,it} - \Theta^0_{j,it} \right| \leq c_5 \eta_{N,2} \quad \text{w.p.a.1}, \end{aligned} \tag{B.12}$$

with c_5 being some positive large enough constant, and the last inequality holds by Theorem 4.1(iii).

Third, we also observe that

$$\min_{1\leq k_s < k_{s^*} \leq K^{(\ell)}} \frac{1}{\sqrt{T_\ell}} \left\| \alpha^{(\ell)}_{k_s} \otimes \iota_{T_\ell} - \alpha^{(\ell)}_{k_{s^*}} \otimes \iota_{T_\ell} \right\| = \min_{1\leq k_s < k_{s^*} \leq K^{(\ell)}} \sqrt{\sum_{j=1}^{p} (\alpha^{(\ell)}_{k_s,j} - \alpha^{(\ell)}_{k_{s^*},j})^2} \geq C_5, \tag{B.13}$$

where the last inequality holds by Assumption 7(i).

Combining (B.11), (B.12) and (B.13), we obtain that $\mathbb{P}\left(\max_i \mathbf{1}\{\hat{g}^{0,(\ell)}_{i,K^{(\ell)}} \neq g^{(\ell)}_i\} = 0\right) \to 1$ once we can ensure Assumption 4.3 in Su et al. (2020) holds with $c_{1n} = C_5$, $c_{2n} = c_5\eta_{N,2}$, $K = K^{(1)}$, and with their c_1 and M being replaced by $\underline{c}$ and and $\sqrt{p}M$ here. Under Assumption 7, Assumption 4.3 in Su et al. (2020) holds. This completes the proof of the lemma. ■

To study the NSP property of our group structure estimator, we introduce some notation in the following definition.

Definition B.1 *Fix* $K^{(\ell)} > 1$ *and* $1 < m \leq K^{(\ell)}$. *Define a* $K^{(\ell)} \times p$ *matrix* $\alpha^{(\ell)} = (\alpha^{(\ell)}_1, \cdots, \alpha^{(\ell)}_K)'$. *Let* $d_{K^{(\ell)}}(\alpha^{(\ell)})$ *be the minimum pairwise distance of all* $K^{(\ell)}$ *rows and* $\alpha^{(\ell)}_k$ *and* $\alpha^{(\ell)}_l$ *be the pair that satisfies* $||\alpha^{(\ell)}_k - \alpha^{(\ell)}_l|| = d_{K^{(\ell)}}(\alpha^{(\ell)})$ *(if this holds for multiple pairs, pick the first pair in lexicographical order). Remove row* l *from matrix* $\alpha^{(\ell)}$ *and let* $d_{K^{(\ell)}-1}(\alpha^{(\ell)})$ *be the minimum pairwise distance for the remaining* $(K^{(\ell)} - 1)$ *rows. Repeat this step and define* $d_{K^{(\ell)}-2}(\alpha^{(\ell)}), \cdots, d_2(\alpha^{(\ell)})$ *recursively.*

Lemma B.10 *Recall that* $\hat{\mathcal{G}}^{(\ell)}_m$ *is the estimated group structure from K-means algorithm with* m *groups. Under Assumptions 1-7 and the event* $\{\hat{T}_1 = T_1\}$, *w.p.a.1, for each* $1 < m < K^{(\ell)}$, $\hat{\mathcal{G}}^{(\ell)}_m$ *enjoys the NSP defined in Definition 4.1.*

Proof. By Theorem 4.1 in Jin et al. (2022), Lemma B.10 is proved if we ensure all conditions in their Theorem 4.1 hold. We now apply their Theorem 4.1 with $\hat{x}_i = \dot{\beta}_i^{0,(\ell)}$, $x_i = \beta_i^{0,(\ell)}$ and $u_k = \frac{1}{\sqrt{T_\ell}}\alpha_k^{(\ell)} \otimes \iota_{T_\ell}$ for $k \in [K^{(\ell)}]$. By the definition of $d_m\left(\alpha^{(\ell)}\right)$ in Definition B.1, we notice that $d_m\left(\alpha^{(\ell)}\right) \geq d_{K^{(\ell)}}\left(\alpha^{(\ell)}\right)$ such that $d_{K^{(\ell)}}\left(\alpha^{(\ell)}\right) \geq C_5$ by Assumption 7(i). With (B.12) shown above and Assumption 2, we have

$$\max_{k\in[K^{(\ell)}]} \|u_k\| \leq M, \quad \max_i \|\hat{x}_i - x_i\| = O_p(\eta_{N,2}),$$

which satisfy the Theorem 4.1 in Jin et al. (2022), i.e., $\max_{k\in[K^{(\ell)}]}\|u_k\| \lesssim d_m\left(\alpha^{(\ell)}\right)$ and $\max_i \|\hat{x}_i - x_i\| \lesssim d_m\left(\alpha^{(\ell)}\right)$. Consequently, it leads to the NSP of $\hat{\mathcal{G}}_m^{(\ell)}$ for $1 < m < K^{(\ell)}$ w.p.a.1 under the event $\{\hat{T}_1 = T_1\}$. ■

Lemma B.11 *Under Assumptions 1, 6(ii), 7(ii), 8 and 9(i)-(iii), for $\ell = \{1,2\}$ and $k \in [K^{(\ell)}]$, we have $\hat{\alpha}_k^{(\ell)} \xrightarrow{p} \alpha_k^{(\ell)}$ and $\sqrt{N_k^{(\ell)}T_\ell}(\hat{\alpha}_k^{(\ell)} - \alpha_k^{(\ell)}) = \mathbb{W}_{NT,k}^{(\ell)-1}\mathbb{C}_{NT,k}^{(\ell)} + o_p(1)$.*

Proof. The result in the lemma combines those in Theorem 4.1 and Corollary 4.2 in Moon and Weidner (2017) under their Assumptions 1-4. Hence, we only need to verify the conditions in their Assumptions 2 and 3 since our Assumptions 8 and 9(ii)-(iii) are the same as their Assumptions 1 and 4.

Notice that the Assumption 2 in Moon and Weidner (2017) holds if we can show that

$$\frac{1}{N_k^{(\ell)}T_\ell}\sum_{i\in G_k^{(\ell)}}\sum_{t\in\mathcal{T}_\ell} X_{j,it}e_{it} \xrightarrow{p} 0,\ \forall k \in [K^{(\ell)}],\ \ell \in \{1,2\}.$$

Fix a specific k and ℓ. We can show that

$$\begin{aligned}
&\mathbb{E}\left(\frac{1}{N_k^{(\ell)}T_\ell}\sum_{i\in G_k^{(\ell)}}\sum_{t\in\mathcal{T}_\ell} X_{j,it}e_{it}\middle|\mathscr{D}\right)^2 \\
&= \frac{1}{\left(N_k^{(\ell)}T_\ell\right)^2}\sum_{i_1\in G_k^{(\ell)}}\sum_{i_2\in G_k^{(\ell)}}\sum_{t_1\in\mathcal{T}_\ell}\sum_{t_2\in\mathcal{T}_\ell}\mathbb{E}\left(X_{j,i_1t_1}X_{j,i_2t_2}e_{i_1t_1}e_{i_2t_2}|\mathscr{D}\right) \\
&= \frac{1}{\left(N_k^{(\ell)}T_\ell\right)^2}\sum_{i\in G_k^{(\ell)}}\sum_{t_1\in\mathcal{T}_\ell}\sum_{t_2\in\mathcal{T}_\ell}\mathbb{E}\left(X_{j,it_1}X_{j,it_2}e_{it_1}e_{it_2}|\mathscr{D}\right) \\
&= \frac{1}{\left(N_k^{(\ell)}T_\ell\right)^2}\sum_{i\in G_k^{(\ell)}}\sum_{t\in\mathcal{T}_\ell}\mathbb{E}\left(X_{j,it}^2e_{it}^2|\mathscr{D}\right) + \frac{2}{\left(N_k^{(\ell)}T_\ell\right)^2}\sum_{i\in G_k^{(\ell)}}\sum_{t_1\in\mathcal{T}_\ell}\sum_{t_2\in\mathcal{T}_\ell,t_2>t_1}\mathbb{E}\left(X_{j,it_1}X_{j,it_2}e_{it_1}e_{it_2}|\mathscr{D}\right) \\
&\leq \frac{M}{N_k^{(\ell)}T_\ell} + \frac{16}{\left(N_k^{(\ell)}T_\ell\right)^2}\max_{i\in G_k^{(\ell)}}\max_{t\in\mathcal{T}_L}\left(\mathbb{E}\left|X_{j,it}e_{it}\right|^q\right)^{2/q}\sum_{i\in G_k^{(\ell)}}\sum_{t_1\in\mathcal{T}_\ell}\sum_{t_2\in\mathcal{T}_\ell,t_2>t_1}\left[\alpha(t_2-t_1)\right]^{1-2/q} \\
&= O\left((NT)^{-1}\right),
\end{aligned} \tag{B.14}$$

where the second equality holds by Assumption 1(i) with the conditional independence sequence for $i_1 \neq i_2$, the first inequality combines Assumption 1(ii), (iii), (v), and the Davydov's inequality for strong mixing sequence in Lemma 4.3, Su and Chen (2013), and the last equality is by Assumption 1(iii), (v), Assumption 6(ii) and Assumption 7(ii). Following this, it yields that

$$\frac{1}{N_k^{(\ell)}T_\ell}\sum_{i\in G_k^{(\ell)}}\sum_{t\in\mathcal{T}_\ell} X_{j,it}e_{it} = O_p((NT)^{-1/2}).$$

By similar arguments to those used in the proof of Lemma B.3, we can show that

$$\left\|E_k^{(\ell)}\right\|_{op} = O_p(\sqrt{N} + \sqrt{T\log T}), \tag{B.15}$$

which, in conjunction with Assumption 9(i), implies that Assumption 3* in Moon and Weidner (2017) is satisfied. ■

For $j \in [p]$, recall that $X_{j,i}^{(1)} = (X_{j,i1}, \cdots, X_{j,iT_1})'$, $X_{j,i}^{(2)} = \left(X_{j,i(T_1+1)}, \cdots, X_{j,iT}\right)'$, $e_i^{(1)} = (e_{i1}, \cdots, e_{iT_1})'$, $e_i^{(2)} = \left(e_{i(T_1+1)}, \cdots, e_{iT}\right)'$, $\tilde{X}_{j,it} = X_{j,it} - \mathbb{E}\left(X_{j,it}|\mathscr{D}\right)$. Besides, let $\mathbb{X}_{j,k}^{(\ell)} \in \mathbb{R}^{N_k^{(\ell)} \times T_\ell}$ and $E_k^{(\ell)} \in \mathbb{R}^{N_k^{(\ell)} \times T_\ell}$ denote the regressor and error matrix for subgroup $k \in [K^{(\ell)}]$ with a typical row being $X_{j,i}^{(\ell)}$ and $e_i^{(\ell)}$, respectively. For $\ell \in \{1, 2\}$ and $k \in [K^{(\ell)}]$, we also define

$$\bar{\mathbb{X}}_{j,k}^{(\ell)} = \mathbb{E}(\mathbb{X}_{j,k}^{(\ell)}|\mathscr{D}), \quad \tilde{\mathbb{X}}_{j,k}^{(\ell)} = \mathbb{X}_{j,k}^{(\ell)} - \bar{\mathbb{X}}_{j,k}^{(\ell)}, \quad \mathfrak{X}_{j,k}^{(\ell)} = M_{\Lambda_k^{0,(\ell)}} \bar{\mathbb{X}}_{j,k}^{(\ell)} M_{F^{0,(\ell)}} + \tilde{\mathbb{X}}_{j,k}^{(\ell)},$$

with $\mathfrak{X}_{j,k,it}^{(\ell)}$ being each entry of $\mathfrak{X}_{j,k}^{(\ell)}$. Further let $\mathfrak{X}_{k,it}^{(\ell)} = (\mathfrak{X}_{1,k,it}^{(\ell)}, \cdots, \mathfrak{X}_{p,k,it}^{(\ell)})'$.

Lemma B.12 *Under Assumptions 1, 2, 6(ii), 7(ii), 8 and 9, for $j \in [p]$, $\ell \in \{1, 2\}$ and $k \in [K^{(\ell)}]$, we have*

(i) $\frac{1}{\sqrt{N_k^{(\ell)} T_\ell}} tr\left(P_{F^{0,(\ell)}} E_k^{(\ell)\prime} P_{\Lambda_k^{0,(\ell)}} \tilde{\mathbb{X}}_{j,k}^{(\ell)}\right) = o_p(1)$,

(ii) $\frac{1}{\sqrt{N_k^{(\ell)} T_\ell}} tr\left(P_{\Lambda_k^{0,(\ell)}} E_k^{(\ell)\prime} \tilde{\mathbb{X}}_{j,k}^{(\ell)}\right) = o_p(1)$,

(iii) $\frac{1}{\sqrt{N_k^{(\ell)} T_\ell}} tr\left\{P_{F^{0,(\ell)}} \left[E_k^{(\ell)\prime} \tilde{\mathbb{X}}_{j,k}^{(\ell)} - \mathbb{E}\left(E_k^{(\ell)\prime} \tilde{\mathbb{X}}_{j,k}^{(\ell)}|\mathscr{D}\right)\right]\right\} = o_p(1)$,

(iv) $\frac{1}{\sqrt{N_k^{(\ell)} T_\ell}} tr\left[E_k^{(\ell)} P_{F^{0,(\ell)}} E_k^{(\ell)\prime} M_{\Lambda_k^{0,(\ell)}} \mathbb{X}_{j,k}^{(\ell)} F^{0,(\ell)} \left(F^{0,(\ell)\prime} F^{0,(\ell)}\right)^{-1} \left(\Lambda_k^{0,(\ell)\prime} \Lambda_k^{0,(\ell)}\right)^{-1} \Lambda_k^{0,(\ell)\prime}\right] = o_p(1)$,

(v) $\frac{1}{\sqrt{N_k^{(\ell)} T_\ell}} tr\left[E_k^{(\ell)\prime} P_{\Lambda_k^{0,(\ell)}} E_k^{(\ell)} M_{F^{0,(\ell)}} \mathbb{X}_{j,k}^{(\ell)\prime} \Lambda_k^{0,(\ell)} \left(\Lambda_k^{0,(\ell)\prime} \Lambda_k^{0,(\ell)}\right)^{-1} \left(F^{0,(\ell)\prime} F^{0,(\ell)}\right)^{-1} F^{0,(\ell)\prime}\right] = o_p(1)$,

(vi) $\frac{1}{\sqrt{N_k^{(\ell)} T_\ell}} tr\left[E_k^{(\ell)\prime} M_{\Lambda_k^{0,(\ell)}} \mathbb{X}_{j,k}^{(\ell)} M_{F^{0,(\ell)}} E_k^{(\ell)\prime} \Lambda_k^{0,(\ell)} \left(\Lambda_k^{0,(\ell)\prime} \Lambda_k^{0,(\ell)}\right)^{-1} \left(F^{0,(\ell)\prime} F^{0,(\ell)}\right)^{-1} F^{0,(\ell)\prime}\right] = o_p(1)$,

(vii) $\frac{1}{\sqrt{N_k^{(\ell)} T_\ell}} tr\left\{\left[E_k^{(\ell)} E_k^{(\ell)\prime} - \mathbb{E}\left(E_k^{(\ell)} E_k^{(\ell)\prime}|\mathscr{D}\right)\right] M_{\Lambda_k^{0,(\ell)}} \mathbb{X}_{j,k}^{(\ell)} F^{0,(\ell)} \left(F^{0,(\ell)\prime} F^{0,(\ell)}\right)^{-1} \left(\Lambda_k^{0,(\ell)\prime} \Lambda_k^{0,(\ell)}\right)^{-1} \Lambda_k^{0,(\ell)\prime}\right\} = o_p(1)$,

(viii) $\frac{1}{\sqrt{N_k^{(\ell)} T_\ell}} tr\left\{\left[E_k^{(\ell)\prime} E_k^{(\ell)} - \mathbb{E}\left(E_k^{(\ell)\prime} E_k^{(\ell)}|\mathscr{D}\right)\right] M_{\Lambda_k^{0,(\ell)}} \mathbb{X}_{j,k}^{(\ell)\prime} \Lambda_k^{0,(\ell)} \left(\Lambda_k^{0,(\ell)\prime} \Lambda_k^{0,(\ell)}\right)^{-1} \left(F^{0,(\ell)\prime} F^{0,(\ell)}\right)^{-1} F^{0,(\ell)\prime}\right\} = o_p(1)$,

(ix) $\frac{1}{N_k^{(\ell)} T_\ell} \sum_{i \in G_k^{(\ell)}} \sum_{t \in \mathcal{T}_\ell} \left[e_{it}^2 \mathfrak{X}_{k,it}^{(\ell)} \mathfrak{X}_{k,it}^{(\ell)\prime} - \mathbb{E}\left(e_{it}^2 \mathfrak{X}_{k,it}^{(\ell)} \mathfrak{X}_{k,it}^{(\ell)\prime}|\mathscr{D}\right)\right] = o_p(1)$,

(x) $\frac{1}{N_k^{(\ell)} T_\ell} \sum_{i \in G_k^{(\ell)}} \sum_{t \in \mathcal{T}_\ell} e_{it}^2 \left(\mathfrak{X}_{k,it}^{(\ell)} \mathfrak{X}_{k,it}^{(\ell)\prime} - \mathcal{X}_{it} \mathcal{X}_{it}'\right) = o_p(1)$.

Proof. (i) We first show that $\left\|F^{0,(\ell)\prime} E_k^{(\ell)\prime} \Lambda_k^{0,(\ell)}\right\| = O_p(\sqrt{NT})$. Note that

$$\mathbb{E}\left[\left(\frac{\left\|F^{0,(\ell)\prime} E_k^{(\ell)\prime} \Lambda_k^{0,(\ell)}\right\|}{\sqrt{N_k^{(\ell)} T_\ell}}\right)^2 \middle| \mathscr{D}\right] = \frac{1}{N_k^{(\ell)} T_\ell} \mathbb{E}\left[\left(\sum_{i \in G_k^{(\ell)}} \sum_{t \in \mathcal{T}_\ell} e_{it} f_t^{0\prime} \lambda_i^0\right)^2 \middle| \mathscr{D}\right]$$

$$
\begin{aligned}
&= \frac{1}{N_k^{(\ell)} T_\ell} \sum_{i_1 \in G_k^{(\ell)}} \sum_{i_2 \in G_k^{(\ell)}} \sum_{t_1 \in \mathcal{T}_\ell} \sum_{t_2 \in \mathcal{T}_\ell} \mathbb{E}\left(e_{i_1 t_1} e_{i_2 t_2} | \mathscr{D}\right) f_{t_1}^{0\prime} \lambda_{i_1}^0 \lambda_{i_2}^{0\prime} f_{t_2}^0 \\
&\leq \max_{i \in G_k^{(\ell)}} \left\|\lambda_i^0\right\|_2^2 \max_{t \in \mathcal{T}_\ell} \left\|f_t^0\right\|_2^2 \frac{1}{N_k^{(\ell)} T_\ell} \sum_{i \in G_k^{(\ell)}} \sum_{t_1 \in \mathcal{T}_\ell} \sum_{t_2 \in \mathcal{T}_\ell} \left|\mathbb{E}\left(e_{it_1} e_{it_2} | \mathscr{D}\right)\right| \\
&\lesssim \frac{1}{N_k^{(\ell)} T_\ell} \sum_{i \in G_k^{(\ell)}} \sum_{t \in \mathcal{T}_\ell} \left|Var\left(e_{it} | \mathscr{D}\right)\right| + \frac{2}{N_k^{(\ell)} T_\ell} \sum_{i \in G_k^{(\ell)}} \sum_{t_1 \in \mathcal{T}_\ell} \sum_{t_2 \in \mathcal{T}_\ell, t_2 > t_1} \left|Cov\left(e_{it_1}, e_{it_2} | \mathscr{D}\right)\right| \\
&= O(1) \ a.s.,
\end{aligned}
$$

where the fourth line is by Lemma B.7(i) and the last line combines Assumption 1(v) and Davydov's inequality for conditional strong mixing sequences, similarly as (B.14). It follows that

$$
\begin{aligned}
\left\|P_{F^{0,(\ell)}} E_k^{(\ell)\prime} P_{\Lambda_k^{0,(\ell)}}\right\| &\leq \left\|F^{0,(\ell)}\right\| \left\|F^{0,(\ell)\prime} F^{0,(\ell)}\right\| \left\|F^{0,(\ell)\prime} E_k^{(\ell)\prime} \Lambda_k^{0,(\ell)}\right\| \left\|(\Lambda_k^{0,(\ell)\prime} \Lambda_k^{0,(\ell)})^{-1}\right\| \left\|\Lambda_k^{0,(\ell)\prime}\right\| \\
&= O(T^{1/2}) O_p(T^{-1}) O_p(\sqrt{NT}) O_p(N^{-1}) O(N^{1/2}) = O_p(1), \qquad \text{(B.16)}
\end{aligned}
$$

where the first equality holds by Assumptions 2 and 8.

Moreover, we have

$$
\left\|P_{\Lambda_k^{0,(\ell)}} \tilde{\mathbb{X}}_{j,k}^{(\ell)}\right\| \leq \left\|\Lambda_k^{0,(\ell)}\right\| \left\|(\Lambda_k^{0,(\ell)\prime} \Lambda_k^{0,(\ell)})^{-1}\right\| \left\|\Lambda_k^{0,(\ell)\prime} \tilde{\mathbb{X}}_{j,k}^{(\ell)}\right\| = O(N^{1/2}) O_p(N^{-1}) O_p(\sqrt{NT}) = O_p(T^{1/2}), \quad \text{(B.17)}
$$

where the first equality holds by Assumptions 2 and 8(i) and the fact that

$$
\begin{aligned}
\mathbb{E}\left(\left\|\Lambda_k^{0,(\ell)\prime} \tilde{\mathbb{X}}_{j,k}^{(\ell)}\right\|^2 | \mathscr{D}\right) &= \sum_{r=1}^{r_0} \sum_{t \in \mathcal{T}_\ell} \mathbb{E}\left[\left(\sum_{i \in G_k^{(\ell)}} \lambda_{i,r}^0 \tilde{X}_{j,it}\right)^2 \middle| \mathscr{D}\right] \\
&= \sum_{r=1}^{r_0} \sum_{t \in \mathcal{T}_\ell} \sum_{i \in G_k^{(\ell)}} \sum_{i^* \in G_k^{(\ell)}} \lambda_{i,r}^0 \lambda_{i^*,r}^0 \mathbb{E}[\tilde{X}_{j,it} \tilde{X}_{j,i^*t} | \mathscr{D}] \\
&= \sum_{r=1}^{r_0} \sum_{t \in \mathcal{T}_\ell} \sum_{i \in G_k^{(\ell)}} \left(\lambda_{i,r}^0\right)^2 \mathbb{E}\left[(\tilde{X}_{j,it})^2 | \mathscr{D}\right] = O_p(NT).
\end{aligned}
$$

Then we are ready to show that

$$
\begin{aligned}
\left|\frac{1}{\sqrt{N_k^{(\ell)} T_\ell}} tr\left(P_{F^{0,(\ell)}} E_k^{(\ell)\prime} P_{\Lambda_k^{0,(\ell)}} \tilde{\mathbb{X}}_{j,k}^{(\ell)}\right)\right| &= \left|\frac{1}{\sqrt{N_k^{(\ell)} T_\ell}} tr\left(P_{F^{0,(\ell)}} E_k^{(\ell)\prime} P_{\Lambda_k^{0,(\ell)}} P_{\Lambda_k^{0,(\ell)}} \tilde{\mathbb{X}}_{j,k}^{(\ell)}\right)\right| \\
&\leq \frac{1}{\sqrt{N_k^{(\ell)} T_\ell}} \left\|P_{F^{0,(\ell)}} E_k^{(\ell)\prime} P_{\Lambda_k^{0,(\ell)}}\right\| \left\|P_{\Lambda_k^{0,(\ell)}} \tilde{\mathbb{X}}_{j,k}^{(\ell)}\right\| \\
&= \frac{1}{\sqrt{N_k^{(\ell)} T_\ell}} O_p(1) O_p(T^{1/2}) = O(N^{-1/2}) = o_p(1).
\end{aligned}
$$

(ii) Let $[A]_{jl}$ denote the (j,l)-th element of A. Note that

$$
\left|\frac{1}{\sqrt{N_k^{(\ell)} T_\ell}} tr\left(P_{\Lambda_k^{0,(\ell)}} E_k^{(\ell)\prime} \tilde{\mathbb{X}}_{j,k}^{(\ell)}\right)\right|
$$

$$= \left| \sum_{j_1,j_2=1}^{r_0} \left[\left(\frac{1}{N_k^{(\ell)}} \Lambda_k^{0,(\ell)\prime} \Lambda_k^{0,(\ell)} \right)^{-1} \right]_{j_1 j_2} \frac{1}{N_k^{(\ell)} \sqrt{N_k^{(\ell)} T_\ell}} \sum_{t \in \mathcal{T}_\ell} \sum_{i_1 \in G_k^{(\ell)}} \sum_{i_2 \in G_k^{(\ell)}} \lambda_{i_1,j_1}^0 \lambda_{i_2,j_2}^0 e_{i_1 t} \tilde{X}_{j,i_2 t}^{(\ell)} \right|$$

$$\lesssim \max_{j_1,j_2 \in [r_0]} \left| \frac{1}{N_k^{(\ell)} \sqrt{N_k^{(\ell)} T_\ell}} \sum_{t \in \mathcal{T}_\ell} \sum_{i_1 \in G_k^{(\ell)}} \sum_{i_2 \in G_k^{(\ell)}} \lambda_{i_1,j_1}^0 \lambda_{i_2,j_2}^0 e_{i_1 t} \tilde{X}_{j,i_2 t}^{(\ell)} \right| = O_p(N^{-1/2}),$$

where the last line holds by the fact that

$$\mathbb{E} \left(\left| \frac{1}{N_k^{(\ell)} \sqrt{N_k^{(\ell)} T_\ell}} \sum_{t \in \mathcal{T}_\ell} \sum_{i_1 \in G_k^{(\ell)}} \sum_{i_2 \in G_k^{(\ell)}} \lambda_{i_1,j_1}^0 \lambda_{i_2,j_2}^0 e_{i_1 t} \tilde{X}_{j,i_2 t}^{(\ell)} \right|^2 \middle| \mathscr{D} \right)$$
$$= \frac{1}{\left(N_k^{(\ell)}\right)^3 T_\ell} \sum_{t \in \mathcal{T}_\ell} \sum_{s \in \mathcal{T}_\ell} \sum_{i_1 \in G_k^{(\ell)}} \sum_{i_2 \in G_k^{(\ell)}} \sum_{m_1 \in G_k^{(\ell)}} \sum_{m_2 \in G_k^{(\ell)}} \lambda_{i_1,j_1}^0 \lambda_{i_2,j_2}^0 \lambda_{m_1,j_1}^0 \lambda_{m_2,j_2}^0 \mathbb{E} \left(e_{i_1 t} \tilde{X}_{j,i_2 t}^{(\ell)} e_{m_1 s} \tilde{X}_{j,m_2 s}^{(\ell)} \middle| \mathscr{D} \right)$$
$$= \frac{1}{\left(N_k^{(\ell)}\right)^3 T_\ell} \sum_{t \in \mathcal{T}_\ell} \sum_{s \in \mathcal{T}_\ell} \sum_{i_1 \in G_k^{(\ell)}} \sum_{i_2 \in G_k^{(\ell)}} \left(\lambda_{i_1,j_1}^0\right)^2 \left(\lambda_{i_2,j_2}^0\right)^2 \mathbb{E} \left(e_{i_1 t} e_{i_1 s} \tilde{X}_{j,i_2 t}^{(\ell)} \tilde{X}_{j,i_2 s}^{(\ell)} \middle| \mathscr{D} \right)$$
$$= \frac{1}{\left(N_k^{(\ell)}\right)^3 T_\ell} \sum_{t \in \mathcal{T}_\ell} \sum_{i_1 \in G_k^{(\ell)}} \sum_{i_2 \in G_k^{(\ell)}} \left(\lambda_{i_1,j_1}^0\right)^2 \left(\lambda_{i_2,j_2}^0\right)^2 \mathbb{E} \left(e_{i_1 t}^2 \left(\tilde{X}_{j,i_2 t}^{(\ell)} \right)^2 \middle| \mathscr{D} \right)$$
$$+ \frac{2}{\left(N_k^{(\ell)}\right)^3 T_\ell} \sum_{t \in \mathcal{T}_\ell} \sum_{s \in \mathcal{T}_\ell, s>t} \sum_{i_1 \in G_k^{(\ell)}} \sum_{i_2 \in G_k^{(\ell)}} \left(\lambda_{i_1,j_1}^0\right)^2 \left(\lambda_{i_2,j_2}^0\right)^2 \mathbb{E} \left(e_{i_1 t} e_{i_1 s} \tilde{X}_{j,i_2 t}^{(\ell)} \tilde{X}_{j,i_2 s}^{(\ell)} \middle| \mathscr{D} \right)$$
$$= O_p(N^{-1}),$$

where the second equality is by Assumption 1(i) and the last line holds by Assumption 1(iii) and (v), and Davydov's inequality.

(iii) Define $\zeta_{j,its}^{(\ell)} := e_{it} \tilde{\mathbb{X}}_{j,is}^{(\ell)} - \mathbb{E}(e_{it} \tilde{\mathbb{X}}_{j,is}^{(\ell)} | \mathscr{D})$. As above, we have

$$\mathbb{E} \left\{ \left| \frac{1}{T_\ell \sqrt{N_k^{(\ell)} T_\ell}} \sum_{i_1 \in G_k^{(\ell)}} \sum_{t_1 \in \mathcal{T}_\ell} \sum_{t_2 \in \mathcal{T}_\ell} f_{t_1,j_1}^0 f_{t_2,j_2}^0 \zeta_{j,i_1 t_1 t_2}^{(\ell)} \right|^2 \middle| \mathscr{D} \right\}$$
$$= \frac{1}{T_\ell^3 N_k^{(\ell)}} \sum_{i_1 \in G_k^{(\ell)}} \sum_{i_2 \in G_k^{(\ell)}} \sum_{t_1 \in \mathcal{T}_\ell} \sum_{t_2 \in \mathcal{T}_\ell} \sum_{s_1 \in \mathcal{T}_\ell} \sum_{s_2 \in \mathcal{T}_\ell} f_{t_1,j_1}^0 f_{t_2,j_2}^0 f_{s_1,j_1}^0 f_{s_2,j_2}^0 \mathbb{E} \left(\zeta_{j,i_1 t_1 t_2}^{(\ell)} \zeta_{j,i_2 s_1 s_2}^{(\ell)} \middle| \mathscr{D} \right\}$$
$$\lesssim \frac{1}{T_\ell^3 N_k^{(\ell)}} \sum_{i_1 \in G_k^{(\ell)}} \sum_{i_2 \in G_k^{(\ell)}} \sum_{t_1 \in \mathcal{T}_\ell} \sum_{t_2 \in \mathcal{T}_\ell} \sum_{s_1 \in \mathcal{T}_\ell} \sum_{s_2 \in \mathcal{T}_\ell} \left| Cov \left(e_{i_1 t_1} \tilde{X}_{j,i_1 t_2}^{(\ell)}, e_{i_2 s_1} \tilde{X}_{j,i_2 s_2}^{(\ell)} \middle| \mathscr{D} \right) \right|$$
$$= \frac{1}{T_\ell^3 N_k^{(\ell)}} \sum_{i \in G_k^{(\ell)}} \sum_{t_1 \in \mathcal{T}_\ell} \sum_{t_2 \in \mathcal{T}_\ell} \sum_{s_1 \in \mathcal{T}_\ell} \sum_{s_2 \in \mathcal{T}_\ell} \left| Cov \left(e_{it_1} \tilde{X}_{j,it_2}^{(\ell)}, e_{is_1} \tilde{X}_{j,is_2}^{(\ell)} \middle| \mathscr{D} \right) \right| = O_p(T^{-1}),$$

where the last equality holds by Assumption 9(iv). It follows that

$$\left| \frac{1}{\sqrt{N_k^{(\ell)} T_\ell}} tr \left\{ P_{F^{0,(\ell)}} \left[E_k^{(\ell)\prime} \tilde{\mathbb{X}}_{j,k}^{(\ell)} - \mathbb{E} \left(E_k^{(\ell)\prime} \tilde{\mathbb{X}}_{j,k}^{(\ell)} \middle| \mathscr{D} \right) \right] \right\} \right|$$
$$= \left| \sum_{j_1,j_2=1}^{r_0} \left[\left(\frac{1}{T_\ell} F^{0,(\ell)\prime} F^{0,(\ell)} \right)^{-1} \right]_{j_1 j_2} \frac{1}{T_\ell \sqrt{N_k^{(\ell)} T_\ell}} \sum_{i_1 \in G_k^{(\ell)}} \sum_{t_1 \in \mathcal{T}_1} \sum_{t_2 \in \mathcal{T}_1} f_{t_1,j_1}^0 f_{t_2,j_2}^0 \zeta_{j,i_1 t_1 t_2}^{(\ell)} \right| = O_p(T^{-1/2}).$$

(iv) As in Moon and Weidner (2017), it is clear that

$$
\begin{aligned}
&\left|\frac{1}{\sqrt{N_k^{(\ell)}T_\ell}} tr\left[E_k^{(\ell)} P_{F^{0,(\ell)}} E_k^{(\ell)\prime} M_{\Lambda_k^{0,(\ell)}} \mathbb{X}_{j,k}^{(\ell)} F^{0,(\ell)}\left(F^{0,(\ell)\prime}F^{0,(\ell)}\right)^{-1}\left(\Lambda_k^{0,(\ell)\prime}\Lambda_k^{0,(\ell)}\right)^{-1}\Lambda_k^{0,(\ell)\prime}\right]\right| \\
&\lesssim \frac{1}{\sqrt{N_k^{(\ell)}T_\ell}}\left\|P_{\Lambda_k^{0,(\ell)}} E_k^{(\ell)} P_{F^{0,(\ell)}}\right\|\left\|E_k^{(\ell)}\right\|_{op}\left\|\mathbb{X}_{j,k}^{(\ell)}\right\|\left\|F^{0,(\ell)}\left(F^{0,(\ell)\prime}F^{0,(\ell)}\right)^{-1}\left(\Lambda_k^{0,(\ell)\prime}\Lambda_k^{0,(\ell)}\right)^{-1}\Lambda_k^{0,(\ell)\prime}\right\| \\
&= \frac{1}{\sqrt{N_k^{(\ell)}T_\ell}} O_p(1) O_p\left(\sqrt{N}+\sqrt{T\log T}\right) O_p((NT)^{1/2}) O_p((NT)^{-1/2}) = o_p(1),
\end{aligned}
$$

where the last line combines (B.15),(B.16), the fact that $||\mathbb{X}_{j,k}^{(\ell)}|| = O_p((NT)^{1/2})$ by Assumption 8(ii), and $||F^{0,(\ell)}(F^{0,(\ell)\prime}F^{0,(\ell)})^{-1}(\Lambda_k^{0,(\ell)\prime}\Lambda_k^{0,(\ell)})^{-1}\Lambda_k^{0,(\ell)\prime}|| = O_p((NT)^{-1/2})$ by Assumptions 2 and 8(i).

(v) The proof of (v) is analogous to that of (iv) and is omitted for brevity.

(vi) First, we note that

$$
\begin{aligned}
\mathbb{E}\left(\left\|\Lambda_k^{0,(\ell)\prime} E_k^{(\ell)} \mathbb{X}_{j,k}^{(\ell)\prime}\right\| \middle| \mathscr{D}\right) &= \mathbb{E}\left[\sum_{j_1=1}^{r_0}\sum_{m\in G_k^{(\ell)}}\left(\sum_{i_1\in G_k^{(\ell)}}\sum_{t_1\in\mathcal{T}_\ell}\lambda_{i_1,j_1}^0 e_{i_1t_1} X_{j,mt_1}\right)^2 \middle| \mathscr{D}\right] \\
&= \sum_{j_1=1}^{r_0}\sum_{m\in G_k^{(\ell)}}\sum_{i_1\in G_k^{(\ell)}}\sum_{i_2\in G_k^{(\ell)}}\sum_{t_1\in\mathcal{T}_\ell}\sum_{t_2\in\mathcal{T}_\ell}\lambda_{i_1,j_1}^0\lambda_{i_2,j_1}^0\mathbb{E}\left(e_{i_1t_1}X_{j,mt_1}e_{i_2t_2}X_{j,mt_2}|\mathscr{D}\right) \\
&= \sum_{j_1=1}^{r_0}\sum_{m\in G_k^{(\ell)}}\sum_{i_1\in G_k^{(\ell)}}\sum_{t_1\in\mathcal{T}_\ell}\sum_{t_2\in\mathcal{T}_\ell}\left(\lambda_{i_1,j_1}^0\right)^2\mathbb{E}\left(e_{i_1t_1}X_{j,mt_1}e_{i_1t_2}X_{j,mt_2}|\mathscr{D}\right) \\
&\lesssim \sum_{m\in G_k^{(\ell)}}\sum_{i_1\in G_k^{(\ell)}}\sum_{t\in\mathcal{T}_\ell}\mathbb{E}\left(e_{i_1t}^2X_{j,mt}^2|\mathscr{D}\right) \\
&+ 2\sum_{m\in G_k^{(\ell)}}\sum_{i_1\in G_k^{(\ell)}}\sum_{t_1\in\mathcal{T}_\ell}\sum_{t_2\in\mathcal{T}_\ell, t_2>t_1}\left|Cov\left(e_{i_1t_1}X_{j,mt_1}, e_{i_1t_2}X_{j,mt_2}|\mathscr{D}\right)\right| \\
&= O_p(N^2T),
\end{aligned}
$$

which leads to the result that

$$
\left\|P_{\Lambda_k^{0,(\ell)}} E_k^{(\ell)} \mathbb{X}_{j,k}^{(\ell)\prime}\right\| \leq \left\|\Lambda_k^{0,(\ell)}\right\|\left\|(\Lambda_k^{0,(\ell)\prime}\Lambda_k^{0,(\ell)})^{-1}\right\|\left\|\Lambda_k^{0,(\ell)\prime}E_k^{(\ell)}\mathbb{X}_{j,k}^{(\ell)\prime}\right\| = O(N^{-1/2})O_p(N\sqrt{T}) = O_p(\sqrt{NT}).
$$

As in the proof of part (iv), it yields that

$$
\begin{aligned}
&\left|\frac{1}{\sqrt{N_k^{(\ell)}T_\ell}} tr\left[E_k^{(\ell)\prime} M_{\Lambda_k^{0,(\ell)}} \mathbb{X}_{j,k}^{(\ell)} M_{F^{0,(\ell)}} E_k^{(\ell)\prime}\Lambda_k^{0,(\ell)}\left(\Lambda_k^{0,(\ell)\prime}\Lambda_k^{0,(\ell)}\right)^{-1}\left(F^{0,(\ell)\prime}F^{0,(\ell)}\right)^{-1}F^{0,(\ell)\prime}\right]\right| \\
&\leq \left|\frac{1}{\sqrt{N_k^{(\ell)}T_\ell}} tr\left[E_k^{(\ell)\prime} M_{\Lambda_k^{0,(\ell)}} \mathbb{X}_{j,k}^{(\ell)} E_k^{(\ell)\prime} P_{\Lambda_k^{0,(\ell)}}\Lambda_k^{0,(\ell)}\left(\Lambda_k^{0,(\ell)\prime}\Lambda_k^{0,(\ell)}\right)^{-1}\left(F^{0,(\ell)\prime}F^{0,(\ell)}\right)^{-1}F^{0,(\ell)\prime}\right]\right| \\
&+ \left|\frac{1}{\sqrt{N_k^{(\ell)}T_\ell}} tr\left[E_k^{(\ell)\prime} M_{\Lambda_k^{0,(\ell)}} \mathbb{X}_{j,k}^{(\ell)} P_{F^{0,(\ell)}} E_k^{(\ell)\prime} P_{\Lambda_k^{0,(\ell)}}\Lambda_k^{0,(\ell)}\left(\Lambda_k^{0,(\ell)\prime}\Lambda_k^{0,(\ell)}\right)^{-1}\left(F^{0,(\ell)\prime}F^{0,(\ell)}\right)^{-1}F^{0,(\ell)\prime}\right]\right| \\
&\lesssim \frac{1}{\sqrt{N_k^{(\ell)}T_\ell}}\left\|E_k^{(\ell)}\right\|_{op}\left\|P_{\Lambda_k^{0,(\ell)}} E_k^{(\ell)}\mathbb{X}_{j,k}^{(\ell)\prime}\right\|\left\|\Lambda_k^{0,(\ell)}\left(\Lambda_k^{0,(\ell)\prime}\Lambda_k^{0,(\ell)}\right)^{-1}\left(F^{0,(\ell)\prime}F^{0,(\ell)}\right)^{-1}F^{0,(\ell)\prime}\right\|
\end{aligned}
$$

$$+\frac{1}{\sqrt{N_k^{(\ell)}T_\ell}}\left\|E_k^{(\ell)}\right\|_{op}\left\|\mathbb{X}_{j,k}^{(\ell)}\right\|\left\|\Lambda_k^{0,(\ell)}\left(\Lambda_k^{0,(\ell)\prime}\Lambda_k^{0,(\ell)}\right)^{-1}\left(F^{0,(\ell)\prime}F^{0,(\ell)}\right)^{-1}F^{0,(\ell)\prime}\right\|\left\|P_{\Lambda_k^{0,(\ell)}}E_k^{(\ell)}P_{F^{0,(\ell)}}\right\|$$
$$=o_p(1).$$

(vii) For this statement, we sketch the proof because Lu and Su (2016) have already proved a similar result.

$$\left|\frac{1}{\sqrt{N_k^{(\ell)}T_\ell}}tr\left\{\left[E_k^{(\ell)}E_k^{(\ell)\prime}-\mathbb{E}\left(E_k^{(\ell)}E_k^{(\ell)\prime}|\mathscr{D}\right)\right]M_{\Lambda_k^{0,(\ell)}}\mathbb{X}_{j,k}^{(\ell)}F^{0,(\ell)}\left(F^{0,(\ell)\prime}F^{0,(\ell)}\right)^{-1}\left(\Lambda_k^{0,(\ell)\prime}\Lambda_k^{0,(\ell)}\right)^{-1}\Lambda_k^{0,(\ell)\prime}\right\}\right|$$
$$\lesssim\frac{1}{\left(N_k^{(\ell)}\right)^{3/2}}\left\|\Lambda_k^{0,(\ell)\prime}\frac{1}{T_\ell}\left[E_k^{(\ell)}E_k^{(\ell)\prime}-\mathbb{E}(E_k^{(\ell)}E_k^{(\ell)\prime}|\mathscr{D})\right]M_{\Lambda_k^{0,(\ell)}}\mathbb{X}_{j,k}^{(\ell)}\right\|=o_p(1),$$

where the last equality holds by the fact that

$$\left(N_k^{(\ell)}\right)^{-3/2}\mathbb{E}\left\{\left\|\Lambda_k^{0,(\ell)\prime}\frac{1}{T_\ell}\left[E_k^{(\ell)}E_k^{(\ell)\prime}-\mathbb{E}\left(E_k^{(\ell)}E_k^{(\ell)\prime}|\mathscr{D}\right)\right]M_{\Lambda_k^{0,(\ell)}}\mathbb{X}_{j,k}^{(\ell)}\right\|\Bigg|\mathscr{D}\right\}=o_p(1)$$

which follows by similar arguments as used in the proof of Lemma D.3(vi) in Lu and Su (2016).

(viii) Analogous to the previous statement, we have

$$(T_\ell)^{-3/2}\,\mathbb{E}\left\{\left\|F^{0,(\ell)\prime}\frac{1}{N_k^{(\ell)}}\left[E_k^{(\ell)\prime}E_k^{(\ell)}-\mathbb{E}\left(E_k^{(\ell)\prime}E_k^{(\ell)}|\mathscr{D}\right)\right]M_{\Lambda_k^{0,(\ell)}}\mathbb{X}_{j,k}^{(\ell)\prime}\right\|\Bigg|\mathscr{D}\right\}=o_p(1)$$

by similar arguments as used in the proof of Lemma D.4(iii) in Lu and Su (2016). Then we are ready to show that

$$\left|\frac{1}{\sqrt{N_k^{(\ell)}T_\ell}}tr\left\{\left[E_k^{(\ell)\prime}E_k^{(\ell)}-\mathbb{E}\left(E_k^{(\ell)\prime}E_k^{(\ell)}|\mathscr{D}\right)\right]M_{\Lambda_k^{0,(\ell)}}\mathbb{X}_{j,k}^{(\ell)\prime}\Lambda_k^{0,(\ell)}\left(\Lambda_k^{0,(\ell)\prime}\Lambda_k^{0,(\ell)}\right)^{-1}\left(F^{0,(\ell)\prime}F^{0,(\ell)}\right)^{-1}F^{0,(\ell)\prime}\right\}\right|$$
$$\lesssim(T_\ell)^{-3/2}\left\|F^{0,(\ell)\prime}\frac{1}{N_k^{(\ell)}}\left[E_k^{(\ell)\prime}E_k^{(\ell)}-\mathbb{E}\left(E_k^{(\ell)\prime}E_k^{(\ell)}|\mathscr{D}\right)\right]M_{\Lambda_k^{0,(\ell)}}\mathbb{X}_{j,k}^{(\ell)\prime}\right\|=o_p(1).$$

(ix) This statement can be proved owing to the fact that the second moment of the term on the left side of the equality conditioning on $\mathscr{D}$ is $O_p(N^{-1})$. See the proof of Lemma B.1(i) for detail.

(x) Similarly to (B.17), we can also show that $||\tilde{\mathbb{X}}_{j,k}^{(\ell)}P_{F^{0,(\ell)}}||=O_p(N^{1/2})$. Then, following the same arguments as used in the proof of Lemma B.1(j) in Moon and Weidner (2017), we can finish the proof. ■

C Estimation of Panels with IFEs and Heterogeneous Slopes

For $\forall i\in\mathcal{N}:=\{n_1,\cdots,n_n\}$ and $t\in[T]$, consider the model

$$Y_{it}=\begin{cases}\lambda_i^{0\prime}f_t^0+X_{it}'\theta_i^{0,(1)}+e_{it},\ t\in\{1,\cdots,T_1\},\\ \lambda_i^{0\prime}f_t^0+X_{it}'\theta_i^{0,(2)}+e_{it},\ t\in\{T_1+1,\cdots,T\}.\end{cases}\tag{C.1}$$

Here $\mathcal{N}$ is a subset of $[N]$ and $n\asymp N$. To distinguish from the notation Λ^0 in the paper, we define $\Lambda_n^0:=(\lambda_{n_1}^0,\cdots,\lambda_{n_n}^0)'$.

Let $X_i^{(1)} = (X_{i1}, \cdots, X_{iT_1})'$, $X_i^{(2)} = \left(X_{i(T_1+1)}, \cdots, X_{iT}\right)'$, $e_i^{(1)} = (e_{i1}, \cdots, e_{iT_1})'$, $e_i^{(2)} = \left(e_{i(T_1+1)}, \cdots, e_{iT}\right)'$, $F^{0,(1)} = \left(f_1^0, \cdots, f_{T_1}^0\right)'$, and $F^{0,(2)} = \left(f_{T_1+1}^0, \cdots, f_T^0\right)'$. To estimate $\theta_i^{0,(\ell)}$, λ_i^0 and f_t^0, we follow the lead of Bai (2009) and consider the PCA for heterogeneous panels. For $\forall \ell \in \{1, 2\}$, let

$$\left(\left\{\hat{\theta}_i^{(\ell)}\right\}_{i\in\mathcal{N}}, \hat{F}^{(\ell)}\right) = \underset{F^{(\ell)}, \{\theta_i\}_{i\in\mathcal{N}}}{\arg\min} \frac{1}{nT_\ell} \sum_{i\in\mathcal{N}} \left(Y_i^{(\ell)} - X_i^{(\ell)}\theta_i\right)' M_{F^{(\ell)}} \left(Y_i^{(\ell)} - X_i^{(\ell)}\theta_i\right), \tag{C.2}$$

where $T_2 = T - T_1$, $W_i^{(1)} = (W_{i1}, \cdots, W_{iT_1})'$, $W_i^{(2)} = \left(W_{i(T_1+1)}, \cdots, W_{iT}\right)'$ for W_i denotes Y_i or X_i, $F^{(\ell)}$ is any $T_\ell \times r_0$ matrix such that $\frac{F^{(\ell)\prime} F^{(\ell)}}{T_\ell} = I_{r_0}$ and $M_{F^{(\ell)}} = I_{T_\ell} - \frac{F^{(\ell)} F^{(\ell)\prime}}{T_\ell}$. Note that we consider the concentrated objective function here by concentrating out the factor loadings. The solutions to the minimization problem in (C.2) solve the following nonlinear system of equations:

$$\hat{\theta}_i^{(\ell)} = \left(X_i^{(\ell)\prime} M_{\hat{F}^{(\ell)}} X_i^{(\ell)}\right)^{-1} X_i^{(\ell)\prime} M_{\hat{F}^{(\ell)}} Y_i^{(\ell)}, \tag{C.3}$$

$$\left[\frac{1}{nT_\ell} \sum_{i\in\mathcal{N}} \left(Y_i^{(\ell)} - X_i^{(\ell)}\hat{\theta}_i^{(\ell)}\right)' \left(Y_i^{(\ell)} - X_i^{(\ell)}\hat{\theta}_i^{(\ell)}\right)\right] \hat{F}^{(\ell)} = \hat{F}^{(\ell)} \hat{V}_{NT}^{(\ell)}, \tag{C.4}$$

where $\hat{V}_{NT}^{(\ell)}$ is a diagonal matrix that contains the r_0 largest eigenvalues of the matrix in the square brackets in (C.4). Let $\hat{\lambda}_i^{(\ell)} = \frac{1}{T} \hat{F}^{(\ell)\prime}(Y_i^{(\ell)} - X_i^{(\ell)}\hat{\theta}_i^{(\ell)})$, which are estimates of λ_i^0. Let $\hat{\Lambda}_n^{(\ell)} := (\hat{\lambda}_{n_1}^{(\ell)}, \cdots, \hat{\lambda}_{n_n}^{(\ell)})'$, and $\hat{a}_{ii}^{(\ell)} := \hat{\lambda}_i^{(\ell)\prime} \left(\frac{\hat{\Lambda}_n^{(\ell)\prime}\hat{\Lambda}_n^{(\ell)}}{n}\right)^{-1} \hat{\lambda}_i^{(\ell)}$.

Let $\theta_i^{0,(\ell)} = \bar{\theta}^{0,(\ell)} + c_i^{(\ell)}$, where $\bar{\theta}^{0,(\ell)} = \frac{1}{n}\sum_{i\in\mathcal{N}} \theta_i^{0,(\ell)}$. Here, we consider testing the slope homogeneity for $i \in \mathcal{N}$. The null and alternative hypotheses are respectively given by

$$H_0 : c_i^{(\ell)} = 0 \ \forall i \in \mathcal{N} \text{ and } H_1 : c_i^{(\ell)} \neq 0 \text{ for some } i \in \mathcal{N}.$$

Following Pesaran and Yamagata (2008) and Ando and Bai (2016), we define

$$\hat{\Gamma}^{(\ell)} = \sqrt{n} \cdot \frac{\frac{1}{n}\sum_{i\in\{n_1\cdots,n_n\}} \hat{\mathbb{S}}_i^{(\ell)} - p}{\sqrt{2p}} \tag{C.5}$$

where

$$\hat{\mathbb{S}}_i^{(\ell)} = T_\ell(\hat{\theta}_i^{(\ell)} - \hat{\bar{\theta}}^{(\ell)})' \hat{S}_{ii}^{(\ell)} (\hat{\Omega}_i^{(\ell)})^{-1} \hat{S}_{ii}^{(\ell)} (\hat{\theta}_i^{(\ell)} - \hat{\bar{\theta}}^{(\ell)})(1 - \hat{a}_{ii}^{(\ell)}/n)^2, \quad \hat{\bar{\theta}}^{(\ell)} = \frac{1}{n}\sum_{i\in\mathcal{N}} \hat{\theta}_i^{(\ell)},$$

$$M_{\hat{F}^{(\ell)}} = I_{T_\ell} - \frac{\hat{F}^{(\ell)}\hat{F}^{(\ell)\top}}{T_\ell}, \quad \hat{S}_{ii}^{(\ell)} = \frac{X_i^{(\ell)\prime} M_{\hat{F}^{(\ell)}} X_i^{(\ell)}}{T_\ell}, \quad (\hat{\mathfrak{x}}_{it}^{(\ell)})' \text{ is the } t\text{-th row of } M_{\hat{F}^{(\ell)}} X_i^{(\ell)},$$

$$\hat{\Omega}_i^{(\ell)} = \frac{1}{T_\ell}\sum_{t\in\mathcal{T}_\ell} \hat{\mathfrak{x}}_{it}^{(\ell)} \hat{\mathfrak{x}}_{it}^{(\ell)\prime} \hat{e}_{it}^2 + \frac{1}{T_\ell} \sum_{j\in\mathcal{T}_{\ell,-1}} k(j/S_T) \sum_{t\in\mathcal{T}_{\ell,j}} [\hat{\mathfrak{x}}_{it}^{(\ell)} \hat{\mathfrak{x}}_{i,t+j}^{(\ell)\prime} \hat{e}_{it}\hat{e}_{i,t+j} + \hat{\mathfrak{x}}_{i,t-j}^{(\ell)} \hat{\mathfrak{x}}_{it}^{(\ell)\prime} \hat{e}_{i,t-j}\hat{e}_{it}]$$

and recall that $\mathcal{T}_1 = [T_1]$, $\mathcal{T}_2 = [T]\backslash[T_1]$, $\mathcal{T}_{1,-1} = \mathcal{T}_1\backslash\{T_1\}$, $\mathcal{T}_{2,-1} = \mathcal{T}_2\backslash\{T\}$, $\mathcal{T}_{1,j} = \{1 + j, \cdots, T_1\}$, and $\mathcal{T}_{2,j} = \{T_1 + 1 + j, \cdots, T\}$ for some specific $j \in \mathcal{T}_{\ell,-1}$.

In the next section, we study the asymptotic distribution of $\hat{\theta}_i^{(\ell)}$, the uniform convergence rates for the estimators of factors and factor loadings, and the asymptotic behavior for $\hat{\Gamma}^{(\ell)}$ under H_0 and H_1, respectively.

D Lemmas for Panel IFEs Model with Heterogeneous Slope

Below we derive the asymptotic distribution for the slope estimators in our heterogeneous panel models which allow for dynamics. To allow the dynamic panel, we focus on Assumption 1* where the error process is an m.d.s.. If we focus on Assumption 1, we can obtain similar results by using Davydov's inequality for strong mixing errors. Here we skip the analyses for static panels with serially correlated errors for brevity. Let M be a generic large positive constant and $\mathcal{F}^{(\ell)} := \left\{F^{(\ell)} \in \mathbb{R}^{T_\ell \times r_0} : \frac{F^{(\ell)\prime}F^{(\ell)}}{T_\ell} = I_{r_0}\right\}$.

Lemma D.1 *Under Assumptions 1* , 2 and 8, we have*

(i) $\left|\frac{1}{nT_\ell}\sum_{i\in\mathcal{N}} e_i^{(\ell)\prime} P_{F^{0,(\ell)}} e_i^{(\ell)}\right| = o_p(1)$,

(ii) $\sup_{F^{(\ell)}\in\mathcal{F}^{(\ell)}}\left|\frac{1}{nT_\ell}\sum_{i\in\mathcal{N}} e_i^{(\ell)\prime} P_{F^{(\ell)}} e_i^{(\ell)}\right| = o_p(1)$,

(iii) $\sup_{F^{(\ell)}\in\mathcal{F}^{(\ell)}}\left|\frac{1}{nT_\ell}\sum_{i\in\mathcal{N}} \lambda_i^{0\prime} F^{0,(\ell)\prime} M_{F^{(\ell)}} e_i^{(\ell)}\right| = o_p(1)$,

(iv) $\sup_{\left\{\max_i\|\theta_i\|_{\max}\le M\right\}, F^{(\ell)}\in\mathcal{F}^{(\ell)}}\left|\frac{1}{n}\sum_{i\in\mathcal{N}}(\theta_i-\theta_i^{0,(\ell)})'\frac{X_i^{(\ell)\prime}M_{F^{(\ell)}}e_i^{(\ell)}}{T_\ell}\right| = o_p(1)$.

Proof. (i) We notice that

$$\left|\frac{1}{nT_\ell}\sum_{i\in\mathcal{N}} e_i^{(\ell)\prime}P_{F^{0,(\ell)}}e_i^{(\ell)}\right| \le \frac{1}{T_\ell}\left(\frac{1}{n}\sum_{i\in\mathcal{N}}\left\|\frac{1}{\sqrt{T_\ell}}\sum_{t\in\mathcal{T}_\ell} f_t^0 e_{it}\right\|^2\right)\left\|\left(\frac{F^{0,(\ell)\prime}F^{0,(\ell)}}{T_\ell}\right)^{-1}\right\| \lesssim \frac{1}{T_\ell}\left(\frac{1}{n}\sum_{i\in\mathcal{N}}\left\|\frac{1}{\sqrt{T_\ell}}\sum_{t\in\mathcal{T}_\ell} f_t^0 e_{it}\right\|^2\right).$$

Recall that $\mathscr{D}$ denotes the minimum σ-fields generated by $\left\{V_j^0\right\}_{j\in[p]\cup\{0\}}$. Furthermore, we observe that

$$\begin{aligned}
\mathbb{E}\left(\frac{1}{n}\sum_{i\in\mathcal{N}}\left\|\frac{1}{\sqrt{T_\ell}}\sum_{t\in\mathcal{T}_\ell} f_t^0 e_{it}\right\|^2\Bigg|\mathscr{D}\right) &\le \frac{1}{nT_\ell}\sum_{i\in\mathcal{N}}\sum_{t\in\mathcal{T}_\ell}\sum_{s\in\mathcal{T}_\ell}\left\|\mathbb{E}\left(f_t^0 f_s^{0\prime} e_{it}e_{is}|\mathscr{D}\right)\right\| \\
&\le \frac{1}{nT_\ell}\sum_{i\in\mathcal{N}}\sum_{t\in\mathcal{T}_\ell}\sum_{s\in\mathcal{T}_\ell}\left\|f_t^0\right\|\left\|f_s^{0\prime}\right\|\left|\mathbb{E}\left(e_{it}e_{is}|\mathscr{D}\right)\right| \\
&\lesssim \frac{1}{nT_\ell}\sum_{i\in\mathcal{N}}\sum_{t\in\mathcal{T}_\ell}\mathbb{E}(e_{it}^2|\mathscr{D}) \le M \ a.s.,
\end{aligned} \tag{D.1}$$

where the fourth line holds by the boundedness of factors shown in Lemma B.7(i) and the conditional independence of e_{it} under Assumption 1* (i) and (iii). It follows that $\frac{1}{n}\sum_{i\in\mathcal{N}}||\frac{1}{\sqrt{T_\ell}}\sum_{t\in\mathcal{T}_\ell} f_t^0 e_{it}||^2 = O_p(1)$ and $|\frac{1}{nT_\ell}\sum_{i\in\mathcal{N}} e_i^{(\ell)\prime}P_{F^{0,(\ell)}}e_i^{(\ell)}| = O_p(T^{-1})$.

(ii) Noting that $P_{F^{(\ell)}} = F^{(\ell)}\left(F^{(\ell)\prime}F^{(\ell)}\right)^{-1}F^{(\ell)\prime} = T^{-1}F^{(\ell)}F^{(\ell)\prime}$ for $F^{(\ell)}\in\mathcal{F}^{(\ell)}$, we have $\sup_{F^{(\ell)}\in\mathcal{F}^{(\ell)}}|\frac{1}{nT_\ell}\sum_{i\in\mathcal{N}} e_i^{(\ell)\prime}P_{F^{(\ell)}}e_i^{(\ell)}| \le \sup_{F^{(\ell)}\in\mathcal{F}^{(\ell)}}\frac{1}{n}\sum_{i\in\mathcal{N}}||\frac{1}{T_\ell}\sum_{t\in\mathcal{T}_\ell} f_t e_{it}||^2$. Next,

$$\begin{aligned}
&\sup_{F^{(\ell)}\in\mathcal{F}^{(\ell)}}\frac{1}{n}\sum_{i\in\mathcal{N}}\left\|\frac{1}{T_\ell}\sum_{t\in\mathcal{T}_\ell} f_t e_{it}\right\|^2 = \sup_{F^{(\ell)}\in\mathcal{F}^{(\ell)}} tr\left(\frac{1}{n}\sum_{i\in\mathcal{N}}\frac{1}{T_\ell^2}\sum_{t\in\mathcal{T}_\ell}\sum_{s\in\mathcal{T}_\ell} f_t f_s' e_{it}e_{is}\right) \\
&= \sup_{F^{(\ell)}\in\mathcal{F}^{(\ell)}} tr\left\{\frac{1}{T_\ell^2}\sum_{t\in\mathcal{T}_\ell}\sum_{s\in\mathcal{T}_\ell} f_t f_s'\frac{1}{n}\sum_{i\in\mathcal{N}}\left[e_{it}e_{is}-\mathbb{E}\left(e_{it}e_{is}|\mathscr{D}\right)\right]\right\} + \sup_{F^{(\ell)}\in\mathcal{F}^{(\ell)}} tr\left\{\frac{1}{T_\ell^2}\sum_{t\in\mathcal{T}_\ell}\sum_{s\in\mathcal{T}_\ell} f_t f_s'\frac{1}{n}\sum_{i\in\mathcal{N}}\mathbb{E}\left(e_{it}e_{is}|\mathscr{D}\right)\right\}.
\end{aligned} \tag{D.2}$$

For the first term on the second line of (D.2), we have $\max_{t,s}|\frac{1}{n}\sum_{i\in\mathcal{N}}[e_{it}e_{is}-\mathbb{E}(e_{it}e_{is}|\mathscr{D})]| = O_p(\sqrt{(\log T)/N})$ by conditional Bernstein's inequality for independent sequence combining the fact that $e_{it}e_{is}$ is independent across i given $\mathscr{D}$ by Assumption 1* (i). Then

$$\sup_{F^{(\ell)}\in\mathcal{F}^{(\ell)}} tr\left\{\frac{1}{T_\ell^2}\sum_{t\in\mathcal{T}_\ell}\sum_{s\in\mathcal{T}_\ell} f_t f_s'\frac{1}{n}\sum_{i\in\mathcal{N}}\left[e_{it}e_{is}-\mathbb{E}\left(e_{it}e_{is}|\mathscr{D}\right)\right]\right\}$$

$$= O_p(\sqrt{(\log T)/N}) \sup_{F^{(\ell)} \in \mathcal{F}^{(\ell)}} \left(\frac{1}{T_\ell} \sum_{t \in \mathcal{T}_\ell} \|f_t\| \right)^2 = O_p(\sqrt{(\log T)/N}). \tag{D.3}$$

For the second term on the second line of (D.2), we have

$$\begin{aligned}
&\sup_{F^{(\ell)} \in \mathcal{F}^{(\ell)}} tr \left\{ \frac{1}{T_\ell^2} \sum_{t \in \mathcal{T}_\ell} \sum_{s \in \mathcal{T}_\ell} f_t f_s' \frac{1}{n} \sum_{i \in \mathcal{N}} \mathbb{E}\left(e_{it} e_{is} | \mathscr{D}\right) \right\} \\
&\leq \sup_{F^{(\ell)} \in \mathcal{F}^{(\ell)}} \frac{1}{n T_\ell^2} \sum_{i \in \mathcal{N}} \sum_{t \in \mathcal{T}_\ell} \sum_{s \in \mathcal{T}_\ell} \|f_t\| \|f_s\| \mathbb{E}\left(e_{it} e_{is} | \mathscr{D}\right) \\
&\lesssim \frac{1}{n T_\ell^2} \sum_{i \in \mathcal{N}} \sum_{t \in \mathcal{T}_\ell} \left| \mathbb{E}\left(e_{it}^2 | \mathscr{D}\right) \right| = O_p(T^{-1}),
\end{aligned} \tag{D.4}$$

where the first inequality is by Cauchy's inequality, the third line is by the definition of $\mathcal{F}^{(\ell)}$ and similar arguments as in (D.1).

Combining (D.2)-(D.4), we have shown that $\sup_{F^{(\ell)} \in \mathcal{F}^{(\ell)}} |\frac{1}{nT_\ell} \sum_{i \in \mathcal{N}} e_i^{(\ell)\prime} P_{F^{(\ell)}} e_i^{(\ell)}| = O_p(\sqrt{(\log T)/N})$.

(iii) Note that

$$\sup_{F^{(\ell)} \in \mathcal{F}^{(\ell)}} \left| \frac{1}{nT_\ell} \sum_{i \in \mathcal{N}} \lambda_i^{0\prime} F^{0,(\ell)\prime} M_{F^{(\ell)}} e_i^{(\ell)} \right| \leq \left| \frac{1}{nT_\ell} \sum_{i \in \mathcal{N}} \lambda_i^{0\prime} F^{0,(\ell)\prime} e_i^{(\ell)} \right| + \sup_{F^{(\ell)} \in \mathcal{F}^{(\ell)}} \left| \frac{1}{nT_\ell} \sum_{i \in \mathcal{N}} \lambda_i^{0\prime} F^{0,(\ell)\prime} P_{F^{(\ell)}} e_i^{(\ell)} \right|.$$

We show the convergence rate for the two terms on the right side of above inequality. For the first term, we note that $\mathbb{E}\left(\lambda_i^{0\prime} f_t^0 e_{it} | \mathscr{D}\right) = 0$ and e_{it} is independent across i and strong mixing across t given $\mathscr{D}$. Then we have

$$\left| \frac{1}{nT_\ell} \sum_{i \in \mathcal{N}} \lambda_i^{0\prime} F^{0,(\ell)\prime} e_i^{(\ell)} \right| = \left| \frac{1}{nT_\ell} \sum_{i \in \mathcal{N}} \sum_{t \in \mathcal{T}_\ell} \lambda_i^{0\prime} f_t^0 e_{it} \right| = O_p((NT)^{-1/2}),$$

by Lemma B.6(ii). For the second term, we note that

$$\begin{aligned}
&\sup_{F^{(\ell)} \in \mathcal{F}^{(\ell)}} \left| \frac{1}{nT_\ell} \sum_{i \in \mathcal{N}} \lambda_i^{0\prime} F^{0,(\ell)\prime} P_{F^{(\ell)}} e_i^{(\ell)} \right| \\
&= \sup_{F^{(\ell)} \in \mathcal{F}^{(\ell)}} \left| \frac{1}{nT_\ell} \sum_{i \in \mathcal{N}} \lambda_i^{0\prime} F^{0,(\ell)\prime} F^{(\ell)} \left(F^{(\ell)\prime} F^{(\ell)} \right)^{-1} \sum_{t \in \mathcal{T}_\ell} f_t e_{it} \right| \\
&\leq \sup_{F^{(\ell)} \in \mathcal{F}^{(\ell)}} \sqrt{\frac{1}{n} \sum_{i \in \mathcal{N}} \|\lambda_i^0\|^2} \sqrt{\sup_{F^{(\ell)} \in \mathcal{F}^{(\ell)}} \frac{1}{n} \sum_{i \in \mathcal{N}} \left\| \frac{1}{T_\ell} \sum_{t \in \mathcal{T}_\ell} f_t e_{it} \right\|^2} \sup_{F^{(\ell)} \in \mathcal{F}^{(\ell)}} \left\| \left(\frac{F^{(\ell)\prime} F^{(\ell)}}{T_\ell} \right)^{-1} \right\| \left\| \frac{F^{0,(\ell)\prime} F^{(\ell)}}{T_\ell} \right\| \\
&\lesssim \sqrt{\sup_{F^{(\ell)} \in \mathcal{F}^{(\ell)}} \left\| \frac{1}{T_\ell} \sum_{t \in \mathcal{T}_\ell} f_t e_{it} \right\|^2} = O_p[((\log T)/N)^{1/4}] = o_p(1),
\end{aligned}$$

where the third line is by Cauchy's inequality and the last line is by arguments in (D.3) and (D.4). Combining the above results completes the proof.

(iv) We first observe that

$$\begin{aligned}
&\mathbb{E}\left(\frac{1}{n} \sum_{i \in \mathcal{N}} \left\| \frac{1}{\sqrt{T_\ell}} \sum_{t \in \mathcal{T}_\ell} X_{it} e_{it} \right\|^2 \right) = \frac{1}{n} \sum_{i \in \mathcal{N}} \mathbb{E}\left(\left\| \frac{1}{\sqrt{T_\ell}} \sum_{t \in \mathcal{T}_\ell} X_{it} e_{it} \right\|^2 \right) \\
&\leq \frac{1}{nT_\ell} \sum_{i \in \mathcal{N}} \sum_{t \in \mathcal{T}_\ell} \sum_{s \in \mathcal{T}_\ell} \|\mathbb{E}\left(X_{it} X_{is}' e_{it} e_{is}\right)\| = \frac{1}{nT_\ell} \sum_{i \in \mathcal{N}} \sum_{t \in \mathcal{T}_\ell} \left\| \mathbb{E}\left(X_{it} X_{it}' e_{it}^2\right) \right\| \leq M \ a.s.,
\end{aligned} \tag{D.5}$$

where the second equality is by Assumption 1* (ii) and the law of iterated expectations, and the last inequality is by Assumption 1* (v). It follows that

$$\begin{aligned}
&\sup_{\left\{\max_i\|\theta_i\|_{\max}\le M\right\},F^{(\ell)}\in\mathcal{F}^{(\ell)}}\left\|\frac{1}{n}\sum_{i\in\mathcal{N}}\left(\theta_i-\theta_i^{0,(\ell)}\right)'\frac{X_i^{(\ell)\prime}M_{F^{(\ell)}}e_i^{(\ell)}}{T_\ell}\right\| \\
&\le \sup_{\left\{\max_i\|\theta_i\|_{\max}\le M\right\}}\left|\frac{1}{n}\sum_{i\in\mathcal{N}}\left(\theta_i-\theta_i^{0,(\ell)}\right)'\frac{X_i'e_i^{(\ell)}}{T_\ell}\right| + \sup_{\left\{\max_i\|\theta_i\|_{\max}\le M\right\},F^{(\ell)}\in\mathcal{F}^{(\ell)}}\left|\frac{1}{n}\sum_{i\in\mathcal{N}}\left(\theta_i-\theta_i^{0,(\ell)}\right)'\frac{X_i^{(\ell)\prime}P_{F^{(\ell)}}e_i^{(\ell)}}{T_\ell}\right| \\
&\le \sup_{\left\{\max_i\|\theta_i\|_{\max}\le M\right\}}\frac{1}{\sqrt{T_\ell}}\left(\frac{1}{n}\sum_{i\in\mathcal{N}}\left\|\theta_i-\theta_i^{0,(\ell)}\right\|^2\right)^{1/2}\left(\frac{1}{n}\sum_{i\in\mathcal{N}}\left\|\frac{1}{\sqrt{T_\ell}}\sum_{t\in\mathcal{T}_\ell}X_{it}e_{it}\right\|^2\right)^{1/2} \\
&+ \sup_{\left\{\max_i\|\theta_i\|_{\max}\le M\right\},F^{(\ell)}\in\mathcal{F}^{(\ell)}}\frac{1}{n}\sum_{i\in\mathcal{N}}\left\|\theta_i-\theta_i^{0,(\ell)}\right\|\left\|\frac{X_i^{(\ell)\prime}F^{(\ell)}}{T_\ell}\right\|\left\|\frac{1}{T_\ell}\sum_{t\in\mathcal{T}_\ell}f_te_{it}\right\|\left\|\left(\frac{F^{(\ell)\prime}F^{(\ell)}}{T_\ell}\right)^{-1}\right\| \\
&\lesssim O_p(T^{-1/2}) + \sup_{F^{(\ell)}\in\mathcal{F}^{(\ell)}}\left(\frac{1}{n}\sum_{i\in\mathcal{N}}\left\|\frac{X_i^{(\ell)\prime}F^{(\ell)}}{T_\ell}\right\|^2\right)^{1/2}\sup_{F^{(\ell)}\in\mathcal{F}^{(\ell)}}\left(\frac{1}{n}\sum_{i\in\mathcal{N}}\left\|\frac{1}{T_\ell}\sum_{t\in\mathcal{T}_\ell}f_te_{it}\right\|^2\right)^{1/2} \\
&= O_p(T^{-1/2}) + O_p[((\log T)/N)^{1/4}] = o_p(1),
\end{aligned}$$

where the second inequality is by Cauchy's inequality, the fifth line holds by the fact that both θ_i and $\theta_i^{0,(\ell)}$ are bounded and $\frac{1}{n}\sum_{i\in\mathcal{N}}||\frac{1}{\sqrt{T_\ell}}\sum_{t\in\mathcal{T}_\ell}X_{it}e_{it}||^2 = O_p(1)$ by (D.5), and the last line is due to (D.2) and the fact that

$$\sup_{F^{(\ell)}\in\mathcal{F}^{(\ell)}}\left(\frac{1}{n}\sum_{i\in\mathcal{N}}\left\|\frac{X_i^{(\ell)\prime}F^{(\ell)}}{T_\ell}\right\|^2\right)^{1/2} \lesssim \max_{i\in\mathcal{N}}\frac{1}{T_\ell}\sum_{t\in\mathcal{T}_\ell}\|X_{it}\| = O_p(1).$$

by Assumption 8(ii). ■

Lemma D.2 *Under Assumptions 1* , 2 and 8, we have* $\hat{\theta}_i^{(\ell)} - \theta_i^{0,(\ell)} \xrightarrow{p} 0$ *and* $||P_{\hat{F}^{(\ell)}} - P_{F^{0,(\ell)}}|| \xrightarrow{p} 0$.

Proof. Let

$$S_{NT}\left(\{\theta_i\},F^{(\ell)}\right) = \frac{1}{nT_\ell}\sum_{i\in\mathcal{N}}(Y_i^{(\ell)}-X_i^{(\ell)}\theta_i)'M_{F^{(\ell)}}(Y_i^{(\ell)}-X_i^{(\ell)}\theta_i) - \frac{1}{nT_\ell}\sum_{i\in\mathcal{N}}e_i^{(\ell)\prime}M_{F^{0,(\ell)}}e_i^{(\ell)}.$$

Recall from (C.2) that $(\{\hat{\theta}_i^{(\ell)}\},\hat{F}^{(\ell)})$ is the minimizer of $S_{NT}\left(\{\theta_i\},F^{(\ell)}\right)$. By (C.1) and Lemma D.1, we have

$$\begin{aligned}
S_{NT}(\{\theta_i\},F^{(\ell)}) &= \tilde{S}_{NT}(\{\theta_i\}_{\forall i},F^{(\ell)}) + \frac{2}{n}\sum_{i\in\mathcal{N}}(\theta_i-\theta_i^{0,(\ell)})'\frac{X_i^{(\ell)\prime}M_{F^{(\ell)}}e_i^{(\ell)}}{T_\ell} \\
&+ \frac{2}{nT_\ell}\sum_{i\in\mathcal{N}}\lambda_i^{0\prime}F^{0,(\ell)\prime}M_{F^{(\ell)}}e_i^{(\ell)} + \frac{1}{nT_\ell}\sum_{i\in\mathcal{N}}e_i^{(\ell)\prime}P_{F^{(\ell)}}e_i^{(\ell)} - \frac{1}{nT_\ell}\sum_{i\in\mathcal{N}}e_i^{(\ell)\prime}P_{F^{0,(\ell)}}e_i^{(\ell)} \\
&= \tilde{S}_{NT}(\{\theta_i\},F^{(\ell)}) + o_p(1),
\end{aligned}$$

where

$$\begin{aligned}
\tilde{S}_{NT}(\{\theta_i\},F^{(\ell)}) &= \frac{1}{n}\sum_{i\in\mathcal{N}}(\theta_i-\theta_i^{0,(\ell)})'\frac{X_i^{(\ell)\prime}M_{F^{(\ell)}}e_i^{(\ell)}}{T_\ell}(\theta_i-\theta_i^{0,(\ell)}) + \frac{1}{nT_\ell}\sum_{i\in\mathcal{N}}\lambda_i^{0\prime}F^{0,(\ell)\prime}M_{F^{(\ell)}}F^{0,(\ell)}\lambda_i^0 \\
&+ \frac{2}{n}\sum_{i\in\mathcal{N}}(\theta_i-\theta_i^{0,(\ell)})'\frac{X_i^{(\ell)\prime}M_{F^{(\ell)}}e_i^{(\ell)}}{T_\ell}F^{0,(\ell)}\lambda_i^0.
\end{aligned}$$

Following Song (2013) and Bai (2009), we can show that $\tilde{S}_{NT}\left(\{\theta_i\}_{\forall i}, F^{(\ell)}\right)$ is uniquely minimized at $(\{\theta_i^{0,(\ell)}\}, F^{0,(\ell)}H^{(\ell)})$, where $H^{(\ell)}$ is a rotation matrix. Hence, we conclude that $\hat{\theta}_i^{(\ell)} - \theta_i^{0,(\ell)} \xrightarrow{p} 0$. Following the proof of Proposition 1 of Bai (2009), we can show that $||P_{\hat{F}^{(\ell)}} - P_{F^{0,(\ell)}}|| \xrightarrow{p} 0$. ■

Let B_N denote the uniform convergence rate for $\hat{\theta}_i^{(\ell)}$. That is, $\max_{i\in\mathcal{N}} ||\hat{\theta}_i^{(\ell)} - \theta_i^{0,(\ell)}|| = O_p(B_N)$.

Lemma D.3 *Under Assumptions 1*, 2 and 8, we have* $\frac{1}{\sqrt{T_\ell}}\left\|\hat{F}^{(\ell)} - F^{0,(\ell)}H^{(\ell)}\right\| = O_p(B_N + \frac{1}{\sqrt{N\wedge T}})$, *where* $H^{(\ell)} := \left(\frac{\Lambda_n^{0\prime}\Lambda_n^0}{n}\right)\left(\frac{F^{0,(\ell)\prime}\hat{F}^{(\ell)}}{T_\ell}\right)V_{NT}^{(\ell)-1}$.

Proof. Recall that $V_{NT}^{(\ell)}$ is the diagonal matrix that contains the eigenvalues of $\frac{1}{nT_\ell}\sum_{i\in\mathcal{N}}(Y_i^{(\ell)} - X_i^{(\ell)}\theta_i)'(Y_i^{(\ell)} - X_i^{(\ell)}\theta_i)$ along its diagonal line. By inserting (C.1) into (C.4), we obtain that

$$\hat{F}^{(\ell)}V_{NT}^{(\ell)} - F^{0,(\ell)}\left(\frac{\Lambda_n^{0\prime}\Lambda_n^0}{n}\right)\left(\frac{F^{0,(\ell)\prime}\hat{F}^{(\ell)}}{T_\ell}\right) = \sum_{m\in[8]} J_m^{(\ell)}, \tag{D.6}$$

where

$$\begin{aligned}
J_1^{(\ell)} &= \frac{1}{nT_\ell}\sum_{i\in\mathcal{N}} X_i^{(\ell)}(\theta_i^{0,(\ell)} - \hat{\theta}_i^{(\ell)})(\theta_i^{0,(\ell)} - \hat{\theta}_i^{(\ell)})'X_i^{(\ell)\prime}\hat{F}^{(\ell)}, J_2^{(\ell)} = \frac{1}{nT_\ell}\sum_{i\in\mathcal{N}} X_i^{(\ell)}(\theta_i^{0,(\ell)} - \hat{\theta}_i^{(\ell)})\lambda_i^{0\prime}F^{0,(\ell)\prime}\hat{F}^{(\ell)},\\
J_3^{(\ell)} &= \frac{1}{nT_\ell}\sum_{i\in\mathcal{N}} X_i^{(\ell)}(\theta_i^{0,(\ell)} - \hat{\theta}_i^{(\ell)})e_i^{(\ell)\prime}\hat{F}^{(\ell)},\ J_4^{(\ell)} = \frac{1}{nT_\ell}\sum_{i\in\mathcal{N}} F^{0,(\ell)}\lambda_i^0(\theta_i^{0,(\ell)} - \hat{\theta}_i^{(\ell)})'X_i^{(\ell)\prime}\hat{F}^{(\ell)},\\
J_5^{(\ell)} &= \frac{1}{nT_\ell}\sum_{i\in\mathcal{N}} e_i^{(\ell)}(\theta_i^{0,(\ell)} - \hat{\theta}_i^{(\ell)})'X_i^{(\ell)\prime}\hat{F}^{(\ell)}, \quad J_6^{(\ell)} = \frac{1}{nT_\ell}\sum_{i\in\mathcal{N}} F^{0,(\ell)}\lambda_i^0 e_i^{(\ell)\prime}\hat{F}^{(\ell)},\\
J_7^{(\ell)} &= \frac{1}{nT_\ell}\sum_{i\in\mathcal{N}} e_i^{(\ell)}\lambda_i^{0\prime}F^{0,(\ell)\prime}\hat{F}^{(\ell)}, \quad \text{and} \quad J_8^{(\ell)} = \frac{1}{nT_\ell}\sum_{i\in\mathcal{N}} e_i^{(\ell)}e_i^{(\ell)\prime}\hat{F}^{(\ell)}.
\end{aligned}$$

We show the convergence rate for $J_m^{(\ell)}$ $\forall m \in [8]$ below..

For $J_1^{(\ell)}$, we notice that

$$\frac{1}{\sqrt{T_\ell}}\left\|J_1^{(\ell)}\right\| \leq \max_{i\in\mathcal{N}}\left\|\hat{\theta}_i^{(\ell)} - \theta_i^{0,(\ell)}\right\|^2 \frac{1}{\sqrt{T_\ell}}\left\|\hat{F}^{(\ell)}\right\| \frac{1}{nT_\ell}\sum_{i\in\mathcal{N}}\left\|X_i^{(\ell)}\right\|^2 = O_p(B_N^2), \tag{D.7}$$

where the equality holds by Assumption 8(ii) and normalization of the factor vector. Similarly, we have

$$\begin{aligned}
\frac{1}{\sqrt{T_\ell}}\left\|J_2^{(\ell)}\right\| &\leq \max_{i\in\mathcal{N}}\left\|\hat{\theta}_i^{(\ell)} - \theta_i^{0,(\ell)}\right\| \max_{i\in\mathcal{N}}\|\lambda_i^0\|_2 \frac{\|F^{0,(\ell)}\|}{\sqrt{T_\ell}}\frac{\left\|\hat{F}^{(\ell)}\right\|}{\sqrt{T_\ell}}\frac{1}{n\sqrt{T_\ell}}\sum_{i\in\mathcal{N}}\left\|X_i^{(\ell)}\right\| = O_p(B_N),\\
\frac{1}{\sqrt{T_\ell}}\left\|J_3^{(\ell)}\right\| &\leq \max_{i\in\mathcal{N}}\left\|\hat{\theta}_i^{(\ell)} - \theta_i^{0,(\ell)}\right\| \frac{\left\|\hat{F}^{(\ell)}\right\|}{\sqrt{T_\ell}}\sqrt{\frac{1}{nT_\ell}\sum_{i\in\mathcal{N}}\left\|X_i^{(\ell)}\right\|^2}\sqrt{\frac{1}{nT_\ell}\sum_{i\in\mathcal{N}}\left\|e_i^{(\ell)}\right\|^2} = O_p(B_N),\\
\frac{1}{\sqrt{T_\ell}}\left\|J_6^{(\ell)}\right\| &\leq \frac{1}{\sqrt{n}}\frac{\|F^{0,(\ell)}\|}{\sqrt{T_\ell}}\frac{\left\|\hat{F}^{(\ell)}\right\|}{\sqrt{T_\ell}}\sqrt{\frac{1}{T_\ell}\sum_{t\in\mathcal{T}_\ell}\left\|\frac{1}{\sqrt{n}}\sum_{i\in\mathcal{N}}\lambda_i^0 e_{it}\right\|^2} = O_p\left(N^{-1/2}\right)\\
\frac{1}{\sqrt{T_\ell}}\left\|J_8^{(\ell)}\right\| &\leq \frac{\left\|\hat{F}^{(\ell)}\right\|}{\sqrt{T_\ell}}\frac{1}{nT_\ell}\left\|\sum_{i\in\mathcal{N}} e_i^{(\ell)}e_i^{(\ell)\prime}\right\| = O_p((N\wedge T)^{-1/2}),
\end{aligned}$$

where the third line is by the fact that $\frac{1}{T_\ell}\sum_{t\in\mathcal{T}_\ell}\left\|\frac{1}{\sqrt{n}}\sum_{i\in\mathcal{N}}\lambda_i^0 e_{it}\right\| = O_p(1)$ by similar arguments as in (D.1) and the last line is due to the fact that

$$
\begin{aligned}
&\mathbb{E}\left(\left\|\frac{1}{nT_\ell}\sum_{i\in\mathcal{N}} e_i^{(\ell)} e_i^{(\ell)\prime}\right\|^2 \Big| \mathscr{D}\right) = \frac{1}{(nT_\ell)^2}\sum_{i\in\mathcal{N}}\sum_{i^*\in\mathcal{N}}\sum_{t\in\mathcal{T}_\ell}\sum_{t^*\in\mathcal{T}_\ell}\mathbb{E}\left(e_{it}e_{it^*}e_{i^*t}e_{i^*t^*}|\mathscr{D}\right) \\
&= \frac{1}{(nT_\ell)^2}\sum_{i\in\mathcal{N}}\sum_{t\in\mathcal{T}_\ell}\sum_{t^*\in\mathcal{T}_\ell}\mathbb{E}\left(e_{it}^2 e_{it^*}^2|\mathscr{D}\right) + \frac{1}{(nT_\ell)^2}\sum_{i\in\mathcal{N}}\sum_{i^*\neq i}\sum_{t\in\mathcal{T}_\ell}\sum_{t^*\in\mathcal{T}_\ell}\mathbb{E}\left(e_{it}e_{it^*}|\mathscr{D}\right)\mathbb{E}\left(e_{i^*t}e_{i^*t^*}|\mathscr{D}\right) \\
&= \frac{1}{(nT_\ell)^2}\sum_{i\in\mathcal{N}}\sum_{t\in\mathcal{T}_\ell}\mathbb{E}\left(e_{it}^4|\mathscr{D}\right) + \frac{1}{(nT_\ell)^2}\sum_{i\in\mathcal{N}}\sum_{t\in\mathcal{T}_\ell}\sum_{t^*\in\mathcal{T}_\ell, t^*\neq t}\mathbb{E}\left(e_{it}^2|\mathscr{D}\right)\mathbb{E}\left(e_{it^*}^2|\mathscr{D}\right) \\
&+ \frac{1}{(nT_\ell)^2}\sum_{i\in\mathcal{N}}\sum_{i^*\neq i}\sum_{t\in\mathcal{T}_\ell}\mathbb{E}\left(e_{it}^2|\mathscr{D}\right)\mathbb{E}\left(e_{i^*t}^2|\mathscr{D}\right) \\
&= O\left((N)^{-1} + (T)^{-1}\right) \ a.s.
\end{aligned}
\tag{D.8}
$$

by Assumption 1* (i), (ii), (iii), and (v). Besides, we have $T_\ell^{-1/2}||J_4^{(\ell)}|| = O_p(B_N)$, $T_\ell^{-1/2}||J_5^{(\ell)}|| = O_p(B_N)$, and $T_\ell^{-1/2}||J_4^{(\ell)}|| = O_p(N^{-1/2})$ by similar analyses as used for $J_2^{(\ell)}$, $J_3^{(\ell)}$ and $J_6^{(\ell)}$, respectively.

Combining the above arguments, premultiplying both sides of (D.6) by $\hat{F}^{(\ell)\prime}$ and using the fact that $\hat{F}^{(\ell)\prime}\hat{F}^{(\ell)} = T_\ell I_r$, we have

$$
\frac{1}{T_\ell}\left\|\hat{F}^{(\ell)}V_{NT}^{(\ell)} - F^{0,(\ell)}\frac{\Lambda_n^{0\prime}\Lambda_n^0}{n}\frac{F^{0,(\ell)\prime}\hat{F}^{(\ell)}}{T_\ell}\right\| = O_p(B_N) + O_p((N\wedge T)^{-1/2}),
\tag{D.9}
$$

and

$$
V_{NT}^{(\ell)} = \frac{F^{0,(\ell)\prime}\hat{F}^{(\ell)}}{T_\ell}\frac{\Lambda_n^{0\prime}\Lambda_n^0}{n}\frac{F^{0,(\ell)\prime}\hat{F}^{(\ell)}}{T_\ell} + \frac{\hat{F}^{(\ell)\prime}}{\sqrt{T_\ell}}\frac{\sum_{m\in[8]}J_m^{(\ell)}}{\sqrt{T_\ell}} = \frac{F^{0,(\ell)\prime}\hat{F}^{(\ell)}}{T_\ell}\frac{\Lambda_n^{0\prime}\Lambda_n^0}{n}\frac{F^{0,(\ell)\prime}\hat{F}^{(\ell)}}{T_\ell} + o_p(1).
$$

Then $V_{NT}^{(\ell)}$ is invertible and $||V_{NT}^{(\ell)}|| = O_p(1)$. By the definition of $H^{(\ell)}$ and (D.9), we have $\frac{1}{\sqrt{T_\ell}}||\hat{F}^{(\ell)} - F^{0,(\ell)}H^{(\ell)}|| = O_p(B_N + (N\wedge T)^{-1/2})$. ■

Lemma D.4 *Under Assumptions 1* , 2 and 8, we have*

(i) $\frac{X_i^{(\ell)\prime}M_{\hat{F}^{(\ell)}}X_{i^*}^{(\ell)}}{T_\ell} = \frac{X_i^{(\ell)\prime}M_{F^{0,(\ell)}}X_{i^*}^{(\ell)}}{T_\ell} + O_p\left(B_N + \frac{1}{\sqrt{N\wedge T}}\right)$ *uniformly in* $i, i^* \in \mathcal{N}$,

(ii) $\frac{X_i^{(\ell)\prime}M_{\hat{F}^{(\ell)}}e_i^{(\ell)}}{T_\ell} = \frac{X_i^{(\ell)\prime}M_{F^{0,(\ell)}}e_i^{(\ell)}}{T_\ell} + O_p\left(B_N^2 + B_N\sqrt{\frac{\log N}{T}} + \frac{1}{N\wedge T} + \sqrt{\frac{\log N}{(N\wedge T)T}}\right)$ *uniformly in* $i \in \mathcal{N}$,

(iii) $\frac{1}{nT_\ell^2}\sum_{i\in\mathcal{N}}\left\|e_i^{(\ell)\prime}\hat{F}^{(\ell)}\right\|^2 = O_p\left(B_N^2 + \frac{1}{N\wedge T}\right)$,

(iv) $\max_{i\in\mathcal{N}}\left\|\frac{1}{T_\ell}X_i^{(\ell)\prime}M_{\hat{F}^{(\ell)}}F^{0,(\ell)}\right\| = O_p\left(B_N + \frac{1}{\sqrt{N\wedge T}}\right)$,

(v) $\frac{1}{nT_\ell^2}\sum_{i^*\in\mathcal{N}}\left\|X_i^{(\ell)\prime}e_{i^*}^{(\ell)}\right\|^2 = O_p\left(\frac{\log N}{T}\right)$ *uniformly in* $i \in \mathcal{N}$,

(vi) $\left\|\frac{1}{nT_\ell}\sum_{i\in\mathcal{N}}\lambda_i^0 e_i^{(\ell)\prime}\hat{F}^{(\ell)}\right\| = O_p\left(\frac{B_N}{\sqrt{N}} + \frac{1}{N} + \frac{1}{\sqrt{NT}}\right)$,

(vii) $\max_{i\in\mathcal{N}}\left\|\frac{1}{nT_\ell}\sum_{i^*\in\mathcal{N}}X_i^{(\ell)\prime}M_{\hat{F}^{(\ell)}}e_{i^*}^{(\ell)}\lambda_{i^*}^{0\prime}\right\| = O_p\left(\frac{B_N}{\sqrt{N}} + \frac{1}{N} + \sqrt{\frac{\log N}{NT}}\right)$,

(viii) $\max_{i\in\mathcal{N}}\left\|\frac{1}{nT_\ell^2}\sum_{i^*\in\mathcal{N}}X_i^{(\ell)\prime}M_{\hat{F}^{(\ell)}}e_{i^*}^{(\ell)}e_{i^*}^{(\ell)\prime}\hat{F}^{(\ell)}\right\| = O_p\left(B_N^2+B_N\sqrt{\frac{\log N}{T}}+\frac{\sqrt{\log N}}{N\wedge T}\right)$.

Proof. (i) Let $\delta_1^{(\ell)}:=(T_\ell^{-1}\hat{F}^{(\ell)\prime}\hat{F}^{(\ell)})^{-1}-(T_\ell^{-1}F^{0,(\ell)\prime}F^{0,(\ell)})^{-1}$ and $\delta_2^{(\ell)}:=T_\ell^{-1/2}(\hat{F}^{(\ell)\prime}-F^{0,(\ell)\prime}H^{(\ell)})$. Noting that $M_{\hat{F}^{(\ell)}}=I_{T_\ell}-\hat{F}^{(\ell)}(\hat{F}^{(\ell)\prime}\hat{F}^{(\ell)})^{-1}\hat{F}^{(\ell)\prime}$ and $M_{F^{0,(\ell)}}=I_{T_\ell}-F^{0,(\ell)}(F^{0,(\ell)\prime}F^{0,(\ell)})^{-1}F^{0,(\ell)\prime}$, we can show that

$$\begin{aligned}
M_{F^{0,(\ell)}}-M_{\hat{F}^{(\ell)}} &= \frac{\hat{F}^{(\ell)}}{\sqrt{T_\ell}}\left(\frac{\hat{F}^{(\ell)\prime}\hat{F}^{(\ell)}}{T_\ell}\right)^{-1}\frac{\hat{F}^{(\ell)\prime}}{\sqrt{T_\ell}}-\frac{F^{0,(\ell)}}{\sqrt{T_\ell}}\left(\frac{F^{0,(\ell)\prime}F^{0,(\ell)}}{T_\ell}\right)^{-1}\frac{F^{0,(\ell)\prime}}{\sqrt{T_\ell}}\\
&=\delta_2^{(\ell)}\delta_1^{(\ell)}\delta_2^{(\ell)\prime}+\delta_2^{(\ell)}\delta_1^{(\ell)}\left(\frac{F^{0,(\ell)}H^{(\ell)}}{\sqrt{T_\ell}}\right)^{\prime}+\delta_2^{(\ell)}\left(\frac{F^{0,(\ell)\prime}F^{0,(\ell)}}{T_\ell}\right)^{-1}\delta_2^{(\ell)\prime}+\frac{F^{0,(\ell)}H^{(\ell)}}{\sqrt{T_\ell}}\delta_1^{(\ell)}\delta_2^{(\ell)\prime}\\
&+\delta_2^{(\ell)}\left(\frac{F^{0,(\ell)\prime}F^{0,(\ell)}}{T_\ell}\right)^{-1}\left(\frac{F^{0,(\ell)}H^{(\ell)}}{\sqrt{T_\ell}}\right)^{\prime}+\frac{F^{0,(\ell)}H^{(\ell)}}{\sqrt{T_\ell}}\delta_1^{(\ell)}\left(\frac{F^{0,(\ell)}H^{(\ell)}}{\sqrt{T_\ell}}\right)^{\prime}\\
&+\frac{F^{0,(\ell)}H^{(\ell)}}{\sqrt{T_\ell}}\left(\frac{F^{0,(\ell)\prime}F^{0,(\ell)}}{T_\ell}\right)^{-1}\delta_2^{(\ell)\prime}.
\end{aligned}\tag{D.10}$$

By Lemma D.3, Assumption 8, the normalization for the factor space, and the fact that

$$\left\|\delta_1^{(\ell)}\right\|\leq\frac{\left\|\hat{F}^{(\ell)}-F^{0,(\ell)}H^{(\ell)}\right\|^2}{T_\ell}+2\frac{\left\|F^{0,(\ell)}H^{(\ell)}\right\|\left\|\hat{F}^{(\ell)}-F^{0,(\ell)}H^{(\ell)}\right\|}{T_\ell}=O_p(B_N+(N\wedge T)^{-1/2}),\tag{D.11}$$

we can readily show that

$$\left\|M_{F^{0,(\ell)}}-M_{\hat{F}^{(\ell)}}\right\|=O_p(B_N+(N\wedge T)^{-1/2}).\tag{D.12}$$

Then $\max_{i,i^*\in\mathcal{N}}\left\|\frac{X_i^{(\ell)\prime}M_{\hat{F}^{(\ell)}}X_{i^*}^{(\ell)}}{T_\ell}-\frac{X_i^{(\ell)\prime}M_{F^{0,(\ell)}}X_{i^*}^{(\ell)}}{T_\ell}\right\|\leq\max_{i\in\mathcal{N}}\frac{1}{T_\ell}\left\|X_i^{(\ell)}\right\|^2\left\|M_{F^{0,(\ell)}}-M_{\hat{F}^{(\ell)}}\right\|=O_p(B_N+(N\wedge T)^{-1/2})$, where recall that $\mathcal{N}:=\{n_1,\cdots,n_n\}$.

(ii) By (D.10), we notice that

$$\begin{aligned}
&\max_{i\in\mathcal{N}}\left\|\frac{X_i^{(\ell)\prime}M_{\hat{F}^{(\ell)}}e_i^{(\ell)}}{\sqrt{T_\ell}}-\frac{X_i^{(\ell)\prime}M_{F^{0,(\ell)}}e_i^{(\ell)}}{\sqrt{T_\ell}}\right\|\\
&=\sqrt{T_\ell}\max_{i\in\mathcal{N}}\frac{\left\|X_i^{(\ell)}\right\|}{\sqrt{T_\ell}}\frac{\left\|e_i^{(\ell)}\right\|}{\sqrt{T_\ell}}O_p\left[(B_N+(N\wedge T)^{-1/2})^3+(B_N+(N\wedge T)^{-1/2})^2\right]\\
&+\sqrt{T_\ell}\max_{i\in\mathcal{N}}\frac{\left\|X_i^{(\ell)}\right\|}{\sqrt{T_\ell}}\frac{\left\|\hat{F}^{(\ell)}-F^{0,(\ell)}H^{(\ell)}\right\|}{\sqrt{T_\ell}}\left\|\left(\frac{F^{0,(\ell)\prime}F^{0,(\ell)}}{T_\ell}\right)^{-1}\right\|\frac{\left\|\left(F^{0,(\ell)}H^{(\ell)}\right)^{\prime}e_i^{(\ell)}\right\|}{T_\ell}\\
&+\sqrt{T_\ell}\max_{i\in\mathcal{N}}\frac{\left\|X_i^{(\ell)}\right\|}{\sqrt{T_\ell}}\frac{\left\|F^{0,(\ell)}H^{(\ell)}\right\|}{\sqrt{T_\ell}}\left\|\delta_1^{(\ell)}\right\|\frac{\left\|\left(F^{0,(\ell)}H^{(\ell)}\right)^{\prime}e_i^{(\ell)}\right\|}{T_\ell}\\
&=\sqrt{T_\ell}\left[O_p(B_N^2+(N\wedge T)^{-1})+O_p(B_N+(N\wedge T)^{-1/2})O_p(\sqrt{(\log N)/T})\right],
\end{aligned}$$

where the last line holds by combining Assumption 8(ii), (D.11), Lemma D.3 and the fact that

$$\max_{i\in\mathcal{N}}\frac{\left\|\left(F^{0,(\ell)}H^{(\ell)}\right)^{\prime}e_i^{(\ell)}\right\|}{T_\ell}\lesssim\max_{i\in\mathcal{N}}\left\|\frac{1}{T_\ell}\sum_{t\in\mathcal{T}_\ell}f_t^0e_{it}\right\|=O_p(\sqrt{(\log N)/T}).$$

We will show the last equality by using the Bernstein's inequality in Lemma B.5(i).

Note that

$$\max_{i\in\mathcal{N},t\in\mathcal{T}_\ell}\left\|Var\left(e_{it}f_t^0|\mathscr{D}\right)\right\|=\max_{i\in\mathcal{N},t\in\mathcal{T}_\ell}\left\|\mathbb{E}\left(e_{it}^2|\mathscr{D}\right)f_t^0f_t^{0\prime}\right\|=O_p(1)\quad\text{and}$$

$$\max_{i\in\mathcal{N},t\in\mathcal{T}_\ell}\left\|e_{it}f_t^0\right\| = O_p\left((NT)^{1/q}\right), \tag{D.13}$$

where the second line is by Assumption 1* (v) and the last line is by Assumption 1* (v). Define events $\mathscr{A}_{4,N}(M) = \left\{\max_{i\in\mathcal{N},t\in\mathcal{T}_\ell}\left\|e_{it}f_t^0\right\| \le M(NT)^{1/q}\right\}$ and $\mathscr{A}_{4,N,i}(M) = \left\{\max_{t\in\mathcal{T}_\ell}\left\|e_{it}f_t^0\right\| \le M(NT)^{1/q}\right\}$ for a large enough constant M. Then for some large positive constants c_6 and c_7, we have $\mathbb{P}\left(\mathscr{A}_{4,N}^c(M)\right)\to 0$ and

$$\begin{aligned}
&\mathbb{P}\left(\max_{i\in\mathcal{N}}\left\|\frac{1}{T_\ell}\sum_{t\in\mathcal{T}_\ell}e_{it}f_t^0\right\| > c_6\sqrt{\frac{\log N}{T}}\right)\\
&\le \mathbb{P}\left(\max_{i\in\mathcal{N}}\left\|\frac{1}{T_\ell}\sum_{t\in\mathcal{T}_\ell}e_{it}f_t^0\right\| > c_6\sqrt{\frac{\log N}{T}}, \mathscr{A}_{4,N}(M)\right) + \mathbb{P}\left(\mathscr{A}_{4,N}^c(M)\right)\\
&\le \sum_{i\in\mathcal{N}}\mathbb{P}\left(\left\|\frac{1}{T_\ell}\sum_{t\in\mathcal{T}_\ell}e_{it}f_t^0\right\| > c_6\sqrt{\frac{\log N}{T}}, \mathscr{A}_{4,N}(M)\right) + \mathbb{P}\left(\mathscr{A}_{4,N}^c(M)\right)\\
&\le \sum_{i\in\mathcal{N}}\mathbb{P}\left(\left\|\frac{1}{T_\ell}\sum_{t\in\mathcal{T}_\ell}e_{it}f_t^0\right\| > c_6\sqrt{\frac{\log N}{T}}, \mathscr{A}_{4,N,i}(M)\right) + \mathbb{P}\left(\mathscr{A}_{4,N}^c(M)\right)\\
&\le \sum_{i\in\mathcal{N}}\mathbb{E}\mathbb{P}\left(\left\|\frac{1}{T_\ell}\sum_{t\in\mathcal{T}_\ell}e_{it}f_t^0\right\| > c_6\sqrt{\frac{\log N}{T}}\middle|\mathscr{D}\right)\mathbf{1}\left\{\mathscr{A}_{4,N,i}(M)\right\} + \mathbb{P}\left(\mathscr{A}_{4,N}^c(M)\right)\\
&\le \sum_{i\in\mathcal{N}}\exp\left\{-\frac{c_4c_6^2T\log N/2}{c_7T + c_6M\sqrt{T\log N}(NT)^{1/q}(\log T)^2/3}\right\} + o(1)\\
&= o(1),
\end{aligned} \tag{D.14}$$

where the last inequality holds by Lemma B.5(i), (D.13), and the definition of event $\mathscr{A}_{4,N,i}$, and the last line holds by Assumption 1* (vi) and the fact that $q > 8$.

(iii) By the fact that

$$\begin{aligned}
\frac{1}{nT_\ell^2}\sum_{i\in\mathcal{N}}\left\|e_i^{(\ell)\prime}\hat{F}^{(\ell)}\right\|^2 &\le \frac{1}{nT_\ell^2}\sum_{i\in\mathcal{N}}\left\|e_i^{(\ell)\prime}F^{0,(\ell)}H^{(\ell)}\right\|^2 + \frac{1}{nT_\ell^2}\sum_{i\in\mathcal{N}}\left\|e_i^{(\ell)\prime}\left(\hat{F}^{(\ell)} - F^{0,(\ell)}H^{(\ell)}\right)\right\|^2\\
&\lesssim \frac{1}{nT_\ell^2}\sum_{i\in\mathcal{N}}\left\|e_i^{(\ell)\prime}F^{0,(\ell)}\right\|^2 + \frac{1}{nT_\ell}\sum_{i\in\mathcal{N}}\left\|e_i^{(\ell)\prime}\right\|^2\frac{1}{T_\ell}\left\|\hat{F}^{(\ell)} - F^{0,(\ell)}H^{(\ell)}\right\|^2\\
&\le \frac{1}{nT_\ell^2}\sum_{i\in\mathcal{N}}\left\|e_i^{(\ell)\prime}F^{0,(\ell)}\right\|^2 + O_p(B_N^2 + (N\wedge T)^{-1}) = O_p(B_N^2 + (N\wedge T)^{-1}),
\end{aligned} \tag{D.15}$$

where the last inequality holds by Assumption 8(ii) and Lemma D.3, and last equality holds by the fact that

$$\frac{1}{nT_\ell^2}\sum_{i\in\mathcal{N}}\left\|e_i^{(\ell)\prime}F^{0,(\ell)}\right\|_2^2 = \frac{1}{nT_\ell^2}\sum_{i\in\mathcal{N}}\left\|\sum_{t\in\mathcal{T}_\ell}e_{it}f_t^0\right\|_2^2 = \frac{1}{T_\ell}\left[\frac{1}{n}\sum_{i\in\mathcal{N}}\left\|\frac{1}{\sqrt{T_\ell}}\sum_{t\in\mathcal{T}_\ell}e_{it}f_t^0\right\|_2^2\right] = O_p(T^{-1})$$

by (D.1).

(iv) Noting that $M_{\hat{F}^{(\ell)}}\hat{F}^{(\ell)} = 0$, we have

$$\begin{aligned}
\max_{i\in\mathcal{N}}\left\|\frac{1}{T_\ell}X_i^{(\ell)\prime}M_{\hat{F}^{(\ell)}}F^{0,(\ell)}\right\| &= \max_{i\in\mathcal{N}}\left\|\frac{1}{T_\ell}X_i^{(\ell)\prime}M_{\hat{F}^{(\ell)}}\left(F^{0,(\ell)} - \hat{F}^{(\ell)}H^{(\ell)-1}\right)\right\|\\
&\le \max_{i\in\mathcal{N}}\left\|\frac{1}{\sqrt{T_\ell}}X_i^{(\ell)}M_{\hat{F}^{(\ell)}}\right\|O_p(B_N + (N\wedge T)^{-1/2}) = O_p(B_N + (N\wedge T)^{-1/2}),
\end{aligned}$$

where the last inequality is by Lemma D.3 and the last equality is by the fact that

$$\max_{i\in\mathcal{N}}\left\|\frac{1}{\sqrt{T_\ell}}X_i^{(\ell)\prime}M_{\hat{F}^{(\ell)}}\right\| \le \max_{i\in\mathcal{N}}\left\|\frac{1}{\sqrt{T_\ell}}X_i^{(\ell)\prime}\right\| = O_p(1) \tag{D.16}$$

by Assumption 8(ii).

(v) Note that $\frac{1}{nT_\ell^2}\sum_{i^*\in\mathcal{N}}\left\|X_i^{(\ell)\prime}e_{i^*}^{(\ell)}\right\|^2=\frac{1}{n}\sum_{i^*\in\mathcal{N}}\left\|\frac{1}{T_\ell}\sum_{t\in\mathcal{T}_\ell}X_{it}e_{i^*t}\right\|^2\le\max_{i,i^*\in\mathcal{N}}\left\|\frac{1}{T_\ell}\sum_{t\in\mathcal{T}_\ell}X_{it}e_{i^*t}\right\|^2$. Under Assumptions 1* and Assumption 8(ii)

$$\max_{i,i^*\in\mathcal{N},t\in\mathcal{T}_\ell}\|X_{it}e_{i^*t}\|=O_p((NT)^{1/q}),$$
$$\max_{i,i^*\in\mathcal{N}}\left\|\sum_{t\in\mathcal{T}_\ell}\mathbb{E}(X_{it}X_{it}'e_{i^*t}^2|\mathscr{G}_{t-1})\right\|\le\max_{i^*\in\mathcal{N},t}\mathbb{E}(e_{i^*t}^2|\mathscr{G}_{t-1})\max_{i\in\mathcal{N}}\sum_{t\in\mathcal{T}_\ell}\|X_{it}\|^2\le c_8T\ a.s.$$

Define events $\mathscr{A}_{5,N}(M)=\{\max_{i,i^*\in\mathcal{N},t\in\mathcal{T}_\ell}\|X_{it}e_{i^*t}\|\le M(NT)^{1/q}\}$ and $\mathscr{A}_{5,N,i,i^*}(M)=\{\max_{t\in\mathcal{T}_\ell}\|X_{it}e_{i^*t}\|\le M(NT)^{1/q}\}$ for a large enough constant M such that $\mathbb{P}\left(\mathscr{A}_{5,N}^c(M)\right)\to0$. Then we have

$$\begin{aligned}
&\mathbb{P}\left(\max_{i,i^*\in\mathcal{N}}\left\|\frac{1}{T_\ell}\sum_{t\in\mathcal{T}_\ell}X_{it}e_{i^*t}\right\|>c_6\sqrt{\frac{\log N}{T}}\right)\\
&\le\mathbb{P}\left(\max_{i,i^*\in\mathcal{N}}\left\|\frac{1}{T_\ell}\sum_{t\in\mathcal{T}_\ell}X_{it}e_{i^*t}\right\|>c_6\sqrt{\frac{\log N}{T}},\mathscr{A}_{5,N}(M)\right)+\mathbb{P}\left(\mathscr{A}_{5,N}^c(M)\right)\\
&\le\sum_{i\in\mathcal{N}}\sum_{i^*\in\mathcal{N}}\mathbb{P}\left(\left\|\frac{1}{T_\ell}\sum_{t\in\mathcal{T}_\ell}X_{it}e_{i^*t}\right\|>c_6\sqrt{\frac{\log N}{T}},\mathscr{A}_{5,N}(M)\right)+\mathbb{P}\left(\mathscr{A}_{5,N}^c(M)\right)\\
&\le\sum_{i\in\mathcal{N}}\sum_{i^*\in\mathcal{N}}\mathbb{P}\left(\left\|\frac{1}{T_\ell}\sum_{t\in\mathcal{T}_\ell}X_{it}e_{i^*t}\right\|>c_6\sqrt{\frac{\log N}{T}},\mathscr{A}_{5,N,i,i^*}(M)\right)+\mathbb{P}\left(\mathscr{A}_{5,N}^c(M)\right)\\
&\le\sum_{i\in\mathcal{N}}\sum_{i^*\in\mathcal{N}}\exp\left\{\frac{-c_6^2T\log N/2}{c_8T+Mc_6(NT)^{1/q}\sqrt{T\log N}/3}\right\}+o(1)\\
&=o(1),
\end{aligned}\tag{D.17}$$

where the last inequality holds by Lemma B.5(ii) and the last line is by Assumption 1* (vi).

(vi) Noted that

$$\begin{aligned}
&\mathbb{E}\left[\left\|\frac{1}{\sqrt{nT_\ell}}\sum_{i\in\mathcal{N}}\lambda_i^0e_i^{(\ell)\prime}F^{0,(\ell)}\right\|^2\middle|\mathscr{D}\right]=\frac{1}{nT_\ell}\sum_{i\in\mathcal{N}}\sum_{i^*\in\mathcal{N}}\sum_{t\in\mathcal{T}_\ell}\sum_{t^*\in\mathcal{T}_\ell}\mathbb{E}\left(\lambda_i^0f_t^{0\prime}f_{t^*}^0\lambda_{i^*}^{0\prime}e_{it}e_{i^*t^*}|\mathscr{D}\right)\\
&\le\max_{i\in\mathcal{N}}\|\lambda_i^0\|_2^2\max_{t\in\mathcal{T}_\ell}\|f_t^0\|_2^2\frac{1}{nT_\ell}\sum_{i\in\mathcal{N}}\sum_{i^*\in\mathcal{N}}\sum_{t\in\mathcal{T}_\ell}\sum_{t^*\in\mathcal{T}_\ell}\left|\mathbb{E}\left(e_{it}e_{i^*t^*}|\mathscr{D}\right)\right|\\
&\lesssim\frac{1}{nT_\ell}\sum_{i\in\mathcal{N}}\sum_{t\in\mathcal{T}_\ell}\left|\mathbb{E}\left(e_{it}^2|\mathscr{D}\right)\right|=O(1)\ a.s.,
\end{aligned}$$

where the last line holds by Lemma B.7(i) and Assumption 1* . Similarly as above, we can also show that $\mathbb{E}\left[\left\|\frac{1}{\sqrt{nT_\ell}}\sum_{i\in\mathcal{N}}\lambda_i^0e_i^{(\ell)\prime}\right\|^2\middle|\mathscr{D}\right]=O_p(1)$. Then

$$\left\|\frac{1}{\sqrt{nT_\ell}}\sum_{i\in\mathcal{N}}\lambda_i^0e_i^{(\ell)\prime}F^{0,(\ell)}\right\|=O_p(1)\text{ and }\left\|\frac{1}{\sqrt{nT_\ell}}\sum_{i\in\mathcal{N}}\lambda_i^0e_i^{(\ell)\prime}\right\|=O_p(1).$$

Furthermore, we have

$$\left\|\frac{1}{nT_\ell}\sum_{i\in\mathcal{N}}\lambda_i^0e_i^{(\ell)\prime}\hat{F}^{(\ell)}\right\|\le\frac{1}{\sqrt{n}}\frac{1}{\sqrt{T_\ell}}\left\|\hat{F}^{(\ell)}-F^{0,(\ell)}H^{(\ell)}\right\|\left\|\frac{1}{\sqrt{nT_\ell}}\sum_{i\in\mathcal{N}}\lambda_i^0e_i^{(\ell)\prime}\right\|+\frac{1}{\sqrt{nT_\ell}}\left\|\frac{1}{\sqrt{nT_\ell}}\sum_{i\in\mathcal{N}}\lambda_i^0e_i^{(\ell)\prime}F^{0,(\ell)}\right\|\left\|H^{(\ell)}\right\|$$

$$= O_p\left(\frac{B_N}{\sqrt{N}} + \frac{1}{N+\sqrt{NT}}\right) + O_p\left(\frac{1}{\sqrt{NT}}\right) = O_p\left(\frac{B_N}{\sqrt{N}} + \frac{1}{N} + \frac{1}{\sqrt{NT}}\right).$$

(vii) We first notice that

$$\max_{i\in\mathcal{N}}\left\|\frac{1}{nT_\ell}\sum_{i^*\in\mathcal{N}} X_i^{(\ell)\prime} e_{i^*}^{(\ell)}\lambda_{i^*}^{0\prime}\right\| = \max_{i\in\mathcal{N}}\left\|\frac{1}{nT_\ell}\sum_{i^*\in\mathcal{N}}\sum_{t\in\mathcal{T}_\ell} X_{it}e_{i^*t}\lambda_{i^*}^{0\prime}\right\| = O_p\left(\sqrt{\frac{\log N}{NT}}\right) \tag{D.18}$$

by similar arguments as used to obtain (D.17). This result, in conjunction with Lemma D.4(vi), implies that

$$\begin{aligned}
&\max_{i\in\mathcal{N}}\left\|\frac{1}{nT_\ell}\sum_{i^*\in\mathcal{N}} X_i^{(\ell)\prime} M_{\hat{F}^{(\ell)}} e_{i^*}^{(\ell)}\lambda_{i^*}^{0\prime}\right\| \\
&\le \max_{i\in\mathcal{N}}\left\|\frac{1}{nT_\ell}\sum_{i^*\in\mathcal{N}} X_i^{(\ell)\prime} e_{i^*}^{(\ell)}\lambda_{i^*}^{0\prime}\right\| + \max_{i\in\mathcal{N}}\left\|\frac{1}{nT_\ell}\sum_{i^*\in\mathcal{N}} X_i^{(\ell)\prime}\frac{\hat{F}^{(\ell)}\hat{F}^{(\ell)\prime}}{T_\ell} e_{i^*}^{(\ell)}\lambda_{i^*}^{0\prime}\right\| \\
&\le \max_{i\in\mathcal{N}}\left\|\frac{1}{nT_\ell}\sum_{i^*\in\mathcal{N}} X_i^{(\ell)\prime} e_{i^*}^{(\ell)}\lambda_{i^*}^{0\prime}\right\| + \max_{i\in\mathcal{N}}\frac{\left\|X_i^{(\ell)}\right\|}{\sqrt{T_\ell}}\frac{\hat{F}^{(\ell)}}{\sqrt{T_\ell}}\left\|\frac{1}{nT_\ell}\sum_{i^*\in\mathcal{N}}\hat{F}^{(\ell)\prime} e_{i^*}^{(\ell)}\lambda_{i^*}^{0\prime}\right\| \\
&= O_p\left(\sqrt{\frac{\log N}{NT}}\right) + O_p\left(\frac{B_N}{\sqrt{N}} + \frac{1}{N} + \frac{1}{\sqrt{NT}}\right) = O_p\left(\frac{B_N}{\sqrt{N}} + \frac{1}{N} + \sqrt{\frac{\log N}{NT}}\right).
\end{aligned}$$

(viii) By (D.17) and Lemma D.4(iii), we have

$$\begin{aligned}
&\max_{i\in\mathcal{N}}\left\|\frac{1}{nT_\ell^2}\sum_{i^*\in\mathcal{N}} X_i^{(\ell)\prime} M_{\hat{F}^{(\ell)}} e_{i^*}^{(\ell)} e_{i^*}^{(\ell)\prime}\hat{F}^{(\ell)}\right\| \\
&\le \max_{i\in\mathcal{N}}\left\|\frac{1}{nT_\ell^2}\sum_{i^*\in\mathcal{N}} X_i^{(\ell)\prime} e_{i^*}^{(\ell)} e_{i^*}^{(\ell)\prime}\hat{F}^{(\ell)}\right\| + \max_{i\in\mathcal{N}}\left\|\frac{1}{nT_\ell^2}\sum_{i^*\in\mathcal{N}} X_i^{(\ell)\prime}\frac{\hat{F}^{(\ell)}\hat{F}^{(\ell)\prime}}{T_\ell} e_{i^*}^{(\ell)} e_{i^*}^{(\ell)\prime}\hat{F}^{(\ell)}\right\| \\
&\le \max_{i\in\mathcal{N}}\sqrt{\frac{1}{n}\sum_{i^*\in\mathcal{N}}\left\|\frac{1}{T_\ell}\sum_{t\in\mathcal{T}_\ell} X_{it}e_{i^*t}\right\|_2^2}\sqrt{\frac{1}{nT_\ell^2}\sum_{i\in\mathcal{N}}\left\|e_i^{(\ell)\prime}\hat{F}^{(\ell)}\right\|_2^2} + \frac{1}{nT_\ell^2}\sum_{i\in\mathcal{N}}\left\|e_i^{(\ell)\prime}\hat{F}^{(\ell)}\right\|_2^2\max_{i\in\mathcal{N}}\frac{\left\|X_i^{(\ell)}\right\|}{\sqrt{T_\ell}}\frac{\left\|\hat{F}^{(\ell)}\right\|}{\sqrt{T_\ell}} \\
&= O_p\left(\sqrt{\frac{\log N}{T}}\right) O_p\left(B_N + \frac{1}{\sqrt{N\wedge T}}\right) + O_p\left(B_N^2 + \frac{1}{N\wedge T}\right) = O_p\left(B_N^2 + B_N\sqrt{\frac{\log N}{T}} + \frac{\sqrt{\log N}}{N\wedge T}\right).
\end{aligned}$$

■

Define

$$\begin{aligned}
&\xi_i^{0,(\ell)} := \frac{X_i^{(\ell)\prime} M_{F^{0,(\ell)}} e_i^{(\ell)}}{T_\ell}, \quad S_{ii^*}^{0,(\ell)} := \frac{X_i^{(\ell)\prime} M_{F^{0,(\ell)}} X_{i^*}^{(\ell)}}{T_\ell}, \quad a_{ii^*}^0 := \lambda_i^{0\prime}\left(\frac{\Lambda_n^{0\prime}\Lambda_n^0}{n}\right)^{-1}\lambda_{i^*}^0, \\
&G_{ii^*}^{0,(\ell)} := S_{ii^*}^{0,(\ell)} a_{ii^*}^0, \quad \text{and } \Omega_i^{0,(\ell)} := Var(\xi_i^{0,(\ell)}).
\end{aligned}$$

Lemma D.5 *Under Assumptions 1* , 2 and 8, we have*

(i) $\mathbb{E}(S_{ii}^{0,(\ell)}|\mathscr{D})(\hat{\theta}_i^{(\ell)} - \theta_i^{0,(\ell)})(1 - \frac{a_{ii}^0}{n}) = \xi_i^{0,(\ell)} + \mathcal{R}_i^{(\ell)}$ *such that* $\max_{i\in\mathcal{N}}||\mathcal{R}_i^{(\ell)}|| = O_p(\log N/(N\wedge T))$,

(ii) $\sqrt{T_\ell}(\Omega_i^{0,(\ell)})^{-1/2}\mathbb{E}(S_{ii}^{0,(\ell)}|\mathscr{D})(\hat{\theta}_i^{(\ell)} - \theta_i^{0,(\ell)})(1 - \frac{a_{ii}^0}{n}) \rightsquigarrow \mathbb{N}(0,1)$,

(iii) $\max_{i\in\mathcal{N}}||\hat{\theta}_i^{(\ell)}-\theta_i^{0,(\ell)}||=O_p(\sqrt{(\log N)/T})$.

Proof. (i) Noting from (C.3) that $\hat{\theta}_i^{(\ell)}=(\hat{S}_i^{(\ell)})^{-1}T_\ell^{-1}X_i^{(\ell)\prime}M_{\hat{F}^{(\ell)}}Y_i^{(\ell)}$ with $\hat{S}_{ii}^{(\ell)}{=}T_\ell^{-1}X_i^{(\ell)\prime}M_{\hat{F}^{(\ell)}}X_i^{(\ell)}$, we have

$$
\begin{aligned}
\hat{\theta}_i^{(\ell)}-\theta_i^{0,(\ell)} &= (\hat{S}_{ii}^{(\ell)})^{-1}\left[\frac{1}{T_\ell}X_i^{(\ell)\prime}M_{\hat{F}^{(\ell)}}e_i^{(\ell)}+\frac{1}{T_\ell}X_i^{(\ell)\prime}M_{\hat{F}^{(\ell)}}F^{0,(\ell)}\lambda_i^0\right]\\
&= (\hat{S}_{ii}^{(\ell)})^{-1}\frac{X_i^{(\ell)\prime}M_{\hat{F}^{(\ell)}}e_i^{(\ell)}}{T_\ell}+(\hat{D}_i^{(\ell)})^{-1}\frac{1}{T_\ell}X_i^{(\ell)\prime}M_{\hat{F}^{(\ell)}}\left[\hat{F}^{(\ell)}H^{(\ell)-1}-\sum_{m\in[8]}J_m^{(\ell)}\left(\frac{F^{0,(\ell)\prime}\hat{F}^{(\ell)}}{T_\ell}\right)^{-1}\left(\frac{\Lambda_n^{0\prime}\Lambda_n^0}{n}\right)^{-1}\right]\lambda_i^0\\
&= (\hat{S}_{ii}^{(\ell)})^{-1}\frac{X_i^{(\ell)\prime}M_{\hat{F}^{(\ell)}}e_i^{(\ell)}}{T_\ell}-(\hat{D}_i^{(\ell)})^{-1}\frac{1}{T_\ell}X_i^{(\ell)\prime}M_{\hat{F}^{(\ell)}}\sum_{m\in[8]}J_m^{(\ell)}\left(\frac{F^{0,(\ell)\prime}\hat{F}^{(\ell)}}{T_\ell}\right)^{-1}\left(\frac{\Lambda_n^{0\prime}\Lambda_n^0}{n}\right)^{-1}\lambda_i^0, \qquad \text{(D.19)}
\end{aligned}
$$

where the second equality is from (D.6). Note that

$$
\max_{i\in\mathcal{N}}\left\|\frac{1}{T_\ell}X_i^{(\ell)\prime}M_{\hat{F}^{(\ell)}}\sum_{m\in[8]}J_m^{(\ell)}\left(\frac{F^{0,(\ell)\prime}\hat{F}^{(\ell)}}{T_\ell}\right)^{-1}\left(\frac{\Lambda_n^{0\prime}\Lambda_n^0}{n}\right)^{-1}\lambda_i^0\right\|\lesssim\sum_{m\in[8]}\max_{i\in\mathcal{N}}\left\|\frac{1}{T_\ell}X_i^{(\ell)\prime}M_{\hat{F}^{(\ell)}}J_m^{(\ell)}\right\|=:\sum_{m\in[8]}II_m,
$$

by Lemmas B.7(i) and D.3, and the normalization of the factor and factor loadings. Hence, it suffices to show the uniform convergence rate II_m for $m\in[8]\backslash\{2\}$. The term associated with II_2 needs to be kept.

For II_1, we have $II_1\le\max_{i\in\mathcal{N}}\left\|\frac{1}{\sqrt{T_\ell}}X_i^{(\ell)\prime}\right\|\left\|J_1^{(\ell)}\right\|=O_p(B_N^2)$ by (D.7) and (D.16). Next, noting that

$$
\begin{aligned}
II_{2,i} &:= \frac{1}{T_\ell}X_i^{(\ell)\prime}M_{\hat{F}^{(\ell)}}J_2^{(\ell)}\left(\frac{F^{0,(\ell)\prime}\hat{F}^{(\ell)}}{T_\ell}\right)^{-1}\left(\frac{\Lambda_n^{0\prime}\Lambda_n^0}{n}\right)^{-1}\lambda_i^0\\
&= \frac{1}{T_\ell}X_i^{(\ell)\prime}M_{\hat{F}^{(\ell)}}\frac{1}{nT_\ell}\sum_{i^*\in\mathcal{N}}X_{i^*}^{(\ell)}\left(\theta_{i^*}^{0,(\ell)}-\hat{\theta}_{i^*}^{(\ell)}\right)\lambda_{i^*}^{0\prime}F^{0,(\ell)\prime}\hat{F}^{(\ell)}\left(\frac{F^{0,(\ell)\prime}\hat{F}^{(\ell)}}{T_\ell}\right)^{-1}\left(\frac{\Lambda_n^{0\prime}\Lambda_n^0}{n}\right)^{-1}\lambda_i^0\\
&= \frac{1}{n}\sum_{i^*\in\mathcal{N}}\frac{X_i^{(\ell)\prime}M_{\hat{F}^{(\ell)}}X_{i^*}^{(\ell)}}{T_\ell}\left(\theta_{i^*}^{0,(\ell)}-\hat{\theta}_{i^*}^{(\ell)}\right)\lambda_{i^*}^{0\prime}\left(\frac{\Lambda_n^{0\prime}\Lambda_n^0}{n}\right)^{-1}\lambda_i^0,
\end{aligned}
$$

we have $\max_{i\in\mathcal{N}}\|II_{2,i}\|=O_p(B_N)$, and this term will be kept in the linear expansion for $\hat{\theta}_i^{(\ell)}$. For II_3, we have

$$
\begin{aligned}
II_3 &= \max_{i\in\mathcal{N}}\left\|\frac{1}{T_\ell}X_i^{(\ell)\prime}M_{\hat{F}^{(\ell)}}\frac{1}{nT_\ell}\sum_{i^*\in\mathcal{N}}X_{i^*}^{(\ell)}\left(\theta_{i^*}^{0,(\ell)}-\hat{\theta}_{i^*}^{(\ell)}\right)e_{i^*}^{(\ell)\prime}\hat{F}^{(\ell)}\right\|\\
&\le \max_{i\in\mathcal{N}}\left\|\frac{1}{\sqrt{T_\ell}}X_i^{(\ell)\prime}\right\|\max_{i\in\mathcal{N}}\left\|\theta_{i^*}^{0,(\ell)}-\hat{\theta}_{i^*}^{(\ell)}\right\|\sqrt{\frac{1}{nT_\ell}\sum_{i^*\in\mathcal{N}}\left\|X_{i^*}^{(\ell)}\right\|^2}\sqrt{\frac{1}{nT_\ell^2}\sum_{i^*\in\mathcal{N}}\left\|e_{i^*}^{(\ell)\prime}\hat{F}^{(\ell)}\right\|}\\
&= O_p(B_N^2+B_N(N\wedge T)^{-1/2})
\end{aligned}
$$

by (D.16), Assumption 8(ii) and Lemma D.4(iii). For II_4, we have

$$
\begin{aligned}
II_4 &= \max_{i\in\mathcal{N}}\left\|\frac{1}{T_\ell}X_i^{(\ell)\prime}M_{\hat{F}^{(\ell)}}\frac{1}{nT_\ell}\sum_{i^*\in\mathcal{N}}F^{0,(\ell)}\lambda_{i^*}^0\left(\theta_{i^*}^{0,(\ell)}-\hat{\theta}_{i^*}^{(\ell)}\right)'X_{i^*}^{(\ell)\prime}\hat{F}^{(\ell)}\right\|\\
&\le \max_{i\in\mathcal{N}}\left\|\frac{1}{T_\ell}X_i^{(\ell)\prime}M_{\hat{F}^{(\ell)}}F^{0,(\ell)}\right\|\max_{i\in\mathcal{N}}\left\|\lambda_i^0\right\|\max_{i\in\mathcal{N}}\left\|\theta_i^{0,(\ell)}-\hat{\theta}_i^{(\ell)}\right\|\frac{1}{n\sqrt{T_\ell}}\sum_{i^*\in\mathcal{N}}\left\|X_{i^*}^{(\ell)}\right\|\frac{\left\|\hat{F}^{(\ell)}\right\|}{\sqrt{T_\ell}}\\
&= O_p(B_N+(N\wedge T)^{-1/2})O_p(B_N)=O_p(B_N^2+B_N(N\wedge T)^{-1/2}),
\end{aligned}
$$

where the last line holds by Lemma D.4(iv), the normalization of factors and the fact that $\frac{1}{n\sqrt{T_\ell}}\sum_{i\in\mathcal{N}}||X_i^{(\ell)}||=\frac{1}{n}\sum_{i\in\mathcal{N}}\sqrt{\frac{1}{T_\ell}\sum_{t\in\mathcal{T}_\ell}\|X_{it}\|^2}=O_p(1)$ by Assumption 8(ii). For II_5, we have

$$\begin{aligned}
II_5 &= \max_{i\in\mathcal{N}}\left\|\frac{1}{T_\ell}X_i^{(\ell)\prime}M_{\hat{F}^{(\ell)}}\frac{1}{nT_\ell}\sum_{i^*\in\mathcal{N}}e_{i^*}^{(\ell)}\left(\theta_{i^*}^{0,(\ell)}-\hat{\theta}_{i^*}^{(\ell)}\right)'X_{i^*}^{(\ell)\prime}\hat{F}^{(\ell)}\right\| \\
&\lesssim \max_{i\in\mathcal{N}}\left\|\frac{1}{nT_\ell\sqrt{T_\ell}}\sum_{i^*\in\mathcal{N}}X_i^{(\ell)\prime}e_{i^*}^{(\ell)}\left(\theta_{i^*}^{0,(\ell)}-\hat{\theta}_{i^*}^{(\ell)}\right)'X_{i^*}^{(\ell)\prime}\right\| \\
&\quad+\max_{i\in\mathcal{N}}\left\|\frac{1}{T_\ell}X_i^{(\ell)\prime}\hat{F}^{(\ell)}\frac{1}{n\sqrt{T_\ell}}\sum_{i^*\in\mathcal{N}}\frac{1}{T_\ell}\hat{F}^{(\ell)\prime}e_{i^*}^{(\ell)}\left(\theta_{i^*}^{0,(\ell)}-\hat{\theta}_{i^*}^{(\ell)}\right)'X_{i^*}^{(\ell)\prime}\right\| \\
&\le \max_{i\in\mathcal{N}}\left\|\theta_i^{0,(\ell)}-\hat{\theta}_i^{(\ell)}\right\|_2\sqrt{\frac{1}{nT_\ell^2}\sum_{i^*\in\mathcal{N}}\left\|X_i^{(\ell)\prime}e_{i^*}^{(\ell)}\right\|^2}\sqrt{\frac{1}{nT_\ell}\sum_{i^*\in\mathcal{N}}\left\|X_{i^*}^{(\ell)\prime}\right\|^2} \\
&\quad+\max_{i\in\mathcal{N}}\left\|\theta_i^{0,(\ell)}-\hat{\theta}_i^{(\ell)}\right\|_2\max_{i\in\mathcal{N}}\frac{\left\|X_i^{(\ell)}\right\|}{\sqrt{T_\ell}}\frac{\left\|\hat{F}^{(\ell)}\right\|}{\sqrt{T_\ell}}\sqrt{\frac{1}{nT_\ell^2}\sum_{i^*\in\mathcal{N}}\left\|e_{i^*}^{(\ell)\prime}\hat{F}^{(\ell)}\right\|^2}\sqrt{\frac{1}{nT_\ell}\sum_{i^*\in\mathcal{N}}\left\|X_{i^*}^{(\ell)}\right\|^2} \\
&= O_p(B_N)O_p(\sqrt{(\log N)/T})+O_p(B_N)O_p\left(B_N+(N\wedge T)^{-1/2}\right)=O_p(B_N^2+B_N\sqrt{(\log N)/(N\wedge T)}),
\end{aligned}$$

where the last line holds by Lemma D.4(iii), D.4(v) and the fact that $\frac{1}{nT_\ell}\sum_{i^*\in\mathcal{N}}||X_{i^*}^{(\ell)}||^2=O_p(1)$ under Assumption 8(ii). Next,

$$\begin{aligned}
II_6 &= \max_{i\in\mathcal{N}}\left\|\frac{1}{T_\ell}X_i^{(\ell)\prime}M_{\hat{F}^{(\ell)}}\frac{1}{nT_\ell}\sum_{i^*\in\mathcal{N}}F^{0,(\ell)}\lambda_{i^*}^0e_{i^*}^{(\ell)\prime}\hat{F}^{(\ell)}\right\|\le\max_{i\in\mathcal{N}}\left\|\frac{1}{T_\ell}X_i^{(\ell)\prime}M_{\hat{F}^{(\ell)}}F^{0,(\ell)}\right\|\left\|\frac{1}{nT_\ell}\sum_{i^*\in\mathcal{N}}\lambda_{i^*}^0e_{i^*}^{(\ell)\prime}\hat{F}^{(\ell)}\right\| \\
&= O_p((B_N+(N\wedge T)^{-1/2})O_p(B_NN^{-1/2}+N^{-1}+(NT)^{-1/2})
\end{aligned}$$

by Lemma D.4(iv) and D.4(vi). By Lemma D.4(vii),

$$\begin{aligned}
II_7 &= \max_{i\in\mathcal{N}}\left\|\frac{1}{T_\ell}X_i^{(\ell)\prime}M_{\hat{F}^{(\ell)}}\frac{1}{nT_\ell}\sum_{i^*\in\mathcal{N}}e_{i^*}^{(\ell)}\lambda_{i^*}^{0\prime}F^{0,(\ell)\prime}\hat{F}^{(\ell)}\right\| \\
&\lesssim \max_{i\in\mathcal{N}}\left\|\frac{1}{nT_\ell}\sum_{i^*\in\mathcal{N}}X_i^{(\ell)\prime}M_{\hat{F}^{(\ell)}}e_{i^*}^{(\ell)}\lambda_{i^*}^{0\prime}\right\|=O_p\left(B_NN^{-1/2}+N^{-1}+(NT/\log N)^{-1/2}\right). \qquad \text{(D.20)}
\end{aligned}$$

By Lemma D.4(viii),

$$II_8=\max_{i\in\mathcal{N}}\left\|\frac{1}{T_\ell}X_i^{(\ell)\prime}M_{\hat{F}^{(\ell)}}\frac{1}{nT_\ell}\sum_{i^*\in\mathcal{N}}e_{i^*}^{(\ell)}e_{i^*}^{(\ell)\prime}\hat{F}^{(\ell)}\right\|=O_p\left(B_N^2+B_N\sqrt{(\log N)/T}+\sqrt{\log N}(N\wedge T)^{-1}\right).$$

In sum, by (D.19) and the above analyses, we have

$$\begin{aligned}
\hat{\theta}_i^{(\ell)}-\theta_i^{0,(\ell)} &= (\hat{S}_{ii}^{(\ell)})^{-1}\frac{X_i^{(\ell)\prime}M_{\hat{F}^{(\ell)}}e_i^{(\ell)}}{T_\ell}+(\hat{S}_{ii}^{(\ell)})^{-1}\frac{1}{n}\sum_{i^*\in\mathcal{N}}\frac{X_i^{(\ell)\prime}M_{\hat{F}^{(\ell)}}X_{i^*}^{(\ell)}}{T_\ell}\left(\hat{\theta}_{i^*}^{(\ell)}-\theta_{i^*}^{0,(\ell)}\right)\lambda_{i^*}^{0\prime}\left(\frac{\Lambda_n^{0\prime}\Lambda_n^0}{n}\right)^{-1}\lambda_i^0 \\
&\quad+O_p\left(B_N^2+B_N\sqrt{\frac{\log N}{N\wedge T}}+\frac{\sqrt{\log N}}{N\wedge T}\right) \qquad \text{(D.21)}
\end{aligned}$$

uniformly in $i\in\mathcal{N}$. Then by (D.19) and Lemma D.4(i)-(ii), we have

$$\hat{\theta}_i^{(\ell)}-\theta_i^{0,(\ell)}=\left(S_{ii}^{0,(\ell)}\right)^{-1}\frac{X_i^{(\ell)\prime}M_{F^{0,(\ell)}}e_i^{(\ell)}}{T_\ell}+\left(S_{ii}^{0,(\ell)}\right)^{-1}\frac{1}{n}\sum_{i^*\in\mathcal{N}}\frac{X_i^{(\ell)\prime}M_{F^{0,(\ell)}}X_{i^*}^{(\ell)}}{T_\ell}\left(\hat{\theta}_{i^*}^{(\ell)}-\theta_{i^*}^{0,(\ell)}\right)\lambda_{i^*}^{0\prime}\left(\frac{\Lambda_n^{0\prime}\Lambda_n^0}{n}\right)^{-1}\lambda_i^0$$

$$+ O_p\left(B_N^2 + B_N\sqrt{\frac{\log N}{N\wedge T}} + \frac{\sqrt{\log N}}{N\wedge T}\right).$$

where recall that $S_{ii^*}^{0,(\ell)} := T_\ell^{-1} X_i^{(\ell)\prime} M_{F^{0,(\ell)}} X_{i^*}^{(\ell)}$. Let $\mathfrak{x}_{it}^{0,(\ell)}$ be the t-th row of matrix $M_{F^{0,(\ell)}} X_i^{(\ell)}$ and note that $\mathfrak{x}_{it}^{0,(\ell)}$ is strong mixing across t and independent across i conditional on $\mathscr{D}$ by Assumption 1* (i), (iii). Then we can show that

$$S_{ii^*}^{0,(\ell)} - \mathbb{E}(S_{ii^*}^{0,(\ell)}|\mathscr{D}) = \frac{1}{T_\ell}\sum_{t\in\mathcal{T}_\ell}[\mathfrak{x}_{it}^{0,(\ell)}\mathfrak{x}_{i^*t}^{0,(\ell)\prime} - \mathbb{E}(\mathfrak{x}_{it}^{0,(\ell)}\mathfrak{x}_{i^*t}^{0,(\ell)\prime}|\mathscr{D})] = O_p\left((T/\log N)^{-1/2}\right)$$

uniformly over $i, i^* \in \mathcal{N}$ by similar arguments as in (D.14). Then by the fact that

$$\max_{i,i^*\in\mathcal{N}}\left\|\frac{X_i^{(\ell)\prime}M_{F^{0,(\ell)}}e_i^{(\ell)}}{T_\ell}\right\| = \max_{i,i^*\in\mathcal{N}}\left\|\frac{1}{T_\ell}\sum_{t\in\mathcal{T}_\ell}X_{it}e_{it}\right\| + \max_{i,i^*\in\mathcal{N}}\left\|\frac{X_i^{(\ell)\prime}F^{0,(\ell)}}{T_\ell}\right\|\left\|\frac{1}{T_\ell}\sum_{t\in\mathcal{T}_\ell}f_t^0 e_{it}\right\| = O_p\left((T/\log N)^{-1/2}\right)$$

by (D.14) and (D.17), we obtain that

$$\hat{\theta}_i^{(\ell)} - \theta_i^{0,(\ell)} = \left[\mathbb{E}\left(S_{ii}^{0,(\ell)}|\mathscr{D}\right)\right]^{-1}\xi_i^{0,(\ell)} + \left[\mathbb{E}\left(S_{ii}^{0,(\ell)}|\mathscr{D}\right)\right]^{-1}\frac{1}{n}\sum_{i^*\in\mathcal{N}}\mathbb{E}\left(G_{ii^*}^{0,(\ell)}|\mathscr{D}\right)\left(\hat{\theta}_{i^*}^{(\ell)} - \theta_{i^*}^{0,(\ell)}\right)$$
$$+ O_p\left(B_N^2 + B_N\sqrt{\frac{\log N}{N\wedge T}} + \frac{\sqrt{\log N}}{N\wedge T}\right). \tag{D.22}$$

For the second term on the right side of (D.22), we observe that

$$\left[\mathbb{E}\left(S_{ii}^{0,(\ell)}|\mathscr{D}\right)\right]^{-1}\frac{1}{n}\sum_{i^*\in\mathcal{N}}\mathbb{E}\left(G_{ii^*}^{0,(\ell)}|\mathscr{D}\right)\left(\hat{\theta}_{i^*}^{(\ell)} - \theta_{i^*}^{0,(\ell)}\right)$$
$$= \left[\mathbb{E}\left(S_{ii}^{0,(\ell)}|\mathscr{D}\right)\right]^{-1}\mathbb{E}\left(G_{ii}^{0,(\ell)}|\mathscr{D}\right)\frac{1}{n}\left(\hat{\theta}_i^{(\ell)} - \theta_i^{0,(\ell)}\right) + \left[\mathbb{E}\left(S_{ii}^{0,(\ell)}|\mathscr{D}\right)\right]^{-1}\frac{1}{n}\sum_{i^*\neq i}\mathbb{E}\left(G_{ii^*}^{0,(\ell)}|\mathscr{D}\right)\left(\hat{\theta}_{i^*}^{(\ell)} - \theta_{i^*}^{0,(\ell)}\right)$$
$$= \frac{a_{ii}^0}{n}\left(\hat{\theta}_i^{(\ell)} - \theta_i^{0,(\ell)}\right) + \left[\mathbb{E}\left(S_{ii}^{0,(\ell)}|\mathscr{D}\right)\right]^{-1}\frac{1}{n}\sum_{i^*\neq i}\mathbb{E}\left(G_{ii^*}^{0,(\ell)}|\mathscr{D}\right)\left(\hat{\theta}_{i^*}^{(\ell)} - \theta_{i^*}^{0,(\ell)}\right).$$

By (D.22), it's clear that

$$\frac{1}{n}\sum_{i^*\neq i}\mathbb{E}\left(G_{ii^*}^{0,(\ell)}|\mathscr{D}\right)\left(\hat{\theta}_{i^*}^{(\ell)} - \theta_{i^*}^{0,(\ell)}\right) = \frac{1}{n}\sum_{i^*\neq i}\mathbb{E}\left(G_{ii^*}^{0,(\ell)}|\mathscr{D}\right)\left[\mathbb{E}\left(S_{i^*i^*}^{0,(\ell)}|\mathscr{D}\right)\right]^{-1}\xi_{i^*}^{0,(\ell)}$$
$$+ \frac{1}{n}\sum_{i^*\neq i}\mathbb{E}\left(G_{ii^*}^{0,(\ell)}|\mathscr{D}\right)\left[\mathbb{E}\left(S_{i^*i^*}^{0,(\ell)}|\mathscr{D}\right)\right]^{-1}\frac{1}{n}\sum_{j=n_1}^{n_n}\mathbb{E}\left(G_{i^*j}^{0,(\ell)}|\mathscr{D}\right)\left(\hat{\theta}_j^{(\ell)} - \theta_j^{0,(\ell)}\right)$$
$$+ O_p\left(B_N^2 + B_N\sqrt{\frac{\log N}{N\wedge T}} + \frac{\sqrt{\log N}}{N\wedge T}\right)$$

where the second term on the right side of the above equality gives the recursive form and shrinks to zero quickly owing to the $\frac{1}{n^k}$ term, and we only need to show the rate of the first term, i.e.,

$$\frac{1}{n}\sum_{i^*\neq i}\mathbb{E}\left(G_{ii^*}^{0,(\ell)}|\mathscr{D}\right)\left[\mathbb{E}\left(S_{i^*i^*}^{0,(\ell)}|\mathscr{D}\right)\right]^{-1}\xi_{i^*}^{0,(\ell)} = \frac{1}{nT_\ell}\sum_{i^*\neq i}\sum_{t\in\mathcal{T}_\ell}\mathbb{E}\left(G_{ii^*}^{0,(\ell)}|\mathscr{D}\right)\left[\mathbb{E}\left(S_{i^*i^*}^{0,(\ell)}|\mathscr{D}\right)\right]^{-1}\mathfrak{x}_{i^*t}e_{i^*t}$$
$$= O_p\left(\sqrt{\frac{\log N}{NT}}\right) \text{ uniformly over } i\in\mathcal{N},$$

similarly to the result in (D.18). This yields

$$\max_{i\in\mathcal{N}}\left\|\frac{1}{n}\sum_{i^*\neq i}\mathbb{E}\left(G_{ii^*}^{0,(\ell)}|\mathscr{D}\right)\left(\hat{\theta}_{i^*}^{(\ell)}-\theta_{i^*}^{0,(\ell)}\right)\right\|=O_p\left(B_N^2+B_N\sqrt{\frac{\log N}{N\wedge T}}+\frac{\sqrt{\log N}}{N\wedge T}\right)$$

and further gives

$$\left(\hat{\theta}_i^{(\ell)}-\theta_i^{0,(\ell)}\right)\left(1-\frac{a_{ii}^0}{n}\right)=\left[\mathbb{E}\left(S_{ii}^{0,(\ell)}|\mathscr{D}\right)\right]^{-1}\xi_i^{0,(\ell)}+O_p\left(B_N^2+B_N\sqrt{\frac{\log N}{N\wedge T}}+\frac{\sqrt{\log N}}{N\wedge T}\right)$$

with $B_N=\max_{i\in\mathcal{N}}||\hat{\theta}_i^{(\ell)}-\theta_i^{0,(\ell)}||=O_p(\sqrt{(\log N)/T})$. Finally, we obtain that

$$\mathbb{E}\left(S_{ii}^{0,(\ell)}|\mathscr{D}\right)\left(\hat{\theta}_i^{(\ell)}-\theta_i^{0,(\ell)}\right)\left(1-\frac{a_{ii}^0}{n}\right)=\xi_i^{0,(\ell)}+\mathcal{R}_i^{(\ell)}$$

such that $\max_{i\in\mathcal{N}}\left\|\mathcal{R}_i^{(\ell)}\right\|_2=O_p\left(\frac{\log N}{N\wedge T}\right)$.

(ii) Given the definition of $\Omega_i^{0,(\ell)}$ and by the central limit theorem for m.d.s., we can easily obtain (ii).

(iii) The proof has already been done in the proof of (i). ■

Lemma D.6 *Under Assumptions 1* , 2 and 8, we have*

(i) $\frac{1}{\sqrt{T_\ell}}\left\|\hat{F}^{(\ell)}-F^{0,(\ell)}H^{(\ell)}\right\|=O_p\left(\sqrt{\frac{\log N}{N\wedge T}}\right)$,

(ii) $\left\|M_{\hat{F}^{(\ell)}}-M_{F^{0,(\ell)}}\right\|=O_p\left(\sqrt{\frac{\log N}{N\wedge T}}\right)$,

(iii) $\max_{i\in\mathcal{N}}\left\|\hat{\lambda}_i^{(\ell)}-H^{(\ell)-1}\lambda_i^0\right\|=O_p\left(\sqrt{\frac{\log N}{N\wedge T}}\right)$,

(iv) $\max_t\left\|\hat{f}_t^{(\ell)}-H^{(\ell)\prime}f_t^0\right\|=O_p\left(\sqrt{\frac{\log(N\vee T)}{N\wedge T}}\right)$.

Proof. (i) We obtain the result by combining Lemma D.3 and Lemma D.5(iii).

(ii) We obtain the result by combining (D.12) and Lemma D.5(iii).

(iii) Recall that

$$\begin{aligned}\hat{\lambda}_i^{(\ell)}&=\left(\hat{F}^{(\ell)\prime}\hat{F}^{(\ell)}\right)^{-1}\hat{F}^{(\ell)\prime}\left(Y_i^{(\ell)}-X_i^{(\ell)\prime}\hat{\theta}_i^{(\ell)}\right)=\frac{1}{T_\ell}\hat{F}^{(\ell)\prime}\left[Y_i^{(\ell)}-X_i^{(\ell)\prime}\theta_i^{0,(\ell)}-X_i^{(\ell)\prime}\left(\hat{\theta}_i^{(\ell)}-\theta_i^{0,(\ell)}\right)\right]\\&=\frac{1}{T_\ell}\hat{F}^{(\ell)\prime}\left[F^{0,(\ell)}\lambda_i^0+e_i^{(\ell)}-X_i^{(\ell)\prime}\left(\hat{\theta}_i^{(\ell)}-\theta_i^{0,(\ell)}\right)\right]\\&=\frac{1}{T_\ell}\hat{F}^{(\ell)\prime}\left(F^{0,(\ell)}-\hat{F}^{(\ell)}H^{(\ell)-1}\right)\lambda_i^0+\frac{1}{T_\ell}\hat{F}^{(\ell)\prime}\hat{F}^{(\ell)}H^{(\ell)-1}\lambda_i^0+\frac{1}{T_\ell}\hat{F}^{(\ell)\prime}e_i^{(\ell)}-\frac{1}{T_\ell}\hat{F}^{(\ell)\prime}X_i^{(\ell)\prime}\left(\hat{\theta}_i^{(\ell)}-\theta_i^{0,(\ell)}\right)\\&=H^{(\ell)-1}\lambda_i^0+\frac{1}{T_\ell}\hat{F}^{(\ell)\prime}\left(F^{0,(\ell)}-\hat{F}^{(\ell)}H^{(\ell)-1}\right)\lambda_i^0+\frac{1}{T_\ell}\hat{F}^{(\ell)\prime}e_i^{(\ell)}-\frac{1}{T_\ell}\hat{F}^{(\ell)\prime}X_i^{(\ell)\prime}\left(\hat{\theta}_i^{(\ell)}-\theta_i^{0,(\ell)}\right)\end{aligned}$$

where the second and fifth equalities are by the normalization that $\frac{\hat{F}^{(\ell)\prime}\hat{F}^{(\ell)}}{T_\ell}=I_{r_0}$. It follows that

$$\begin{aligned}\hat{\lambda}_i^{(\ell)}-H^{(\ell)-1}\lambda_i^0&=\frac{1}{T_\ell}\hat{F}^{(\ell)\prime}\left(F^{0,(\ell)}-\hat{F}^{(\ell)}H^{(\ell)-1}\right)\lambda_i^0+\frac{1}{T_\ell}\hat{F}^{(\ell)\prime}e_i^{(\ell)}-\frac{1}{T_\ell}\hat{F}^{(\ell)\prime}X_i^{(\ell)\prime}\left(\hat{\theta}_i^{(\ell)}-\theta_i^{0,(\ell)}\right)\\&=\frac{1}{T_\ell}\hat{F}^{(\ell)\prime}\left(F^{0,(\ell)}-\hat{F}^{(\ell)}H^{(\ell)-1}\right)\lambda_i^0+\frac{1}{T_\ell}\left(\hat{F}^{(\ell)}-F^{0,(\ell)}H^{(\ell)}\right)^{\prime}e_i^{(\ell)}\end{aligned}$$

$$+ H^{(\ell)\prime}\frac{1}{T_\ell}F^{0,(\ell)}e_i^{(\ell)} - \frac{1}{T_\ell}\hat{F}^{(\ell)\prime}X_i^{(\ell)\prime}\left(\hat{\theta}_i^{(\ell)} - \theta_i^{0,(\ell)}\right) := I_{i,1}^{(\ell)} + I_{i,2}^{(\ell)} + I_{i,3}^{(\ell)} - I_{i,4}^{(\ell)}.$$

First, by Lemmas D.6(i) and B.7(i) and Assumption 8(ii),

$$\max_{i\in\mathcal{N}}\left\|I_{i,1}^{(\ell)}\right\| \leq \left\|\frac{\hat{F}^{(\ell)}}{\sqrt{T_\ell}}\right\|\frac{\left\|F^{0,(\ell)} - \hat{F}^{(\ell)}H^{(\ell)-1}\right\|}{\sqrt{T_\ell}}\max_{i\in\mathcal{N}}\left\|\lambda_i^0\right\| = O_p\left(\sqrt{\frac{\log N}{N\wedge T}}\right), \text{ and}$$

$$\max_{i\in\mathcal{N}}\left\|I_{i,2}^{(\ell)}\right\| \leq \frac{1}{\sqrt{T_\ell}}\left\|\hat{F}^{(\ell)} - F^{0,(\ell)}H^{(\ell)}\right\|\frac{\left\|e_i^{(\ell)}\right\|}{\sqrt{T_\ell}} = O_p\left(\sqrt{\frac{\log N}{N\wedge T}}\right).$$

Similarly $\max_{i\in\mathcal{N}}||I_{i,3}^{(\ell)}|| = O_p(\sqrt{(\log N)/T})$ by (D.14). Now,

$$\max_{i\in\mathcal{N}}\left\|I_{i,4}^{(\ell)}\right\| \leq \left\|\frac{\hat{F}^{(\ell)}}{\sqrt{T_\ell}}\right\|\frac{\left\|X_i^{(\ell)}\right\|}{\sqrt{T_\ell}}\max_{i\in\mathcal{N}}\left\|\hat{\theta}_i^{(\ell)} - \theta_i^{0,(\ell)}\right\| = O_p\left(\sqrt{\frac{\log N}{T}}\right),$$

by Lemma D.5(iii). Combining the above results yields $\max_{i\in\mathcal{N}}||\hat{\lambda}_i - H^{(\ell)-1}\lambda_i|| = O_p(\sqrt{(\log N)/(N\wedge T)})$.

(iv) Recall from (D.6) that $\hat{F}^{(\ell)\prime} - H^{(\ell)\prime}F' = V_{NT}^{(\ell)-1}\sum_{m\in[8]}J_m^{(\ell)\prime}$, where $J_m^{(\ell)}$, $m\in[8]$, are defined in the proof of Lemma D.3. Let $J_{m,t}^{(\ell)}$ be the t-th column of $V_{NT}^{(\ell)-1}J_{m,t}^{(\ell)}$ for $m\in[8]$. We observe that $\hat{f}_t - H^{(\ell)\prime}f_t^0$ is the t-th column of $\hat{F}^{(\ell)\prime} - H^{(\ell)\prime}F'$. It remains to show the convergence rate for $J_{m,t}^{(\ell)}, m\in[8]$.

For $J_{1,t}^{(\ell)}$, we notice that

$$\begin{aligned}
\max_{t\in\mathcal{T}_\ell}\left\|J_{1,t}^{(\ell)}\right\| &= \max_{t\in\mathcal{T}_\ell}\left\|V_{NT}^{(\ell)-1}\hat{F}^{(\ell)\prime}\left[\frac{1}{NT_\ell}\sum_{i\in\mathcal{N}}X_i^{(\ell)}\left(\theta_i^{0,(\ell)} - \hat{\theta}_i^{(\ell)}\right)\left(\theta_i^{0,(\ell)} - \hat{\theta}_i^{(\ell)}\right)'X_{it}\right]\right\| \\
&\leq \left\|V_{NT}^{(\ell)-1}\right\|\frac{\left\|\hat{F}^{(\ell)}\right\|}{\sqrt{T_\ell}}\max_{i\in\mathcal{N}}\left\|\hat{\theta}_i^{(\ell)} - \theta_i^{0,(\ell)}\right\|^2\max_{i\in\mathcal{N}}\frac{\left\|X_i^{(\ell)}\right\|}{\sqrt{T_\ell}}\max_{t\in\mathcal{T}_\ell}\frac{1}{n}\sum_{i\in\mathcal{N}}\|X_{it}\| \\
&\lesssim \max_{i\in\mathcal{N}}\left\|\hat{\theta}_i^{(\ell)} - \theta_i^{0,(\ell)}\right\|^2 = O_p((\log N)/T),
\end{aligned}$$

by Lemma D.5(iii) and Assumption 8(ii). Similarly, by Lemma B.7(i),

$$\begin{aligned}
\max_{t\in\mathcal{T}_\ell}\left\|J_{2,t}^{(\ell)}\right\| &= \max_{t\in\mathcal{T}_\ell}\left\|V_{NT}^{(\ell)-1}\hat{F}^{(\ell)\prime}F^{0,(\ell)}\left[\frac{1}{nT_\ell}\sum_{i\in\mathcal{N}}\lambda_i^0\left(\theta_i^{0,(\ell)} - \hat{\theta}_i^{(\ell)}\right)'X_{it}\right]\right\| \\
&\lesssim \max_{i\in\mathcal{N}}\left\|\hat{\theta}_i^{(\ell)} - \theta_i^{0,(\ell)}\right\| = O_p(\sqrt{(\log N)/T}), \\
\max_{t\in\mathcal{T}_\ell}\left\|J_{3,t}^{(\ell)}\right\| &= \max_{t\in\mathcal{T}_\ell}\left\|V_{NT}^{(\ell)-1}\hat{F}^{(\ell)\prime}\left[\frac{1}{nT_\ell}\sum_{i\in\mathcal{N}}e_i^{(\ell)}\left(\theta_i^{0,(\ell)} - \hat{\theta}_i^{(\ell)}\right)'X_{it}\right]\right\| \\
&\lesssim \max_{i\in\mathcal{N}}\left\|\hat{\theta}_i^{(\ell)} - \theta_i^{0,(\ell)}\right\|_2 = O_p(\sqrt{(\log N)/T}) \\
\max_{t\in\mathcal{T}_\ell}\left\|J_{4,t}^{(\ell)}\right\| &= \max_{t\in\mathcal{T}_\ell}\left\|V_{NT}^{(\ell)-1}\hat{F}^{(\ell)\prime}\left[\frac{1}{nT_\ell}\sum_{i\in\mathcal{N}}X_i^{(\ell)}\left(\theta_i^{0,(\ell)} - \hat{\theta}_i^{(\ell)}\right)\lambda_i^{0\prime}\right]f_t^0\right\| \\
&\lesssim \max_{i\in\mathcal{N}}\left\|\hat{\theta}_i^{(\ell)} - \theta_i^{0,(\ell)}\right\|_2 = O_p(\sqrt{(\log N)/T}), \\
\max_{t\in\mathcal{T}_\ell}\left\|J_{5,t}^{(\ell)}\right\|_2 &= \max_{t\in\mathcal{T}_\ell}\left\|V_{NT}^{(\ell)-1}\hat{F}^{(\ell)\prime}\left[\frac{1}{nT_\ell}\sum_{i\in\mathcal{N}}X_i^{(\ell)}\left(\theta_i^{0,(\ell)} - \hat{\theta}_i^{(\ell)}\right)e_{it}\right]\right\|
\end{aligned}$$

$$\lesssim \max_{i\in\mathcal{N}}\left\|\hat{\theta}_i^{(\ell)}-\theta_i^{0,(\ell)}\right\|_2 = O_p(\sqrt{(\log N)/T}).$$

Next,

$$\max_{t\in\mathcal{T}_\ell}\left\|J_{6,t}^{(\ell)}\right\| = \max_{t\in\mathcal{T}_\ell}\left\|V_{NT}^{(\ell)-1}\hat{F}^{(\ell)\prime}\left[\frac{1}{nT_\ell}\sum_{i\in\mathcal{N}}e_i^{(\ell)}\lambda_i^{0\prime}\right]f_t^0\right\| \lesssim \frac{1}{n\sqrt{T_\ell}}\left\|\sum_{i\in\mathcal{N}}e_i^{(\ell)}\lambda_i^{0\prime}\right\| = O_p(N^{-1/2}),$$

by the fact that

$$\mathbb{E}\left(\frac{1}{nT_\ell}\left\|\sum_{i\in\mathcal{N}}e_i^{(\ell)}\lambda_i^{0\prime}\right\|^2\middle|\mathscr{D}\right) = \mathbb{E}\left(\frac{1}{T_\ell}\sum_{t\in\mathcal{T}_\ell}\left\|\frac{1}{\sqrt{n}}\sum_{i\in[N]}e_{it}\lambda_i^0\right\|^2\middle|\mathscr{D}\right) = O_p(1)$$

with the same manner as (D.1). Similarly,

$$\max_{t\in\mathcal{T}_\ell}\left\|J_{7,t}^{(\ell)}\right\| = \max_{t\in\mathcal{T}_\ell}\left\|V_{NT}^{(\ell)-1}\hat{F}^{(\ell)\prime}F^{0,(\ell)}\left[\frac{1}{nT_\ell}\sum_{i\in\mathcal{N}}\lambda_i^0 e_{it}\right]\right\| \lesssim \max_{t\in\mathcal{T}_\ell}\left\|\frac{1}{n}\sum_{i\in\mathcal{N}}\lambda_i^0 e_{it}\right\| = O_p(\sqrt{(\log T)/N}),$$

by using the Bernstein's inequality for the independent sequence in Lemma B.5(i). For $J_{8,t}^{(\ell)}$, we have

$$\begin{aligned}\max_{t\in\mathcal{T}_\ell}\left\|J_{8,t}^{(\ell)}\right\| &\lesssim \frac{1}{\sqrt{n}}\max_{t\in\mathcal{T}_\ell}\left\|\frac{1}{\sqrt{nT_\ell}}\sum_{i\in\mathcal{N}}e_i^{(\ell)}e_{it}\right\| = \frac{1}{\sqrt{n}}\sqrt{\max_{t\in\mathcal{T}_\ell}\frac{1}{T_\ell}\sum_{s\in[T_\ell]}\left(\frac{1}{\sqrt{n}}\sum_{i\in\mathcal{N}}e_{is}e_{it}\right)^2} \\ &\le \frac{1}{\sqrt{n}}\sqrt{\frac{1}{T_\ell}\sum_{t\in\mathcal{T}_\ell}\sum_{s\in[T_\ell]}\left(\frac{1}{\sqrt{n}}\sum_{i\in\mathcal{N}}e_{is}e_{it}\right)^2} = O_p(N^{-1/2}),\end{aligned}$$

where the last equality holds by the fact that $\mathbb{E}[\frac{1}{T_\ell}\sum_{t\in\mathcal{T}_\ell}\sum_{s\in[T_\ell]}(\frac{1}{\sqrt{N}}\sum_{i\in\mathcal{N}}e_{is}e_{it})^2] = O(1)$ by (D.8). In sum, we have $\max_t ||\hat{f}_t - H^{(\ell)\prime}f_t^0|| = O_p(\sqrt{(\log N\vee T)/(N\wedge T)})$. ■

Lemma D.7 *Under Assumptions 1* , 2 and 8, we have*

$$\max_{i\in\mathcal{N}}\left\|\hat{S}_{ii}^{(\ell)} - \mathbb{E}\left(S_{ii}^{0,(\ell)}|\mathscr{D}\right)\right\| = O_p\left(\sqrt{\frac{\log N\vee T}{N\wedge T}}\right) \text{ and } \max_{i\in\mathcal{N}}\left\|\hat{\Omega}_i^{(\ell)} - \Omega_i^{0,(\ell)}\right\| = o_p(1).$$

Proof. Recall that $S_{ii}^{0,(\ell)} = \frac{X_i^{(\ell)\prime}M_{F^{0,(\ell)}}X_i^{(\ell)}}{T_\ell}$ and $\hat{S}_{ii}^{(\ell)} = \frac{X_i^{(\ell)\prime}M_{\hat{F}^{(\ell)}}X_i^{(\ell)}}{T_\ell}$. Combining (D.20), Lemma D.4(i) and Lemma D.5(iii), we have

$$\max_{i\in\mathcal{N}}\left\|\hat{S}_{ii}^{(\ell)} - \mathbb{E}\left(S_{ii}^{0,(\ell)}|\mathscr{D}\right)\right\| = O_p\left(\sqrt{\frac{\log N\vee T}{N\wedge T}}\right).$$

Recall that $\mathfrak{x}_{it}^{(\ell)\prime}$ is the t-th row of $M_{F^{0,(\ell)}}X_i^{(\ell)}$ and let $\hat{\mathfrak{x}}_{it}^{(\ell)\prime}$ be the t-th row of $M_{\hat{F}^{(\ell)}}X_i^{(\ell)}$, respectively. Under Assumption 1* (iii), we have $\hat{\Omega}_i^{(\ell)} = \frac{1}{T_\ell}\sum_{t\in\mathcal{T}_\ell}\hat{\mathfrak{x}}_{it}^{(\ell)}\hat{\mathfrak{x}}_{it}^{(\ell)\prime}\hat{e}_{it}^2$ and $\Omega_i^{0,(\ell)} = \frac{1}{T_\ell}\sum_{t\in\mathcal{T}_\ell}\mathbb{E}(\mathfrak{x}_{it}^{(\ell)}\mathfrak{x}_{it}^{(\ell)\prime}e_{it}^2)$. It remains to show

$$\max_{i\in\mathcal{N}}\left\|\frac{1}{T_\ell}\sum_{t\in\mathcal{T}_\ell}\left[\hat{\mathfrak{x}}_{it}^{(\ell)}\hat{\mathfrak{x}}_{it}^{(\ell)\prime}\hat{e}_{it}^2 - \mathbb{E}(\mathfrak{x}_{it}^{(\ell)}\mathfrak{x}_{it}^{(\ell)\prime}e_{it}^2)\right]\right\| = o_p(1). \tag{D.23}$$

From the definitions of $\mathfrak{x}_{it}^{(\ell)}$ and $\hat{\mathfrak{x}}_{it}^{(\ell)}$, we notice that $\mathfrak{x}_{it}^{(\ell)} = X_{it} - \frac{1}{T_\ell}X_i^{(\ell)\prime}F^{0,(\ell)}f_t^0$ and $\hat{\mathfrak{x}}_{it}^{(\ell)} = X_{it} - \frac{1}{T_\ell}X_i^{(\ell)\prime}\hat{F}^{(\ell)}\hat{f}_t^0$, which gives

$$\mathfrak{x}_{it}^{(\ell)} - \hat{\mathfrak{x}}_{it}^{(\ell)} = \frac{1}{T_\ell}X_i^{(\ell)\prime}(\hat{F}^{(\ell)}\hat{f}_t^0 - F^{0,(\ell)}f_t^0). \tag{D.24}$$

Note that

$$\max_{t\in\mathcal{T}_\ell}\frac{1}{\sqrt{T_\ell}}\left\|\hat{F}^{(\ell)}\hat{f}_t^0-F^{0,(\ell)}f_t^0\right\|_2\leq\frac{\left\|\hat{F}^{(\ell)}-F^{0,(\ell)}H^{(\ell)}\right\|}{\sqrt{T_\ell}}\max_t\left\|\hat{f}_t^{(\ell)}-H^{(\ell)\prime}f_t^0\right\|+\frac{\left\|F^{0,(\ell)}H^{(\ell)}\right\|}{\sqrt{T_\ell}}\max_t\left\|\hat{f}_t^{(\ell)}-H^{(\ell)\prime}f_t^0\right\|$$
$$+\frac{\left\|\hat{F}^{(\ell)}-F^{0,(\ell)}H^{(\ell)}\right\|}{\sqrt{T_\ell}}\max_t\left\|H^{(\ell)\prime}f_t^0\right\|=O_p\left(\sqrt{\frac{\log N\vee T}{N\wedge T}}\right),$$

by Lemma D.6(i) and (iv) and Lemma B.7(i). Then by (D.24) and Assumption 8(ii), we have

$$\max_{i\in\mathcal{N},t\in\mathcal{T}_\ell}\left\|\mathfrak{x}_{it}^{(\ell)}-\hat{\mathfrak{x}}_{it}^{(\ell)}\right\|\leq\frac{1}{\sqrt{T_\ell}}\left\|X_i\right\|O_p\left(\sqrt{\frac{\log N\vee T}{N\wedge T}}\right)=O_p\left(\sqrt{\frac{\log N\vee T}{N\wedge T}}\right).\tag{D.25}$$

Next, for $i\in\mathcal{N},t\in\mathcal{T}_\ell$, note that

$$\begin{aligned}\hat{e}_{it}&=Y_{it}-X_{it}'\hat{\theta}_i^{(\ell)}-\hat{\lambda}_i^{(\ell)\prime}\hat{f}_t^{(\ell)}\\&=e_{it}-\Big[X_{it}'\left(\theta_i^{0,(\ell)}-\hat{\theta}_i^{(\ell)}\right)+\left(\hat{\lambda}_i^{(\ell)}-H^{(\ell)-1}\lambda_i^0\right)'\left(\hat{f}_t^{(\ell)}-H^{(\ell)\prime}f_t^0\right)+\left(\hat{\lambda}_i^{(\ell)}-H^{(\ell)-1}\lambda_i^0\right)'H^{(\ell)\prime}f_t^0\\&+\left(H^{(\ell)-1}\lambda_i^0\right)'\left(\hat{f}_t^{(\ell)}-H^{(\ell)\prime}f_t^0\right)\Big].\end{aligned}$$

Then

$$\begin{aligned}&\max_{i\in\mathcal{N},t\in\mathcal{T}_\ell}\left|\hat{e}_{it}-e_{it}\right|=O_p\left(\sqrt{\frac{\log N}{T}}(NT)^{1/q}\right)+O_p\left(\sqrt{\frac{\log N\vee T}{N\wedge T}}\right),\quad\text{and}\\&\hat{e}_{it}^2-e_{it}^2=e_{it}\left(\hat{e}_{it}-e_{it}\right)+\left(\hat{e}_{it}-e_{it}\right)^2=e_{it}X_{it}'R_{1,it}+R_{2,it}\\&\text{s.t.}\max_{i\in\mathcal{N},t\in\mathcal{T}_\ell}\left\|R_{1,it}\right\|_2=O_p\left(\sqrt{\frac{\log N}{T}}\right),\max_{i\in[N],t\in\mathcal{T}_\ell}\left|R_{2,it}\right|=O_p\left(\sqrt{\frac{\log N\vee T}{N\wedge T}}\right)\end{aligned}\tag{D.26}$$

by Lemmas D.5(iii), D.6(iii), and D.6(iv). It follows that

$$\begin{aligned}\frac{1}{T_\ell}\sum_{t\in\mathcal{T}_\ell}\hat{\mathfrak{x}}_{it}^{(\ell)}\hat{\mathfrak{x}}_{it}^{(\ell)\prime}\hat{e}_{it}^2&=\frac{1}{T_\ell}\sum_{t\in\mathcal{T}_\ell}\mathfrak{x}_{it}^{(\ell)}\mathfrak{x}_{it}^{(\ell)\prime}e_{it}^2+\frac{1}{T_\ell}\sum_{t\in\mathcal{T}_\ell}\left(\mathfrak{x}_{it}^{(\ell)}-\hat{\mathfrak{x}}_{it}^{(\ell)}\right)\left(\mathfrak{x}_{it}^{(\ell)}-\hat{\mathfrak{x}}_{it}^{(\ell)}\right)'\left(\hat{e}_{it}^2-e_{it}^2\right)\\&+\frac{1}{T_\ell}\sum_{t\in\mathcal{T}_\ell}\left(\mathfrak{x}_{it}^{(\ell)}-\hat{\mathfrak{x}}_{it}^{(\ell)}\right)\left(\mathfrak{x}_{it}^{(\ell)}-\hat{\mathfrak{x}}_{it}^{(\ell)}\right)'e_{it}^2+\frac{1}{T_\ell}\sum_{t\in\mathcal{T}_\ell}\mathfrak{x}_{it}^{(\ell)}\left(\mathfrak{x}_{it}^{(\ell)}-\hat{\mathfrak{x}}_{it}^{(\ell)}\right)'\left(\hat{e}_{it}^2-e_{it}^2\right)\\&+\frac{1}{T_\ell}\sum_{t\in\mathcal{T}_\ell}\left(\mathfrak{x}_{it}^{(\ell)}-\hat{\mathfrak{x}}_{it}^{(\ell)}\right)\mathfrak{x}_{it}^{(\ell)\prime}\left(\hat{e}_{it}^2-e_{it}^2\right)+\frac{1}{T_\ell}\sum_{t\in\mathcal{T}_\ell}\left(\mathfrak{x}_{it}^{(\ell)}-\hat{\mathfrak{x}}_{it}^{(\ell)}\right)\mathfrak{x}_{it}^{(\ell)\prime}e_{it}^2\\&+\frac{1}{T_\ell}\sum_{t\in\mathcal{T}_\ell}\mathfrak{x}_{it}^{(\ell)}\left(\mathfrak{x}_{it}^{(\ell)}-\hat{\mathfrak{x}}_{it}^{(\ell)}\right)'e_{it}^2+\frac{1}{T_\ell}\sum_{t\in\mathcal{T}_\ell}\mathfrak{x}_{it}^{(\ell)}\mathfrak{x}_{it}^{(\ell)\prime}\left(\hat{e}_{it}^2-e_{it}^2\right)\\&=\frac{1}{T_\ell}\sum_{t\in\mathcal{T}_\ell}\mathfrak{x}_{it}^{(\ell)}\mathfrak{x}_{it}^{(\ell)\prime}e_{it}^2+O_p\left(\sqrt{\frac{\log N\vee T}{N\wedge T}}\right)\quad\text{uniformly over } i\in\mathcal{N},\end{aligned}\tag{D.27}$$

where the last line holds by (D.25), (D.26), and Assumptions 8(ii) and 1* (iv). Using similar arguments as used to derive (D.17) by the Bernstein's inequality for m.d.s., for a positive constant c_9, we have

$$\mathbb{P}\left\{\max_{i\in\mathcal{N}}\left\|\frac{1}{T_\ell}\sum_{t\in\mathcal{T}_\ell}\left[\mathfrak{x}_{it}^{(\ell)}\mathfrak{x}_{it}^{(\ell)\prime}e_{it}^2-\mathbb{E}\left(\mathfrak{x}_{it}^{(\ell)}\mathfrak{x}_{it}^{(\ell)\prime}e_{it}^2\right)\right]\right\|>c_9\sqrt{\frac{\log N}{T}}\right\}=o(1).\tag{D.28}$$

Combining (D.27) and (D.28), we obtain (D.23). ■

Lemma D.8 *Let $\hat{\Gamma}^{(\ell)}$ be as defined in (C.5). Under Assumptions 1* , 2 and 8, we have $\hat{\Gamma}^{(\ell)} \rightsquigarrow \mathbb{N}(0,1)$ under H_0.*

Proof. Under the null that $\theta_i^{0,(\ell)} = \theta^{0,(\ell)}$ for $\forall i \in \mathcal{N}$, we use Lemma D.5(i) to obtain that

$$\begin{aligned}\hat{\bar{\theta}}^{(\ell)} - \theta^{0,(\ell)} &= \frac{1}{n}\sum_{i\in\mathcal{N}}\left[\mathbb{E}\left(S_{ii}^{0,(\ell)}|\mathscr{D}\right)\right]^{-1}\xi_i^{(\ell)} \\ &+ \frac{1}{n^2}\sum_{i\in\mathcal{N}} a_{ii}^0\left[\mathbb{E}\left(S_{ii}^{0,(\ell)}|\mathscr{D}\right)\right]^{-1}\left(\hat{\theta}_i^{(\ell)} - \theta_i^{0,(\ell)}\right) + O_p\left(\frac{\log N}{N\wedge T}\right),\end{aligned} \tag{D.29}$$

such that

$$\left\|\frac{1}{n^2}\sum_{i\in\mathcal{N}} a_{ii}^0(\hat{\theta}_i^{(\ell)} - \theta_i^{0,(\ell)})\right\| \le \frac{1}{n}\max_{i\in\mathcal{N}}\left|a_{ii}^{(\ell)}\right|\left\|\hat{\theta}_i^{(\ell)} - \theta_i^{0,(\ell)}\right\| = O_p\left(\frac{\sqrt{\log N}}{N\sqrt{T}}\right),$$

with Lemma D.5(iii) and the fact that $\max_{i\in\mathcal{N}}\left|a_{ii}^0\right| = O(1)$. For the first term on the right side of (D.29), we have

$$\frac{1}{n}\sum_{i\in\mathcal{N}}\left[\mathbb{E}\left(S_{ii}^{0,(\ell)}|\mathscr{D}\right)\right]^{-1}\xi_i^{(\ell)} = \frac{1}{nT_\ell}\sum_{i\in\mathcal{N}}\sum_{t\in\mathcal{T}_\ell}\left[\mathbb{E}\left(S_{ii}^{0,(\ell)}|\mathscr{D}\right)\right]^{-1}\mathfrak{x}_{it}e_{it} = O_p(\frac{1}{\sqrt{nT}}),$$

by the central limit theorem for m.d.s., which yields that

$$\left\|\hat{\bar{\theta}}^{0,(\ell)} - \theta^{0,(\ell)}\right\| = O_p\left(\frac{\log N}{N\wedge T}\right). \tag{D.30}$$

Recall from (C.5) that $\hat{\Gamma}^{(\ell)} = \sqrt{n}\cdot\frac{\frac{1}{n}\sum_{i\in\mathcal{N}}\hat{\mathbb{S}}_i^{(\ell)} - p}{\sqrt{2p}}$. Note that

$$\begin{aligned}\hat{\mathbb{S}}_i^{(\ell)} &= T_\ell\left(\hat{\theta}_i^{(\ell)} - \hat{\bar{\theta}}^{(\ell)}\right)'\hat{S}_{ii}^{(\ell)}\left(\hat{\Omega}_i^{(\ell)}\right)^{-1}\hat{S}_{ii}^{(\ell)}\left(\hat{\theta}_i^{(\ell)} - \hat{\bar{\theta}}^{(\ell)}\right)\left(1 - \hat{a}_{ii}^{(\ell)}/n\right)^2 \\ &= T_\ell\left(\hat{\theta}_i^{(\ell)} - \theta^{0,(\ell)}\right)'\hat{S}_{ii}^{(\ell)}\left(\hat{\Omega}_i^{(\ell)}\right)^{-1}\hat{S}_{ii}^{(\ell)}\left(\hat{\theta}_i^{(\ell)} - \theta^{0,(\ell)}\right)\left(1 - \hat{a}_{ii}^{(\ell)}/n\right)^2 \\ &+ T_\ell\left(\hat{\bar{\theta}}^{(\ell)} - \theta^{0,(\ell)}\right)'\hat{S}_{ii}^{(\ell)}\left(\hat{\Omega}_i^{(\ell)}\right)^{-1}\hat{S}_{ii}^{(\ell)}\left(\hat{\bar{\theta}}^{(\ell)} - \theta^{0,(\ell)}\right)\left(1 - \hat{a}_{ii}^{(\ell)}/n\right)^2 \\ &- 2T_\ell\left(\hat{\theta}_i^{(\ell)} - \theta^{0,(\ell)}\right)'\hat{S}_{ii}^{(\ell)}\left(\hat{\Omega}_i^{(\ell)}\right)^{-1}\hat{S}_{ii}^{(\ell)}\left(\hat{\bar{\theta}}^{(\ell)} - \theta^{0,(\ell)}\right)\left(1 - \hat{a}_{ii}^{(\ell)}/n\right)^2 := \hat{\mathbb{S}}_{i,1}^{(\ell)} + \hat{\mathbb{S}}_{i,2}^{(\ell)} - \hat{\mathbb{S}}_{i,3}^{(\ell)}.\end{aligned}$$

Below we show that $\frac{1}{\sqrt{n}}\sum_{i\in\mathcal{N}}\hat{\mathbb{S}}_{i,2}^{(\ell)}$ and $\frac{1}{\sqrt{n}}\sum_{i\in\mathcal{N}}\hat{\mathbb{S}}_{i,3}^{(\ell)}$ are smaller terms and $\frac{1}{\sqrt{n}}\sum_{i\in\mathcal{N}}\hat{\mathbb{S}}_{i,1}^{(\ell)} \rightsquigarrow \mathcal{N}(0,1)$.

First, noted that

$$\begin{aligned}&\left|\frac{1}{\sqrt{n}}\sum_{i\in\mathcal{N}}\hat{\mathbb{S}}_{i,2}^{(\ell)}\right| \le \sqrt{n}T_\ell\left\|\hat{\bar{\theta}}^{(\ell)} - \theta^{0,(\ell)}\right\|^2\max_{i\in\mathcal{N}}\lambda_{\max}\left(\hat{S}_{ii}^{(\ell)}\left(\hat{\Omega}_i^{(\ell)}\right)^{-1}\hat{S}_{ii}^{(\ell)}\right)\max_{i\in\mathcal{N}}\left(1 - \hat{a}_{ii}^{(\ell)}/n\right)^2 \\ &= \sqrt{n}T_\ell O_p\left(\frac{(\log N)^2}{N^2\wedge T^2}\right)\max_{i\in\mathcal{N}}\left\|\hat{S}_{ii}^{(\ell)}\right\|^2\max_{i\in\mathcal{N}}\left\|\hat{\Omega}_i^{(\ell)}\right\|\max_{i\in\mathcal{N}}\left(1 - \hat{a}_{ii}^{(\ell)}/n\right)^2 \\ &= \sqrt{n}T_\ell O_p\left(\frac{(\log N)^2}{N^2\wedge T^2}\right)\left[\max_{i\in\mathcal{N}}\left\|S_{ii}^{0,(\ell)}\right\|^2\max_{i\in\mathcal{N}}\left\|\Omega_i^{0,(\ell)}\right\|\max_{i\in\mathcal{N}}\left(1 - a_{ii}^0/N\right)^2 + o_p(1)\right] = o_p(1),\end{aligned}$$

where the first equality is by (D.30), the second equality is by Lemma D.7 and the fact that $\max_{i\in\mathcal{N}}|\hat{a}_{ii}^{(\ell)} - a_{ii}^0| = o_p(1)$ owing to Lemma D.6(iii), and the last equality holds by the fact that $\max_{i\in\mathcal{N}}||S_{ii}^{0,(\ell)}|| = O(1)$, $\max_{i\in\mathcal{N}}||\Omega_i^{0,(\ell)}|| = O(1)$, and $\max_{i\in\mathcal{N}}|a_{ii}^{(\ell)}| = O(1)$ and Assumption 1* (vi).

Second, by analogous arguments as used above, we have

$$\left|\frac{1}{\sqrt{n}}\sum_{i\in\mathcal{N}}\hat{\mathbb{S}}_{i,3}^{(\ell)}\right| \le 2\sqrt{n}T_\ell\max_{i\in\mathcal{N}}\left\|\hat{\theta}_i^{(\ell)} - \theta^{0,(\ell)}\right\|\left\|\hat{\bar{\theta}}^{(\ell)} - \theta^{0,(\ell)}\right\|\max_{i\in\mathcal{N}}\lambda_{\max}\left(\hat{S}_{ii}^{(\ell)}(\hat{\Omega}_i^{(\ell)})^{-1}\hat{S}_{ii}^{(\ell)}\right)\max_{i\in\mathcal{N}}(1 - \hat{a}_{ii}^{(\ell)}/n)^2$$

$$= \sqrt{n} T_\ell O_p \left(\sqrt{\frac{\log N}{T}} \right) O_p \left(\frac{\log N}{N \wedge T} \right) = o_p(1).$$

At last for $\hat{\mathbb{S}}_{i,1}^{(\ell)}$, it's clear that

$$\frac{1}{\sqrt{n}} \sum_{i \in \mathcal{N}} \hat{\mathbb{S}}_{i,1}^{(\ell)} = \frac{1}{\sqrt{n}} \sum_{i \in \mathcal{N}} T_\ell (\hat{\theta}_i^{(\ell)} - \theta^{0,(\ell)})' \hat{S}_{ii}^{(\ell)} (\hat{\Omega}_i^{(\ell)})^{-1} \hat{S}_{ii}^{(\ell)} (\hat{\theta}_i^{(\ell)} - \theta^{0,(\ell)}) (1 - \hat{a}_{ii}^{(\ell)}/n)^2 = \frac{1}{\sqrt{n}} \sum_{i \in \mathcal{N}} \hat{z}_i^{(\ell)} + o_p(1),$$

$$\text{where} \quad \hat{z}_i^{(\ell)} = T_\ell (\hat{\theta}_i^{(\ell)} - \theta^{0,(\ell)})' S_{ii}^{0,(\ell)} (\Omega_i^{0,(\ell)})^{-1} S_{ii}^{0,(\ell)} (\hat{\theta}_i^{(\ell)} - \theta^{0,(\ell)}) \left(1 - a_{ii}^0/n\right)^2.$$

Then by the central limit theorem, we have $\hat{\Gamma}^{(\ell)} = \frac{1}{\sqrt{n}} \sum_{i \in \mathcal{N}} \frac{\hat{z}_i^{(\ell)} - p}{\sqrt{2p}} + o_p(1) \rightsquigarrow \mathbb{N}(0,1)$. ■

Lemma D.9 *Under Assumptions 1* , 2 and 8, we have* $|\hat{\Gamma}^{(\ell)}|/(\log N)^{1/2} \to \infty$ *under* H_1 *if* $\frac{T_\ell}{\sqrt{n}} \sum_{i \in \mathcal{N}} ||c_i^{(\ell)}||^2/(\log N)^{1/2} \to \infty$.

Proof. Noting that $\theta_i^{0,(\ell)} = \theta^{0,(\ell)} + c_i^{(\ell)}$, we have

$$\hat{\theta}_i^{(\ell)} - \hat{\bar{\theta}}^{(\ell)} = (\hat{\theta}_i^{(\ell)} - \theta_i^{0,(\ell)}) - (\hat{\bar{\theta}}^{(\ell)} - \bar{\theta}^{0,(\ell)}) + \theta_i^{0,(\ell)} - \theta^{0,(\ell)} = (\hat{\theta}_i^{(\ell)} - \theta_i^{0,(\ell)}) - (\hat{\bar{\theta}}^{(\ell)} - \bar{\theta}^{0,(\ell)}) + c_i^{(\ell)}.$$

Then

$$\begin{aligned}
\hat{\mathbb{S}}_i^{(\ell)} &= T_\ell (\hat{\theta}_i^{(\ell)} - \theta_i^{0,(\ell)})' \hat{S}_{ii}^{(\ell)} (\hat{\Omega}_i^{(\ell)})^{-1} \hat{S}_{ii}^{(\ell)} (\hat{\theta}_i^{(\ell)} - \theta_i^{0,(\ell)}) (1 - \hat{a}_{ii}^{(\ell)}/n)^2 \\
&\quad + T_\ell (\hat{\bar{\theta}}^{(\ell)} - \bar{\theta}^{0,(\ell)})' \hat{S}_{ii}^{(\ell)} (\hat{\Omega}_i^{(\ell)})^{-1} \hat{S}_{ii}^{(\ell)} (\hat{\bar{\theta}}^{(\ell)} - \bar{\theta}^{0,(\ell)}) (1 - \hat{a}_{ii}^{(\ell)}/n)^2 \\
&\quad - 2 T_\ell (\hat{\theta}_i^{(\ell)} - \theta_i^{0,(\ell)})' \hat{S}_{ii}^{(\ell)} (\hat{\Omega}_i^{(\ell)})^{-1} \hat{S}_{ii}^{(\ell)} (\hat{\bar{\theta}}^{(\ell)} - \bar{\theta}^{0,(\ell)}) (1 - \hat{a}_{ii}^{(\ell)}/n)^2 \\
&\quad + T_\ell c_i^{(\ell)\prime} \hat{S}_{ii}^{(\ell)} (\hat{\Omega}_i^{(\ell)})^{-1} \hat{S}_{ii}^{(\ell)} c_i^{(\ell)} (1 - \hat{a}_{ii}^{(\ell)}/n)^2 + 2 T_\ell (\hat{\theta}_i^{(\ell)} - \theta_i^{0,(\ell)})' \hat{S}_{ii}^{(\ell)} (\hat{\Omega}_i^{(\ell)})^{-1} \hat{S}_{ii}^{(\ell)} c_i^{(\ell)} (1 - \hat{a}_{ii}^{(\ell)}/n)^2 \\
&\quad - 2 T_\ell (\hat{\bar{\theta}}^{(\ell)} - \bar{\theta}^{0,(\ell)})' \hat{S}_{ii}^{(\ell)} (\hat{\Omega}_i^{(\ell)})^{-1} \hat{S}_{ii}^{(\ell)} c_i^{(\ell)} (1 - \hat{a}_{ii}^{(\ell)}/n)^2 := \sum_{m=4}^{9} \hat{\mathbb{S}}_{i,m}^{(\ell)}.
\end{aligned}$$

In the proof of Lemma D.8, we have already shown that

$$\frac{1}{\sqrt{n}} \sum_{i \in \mathcal{N}} \frac{\hat{\mathbb{S}}_{i,4}^{(\ell)} - p}{\sqrt{2p}} \rightsquigarrow \mathbb{N}(0,1) \quad \text{and} \quad \left| \frac{1}{\sqrt{n}} \sum_{i \in \mathcal{N}} \hat{\mathbb{S}}_{i,m}^{(\ell)} \right| = o_p(1), \; m = 5, 6.$$

As for $\hat{\mathbb{S}}_{i,7}^{(\ell)}$, we can show that

$$\begin{aligned}
\frac{1}{\sqrt{n}} \sum_{i \in \mathcal{N}} \hat{\mathbb{S}}_{i,7}^{(\ell)} &= \frac{T_\ell}{\sqrt{n}} \sum_{i \in \mathcal{N}} c_i^{(\ell)\prime} \hat{S}_{ii}^{(\ell)} (\hat{\Omega}_i^{(\ell)})^{-1} \hat{S}_{ii}^{(\ell)} c_i^{(\ell)} (1 - \hat{a}_{ii}^{(\ell)}/n)^2 \\
&= \frac{T_\ell}{\sqrt{n}} \sum_{i \in \mathcal{N}} \left[c_i^{(\ell)\prime} S_{ii}^{0,(\ell)} (\Omega_i^{0,(\ell)})^{-1} S_{ii}^{0,(\ell)} c_i^{(\ell)} \left(1 - a_{ii}^0/n\right)^2 + o_p(1) \right] \\
&\geq \frac{T_\ell}{\sqrt{n}} \sum_{i \in \mathcal{N}} \left[[\lambda_{\max}(S_{ii}^{0,(\ell)-1} \Omega_i^{0,(\ell)} S_{ii}^{0,(\ell)-1})]^{-1} \left\| c_i^{(\ell)} \right\|_2^2 \left(1 - a_{ii}^0/n\right)^2 + o_p(1) \right] \\
&\geq \frac{T_\ell}{\sqrt{n}} \sum_{i \in \mathcal{N}} \left[\frac{\left\| c_i^{(\ell)} \right\|_2^2 \left(1 - a_{ii}^0/n\right)^2}{\left\| S_{ii}^{0,(\ell)-1} \right\|^2 \left\| \Omega_i^{0,(\ell)} \right\|} + o_p(1) \right] \\
&= \frac{T_\ell}{\sqrt{n}} \sum_{i \in \mathcal{N}} \left[\frac{\left\| c_i^{(\ell)} \right\|_2^2}{\left\| S_{ii}^{0,(\ell)-1} \right\|^2 \left\| \Omega_i^{0,(\ell)} \right\|} + o_p(1) \right]
\end{aligned}$$

$$\geq \frac{1}{\max_{i\in\mathcal{N}}\left\|S_{ii}^{0,(\ell)-1}\right\|^2\left\|\Omega_i^{0,(\ell)}\right\|}\frac{T_\ell}{\sqrt{n}}\left[\sum_{i\in\mathcal{N}}\left\|c_i^{(\ell)}\right\|_2^2+o_p(1)\right]\to\infty \text{ at a rate faster than } (\log N)^{1/2},$$

where the second line is by the uniform convergence of $\hat{S}_{ii}^{(\ell)}$, $\hat{\Omega}_i^{(\ell)}$ and $\hat{a}_{ii}^{(\ell)}$, the fifth line is by the fact that $\max_{i\in\mathcal{N}}|\frac{a_{ii}^0}{n}| = o_p(1)$. and the last line draws from the assumption that $\frac{T_\ell}{\sqrt{n}}\sum_{i\in\mathcal{N}}||c_i^{(\ell)}||^2/(\log N)^{1/2}\to\infty$. By Cauchy's inequality, we observe that

$$\left|\frac{1}{\sqrt{n}}\sum_{i\in\mathcal{N}}\hat{\mathbb{S}}_{i,8}^{(\ell)}\right| \leq 2\sqrt{\frac{1}{\sqrt{n}}\sum_{i\in\mathcal{N}}\hat{\mathbb{S}}_{i,4}^{(\ell)}}\sqrt{\frac{1}{\sqrt{n}}\sum_{i\in\mathcal{N}}\hat{\mathbb{S}}_{i,7}^{(\ell)}} = o_p\left(\frac{1}{\sqrt{n}}\sum_{i\in\mathcal{N}}\hat{\mathbb{S}}_{i,7}^{(\ell)}\right),$$

$$\left|\frac{1}{\sqrt{n}}\sum_{i\in\mathcal{N}}\hat{\mathbb{S}}_{i,9}^{(\ell)}\right| \leq 2\sqrt{\frac{1}{\sqrt{n}}\sum_{i\in\mathcal{N}}\hat{\mathbb{S}}_{i,5}^{(\ell)}}\sqrt{\frac{1}{\sqrt{n}}\sum_{i\in\mathcal{N}}\hat{\mathbb{S}}_{i,7}^{(\ell)}} = o_p\left(\frac{1}{\sqrt{n}}\sum_{i\in\mathcal{N}}\hat{\mathbb{S}}_{i,7}^{(\ell)}\right).$$

Combining the above results yields that $|\hat{\Gamma}^{(\ell)}|/(\log N)^{1/2}\to\infty$. ■

E Algorithm for Nuclear Norm Regularization

To solve the optimization problem in (3.1), there are different algorithms in the literature such as the Alternating Direction Method of Multipliers (ADMM) algorithm and the singular value thresholding (SVT) procedure. Wang et al. (2022) provide the ADMM algorithm based on a quantile regression framework, which can be easily extended to the linear conditional mean regression framework. In this section, we focus on the SVT procedure for the case of low-rank estimation with two regressors. The case of more than two regressors is self-evident.

We can iteratively use SVT estimation to obtain the nuclear norm regularized regression estimates. Specifically, given Θ_1 and Θ_2, we solve for Θ_0 with

$$\Theta_0(\Theta_1,\Theta_2) = \arg\min_{\Theta_0}\|Y - X_1\odot\Theta_1 - X_2\odot\Theta_2 - \Theta_0\|_F + \nu_0 NT\|\Theta_0\|_*.$$

Given Θ_0 and Θ_2, we solve for Θ_1 with

$$\Theta_1(\Theta_0,\Theta_2) = \arg\min_{\Theta_1}\|Y - \Theta_0 - X_2\odot\Theta_2 - X_1\odot\Theta_1\|_F + \nu_1 NT\|\Theta_1\|_*.$$

Given Θ_0 and Θ_1, we solve for Θ_2 with

$$\Theta_2(\Theta_0,\Theta_1) = \arg\min_{\Theta_2}\|Y - \Theta_0 - X_1\odot\Theta_1 - X_2\odot\Theta_2\|_F + \nu_2 NT\|\Theta_2\|_*.$$

Specifically, the algorithm goes as follows:
Step 1: initialize Θ_0, Θ_1 and Θ_1 to be Θ_0^1, Θ_1^1 and Θ_1^1 and set $k=1$.
Step 2: let

$$\begin{aligned}
\Theta_0^{k+1} &= S_{\frac{\nu_0 NT}{2}}\left(Y - X_1\odot\Theta_1^k - X_2\odot\Theta_2^k\right),\\
\Theta_1^{k+1} &= S_{\frac{\tau\nu_1 NT}{2}}\left(\Theta_1^k - \tau X_1\odot\left(X_1\odot\Theta_1^k - Y + \Theta_0^{k+1} + X_2\odot\Theta_2^k\right)\right),\\
\Theta_2^{k+1} &= S_{\frac{\tau\nu_2 NT}{2}}\left(\Theta_2^k - \tau X_2\odot\left(X_2\odot\Theta_2^k - Y + \Theta_0^{k+1} + X_1\odot\Theta_1^{k+1}\right)\right),\\
k &= k+1,
\end{aligned}$$

where τ is the step size, and $S_\lambda(M)$ is the singular value operator for any matrix M and fixed parameter λ. By SVD, we have $M = U_M D_M V_M'$. Define $D_{M,\lambda}$ by replacing the diagonal entry $D_{M,ii}$ of D_M by $\max(D_{M,ii}-\lambda,0)$,

and then let $S_\lambda(M) = U_M D_{M,\lambda} V_M'$.
Step 3: repeat step 2 until convergence.

We can follow Chernozhukov et al. (2020), which gives the expression to pin down the step size τ. In addition, Proposition 2.1 of Chernozhukov et al. (2020) shows the convergence of the above algorithm.